Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section)

Tuesday, 2 June 2009 — 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

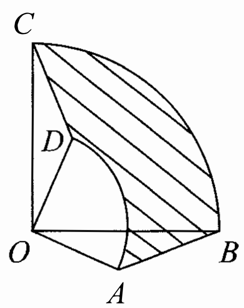
PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1 Let C_1 and C_2 be distinct circles of radius 7 cm that are in the same plane and tangent to each other. Find the number of circles of radius 26 cm in this plane that are tangent to both C_1 and C_2.

(A) 2
(B) 4
(C) 6
(D) 8
(E) none of the above

2 In the diagram below, the radius of quadrant OAD is 4 and the radius of quadrant OBC is 8. Given that $\angle COD = 30°$, find the area of the shaded region $ABCD$.



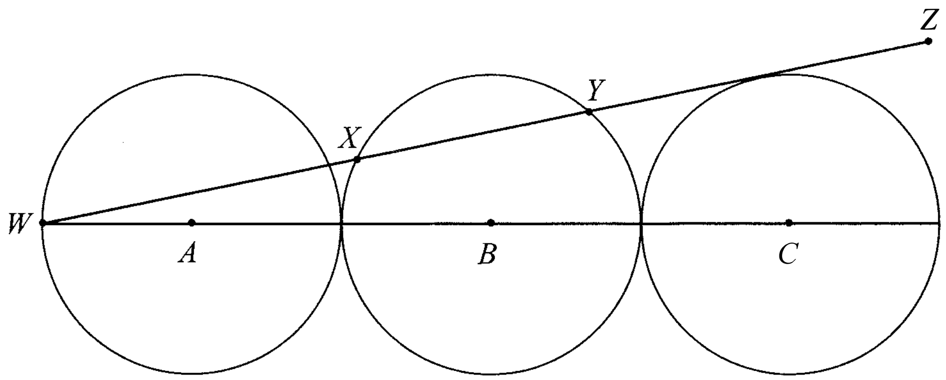
(A) 12π
(B) 13π
(C) 15π
(D) 16π
(E) none of the above

3 Let k be a real number. Find the maximum value of k such that the following inequality holds:

$$\sqrt{x-2} + \sqrt{7-x} \geq k.$$

(A) $\sqrt{5}$
(B) 3
(C) $\sqrt{2}+\sqrt{3}$
(D) $\sqrt{10}$
(E) $2\sqrt{3}$

4 Three circles of radius 20 are arranged with their respective centres A, B and C in a row. If the line WZ is tangent to the third circle, find the length of XY.



(A) 30
(B) 32
(C) 34
(D) 36
(E) 38

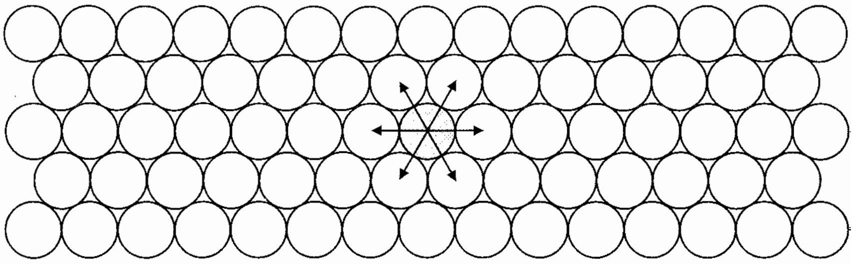
5 Given that x and y are both negative integers satisfying the equation $y = \dfrac{10x}{10-x}$, find the maximum value of y.

(A) −10
(B) −9
(C) −6
(D) −5
(E) None of the above

6 The sequence a_n satisfy $a_n = a_{n-1} + n^2$ and $a_0 = 2009$. Find a_{50}.

(A) 42434
(B) 42925
(C) 44934
(D) 45029
(E) 45359

7 Coins of the same size are arranged on a very large table (the infinite plane) such that each coin touches six other coins. Find the percentage of the plane that is covered by the coins.



(A) $\frac{20}{\sqrt{3}}\pi\ \%$

(B) $\frac{50}{\sqrt{3}}\pi\ \%$

(C) $16\sqrt{3}\,\pi\ \%$

(D) $17\sqrt{3}\,\pi\ \%$

(E) $18\sqrt{3}\,\pi\ \%$

8 Given that x and y are real numbers satisfying the following equations:

$$x + xy + y = 2 + 3\sqrt{2} \quad \text{and} \quad x^2 + y^2 = 6,$$

find the value of $|x + y + 1|$.

(A) $1 + \sqrt{3}$

(B) $2 - \sqrt{3}$

(C) $2 + \sqrt{3}$

(D) $3 - \sqrt{2}$

(E) $3 + \sqrt{2}$

9 Given that $y = (x - 16)(x - 14)(x + 14)(x + 16)$, find the minimum value of y.

(A) -896

(B) -897

(C) -898

(D) -899

(E) -900

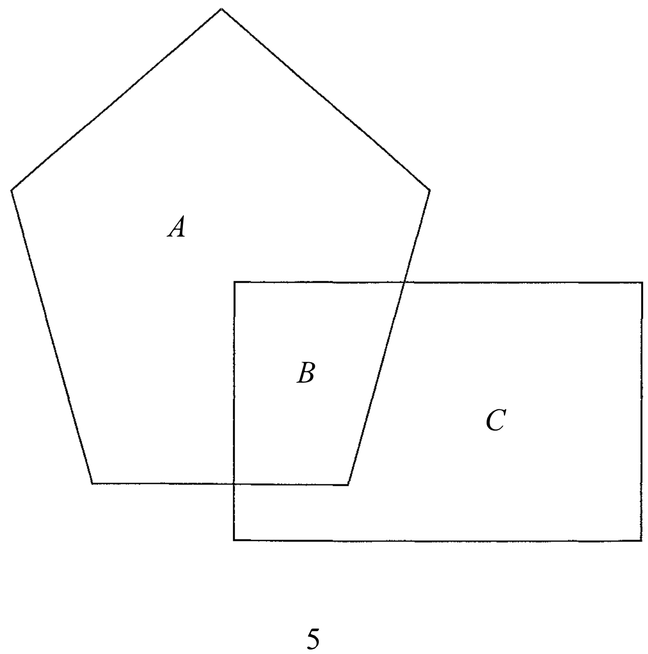
10 The number of positive integral solutions (a, b, c, d) satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1$ with the condition that $a<b<c<d$ is

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

Short Questions

11 There are two models of LCD television on sale. One is a '20 inch' standard model while the other is a '20 inch' widescreen model. The ratio of the length to the height of the standard model is 4 : 3, while that of the widescreen model is 16 : 9. Television screens are measured by the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is A : 300, find the value of A.

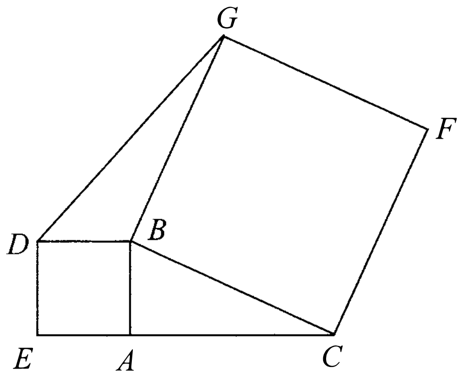
12 The diagram below shows a pentagon (made up of region A and region B) and a rectangle (made up of region B and region C) that overlaps. The overlapped region B is $\frac{3}{16}$ of the pentagon and $\frac{2}{9}$ of the rectangle. If the ratio of region A of the pentagon to region C of the rectangle is $\frac{m}{n}$ in its lowest term, find the value of $m+n$.



13 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with at least nine identical answers.

14 The number of ways to arrange 5 boys and 6 girls in a row such that girls can be adjacent to other girls but boys cannot be adjacent to other boys is $6! \times k$. Find the value of k.

15 ABC is a right-angled triangle with $\angle BAC = 90°$. A square is constructed on the side AB and BC as shown. The area of the square $ABDE$ is 8 cm^2 and the area of the square $BCFG$ is 26 cm^2. Find the area of triangle DBG in cm^2.



16 The sum of $\frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \frac{1}{4 \times 5 \times 6} + \cdots + \frac{1}{13 \times 14 \times 15} + \frac{1}{14 \times 15 \times 16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

17 Given that $a + \frac{1}{a+1} = b + \frac{1}{b-1} - 2$ and $a - b + 2 \neq 0$, find the value of $ab - a + b$.

18 If $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y + 2009$.

19 Let p and q represent two consecutive prime numbers. For some fixed integer n, the set $\{n - 1, 3n - 19, 38 - 5n, 7n - 45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of n.

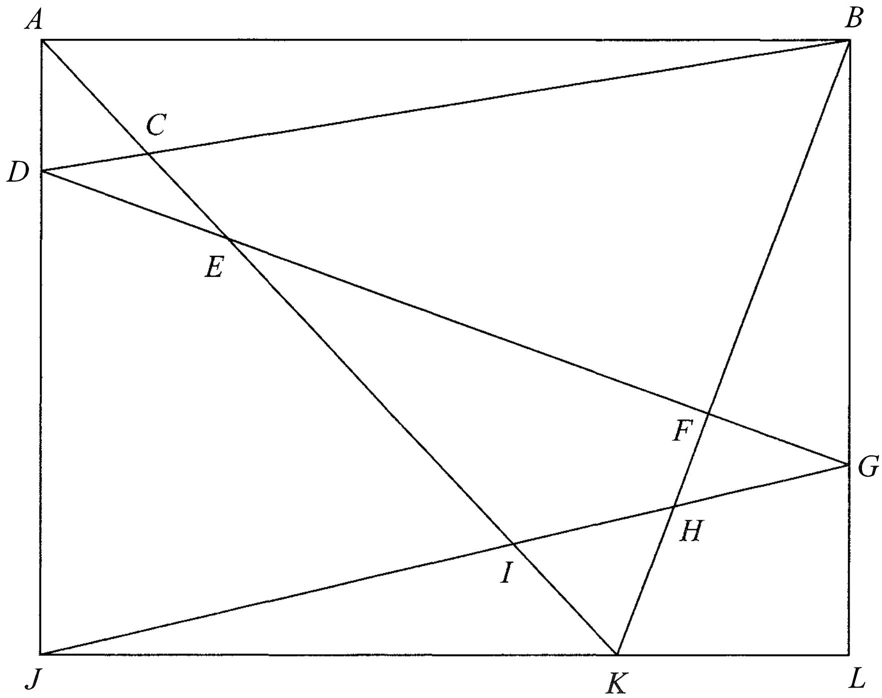
20 Find the number of ordered pairs of positive integers (x, y) that satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2009xy} - \sqrt{2009x} - \sqrt{2009y} - 2009 = 0.$$

21 Find the integer part of

$$\frac{1}{\frac{1}{2003} + \frac{1}{2004} + \frac{1}{2005} + \frac{1}{2006} + \frac{1}{2007} + \frac{1}{2008} + \frac{1}{2009}}.$$

22 The diagram below shows a rectangle $ABLJ$, where the area of ACD, $BCEF$, $DEIJ$ and FGH are 22 cm^2, 500 cm^2, 482 cm^2 and 22 cm^2 respectively. Find the area of HIK in cm^2.



23 Evaluate $\sqrt[3]{77 - 20\sqrt{13}} + \sqrt[3]{77 + 20\sqrt{13}}$.

24 Find the number of integers in the set $\{1, 2, 3, \ldots, 2009\}$ whose sum of the digits is 11.

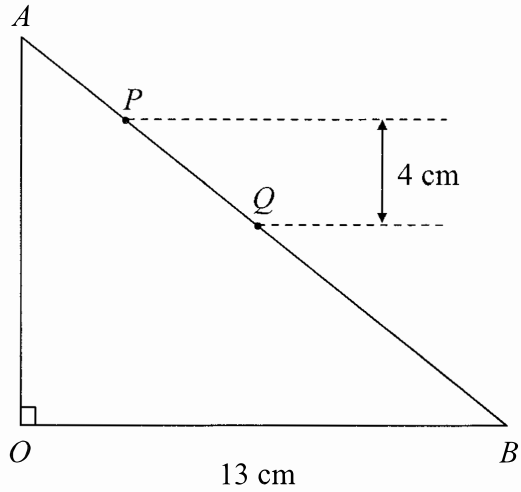
25 Given that

$$x + (1+x)^2 + (1+x)^3 + \dots + (1+x)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n,$$

where each a_r is an integer, $r = 0, 1, 2, \dots, n$.

Find the value of n such that $a_0 + a_2 + a_3 + a_4 + \dots + a_{n-2} + a_{n-1} = 60 - \dfrac{n(n+1)}{2}$.

26 In the diagram, OAB is a triangle with $\angle AOB = 90°$ and $OB = 13$ cm. P & Q are 2 points on AB such that $26AP = 22PQ = 11QB$. If the vertical height of $PQ = 4$ cm, find the area of the triangle OPQ in cm^2.



27 Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 + kx^2 + 90x - 2009 = 0.$$

If $x_1 x_2 = 49$, find the value of k.

28 Three sides OAB, OAC and OBC of a tetrahedron $OABC$ are right-angled triangles, i.e. $\angle AOB = \angle AOC = \angle BOC = 90°$. Given that $OA = 7$, $OB = 2$ and $OC = 6$, find the value of

$$(\text{Area of } \Delta OAB)^2 + (\text{Area of } \Delta OAC)^2 + (\text{Area of } \Delta OBC)^2 + (\text{Area of } \Delta ABC)^2.$$

29 Find the least positive integer n for which $\dfrac{n-10}{9n+11}$ is a non-zero reducible fraction.

30 Find the value of the smallest positive integer m such that the equation

$$x^2 + 2(m+5)x + (100m+9) = 0$$

has only integer solutions.

31 In a triangle ABC, the length of the altitudes AD and BE are 4 and 12 respectively. Find the largest possible integer value for the length of the third altitude CF.

32 A four digit number consists of two distinct pairs of repeated digits (for example 2211, 2626 and 7007). Find the total number of such possible numbers that are divisible by 7 or 101 but not both.

33 m and n are two positive integers satisfying $1 \leq m \leq n \leq 40$. Find the number of pairs of (m, n) such that their product mn is divisible by 33.

34 Using the digits 0, 1, 2, 3 and 4, find the number of 13-digit sequences that can be written so that the difference between any two consecutive digits is 1.

Examples of such 13-digit sequences are 0123432123432, 2323432321234 and 3210101234323.

35 m and n are two positive integers of reverse order (for example 123 and 321) such that $mn = 1446921630$. Find the value of $m + n$.

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Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2)

1. In $\triangle ABC$, $\angle A = 2\angle B$. Let a, b, c be the lengths of its sides BC, CA, AB, respectively. Prove that

$$a^2 = b(b+c).$$

2. The set of 2000-digit integers are divided into two sets: the set M consisting all integers each of which can be represented as the product of two 1000-digit integers, and the set N which contains the other integers. Which of the sets M and N contains more elements?

3. Suppose $\overline{a_1a_2\ldots a_{2009}}$ is a 2009-digit integer such that for each $i = 1, 2, \ldots, 2007$, the 2-digit integer $\overline{a_ia_{i+1}}$ contains 3 distinct prime factors. Find a_{2008}. (Note: $\overline{xyz\ldots}$ denotes an integer whose digits are $x, y, z, \ldots$.)

4. Let S be the set of integers that can be written in the form $50m + 3n$ where m and n are non-negative integers. For example 3, 50, 53 are all in S. Find the sum of all positive integers not in S.

5. Let a, b be positive real numbers satisfying $a + b = 1$. Show that if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1x_2\ldots x_5 = 1$, then

$$(ax_1 + b)(ax_2 + b)\cdots(ax_5 + b) \geq 1.$$