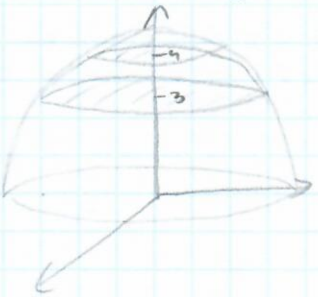
Zadanie 7

Oblicz pole powierzchni części sfery $x^2+y^2+z^2=25$ leżącej pomiędzy płaszczyznami $z=3$ i $z=4$



$$z=\sqrt{25-x^2-y^2}$$

$$z'_y=\frac{1}{\cancel{2}\sqrt{25-x^2-y^2}}\cdot(-\cancel{2}y)=\frac{-y}{\sqrt{25-x^2-y^2}}$$

$$z'_x=\frac{1}{\cancel{2}\sqrt{25-x^2-y^2}}\cdot(-2x)=\frac{-x}{\sqrt{25-x^2-y^2}}$$

$$\sqrt{1+(z'_x)^2+(z'_y)^2}=\sqrt{1+\left(\frac{-x}{\sqrt{25-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{25-x^2-y^2}}\right)^2}=\sqrt{1+\frac{x^2}{25-x^2-y^2}+\frac{y^2}{25-x^2-y^2}}=$$

$$=\sqrt{\frac{25-x^2-y^2+x^2+y^2}{25-x^2-y^2}}=\sqrt{\frac{25}{25-x^2-y^2}}=\frac{5}{\sqrt{25-x^2-y^2}}$$

gdy $z=3 \Rightarrow x^2+y^2+9=25$
$x^2+y^2=16$
$x^2+y^2=4^2$

gdy $z=4 \Rightarrow x^2+y^2+16=25$
$x^2+y^2=9$
$x^2+y^2=3^2$

(czyli granice mamy od 3 do 4)

$$\iint_D \frac{5}{\sqrt{25-x^2-y^2}}\,dx\,dy=\int_3^4\left[\int_0^{2\pi}\frac{5}{\sqrt{25-(\varrho\cos\varphi)^2-(\varrho\sin\varphi)^2}}\,\varrho\,d\varphi\right]d\varrho=$$

$$=2\pi\int_3^4\frac{5}{\sqrt{25-\varrho^2}}\varrho\,d\varrho=\left\{\begin{array}{l}t=25-\varrho^2\\ dt=-2\varrho\,d\varrho\\ \frac{dt}{-2}=\varrho\,d\varrho\end{array}\right\}=2\pi\cdot5\int_{16}^{9}t^{-\frac12}\frac{dt}{-2}=-5\pi\frac{t^{1/2}}{1/2}\Big|_{16}^{9}=$$

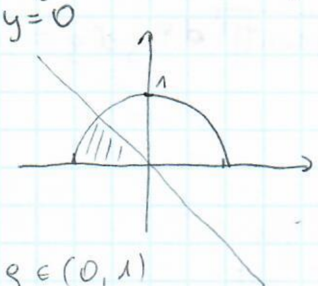
bo $\int_0^{2\pi}d\varphi=\varphi\Big|_0^{2\pi}=2\pi$

$\varrho=3\Rightarrow t=16$
$\varrho=4\Rightarrow t=9$

$$=-5\pi\cdot\frac{2}{1}\left(\sqrt{9}-\sqrt{16}\right)=-10\pi(3-4)=10\pi.$$

Zadanie 8

Oblicz całkę podwójną $\iint\sqrt{4-x^2-y^2}\,dx\,dy$ jeżeli D jest obszarem ograniczonym przez okrąg $x^2+y^2=1$, prostą $y=-x$ i prostą $y=0$



$\varrho\in(0,1)$
$\varphi\in(\frac34\pi;\pi)$

$$\int_0^1\left[\int_{\frac{3\pi}{4}}^{\pi}\sqrt{4-(\varrho\cos\varphi)^2-(\varrho\sin\varphi)^2}\,\varrho\,d\varphi\,d\varrho\right]=$$

$$=\int_0^1\left[\int_{\frac{3\pi}{4}}^{\pi}\sqrt{4-\varrho^2}\,d\varphi\right]\varrho\,d\varrho=\int_0^1\sqrt{4-\varrho^2}\,\varphi\Big|_{\frac{3\pi}{4}}^{\pi}\varrho\,d\varrho=$$

$$=\int_0^1\sqrt{4-\varrho^2}\left(\pi-\tfrac34\pi\right)\varrho\,d\varrho=\frac{\pi}{4}\int_0^1\sqrt{4-\varrho^2}\,\varrho\,d\varrho=\left\{\begin{array}{l}t=4-\varrho^2\\ dt=-2\varrho\,d\varrho\\ \frac{dt}{-2}=\varrho\,d\varrho\end{array}\right\}$$

$\varrho=0\Rightarrow t=4$
$\varrho=1\Rightarrow t=3$

$$=\frac{\pi}{4}\int_4^3 t^{\frac12}\frac{dt}{-2}=-\frac{\pi}{8}\int_4^3 t^{\frac12}\,dt=-\frac{\pi}{8_4}\cdot\frac{2}{3}t^{3/2}\Big|_4^3=$$

$$=-\frac{\pi}{12}\left(3\sqrt3-8\right)=\frac{\pi}{12}\left(8-3\sqrt3\right)$$