AZZAM LABIB HAKIM

A Special Point on the Median

RWK

Malang, 1 Oktober 2018

Pada segitiga ABC dengan titik tinggi H, lingkaran dengan diameter AH memotong $\odot BHC$ pada X_A.

√ 1 • X_A berada pada *median* A.

Definisikan ω_B merupakan sebuah lingkaran yang melewatii A, B dan menyinggung garis BC, dengan cara yang sama definisikan ω_C.

√ 2 • ω_B dan ω_C berpotongan pada X_A.

Jika K titik pada $\odot ABC$ sehingga AK merupakan *symedian* dan P adalah titik tengah AK

3 • X_A merupakan *isogonal conjugate* dari P.

4 • X_A adalah pencerminan K terhadap BC.

Garis AK memotong BC di D. Garis tegak lurus dari D terhadap BC memotong *median* A pada S. Garis melalui S pararel terhadap BC memotong AB dan AC berturut-turut pada E dan F.

5 • X_A meurpakan perpotongan CE dan BF.

Lingkaran luar segitiga ABC berpusat di O dan titik tinggi dari segitiga DEF adalah H.

6 • O, H, dan X_A segaris.

Problems

1. In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

2. Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

3. Let $\triangle ABC$ be a scalene triangle and X, Y and Z be points on the lines BC, AC and AB, respectively, such that $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. The circumcircles of BXZ and CXY intersect at P. Prove that P lies on the circle with diameter HG where H and G are the orthocenter and the centroid, respectively, of triangle ABC.

4. A semicircle has center O and diameter AB. Let M be a point on AB extended past B. A line through M intersects the semicircle at C and D, so that D is closer to M than C. The circumcircles of triangles AOC and DOB intersect at O and K. Show that $\angle MKO = 90°$.

5. In $\triangle ABC$ with orthocenter H, suppose P is the projection of H onto the C-median; let the second intersections of AP, BP, CP with circumcircle $\triangle ABC$ be K, L, M respectively. Show that $MK = ML$.

6. Two circles ω_1 and ω_2, of equal radius intersect at different points X_1 and X_2. Consider a circle ω externally tangent to ω_1 at T_1 and internally tangent to ω_2 at point T_2. Prove that lines X_1T_1 and X_2T_2 intersect at a point lying on ω.

7. Let P be a point in the interior of an acute triangle ABC, and let Q be its isogonal conjugate. Denote by ω_P and ω_Q the circumcircles of triangles BPC and BQC, respectively. Suppose the circle with diameter $\overline{AP}$ intersects ω_P again at M, and line AM intersects ω_P again at X. Similarly, suppose the circle with diameter $\overline{AQ}$ intersects ω_Q again at N, and line AN intersects ω_Q again at Y.

   Prove that lines MN and XY are parallel.

AZZAM LABIB HAKIM Pak Suwono Senin, 1 Oktober 2019

TIM OLIMPIADE MATEMATIKA INDONESIA

PELATIHAN TAHAP SATU CALON PESERTA IMO

2019

Bidang Geometri

√ 1. Let P and Q be points of the side AB of the triangle ABC (with P between A and Q) such that $\angle ACP = \angle PCQ = \angle QCB$, and let AD be the angle bisector of $\angle BAC$. Line AD meets lines CP and CQ at M and N respectively. Given that $PN = CD$ and $3\angle BAC = 2\angle BCA$. Prove that triangles CQD and QNB have the same area.

√ 2. Let O be the center of circle Γ. Two equal chords AB and CD of Γ intersect at L such that $AL > LB$ and $DL > LC$. Let M and N be points on AL and DL respectively such that $\angle ALC = 2\angle MON$. Prove that the chord of Γ passing through M and N is equal to AB and CD.

√ 3. Let O be the center of the excircle of triangle ABC opposite A. Let M be the midpoint of AC, and let P be the intersection of lines MO and BC. Prove that if $\angle BAC = 2\angle ACB$ then $AB = BP$.

√ 4. In a right triangle ABC, we have $\angle A = 90°, \angle C = 30°$. Denote by Γ the circle passing through A which is tangent to BC at the midpoint. Assume that Γ intersects AC and the circumcircle of ABC at N and M respectively. Prove that $MN \perp BC$.

√ 5. The inscribed circle of triangle ABC touches BC, CA and AB at D, E and F respectively. Denote the perpendiculat feet from F, E to BC by K, L respectively. Let the second intersection of these perpendiculars with the incircle be M, N respectively. Show that

$$\frac{[BMD]}{[CND]} = \frac{DK}{DL}.$$

√ 6. In a triangle ABC, we have $\angle C = \angle A + 90°$. The point D on the continuation of BC is given such that $AC = AD$. A point E in the side of BC in which A does not lie is chosen such that $\angle EBC = \angle A, \angle EDC = \frac{1}{2}\angle A$. Prove that $\angle CED = \angle ABC$.

AZZAM LABIB HAKIM Pak Suwono Senin, 1 oktober 2018

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7. Two points X, Y lie on the arc BC of the circumcircle of triangle ABC (this arc does not contain A) such that $\angle BAX = \angle CAY$. Let M denotes the midpoint of the chord AX. Show that $BM + CM > AY$.

8. In a quadrilateral $ABCD$ we have $\angle B = \angle D = 60°$. Consider the line which is drawn from M, the midpoint of AD, parallel to CD. Assume this line intersects BC at P. A point X lies on CD such that $BX = CX$. Prove that $AB = BP \Leftrightarrow \angle MXB = 60°$.

9. An acuted-angled triangle ABC is given. The circle with diameter BC intersects AB, AC at E, F respectively. Let M be the midpoint of BC and P the intersection point of AM and EF. X is a point on the arc EF and Y the second intersection point of XP with circle mentioned above. Show that $\angle XAY = \angle XYM$.

10. The tangent line to circumcircle of the acute-angled triangle ABC ($AC > AB$) at A intersects the continuation of BC at P. We denote by O the circumcenter of ABC. X is a point OP such that $\angle AXP = 90°$. Two points E, F respectively on AB, AC at the same side of OP are chosen such that $\angle EXP = \angle ACX, \angle FXO = \angle ABX$. If K, L denote the intersection points of EF with the circumcircle of triangle ABC, show that OP is tangent to the circumcircle of triangle KLX.

11. A convex quadrilateral $ABCD$ is inscribed in a circle whose center O is inside the quadrilateral. Let $MNPQ$ be the quadrilateral whose vertices are the projections of the intersection point of the diagonals AC and BD onto the sides of $ABCD$. Prove that $2[MNPQ] \leq [ABCD]$.

12. Let B_1 and C_1 be points on the sides AC and AB of triangle ABC. Lines BB_1 and CC_1 intersect at point D. Prove that a circle can be inscribed inside quadrilateral AB_1DC_1 if and only if the incircles of the triangles ABD and ACD are tangent to each other.

Azzam Labib Hakim   Pak Suwono.   Senin, 1 oktober 2018

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13. The vertices A, B and C of an acute-angled triangle ABC lie on the sides B_1C_1, C_1A_1 and A_1B_1 of triangle $A_1B_1C_1$ such that $\angle ABC = \angle A_1B_1C_1, \angle BCA = \angle B_1C_1A_1$ and $\angle CAB = \angle C_1A_1B_1$. Prove that the orthocenters of the triangle ABC and triangle $A_1B_1C_1$ are equidistant from the circumcenter of triangle ABC.

14. Let ABC be a equilateral triangle of altitude 1. A circle, with radius 1 and the center on the same side of AB as C, rolls along the segment AB; as it rolls, it always intersects both AC and BC. Prove that the length of the arc of the circle that is inside the triangle remains constant.

√ 15. Let ABC be a triangle and D a point on the side AB. The incircles of the triangles ACD and CDB touch each other on CD. Prove that the incircle of ABC touches AB at D.

16. Two points P, Q lie on the side BC of triangle ABC and have the same distance to the midpoint. The perpendiculars from P, Q to BC intersects AC, AB at E, F respectively. Let M be the intersection point of PF and EQ. If H_1 and H_2 denote the orthocenters of the triangles BFP and CEQ respectively, show that $AM \perp H_1H_2$.

17. Let ABC be a triangle with $BC > CA > AB$. Choose points D on BD and E on BA such that $BD = BE = AC$. The circumcircle of triangle BED inersects AC at P and the line BP intersects the circumcircle of triangle ABC again at C. Prove that $AQ + QC = BP$.

√ 18. Points D and E are given on the sides AB and AC of triangle ABC such that DE parallel to BC and DE is tangent to the incircle of ABC. Prove that

$$DE \leq \frac{AB + BC + CA}{8}.$$

19. Let D be a point on side BC of triangle ABC such that $AD > BC$. Point E on side AC is defined by equation $\dfrac{AE}{EC} = \dfrac{BD}{AD - BC}$. Show that $AD > BE$.

AZZAM LABIB HAKIM Pak ~~Sugiono~~ Suwono Senin, 1 oktober 2018

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20. Let $ABCD$ be a convex quadrilateral, with $\angle BAC = \angle CAD$, and $\angle ABC = \angle ACD$. Rays AD and BC meet at E and rays AB and DC meet at F. Prove that
    (a) $AB \cdot DE = BC \cdot CE$
    (b) $AC^2 < \frac{1}{2}(AD \cdot AF + AB \cdot AE)$.

21. The point O is the centre of the circumcircle of triangle ABC. The line AO intersects the side BC in point N, and the line BO intersect the side AC in point M. Prove that, if $CM = CN$, then $AC = BC$.

22. Let ABC be triangle and let P denote the midpoint of side BC. Suppose that there exist two points M and N interior to the sides AB and AC respectively, such that $AD = DM = 2DN$, where D is the intersection of the lines MN and AP. Show that $AC = BC$.

√ 23. In the triangle ABC, X and Y are the midpoints of AB and AC respectively. On BC there is a point D, which is not the midpoint of BC. Prove that $\angle XDY = \angle BAC$ implies $AD \perp BC$.

24. Points K and M are chosen on the sides AB and BC of triangle ABC in such a way, that $AK = KM = MC$. Let N be the point of intersection of AM and CK, P is the foot of perpendicular from point N to the line KM, and Q is such point inside segment KM, that $MQ = KP$. Prove that incircle of KMB touches KM at point Q.

25. In a convex quadrilateral $ABCD, \angle ABC$ and $\angle BCD$ are not less than $120°$. Prove that $AC + BD > AB + BC + CD$.

26. Let the inner bisector of the angle A of triangle ABC intersects BC and the circumcircle of ABC in D and M respectively. We draw a line through D so that it intersects the rays MB and MC in P and Q. Show that $\angle PAQ \geq \angle A$.

AZZAM LABIB HAKIM Pak Suwono Selasa, 2 Oktober 2018

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27. Two circles Γ_1 and Γ_2 intersect at P and K. XY is the common tangent of them near to P, such that X is on Γ_1 and Y is on Γ_2. XP intersects Γ_2 for the second time at C, and YP intersects Γ_1 for the second time at B. A is the intersection of BX and CY. If Q is the second intersection point of circumcircle ABC and circumcircle AXY, prove that $\angle QXA = \angle QKP$.

28. Let $ABCD$ be a cyclic quadrilateral, and let E be the point of intersection of its diagonal AC and BD. Suppose AD and BC meet in F. Let the midpoints of AB and CD be G and H respectively. If Γ is the circumcircle of triangle EGH, prove that EF is tangent to Γ.

29. Let ABC be an arbitrary triangle. On the perpendicular bisector of AB, there is a point P inside of triangle ABC. On the sides BC and CA, triangles BQC and CRA are placed externally. These triangles satisfy $\triangle BPA \sim \triangle BQC \sim \triangle CRA$. So Q and A lie on opposite sides of BC, and B and R lie on opposite sides of AC. Show that the points P, Q, C, and R form a parallelogram.

30. The point P lies inside triangle ABC. Denote by O_A, O_B, O_C circumcenters of triangles PBC, PAC, PAB respectively. Let O_P be circumcenter of triangle $O_AO_BO_C$. Prove that the point P satisfy the condition $O_P = P$ if P is orthocenter of triangle ABC.

31. Acute-angled triangle ABC is given. On the perpendicular bisector to sides AB and BC respectively, points P and Q are chosen. Let M and N be the projections of P and Q onto AC. It turns out that $2MN = AC$. Prove that circumcircle of triangle PBQ passes through the circumcenter of ABC.

32. Point P lies inside triangle ABC. Let I_A, I_B, I_C be incenters of triangles PBC, PAC, PAB respectively. Prove that for point P which satisfy the condition $I_P = P$, where I_P denotes the incenter of triangle $I_AI_BI_C$, the following equalities hold: $AP - BP = AC - BC; BP - CP = BA - CA; CP - AP = CB - AB$.

AZZAM LABIB HAKIM Pak Soewono Selasa, 2 oktober 2018

TIM OLIMPIADE MATEMATIKA INDONESIA

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33. Let P and Q are two points inside parallelogram $ABCD$, that are symmetric with respect to the intersection point of the diagonals. Prove that circumcircles of triangles ABP, CDP, BCQ and ADQ have common point.

34. Given $ABCD$ be a quadrilateral inscribed in a circle with center O, and let $P = AC \cap BD$, $BC \nparallel= AD$. Rays AB and DC intersect in the point E. A circle with the center I inscribed in the triangle EBC and tangents to the line BC in point T_1. The excircle of the triangle EAD with center J tangents to the side AD in point T_2. The lines IT_1 and JT_2 intersect in the point Q. Prove that points O, P, Q are collinear.

35. Let h be the altitude of triangle ABC passing through the vertex A and $\alpha = \angle BAC$. Prove that the following inequality holds
$$AB + AC \geq BC \cos\alpha + 2h \sin\alpha$$
In what triangles does the equality hold?

36. Let O be the center of the incircle of triangle ABC. The points K and L are the intersection points of the circumcircles of triangles BOC and AOC respectively with the bisectors of the angles at A and B. P is the middle point of KL. M is symmetrical to O with respect to P, and N is symmetrical to O with respect to the line KL. Prove that the quadrilateral $KLMN$ is inscribed.

Sama kayak no. 21

37. The point O is the center of the circumcircle of triangle ABC. The line AO intersects the side BC in point N, and the line BO intersects the side AC in point M. Prove that if $CM = CN$, then $AC = BC$.

38. Let H be the orthocenter of triangle ABC, D be the midpoint of side BC. A line passing through the point H meets the sides AB, AC at points F, E respectively such that $AE = AF$. The ray DH meets the circumcircle of triangle ABC at point P. Prove that P, A, E, F are concyclic.

AZZAM LABIB HAKIM

RWK

Malang, 1st October 2018

1. Determine all pairs (a, b) of real numbers such that $a\lfloor bn \rfloor = b\lfloor an \rfloor$ for all positive integers n. (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

√ 2. Let m and n are natural numbers such that $m > 1$ and $2^{2m+1} \geq n^2$. Prove that
$$2^{2m+1} \geq n^2 + 7$$
Tinjau mod 8 untuk $2^{2m+1} = n^2, n^2+1, \ldots, n^2+3, n^2+5, \ldots, n^2+6$ (agar kontra)
Tinjau mod 16 untuk $2^{2m+1} = n^2+4$

√ 3. Denote by $P(n)$ the greatest prime divisor of n. Find all integers $n \geq 2$ for which
$$P(n) + \lfloor \sqrt{n} \rfloor = P(n+1) + \lfloor \sqrt{n+1} \rfloor$$

4. Find all function $f : \mathbb{N} \to \mathbb{N}$ such that for every natural numbers m, n the following condition hold
$$\gcd(f(m), n) = \gcd(m, f(n)).$$

5. Let $x, y, z \in \mathbb{R}^+$ be non-negative real numbers such that $0 \leq x, y, z \leq 1$. Find the maximum possible value of
$$x + y + z - xy - yz - zx.$$
Determine all triples (x, y, z) for which this maximum is attained.

6. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$. Prove that $f(n) = n$ for all natural number n.

7. Find all function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(f(n) - 1)) = n + 2 \ \forall n \in \mathbb{N}$$

√ 8. Let a, b, and c denote three distinct integers, and let P denote a polynomial having integer coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$. $b-c \mid P(a)-P(b)$ padahal $a-b \mid P(a)-P(b) \Rightarrow a=b=c$. ※

√ 9. Let $P(x)$ be a polynomial with integer coefficients. Prove that if there is an integer k such that none of the integers $P(1), P(2), \ldots, P(k)$ is divisible by k, then $P(x)$ does not have integer roots.

ambil a,b $P(a) \equiv P(b) \bmod k$, $P(a) \equiv P(b) \not\equiv 0 \bmod k$. Berarti misal $a = kn+r$ $k \mid P(kn+r) - P(r)$ padahal $P(r) \not\equiv 0 \bmod k$. dan $kn+r$ mencakup semua bilangan. Jadi tdk ada $P(x) = 0$.

√ 10. There are some cities in a country; one of them is the capital. For any two cities A and B there is a direct flight from A to B and a direct flight from B to A, both having the same price. Suppose that all round trips with exactly one landing in every city have the same total cost. Prove that all round trips that miss the capital and with exactly one landing in every remaining city cost the same.

11. In a club with 30 members, every member initially had a hat. One day each member sent his hat to a different member (a member could have received more than one hat). Prove that there exists a group of 10 members such that no one in the group has received a hat from another one in the group.

AZZAM LABIB HAKIM Pak Soewono Selasa, 2 oktober 2018

1 Teorema van Aubel

Misalkan diberikan segitiga ABC, dan misalkan pula bahwa P titik di dalam segitiga tersebut. Garis-garis AP, BP, CP memotong segmen BC, CA, AB berturut-turut di A_1, B_1, C_1. Maka buktikan bahwa:

a $\frac{AP}{PA_1} = \frac{AC_1}{C_1B} + \frac{AB_1}{B_1C}$

b $\frac{BP}{PB_1} = \frac{BA_1}{A_1C} + \frac{BC_1}{C_1A}$

c $\frac{CP}{PC_1} = \frac{CB_1}{B_1A} + \frac{CA_1}{A_1B}$

2 Theorema Gergonne - Euler

Misalkan diberikan segitiga ABC, dengan titik P di dalam segitiga tersebut. Garis - garis AP, BC, CP memotong segmen BC, CA, AB berturut - turut di A_1, B_1, C_1. Maka buktikan bahwa:

$$\frac{PA_1}{AA_1} + \frac{PB_1}{BB_1} + \frac{PC_1}{CC_1} = 1$$

AZZAM LABIB HAKIM

SESI MANDIRI

PEMBINAAN TAHAP I CALON PESERTA IMO 2019
@muhammadfaikar
2 Oktober 2018

Latihan Soal.

1. For a positive integer n let $p(n)$ denote the product of all primes less than or equal to n (or 1 if there are no such primes), and let $F(n)$ denote the largest integer k for which $p(k)$ divides n. Find the reminder when $F(1)+F(2)+...+F(p(2015)-1)+F(p(2015))$ is divided by 3999991.

2. Let $V_0 = \emptyset$ be the empty set and recursively define V_{n+1} to be the set of all $2^{|V_n|}$ subsets of V_n for each $n = 0, 1, ...$ For example
$$V_2 = \{\emptyset, \{\emptyset\}\}, V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$
A set $x \in V_5$ is called *transitive* if each element of x is a subset of x. How many such transitive sets are there?

3. Let k be a fixed natural number. A bijection $f : \mathbb{Z} \to \mathbb{Z}$ is such that if i and j are any integers satisfying $|i - j| \le k$, then $|f(i) - f(j)| \le k$. Prove that for any $i, j \in \mathbb{Z}$,
$$|f(i) - f(j)| = |i - j|.$$

4. Let x, y, z, t be the real numbers satisfying $xy + yz + zt + tx = 1$. Find the minimum of the expression $5x^2 + 4y^2 + 5z^2 + t^2$. $(2y^2+x^2)+(2y^2+z^2)+(4z^2+\frac{1}{2}t^2)+(4x^2+\frac{1}{2}t^2) \ge 2\sqrt{2}(\Sigma)=2\sqrt{2}.$

5. Let ABC be an acute-angled triangle with $\angle BAC = 60^o$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that
$$2\angle AHI = 3\angle ABC.$$
reflect A over BC to A'. $\angle AHI = \angle IBA'$.
BIHC cyclic.

6. Find all $n \in \mathbb{N}$ for which it is possible to partition the set $\{1, 2, ..., 3n\}$ into n three-element subsets a, b, c in which $b - a$ and $c - b$ are different numbers from the set $\{n - 1, n, n + 1\}$.

7. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote M and N the feet of the altitudes from B and C respectively. Denote by ω_1 the circumcircle of BWN, let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y, and H are collinier.

8. Find all positive integers m such that the fourth power of the number of positive divisors of m equals m.

9. Define $S_n = \left\{\binom{n}{n}, \binom{2n}{n}, \binom{3n}{n}, ..., \binom{n^2}{n}\right\}$, for $n \in \mathbb{N}$.

   a. Prove that there exist infinitely many composite natural numbers n such that S_n is not complete set of residues modulo n.

   b. Prove that there exist infinitely many composite natural numbers n such that S_n is complete set of residues modulo n.

10. We call polynomials $A(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ and $B(x) = b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0$, $(a_n b_m \neq 0)$ similar if the following conditions hold:

    i. $n = m$.

    ii. There is a permutation π of the set $\{0, 1, ..., n\}$ such that $b_i = a_{\pi(i)}$ for each $i \in 0, 1, ..., n$.

    Let $P(x)$ and $Q(x)$ be similar polynomials with integer coefficients. Given that $P(16) = 3^{2012}$, find the smallest possible value of $|Q(3^{2012})|$.

Azzam Labib Hakim

Rabu, 3 oktober 2018

SISTEM PERSAMAAN

Hery Susanto
Jurusan Matematika FMIPA UM
Jl. Semarang 5 Malang 65145
Email: susanto.um@gmail.com

Bentuk yang melibatkan variabel $x_1, x_2, \ldots, x_n$, yaitu

$$f(x_1, x_2, \ldots, x_n) = c$$

disebut persamaan dengan n variabel. Sistem persamaan adalah suatu sistem yang terdiri dari dua atau lebih persamaan, yaitu

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= c_1 \\ f_2(x_1, x_2, \ldots, x_n) &= c_2 \\ &\vdots \\ f_m(x_1, x_2, \ldots, x_n) &= c_m \end{aligned}$$

Sistem persamaan di atas disebut sistem persamaan dengan n variabel dan m persamaan. Solusi dari suatu sistem persamaan adalah solusi secara simultan dari semua persamaan di dalam sistem itu. Untuk soal-soal rutin disekolah, solusi suatu sistem persamaan biasanya dicari dengan menggunakan eliminasi dan atau substitusi. Cara ini tidak cukup untuk soal-soal nonrutin. Berikut diberikan contoh soal sistem persamaan yang dapat diselesaikan dengan menggunakan teori ketaksamaan.

Contoh 1 *Cari semua solusi real dari sistem persamaan*

$$\begin{aligned} x + \frac{2}{x} &= 2y \\ y + \frac{2}{y} &= 2z \\ z + \frac{2}{z} &= 2x. \end{aligned}$$

Penyelesaian: Misalkan (x, y, z) solusi sistem persamaan di atas. Diantara x, y, dan z tidak mungkin ada yang nol. Perhatikan bahwa jika salah satu positif maka dua yang lain juga positif. Selanjutnya, dengan mengalikan dengan -1 akan diperoleh solusi yang lain. Asumsikan $x, y, z > 0$.

e

Dengan menggunakan ketaksamaan AM-GM untuk masing-masing persamaan diperoleh

$$\begin{aligned} 2y &= x+\frac{2}{x} \geq 2\sqrt{x\left(\frac{2}{x}\right)} = 2\sqrt{2} \Longleftrightarrow y \geq \sqrt{2}, \\ 2z &= y+\frac{2}{y} \geq 2\sqrt{y\left(\frac{2}{y}\right)} = 2\sqrt{2} \Longleftrightarrow z \geq \sqrt{2}, \\ 2x &= z+\frac{2}{z} \geq 2\sqrt{z\left(\frac{2}{z}\right)} = 2\sqrt{2} \Longleftrightarrow x \geq \sqrt{2}. \end{aligned}$$

Dengan menambahkan semua persamaan dari sistem persamaan semula dan hasil di atas, diperoleh

$$3\sqrt{2} \leq x+y+z = 2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \leq 3\sqrt{2}.$$

Dengan demikan haruslah $x = y = z = \sqrt{2}$. Selanjutnya dapat ditunjukkan (dengan mengecek langsung pada soal) bahwa $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ dan $(-\sqrt{2}, -\sqrt{2}, -\sqrt{2})$ ketiga persamaan pada soal. Jadi $(\sqrt{2}, \sqrt{2}, \sqrt{2})$ dan $(-\sqrt{2}, -\sqrt{2}, -\sqrt{2})$ adalah solusi sistem persamaan pada soal.

Contoh 2 *Tentukan semua solusi real dari sistem persamaan*

$$\begin{cases} x^2+y^2+z^2 &= 1 \\ x^2y^2+y^2z^2+z^2x^2 &= 9x^2y^2z^2 \end{cases}$$

Penyelesaian: Dari persamaan pertama, tidak mungkin $x = y = z = 0$. Dari persamaan kedua tidak mungkin jika satu variabel nol dan dua variabel tidak nol.

Kasus I: Jika dua variabel nol dan satu variabel tidak nol. Misalkan $x = y = 0$, $z \neq 0$. Diperoleh $z = \pm 1$. Dengan demikian $(0,0,\pm 1)$ merupakan solusi. Dengan cara yang sama diperoleh $(0,\pm 1,0)$ dan $(\pm 1,0,0)$ juga merupakan solusi.

Kasus II: Jika ketiga variabel tidak nol. Persamaan kedua ekivalen dengan

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = 9.$$

Digunakan AM-HM, persamaan pertama, dan persamaan di atas diperoleh

$$\frac{1}{3} = \frac{x^2+y^2+z^2}{3} \geq \frac{3}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}} = \frac{3}{9} = \frac{1}{3}.$$

Jadi $\frac{x^2+y^2+z^2}{3} = \frac{3}{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}$. Oleh karena itu $x^2 = y^2 = z^2$. Diperoleh $(\pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{3}\sqrt{3})$ solusi sistem persamaan di atas.

Jadi solusi sistem persamaan pada soal adalah $(0,0,\pm 1)$, $(0,\pm 1,0)$, $(\pm 1,0,0)$, $(\pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{3}\sqrt{3}, \pm\frac{1}{3}\sqrt{3})$.

Contoh 3 *Cari semua solusi real dari sistem persamaan*

$$\begin{aligned} \frac{4x^2}{4x^2+1} &= y \\ \frac{4y^2}{4y^2+1} &= z \\ \frac{4z^2}{4z^2+1} &= x. \end{aligned}$$

Penyelesaian: Perhatikan fungsi $f : [0, \infty) \longrightarrow [0, \infty)$,

$$f(t) = \frac{4t^2}{4t^2 + 1},$$

merupakan fungsi monoton naik murni.Oleh karena itu jika $x < y$ maka $y = f(x) < f(y) = z$. Akibatnya $z = f(y) < f(z) = x$. Sehingga $x < y < z < x$, suatu yang tidak mungkin. Dengan cara yang sama jika $x > y$ maka akan diperoleh suatu kontradiksi. Jadi $x = y$. Dengan menggunakan argumen yang sama diperoleh $y = z$. Jadi $x = y = z$. Dengan menyelesaikan persamaan

$$\frac{4t^2}{4t^2 + 1} = t$$

diperoleh $t = 0$ atau $t = \frac{1}{2}$. Selanjutnya dapat ditunjukkan (dengan mengecek langsung pada soal) bahwa $(0, 0, 0)$ dan $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ ketiga persamaan pada soal. Jadi solusi dari sistem persamaan di atas hanyalah tripel $(0, 0, 0)$ dan $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Soal-soal Latihan

√ 1. Selesaikan sistem persamaan berikut.

$$\begin{aligned} x + y + u &= 4 \\ y + u + v &= -5 \\ u + v + x &= 0 \\ v + x + y &= -8 \end{aligned}$$

$x+y+u+v = \frac{4+(-5)+(-8)}{3} = -3$

$\Rightarrow (x,y,u,v) = (2,-3,5,-7)$

√ 2. Selesaikan sistem persamaan berikut.

$$\begin{aligned} (x + y)(x + z) &= 30 \\ (y + z)(y + x) &= 15 \\ (z + x)(z + y) &= 18 \end{aligned}$$

$\prod (x+y)^2 = 30 \cdot 15 \cdot 18 = 3^4 \cdot 2^2 \cdot 5^2 \Leftrightarrow \prod (x+y) = 3^2 \cdot 2 \cdot 5$

$\Rightarrow x+y=5,\ y+z=3,\ x+z=6$

$\Rightarrow x=4,\ y=1,\ z=2$

√ 3. Cari semua solusi real dari sistem persamaan berikut.

$$\begin{aligned} x^3 + y^3 &= 1 \\ x^4 + y^4 &= 1 \end{aligned}$$

$(1,0), (0,1)$

0. $(1,0), (0,1)$

I. $0 < x, y < 1$

$x^3 > x^4,\ y^3 > y^4$.

II. WLOG $-1 < x < 0 < y < 1$

$\cancel{X}$. misalin $a = -x$.

√ 4. Cari semua tripel bilangan bulat (x, y, z) yang memenuhi $x^3 = 2y^3 + 4z^3$.

$(0,0,0)$. ~~tidak ada~~. $x=2a, y=2b, z=2c$. trus infinite descent.

√ 5. Cari semua solusi real dari sistem persamaan berikut.

$$\begin{aligned} x^3 + y &= 3x + 4 \\ 2y^3 + z &= 6y + 6 \\ 3z^3 + x &= 9z + 8 \end{aligned}$$

$(x+1)^2(x-2) = 2-y$

$2(y+1)^2(y-2) = 2-z$

$3(z+1)^2(z-2) = 2-x$

$x=y=z=2$. if $x,y,z \notin \{2\}$ then $6(x+1)^2(y+1)^2(z+1)^2 = -1$ which is impossible.

√ 6. Cari semua tripel bilangan real (x, y, z) yang memenuhi $x^4 + y^4 + z^4 - 4xyz = -1$.

$x, y, z \notin \{0\}$. $x^4+y^4+z^4 = 4xyz - 1 > 0$. tiga, atau satu pasti real +. Boleh AM-GM.

$\sum x^4 \overset{\text{soal}}{<} 4xyz$. ~~$\sum x^4 \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{xyz}\,xyz$~~ $4xyz = \sum x^4 + 1 \overset{\text{AM-GM}}{\geq} 4xyz \Rightarrow x=y=z$ (asumsi +)

$(x,y,z) = (1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$.

7. Cari semua solusi real dari sistem persamaan

$$\begin{aligned} x+y &= \sqrt{4z-1} \\ y+z &= \sqrt{4x-1} \\ z+x &= \sqrt{4y-1} \end{aligned}$$

WLOG $x \geq y \geq z$

$0 \leq x - z = \sqrt{4z-1} - \sqrt{4x-1} \leq 0$, dst.

$\Rightarrow x = y = z$

$\Rightarrow (x,y,z) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

8. Buktikan tidak ada belangan real x, y, x yang memenuhi

~~trivial (0,0~~ $x, y, z \neq 0$.

$x^2, 2z^2, y^2 > 0$

$$\begin{aligned} x^2 + 4yz + 2z &= 0 \\ x + 2xy + 2z^2 &= 0 \\ 2xz + y^2 + y + 1 &= 0 \end{aligned}$$

$4yz + 2z = 2z(2y+1) < 0$

$x + 2xy = x(2y+1) < 0$

$y^2 + y + 1 = (y+1)^2 - y$

I. $2y+1 < 0 \Rightarrow y^2 + y + 1 > 0$ ※

II. $2y+1 > 0 \Rightarrow 2xz > 0$, $y^2 + y + 1 > \frac{7}{4} > 0$ ※

9. Carilah semua tripel bilangan real non-negatif (x, y, z) yang memenuhi sistem persamaan

$x, y, z \notin \{0\}$.

$$\begin{aligned} x^2y^2 + 1 &= x^2 + xy \\ y^2z^2 + 1 &= y^2 + yz \\ z^2x^2 + 1 &= z^2 + zx. \end{aligned}$$

$\rightarrow$ LHS $>$ RHS ※

eliminasi,
ketemu $y(x-z)[y(x+z-1)] = (x-y)(x+y)$ dan pers. cyc. lainnya.

kalo $x = y \Rightarrow y = z \Rightarrow x = y = z = 1$.

kalo ngga, $xyz \prod_{cyc} [y(x+z)-1] = -\prod_{cyc}(x+y) \Rightarrow$ WLOG $yx + yz - 1 < 0 \Rightarrow xy < 1 \Rightarrow x, y < 1$

10. Tentukan semua pasangan bilangan real (a, b, c) yang memenuhi sistem persamaan

$$\begin{aligned} 2a + \sqrt{(1-4b(c+1)} &= 1 \\ 2b + \sqrt{(1-4c(a+1)} &= 1 \\ 2c + \sqrt{(1-4a(b+1)} &= 1. \end{aligned}$$

$\Rightarrow -a^2 + a = bc + b$

$-b^2 + b = ac + c$

$-c^2 + c = ab + a$

$\Rightarrow (a+b)^2 + (b+c)^2 + (c+a)^2 = 0$

$\Rightarrow a = b = c = 0$.

11. Tentukan semua tripel bilangan real tak negatif (a, b, c) yang memenuhi sistem persamaan

$(1,1,0)$

$$\begin{cases} \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{2c}{a+b} = 2 \\ bc + ca + 2ab = a^2 + b^2 + c^2. \end{cases}$$

AZZAM LABIB HAKIM

Rabu, 3 oktober 2018

Pelatihan Olimpiade Matematika SMA
Fasilitator : Darmajid (DJ)
djdarmajid@gmail.com

THE EXTREMAL PRINCIPLE

Basic Fact :

- Every finite nonempty set A of nonnegative integers or real numbers has a minimal element min A and a maximal element max A, which need not be unique.
- Every nonempty subset of positive integers has a smallest element. This is called the well ordering principle, and it is equivalent to the principle of mathematical induction.
- An infinite set A of real numbers need not have a minimal or maximal element. If A is bounded above, then it has a smallest upper bound sup A. Read: supremum of A. If A is bounded below, then it has a largest lower bound inf A. Read: infimum of A. If sup $A \in A$, then sup A = max A, and if inf $A \in A$, then inf A = min A.

1. Into how many parts at most is a plane cut by n lines? $F_{n-1}+n=F_n$. $F_n=\frac{n^2+n+2}{2}$.
2. Into how many parts is space divided by n planes in general position? $F_n=2F_{n-1}$ $F_n=2^n$
3. (HMO 1973) Let $n \geq 5$. Show that, among the s_n space parts, there at least $(2n-3)/4$ tetrahedra.
4. There are n points given in the plane. Any three of the points form a triangle of area ≤ 1. Show that all n points lie in a triangle of area ≤ 4. asumsikan luas terbesar.
5. $2n$ points are given in the plane, no three collinear. Exactly n of these points are *farms* $F = \{F_1, F_2, \ldots, F_n\}$. The remaining n points are *wells*: $W = \{W_1, W_2, \ldots, W_n\}$. It is intended to build a straight line road from each farm to one well. Show that the wells can be assigned bijectively to the farms, so that none of the roads intersect.

   Perhatiin yang terdekat. kalo ada yang berpotongan bisa dituker

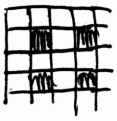
   F_1 F_2 W_1 W_2 $\to$ F_1 F_2 W_1 W_2
6. Let Ω be a set of points in the plane. Each point in Ω is a midpoint of two points in Ω. Show that Ω is an infinite set. tinjau titik terluar .
7. Prove that, in each convex pentagon, we can choose three diagonals fromwhich a triangle can be constructed.
8. Prove that in every tetrahedron, there are three edges meeting at the same vertex from which a triangle can be constructed.
9. Prove that each lattice point of the plane is labeled by a positive integer. Each of these numbers is the arithmetic mean of its four neighbors (above, below, left, right). Show that all the labels are equal.
10. There is no quadruple of positive integers (d, r, m, j) satisfying
    $$d^2 + j^2 = 3(r^2 + m^2).$$
    0,1 mod 3 ~~extremal~~ infinite descent
11. A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line. kalo nggak, pasti $|S|=\infty$.
12. Every road in Sikinia is one-way. Every pair of cities is connected exactly by one direct road. Show that there exists a city which can be re ached from every city directly or via at most one other city.

13. Rooks on an $n \times n \times n$ chessboard. Obviously n is the smallest number of rooks which can dominate an $n \times n$ chessboard. But what is the number R_n of rooks, which can dominate an $n \times n \times n$ -chessboard?
14. (AUO 1977, grade 8). Seven dwarfs are sitting around a circular table. There is a cup in front of each. There is milk in some cups, altogether 3 liters. One of the dwarfs shares his milk uniformly with the other cups. Proceeding counter-clockwise, each of the other dwarfs, in turn, does the same. After the seventh dwarf has shared his milk, the initial content of each cup is restored. Find the initial amount of milk in each cup.
15. The Sikinian Parliament consists of one house. Every member has three enemies at most among the remaining members. Show that one can split the house into two houses so that every member has one enemy at most in his house.
16. (IMO 1983). Can you choose 1983 pairwise distinct positive integers < 100000, such that no three are in arithmetic progression?
17. There exist three consecutive vertices D, M, J in every convex n-gon with $n \geq 3$, such that the circumcircle of ΔDMJ covers the whole n-gon.
18. Prove that $n\sqrt{2}$ is not an integer for any positive intege n.
19. Prove that there are at least $(2n—2)/3$ triangles among the p_n parts of the plane in Problem 1.
20. In the plane, n lines are given ($n \geq 3$), no two of them parallel. Through every intersection of two lines there passes at least an additional line. Prove that all lines pass through one point. andaikan ngga, nanti ada ∞ intersection . X
21. If n points of the plane do not lie on the same line, then there exists a line passing through exactly two points.
22. Start with several piles of chips. Two players move alternately. A move consists in splitting every pile with more than one chip into two piles. The one who makes the last move wins. For what initial conditions does the first player win and what is his winning strategy?
23. Does there exist a tetrahedron, so that every edge is the side of an obtuse angle of a face?
24. Prove that every convex polyhedron has at least two faces with the same number of sides.
25. $(2n + 1)$ persons are placed in the plane so that their mutual distances are different. Then everybody shoots his nearest neighbor. Prove that
(a) at least one person survives;
(b) nobody is hit by more then five bullets;
(c) the paths of the bullets do not cross; (no. 5)
(d) the set of segments formed by the bullet paths does not contain a closed polygon.

AZZAM LABIB HAKIM Rabu, 3 oktober 2018

Pelatihan Olimpiade Matematika SMA
Fasilitator : Darmajid (DJ)
djdarmajid@gmail.com

COLORING PROOF



2x2 tutup 1 item
1x4 tutup 0 atau 2 item

1. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.
2. Is it possible to form a rectangle with the five tetrominoes in Fig. 2.1? NO (warnain kayak catur)
3. Prove that a 10×10 chessboard cannot be covered by 25 T-tetrominoes in Fig. 2.1. These tiles are called from left to right: straight tetromino, T-tetromino, square tetromino, L-tetromino, and skew tetromino.

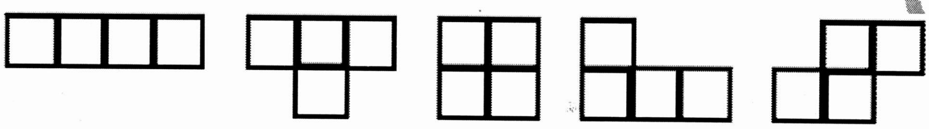


Fig. 2.1

4. An 8×8 chessboard cannot be coveredby 15 T-tetrominoes and one square tetromino.
5. A 10×10 board cannot be coveredby 25 straight tetrominoes (Fig. 2.1). $10 \equiv 2 \mod 4$. 4 warna
6. Consider an $n \times n$ chessboard with the four corners removed. For which values of n can you cover the board with L-tetrominoes as in Fig. 2.2?
7. Is there a way to pack 250 $1 \times 1 \times 4$-bricks into a $10 \times 10 \times 10$ box? no
8. An $a \times b$ rectangle can be covered by $1 \times n$ rectangles iff $n|a$ or $n|b$. QED a atau b warna
9. One comer of a $(n+1) \times (n+1)$ chessboard is cut off. For which n can you cover the remaining squares by 2×1 dominoes, so that half of the dominoes are horizontal?
10. Fig. 2.3 shows five heavy boxes which can be displaced only by rolling them about one of their edges. Their tops are labeled by the letter T. Fig. 2.4 shows the same five boxes rolled into a new position. Which box in this row was originally at the center of the cross?
11. Fig. 2.5 shows a road map connecting 14 cities. Is there a path passing through each city exactly once? No odd degree = 1, even degree = 0

jalannya 101010... atau 010101...
padahal ada 5, 0
9, 1

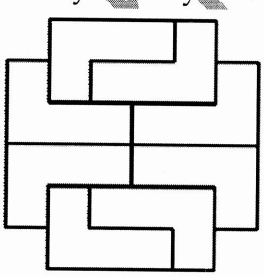


Fig. 2.2

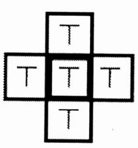


Fig. 2.3

Fig. 2.4

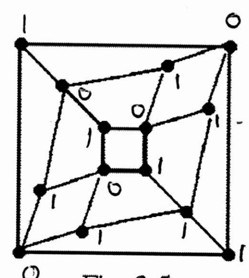


Fig. 2.5

12. A beetle sits on each square of a 9 x 9 chessboard. At a signal each beetle crawls diagonally onto a neighboring square. Then it may happen that several beetles will sit on some squares and none on others. Find the minimal possible number of free squares.

13. Every point of the plane is colored red or blue. Show that there exists a rectangle with vertices of the same color. Generalize! ramsey
14. Every space point is colored either red or blue. Show that among the squares with side 1 in this space there is at least one with three red vertices or at least one with four blue vertices.
15. Show that there is no curve which intersects every segment in Fig. 2.6 exactly once.

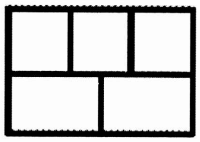


Fig. 2.6

16. On one square of a 5×5 chessboard, we write -1 and on the other 24 squares $+1$. In one move, you may reverse the signs of one ay: a subsquare with $a > 1$. My goal is to reach $+1$ on each square. On which squares should -1 be to reach the goal?
17. The points of a plane are colored red or blue. Then one of the two colors contains points with any distance.
18. The points of a plane are colored with three colors. Show that there exist two points with distance 1 both having the same color.
19. All vertices of a convex pentagon are lattice points, andits sides have integral length. Show that its perimeter is even.
20. $n \geq 5$ points of the plane can be colored by two colors so that no line can separate the points of one color from those of the other color.
21. You have many 1×1 squares. You may color their edges with one of four colors and glue them together along edges of the same color. Your aim is to get an $m \times n$ rectangle. For which m and n is this possible?
22. You have many unit cubes and six colors. You may color each cube with 6 colors and glue together faces of the same color. Your aim is to get a $r \times s \times t$ box, each face having different color. For which r, s, t is this possible?
23. Consider three vertices $A(0,0), B(0,1), C(1,0)$ in a plane lattice. Can you reach the fourth vertex $D(1,1)$ of the square by reflections at A, B, C or at points previously reflected?
24. Every space point is colored with exactly one of the colors red, green, or blue. The sets R, G, B consist of the lengths of those segments in space with both endpoints red, green, and blue, respectively. Show that at least one of these sets contains all nonnegative real numbers.
25. The Art Gallery Problem. An art gallery has the shape of a simple n-gon. Find the minimum number of watchmen needed to survey the building, no matter how complicated its shape.
26. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?
27. The vertices of a regular n-gon $A_1, \ldots, A_{2n}$ are partitioned into n pairs. Prove that, if $n = 4m + 2$ or $n = 4m + 3$, then two pairs of vertices are endpoints of congruent segments.
28. A 6×6 rectangle is tiled by 2×1 dominoes. Then it has always at least one *fault-line,* i.e., a line cutting the rectangle without cutting any domino.

SESI MANDIRI

PEMBINAAN TAHAP I CALON PESERTA IMO 2019
@muhammadfaikar
3 Oktober 2018

Latihan Soal.

1. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio $\frac{MN}{BC}$ in terms of R and r.

2. Prove that for any four nonnegative reals a, b, c, d the following inequaliy hold

$$(ab)^{1/3} + (cd)^{1/3} \leq ((a+c+b)(a+c+d))^{1/3}.$$

3. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2+b+c, b^2+c+a, c^2+a+b$ to be perfect squares. ~~kali mod 4~~ $a \geq b \geq c$

$(a+1)^2 = a^2+2a+1 > a^2+2a \geq a^2+b+c \geq a^2$ no squares between two squares.

4. Let k be any positive integer. Prove that there exist integers x and y, neither of which is divisible by 3, such that $x^2+2y^2 = 3^k$. $k=$ $x=y=1$. $k\geq 2$ ~~$x=(3k+1)^m, y=(3n+1)^m$~~

induction $k=n$, $x=x_0$, $y=y_0$. $k=n+1$, $x=x_0+2y_0$, $y=x_0-y_0$.

5. Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \text{ and } \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

$ABCD \sim ACDE \rightarrow BCD \sim CDE \rightarrow \angle PCD = \angle PBC \wedge \angle ABC = \angle ACD \rightarrow \angle ABP = \angle ACP \rightarrow ABCP$ cyclic tangent at C. $\rightarrow APDE$ tangent at D $\rightarrow$ POP $\rightarrow$ done.

6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers such that $a_0 = 1$, $a_n = 2 + \sqrt{a_{n-1}} - 2\sqrt{1+\sqrt{a_{n-1}}}$. Calculate $a_n = \left(\sqrt{1+\sqrt{a_{n-1}}} - 1\right)^2 \Rightarrow a_n = \left(\sqrt[n]{1+\sqrt{a_0}} - 1\right)^2 = \left(\sqrt[n]{2} - 1\right)^2$

$b_n = \sqrt[n]{1+\sqrt{a_n}} = \sqrt[n]{2} \Rightarrow a_n = (b_n - 1)^2 = b_n^2 - 2b_n + 1 = 2^{\frac{2}{n}} - 2^{\frac{2}{n}+1} + 1$

$$\sum_{n=0}^{2019} a_n 2^n = \sum_{n=0}^{2019} \left(2^2 - 2^{\frac{n^2+2n}{2}} + 2^n\right) + 1 =$$

7. Find the greatest positive integer n such that there are n different real numbers $x_1, x_2, \ldots, x_n$ satisfying that for any $1 \leq i < j \leq n$, the inequality $(1 + x_i x_j)^2 \leq 0.99(1+x_i^2)(1+x_j^2)$ hold.

8. For a positive integer m denote by $S(m)$ and $P(m)$ the sum and the product, respectively, of the digits od m. Show that for each positive integer n, the exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following conditions:

$$S(a_1) < S(a_2) < \ldots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \qquad (i = 1, 2, . , n)$$

(we let $a_{n+1} = a_1$).

9. Let $S = \{2, 3, 4, ...\}$ denote the set of integers that are greater than or equal to 2. Does there exist a function $f : S \to S$ such that $f(a)f(b) = f(a^2b^2)$ for all $a, b \in S$,with $a \neq b$?

10. We have $n \geq 2$ lamps $L_1, ..., L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows:
if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off;
otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on.

    a. Prove that there are infinitely many integers n for which all the lamps will eventually be off. ~~A genap~~ $n=2^k$ $10\ldots0 \to 00\ldots0110\ldots0 \to 111\ldots111 \to 000\ldots00$
    operasi pencerminan

    b. Prove that there are infinitely many integers n for which the lamps will never be all off. n ganjil mirip a.

a. induksi, operasi pencerminan untuk $n=2$, ...

AZZAM LABIB HAKIM   Kamis, 4 oktober 2018   Pak Purwanto Hal. 1

• Ada berapa barisan biner m suku yang memuat tepat n urutan 01?

$1 a_1 a_2 \dots a_m 0$ $1 \dots \underbrace{01 \dots 01 \dots 01}_{n} \dots 0$ banyaknya $\binom{m+1}{2n+1}$

hitung sequence naik turun. Tempatin 1 di depan, 0 dibelakang. No matter what, pasti pertamanya turun, yang terakhirnya naik. Jadi sekarang kita hitung cara penempatan 2n+2 naik-turunnya. Karena tepat ada m+2 suku ~~naik dan m+2 turun~~, make the moivre $\binom{m+2-1}{2n+2-1} = \binom{m+1}{2n+1}$

• Susun 15 huruf. 5 A, B, C. A,B,C ngga boleh di 5 depan, 5 tengah, 5 akhir.

$\binom{5}{5-x} = \binom{5}{x} = \binom{5}{y} = \binom{5}{5-y}$

y b's z c's x a's 5-z c's 5-x a's 5-y b's

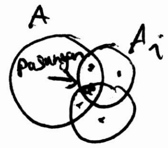
$\to \sum_{i=0}^{5} \binom{5}{i}^3$ ✓

• Ambil 3 chip merah dan 2 putih. Peluang terambil 2 putih (habis itu ngga jalan lagi) (jadi masih kesisa merah)? Peluang terakhir merah ---- R sama kayak peluang pertama merah R----. Jadi sama aja kayak peluang ngambil 3 merah diantara 5 chip $\to \frac{3}{5}$.

• $\exists$ n boys n girls. Each boys and girls like x girls and y boys. (x,y) such that $\exists$ like each other? $x+y>n$. n^2 semua pasangan 1 arah kalo udah $>n^2$ ada yang saling suka. $na+nb$ banyaknya semua pasangan satu arah. Haruslah $na+nb > n^2 \Rightarrow a+b > n$.

• X himpunan. $|X|=n$. $A_1, A_2, \dots, A_n \subseteq X$. $|A_i|=3$. $|A_i \cap A_j| \le 1$. Prove $\exists A \subseteq X$, $|A| \ge \lfloor\sqrt{2n}\rfloor$, $A_i \nsubseteq X$.

Misal $|A|=k$, $|A'|=n-k$.



Misal k maksimal. Agar k maksimal, haruslah 1 elemen A_i di luar, dua $\in A$. Maka, untuk setiap satu anggota $A_i \notin A$, ada sepasang anggota, sehingga

$n-k \le \binom{k}{2} \to 2n \le k^2+k < (k+1)^2 \to \sqrt{2n} < k+1 \to \lfloor\sqrt{2n}\rfloor \le k$. □

AZZAM LABIB HAKIM sesi malam Vasya

Harmonic Division
RWK-MYBU
Malang, 4th Oktober 2018

1. Let ABC be a triangle, and let D, E, F be the points of tangency of the incircle of triangle ABC with the sides BC, CA and AB respectively. Let X be in the interior of ABC such that the incircle of XBC touches XB, XC and BC in Z, Y and D respectively. Prove that $EFZY$ is cyclic.

   solution

   Denote $T = BC \cap EF$. Because of the concurency of the lines AD, BE, CF of triangle ABC (*Gergonne* point), we deduce that the division (T, B, D, C) is harmonic. Similarly, the lines XD, BY and CZ are concurrent in the Gergonne point of triangle XBC, so $T \in YZ$.

   Now expressing the power of point T with respect to the incircle of triangle ABC and the incircle of triangle XBC we have that $TD^2 = TE.TF$ and $TD^2 = TZ.TY$. So $TE.TF = TZ.TY$, therefore the quadrilateral $EFZY$ is cyclic.

2. Let $ABCD$ be a convex quadrilateral. Let $E = AB \cap CD$, $F = AD \cap BC$, $P = AC \cap BD$, and let O the foot of the perpendicular from P to the line EF. Prove that $\angle BOC = \angle AOD$.

   solution

   Denote $S = AC \cap EF$ and $T = BD \cap EF$. We know that the division (E, T, F, S) is harmonic. Furthermore, the division (A, P, C, S) is also harmonic, due to the pencil $B(E, T, F, S)$.

   But now, the pencil $E(A, P, C, S)$ is harmonic, so by intersecting it with the line BD, it follows that the four-point (B, P, D, T) is harmonic. Therefore, the pencil $O(A, P, C, S)$ is harmonic and $OP \perp OS$, thus by *harmonic-lemma*, $\angle POA = \angle POC$. Similarly, the pencil $O(B, P, D, T)$ is harmonic and $OP \perp OT$, thus again by *harmonic-lemma*, $\angle POB = \angle POD$. It follows that $\angle AOD = \angle BOC$.

3. Let ABC be a right triangle with $\angle A = 90°$ and let D be a point on side AC. Denote by E the reflection of A across the line BD and F the intersection point of CE with the perpendicular to BC at D . Prove that AF, DE and BC are concurrent.

   solution

Denote the points $X = AE \cap BD, Y = AE \cap BC, Z = AE \cap DF$ and $T = DF \cap BC$. In triangle AEC and for the cevians AF and ED, we observe that the lines $AF, DEandBC$ are concurrent if and only if the division (A, Y, E, Z) is harmonic.

Since the quadrilateral $XYTD$ is cyclic, $tan(XYB) = tan(XDZ)$, which is equivalent to $\frac{XB}{XY} = \frac{XZ}{XD}$. So $XB.XD = XY.XZ$.

Since triangles XAB and XDA are similar, we have that $XA^2 = XB.XD$, so $XA^2 = XY.XZ$. Using $XA = XE$, we obtain that $\frac{YA}{YE} = \frac{ZA}{ZE}$, and thus the division (A, Y, E, Z) is harmonic.

11. In $\triangle ABC$, I is the incentre, and E is the excentre opposite A. Suppose the excircle opposite A touches BC at F. Let AD be the altitude from A, with midpoint M. Prove that F, I, and M are collinear.

12. Given that AB is a diameter of a circle and the lines CD, CB are tangents to the circle, prove that $DE = EF$, where F is the foot of perpendicular from D onto AB and $E = DF \cap CA$.

13. Let ABC be a triangle with orthocenter H and let D, E, F be the feet of the altitudes lying on sides BC, CA, AB respectively. Let P be the intersection of the lines EF and BC. Suppose M is the midpoint of BC. Prove that the line PH is perpendicular to AM.

14. Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

15. In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

16. Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

17. Let $\triangle ABC$ be isosceles triangle with $AC = BC$. The point D lies on the extension of AC beyond C and is that $AC > CD$. The angular bisector of $\angle BCD$ intersects BD at point N and let M be the midpoint of BD. The tangent at M to the circumcircle of triangle AMD intersects the side BC at point P. Prove that points A, P, M and N lie on a circle.

Problems

1. **[Useful]** M is the midpoint of a line segment AB. Let $P\infty$ be a point at infinity on line AB. Prove that $(M, P\infty; A, B)$ is harmonic.

2. **[Useful]** Points A, C, B, D are on a line in this order, so that $(A, B; C, D)$ is harmonic. Let M be the midpoint of AB. Prove that $AM^2 = MC.MD$.

   (There is a purely algebraic way to do this, as well as a way using poles and polars. Try to find the latter).

3. The tangents to the circumcircle of $\triangle ABC$ at B and C intersect at D. Prove that AD is the symmedian of $\triangle ABC$.

4. AD is the altitude of an acute $\triangle ABC$. Let P be an arbitrary point on AD. BP, CP meet AC, AB at M, N, respectively. MN intersects AD at Q. F is an arbitrary point on side AC. FQ intersects line CN at E. Prove that $\angle FDA = \angle EDA$.

5. Let $ABCD$ be a convex quadrilateral. If $K = AD \cap BC, M = AC \cap BD, P = AB \cap KM, Q = DC \cap KM$, then we must have K, M, P, Q to form a harmonic division.

6. If AD, BE, CF are the altitudes of a triangle $\triangle ABC$, and if DE, EF meet AB, BC at F', D' respectively, then show that FD and $F'D'$ intersect on a point lying on AC.

7. The tangent at a point P of a circle cuts a diametre AB at T; and PN is the perpendicular to AB, the line joining B to the midpoint of TP cuts PN at Q. Prove that $AQ \parallel TP$.

8. Let ABC be a right angled triangle at A. D is a point on CB. Let M be the midpoint of AD. CM intersects the perpendicular bisector of AB at E. Prove that $BE \parallel DA$.

9. Consider triangle ABC with altitudes AD, BE, CF, and orthocentre H. Let lines EF and BC intersect at G, and let M be the midpoint of BC. Prove that $AM \perp GH$.

10. Consider a point P inside a triangle ABC. Let AD, BE, CF be cevians through P. The midpoint M of BC dierent from D, and T is the intersection of AD and EF. Prove that if the circumcircle of BTC is tangent to the line EF, then $\angle BTD = \angle MTC$.

AZZAM LABIB HAKIM Sesi Malam Yasya

Mandiri Malam
Kamis, 4 Oktober 2018

√ 1. Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of the triangle ABC must be 6 if the area of each of the triangles PFA, PDB and PEC is 1. Algebraic

√ 2. Let ABC be an acute triangle with altitudes AD, BE, and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas. cari yang sebangun terus perbandingan luas

3. Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r. x^2+2 $\lfloor\sqrt{n}\rfloor = x$, $n = x^2+y$

4. Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes.

5. For $2k$ real numbers $a_1, a_2, ..., a_k$, $b_1, b_2, ..., b_k$ define a sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, ...).$$

If the sequence X_N forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

6. Let a and b be positive integers, and let A and B be finite sets of integers satisfying (i) A and B are disjoint; (ii) if an integer i belongs to either to A or to B, then either $i+a$ belongs to A or $i-b$ belongs to B. Prove that $a|A| = b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

7. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

√ 8. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear. harmonic

9. Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold. reflects H over M to H' $\angle AEH_1 = \angle ABH_1$.

(Here we denote XY the length of the line segment XY.)

10. Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \le i < j \le n} \frac{1}{n - a_i a_j} \le \frac{n}{2}$$

must hold.

AZZAM LABIB HAKIM   Sesi Malam Yasya   Kamis, 4 oktober 2018

1 Reduction to Absurdity

Pada permasalahan tertentu terkadang menyusahkan ketika membuktikan secara langsung. Gunakan pembuktian dengan memisalkan atau mengandaikan konklusi pada soal salah kemudian mendapatkan kontradiksi yang mengakibatkan konklusi pada soal adalah benar.

Contoh 1. Ketiga titik sudut $\triangle ABC$ diwarnai merah, biru dan hijau. Di dalam $\triangle ABC$ diberi beberapa titik kemudian $\triangle ABC$ dipartisi menjadi segitiga-segitiga kecil yang dibentuk oleh titik-titik di dalam $\triangle ABC$ dan titik sudut $\triangle ABC$. Apabila titik-titik didalam $\triangle ABC$ diwarnai merah, biru atau hijau. Buktikan bahwa terdapat segitiga kecil dengan titik-titik sudut dengan warna berbeda-beda.

Petunjuk. :

1. Andaikan tidak ada segitiga kecil dengan warna titik sudut berbeda-beda
2. Perhatikan banyaknya titik berwarna merah dilihat dari banyaknya segitiga merah-biru-biru atau merah-merah-biru dan dilihat dari banyaknya segmen merah dan biru (kalo ga salah, aku juga lupa)
3. Dapatkan kontradiksinya

Contoh 2. Terdapat 2004 anak perempuan duduk pada meja bundar. Pada mulanya ada seorang anak memegang 2004 kerupuk. Kemudian anak yang memiliki kerupuk lebih dari 1 dapat membagikan masing-masing 1 kerupuk ke anak di kanan dan di kirinya. Buktikan tak akan pernah semua anak tersebut mendapat tepat 1 kerupuk.

Petunjuk. :   invarian paritas   Awal → akhir harus genap

1. Andaikan mungkin pernah terjadi
2. Gunakan Invariant Principle

Contoh 3. Misal 2004 titik $A_1, A_2, ..., A_{2004}$ disusun melingkar pada sebuah lingkaran searah dengan arah jarum jam. Pada mulanya, A_1 diberi label 0 dan $A_2, A_3, ..., A_{2004}$ diberi label 1. Operasi yang dilakukan adalah memilih tiga titik berurutan A_{i-1}, A_i, A_{i+1} mengganti label a, b, c dengan $1-a, 1-b, 1-c$. Apakah mungkin mengganti label semua titik menjadi 0?

Petunjuk. :

1. Andaikan mungkin mengganti semua label
2. Gunakan Invariant Principle

Contoh 4. Sebuah segilima konveks $ABCDE$ terletak pada bidang kartesian dan semua titik sudutnya adalah titik lattice (absis dan ordinatnya berupa bilangan bulat). Kelima diagonalnya berpotongan membentuk segilima $A_1B_1C_1D_1E_1$. Buktikan terdapat titik lattice di dalam atau di sisi segilima $A_1B_1C_1D_1E_1$.

Petunjuk. :

1. Andaikan tidak titik di dalam segilima $A_1B_1C_1D_1E_1$
2. Gunakan kasus ekstrem sehingga tidak ada titik lattice di dalam $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, $\triangle EAB$
3. Gunakan WLOG memisalkan $\triangle ABC$ adalah segitiga dengan luas terkecil diantara $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, $\triangle EAB$
4. Perhatikan ada titik lattice di dalam $A_1B_1C_1D_1E_1$ dan di dekat $\triangle ABC$

Contoh 5. Apakah mungkin himpunan bilangan bulat dipartisi menjadi tiga himpunan sehingga untuk setiap bilangan bulat n, bilangan $n, n-50, n+2005$ terletak pada himpunan yang berbeda-beda?

Petunjuk. :

1. Andaikan mungkin bilangan bulat dapat dipartisi
2. Lakukan substitusi terhadap n dengan $n - 50$ atau $n + 2005$
3. Lakukan substitusi terhadap n seterusnya sampai menemukan kontradiksi

1.1 Latihan Soal

1. Terdapat n anggota pada sebuah komite. Apabila terdapat $n + 1$ kelompok yang beranggotakan tiga orang dan tidak ada dua kelompok yang anggotanya semuanya sama. Buktikan terdapat dua kelompok yang memiliki tepat satu anggota yang sama
2. Pada papan catur 8×8 terdapat 9 pion terletak pada 3×3 bagian papan catur di pojok kiri bawah. Langkah yang diperbolehkan pion adalah melompati di sebelahnya kearah horizontal, vertikal atau diagonal. Mungkinkah pion-pion ini menutupi 3×3 bagian papan catur di pojok kiri atas?
3. Misal $n > 1$ bilangan ganjil dan diberikan bilangan bulat positif $k_1, k_2, ..., k_n$. Untuk setiap permutasi $a = (a_1, a_2, ..., a_n)$ dari bilangan $1, 2, ..., n$ kita definisikan $S(a) = \sum_{i=1}^{n} k_i a_i$. Buktikan terdapat dua buah permutasi b dan c sedemikian sehingga $S(b) - S(c)$ habis dibagi n!
4. Terdapat 17 Orang pada sebuah konferensi dan setiap orang memiliki 4 teman dalam konferensi tersebut. Buktikan terdapat dua orang yang tidak berteman dan tidak memiliki teman yang sama dalam konferensi tersebut.
5. Terdapat n Orang yang hadir pada sebuah konferensi dan setiap orang memiliki 8 teman dalam konferensi tersebut. Misalkan setiap dua orang yang berteman memiliki tepat 4 teman yang sama dan setiap dua orang yang tidak berteman memiliki tepat 2 teman yang sama. Cari semua n yang mungkin yang memenuhi kondisi tersebut

AZZAM LABIB HAKIM

Sesi malam Yasya

Mandiri Malam

Jumat, 5 Oktober 2018

√ 1. For pairwise distinct nonnegative reals a, b, c, prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2$$

WLOG $a > b > c \geq 0$. $c = 0$ jelas (AM-GM).

$c > 0$. $\min\{a, b, c\} = c$

$a = c + m$

$b = c + n$

$$\frac{(c+m)^2}{n^2} + \frac{(c+n)^2}{m^2} + \frac{c^2}{(m-n)^2}$$

↳ bongkar ada $\frac{m^2}{n^2} + \frac{n^2}{m^2} \geq 2$.

2. Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

   We say that an arrangement of n tokens is collapsible if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

√ 3. Define a function $f(n)$ from the positive integers to the positive integers such that $f(f(n))$ is the number of positive integer divisors of n. Prove that if p is a prime, then $f(p)$ is prime. Bukti kontradiksi

√ 4. Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF, assume it does not lie on line AB. Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ. harmonik

5. In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a box. Two boxes intersect if they have a common point in their interior or on their boundary. Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

√ 6. Find all positive integers m and n such that $(2^{2^n} + 1)(2^{2^m} + 1)$ is divisible by $m \cdot n$ order

√ 7. Point O is inside $\triangle ABC$. The feet of perpendicular from O to BC, CA, AB are D, E, F. Perpendiculars from A and B respectively to EF and FD meet at P. Let H be the foot of perpendicular from P to AB. Prove that D, E, F, H are concyclic.

8. Determine whether there exist a positive integer $n < 10^9$, such that n can be expressed as a sum of three squares of positive integers by more than 1000 distinct ways?

9. Let k be a given circle and A is a fixed point outside k. BC is a diameter of k. Find the locus of the orthocentre of $\triangle ABC$ when BC varies.

10. Find all functions $f : \mathbb{Q}^+ \to \mathbb{R}^+$ with the property:

$$f(xy) = f(x+y)(f(x) + f(y)), \forall x, y \in \mathbb{Q}^+$$

AZZAM LABIB HAKIM sesi malam Yasya Jumat, 5 oktober 2018

Projective Geometry

Alexander Remorov
alexanderrem@gmail.com

Harmonic Division

Given four collinear points A, B, C, D, we define their *cross-ratio* as:

$$(A, B; C, D) = \frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}} \tag{1}$$

Note that **the lengths are directed**. When the cross-ratio is equal to -1, we say that $(A, B; C, D)$ is a *harmonic bundle*. A particular case which occurs in many problems is when A, C, B, D are on a line in this order.
Let P be a point not collinear with A, B, C, D; we define the *pencil* $P(A, B, C, D)$ to be made up of 4 lines PA, PB, PC, PD. $P(A, B, C, D)$ is called *harmonic* when $(A, B; C, D)$ is harmonic.

Lemma 1: A pencil $P(A, B, C, D)$ is given. The lines PA, PB, PC, PD intersect a line l at A', B', C', D' respectively. Then $(A', B'; C', D') = (A, B; C, D)$.
Proof: Wolog A, C, B, D are collinear in this order. Using Sine Law in $\triangle CPA, \triangle CPB, \triangle DPA$, $\triangle DPB$, we get (the lengths and angles are directed):

$$\frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}} = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} \tag{2}$$

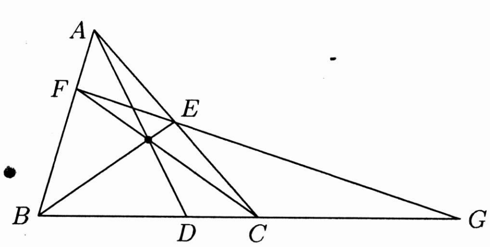
This gives a "trigonometric definition" corresponding to a cross ratio. Lemma 1 follows from (2).

Therefore for any pencil $P(A, B, C, D)$, we can define its cross ratio to be:
$(PA, PB; PC, PD) = (A, B; C, D)$. This definition is ok because of lemma 1.

Corollary 1: **This is extremely useful**. Using the same notation as in lemma 1, if $(A, B; C, D)$ is harmonic then so is $(A', B'; C', D')$.

Lemma 2: In $\triangle ABC$, points D, E, F are on sides BC, CA, AB. Let FE intersect BC at G. Then $(B, C; D, G)$ is harmonic iff AD, BE, CF are concurrent.

Proof: Use Ceva and Menelaus.

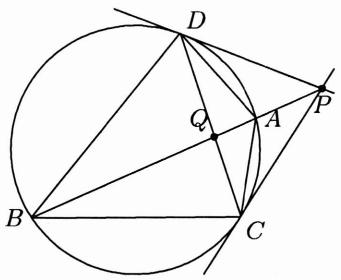


Lemma 3: Consider points A, B, C, D on a circle. Let P be any point on the circle. Then the cross ratio $(PA, PB; PC, PD)$ does not depend on P.
Proof: From (2) it follows that

$$|(PA, PB; PC, PD)| = |\frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)}| = |\frac{CA}{CB} : \frac{DA}{DB}| \quad (3)$$

If $(PA, PB; PC, PD) = -1$, the quadrilateral $ACBD$ is called *harmonic.* From lemma 3, if A, C, B, D are on a circle in this order, and $|\frac{CA}{CB}| = |\frac{DA}{DB}|$, then $ACBD$ is harmonic. The nice thing about lemma 3 is that it allows you to use harmonic pencils for circles.

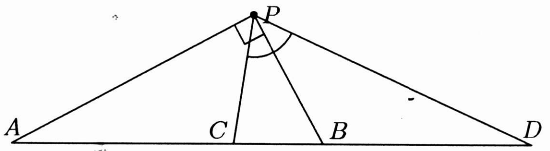
Lemma 4: A point P is outside or on a circle ω. Let PC, PD be tangents to ω, and l be a line through P intersecting ω at A, B (so that P, A, B are collinear in this order). Let AB intersect CD at Q. Then $ACBD$ is a harmonic quadrilateral and $(P, Q; A, B)$ is harmonic.



Proof: $\triangle PAD \sim \triangle PDB \implies \frac{AD}{DB} = \frac{PA}{PD}$. Similarly $\frac{AC}{CB} = \frac{PA}{PC}$. Because $PC = PD$, it follows that $ACBD$ is harmonic.
We can now apply lemma 3. **We take P=C (!)** and consider the intersection of $C(A, B, C, D)$ with line l. Since $ACBD$ is harmonic, the resulting 4 points of intersection form a harmonic bundle, hence $(P, Q; A, B)$ is harmonic.

Corollary 2: Points A, C, B, D lie on a line in this order, and M is the midpoint of CD. Then $(A, B; C, D)$ is harmonic iff $AC \cdot AD = AB \cdot AM$.
Proof: Whenever you see things like $AC \cdot AD$ and circles, trying Power of a Point is a good idea. Assume $AC \cdot AD = AB \cdot AM$. Consider the circle centred at M passing through C, D. Let AT be a tangent from A to this circle. Then $AC \cdot AD = AT^2$. Hence $AB \cdot AM = AT^2$ and $\triangle ATM \sim \triangle ABT$. Since $\angle ATM = 90°$ it follows that $\angle ABT = 90°$. By lemma 4 $(A, B; C, D)$ is harmonic. The converse of the corollary is proved in the same way.

Lemma 5: Points A, C, B, D lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:
1. $(A, B; C, D)$ is harmonic.
2. PB is the angle bisector of $\angle CPD$.
3. $AP \perp PB$.



Proof: Straightforward application of Sine Law.

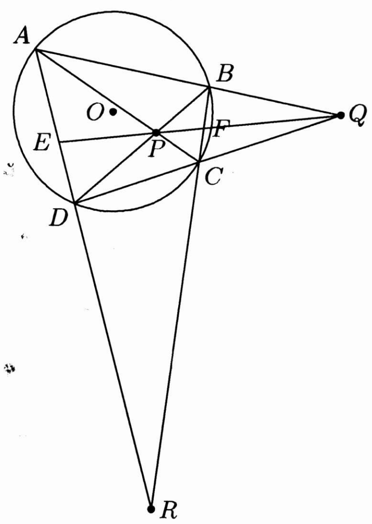
Poles and Polars

Given a circle ω with center O and radius r and any point $A \neq O$. Let A' be the point on ray OA such that $OA \cdot OA' = r^2$. The line l through A' perpendicular to OA is called the *polar* of A with respect to ω. A is called the *pole* of l with respect to ω.
Lemma 6: Consider a circle ω and a point P outside it. Let PC and PD be the tangents from P to ω. Then ST is the polar of P with respect to ω.
Proof: Straightforward.

Note: Using the same notation as in lemma 4, it follows that Q lies on the polar of P with respect to ω.

La Hire's Theorem: **This is extremely useful.** A point X lies on the polar of a point Y with respect to a circle ω. Then Y lies on the polar of X with respect to ω.
Proof: Straightforward.

Brokard's Theorem: The points A, B, C, D lie in this order on a circle ω with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to ω.
Proof: Let QP intersect BC, AD at F, E, respectively. From lemma 2 it follows that $(R, E; A, D)$ and $(R, F; B, C)$ is harmonic. From lemma 4 it follows that EF is the polar of R. Hence PQ is the polar of R. Similarly PR is the polar of Q and RQ is the polar of P. The fact that O is the orthocenter of $\triangle PQR$ follows from properties of poles and polars.



The configurations in the above lemmas and theorems come up in olympiad problems over and over again. You have to learn to recognize these configurations. **Sometimes you need to complete the diagram by drawing extra lines and sometimes even circles to arrive at a "standard" configuration.**

Projective Geometry - Part 2

Alexander Remorov
alexanderrem@gmail.com

Review

- The harmonic bundle $(A, B; C, D)$ is harmonic when $\frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}}$.
- A pencil $P(A, B, C, D)$ is the set of four lines PA, PB, PC, PD. It is harmonic iff $(A, B; C, D)$ is harmonic. Intersecting a harmonic pencil with any line produces a harmonic bundle.
- In $\triangle ABC$, points D, E, F are on sides BC, CA, AB. Let FE intersect BC at G. Then $(B, C; D, G)$ is harmonic iff AD, BE, CF are concurrent.
- A point P is outside or on a circle ω. Let PC, PD be tangents to ω, and l be a line through P intersecting ω at A, B (so that P, A, B are collinear in this order). Let AB intersect CD at Q. Then $ACBD$ is a harmonic quadrilateral (i.e. $\frac{AC}{CB} = \frac{AD}{DB}$) and $(P, Q; A, B)$ is harmonic.
- Points A, C, B, D lie on a line in this order, and M is the midpoint of CD. Then $(A, B; C, D)$ is harmonic iff $AC \cdot AD = AB \cdot AM$. Furthermore, if $(A, B; C, D)$ is harmonic then $MD^2 = MA \cdot MB$.
- Points A, C, B, D lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:
  1. $(A, B; C, D)$ is harmonic.
  2. PB is the angle bisector of $\angle CPD$.
  3. $AP \perp PB$.
- Given a circle ω with center O and radius r and any point $A \neq O$. Let A' be the point on ray OA such that $OA \cdot OA' = r^2$. The line l through A' perpendicular to OA is called the *polar* of A with respect to ω. A is called the *pole* of l with respect to ω.
- Consider a circle ω and a point P outside it. Let PC and PD be the tangents from P to ω. Then ST is the polar of P with respect to ω.
- **La Hire's Theorem:** A point X lies on the polar of a point Y with respect to a circle ω. Then Y lies on the polar of X with respect to ω.
- **Brokard's Theorem:** The points A, B, C, D lie in this order on a circle ω with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to ω.
- M is the midpoint of a line segment AB. Let P_∞ be a point at infinity on line AB. Then $(M, P_\infty; A, B)$ is harmonic.

AZZAM LABIB HAKIM   Sesi Pak Barra   Sabtu, 6 oktober 2018

PCMI
Tuesday, July 12, 2005

Francis E. Su
Harvey Mudd College

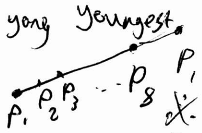
Pizza & Problem Solving # 2

The Extreme Principle

This problem session is modelled after the Harvey Mudd College Putnam Problem Solving Seminar, which runs every Tuesday night in the fall semester in preparation for the annual Putnam Mathematics Competition.

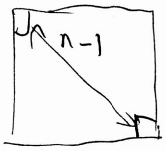
B1: Eight people sit around a lunch table at PCMI. As it happens, each person's age is the average of the two persons' ages on his/her right and left. Show that all their ages are equal.

Usia Angga, ada yang youngest. misal P_1 minimal.
$P_1 < P_2, P_3, \dots,$



B2: Fifteen sheets of paper of various sizes and shapes lie on a desktop covering it completely. The sheets may overlap and may even hang over the edge. Show that five of the sheets may be removed so that the remaining ten sheets cover at least 2/3 of the desktop. (Wagon)

B3: Place the integers $1, 2, 3, \dots, n^2$ (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two (horizontally, vertically, or diagonally) adjacent squares whose values differ by at least $n + 1$. (Zeitz)



$\boxed{a'}\boxed{a}$ $a' \le a + k$
k selisih maks.
$(n-1)k + 1 \ge n^2$
$(n-1)k \ge n^2 - 1$
$k \ge n+1$.

?

B4: Let $p(x)$ be a real polynomial such that for all x, $p(x) + p'(x) \geq 0$. Does it follow that for all x, $p(x) \geq 0$? [For example, $p(x) = x^2 + 1$ satisfies the condition and conclusion.]

$P'(x_{puncak}) = 0 \Rightarrow P(x_{puncak}) \ge 0$

B5: Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Show that all these points lie within the interior or on the boundary of a triangle of area less than 4. (Korea, 1995)

B6: Let A be a set of $2n$ points in the plane, no three of which are collinear. Suppose that n of them are colored red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colors. (Putnam, 1979)

And for a little bit of variety . . .

B7: Strange biology. The DNA of alien creatures on Planet Alpha Lyra IV is very strange; it may be represented as strings of two letters (nucleotides) A and B, it never includes 3 consecutive repetitions of any sequence (of any length) nor does the repetition BB ever occur. What is the longest Lyran DNA sequence? (Hess, variant)

Hints:

1. Who's the youngest?
2. Which piece of paper would you remove first?
3. Any two squares are separated by a path of adjacent squares that is how long?
4. If a polynomial is non-negative, what can be said about its degree? Does $p(x)$ achieve a minimum somewhere?
5. Which triangle should you focus on?
6. Consider the solution that minimizes what quantity?

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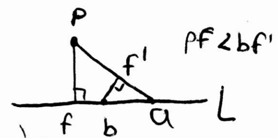
Math 326a, Fall 2006, Problem Set # 5

The Extremal Principle

Here is my favorite example of a problem solved by applying this principle: the Sylvester Problem, posed by Sylvester in 1893 and solved by Gallai in 1933 in a very complicated way.

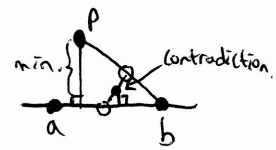
Theorem. *Suppose that a finite set S of points in the plane has the property that any line through two of them passes through a third. Then all the points lie on a single line.*

Proof (Kelly, 1948). Consider the set of pairs $\{(p, L)\}$ where L is a line passing through two different points of S, and $p \in S \setminus L$. If this set is nonempty, choose a pair which miminizes the distance d from p to L. Let f be the foot of the prependicular from p to L. By assumption there are at least three elements a, b, c of $L \cap S$. Choose two of them, say a, b, on the same side of f. Let b be nearer to f than a. Then the distance from b to the line passing through a and p is less than d. Contradiction. □



✓ **0.** Draw a picture and convince yourself that the above proof works. :)

✓ **1.** Let B and W be two finite sets of black and white points, respectively, in the plane, such that every line segment joining two points of the same color contains a point of the other color. Prove that all the points lie on a single line.



✓ **2.** In a circle, a finite set C of chords (segments connecting two points on the circle) is given, with a property that each of the chords from C passes through a midpoint of another chord from C. Prove that all these chords are diameters (that is, connect the antipodal points).

✓ **3.** Prove that it is not possible to find different natural numbers x, y, z, t which are solutions of   WLOG $t > x > y > z \geq 1$

$t \geq x+1 \Rightarrow t^t \geq (x+1)^{x+1} > x^{x+1} = x \cdot x^x \geq 3x^x$

$$x^x + y^y = z^z + t^t .$$

$x^x + y^y = z^z + t^t > z^z + 3x^x \rightarrow y^y > z^z + 2x^x$, kontradiksi karena $2x^x > 2y^y > y^y$.

✓ **4.** Each of the $3n$ members of a parliament of some country slapped one of his/her colleagues. Prove that among them it is possible to choose a committee consisting of n members none of whom slapped each other.

✓ **5.** Every member of a parliament of another country has at most three enemies among the remaining members. Show that one can split the parliament into two houses so that every member has at most one enemy in his/her house.

✓ **6.** 3009 numbers $a_1, \ldots, a_{3009}$ are written along the circle in such a way that each of them is equal to the absolute value of the difference between the next two in the clockwise direction (that is, $a_1 = |a_2 - a_3|$, $a_2 = |a_3 - a_4|$, … , $a_{3008} = |a_{3009} - a_1|$, $a_{3009} = |a_1 - a_2|$. The sum of all of the numbers is equal to 2006. What are they?

1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, …

7. Consider a walk in the plane according to the following rules. From a given point (x, y) we may move in one step to one of the four points $(x, y + 2x)$, $(x, y - 2x)$, $(x - 2y, y)$, $(x + 2y, y)$, with the restriction that a step just made cannot be retraced. Prove that if we start from $(1, \sqrt{2})$, we cannot return to this point anymore.

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GRAF :)

√ **Example 1.** Terdapat 101 orang pada suatu pesta. Setiap orang pada pesta berjabat tangan minimal sekali dengan orang lain. Tunjukkan bahwa ada sedikitnya seorang yang berjabat tangan dengan dua orang berbeda. degree total ≥ 101 padahal harus genap

√ **Example 2.** Apakah mungkin untuk mengubah gambar di tabel pertama menjadi gambar di tabel kedua dengan menggerakkan kuda beberapa kali? (Pada gambar H menyatakan kuda hitam dan P menyatakan kuda putih) lihat jalur perjalanan(?)

P		P
H		H

P		H
H		P

urutannya pasti

(lihat dari kemunculan jalur)

√ **Example 3.** Terdapat n orang $A_1, \ldots, A_n$ yang mengikuti suatu lomba matematika di mana beberapa orang mengenal satu sama lain dan diantara dua orang yang tidak saling mengenal memiliki seorang kenalan yang sama. Misalkan A_1 dan A_2 saling kenal tapi tidak memiliki kenalan yang sama. Buktikan bahwa banyaknya kenalan A_1 dan A_2 persis sama. Counter : $A_1 \to A_2, A_3, \ldots, A_n$

√ **Example 4.** Sembilan orang matematikawan bertemu pada sebuah konferensi matematika internasional. Untuk setiap tiga orang, minimal dua diantaranya dapat berbicara dalam bahasa yang sama. Jika setiap matematikawan dapat berbicara maksimum dalam 3 bahasa, buktikan bahwa setidaknya terdapat tiga matematikawan yang dapat berbicara dalam suatu bahasa yang sama. dibelakang

Example 5. Terdapat n orang, setiap dua diantaranya maksimum berbicara pada telepon maksimal satu kali. Setiap $n-2$ daintara merea berbicara melalui telepon 3^m kali dengan m bilangan asli. Tentukan nilai dari n.

Ada diantara orang yang tidak saling mengenal

√ **Example 6.** Terdapat $n > 3$ orang. Beberapa diantara mereka saling mengenal dan sebagian lagi tidak saling kenal. Berapa nilai terbesar dari banyaknya orang yang mengenal $n-1$ orang lain? $n-2$ Pilih tepat satu pasang yang tidak saling kenal.

Example 7. Misalkan n bilangan bulat positif dan $A_1, A_2, \ldots, A_{2n+1}$ merupakan subhimpunan dari sebuah himpunan B. Misalkan juga

(1) setiap A_i tepat memiliki $2n$ unsur.
(2) setiap $A_i \cap A_j$ untuk i, j berbeda memiliki tepat satu unsur.
(3) setiap unsur di B sedikitnya termuat di A_i, A_j untuk dua indeks i, j yang berbeda.

Untuk nilai n yang manakah kita bisa melabeli unsur-unsur di B dengan 0 atau 1 sedemikian sehingga 0 merupakan label dari tepat n unsur di B

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Exercise.

(1) There are n medicine boxes. Any two medicine boxes have the same kind of medecine inside and every kind of medecine is contained in just two medecine boxes. How many kinds of medicine are there? $\binom{n}{2}$ memilih sepasang titik dari n

(2) There are n professors $A_1, A_2, \ldots, A_n$ in a conference. Prove that these n professors can be divided into two teams such that for each professor the the number of the people whom he has acquanitance with in another team is not less than the number of acquintance in his team.

(3) There are 18 teams in a match. In every round, if one team competes with another team then it does not compete with the same team in another round. Now there have been 8 rounds. Prove that there must be three teams that have never competed with each other in the former 8 rounds.

(4) n representatives attend a conference. For any four represent atives, there is one person who has shaked hands with the other three. Prove that for any four representatives, there must be one person who shakes hands with the rest of the $n-1$ representatives.

(5) There are three middle schools, each of which has n students. Every student has acquaintance with $n+1$ students in the other two schools. Prove that we can choose one student from each school such that the three students know each other.

(6) There are $2n$ red squares on the a big chess board. For any two red squares, we can go from one of them to the other by moving horizontally or vertically to the adjacent red square in one step. Prove that all the red squares can be divided into n rectangules.

(7) There are 2000 people in a tour group. For any four people, there is one person having acquaintance with the other three. What is the least number of people having acquaintance with all the other people in the tour group?

(8) In a carriage, for any m $(m = 3)$ travelers, they have only one common friend. (If A is a friend of B, then B is a friend of A. Anyone is not a friend of himself.) How many people are there in the carriage?

(9) There are five points A, B, C, D, E in the plane, where any three points are not on the same line. Suppose that we join some points with segments, called edges, to form a figure. If there are no above five points in the figure of which any three points are the vertices of a triangle in the figure, then there cannot be seven or more than seven edges.

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good. Hence Cinderella can prevent an overflow by always staying in a good position. ■

Exercises

1. **[Cram]** Open problem Aaaaaarrgghh (announced by Pak Barra)
   A and **B** take turns placing dominoes on an $m \times n$ rectangular grid, where mn is even. **A** must place dominoes vertically and **B** must place dominoes horizontally, and dominoes cannot overlap with each other or stick out of the board. The player who cannot make any legal move loses. Given m and n, determine who has a winning strategy, and find this strategy.

2. **[Double Chess]**
   The game of double chess is played like regular chess, except each player makes two moves in their turn (white plays twice, then black plays twice, and so on). Show that white can always win or draw.

3. **[Russia 1999]**
   There are 2000 devices in a circuit, every two of which were initially joined by a wire. The hooligans Vasya and Petya cut the wires one after another. Vasya, who starts, cuts one wire on his turn, while Petya cuts two or three. A device is said to be disconnected if all wires incident to it have been cut. The player who makes some device disconnected loses. Who has a winning strategy?

4. **[IMO Shortlist 2009, C1]**
   Consider 2009 cards, each having one gold side and one black side, lying on parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over,

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so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

√ a) Does the game necessarily end? Yes

b) Does there exist a winning strategy for the starting player?

5. [Russia 1999]
There are three empty jugs on a table. Winnie the Pooh, Rabbit, and Piglet put walnuts in the jugs one by one. They play successively, with the order chosen by Rabbit in the beginning. Thereby Winnie the Pooh plays either in the first or second jug, Rabbit in the second or third, and Piglet in the first or third. The player after whose move there are exactly 1999 walnuts in some jug loses. Show that Winnie the Pooh and Piglet can cooperate so as to make Rabbit lose.

6. Solve the problem in example 10 with 1,000,000 replaced by n, an arbitrary odd number. Use this complete characterization of positions to provide a complete description of the winning strategy. If you have some programming experience, you could also write a program to play this game against you.
Remark (for programmers): You could also write a program to solve this probem, that is, to determine for each n who has a winning strategy. A simple dynamic programming approach would run in $O(n)$ time. Using this as a subroutine, the program to play the game against you would take $O(n)$ time for each move. However, if you found the characterization of positions on your own first, the program to play the game would only take $O(\log n)$ time for each move.

7. [Bulgaria 2005]
For positive integers t, a, b, a *(t, a, b)-game* is a two player game defined by the following rules. Initially, the number t is written on a blackboard. In his first move, the first player replaces t with either $t - a$ or $t - b$. Then, the second player subtracts either a or b from this number, and writes the result on the blackboard, erasing the old number. After this, the first player once again erases either a or b from the number

6