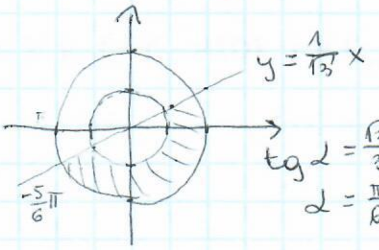
Zadanie 2

Oblicz całkę

a) $\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy \qquad D = \{(x,y):\ 1 \le x^2+y^2 \le 4 \ \text{i}\ y \le \frac{1}{\sqrt{3}}x\}$



$\operatorname{tg}\alpha = \frac{\sqrt{3}}{3}$

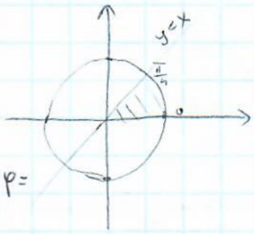
$\alpha = \frac{\pi}{6}$

$$\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy = \int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}}\left[\int_1^2 \sqrt{(\rho\cos\varphi)^2+(\rho\sin\varphi)^2}\,\rho\,d\rho\right]d\varphi =$$

$$= \int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}}\left[\int_1^2 \sqrt{\rho^2(\cos^2\varphi+\sin^2\varphi)}\,\rho\,d\rho\right]d\varphi = \int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}}\left[\int_1^2 \rho^2\,d\rho\right]d\varphi =$$

$$= \int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}}\left[\frac{\rho^3}{3}\right]_1^2 d\varphi = \int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}}\left[\frac{8}{3}-\frac{1}{3}\right]d\varphi = \frac{7}{3}\int_{-\frac{5}{6}\pi}^{\frac{\pi}{6}} d\varphi = \frac{7}{3}\varphi\Big|_{-\frac{5}{6}\pi}^{\frac{\pi}{6}} = \frac{7}{3}\left[\frac{\pi}{6}+\frac{5}{6}\pi\right] = \frac{7}{3}\pi$$

b) $\iint\limits_D e^{-x^2-y^2}\,dx\,dy$, gdzie $D = \{(x,y):\ x^2+y^2 \le 9,\ 0 \le y \le x\}$



$$\iint\limits_D e^{-x^2-y^2}\,dx\,dy = \int_0^{\frac{\pi}{4}}\left[\int_0^3 e^{(-(\rho\cos\varphi)^2-(\rho\sin\varphi)^2)}\,\rho\,d\rho\,d\varphi\right] =$$

$$= \int_0^{\frac{\pi}{4}}\left[\int_0^3 e^{-(\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)}\,\rho\,d\rho\,d\varphi\right] = \int_0^{\frac{\pi}{4}}\left[\int_0^3 e^{-\rho^2}\,\rho\,d\rho\right]d\varphi =$$

$t = -\rho^2$

$dt = -2\rho\,d\rho$

$\frac{dt}{-2} = \rho\,d\rho$

$\int e^{-\rho^2}\rho\,d\rho = \int e^t \frac{dt}{-2} = -\frac{1}{2}\int e^t\,dt = -\frac{1}{2}e^t = -\frac{1}{2}e^{-\rho^2}$

$$= \int_0^{\frac{\pi}{4}}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^3 d\varphi = \int_0^{\frac{\pi}{4}}\left[-\frac{1}{2}e^{-9}+\frac{1}{2}e^0\right]d\varphi =$$

$$= \int_0^{\frac{\pi}{4}}\left[-\frac{1}{2}e^{-9}+\frac{1}{2}\right]d\varphi = \left(-\frac{1}{2}e^{-9}+\frac{1}{2}\right)\varphi\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}\left(\frac{1}{2}-\frac{1}{2}e^{-9}\right)$$

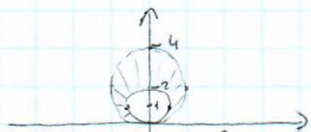
c) $\iint\limits_D xy^2\,dx\,dy$, gdzie D jest obszarem ograniczonym okręgami

$x^2+(y-1)^2 = 1 \quad \text{i} \quad x^2+y^2 = 4y$

$x^2+y^2 = 4y$

$x^2+y^2-4y = 0$

$x^2+(y-2)^2 = 4$



$\rho^2 = 2\rho\sin\varphi \qquad \rho^2 = 4\rho\sin\varphi$

$\rho = 2\sin\varphi \qquad \rho = 4\sin\varphi$

$$\iint\limits_D xy^2\,dx\,dy = \int_0^{\pi}\left[\int_{2\sin\varphi}^{4\sin\varphi}(\rho\cos\varphi)(\rho\sin\varphi)^2\,\rho\,d\rho\right]d\varphi =$$

$$= \int_0^{\pi}\left[\int_{2\sin\varphi}^{4\sin\varphi}\rho^4\cos\varphi\sin^2\varphi\,d\rho\right]d\varphi = \int_0^{\pi}\cos\varphi\sin^2\varphi\int_{2\sin\varphi}^{4\sin\varphi}\rho^4\,d\rho\,d\varphi =$$

$$= \int_0^{\pi}\cos\varphi\sin^2\varphi\left[\frac{1}{5}\rho^5\right]_{2\sin\varphi}^{4\sin\varphi} = \int_0^{\pi}\cos\varphi\sin^2\varphi\left(\frac{1}{5}(4\sin\varphi)^5-\frac{1}{5}(2\sin\varphi)^5\right)d\varphi =$$