SMO Senior 2012 Round 1

1 Questions

1.1 Multiple Choice Questions

1. Suppose α and β are real numbers that satisfy the equation

$$x^2 + (2\sqrt{\sqrt{2}+1})x + (\sqrt{\sqrt{2}+1}).$$

   Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$

2. Find the value of

$$\frac{2011^2 \times 2012 - 2013}{2012!} + \frac{2013^2 \times 2014 - 2015}{2014!}$$

3. The increasing sequence $T = \{2, 3, 5, 6, 7, 8, ...\}$ consists of all positive integers which are not perfect squares. What is the 2012th term of T?

4. Let O be the center of the incircle of triangle $\triangle ABC$ and D be the point of tangency of O with AC. If AB = 10, AC = 9, BC = 11, find CD.

5. Find the value of

$$\frac{\cos^4 75° + \sin^4 75° + 3\sin^2 75° \cos^2 75°}{\cos^6 75° + \sin^6 75° + 4\sin^2 75° \cos^2 75°}$$

6. If the roots of the equation $x^2 + 3x - 1 = 0$ are also the roots of the equation $x^4 + ax^2 + bx + c = 0$, find the value of $a + b + 4c$.

7. Find the sum of the **digits** of all natural numbers from 1 to 1000.

8. Find the number of real solutions to the equation

$$\frac{x}{100} = \sin x$$

   .

9. In the triangle $\triangle ABC$, $AB = AC$, $\angle ABC = 40°$, and the point D is on AC such that BD is the angle bisector of $\angle ABC$. If BD is extended to the point E such that $DE = AD$, find $\angle ECA$.

10. Let m and n be positive integers such that $m > n$. If the last three digits of 2012^m and 2012^n are identical, find the smallest possible value of $m+n$.

1.2 Short Questions

1. Let a, b, c, d be four distinct positive real numbers that satisfy the equations
$$(a^{2012} - c^{2012})(a^{2012} - d^{2012}) = 2011$$
and
$$(b^{2012} - c^{2012})(b^{2012} - d^{2012})$$
. Find the value of $(cd)^{2012} - (ab)^{2012}$.

2. Determine the total number of pairs of integers x and y that satisfy the equation that satisfy the equation
$$\frac{1}{y} - \frac{1}{y+2} = \frac{1}{3 \cdot 2^x}$$

3. Given a set $S = \{1, 2, \ldots, 10\}$, a collection F of subsets of S is said to be *intersecting* if for any two subsets A and B in F, A and B have a common element. What is the maximum size of F?

4. The set M contains all the integral values of m such that the polynomial
$$2(m-1)x^2 - (m^2 - m + 12)x + 6m$$
has either one repeated or two distinct integral roots. Find the number of elements of M.

5. Find the minimum value of
$$\left| \sin x + \cos x + \frac{\cos x - \sin x}{\cos 2x} \right|$$

6. Find the number of ways to arrange the letters A, A, B, B, C, C, D and E in a line, such that there are no consecutive identical letters.

7. Suppose $x = 3^{\sqrt{2+\log_3 x}}$ is an integer. Find x.

8. Let f(x) be the polynomial $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ where a_1, a_2, a_3, a_4, a_5 are distinct integers. Given that $f(104) = 2012$, evaluate $a_1 + a_2 + a_3 + a_4 + a_5$.

9. Suppose that x, y, z, a are positive reals such that
$$yz = 6ax$$
$$xz = 6ay$$
$$xy = 6az$$
$$x^2 + y^2 + z^2 = 1.$$
Find $\frac{1}{xyza}$.

10. Find the least value of the expression $(x+y)(y+z)$, given that x, y, z are positive reals satisfying the equation

$$xyz(x+y+z) = 1.$$

11. For each real number x, let $f(x)$ be the minimum of the numbers $4x + 1, x+2$, and $-2x+4$. Determine the maximum value of $6f(x) + 2012$.

12. Find the number of pairs (A, B) of distinct subsets of $\{1, 2, 3, 4, 5, 6\}$ such that A is a proper subset of B. Note that A can be an empty set.

13. Find the sum of all integral values of x that satisfy

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1.$$

14. Three integers are selected from the set $S = \{1, 2, 3 \ldots, 19, 20\}$. Find the number of selections where the sum of the 3 integers is divisible by 3.

15. ABCD is a cyclic quadrilateral with $AB = AC$. The line FG is tangent to the circle at the point C, and is parallel to BD. If $AB = 6$ and $BC = 4$, find the value of $3AE$

16. Two Wei Qi teams, A and B, each comprising of 7 members, take on each other in a competition. The players on each team are fielded in a fixed sequence. The first game is played by the first player of each team. The losing player is eliminated while the winning player stays on to play with the next player of the opposing team. This continues until one team is completely eliminated and the surviving team emerges as the final winner - thus yielding a possible gaming outcome. Find the total number of possible gaming outcomes.

17. Given that $\mathbf{m} = (\cos\theta)\ \mathbf{i} + (\sin\theta)\ \mathbf{j}$ and $\mathbf{n} = (\sqrt{2} - \sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$, where $\mathbf{i}$, and $\mathbf{j}$ are the usual unit vectors along the x-axis and the y-axis respectively, and $\theta \in (\pi, 2\pi)$. If the magnitude of the vector $\mathbf{m}+\mathbf{n}$ is $\frac{8\sqrt{2}}{5}$, find the value of $5\cos(\frac{\theta}{2} + \frac{\pi}{8}) + 5$.

18. Given that the real numbers x, y, z satisfy the condition $x+y+z = 3$, find the maximum possible value of $f(x, y, z) = \sqrt{2x+3} + \sqrt[3]{3y+5} + \sqrt[4]{8z+12}$.

19. Let P(x) be a polynomial of degree 34 such that $P(k) = \frac{k}{k+1}$ for all integers $k = 0, 1, 2 \ldots, 34$. Evaluate $42840 \times P(35)$.

20. Given that α is an acute angle satisfying

$$\sqrt{369 - 360\cos\alpha} + \sqrt{544 - 480\sin\alpha} - 25 = 0$$

, find the value of $40 tan\alpha$.

21. Given that a, b, c, d, e are reals such that

$$a + b + c + d + e = 8$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = 16$$

. Determine the maximum value of $[e]$. ([e] denotes the floor function)

22. Let L denote the minimum value of the quotient of a 3-digit number formed by three distinct digits divided by the sum of its digits. Determine $[10L]$.

23. Find the last 2 digits of $19^{17^{15^{\cdots^{1}}}}$.

24. Let f(n) be the integer nearest to $\sqrt{n}$. Find the value of

$$\sum_{n=1}^{\infty} \frac{\frac{3}{2}^{f(n)} + \frac{3}{2}^{-f(n)}}{\frac{3}{2}^{n}}$$