Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Junior Section, Special Round)

Saturday, 24 June 2006 0930– 1230

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. Find all integers x, y that satisfy the equation
$$x + y = x^2 - xy + y^2.$$

2. The fraction $\frac{2}{3}$ can be expressed as a sum of two distinct unit fractions: $\frac{1}{2} + \frac{1}{6}$. Show that the fraction $\frac{p-1}{p}$, where $p \geq 5$ is a prime, cannot be expressed as a sum of two distinct unit fractions.

3. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum of phone calls between pairs of people needed for everyone to know everything?

4. In $\triangle ABC$, the bisector of $\angle B$ meets AC at D and the bisector of $\angle C$ meets AB at E. These bisectors intersect at O and $OD = OE$. If $AD \neq AE$, prove that $\angle A = 60°$.

5. You have a large number of congruent equilateral triangular tiles on a table and you want to fit n of them together to make a convex equiangular hexagon (i.e., one whose interior angles are $120°$). Obviously, n cannot be any positive integer. The first three feasible n are 6, 10 and 13. Show that 12 is not feasible but 14 is.

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1. Solving for y, we get:

$$y = \frac{x + 1 \pm \sqrt{-3(x-1)^2 + 4}}{2}.$$

Thus $3(x-1)^2 \leq 4$, i.e.,

$$1 - \frac{2}{\sqrt{3}} \leq x \leq 1 + \frac{2}{\sqrt{3}}.$$

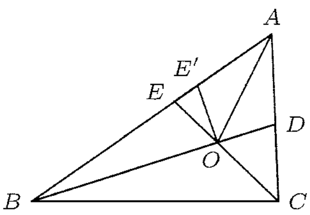
Thus $x = 0, 1, 2$ and $(x, y) = (0,0), (0,1), (1,0), (1,2), (2,1), (2,2)$ are all the solutions.

2. Note that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Since $\frac{1}{2}$ and $\frac{1}{3}$ are the two largest unit fractions and $\frac{p-1}{p} > \frac{5}{6}$ for all $p \geq 7$, the result is true for $p \geq 7$. Suppose $\frac{4}{5} = \frac{1}{a} + \frac{1}{b}$, with $a > b$. Then $\frac{4}{5} = \frac{1}{a} + \frac{1}{b} < \frac{2}{b}$. Therefore $2b < 5$, i.e., $b = 2$ and there is no solution for a.

3. We claim that the minimum of calls needed is $2n - 2$. Let A be a particular person, the $2n - 2$ calls made by A to each of the persons and vice versa will leave everybody informed. Thus at most $2n - 2$ calls are needed.

Next we prove that we need at least $2n - 2$ calls. Suppose that there is a sequence of calls that leaves everybody informed. Let B be the first person to be fully informed and that he receives his last piece of information at the pth call. Then each of the remaining $n - 1$ people must have placed at least one call prior to p so that B can be fully informed. Also these people must received at least one call after p since they were still not fully informed at the pth call. Thus we need at least $2(n-1)$ calls.

4.

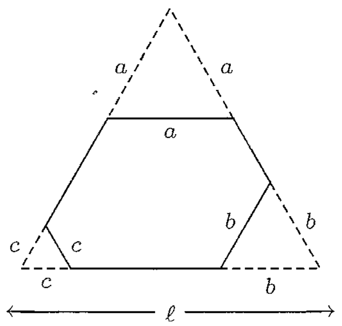


Assume that $AE > AD$. Let $\angle A = 2a$, $\angle B = 2b$, $\angle C = 2c$ and $\angle ADO = x$. Now AO bisects $\angle A$. Let E' be the point on AB such that $OE' = OE$. Since $AE > AD$, E' lies between A and E. We have $\triangle AE'O \equiv \triangle ADO$ (SAS). Thus $OE' = OD = OE$ and $x = \angle ADO = \angle AE'O = \angle BEO$. From $\triangle ABD$ and $\triangle BEC$, we have $2a + x + b = 180°$ and $x + 2b + c = 180°$. Thus $2a = b + c$ and so $\angle BAC = 2a = 60°$.

5. Assume that the tiles are of side length 1. Note the number of tiles required to form an equilateral triangle of length x is $1 + 3 + \cdots + (2x - 1) = x^2$. The triangle formed by extending the alternate sides of the hexagon must be an equilateral triangle of side length say ℓ. The hexagon is formed by removing the three corner equilateral triangles of side lengths a, b, c and $\ell > a + b, a + c, b + c$. An equilateral triangle of side length x contains x^2 tiles. Thus n is feasible if and only if

$$n = \ell^2 - a^2 - b^2 - c^2 \quad \text{and} \quad \ell > a + b,\ a + c,\ b + c$$

Take $\ell = 5$, $a = 3$, $b = c = 1$. Then $n = 14$ and so is feasible.

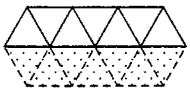


Next we show that $n = 12$ is not feasible. Let $a \geq b \geq c$. For fixed ℓ, we want to find a lower bound for n. For this purpose, we may assume that $a + b = \ell - 1$. Thus $b \leq (\ell - 1)/2$. If $a = \ell - 1 - k$, then $b, c = k \leq (\ell - 1)/2$. Thus

$$n \geq \ell^2 - 2k^2 - (\ell - 1 - k)^2 = (2\ell - 1) - (3k^2 - 2k(\ell - 1)).$$

Since $1 \leq k \leq (\ell - 1)/2$, we see that the maximum value of $3k^2 - 2k(\ell - 1)$ is attained at either $k = 1$ or $k = (\ell - 1)/2$. Thus $n \geq 4\ell - 6 = A$ (when $k = 1$) and $n \geq (\ell^2 + 6\ell - 3)/4 = B$ when $k = (\ell - 1)/2$. Thus $n \geq 6$ for $\ell = 3$, $n \geq 10$ for $\ell = 4$, $n \geq 13$ for $\ell = 5$, $n \geq 18$ for $\ell = 6$. For $\ell \geq 7$, $n \geq 22$. Thus we only have to check the case $\ell = 3, 4$. For $\ell = 3$, we have $a = b = c = 1$. For $\ell = 4$, we have $(a, b, c) = (2, 1, 1), (1, 1, 1)$. These give $n = 6, 10, 13$. Thus 12 is not feasible.

(Note: You can show that 14 is feasible by drawing a hexagon with 14 tiles. It is possible to show that 12 is not feasible by brute force. One of the sides must be at least of length 2. If one side has length 3, we need at least 14 tiles. In Fig. 1, the top side is of length 3 and the 7 tiles in the unshaded region must be present. No matter what you do, the 7 tiles in the shaded region must also be present. In fact this is the smallest hexagon with one side of length 3. If two adjacent sides are of length 2, then we need at least 16 tiles (Fig 2). If three consecutive sides are of lengths 2, 1, 2, then we need at least 13 tiles (Fig 3). The only other case is 2, 1, 1, 2, 1, 1 which gives 10 tiles (Fig 4). Thus 12 is not feasible.)



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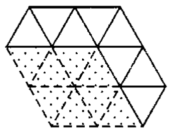

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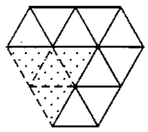

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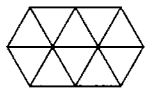

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