5G WIRELESS TECHNOLOGIES

Pulse-shaping Filters

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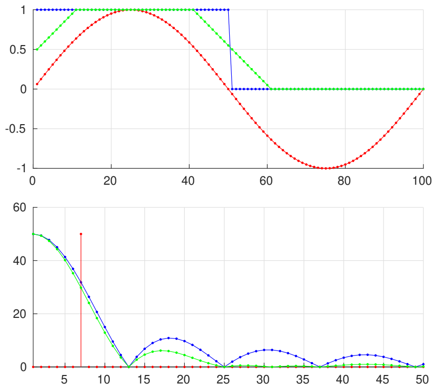
Why they are necessary?

- The sinc() function has multiple lobes and very slowly decaying.
- Spectral mask restrictions.
- Symbol length τ_s is a bottle neck.
- Time scaling property of the Fourier transform:

$$
\begin{aligned}
s(t) \quad &\Leftrightarrow \quad \dot{S}(\omega) \\
s(at) \quad &\Leftrightarrow \quad \frac{1}{a}\dot{S}\left(\frac{\omega}{a}\right)
\end{aligned}
$$

Solution: Symbols wider than τ_s!



Nyquist filtration

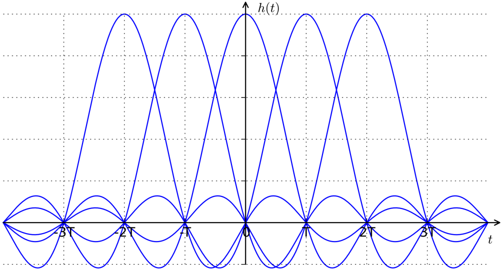
- Nyquist criterion:

$$h(n\tau_s) = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0 \end{cases}$$

- In frequency domain, it means:

$$\frac{1}{\tau_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{\tau_s}\right) = 1$$

- What does it mean?



Nyquist filter derivation

- Let us begin from the time-domain criterion:

$$h(n\tau_s) = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0 \end{cases}.$$

- Sample it—multiplying by the sequency of $\delta(t)$

$$h(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - k\tau_s) = \delta(t).$$

- Fourier transform of both sides:

$$H\left(f\right) * \frac{1}{\tau_s} \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{\tau_s}\right) = 1 \Rightarrow \frac{1}{\tau_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{\tau_s}\right) = 1$$

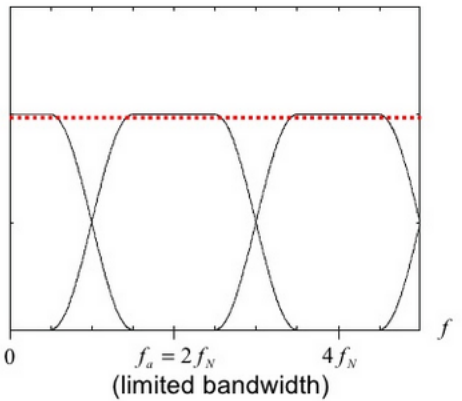
Meaning of the Nyquist criterion

$$\frac{1}{\tau_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{\tau_s}\right) = 1$$

- The sum is constant.
- Bands and symbols should overlay:

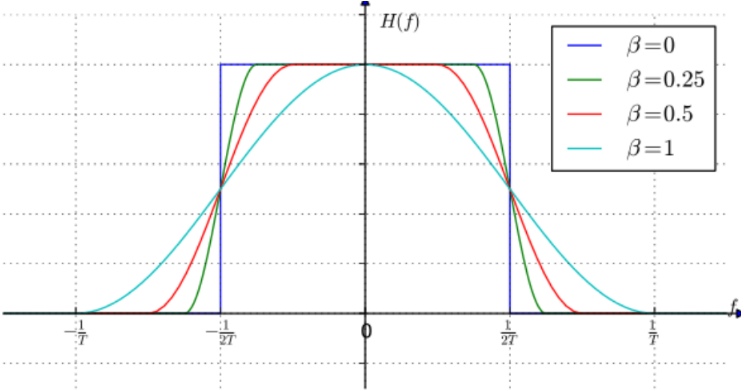
$$\Delta f > \frac{2}{\tau_s}.$$

- Frequency response should have even symmetry.
- Odd symmetry at $\dfrac{1}{2\tau_s}$.



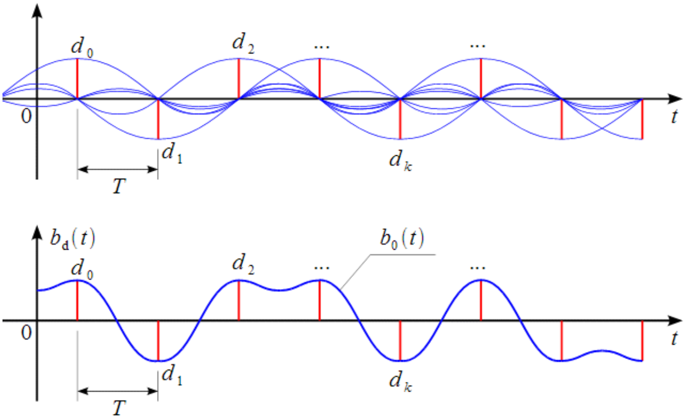
Raised cosine function

$$H(f) = \begin{cases} 1, & |f| \leq \frac{1-\beta}{2T} \\ \frac{1}{2}\left[1 + \cos\left(\frac{\pi T}{\beta}\left[|f| - \frac{1-\beta}{2T}\right]\right)\right], & \frac{1-\beta}{2T} < |f| \leq \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{cases}$$



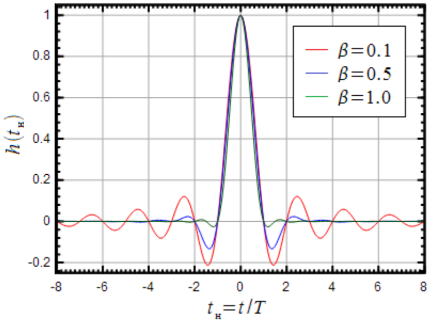
Symbol and sample rate

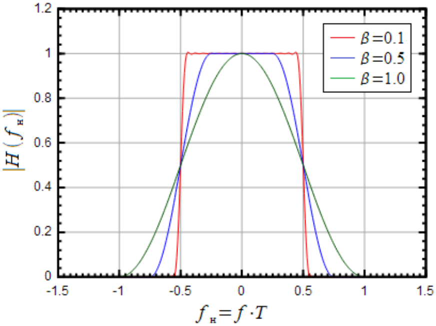
- Additional samples to perfrom filtration.
- Bandwidth $1/(2\tau_s)$.



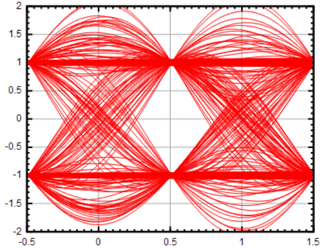
Roll-off factor I

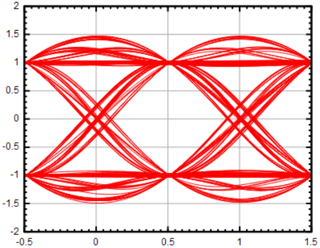
- Length of the filter vs. steepness.

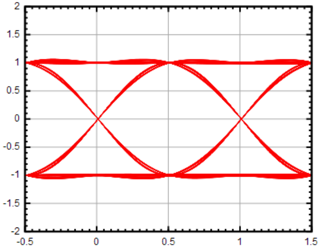


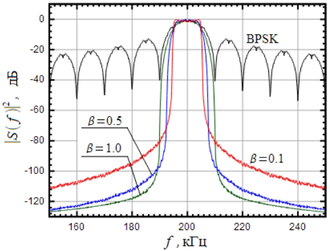


Roll-off factor II



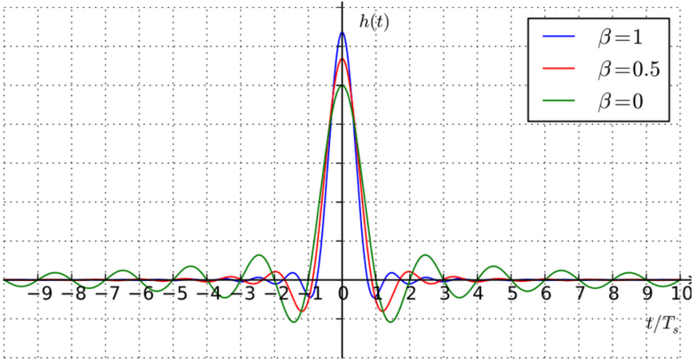






Root-raised-cosine filter

- Split single Raised-cosine filter into two parts: for the Rx and Tx.
- Frequency response $H_{rc}(f) = H_{rrc}(f)H_{rrc}(f)$, and thus $|H_{rrc}(f)| = \sqrt{|H_{rc}(f)|}$.



Filter requirements

Transmitter:

- Pulse-shaping filter;
- Band limitation and spectral mask requirements.

Receiver:

- Matched filter (with the transmitter signal);
- Noise and adjacent channel suppression;
- Should match Nyquist criterion.

Should they necessarily be identical?

Complex baseband

Problem definition

- For the simulation of a small number of symbols, a huge number of samples need to be processed.
- In digital implementations, it is necessary to lower the number of operations and power consumption.
- Sampling theorem: $f_s \geqslant 2f_{\text{max}}$.
- Arbitrary modulated signal:

$$s_{\text{mod}}(t) = S_m(t)\cos\Big(2\pi f_0 t + \varphi(t)\Big) \tag{1}$$

- How to get rid of the $2\pi f_0 t$ component? Fortunately, complex envelope theory is already developed!

Complex baseband

QAM modulation

QAM modulated signal:

$$s_{\mathsf{mod}}(t) = I(t)\cos 2\pi f_0 t - Q(t)\sin 2\pi f_0 t$$

Rewrite as following:

$$\begin{aligned} s_{\mathsf{mod}}(t) &= I(t)\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} - Q(t)\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} = \\ &= \frac{I(t) + jQ(t)}{2}e^{j2\pi f_0 t} + \frac{I(t) - jQ(t)}{2}e^{-j2\pi f_0 t} \end{aligned}$$

Conclusions:

- Both in-phase and quadrature components are real; therefore, spectrum must be symmetric.
- Task: get rid from the $e^{j2\pi f_0 t}$ multiplier.

Reference material:

$$\mathrm{e}^{jx} = \cos x + j\sin x \quad \cos x = \frac{e^{jx} + e^{-jx}}{2} \quad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

Complex baseband

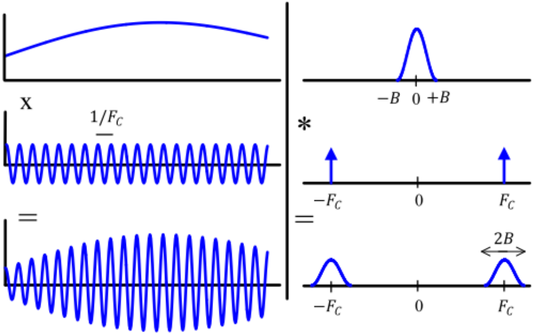
Problem definition II

- Transformation should be recoverable.
- Will not be symmetric—we can add corresponding imaginary part:

$$\acute{s}(t) = s(t) + j\hat{s}(t).$$

- **Idea:** similarly to the phasor (complex amplitude) approach in AC circuit analysis:

$$S_m \cos(\omega t + \varphi) \rightarrow S_m \,\mathrm{e}^{j\varphi}\,\mathrm{e}^{\omega t}\,.$$



Complex baseband

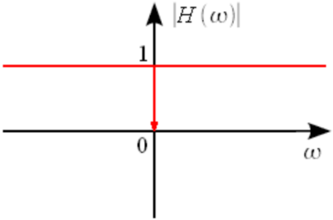
Hilbert filter I

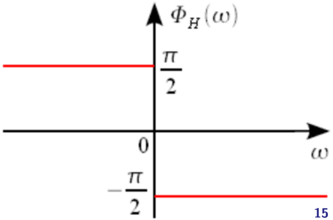
Analytic signal:

$$\begin{aligned}\dot{s}_{\text{mod}}(t) &= \frac{I(t)+jQ(t)}{2}\,\mathrm{e}^{j2\pi f_0 t} + \frac{I(t)-jQ(t)}{2}\,\mathrm{e}^{-j2\pi f_0 t} + \\ &+ jH\left\{\frac{I(t)+jQ(t)}{2}\,\mathrm{e}^{j2\pi f_0 t} + \frac{I(t)-Q(t)}{2}\,\mathrm{e}^{-j2\pi f_0 t}\right\}\end{aligned}$$

By the definition of the imaginary unit, $j^2 = -1$. To double positive frequency components and destroy negative frequency components the transformation should be:

$$H_h(f) = \begin{cases} j, & f < 0 \\ 0, & f = 0 \\ -j, & f > 0 \end{cases}$$





Complex baseband

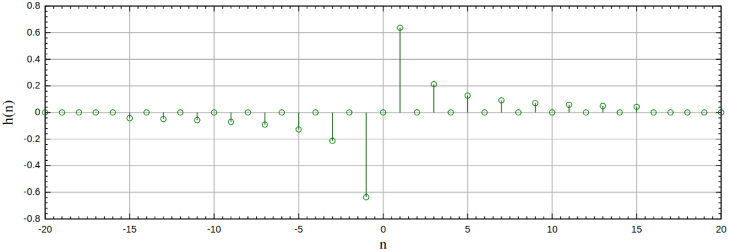
Hilbert filter II

To obtain impulse response we need to calculate Fourier transform from $H_h(f) = -j\text{sign}(f)$:

$$h(t) = -j \int_{-\infty}^{\infty} \text{sign}(f)\, e^{j2\pi f t}\, \mathrm{d}f = \frac{1}{\pi t}.$$

Thus, Hilbert transform:

$$\hat{s}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} \mathrm{d}\tau.$$



Necessary properties:

$$H\{\sin 2\pi f_0 t\} = -\cos 2\pi f_0 t$$

$$H\{S_m(t) \sin 2\pi f_0 t\} = -S_m(t) \cos 2\pi f_0 t$$

Complex baseband

QAM complex baseband

$$
\begin{aligned}
\dot{s}_{\text{mod}}(t) &= \frac{I(t)+jQ(t)}{2}\,\mathrm{e}^{j2\pi f_0 t} + \frac{I(t)-jQ(t)}{2}\,\mathrm{e}^{-j2\pi f_0 t} + \\
&+ \quad jH\left\{\frac{I(t)+jQ(t)}{2}\,\mathrm{e}^{j2\pi f_0 t} + \frac{I(t)-Q(t)}{2}\,\mathrm{e}^{-j2\pi f_0 t}\right\}
\end{aligned}
$$

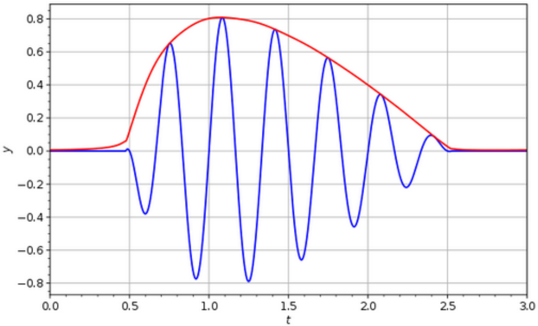
The analytic signal now becomes:

$$
\dot{s}_{\text{mod}}(t) = \Big(I(t)+jQ(t)\Big)\,\mathrm{e}^{j2\pi f_0 t},
$$

And the complex envelope is:

$$
\dot{S}_m(t) = I(t) + jQ(t),
$$

Conclusion: for QAM signals, complex envelope is simply the sum of the in-phase and quadrature channels.



Demodulation analog in complex baseband

To obtain $\dot{S}_m(t)$ from the $\dot{s}_{\mathsf{mod}}(t)$:

$$\dot{s}_{\mathsf{mod}}(t) = \Big(I(t) + jQ(t)\Big)\,\mathrm{e}^{j2\pi f_0 t},$$

it is sufficient to multiply:

$$\dot{s}_{\mathsf{mod}}(t)\,\mathrm{e}^{-j2\pi f_0 t} = \Big(I(t) + jQ(t)\Big)\,\mathrm{e}^{j2\pi f_0 t}\,\mathrm{e}^{-j2\pi f_0 t}\,.$$

- Filtration is not needed.
- If demodulation phase or frequency is not precise,

$$\dot{s}_{\mathsf{mod}}\,\mathrm{e}^{-j2\pi f_1 t} = \Big(I(t) + jQ(t)\Big)\,\mathrm{e}^{j2\pi (f_0 - f_1) t}\,.$$

Spectrum inversion I

The signal in baseband is:

$$\dot{S}_m(t) = \Big(I(t) + jQ(t)\Big) = S_m(t)\,\mathrm{e}^{j\varphi(t)},$$

where the instantaneous phase is :

$$\varphi(t) = \mathsf{atan}\frac{Q(t)}{I(t)}.$$

Assume $Q(t) \to -Q(t)$, then phase $\varphi(t) \to -\varphi(t)$.

- How does this effect the spectrum of the signal?

Fourier transform:

$$\dot{S}(\omega) = \int\limits_{-\infty}^{\infty} S_m(t)\,\mathrm{e}^{j\varphi(t)}\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t \quad \text{vs} \quad \dot{S}(\omega) = \int\limits_{-\infty}^{\infty} S_m(t)\,\mathrm{e}^{-j\varphi(t)}\,\mathrm{e}^{-j\omega t}\,\mathrm{d}t.$$

Spectrum inversion II

$$\dot{S}(\omega) = \int_{-\infty}^{\infty} S_m(t)\, \mathrm{e}^{j(\varphi(t)-\omega t)}\, \mathrm{d}t \quad \text{vs} \quad \dot{S}(-\omega) = \int_{-\infty}^{\infty} S_m(t)\, \mathrm{e}^{-j(\varphi(t)-\omega t)}\, \mathrm{d}t.$$

Split to real and imaginary part $\mathrm{e}^{jx} = \cos x + j \sin x$:

$$\dot{S}(\omega) = \int_{-\infty}^{\infty} S_m(t) \cos(\varphi(t) - \omega t)\mathrm{d}t + j \int_{-\infty}^{\infty} S_m(t) \sin(\varphi(t) - \omega t)\mathrm{d}t$$

$$\dot{S}(-\omega) = \int_{-\infty}^{\infty} S_m(t) \cos(\varphi(t) - \omega t)\mathrm{d}t - j \int_{-\infty}^{\infty} S_m(t) \sin(\varphi(t) - \omega t)\mathrm{d}t$$

Conclusion: Instantaneous phase $\varphi(t)$ sign inversion leads ot the signal spetrum inversion about $f_0 = 0$ frequency axis with complex conjugation:

$$S_m(t)\, \mathrm{e}^{j\varphi(t)} \leftrightarrow \dot{S}(\omega) \Rightarrow S_m(t)\, \mathrm{e}^{-j\varphi(t)} \leftrightarrow \dot{S}^*(-\omega)$$

Spectrum inversion III

What operations in the time domain lead to spectrum inversion?

$$\varphi(t) = \mathsf{atan}\frac{Q(t)}{I(t)}.$$

- In-phase channel inversion $I(t) \to -I(t)$, then phase $\varphi(t) \to -\varphi(t)$.
- Baseband complex conjugation $Q(t) \to -Q(t)$, then phase $\varphi(t) \to -\varphi(t)$.
- Swap of the channel $I(t) \to Q(t)$ and $Q(t) \to I(t)$, then

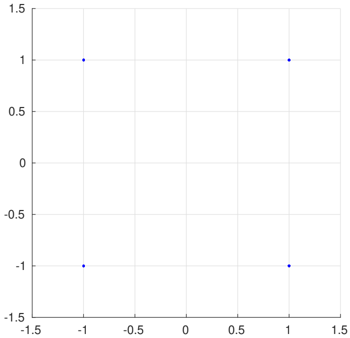
$$\varphi(t) \to \mathsf{atan}\frac{I(t)}{Q(t)} = \mathsf{acot}\frac{Q(t)}{I(t)} = \frac{\pi}{2} - \mathsf{atan}\frac{Q(t)}{I(t)} = \frac{\pi}{2} - \varphi(t)$$

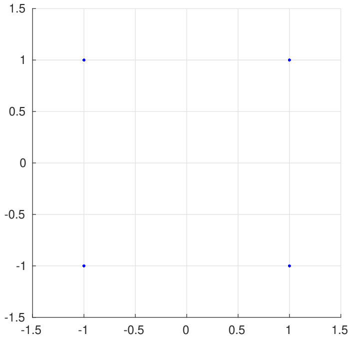
In-phase and quadrature channel swap inverts the spectrum and shifts the phase by $\pi/2$.

QPSK vs 4-QAM

Constellation

- Is it the same modulation?

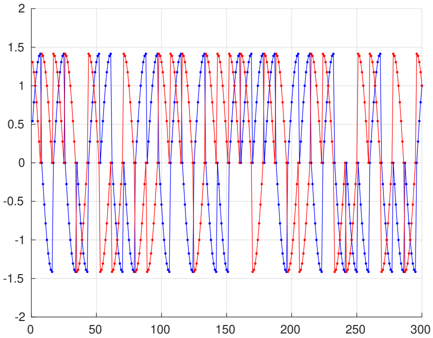


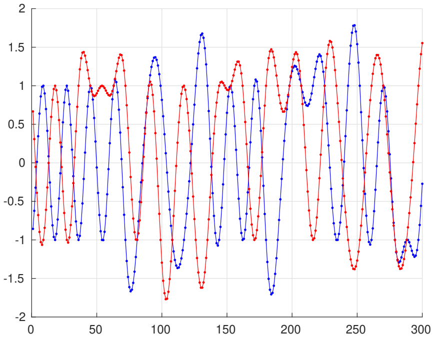


QPSK vs 4-QAM

Time diagrams

- Let's see time domain signals:

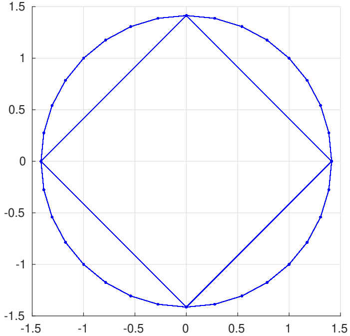


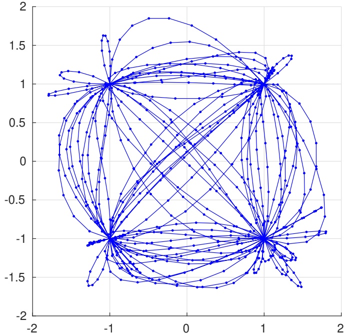


QPSK vs 4-QAM

Scatter plots

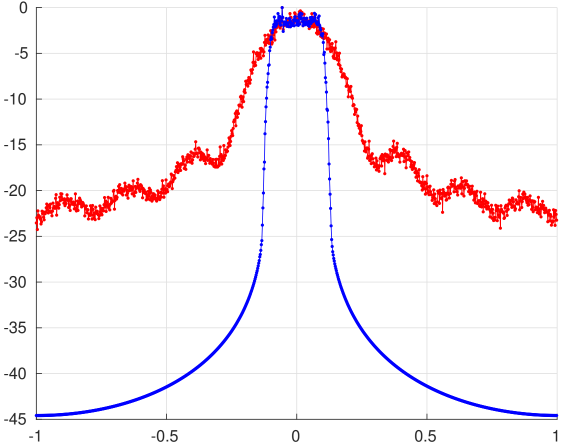
- Let us add to the constellation also sample between the symbols:





QPSK vs 4-QAM

Spectra



QPSK vs 4-QAM

Filters

- The answer is: it is the same modulation with the precision to pulse-shaping filter.