PAPER CODE	EXAMINER	DEPARTMENT	TEL
CPT 107	Ka Lok Man Gabriela Mogos	Discrete Mathematics and Statistics	1509

2023/24 SEMESTER 1 – Assessment II

BACHELOR DEGREE – Year 2

Discrete Mathematics and Statistics

INSTRUCTIONS TO CANDIDATES

1. **Total marks available are 100, accounting for 10% of the overall module marks.**
2. **The number in the column on the right indicates the marks for each question.**
3. **Answer all questions.**
4. **Answers should be written in English.**
5. **Relevant and clear steps should be included in your answers.**

QUESTION I: Functions and PHP (15 marks)

1). Let $f: \mathbb{R} \longrightarrow (-1, 1)$ defined by $f(x) = \frac{x}{1+x^2}$, $x \in \mathbb{R}$.
Find the inverse of above function if exists, where $\mathbb{R}$ is the set of real numbers.

(4 marks)

2). Let f(x) and g(x) are both even functions. Prove that $(f \circ g)$ is also an even function.

(4 marks)

3). Let A = {1, 2, 3, 4} and B = {l, 2, 3, 4, 5, 6}.

(a) How many functions are there from A to B?

(b) How many of these are one-to-one?

(c) How many of these are onto?

(3 marks)

4). Prove that if 101 integers are selected from the set S = {1, 2, 3, . . . , 200}, then there are two integers such that one divides the other.

(4 marks)

QUESTION II: Logic (40 marks)

1). Let p, q and r stand for the following propositions:

p: "*You get a grade B*."

q: "*You score at least 70% on the final exam*."

r: "*The average of your coursework is above 80%*."

Use p, q and r to translate the following into propositional logic:

(a) You get a grade B only if you score at least 70% on the final exam or the average of your coursework is above 80%. (3 marks)

(b) If you score at least 70% on the final exam or the average of your coursework is above 80%, then you will get a grade B. (3 marks)

2). State whether $(a \wedge b)$ and $\neg((a \wedge \neg b) \vee (\neg a \wedge b)) \wedge (a \vee b)$ are logically equivalent. If yes, prove it. If not, explain why.

(4 marks)

3). Consider the signature *S = {Cousin, Male, Female, jessie, carol, paul}* consisting of a binary predicate symbol *Cousin*, two unary predicate symbols *Male* and *Female*, and three constant symbols *jessie*, *carol* and *paul*. Assume that these symbols have the following meaning:

Cousin means "*is a cousin of*" (i.e., *Cousin (a, b)* states *a* is a cousin of *b*).

Male means "*is male*" (i.e., *Male(a)* states *a* is male).

Female means "*is female*" (i.e., *Female(a)* states *a* is female).

jessie, carol and *paul* refer to "Jessie", "Carol" and "Paul", respectively.

Translate the following sentences into S-formulae; that is, for each of the following sentences provide an S-formula that expresses the sentences:

(a) Jessie is a cousin of Carol. (3 marks)

(b) Paul has a female cousin. (3 marks)

(c) All cousins of Paul are also cousins of Carol. (3 marks)

(d) Jessie has at least 2 male cousins. (3 marks)

4). Without using any truth table, prove or disprove the following:

$(p \wedge \neg q) \rightarrow \neg q$ is a tautology (3 marks)

$(\neg p \vee \neg q) \wedge (\neg p \wedge q)$ is a contradiction (3 marks)

5). Translate the following first-order formulae into English.

(a) $\exists x(x \wedge \neg \forall y(x < y))$ (3 marks)

(b) $\forall x \forall y(x < y \rightarrow \exists z(x < z \wedge z < y)$ (3 marks)

(c) $\forall x \exists y (xy > x)$ (3 marks)

(d) $\neg \exists x \forall y(y \rightarrow x > y)$ (3 marks)

We use the mathematical symbols:
$>$ ($\geq$) to express *greater than (or equal to)*
$<$ ($\leq$) to express *less than (or equal to)*
x, y and z are real numbers

QUESTION III: Combinatorics (30 marks)

1). You can travel between the airport and the soccer stadium directly using either the bus, tram, metro or taxi; between the soccer stadium and the opera house directly using the bus, tram or taxi; and between the opera house and airport directly using the ferry or taxi.

(a) How many different ways altogether are there to travel from the airport to the opera house? (3 marks)

(b) How many different ways are there to travel from the airport to the opera house and back? (3 marks)

(c) How many different ways are there to travel from the airport to the opera house if any form of transport may be used at most once? (3 marks)

(d) How many different ways are there to travel from the airport to the soccer stadium if at most two different forms of transport may be used? (3 marks)

2). Find the number m of ways that 7 people can arrange themselves:

(a) In a row of chairs; (3 marks)

(b) Around a circular table. (3 marks)

3). In how many ways can

(a) I hang my 6 shirts and 4 suits in the wardrobe if every shirt must be hung on the left of every suit? (3 marks)

(b) I arrange, in a row, my 8 math books, 5 physics books and 7 chemistry books on my bookshelf if all books of each subject must be together? (3 marks)

4). A mathematics department receives 60 applications for 5 vacant PhD positions. Supposing that every applicant meets the requirements for each of the positions, in how many different ways can the department allocate the positions if:

(a) the positions are all different? (3 marks)

(b) the five positions consist of three identical positions in *Functional Analysis* and two identical positions in *Universal Algebra*? (3 marks)

QUESTION IV: Probability (15 marks)

1) In a committee meeting, there were 5 freshmen, 6 sophomores, 3 juniors, and 2 seniors. If a student is selected at random to be the chairperson, find the probability that the chairperson is a sophomore or a junior. (3 marks)

2) Three dice are rolled. Find the probability of getting three twos (2s) if it is known that the sum of the spots of the three dice was six. (3 marks)

3) Thomas will take three examinations in the coming few months. He will take the first examination in April. If he can pass that exam, then he will take the second examination in May. Similarly, if he can pass the second examination in May, and then he will take the third examination in June. If he fails an examination, he cannot take any others.

The probability that he can pass the first examination is A. If he passes the first examination, then the conditional probability that he can also pass the second examination is B. Finally, if he can pass both the first and the second examination, then the conditional probability that he can pass the third exam is C.

(a) What is the probability that Thomas can pass all three examinations? (3 marks)

(b) What is the probability that Thomas only passes the second examination? (3 marks)

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4) Find the expected number E(X) of correct answers obtained by guessing in a five-question true–false test. (3 marks)

END OF EXAM PAPER