JBMO Shortlist 2016

–	Algebra
1	Let a, b, c be positive real numbers such that $abc = 8$. Prove that $\frac{ab+4}{a+2} + \frac{bc+4}{b+2} + \frac{ca+4}{c+2} \geq 6$.
2	Let a, b, cbe positive real numbers.Prove that $\frac{8}{(a+b)^2+4abc} + \frac{8}{(b+c)^2+4abc} + \frac{8}{(a+c)^2+4abc} + a^2 + b^2 + c^2 \geq \frac{8}{a+3} + \frac{8}{b+3} + \frac{8}{c+3}$.
3	Find all the pairs of integers (m, n) such that $\sqrt{n+\sqrt{2016}}+\sqrt{m-\sqrt{2016}} \in \mathbb{Q}$.
4	If the non-negative reals x, y, z satisfy $x^2 + y^2 + z^2 = x + y + z$. Prove that $$\frac{x+1}{\sqrt{x^5+x+1}} + \frac{y+1}{\sqrt{y^5+y+1}} + \frac{z+1}{\sqrt{z^5+z+1}} \geq 3.$$ When does the equality occur? *Proposed by Dorlir Ahmeti, Albania*
5	Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that $$x+y+z \geq \sqrt{\frac{xy+1}{2}} + \sqrt{\frac{yz+1}{2}} + \sqrt{\frac{zx+1}{2}}\ .$$ *Proposed by Azerbaijan* Let x, y, z be positive real numbers such that $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Prove that $$x+y+z \geq \sqrt{\frac{x^2+1}{2}} + \sqrt{\frac{y^2+1}{2}} + \sqrt{\frac{z^2+1}{2}}\ .$$
–	Combinatorics
1	Let S_n be the sum of reciprocal values of non-zero digits of all positive integers up to (and including) n. For instance, $S_{13} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3}$. Find the least positive integer k making the number $k! \cdot S_{2016}$ an integer.

2 The natural numbers from 1 to 50 are written down on the blackboard. At least how many of them should be deleted, in order that the sum of any two of the remaining numbers is not a prime?

3 A 5×5 table is called regular f each of its cells contains one of four pairwise distinct real numbers,such that each of them occurs exactly one in every 2×2 subtable.The sum of all numbers of a regular table is called the total sum of the table.With any four numbers,one constructs all possible regular tables,computes their total sums and counts the distinct outcomes.Determine the maximum possible count.

4 A splitting of a planar polygon is a fi nite set of triangles whose interiors are pairwise disjoint, and whose union is the polygon in question. Given an integer $n \geq 3$, determine the largest integer m such that no planar n-gon splits into less than m triangles.

– Geometry

1 Let ABC be an acute angled triangle, let O be its circumcentre, and let D, E, F be points on the sides BC, CA, AB, respectively. The circle (c_1) of radius FA, centered at F, crosses the segment OA at A' and the circumcircle (c) of the triangle ABCagain at K. Similarly, the circle (c_2) of radius DB, centered at D, crosses the segment (OB) at B' and the circle (c) again at L. Finally, the circle (c_3) of radius EC, centered at E, crosses the segment (OC)at C' and the circle (c) again at M. Prove that the quadrilaterals $BKFA', CLDB'$ and $AMEC'$ are all cyclic, and their circumcircles share a common point.

Evangelos Psychas (Greece)

2 Let ABC be a triangle with $\angle BAC = 60^{\circ}$. Let D and E be the feet of the perpendiculars from A to the external angle bisectors of $\angle ABC$ and $\angle ACB$, respectively. Let O be the circumcenter of the triangle ABC. Prove that the circumcircles of the triangles $ADE$and BOC are tangent to each other.

3 A trapezoid $ABCD$ $(AB||CF,AB > CD)$ is circumscribed.The incircle of the triangle ABC touches the lines AB and AC at the points M and N,respectively.Prove that the incenter of the trapezoid $ABCD$ lies on the line MN.

4 Let ABC be an acute angled triangle whose shortest side is BC. Consider a variable point P on the side BC, and let D and E be points on AB and AC, respectively, such that $BD = BP$ and $CP = CE$. Prove that, as P traces BC, the circumcircle of the triangle ADE passes through a fixed point.

5 Let ABC be an acute angled triangle with orthocenter H and circumcenter O. Assume the circumcenter X of BHClies on the circumcircle of ABC. Reflect O across X to obtain O', and let the lines $XH$and $O'A$ meet at K. Let L, M and N be the midpoints of $[XB]$, $[XC]$ and $[BC]$, respectively. Prove that the points K, L, M and K, L, M, N are cocyclic.

6 Given an acute triangle ABC, erect triangles ABD and ACE externally, so that $\angle ADB = \angle AEC = 90^o$ and $\angle BAD = \angle CAE$. Let $A_1 \in BC, B_1 \in AC$ and $C_1 \in AB$ be the feet of the altitudes of the triangle ABC, and let K and K, L be the midpoints of $[BC_1]$ and BC_1, CB_1, respectively. Prove that the circumcenters of the triangles $AKL, A_1B_1C_1$ and DEA_1 are collinear.

(Bulgaria)

7 Let AB be a chord of a circle (c) centered at O, and let K be a point on the segment AB such that $AK < BK$. Two circles through K, internally tangent to (c) at A and B, respectively, meet again at L. Let P be one of the points of intersection of the line $KL$and the circle (c), and let the lines $AB$and LOmeet at M. Prove that the line $MP$is tangent to the circle (c).

Theoklitos Paragyiou (Cyprus)

– Number Theory

1 Determine the largest positive integer n that divides $p^6 - 1$ for all primes $p > 7$.

2 Find the maximum number of natural numbers $x_1, x_2, ..., x_m$ satisfying the conditions:
a) No $x_i - x_j, 1 \le i < j \le m$ is divisible by 11, and
b) The sum $x_2x_3...x_m + x_1x_3...x_m + \cdots + x_1x_2...x_{m-1}$ is divisible by 11.

3 Find all positive integers n such that the number $A_n = \frac{2^{4n+2}+1}{65}$ is
a) an integer,
b) a prime.

4 Find all triplets of integers (a, b, c) such that the number

$$N = \frac{(a-b)(b-c)(c-a)}{2} + 2$$

is a power of 2016.

(A power of 2016 is an integer of form 2016^n,where n is a non-negative integer.)

5 Determine all four-digit numbers $\overline{abcd}$ such that $(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) = \overline{abcd}$: