Question 3

Prove Theorem (8.12) and (8.13). Show that Minkowski's inequality for series fails when $p < 1$.

Proof: For Holder's Inequality, when $p = +\infty$, the last inequality is obvious.

Suppose $1 < p < \infty$, from (8.5) we have

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q},$$

here $u, v \geq 0$, $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Next, note that Hölder's inequality is clear when at least one of $\|\{a_k\}\|_p$, $\|\{b_k\}\|_q = 0$. Moreover, it is also clear if one of them if infinite. Thus we may assume $0 < \|\{a_k\}\|_p, \|\{b_k\}\|_q < \infty$.

Let $u = \frac{|a_k|}{(\sum |a_k|^p)^{\frac{1}{p}}}, v = \frac{|b_k|}{(\sum |b_k|^q)^{\frac{1}{q}}}$, then we have

$$uv = \frac{|a_k b_k|}{(\sum |a_k|^p)^{\frac{1}{p}} (\sum |b_k|^q)^{\frac{1}{q}}} \leq \frac{\frac{|a_k|^p}{\sum |a_k|^p}}{p} + \frac{\frac{|b_k|^q}{\sum |b_k|^q}}{q}.$$

So,

$$\sum \frac{|a_k b_k|}{(\sum |a_k|^p)^{\frac{1}{p}} (\sum |b_k|^q)^{\frac{1}{q}}} \leq \sum (\frac{\frac{|a_k|^p}{\sum |a_k|^p}}{p} + \frac{\frac{|b_k|^q}{\sum |b_k|^q}}{q}) = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus

$$\sum |a_k b_k| \leq (\sum |a_k|^p)^{\frac{1}{p}} (\sum |b_k|^q)^{\frac{1}{q}}.$$

For Minkowski's inequality:

If $p = 1$, the result is obvious.

If $p = \infty$, we have $|a_k| \leq \sup |a_k|$ and $|b_k| \leq \sup |b_k|$. Therefore, $|a_k + b_k| \leq \sup |a_k| + \sup |b_k|$, so that $\sup |a_k + b_k| \leq \sup |a_k| + \sup |b_k|$.

If $1 < p < \infty$, by inequality $|u+v|^p \leq 2^p(|u|^p + |v|^p)$, we have $\{a_k + b_k\} \in l^p$, and $\{|a_k + b_k|^{\frac{p}{q}}\} \in l^q$. Using Holder's Inequality, we have

$$\sum |a_k| |a_k + b_k|^{\frac{p}{q}} \leq (\sum |a_k|^p)^{\frac{1}{p}} (\sum |a_k + b_k|^p)^{\frac{1}{q}}$$

and

$$\sum |b_k||a_k+b_k|^{\frac{p}{q}} \le (\sum |b_k|^p)^{\frac{1}{p}}(\sum |a_k+b_k|^p)^{\frac{1}{q}}.$$

Notice that $p-1=\frac{p}{q}$, we obtain

$$\begin{aligned}
\sum |a_k+b_k|^p &= \sum |a_k+b_k||a_k+b_k|^{p-1}\\
&\le \sum |a_k||a_k+b_k|^{\frac{p}{q}} + \sum |b_k||a_k+b_k|^{\frac{p}{q}}\\
&= [(\sum |a_k|^p)^{\frac{1}{p}} + (\sum |b_k|^p)^{\frac{1}{p}}](\sum |a_k+b_k|^p)^{\frac{1}{q}},
\end{aligned}$$

hence $(\sum |a_k+b_k|^p)^{\frac{1}{p}} \le (\sum |a_k|^p)^{\frac{1}{p}} + (\sum |b_k|^p)^{\frac{1}{p}}$ provided the right hand side is finite.

Next, check that when the right hand side is infinite, then the left hand side of the above is also infinite.

We list some example where Minkowski's inequality fails.

If $0<p<1$, then $2<2^{1/p}$. Let $a=(0,1,0,0,\dots,0)$, $\quad b=(1,0,0,\dots,0,\dots)$, then $(\sum |a_k+b_k|^p)^{\frac{1}{p}} = 2^{1/p}$, $(\sum |a_k|^p)^{\frac{1}{p}} = 1$, $(\sum |b_k|^p)^{\frac{1}{p}} = 1$. Since $2<2^{1/p}$, we have

$$(\sum |a_k+b_k|^p)^{\frac{1}{p}} > (\sum |a_k|^p)^{\frac{1}{p}} + (\sum |b_k|^p)^{\frac{1}{p}}.$$

Thus, Minkowski's inequality for series fails when $p<1$.

Question 4

Let f and g be real-valued, and let $1<p<\infty$. Prove that equality holds in the inequality $|\int fg| \le \|f\|_p\|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

If $\|f+g\|_p = \|f\|_p + \|g\|_p$ and $g \ne 0$ in Minkowski's inequality, show that f is a multiple of g.

Proof: From **Theorem8.4** we have

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'},$$

and the equality holds if and only if $b^{p'} = a^p$. Then from the proof of **Theorem8.6**, we know the equality of **Theorem8.6** holds if and only if

$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^{p'}}{\|g\|_{p'}^{p'}}$ a.e.. Thus, the equality of of **Theorem8.6** holds if and only if $|f|^p = \frac{\|f\|_p^p}{\|g\|_{p'}^{p'}}|g|^{p'}$ a.e.

Since $|\int fg| \leq \int |fg|$, then $|\int fg| = \int |fg|$ holds, if and only if fg has constant sign a.e., otherwise the equality cannot hold.

Conversely, if $|f|^p$ is a multiple of $|g|^{p'}$ and fg have constant sign a.e., it is easy to check that the equality holds.

From the above proof, we have that equality holds in the inequality $|\int fg| \leq \|f\|_p \|g\|_{p'}$ if and only if fg has constant sign a.e. and $|f|^p$ is a multiple of $|g|^{p'}$ a.e.

Question 5

For $1 \leq p < \infty$ and $0 < |E| < +\infty$, define $N_p[f] = (\frac{1}{|E|}\int_E |f|^p)^{\frac{1}{p}}$. Prove that if $p_1 < p_2$, then $N_{p_1}[f] \leq N_{p_2}[f]$. prove also that $N_p[f+g] \leq N_p[f] + N_p[g]$, $(\frac{1}{|E|})\int_E |fg| \leq N_p[f]N_{p'}[g]$, $\frac{1}{p} + \frac{1}{p'} + 1$, and that $\lim_{p\to\infty} N_p[f] = \|f\|_\infty$. Thus, N_p behaves like $\|\cdot\|_p$, but has the advantage of being monotone in p.

Proof: By Hölder's inequality, we have

$$\begin{aligned} \int_E |f|^{p_1} = \int_E |f|^{p_1}\chi_E &\leq (\int_E \chi_E^{\frac{p_2}{p_2-p_1}})^{\frac{p_2-p_1}{p_2}} (\int_E |f|^{p_1 \cdot \frac{p_2}{p_1}})^{\frac{p_1}{p_2}} \\ &= |E|^{1-\frac{p_1}{p_2}} (\int_E |f|^{p_2})^{\frac{p_1}{p_2}}. \end{aligned}$$

So,

$$(\int_E |f|^{p_1})^{\frac{1}{p_1}} \leq |E|^{\frac{1}{p_1}-\frac{1}{p_2}} (\int_E |f|^{p_2})^{\frac{1}{p_2}},$$

Thus, $N_{p_1}[f] \leq N_{p_2}[f]$.

Using Minkowski's inequality, we have

$$\begin{aligned} N_p[f+g] &= (\frac{1}{|E|}\int_E |f+g|^p)^{\frac{1}{p}} \\ &\leq (\frac{1}{|E|})^{\frac{1}{p}}[(\int_E |f|^p)^{\frac{1}{p}} + (\int_E |g|^p)^{\frac{1}{p}}] \\ &= N_p[f] + N_p[g], \end{aligned}$$

By Hölder's Inequality

$$(\frac{1}{|E|})\int_E |fg| \leq (\frac{1}{|E|})^{\frac{1}{p}+\frac{1}{p'}} (\int_E |f|^p)^{\frac{1}{p}} (\int_E |g|^{p'})^{\frac{1}{p'}} = N_p[f]N_{p'}[g].$$

Let $M = \|f\|_\infty$. If $L < M$, then the set $A = \{x \in E : |f(x)| > L\}$ has positive measure. Moreover, $N_p[f] = (\frac{1}{|E|}\int_E |f|^p)^{\frac{1}{p}} \geq (\frac{1}{|E|}\int_A |f|^p)^{\frac{1}{p}} > L(\frac{|A|}{|E|})^{\frac{1}{p}}$. Since $(\frac{|A|}{|E|})^{\frac{1}{p}} \to 1$, as $p \to \infty$, $\liminf_{p\to\infty} N_p[f] \geq L$. So $\liminf_{p\to\infty} N_p[f] \geq M$. However, we also have $N_p[f] = (\frac{1}{|E|}\int_E |f|^p)^{\frac{1}{p}} \leq (\frac{1}{|E|}\int_E M^p)^{\frac{1}{p}} = M$, then $\limsup_{p\to\infty} N_p[f] \leq M$. Thus, $\lim\limits_{p\to\infty} N_p[f] = \|f\|_\infty$.

Question 7

Show that when $0 < p < 1$, the neighborhoods $\{f : \|f\|_p < \epsilon\}$ of zero in $L^p(0,1)$ are not convex.

Proof: For any small $\delta > 0$, consider $f = \chi_{(0,\delta^p)}$, $g = \chi_{(\delta^p, 2\delta^p)}$, we have

$$\|1/2f + 1/2g\|_p > 1/2\|f\|_p + 1/2\|g\|_p.$$

So $\{f : \|f\|_p < \epsilon\}$ is not convex for $0 < p < 1$.

Question 8

Prove the following integral version of Minkowski's inequality for $1 \leq p < \infty$: $[\int |\int f(x,y)dx|^p dy]^{\frac{1}{p}} \leq \int [\int |f(x,y)|^p dy]^{\frac{1}{p}} dx$.

Proof. (1)If $p = 1$, the inequality is obvious.

(2)If $1 < p < \infty$, then there exist $q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Using

Hölder's inequality and Toneli's theorem, we have

$$\begin{aligned}
\int|\int f(x,y)dx|^p dy &= \int|\int f(x,y)dx|^{p-1}|\int f(x,y)dx|dy \\
&= \int|\int f(z,y)dz|^{p-1}|\int f(x,y)dx|dy \\
&\le \int|\int f(z,y)dz|^{p-1}\int|f(x,y)|dxdy \\
&= \iint|f(x,y)|\cdot|\int f(z,y)dz|^{p-1}dydx \\
&= \int[\int|f(x,y)|\cdot|\int f(z,y)dz|^{p-1}dy]dx \\
&\le \int(\int|f(x,y)|^p dy)^{\frac{1}{p}}(\int|\int f(z,y)dz|^p dy)^{\frac{1}{q}}dx \\
&= (\int|\int f(z,y)dz|^p dy)^{\frac{1}{q}}\int(\int|f(x,y)|^p dy)^{\frac{1}{p}}dx.
\end{aligned}$$

(i) If $(\int|\int f(x,y)dx|^p dy)^{\frac{1}{q}} \in (0,+\infty)$, by dividing both sides by $(\int|\int f(x,y)dx|^p dy)^{\frac{1}{q}}$, we have

$$[\int|\int f(x,y)dx|^p dy]^{\frac{1}{p}} \le \int[\int|f(x,y)|^p dy]^{\frac{1}{p}}dx.$$

(ii) If $(\int|\int f(x,y)dx|^p dy)^{\frac{1}{q}} = 0$, the inequality obviously holds.

(iii) If $(\int|\int f(x,y)dx|^p dy)^{\frac{1}{q}} = \infty$, consider

$$f_k(x,y) = |f(x,y)|\cdot\chi_{\{|f(x,y)|<k\}}(x,y)\cdot\chi_{\{|x|+|y|<k\}}(x,y),$$

Obviously, for f_k, this Minkowski's inequality holds. By monotone convergence,

$$(\int|\int f_k(x,y)dx|^p dy)^{\frac{1}{q}} \to (\int\int|f(x,y)|dx^p dy)^{\frac{1}{q}} = +\infty,$$

we have

$$\int[\int|f_k(x,y)|^p dy]^{\frac{1}{p}}dx \ge [\int|\int|f_k(x,y)|dx|^p dy]^{\frac{1}{p}} \to +\infty,$$

so

$$\int[\int|f(x,y)|^p dy]^{\frac{1}{p}}dx = \infty,$$

the inequality holds.

Question 9

If f is real-valued and measurable on E, define its essential infimum on E by $ess\inf_E f = \sup\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\}$. If $f \geq 0$, show that $\text{ess}_E \inf f = (\text{ess}_E \sup \frac{1}{f})^{-1}$.

Proof: $\text{ess}_E \sup \frac{1}{f} = \inf\{\alpha : |\{x \in E : \frac{1}{f} > \alpha\}| = 0\}$.

For each $\alpha > 0$ which satisfies $|\{x \in E : \frac{1}{f} > \alpha\}| = 0$, since

$$\{x \in E : \frac{1}{f} > \alpha\} = \{x \in E : f < \frac{1}{\alpha}\},$$

then $|\{x \in E : f < \frac{1}{\alpha}\}| = 0$, and $\frac{1}{\alpha} \leq \text{ess}_E \inf f$. Therefore,

$$(\text{ess}_E \sup \frac{1}{f})^{-1} \leq \text{ess}_E \inf f.$$

For each $\alpha > 0$ which satisfies $|\{x \in E : f(x) < \alpha\}| = 0$, since

$$\{x \in E : f(x) < \alpha\} = \{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\},$$

then $|\{x \in E : \frac{1}{f(x)} > \frac{1}{\alpha}\}| = 0$, then $\frac{1}{\alpha} \geq \text{ess}_E \sup \frac{1}{f}$, $(\text{ess}_E \sup \frac{1}{f})^{-1} \geq \alpha$. Therefore,

$$(\text{ess}_E \sup \frac{1}{f})^{-1} \geq \text{ess}_E \inf f.$$

. Thus,

$$\text{ess}_E \inf f = (\text{ess}_E \sup \frac{1}{f})^{-1}.$$

Question 11

If $f_k \longrightarrow f$ in L^p, $1 \leq p < \infty$, $g_k \longrightarrow g$ pointwise, and $\|g_k\|_\infty \leq M$ for all k, prove that $f_k g_k \longrightarrow fg$ in L^p.

Proof: By assumption, $g_k \to g$ pointwise, and $\|g_k\|_\infty \le M$, so $\|g\|_\infty \le M$. Since

$$f_k g_k - fg = f_k g_k - f g_k + f g_k - fg = (f_k - f)g_k + f(g_k - g),$$

we have

$$\begin{aligned}\|f_k g_k - fg\|_p &\le \|(f_k - f)g\|_p + \|f(g_k - g)\|_p \\ &\le M\|f_k - f\|_p + \|f(g_k - g)\|_p.\end{aligned}$$

Since $|f(g_k - g)|^p = |f|^p|g_k - g|^p \le |f|^p 2^p(|g_k|^p + |g|^p) \le 2^{p+1}M^p|f|^p$, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{k\to\infty}\int |f(g_k - g)|^p = \int \lim_{k\to\infty} |f(g_k - g)|^p = 0,$$

therefore $\|f(g_k - g)\|_p \to 0$ as $k \to \infty$. We also know $f_k \longrightarrow f$ in L^p, hence, $\|f_k g_k - fg\|_p \to 0$ as $k \to \infty$ and so $f_k g_k \to fg$ in L^p.

Question 12

Let $f, \{f_k\} \in L^p$. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $1 \le p < \infty$, show that $\|f - f_k\|_p \to 0$.

Proof:

(1) If $0 < p < 1$, for any $\epsilon > 0$, we have the following inequality

$$|\int |f_k|^p - \int |f|^p| \le \int |f - f_k|^p,$$

which implies $\|f_k\|_p \longrightarrow \|f\|_p$.

If $1 \le p \le \infty$, by Minkowski's inequality, $|||f_k||_p - ||f||_p| \le ||f_k - f||_p \longrightarrow 0$, so $||f_k||_p \longrightarrow ||f||_p$

(2) Since

$$|f_k - f|^p \le 2^p(|f|^p + |f_k|^p),$$

we have $2^p(|f|^p+|f_k|^p)-|f_k-f|^p \geq 0$. By Fatou's lemma,

$$\int \underline{\lim}(2^p(|f|^p+|f_k|^p)-|f_k-f|^p) = \int 2^{p+1}|f|^p$$
$$\leq \underline{\lim}\int (2^p(|f|^p+|f_k|^p)-|f_k-f|^p)$$
$$= \int 2^p|f|^p + 2^p\underline{\lim}\int |f_k|^p - \overline{\lim}\int |f_k-f|^p$$
$$= 2^{p+1}\int |f|^p - \overline{\lim}\int |f_k-f|^p.$$

so $\overline{\lim}\int |f_k-f|^p \leq 0$, so $\int |f_k-f|^p \longrightarrow 0$, so $||f-f_k||_p \longrightarrow 0$.

Question 13

Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p$, $1 < p < \infty$. If $\|f_k\|_p \leq M < \infty$, show that $\int f_k g \to \int fg$ for all $g \in L^{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof: By Fatou's Lemma, we have $\|f\|_p \leq \liminf \|f_k\|_p \leq M$. Thus, from Minkowski's inequality, we know $\|f-f_k\|_p \leq 2M$. By Hölder's inequality, for any measurable set E,

$$\int_E |f-f_k|\cdot|g| \leq (\int_E |f-f_k|^p)^{1/p}(\int_E |g|^{p'})^{1/p'}.$$

For any given $\epsilon > 0$, we can find a ball $B_R = \{|x| < R\}$, $(R > 0)$, , such that

$$\begin{aligned} \|g(1-\chi_{B_R})\|_{p'} &= (\int_{B_R^c} |g|^{p'})^{1/p'} \\ &< \frac{\epsilon}{1+4M}. \quad \text{(monotone convergence)} \end{aligned}$$

In B_R, for any $\delta > 0$, by Egorov's theorem there exits a closed subset $F \subset B_R$ with $|B_R - F| < \delta$ such that $f_k \to f$ uniformly on F. Then, we have

$$\begin{aligned} \lim_{k\to\infty}\int_F |f_k-f||g| &\leq \lim_{k\to\infty}(\int_F |f-f_k|^p)^{1/p}(\int_F |g|^{p'})^{1/p'} \\ &= 0. \end{aligned}$$

Since $g \in L^{p'}$, there exists $\delta_0 > 0$, such that for any measurable set E' with $|E'| < \delta_0$,

$$(\int_{E'} |g|^{p'})^{1/p'} < \frac{\epsilon}{1+4M}.$$

Choose $\delta = \delta_0$, since

$$\int |f_k - f| \cdot |g| = \int_{B_R^c} |f_k - f| \cdot |g| + \int_F |f_k - f| \cdot |g| + \int_{B_R - F} |f_k - f| \cdot |g|,$$

we obtain

$$\begin{aligned} \limsup \int |f_k - f| \cdot |g| &\leq \frac{\epsilon}{1+4M}(\int_{B_R^c} |f - f_k|^p)^{1/p} + \frac{\epsilon}{1+4M}(\int_{B_R - F} |f - f_k|^p)^{1/p} \\ &\leq 2 \cdot \frac{\epsilon}{1+4M} \cdot 2M \\ &< \epsilon, \end{aligned}$$

this implies $\lim_{k \to \infty} \int |f_k - f| \cdot |g| = 0$. The conclusion follows.

Question 14

Verify that the following systems are orthogonal:

(a) $\{\frac{1}{2}, \cos x, \sin x, \ldots, \cos kx, \sin kx, \ldots\}$ on any interval of length 2π;

(b) $\{e^{2\pi ikx/(b-a)} : k = 0, \pm 1, \pm 2, \ldots\}$ on (a, b).

Proof: Notice that (a) can be deduced from (b), so we only need to verify (b). Without loss of generality, we may assume that $a = 0$, $b = 1$.

For $k \neq m$,

$$\begin{aligned} \int_0^1 e^{2\pi ikx} e^{-2\pi imx} dx &= \int_0^1 e^{i2\pi(k-m)x} dx \\ &= \frac{1}{2(k-m)\pi i} e^{i2\pi(k-m)x}\Big|_{x=0}^{x=1} \\ &= 0. \end{aligned}$$

Thus, this system is orthogonal on $(0, 1)$. The conclusions follow.

Question 15

If $f \in L^2(0, 2\pi)$, show that $\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kx dx = \lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kx dx = 0$. Prove that the same is true if $f \in L^1(0, 2\pi)$. To prove it, approximate f in L^1 norm by L^2 functions.

Proof: Since $\int_0^{2\pi} \cos^2 kx dx = \pi$, let $\phi_k = \frac{1}{\sqrt{\pi}}\cos kx$, by the conclusion of **Question14(a)**, we know $\{\phi_k\}$ is an orthonormal system in $L^2(0, 2\pi)$, then $c_k = \int_0^{2\pi} f(x)\cos kx dx$ is the Fourier coefficients of f respect to $\{\phi_k\}$, then by **Theorem8.27** we know $\sum_{k=1}^{\infty} |c_k|^2 < +\infty$, so $|c_k|^2 \longrightarrow 0$ as $k \longrightarrow \infty$, then $c_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, $\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kx dx = 0$.

We can use the similar way to have $\lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kx dx = 0$.

If $f \in L(0, 2\pi)$, write $f = g + h$, where $g \in L^2$ and $\int_0^{2\pi} |h| < \varepsilon$. let $\phi_k = \frac{1}{\sqrt{\pi}}\cos kx$, then $c_k(f) = \int_0^{2\pi}(g+h)\phi_k = c_k(g) + c_k(h)$. $|c_k(h)| = |\int_0^{2\pi} h\phi_k| \le \frac{1}{\sqrt{\pi}}\int_0^{2\pi} |h| < \frac{\varepsilon}{\sqrt{\pi}}$ for all k. Since ε is arbitrary small, then $c_k(h) \longrightarrow 0$ as $k \longrightarrow \infty$. We also know $c_k(g) \longrightarrow 0$ as $k \longrightarrow \infty$, so $c_k(f) \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, $\lim_{k\to\infty} \int_0^{2\pi} f(x)\cos kx dx = 0$.

We can use the similar way to have $\lim_{k\to\infty} \int_0^{2\pi} f(x)\sin kx dx = 0$.

Question 16

A sequence $\{f_k\}$ in L^p is said to converge weakly to a function f in L^p if $\int f_k g \to \int fg$ for all $g \in L^{p'}$. prove that if $f_k \to f$ in L^p norm, $1 \le p \le \infty$, then $\{f_k\}$ converges weakly to f in L^p.

Proof. By Hölder's inequality, if $1 \le p \le \infty$,

$$|\int f_k g - \int fg| \le \int |f_k - f||g| \le (\int |f_k - f|^p)^{\frac{1}{p}} (\int |g|^{p'})^{\frac{1}{p'}} = \|f_k - f\|_p \|g\|_{p'} \to 0$$

as $k \to 0$, since $f_k \to f$ in L^p norm. Thus, $\{f_k\}$ converges weakly to f in L^p.

Conversely, as shown in Exercise 15, let $f_k(x) = \cos kx$ and $g(x) = f(x)$, $\int_0^{2\pi} f(x)\cos kx dx \to 0$ as $k \to \infty$. If the converse is true, we can get $\cos kx \to 0$ weakly in L^2. However,

$$\int_0^{2\pi} (\cos kx)^2 dx = \int_0^{2\pi} \frac{\cos 2kx - 1}{2} dx = \pi \not\to 0 \text{ in } L^2 \text{ norm.}$$

Question 17

Suppose that $f_k, f \in L^2$ and that $\int f_k g \to \int fg$ for all $g \in L^2$. If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2 norm.

Proof: Let $g = f$, then $\int f_k f \to \int f^2$. Since $\|f_k\|_2 \to \|f\|_2$, $\int f_k^2 \to \int f^2$. $\int |f_k - f|^2 = \int f_k^2 - 2f_k f + f^2 = \int f_k^2 - 2\int f_k f + \int f^2 \to \int f^2 - 2\int f^2 + \int f^2 = 0$. Thus, $f_k \to f$ in L^2 norm.

Question 18

Prove the parallelogram law for L^2: $\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$. Is this true for L^p when $p \neq 2$?

Proof:

$$\|f+g\|^2 = \int (f+g)(\overline{f}+\overline{g}) = \int f\overline{f} + \int f\overline{g} + \int g\overline{f} + \int g\overline{g}.$$

$$\|f-g\|^2 = \int (f-g)(\overline{f}-\overline{g}) = \int f\overline{f} - \int f\overline{g} - \int g\overline{f} + \int g\overline{g}.$$

Thus,

$$\|f+g\|^2 + \|f-g\|^2 = 2\int f\overline{f} + 2\int g\overline{g} = 2\|f\|^2 + 2\|g\|^2.$$

If $p \neq 2$, the parallelogram law cannot hold. Let $g = f$ a.e., $\|f+g\|^p = 2^p\|f\|^p$, $\|f-g\|^p = 0$, then $\|f+g\|^p + \|f-g\|^p = 2^p\|f\|^p$, $2\|f\|^p + 2\|g\|^p = 2^2\|f\|^p \neq 2^p\|f\|^p$ provided $\|f\|_p \neq 0$. Thus, the parallelogram law fails.

Question 19

Prove that a finite dimensional Hilbert space is isometric with R^n for some n.

Proof: Let H be a $n-$ dimensional complex Hilbert space and $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is the basis of H, we assume $\{\phi_k\}$ is orthonormal. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, define $T : \mathbb{C}^n \longrightarrow H$:

$$Tx = x_1\phi_1 + x_2\phi_2 + \cdots + x_n\phi_n$$

Since $\{\phi_k\}$ is orthonormal, T is an on-to map. Put $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$, then $\|x-y\|^2 = \sum_{i=1}^{n} |x_i - y_i|^2$.

$\|Tx - Ty\|^2 = \|T(x-y)\|^2 = (\sum_{i=1}^{n}(x_i - y_i)\phi_i, \sum_{j=1}^{n}(x_j - y_j)\phi_j) = \sum_{i=1}^{n} |x_i - y_i|^2$.

So $\|x-y\| = \|Tx - Ty\|$, and H is isometric with $\mathbb{C}^n$, thus is isometric with $\mathbb{R}^{2n}$.

If H is a real Hilbert space, the proof is the same.

Question 21

If $f \in L^p(R^n)$, $0 < p < \infty$, show that $\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy = 0$ a.e.

Proof: Let $\{r_k\}$ be the set of rational numbers, and let Z_k be the set where the formula

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy = |f(x) - r_k|^p$$

is not valid. Since $|f(y) - r_k|^p \leq 2^p(|f(y)|^p + |r_k|^p)$ is locally integrable, by Lebesgue's differential theorem, we have $|Z_k| = 0$. Let $Z = \cup Z_k$; then $|Z| = 0$. For any Q, x and r_k

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy &\leq 2^p \frac{1}{|Q|} \int_Q |f(y) - r_k)|^p dy + 2^p \frac{1}{|Q|} \int_Q |r_k - f(x)|^p dy \\
&= 2^p \frac{1}{|Q|} \int_Q |f(y) - r_k)|^p dy + 2^p |r_k - f(x)|^p.
\end{aligned}$$

Therefor, if $x \notin Z$,

$$\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy \leq 2^{p+1} |f(x) - r_k|^p,$$

for every r_k. For any x at which $f(x)$ is finite (in particular, almost everywhere), we can choose r_k such that $|f(x) - r_k|$ is arbitrarily small. This shows that the left side of the last formula is zero a.e., and completes the proof.