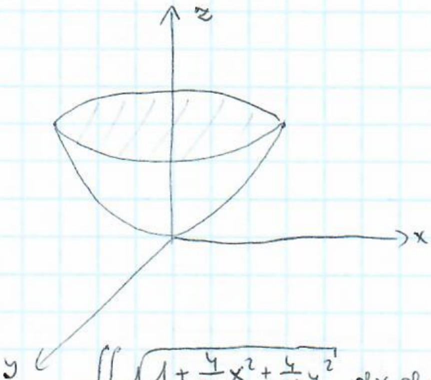
Zadanie 5.

Oblicz pole części paraboloidy $9z = x^2+y^2$ odciętej przez płaszczyznę $z = 1$



$9z = x^2+y^2$

$z = \frac{1}{9}(x^2+y^2)$

$z'_x = \frac{1}{9}\cdot 2x = \frac{2}{9}x$

$z'_y = \frac{1}{9}2y = \frac{2}{9}y$

dla $z=1$ $x^2+y^2=9$ (stąd granica 3)

dla $z=0$ $x^2+y^2=0$ (stąd granica 0)

$\sqrt{1+(z'_x)^2+(z'_y)^2} = \sqrt{1+\frac{4}{81}x^2+\frac{4}{81}y^2}\,dx\,dy$

$$\iint_D \sqrt{1+\frac{4}{81}x^2+\frac{4}{81}y^2}\,dx\,dy = \int_0^{2\pi}\left[\int_0^3 \sqrt{1+\frac{4}{81}\left[(\varrho\cos\varphi)^2+(\varrho\sin\varphi)^2\right]}\,\varrho\,d\varrho\right]d\varphi =$$

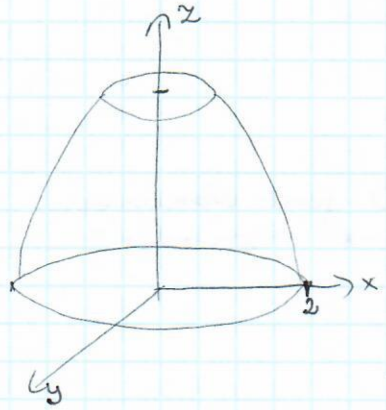
$$= \int_0^{2\pi}\left[\int_0^3 \sqrt{1+\frac{4}{81}\varrho^2}\,\varrho\,d\varrho\right]d\varphi = \left\{\begin{array}{l} t = 1+\frac{4}{81}\varrho^2 \\ dt = \frac{4}{81}\cdot 2\varrho\,d\varrho \\ \frac{81}{8}dt = \varrho\,d\varrho \\ \varrho = 0 \Rightarrow t = 1 \\ \varrho = 3 \Rightarrow t = 1+\frac{4}{81}\cdot 9 = \frac{13}{9} \end{array}\right\} = \int_0^{2\pi}\left(\frac{81}{8}\int_1^{13/9} t^{\frac{1}{2}}\,dt\right)d\varphi =$$

$$= \int_0^{2\pi}\frac{81}{8}\left(\frac{t^{3/2}}{\frac{3}{2}}\right)\Bigg|_1^{\frac{13}{9}}d\varphi = \frac{81}{8}\cdot\frac{2}{3}\cdot 2\pi\left(t^{3/2}\right)\Big|_1^{\frac{13}{9}} = \frac{27}{2}\pi\left(\left(\frac{13}{9}\right)^{3/2}-1\right) = \frac{27}{2}\pi\left(\frac{13\sqrt{13}}{27}-1\right) = \frac{\pi}{2}\left(13\sqrt{13}-27\right)$$

$\int_0^{2\pi} d\varphi = \varphi\Big|_0^{2\pi} = 2\pi$

Zadanie 6

Oblicz pole powierzchni części paraboloidy o równaniu $z = 4-x^2-y^2$ odciętej płaszczyznami $z=0$ i $z=3$



$z = 4-x^2-y^2$

dla $z=0$ $0 = 4-x^2-y^2 \Rightarrow x^2+y^2=4$ stąd gr. 2

dla $z=3$ $3 = 4-x^2-y^2$ $x^2+y^2=1$ stąd granica 1

$z'_x = -2x$

$z'_y = -2y$

$\sqrt{1+(z'_x)^2+(z'_y)^2} = \sqrt{1+4x^2+4y^2}$

tu takie granice bo po całym kole

$$\iint_D \sqrt{1+4x^2+4y^2}\,dx\,dy = \int_1^2\left[\int_0^{2\pi}\sqrt{1+4\left((\varrho\cos\varphi)^2+(\varrho\sin\varphi)^2\right)}\,d\varphi\right]\varrho\,d\varrho =$$

$$= \int_1^2\left[\int_0^{2\pi}\sqrt{1+4\varrho^2}\,d\varphi\right]\varrho\,d\varrho =$$

z tego całka jest $\varphi\Big|_0^{2\pi}$ czyli 2π

$$= 2\pi\int_1^2\sqrt{1+4\varrho^2}\,\varrho\,d\varrho = \left\{\begin{array}{l} 1+4\varrho^2 = t \\ 8\varrho\,d\varrho = dt \\ \varrho\,d\varrho = \frac{dt}{8} \\ \varrho = 1 \Rightarrow t = 5 \\ \varrho = 2 \Rightarrow t = 1+4\cdot4 = 17 \end{array}\right\} = 2\pi\int_5^{17}\frac{1}{8}\sqrt{t}\,dt =$$

$$= 2\pi\,\frac{1}{8}\,\frac{t^{3/2}}{3/2}\Bigg|_5^{17} = \frac{\pi}{4}\cdot\frac{2}{3}\,t^{3/2}\Big|_5^{17} = \frac{\pi}{6}\left(\sqrt{17^3}-\sqrt{5^3}\right) = \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$