SMO Open 2012 Round 1

1 Questions

1. The sum of the squares of 50 consecutive odd natural numbers is 300850. Find the largest odd integer whose square is the last term of this sum.

2. Find the value of $\sum_{k=3}^{1000} [\log_2 k]$.

3. Given that $f(x)$ is a polynomial of degree 2012, and that $f(k) = \frac{2}{k}$ for $k = 1, 2, 3, \ldots, 2013$, find the value of $2014 \times f(2014)$

4. Find the total number of sets of positive integers (x, y, z), where x, y and z are positive integers, with $x < y < z$ such that
$$x + y + z = 203.$$

5. There are a few integers n such that $n^2 + n + 1$ divides $n^2013 + 61$. Find the sum of the squares of these integers.

6. It is given that the sequence $(a_n)_{n=1}^{\infty}$, with $a_1 = a_2 = 2$, is given by the recurrence relation
$$\frac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2}$$
for all integers $n \geq 2$. Find the integer that is closest to the value of $\sum_{k=2}^{2011} \frac{a_{k+1}}{a_k}$.

7. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

8. The lengths of the sides of a triangle are successive terms of a geometric progression. Let A, B and C be the interior angles of the triangle with $A \leq B \leq C$. If the shortest side has length 16 cm and
$$\frac{\sin A - 2\sin B + 3\sin C}{\sin C - 2\sin C + 3\sin A} = \frac{19}{9},$$
find the perimeter of the triangle in centimeters.

9. Find the least positive integral value of n for which the equation
$$x_1^3 + x_2^3 + \cdots + x_n^3 = 2002^{2002}$$
has integer solutions $(x_1, x_2, x_3, \cdots, x_n)$.

10. Let α_n be a real root of the cubic equation $nx^3 + 2x - n = 0$, where n is a positive integer. If $\beta_n = [(n+1)\alpha_n]$ for $n = 2, 3, 4, \ldots$, find the value of $\frac{1}{1006} \sum_{k=2}^{2013} \beta_k$.

11. A point D is in a triangle ABC such that $\angle BAD = \angle BCD$ and $\angle BDC = 90^\circ$. Given that $AB = 5$ and $BC = 6$, and the point M is the midpoint of AC, find the value of $8 \times DM^2$.

12. Suppose the real numbers x and y satisfy the equations

$$x^3 - 3x^2 + 5x = 1, and\ y^3 - 3y^2 + 5y = 5.$$

Find the value of $x + y$.

13. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.

14. Determine the smallest integer n with $n \geq 2$ such that

$$\sqrt{\frac{(n+1)(2n+1)}{6}}$$

is an integer.

15. Given that f is a real-valued function on the set of all real numbers such that for any real numbers a and b,

$$f(af(b)) = ab.$$

Find the value of $|f(2011)|$.

16. The solutions to the equation $x^3 - 4[x] = 5$, where x is a real number, are denoted by $x_1, x_2, x_3, \ldots, x_k$ for some positive integer k. Find $\sum_{i=1}^{k} x_i^3$.

17. Determine the maximum integer solution of the equation $\sum_{i=1}^{1} 0[\frac{x}{i!}] = 1001$.

18. Let A, B, C be the three angles of a triangle. Let L be the maximum value of

$$|\sin 3A + \sin 3B + \sin 3C|.$$

Determine $[10L]$.

19. Determine the number of sets of solutions (x, y, z), where x, y, z are integers, of the equation $x^2 + y^2 + z^2 = x^2y^2$.

20. We can find sets of 13 distinct positive integers that add up to 2142. Find the largest possible greatest common divisor of these 13 distinct positive integers.

21. Determine the maximum number of different sets consisting of three terms which form an arithmetic progressions that can be chosen from a sequence of real numbers $a_1, a_2, \ldots, a_{101}$, where

$$a_1 < a_2 < \ldots < a_{101}.$$

(Note: The sets $\{1,3,5\}$ and $\{5,3,1\}$ are considered as the same set)

22. Find the value of the series $\displaystyle\sum_{k=0}^{\infty}[\frac{20121+2^k}{2^{k+1}}]$.

23. The sequence $(x_n)_{n=1}^{\infty}$ is defined recursively by

$$x_{n+1} = \frac{x_n + (2-\sqrt{3})}{1 - x_n(2-\sqrt{3})},$$

with $x_1 = 1$. Determine the value of $x_{1001} - x_{401}$.

24. Determine the maximum value of the following expression

$$|\cdots|||x_1 - x_2| - x_3| - x_4| \cdots - x_{2014}|$$

where $x_1, x_2, \cdots, x_2014$ are distinct numbers in the set $\{1,2,3,4,\cdots,2014\}$.

25. Evaluate $\frac{-1}{2^{2011}} \displaystyle\sum_{k=0}^{1006} (-1)^k 3^k \binom{2012}{2k}$.

2 Solutions