11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Guts Round

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1. [**5**] Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.

   Answer: $\boxed{(4, -2), (\frac{5}{2}, -5)}$ Since $10, a, b$ is an arithmetic progression, we have $a = \frac{1}{2}(10 + b)$. Also, we have $a + ab = 2b$, and so $a(1 + b) = 2b$. Substituting the expression for a gives $(10 + b)(1 + b) = 4b$. Solving this quadratic equation gives the solutions $b = -2$ and $b = -5$. The corresponding values for a can be found by $a = \frac{1}{2}(10 + b)$.

2. [**5**] Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. What is the radius of ω?

   Answer: $\boxed{\frac{25}{8}}$ Let O be the center of ω, and let M be the midpoint of AC. Since $OA = OC$, $OM \perp AC$. Also, $\angle OCM = \angle BAC$, and so triangles ABC and CMO are similar. Then, $CO/CM = AC/AB$, from which we obtain that the radius of ω is $CO = \frac{25}{8}$.

3. [**5**] How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

   Answer: $\boxed{2 \cdot 3^{2008}}$ Denote the colors A, B, C. The left-most column can be colored in 6 ways. For each subsequent column, if the kth column is colored with AB, then the $(k+1)$th column can only be colored with one of BA, BC, CA. That is, if we have colored the first k columns, then there are 3 ways to color the $(k + 1)$th column. It follows that the number of ways of coloring the board is 6×3^{2007}.

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4. [**6**] Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.

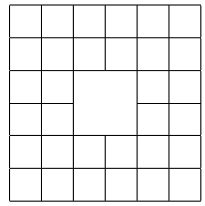
   Answer: $\boxed{(-1, 1)}$ Set $p = x + 1$ and $q = y - 1$, then we get $(p + q)^2 = pq$, which simplifies to $p^2 + pq + q^2 = 0$. Then we have $(p + \frac{q}{2})^2 + \frac{3q^2}{4}$, and so $p = q = 0$. Thus $(x, y) = (-1, 1)$.

5. [**6**] A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?

   Answer: $\boxed{\frac{416}{729}}$ Consider the error that was introduced on day 1. The probability that the Moderator misses this error on all three checks is $1/3^3$, so the probability that this error gets removed is $1 - \frac{1}{3^3}$. Similarly, the probability that the moderator misses the other two errors are $1 - \frac{1}{3^2}$ and $1 - \frac{1}{3}$. So the probability that the article is error-free is

$$\left(1 - \frac{1}{3^3}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{3}\right) = \frac{416}{729}.$$

6. [**6**] Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



Answer: $\boxed{297}$ First, let us count the total number of rectangles in the grid without the hole in the middle. There are $\binom{7}{2} = 21$ ways to choose the two vertical boundaries of the rectangle, and there are 21 ways to choose the two horizontal boundaries of the rectangles. This makes $21^2 = 441$ rectangles. However, we must exclude those rectangles whose boundary passes through the center point. We can count these rectangles as follows: the number of rectangles with the center of the grid lying in the interior of its south edge is $3 \times 3 \times 3 = 27$ (there are three choices for each of the three other edges); the number of rectangles whose south-west vertex coincides with the center is $3 \times 3 = 9$. Summing over all 4 orientations, we see that the total number of rectangles to exclude is $4(27 + 9) = 144$. Therefore, the answer is $441 - 144 = 297$.

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7. [**6**] Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \le y \le \pi/2$, find the value of $x + y$.

   Answer: $\boxed{2007 + \frac{\pi}{2}}$ Subtracting the two equations gives $\sin y - 2008 \cos y = 1$. But since $0 \le y \le \pi/2$, the maximum of $\sin y$ is 1 and the minimum of $\cos y$ is 0, so we must have $\sin y = 1$, so $y = \pi/2$ and $x + y = 2007 + \frac{\pi}{2}$.

8. [**6**] Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?

   Answer: $\boxed{1920}$ We look at the number of cottages after each wave of peasants. Let A_n be the number of cottages remaining after $8n$ minutes. During each 8 minute interval, Trodgor burns a total of $480/5 = 96$ peasants and cottages. Trodgor first burns A_n peasants and spends the remaining time burning $96 - A_n$ cottages. Therefore, as long as we do not reach negative cottages, we have the recurrence relation $A_{n+1} = A_n - (96 - A_n)$, which is equivalent to $A_{n+1} = 2A_n - 96$. Computing the first few terms of the series, we get that $A_1 = 84$, $A_2 = 72$, $A_3 = 48$, and $A_4 = 0$. Therefore, it takes Trodgor 32 minutes, which is 1920 seconds.

9. [**6**] Consider a circular cone with vertex V, and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV.

   Answer: $\boxed{\frac{\pi}{4-\pi}}$ Let R be the radius of the base, H the height of the cone, h the height of the pyramid and let $BL/LV = x/y$. Let $[\cdot]$ denote volume. Then $[\text{cone}] = \frac{1}{3}\pi R^2 H$ and $[ABCL] = \frac{1}{3}\pi R^2 h$ and $h = \frac{x}{x+y} H$. We are given that $[\text{cone}] = 4[ABCL]$, so $x/y = \frac{\pi}{4-\pi}$.

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10. [**7**] Find the number of subsets S of $\{1, 2, \ldots 63\}$ the sum of whose elements is 2008.

    Answer: $\boxed{6}$ Note that $1 + 2 + \cdots + 63 = 2016$. So the problem is equivalent to finding the number of subsets of $\{1, 2, \cdots 63\}$ whose sum of elements is 8. We can count this by hand: $\{8\}$, $\{1, 7\}$, $\{2, 6\}$, $\{3, 5\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$.

11. [**7**] Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

Answer: $\boxed{\frac{2007}{2008}}$ We change the order of summation:

$$\sum_{k=2}^{\infty} \sum_{j=2}^{2008} \frac{1}{j^k} = \sum_{j=2}^{2008} \sum_{k=2}^{\infty} \frac{1}{j^k} = \sum_{j=2}^{2008} \frac{1}{j^2(1-\frac{1}{j})} = \sum_{j=2}^{2008} \frac{1}{j(j-1)} = \sum_{j=2}^{2008} \left(\frac{1}{j-1} - \frac{1}{j} \right) = 1 - \frac{1}{2008} = \frac{2007}{2008}.$$

12. [**7**] Suppose we have an (infinite) cone $\mathcal{C}$ with apex A and a plane π. The intersection of π and $\mathcal{C}$ is an ellipse $\mathcal{E}$ with major axis BC, such that B is closer to A than C, and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of $\mathcal{C}$ cut up by $\mathcal{E}$ with both spheres tangent to $\mathcal{E}$. What is the ratio of the radii of the spheres (smaller to larger)?

Answer: $\boxed{\frac{1}{3}}$ It can be seen that the points of tangency of the spheres with E must lie on its major axis due to symmetry. Hence, we consider the two-dimensional cross-section with plane ABC. Then the two spheres become the incentre and the excentre of the triangle ABC, and we are looking for the ratio of the inradius to the exradius. Let s, r, r_a denote the semiperimeter, inradius, and exradius (opposite to A) of the triangle ABC. We know that the area of ABC can be expressed as both rs and $r_a(s - |BC|)$, and so $\frac{r}{r_a} = \frac{s-|BC|}{s}$. For the given triangle, $s = 6$ and $a = 4$, so the required ratio is $\frac{1}{3}$.

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13. [**8**] Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \ldots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

Answer: $\boxed{2008! - 1}$ Consider the polynomial $Q(x) = P(x) + x - 2007$. The given conditions tell us that $Q(x) = 0$ for $x = 0, 1, 2, \ldots, 2007$, so these are the roots of $Q(x)$. On the other hand, we know that $Q(x)$ is also a polynomial with degree 2008 and leading coefficient 1. It follows that $Q(x) = x(x-1)(x-2)(x-3)\cdots(x-2007)$. Thus

$$P(x) = x(x-1)(x-2)(x-3)\cdots(x-2007) - x + 2007.$$

Setting $x = 2008$ gives the answer.

14. [**8**] Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

Answer: $\boxed{\frac{3}{8}}$ We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{n^4+4} &= \sum_{n=1}^{\infty} \frac{n}{(n^2+2n+2)(n^2-2n+2)} \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{n^2-2n+2} - \frac{1}{n^2+2n+2} \right) \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)^2+1} - \frac{1}{(n+1)^2+1} \right).
\end{aligned}$$

Observe that the sum telescopes. From this we find that the answer is $\frac{1}{4}\left(\frac{1}{0^2+1} + \frac{1}{1^2+1}\right) = \frac{3}{8}$.

15. [**8**] In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

Answer: $\boxed{\frac{5}{21}}$ If Bob initially chooses a door with a prize, then he will not find a prize by switching. With probability 5/7 his original door does not hide the prize. After the host opens the three doors, the remaining three doors have equal probability of hiding the prize. Therefore, the probability that Bob finds the prize is $\frac{5}{7} \times \frac{1}{3} = \frac{5}{21}$.

Remark: This problem can be easily recognized as a variation of the classic Monty Hall problem.

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16. [**9**] Point A lies at $(0,4)$ and point B lies at $(3,8)$. Find the x-coordinate of the point X on the x-axis maximizing $\angle AXB$.

Answer: $\boxed{5\sqrt{2}-3}$ Let X be a point on the x-axis and let $\theta = \angle AXB$. We can easily see that the circle with diameter AB does not meet the x-axis, so $\theta \le \pi$. Thus, maximizing θ is equivalent to maximizing $\sin\theta$. By the Law of Sines, this in turn is equivalent to minimizing the circumradius of triangle ABX. This will occur when the circumcircle of ABX is the smaller of the two circles through A and B tangent to the x-axis. So let X now be this point of tangency. Extend line AB to meet the x-axis at $C = (-3,0)$; by Power of a Point $CX^2 = CA \cdot CB = 50$ so $CX = 5\sqrt{2}$. Clearly X has larger x-coordinate than C, so the x-coordinate of X is $5\sqrt{2}-3$.

17. [**9**] Solve the equation

$$\sqrt{x+\sqrt{4x+\sqrt{16x+\sqrt{\ldots+\sqrt{4^{2008}x+3}}}}}-\sqrt{x}=1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

Answer: $\boxed{\frac{1}{2^{4016}}}$ Rewrite the equation to get

$$\sqrt{x+\sqrt{4x+\sqrt{16x+\sqrt{\ldots+\sqrt{4^{2008}x+3}}}}}=\sqrt{x}+1.$$

Squaring both sides yields

$$\sqrt{4x+\sqrt{\ldots+\sqrt{4^{2008}x+3}}}=2\sqrt{x}+1.$$

Squaring again yields

$$\sqrt{16x+\sqrt{\ldots+\sqrt{4^{2008}x+3}}}=4\sqrt{x}+1.$$

One can see that by continuing this process one gets

$$\sqrt{4^{2008}x+3}=2^{2008}\sqrt{x}+1,$$

so that $2\cdot 2^{2008}\sqrt{x}=2$. Hence $x = 4^{-2008}$. It is also easy to check that this is indeed a solution to the original equation.

18. [**9**] Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135^\circ$, determine BC/AB.

Answer: $\boxed{\frac{\sqrt{13}}{2}}$ Let $\alpha = \angle ADC$ and $\beta = \angle ABE$. By exterior angle theorem, $\alpha = \angle BFD + \beta = 45^\circ + \beta$. Also, note that $\tan\beta = AE/AB = AD/AB = 1/2$. Thus,

$$1 = \tan 45^\circ = \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\tan\alpha - \frac{1}{2}}{1+\frac{1}{2}\tan\alpha}.$$

Solving for $\tan\alpha$ gives $\tan\alpha = 3$. Therefore, $AC = 3AD = \frac{3}{2}AB$. Using Pythagorean Theorem, we find that $BC = \frac{\sqrt{13}}{2}AB$. So the answer is $\frac{\sqrt{13}}{2}$.

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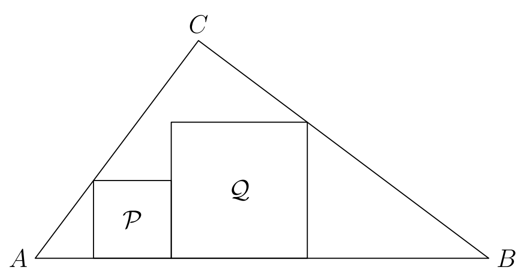
19. [**10**] Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD. Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin\angle PBO$.

Answer: $\boxed{\frac{1}{6}}$ We translate the problem into one about 2-D geometry. Consider the right triangle ABO, and P is some point on AO. Then, the choice of P minimizes $PA + 6PB$. Construct the line ℓ through A but outside the triangle ABO so that $\sin\angle(AO, \ell) = \frac{1}{6}$. For whichever P chosen, let Q be the projection of P onto ℓ, then $PQ = \frac{1}{6}AP$. Then, since $PA + 6PB = 6(PQ + PB)$, it is equivalent to minimize $PQ + PB$. Observe that this sum is minimized when B, P, Q are collinear and the line through them is perpendicular to ℓ (so that $PQ + PB$ is simply the distance from B to ℓ). Then, $\angle AQB = 90°$, and since $\angle AOB = 90°$ as well, we see that A, Q, P, B are concyclic. Therefore, $\angle PBO = \angle OPA = \angle(AO, \ell)$, and the sine of this angle is therefore $\frac{1}{6}$.

20. [**10**] For how many ordered triples (a, b, c) of positive integers are the equations $abc + 9 = ab + bc + ca$ and $a + b + c = 10$ satisfied?

Answer: $\boxed{21}$ Subtracting the first equation from the second, we obtain $1-a-b-c+ab+bc+ca-abc = (1-a)(1-b)(1-c) = 0$. Since a, b, and c are positive integers, at least one must equal 1. Note that $a = b = c = 1$ is not a valid triple, so it suffices to consider the cases where exactly two or one of a, b, c are equal to 1. If $a = b = 1$, we obtain $c = 8$ and similarly for the other two cases, so this gives 3 ordered triples. If $a = 1$, then we need $b + c = 9$, which has 6 solutions for $b, c \neq 1$; a similar argument for b and c gives a total of 18 such solutions. It is easy to check that all the solutions we found are actually solutions to the original equations. Adding, we find $18 + 3 = 21$ total triples.

21. [**10**] Let ABC be a triangle with $AB = 5$, $BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Determine the minimum value of the sum of the areas of the two squares.



Answer: $\boxed{\frac{144}{49}}$ Let the side lengths of $\mathcal{P}$ and $\mathcal{Q}$ be a and b, respectively. Label two of the vertices of $\mathcal{P}$ as D and E so that D lies on AB and E lies on AC, and so that DE is perpendicular to AB. The triangle ADE is similar to ACB. So $AD = \frac{3}{4}a$. Using similar arguments, we find that

$$\frac{3a}{4} + a + b + \frac{4b}{3} = AB = 5$$

so

$$\frac{a}{4} + \frac{b}{3} = \frac{5}{7}.$$

Using Cauchy-Schwarz inequality, we get

$$(a^2 + b^2)\left(\frac{1}{4^2} + \frac{1}{3^2}\right) \geq \left(\frac{a}{4} + \frac{b}{3}\right)^2 = \frac{25}{49}.$$

It follows that

$$a^2 + b^2 \geq \frac{144}{49}.$$

Equality occurs at $a = \frac{36}{35}$ and $b = \frac{48}{35}$.

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22. [**10**] For a positive integer n, let $\theta(n)$ denote the number of integers $0 \leq x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

Answer: $\boxed{335}$ Let us consider the sum $\sum_{n=0}^{2009} n \cdot \theta(n) \pmod{2010}$ in a another way. Consider the sum $0^2 + 1^2 + 2^2 + \cdots + 2007^2 \pmod{2010}$. For each $0 \leq n < 2010$, in the latter sum, the term n appears $\theta(n)$ times, so the sum is congruent to $\sum_{n=0}^{2009} n \cdot \theta(n)$. In other words,

$$\sum_{n=0}^{2009} n \cdot \theta(n) = \sum_{n=0}^{2009} n^2 = \frac{(2009)(2009+1)(2 \cdot 2009 + 1)}{6} \equiv (-1) \cdot \frac{2010}{6} \cdot (-1) = 335 \pmod{2010}.$$

23. [**10**] Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n, and it tells Jason the number of divisors of n. Then, Kelly and Jason have the following dialogue:

Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

Jason: Oh, then I know what the value of n is.

Kelly: Now I also know what n is.

Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n?

Answer: $\boxed{10}$ The only way in which Kelly can know that n is divisible by at least two different primes is if she is given 0 as the unit digit of n, since if she received anything else, then there is some number with that unit digit and not divisible by two primes (i.e., $1, 2, 3, 4, 5, 16, 7, 8, 9$). Then, after Kelly says the first line, Jason too knows that n is divisible by 10.

The number of divisors of $10, 20, 30, 40, 50$ are $4, 6, 8, 8, 6$, respectively. So unless Jason received 4, he cannot otherwise be certain of what n is. It follows that Jason received 4, and thus $n = 10$.

24. [**10**] Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC.

Answer: $\boxed{\frac{9}{10}}$ Let Q be the point on AB so that $AQ = 2BQ$, and let X be the intersection of BP and CQ. The key observation that, as we will show, BX and CX are fixed lengths, and the ratio of areas $[ABC]/[BCX]$ is constant. So, to maximize $[ABC]$, it is equivalent to maximize $[BCX]$.

Using Menelaus' theorem on ABP, we have

$$\frac{BX \cdot PC \cdot AQ}{XP \cdot CA \cdot QB} = 1.$$

Since $PC/CA = 1/3$ and $AQ/QB = 2$, we get $BX/XP = 3/2$. It follows that $BX = 3/5$. By symmetry, $CX = 3/5$.

Also, we have

$$[ABC] = 3[BPC] = 3 \cdot \frac{5}{3}[BXC] = 5[BXC].$$

Note that $[BXC]$ is maximized when $\angle BXC = 90^\circ$ (one can check that this configuration is indeed possible). Thus, the maximum value of $[BXC]$ is $\frac{1}{2}BX \cdot CX = \frac{1}{2}\left(\frac{3}{5}\right)^2 = \frac{9}{50}$. It follows that the maximum value of $[ABC]$ is $\frac{9}{10}$.

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25. [**12**] Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

Answer: $\boxed{35}$ Consider the number of beans Alice has in base 5. Note that $2008 = 31013_5$, $42 = 132_5$, and $100 = 400_5$. Now, suppose Alice has $d_k \cdots d_2 d_1$ beans when she wins; the conditions for winning mean that these digits must satisfy $d_2 d_1 = 32$, $d_k \cdots d_3 \geq 310$, and $d_k \cdots d_3 = 4i + 1$ for some i. To gain these $d_k \cdots d_2 d_1$ beans, Alice must spend at least $5(d_1 + d_2 + \cdots + d_k) + k - 1$ cents (5 cents to get each bean in the "units digit" and $k-1$ cents to promote all the beans). We now must have $k \geq 5$ because $d_k \cdots d_2 d_1 > 2008$. If $k = 5$, then $d_k \geq 3$ since $d_k \cdots d_3 \geq 3100$; otherwise, we have $d_k \geq 1$. Therefore, if $k = 5$, we have $5(d_1 + d_2 + \cdots + d_k) + k - 1 \geq 44 > 36$; if $k > 5$, we have $5(d_1 + d_2 + \cdots + d_k) + k - 1 \geq 30 + k - 1 \geq 35$. But we can attain 36 cents by taking $d_k \cdots d_3 = 1000$, so this is indeed the minimum.

26. [**12**] Let $\mathcal{P}$ be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on $\mathcal{P}$ so that $\angle AV_1B = 90^\circ$. Let $\mathcal{Q}$ be the locus of the midpoint of AB. It turns out that $\mathcal{Q}$ is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio F_1F_2/V_1V_2.

Answer: $\boxed{\frac{7}{8}}$ Since all parabolas are similar, we may assume that $\mathcal{P}$ is the curve $y = x^2$. Then, if $A = (a, a^2)$ and $B = (b, b^2)$, the condition that $\angle AV_1B = 90^\circ$ gives $ab + a^2b^2 = 0$, or $ab = -1$. Then, the midpoint of AB is

$$\frac{A+B}{2} = \left(\frac{a+b}{2}, \frac{a^2+b^2}{2}\right) = \left(\frac{a+b}{2}, \frac{(a+b)^2 - 2ab}{2}\right) = \left(\frac{a+b}{2}, \frac{(a+b)^2}{2} + 1\right).$$

(Note that $a + b$ can range over all real numbers under the constraint $ab = -1$.) It follows that the locus of the midpoint of AB is the curve $y = 2x^2 + 1$.

Recall that the focus of $y = ax^2$ is $(0, \frac{1}{4a})$. We find that $V_1 = (0,0)$, $V_2 = (0,1)$, $F_1 = (0, \frac{1}{4})$, $F_2 = (0, 1 + \frac{1}{8})$. Therefore, $F_1F_2/V_1V_2 = \frac{7}{8}$.

27. [**12**] Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

Answer: $\boxed{7}$ By Pythagoras, $AC = 25$. Since $\overline{AC}$ is a diameter, angles $\angle ADC$ and $\angle AEC$ are also right, so that $CE = 20$ and $AD^2 + CD^2 = AC^2$ as well. Beginning with Ptolemy's theorem,

$$\begin{aligned}
&(AE \cdot CD + AC \cdot DE)^2 = AD^2 \cdot EC^2 = \left(AC^2 - CD^2\right) EC^2 \\
&\quad \Longrightarrow CD^2\left(AE^2 + EC^2\right) + 2 \cdot CD \cdot AE^2 \cdot AC + AC^2\left(DE^2 - EC^2\right) = 0 \\
&\quad \Longrightarrow CD^2 + 2CD\left(\frac{AE^2}{AC}\right) + DE^2 - EC^2 = 0.
\end{aligned}$$

It follows that $CD^2 + 18CD - 175 = 0$, from which $CD = 7$.

Remark: A simple trigonometric solution is possible. One writes $\alpha = \angle ACE = \angle ECD \Longrightarrow \angle DAC = 90^\circ - 2\alpha$ and applies double angle formula.

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28. [**15**] Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

    Answer: $\boxed{\frac{\sqrt{2}}{3}}$ Observe that P is a truncated octahedron, formed by cutting off the corners from a regular octahedron with edge length 3. So, to compute the value of P, we can find the volume of the octahedron, and then subtract off the volume of truncated corners. Given a square pyramid where each triangular face an equilateral triangle, and whose side length is s, the height of the pyramid is $\frac{\sqrt{2}}{2}s$, and thus the volume is $\frac{1}{3} \cdot s^2 \cdot \frac{\sqrt{2}}{2}s = \frac{\sqrt{2}}{6}s^3$. The side length of the octahedron is 3, and noting that the octahedron is made up of two square pyramids, its volume must be is $2 \cdot \frac{\sqrt{2}(3)^3}{6} = 9\sqrt{2}$. The six "corners" that we remove are all square pyramids, each with volume $\frac{\sqrt{2}}{6}$, and so the resulting polyhedron P has volume $9\sqrt{2} - 6 \cdot \frac{\sqrt{2}}{6} = 8\sqrt{2}$.

    Finally, to find the volume of all points closer to one particular vertex than any other vertex, note that due to symmetry, every point in P (except for a set with zero volume), is closest to one of the 24 vertices. Due to symmetry, it doesn't matter which V is picked, so we can just divide the volume of P by 24 to obtain the answer $\frac{\sqrt{2}}{3}$.

29. [**15**] Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2 + y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2 + y^2}.$$

    Determine the value of $|x| + |y|$.

    Answer: $\boxed{\frac{11}{65}}$ Observe that

$$\frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2} = 33x - 56y + (56x + 33y)i = (33 + 56i)(x + yi).$$

    So

$$(x + yi)^2 = \frac{1}{33 + 56i} = \frac{1}{(7 + 4i)^2} = \left(\frac{7 - 4i}{65}\right)^2.$$

    It follows that $(x, y) = \pm\left(\frac{7}{65}, -\frac{4}{65}\right)$.

30. [**15**] Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120°$. Points P and Q lie on segment BC such that

$$AB^2 + BC \cdot CP = BC^2$$
$$3AC^2 + 2BC \cdot CQ = BC^2$$

    Find $\angle PAQ$ in degrees.

    Answer: $\boxed{40°}$ We have $AB^2 = BC(BC - CP) = BC \cdot BP$, so triangle ABC is similar to triangle PBA. Also, $AB^2 = BC(BC-2CQ)+AC^2 = (BC-CQ)^2-CQ^2+AC^2$, which rewrites as $AB^2+CQ^2 = BQ^2+AC^2$. We deduce that Q is the foot of the altitude from A. Thus, $\angle PAQ = 90° - \angle QPA = 90° - \angle ABP - \angle BAP$. Using the similar triangles, $\angle PAQ = 90° - \angle ABC - \angle BCA = \angle BAC - 90° = 40°$.

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31. [**18**] Let $\mathcal{C}$ be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points (P_n) on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing passing

through P_n, then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)

Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

Answer: $\boxed{254}$ Let $P_n = (x_n, 0)$. Then the ℓ_n meet $\mathcal{C}$ at $(x_{n+1}, x_{n+1} - x_n)$. Since this point lies on the hyperbola, we have $(x_{n+1} - x_n)^2 - x_{n+1}^2 = 1$. Rearranging this equation gives

$$x_{n+1} = \frac{x_n^2 - 1}{2x_n}.$$

Choose a $\theta_0 \in (0, \pi)$ with $\cot\theta_0 = x_0$, and define $\theta_n = 2^n\theta_0$. Using the double-angle formula, we have

$$\cot\theta_{n+1} = \cot(2\theta_n) = \frac{\cot^2\theta_n - 1}{2\cot\theta_n}.$$

It follows by induction that $x_n = \cot\theta_n$. Then, $P_0 = P_{2008}$ corresponds to $\cot\theta_0 = \cot\left(2^{2008}\theta_0\right)$ (assuming that P_0 is never at the origin, or equivalently, $2^n\theta$ is never an integer multiple of π). So, we need to find the number of $\theta_0 \in (0, \pi)$ with the property that $2^{2008}\theta_0 - \theta_0 = k\pi$ for some integer k. We have $\theta_0 = \frac{k\pi}{2^{2008}-1}$, so k can be any integer between 1 and $2^{2008} - 2$ inclusive (and note that since the denominator is odd, the sequence never terminates). It follows that the number of starting positions is $N = 2^{2008} - 2$.

Finally, we need to compute the remainder when N is divided by 2008. We have $2008 = 2^3 \times 251$. Using Fermat's Little Theorem with 251, we get $2^{2008} \equiv \left(2^{250}\right)^4 \cdot 256 \equiv 1^4 \cdot 5 = 5 \pmod{251}$. So we have $N \equiv 3 \pmod{251}$ and $N \equiv -2 \pmod{8}$. Using Chinese Remainder Theorem, we get $N \equiv 254 \pmod{2008}$.

32. [**18**] Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

Answer: $\boxed{\frac{225\sqrt{11}}{88}}$ Let C' be the point on minor arc BCD such that $BC' = 12$ and $C'D = 5$, and write $AC' = BD = C'E = x$, $AD = y$, and $BD = z$. Ptolemy applied to quadrilaterals $ABC'D, BC'DE$, and $ABDE$ gives

$$\begin{aligned} x^2 &= 12y + 5^2 \\ x^2 &= 5z + 12^2 \\ yz &= 14x + 5 \cdot 12 \end{aligned}$$

Then

$$(x^2 - 5^2)(x^2 - 12^2) = 5 \cdot 12yz = 5 \cdot 12 \cdot 14x + 5^2 \cdot 12^2,$$

from which $x^3 - 169x - 5 \cdot 12 \cdot 14 = 0$. Noting that $x > 13$, the rational root theorem leads quickly to the root $x = 15$. Then triangle BCD has area $\sqrt{16 \cdot 1 \cdot 4 \cdot 11} = 8\sqrt{11}$ and circumradius $R = \frac{5 \cdot 12 \cdot 15}{4 \cdot 8\sqrt{11}} = \frac{225\sqrt{11}}{88}$.

33. [**18**] Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.

Answer: $\boxed{\frac{6}{5}}$ Let $\sigma_1 = a + b + c$, $\sigma_2 = ab + bc + ca$ and $\sigma_3 = abc$ be the three elementary symmetric polynomials. Since $a^3 + b^3 + c^3$ is a symmetric polynomial, it can be written as a polynomial in σ_1, σ_2 and σ_3. Now, observe that $\sigma_1 = 0$, and so we only need to worry about the terms not containing σ_1. By considering the degrees of the terms, we see that the only possibility is σ_3. That is, $a^3 + b^3 + c^3 = k\sigma_3$ for some constant k. By setting $a = b = 1$, $c = -2$, we see that $k = 3$.

By similar reasoning, we find that $a^5 + b^5 + c^5 = h\sigma_2\sigma_3$ for some constant h. By setting $a = b = 1$ and $c = -2$, we get $h = -5$.

So, we now know that $a+b+c=0$ implies

$$a^3+b^3+c^3=3abc \qquad \text{and} \qquad a^5+b^5+c^5=-5abc(ab+bc+ca)$$

Then $a^3+b^3+c^3=a^5+b^5+c^5$ implies that $3abc=-5abc(ab+bc+ca)$. Given that a,b,c are nonzero, we get $ab+bc+ca=-\frac{3}{5}$. Then, $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=\frac{6}{5}$.

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34. **Who Wants to Be a Millionaire.** In 2000, the Clay Mathematics Institute named seven *Millennium Prize Problems*, with each carrying a prize of \$1 Million for its solution. Write down the name of ONE of the seven Clay Millennium Problems. If your submission is incorrect or misspelled, then your submission is disqualified. If another team wrote down the same Millennium Problem as you, then you get 0 points, otherwise you get 20 points.

    Solution: The seven Millennium Prize Problems are:

    (a) Birch and Swinnerton-Dyer Conjecture
    (b) Hodge Conjecture
    (c) Navier-Stokes Equations
    (d) P vs NP
    (e) Poincaré Conjecture
    (f) Riemann Hypothesis
    (g) Yang-Mills Theory

    More information can be found on its official website *http://www.claymath.org/millennium/*.

    As far as this as an HMMT problem goes, it's probably a good idea to submit something that you think is least likely for another team to think of (or to spell correctly). Though, this may easily turn into a contest of who can still remember the names of the user ranks from the Art of Problem Solving forum.

35. **NUMB3RS.** The RSA Factoring Challenge, which ended in 2007, challenged computational mathematicians to factor extremely large numbers that were the product of two prime numbers. The largest number successfully factored in this challenge was RSA-640, which has 193 decimal digits and carried a prize of \$20,000. The next challenge number carried prize of \$30,000, and contains N decimal digits. Your task is to submit a guess for N. Only the team(s) that have the closest guess(es) receives points. If k teams all have the closest guesses, then each of them receives $\lceil \frac{20}{k} \rceil$ points.

    Answer: $\boxed{212}$ For more information, see the Wikipedia entry at `http://en.wikipedia.org/wiki/RSA_Factoring_Challenge`.

    RSA-640 was factored in November 2005, and the effort took approximately 30 2.2GHz-Opteron-CPU years over five months of calendar time.

36. **The History Channel.** Below is a list of famous mathematicians. Your task is to list a subset of them in the chronological order of their birth dates. Your submission should be a sequence of letters. If your sequence is not in the correct order, then you get 0 points. Otherwise your score will be $\min\{\max\{5(N-4),0\},25\}$, where N is the number of letters in your sequence.

    (A) Niels Abel (B) Arthur Cayley (C) Augustus De Morgan (D) Gustav Dirichlet (E) Leonhard Euler (F) Joseph Fourier (G) Évariste Galois (H) Carl Friedrich Gauss (I) Marie-Sophie Germain (J) Joseph Louis Lagrange (K) Pierre-Simon Laplace (L) Henri Poincaré (N) Bernhard Riemann

    Answer: $\boxed{\text{any subsequence of EJKFIHADCGBNL}}$ The corresponding birth dates are listed below:

    (A) Niels Abel (1802–1829)

(B) Arthur Cayley (1821–1895)
(C) Augustus De Morgan (1806–1871)
(D) Gustav Dirichlet (1805–1859)
(E) Leonhard Euler (1707–1783)
(F) Joseph Fourier (1768–1830)
(G) Évariste Galois (1811–1832)
(H) Carl Friedrich Gauss (1777–1855)
(I) Marie-Sophie Germain (1776–1831)
(J) Joseph Louis Lagrange (1736–1813)
(K) Pierre-Simon Laplace (1749–1827)
(L) Henri Poincaré (1854–1912)
(N) Bernhard Riemann (1826–1866)