Art of Problem Solving

Instructive Olympiad Algebra Problems

At a national Olympiad and higher level.

APMO 2015 Q3 A sequence of real numbers $a_0, a_1, ...$ is said to be good if the following three conditions hold.

(i) The value of a_0 is a positive integer.

(ii) For each non-negative integer i we have $a_{i+1} = 2a_i + 1$ or $a_{i+1} = \frac{a_i}{a_i+2}$

(iii) There exists a positive integer k such that $a_k = 2014$.

Find the smallest positive integer n such that there exists a good sequence $a_0, a_1, ...$ of real numbers with the property that $a_n = 2014$.

Proposed by Wang Wei Hua, Hong Kong

APMO 2009 Q2 Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:
$\frac{a_1}{k^2+1} + \frac{a_2}{k^2+2} + \frac{a_3}{k^2+3} + \frac{a_4}{k^2+4} + \frac{a_5}{k^2+5} = \frac{1}{k^2}$ for $k = 1, 2, 3, 4, 5$

Find the value of $\frac{a_1}{37} + \frac{a_2}{38} + \frac{a_3}{39} + \frac{a_4}{40} + \frac{a_5}{41}$ (Express the value in a single fraction.)

APMO 1999 Q2 Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

APMO 1993 Q3 Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{and} \\ g(x) &= c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x + r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

EGMO 2015 Q4 Determine whether there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers
which satisfies the equality

$$a_{n+2} = a_{n+1} + \sqrt{a_{n+1} + a_n}$$

for every positive integer n.

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