Basics of Olympiad Inequalities

Article · January 2008

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Introduction

The aim of this note is to acquaint students, who want to participate in mathematical Olympiads, to Olympiad level inequalities from the basics. Inequalities are used in all fields of mathematics. They have some very interesting properties and numerous applications. Inequalities are often hard to solve, and it is not always possible to find a nice solution. But it is worth approaching an inequality rather than solving it. Most inequalities need to be transformed into a suitable form by algebraic means before applying some theorem. This is what makes the problem rather difficult. Throughout this little note you will find different ways and approaches to solve an inequality. Most of the problems are recent and thus need a fruitful combination of wisely applied techniques.

It took me around two years to complete this; although I didn't work on it for some months during this period. I have tried to demonstrate how one can use the classical inequalities through different examples that show different ways of applying them. After almost each section there are some exercise problems for the reader to get his/her hands dirty! And at the end of each chapter some harder problems are given for those looking for challenges. Some additional exercises are given at the end of the book for the reader to practice his/her skills. Solutions to some selected problems are given in the last chapter to present different strategies and techniques of solving inequality problems. In conclusion, I have tried to explain that inequalities can be overcome through practice and more practice.

Finally, though this note is aimed for students participating in the Bangladesh Mathematical Olympiad who will be hoping to be in the Bangladesh IMO team I hope it will be useful for everyone. I am really grateful to the MathLinks forum for supplying me with the huge collection of problems.

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28 November, 2008

Contents

Chapter 1

The AM-GM Inequality

1.1 General AM-GM Inequality

The most well-known and frequently used inequality is the Arithmetic mean-Geometric mean inequality or widely known as the AM-GM inequality. The term AM-GM is the combination of the two terms *Arithmetic Mean* and *Geometric Mean*. The arithmetic mean of two numbers a and b is defined by $\frac{a+b}{2}$. Similarly $\sqrt{ab}$ is the geometric mean of a and b. The simplest form of the AM-GM inequality is the following:

Basic AM-GM Inequality. For positive real numbers a, b

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

The proof is simple. Squaring, this becomes

$$(a+b)^2 \geq 4ab,$$

which is equivalent to

$$(a-b)^2 \geq 0.$$

This is obviously true. Equality holds if and only if $a = b$.

Example 1.1.1. For real numbers a, b, c prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

First Solution. By AM-GM inequality, we have

$$a^2 + b^2 \geq 2ab,$$
$$b^2 + c^2 \geq 2bc,$$
$$c^2 + a^2 \geq 2ca.$$

Adding the three inequalities and then dividing by 2 we get the desired result. Equality holds if and only if $a = b = c$.

Second Solution. The inequality is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0,$$

which is obviously true.

However, the general AM-GM inequality is also true for any n positive numbers.

General AM-GM Inequality. For positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds.

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. Here we present the well known Cauchy's proof by induction. This special kind of induction is done by performing the following steps:

i. Base case.
ii. $P_n \implies P_{2n}$.
iii. $P_n \implies P_{n-1}$.

Here P_n is the statement that the AM-GM is true for n variables.

Step 1: We already proved the inequality for $n = 2$. For $n = 3$ we get the following inequality:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

Letting $a = x^3, b = y^3, c = z^3$ we equivalently get

$$x^3 + y^3 + z^3 - 3xyz \geq 0.$$

This is true by Example 1.1.1 and the identity

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

Equality holds for $x = y = z$, that is, $a = b = c$.

Step 2: Assuming that P_n is true, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Now it's not difficult to notice that

$$a_1 + a_2 + \cdots + a_{2n} \geq n\sqrt[n]{a_1 a_2 \cdots a_n} + n\sqrt[n]{a_{n+1} a_{n+2} \cdots a_{2n}} \geq 2n\sqrt[2n]{a_1 a_2 \cdots a_{2n}}$$

implying P_{2n} is true.

Step 3: First we assume that P_n is true i.e.

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

As this is true for all positive a_is, we let $a_n = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}$. So now we have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}} \\
&= \sqrt[n]{(a_1 a_2 \cdots a_{n-1})^{\frac{n}{n-1}}} \\
&= \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}} \\
&= a_n,
\end{aligned}$$

which in turn is equivalent to

$$\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1} \geq a_n = \sqrt[n-1]{a_1 a_2 \cdots a_{n-1}}.$$

The proof is thus complete. It also follows by the induction that equality holds for $a_1 = a_2 = \cdots = a_n$.

Try to understand yourself why this induction works. It can be useful sometimes.

Example 1.1.2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

Solution. By AM-GM,

$$\begin{aligned} 1 + a_1 &\geq 2\sqrt{a_1}, \\ 1 + a_2 &\geq 2\sqrt{a_2}, \\ &\vdots \\ 1 + a_n &\geq 2\sqrt{a_n}. \end{aligned}$$

Multiplying the above inequalities and using the fact $a_1 a_2 \cdots a_n = 1$ we get our desired result. Equality holds for $a_i = 1$, $i = 1, 2, \ldots, n$.

Example 1.1.3. Let a, b, c be nonnegative real numbers. Prove that

$$(a + b)(b + c)(c + a) \geq 8abc.$$

Solution. The inequality is equivalent to

$$\left(\frac{a+b}{\sqrt{ab}}\right)\left(\frac{b+c}{\sqrt{bc}}\right)\left(\frac{c+a}{\sqrt{ca}}\right) \geq 2 \cdot 2 \cdot 2,$$

true by AM-GM. Equality holds if and only if $a = b = c$.

Example 1.1.4. Let $a, b, c > 0$. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

Solution. By AM-GM we deduce that

$$\begin{aligned} \frac{a^3}{bc} + b + c &\geq 3\sqrt[3]{\frac{a^3}{bc} \cdot b \cdot c} = 3a, \\ \frac{b^3}{ca} + c + a &\geq 3\sqrt[3]{\frac{b^3}{ca} \cdot c \cdot a} = 3b, \\ \frac{c^3}{ab} + a + b &\geq 3\sqrt[3]{\frac{c^3}{ab} \cdot a \cdot b} = 3c. \end{aligned}$$

Adding the three inequalities we get

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}+2(a+b+c)\geq 3(a+b+c),$$

which was what we wanted.

Example 1.1.5. (Samin Riasat) Let a, b, c be positive real numbers. Prove that

$$ab(a+b)+bc(b+c)+ca(c+a)\geq \sum_{cyc} ab\sqrt{\frac{a}{b}(b+c)(c+a)}.$$

Solution. By AM-GM,

$$\begin{aligned}
&2ab(a+b)+2ac(a+c)+2bc(b+c)\\
=&\,ab(a+b)+ac(a+c)+bc(b+c)+ab(a+b)+ac(a+c)+bc(b+c)\\
=&\,a^2(b+c)+b^2(a+c)+c^2(a+b)+(a^2b+b^2c+a^2c)+(ab^2+bc^2+a^2c)\\
\geq&\,a^2(b+c)+b^2(a+c)+c^2(a+b)+(a^2b+b^2c+a^2c)+3abc\\
=&\,a^2(b+c)+b^2(a+c)+c^2(a+b)+ab(a+c)+bc(a+b)+ac(b+c)\\
=&\,\left(a^2(b+c)+ab(a+c)\right)+\left(b^2(a+c)+bc(a+b)\right)+\left(c^2(a+b)+ac(b+c)\right)\\
\geq&\,2\sqrt{a^3b(b+c)(a+c)}+2\sqrt{b^3c(a+c)(a+b)}+2\sqrt{c^3a(a+b)(b+c)}\\
=&\,2ab\sqrt{\frac{a}{b}(b+c)(a+c)}+2cb\sqrt{\frac{b}{c}(a+c)(a+b)}+2ac\sqrt{\frac{c}{a}(a+b)(b+c)}.
\end{aligned}$$

Equality holds if and only if $a=b=c$.

Exercise 1.1.1. Let $a, b > 0$. Prove that

$$\frac{a}{b}+\frac{b}{a}\geq 2.$$

Exercise 1.1.2. For all real numbers a, b, c prove the following chain inequality

$$3(a^2+b^2+c^2)\geq (a+b+c)^2\geq 3(ab+bc+ca).$$

Exercise 1.1.3. Let a, b, c be positive real numbers. Prove that

$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a.$$

Exercise 1.1.4. Let a, b, c be positive real numbers. Prove that

$$a^3+b^3+c^3+ab^2+bc^2+ca^2\geq 2(a^2b+b^2c+c^2a).$$

Exercise 1.1.5. Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$a^2+b^2+c^2\geq a+b+c.$$

Exercise 1.1.6. **(a)** Let $a, b, c > 0$. Show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9.$$

(b) For positive real numbers $a_1, a_2, \ldots, a_n$ prove that

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geq n^2.$$

Exercise 1.1.7. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$a^2+b^2+c^2+ab+bc+ca \geq 6.$$

Exercise 1.1.8. Let $a, b, c, d > 0$. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \geq a+b+c+d.$$

1.2 Weighted AM-GM Inequality

The weighted version of the AM-GM inequality follows from the original AM-GM inequality. Suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers and let $m_1, m_2, \ldots, m_n$ be positive integers. Then we have by AM-GM,

$$\frac{\underbrace{a_1+a_1+\cdots+a_1}_{m_1}+\underbrace{a_2+a_2+\cdots+a_2}_{m_2}+\cdots+\underbrace{a_n+a_n+\cdots+a_n}_{m_n}}{m_1+m_2+\cdots+m_n}$$
$$\geq \left(\underbrace{a_1a_1\ldots a_1}_{m_1}\underbrace{a_2a_2\ldots a_2}_{m_2}\cdots\underbrace{a_na_n\ldots a_n}_{m_n}\right)^{\frac{1}{m_1+m_2+\cdots+m_n}}.$$

This can be written as

$$\frac{m_1a_1+m_2a_2+\cdots+m_na_n}{m_1+m_2+\cdots+m_n} \geq \left(a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n}\right)^{\frac{1}{m_1+m_2+\cdots+m_n}}.$$

Or equivalently in symbols

$$\frac{\sum m_ia_i}{\sum m_i} \geq \left(\prod a_i^{m_i}\right)^{\frac{1}{\sum m_i}}.$$

Letting $i_k = \dfrac{m_k}{\sum m_j} = \dfrac{m_k}{m_1+m_2+\cdots+m_n}$ for $k=1,2,\ldots,n$ we can rewrite this as follows:

Weighted AM-GM Inequality. For positive real numbers $a_1, a_2, \ldots, a_n$ and n weights $i_1, i_2, \ldots, i_n$ such that $\displaystyle\sum_{k=1}^{n} i_k = 1$, we have

$$a_1i_1+a_2i_2+\cdots+a_ni_n \geq a_1^{i_1}a_2^{i_2}\cdots a_n^{i_n}.$$

Although we have a proof if $i_1, i_2, \ldots, i_n$ are rational, this inequality is also true if they are positive real numbers. The proof, however, is beyond the scope of this note.

Example 1.2.1. Let a, b, c be positive real numbers such that $a+b+c=3$. Show that

$$a^b b^c c^a \leq 1$$

Solution. Notice that

$$\begin{aligned} 1 &= \frac{a+b+c}{3} \\ &\geq \frac{ab+bc+ca}{a+b+c} \\ &\geq \left(a^b b^c c^a\right)^{\frac{1}{a+b+c}}, \end{aligned}$$

which implies $a^b b^c c^a \leq 1$.

Example 1.2.2. (Nguyen Manh Dung) Let $a, b, c > 0$ such that $a+b+c=1$. Prove that

$$a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq 1.$$

Solution. From weighted AM-GM, we have

$$\frac{a^2+b^2+c^2}{a+b+c} \geq (a^a b^b c^c)^{\frac{1}{a+b+c}} \implies a^2+b^2+c^2 \geq a^a b^b c^c,$$

$$\frac{ab+bc+ca}{a+b+c} \geq (a^b b^c c^a)^{\frac{1}{a+b+c}} \implies ab+bc+ca \geq a^a b^b c^c,$$

$$\frac{ac+ba+cb}{a+b+c} \geq (a^c b^a c^b)^{\frac{1}{a+b+c}} \implies ab+bc+ca \geq a^c b^a c^b.$$

Summing up the three inequalities we get

$$(a+b+c)^2 \geq a^a b^b c^c + a^b b^c c^a + b^a c^b a^c.$$

That is,

$$a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq 1.$$

Very few inequalities can be solved using only the weighted AM-GM inequality. So no exercise in this section.

1.3 More Challenging Problems

Exercise 1.3.1. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c.$$

Exercise 1.3.2. **(Michael Rozenberg)** Let a, b, c and d be non-negative numbers such that $a+b+c+d = 4$. Prove that

$$\frac{4}{abcd} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$

Exercise 1.3.3. **(Samin Riasat)** Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+b^3+c^3}{3} \geq \frac{a+b+c}{3} \cdot \frac{a^2+b^2+c^2}{3} \geq \frac{a^2b+b^2c+c^2a}{3}.$$

Exercise 1.3.4.(a) **(Pham Kim Hung)** Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{3\sqrt[3]{abc}}{a+b+c} \geq 4.$$

(b) **(Samin Riasat)** For real numbers $a, b, c > 0$ and $n \leq 3$ prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+n\left(\frac{3\sqrt[3]{abc}}{a+b+c}\right) \geq 3+n.$$

Exercise 1.3.5. **(Samin Riasat)** Let a, b, c be positive real numbers such that $a+b+c = ab+bc+ca$ and $n \leq 3$. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{3n}{a^2+b^2+c^2} \geq 3+n.$$

Chapter 2

Cauchy-Schwarz and Hölder's Inequalities

2.1 Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality is a very powerful inequality. It is very useful in proving both cyclic and symmetric inequalities. The special equality case also makes it exceptional. The inequality states:

Cauchy-Schwarz Inequality. For any real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ the following inequality holds.

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality if the sequences are proportional. That is if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

First proof. This is the classical proof of Cauchy-Schwarz inequality. Consider the quadratic

$$f(x) = \sum_{i=1}^{n}(a_ix - b_i)^2 = x^2\sum_{i=1}^{n}a_i^2 - x\sum_{i=1}^{n}2a_ib_i + \sum_{i=1}^{n}b_i^2 = Ax^2 + Bx + C.$$

Clearly $f(x) \geq 0$ for all real x. Hence if D is the discriminant of f, we must have $D \leq 0$. This implies

$$B^2 \leq 4AC \Rightarrow \left(\sum_{i=1}^{n}2a_ib_i\right)^2 \leq 4\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right),$$

which is equivalent to

$$\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right) \geq \left(\sum_{i=1}^{n}a_ib_i\right)^2$$

Equality holds when $f(x) = 0$ for some x, which happens if $x = \dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \cdots = \dfrac{b_n}{a_n}$.

Second Proof. By AM-GM, we have

$$\frac{a_1^2}{\sum a_i^2} + \frac{b_1^2}{\sum b_i^2} \geq \frac{2a_1b_1}{\sqrt{\left(\sum a_i^2\right)\left(\sum b_i^2\right)}},$$
$$\frac{a_2^2}{\sum a_i^2} + \frac{b_2^2}{\sum b_i^2} \geq \frac{2a_2b_2}{\sqrt{\left(\sum a_i^2\right)\left(\sum b_i^2\right)}},$$
$$\vdots$$
$$\frac{a_n^2}{\sum a_i^2} + \frac{b_n^2}{\sum b_i^2} \geq \frac{2a_nb_n}{\sqrt{\left(\sum a_i^2\right)\left(\sum b_i^2\right)}}.$$

Summing up the above inequalities, we have

$$2 \geq \sum \frac{2a_ib_i}{\sqrt{\left(\sum a_i^2\right)\left(\sum b_i^2\right)}},$$

which is equivalent to

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \geq \left(\sum_{i=1}^n a_ib_i\right)^2.$$

Equality holds if for each $i \in \{1, 2, \ldots, n\}$, $\dfrac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = \dfrac{b_i^2}{b_1^2 + b_2^2 + \cdots + b_n^2}$, which in turn is equivalent to $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

We could rewrite the above solution as follows

$$\begin{aligned} 2 &= \sum \frac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2} + \sum \frac{b_i^2}{b_1^2 + b_2^2 + \cdots + b_n^2} \\ &\geq \sum \frac{2a_ib_i}{\sqrt{\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)}}. \end{aligned}$$

Here the sigma $\sum$ notation denotes cyclic sum and it will be used everywhere throughout this note. It is recommended that you get used to the summation symbol. Once you get used to it, it makes your life easier and saves your time.

Cauchy-Schwarz in Engel Form. For real numbers $a_i, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n > 0$ the following inequality holds:

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}, \tag{2.1}$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Although this is a direct consequence of the Cauchy-Schwarz inequality, let us prove it in a different way. For $n = 2$ this becomes

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}. \tag{2.2}$$

Clearing out the denominators, this is equivalent to

$$(ay - bx)^2 \geq 0,$$

which is clearly true. For $n = 3$ we have from (2.2)

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b)^2}{x+y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}.$$

A similar inductive process shows that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}.$$

And the case of equality easily follows too.

From (2.1) we deduce another proof of the Cauchy-Schwarz inequality.

Third Proof. We want to show that

$$\left(\sum b_i^2\right)\left(\sum c_i^2\right) \geq \left(\sum b_i c_i\right)^2.$$

Let a_i be real numbers such that $a_i = b_i c_i$. Then the above inequality is equivalent to

$$\frac{a_1^2}{b_1^2} + \frac{a_2^2}{b_2^2} + \cdots + \frac{a_n^2}{b_n^2} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1^2 + b_2^2 + \cdots + b_n^2}.$$

This is just (2.1).

Example 2.1.1. Let a, b, c be real numbers. Show that

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2.$$

Solution. By Cauchy-Schwarz inequality,

$$(1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2) \geq (1 \cdot a + 1 \cdot b + 1 \cdot c)^2.$$

Example 2.1.2. (Nesbitt's Inequality) For positive real numbers a, b, c prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

First Solution. Our inequality is equivalent to

$$\frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \geq \frac{9}{2},$$

or

$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{9}{2}.$$

This can be written as

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) \geq (1 + 1 + 1)^2,$$

where $x = \sqrt{b+c}, y = \sqrt{c+a}, z = \sqrt{a+b}$. Then this is true by Cauchy-Schwarz.

Second Solution. As in the previous solution we need to show that

$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{9}{2},$$

which can be written as

$$\frac{b+c+c+a+a+b}{3}\cdot\frac{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}}{3}\geq\sqrt[3]{(b+c)(c+a)(a+b)}\cdot\sqrt[3]{\frac{1}{(b+c)(c+a)(a+b)}},$$

which is true by AM-GM.

Third Solution. We have

$$\sum\frac{a}{b+c}=\sum\frac{a^2}{ab+ca}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

So it remains to show that

$$(a+b+c)^2\geq 3(ab+bc+ca)\Leftrightarrow(a-b)^2+(b-c)^2+(c-a)^2\geq 0.$$

Example 2.1.3. For nonnegative real numbers x, y, z prove that

$$\sqrt{3x^2+xy}+\sqrt{3y^2+yz}+\sqrt{3z^2+zx}\leq 2(x+y+z).$$

Solution. By Cauchy-Schwarz inequality,

$$\sum\sqrt{x(3x+y)}\leq\sqrt{\left(\sum x\right)\left(\sum(3x+y)\right)}=\sqrt{4(x+y+z)^2}=2(x+y+z).$$

Example 2.1.4. (IMO 1995) Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Solution. Let $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$. Then by the given condition we obtain $xyz=1$. Note that

$$\sum\frac{1}{a^3(b+c)}=\sum\frac{1}{\frac{1}{x^3}\left(\frac{1}{y}+\frac{1}{z}\right)}=\sum\frac{x^2}{y+z}.$$

Now by Cauchy-Schwarz inequality

$$\sum\frac{x^2}{y+z}\geq\frac{(x+y+z)^2}{2(x+y+z)}=\frac{x+y+z}{2}\geq\frac{3\sqrt[3]{xyz}}{2}=\frac{3}{2},$$

where the last inequality follows from AM-GM.

Example 2.1.5. For positive real numbers a, b, c prove that

$$\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}\leq 1.$$

Solution. We have

$$\begin{aligned}&\sum\frac{a}{2a+b}\leq 1\\ \Leftrightarrow\quad&\sum\left(\frac{a}{2a+b}-\frac{1}{2}\right)\leq 1-\frac{3}{2}\\ \Leftrightarrow\quad&-\frac{1}{2}\sum\frac{b}{2a+b}\leq-\frac{1}{2}\\ \Leftrightarrow\quad&\sum\frac{b}{2a+b}\geq 1.\end{aligned}$$

This follows from Cauchy-Schwarz inequality

$$\sum \frac{b}{2a+b} = \frac{b^2}{2ab+b^2} + \frac{c^2}{2bc+c^2} + \frac{a^2}{2ca+a^2} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)+b^2+c^2+a^2} = 1.$$

Example 2.1.6. (Vasile Cirtoaje, Samin Riasat) Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{x}{x+y}} + \sqrt{\frac{y}{y+z}} + \sqrt{\frac{z}{z+x}} \leq \frac{3}{\sqrt{2}}.$$

Solution. Verify that

$$\begin{aligned}
LHS &= \frac{\sqrt{x(y+z)(z+x)} + \sqrt{y(z+x)(x+y)} + \sqrt{z(x+y)(y+z)}}{\sqrt{(x+y)(y+z)(z+x)}} \\
&\leq \sqrt{\frac{(x(y+z)+y(z+x)+z(x+y))\,(z+x+x+y+y+z))}{(x+y)(y+z)(z+x)}} \\
&= \sqrt{4 \cdot \frac{(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}} \\
&= 2 \cdot \sqrt{\frac{(x+y)(y+z)(z+x)+xyz}{(x+y)(y+z)(z+x)}} \\
&= 2 \cdot \sqrt{1 + \frac{xyz}{(x+y)(y+z)(z+x)}} \\
&\leq 2 \cdot \sqrt{1 + \frac{1}{8}} = \frac{3}{\sqrt{2}},
\end{aligned}$$

where the last inequality follows from Example 2.1.3.

Here Cauchy-Schwarz was used in the following form:

$$\sqrt{ax} + \sqrt{by} + \sqrt{cz} \leq \sqrt{(a+b+c)(x+y+z)}.$$

Exercise 2.1.1. Prove Example 1.1.1 and Exercise 1.1.6 using Cauchy-Scwarz inequality.

Exercise 2.1.2. Let a, b, c, d be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+d} + \frac{c}{d+a} + \frac{d}{a+b} \geq 2.$$

Exercise 2.1.3 Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1} \geq a_1 + a_2 + \cdots + a_n.$$

Exercise 2.1.4. (Michael Rozenberg) Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2 = 4$. Show that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq 4.$$

Exercise 2.1.5. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{a+2b}\right)^2 + \left(\frac{b}{b+2c}\right)^2 + \left(\frac{c}{c+2a}\right)^2 \geq \frac{1}{3}.$$

Exercise 2.1.6. (Zhautykov Olympiad 2008) Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{3}{2}.$$

Exercise 2.1.7. If a, b, c and d are positive real numbers such that $a+b+c+d = 4$ prove that

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 2.$$

Exercise 2.1.8. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that

$$\sqrt{a_1^2+b_1^2} + \sqrt{a_2^2+b_2^2} + \cdots + \sqrt{a_n^2+b_n^2} \geq \sqrt{(a_1+a_2+\cdots+a_n)^2 + (b_1+b_2+\cdots+b_n)^2}$$

Exercise 2.1.9. (Samin Riasat) Let a, b, c be the side lengths of a triangle. Prove that

$$\frac{a}{3a-b+c} + \frac{b}{3b-c+a} + \frac{c}{3c-a+b} \geq 1.$$

Exercise 2.1.10. (Pham Kim Hung) Let a, b, c be positive real numbers such that $a+b+c = 1$. Prove that

$$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} < \sqrt{\frac{3}{2}}.$$

Exercise 2.1.11. Let $a, b, c > 0$. Prove that

$$\sqrt{\frac{2a}{b+c}} + \sqrt{\frac{2b}{c+a}} + \sqrt{\frac{2c}{a+b}} \leq \sqrt{3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)}.$$

2.2 Hölder's Inequality

Hölder's inequality is a generalization of the Cauchy-Schwarz inequality. This inequality states:

Hölder's Inequality. Let $a_{i_j}, 1 \leq i \leq m, 1 \leq j \leq n$ be positive real numbers. Then the following inequality holds

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i_j}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} a_{i_j}}\right)^m.$$

It looks kind of difficult to understand. So for brevity a special case is the following: for positive real numbers $a, b, c, p, q, r, x, y, z$,

$$(a^3+b^3+c^3)(p^3+q^3+r^3)(x^3+y^3+z^3) \geq (aqx+bqy+crz)^3.$$

Not only Hölder's inequality is a generalization of Cauchy-Schwarz inequality, it is also a direct consequence of the AM-GM inequality, which is demonstrated in the following proof of the special case: by AM-GM,

$$\begin{aligned} 3 &= \sum \frac{a^3}{a^3+b^3+c^3} + \sum \frac{p^3}{p^3+q^3+r^3} + \sum \frac{x^3}{x^3+y^3+z^3} \\ &\geq 3\sum \frac{apx}{\sqrt[3]{(a^3+b^3+c^3)(p^3+q^3+r^3)(x^3+y^3+z^3)}}, \end{aligned}$$

which is equivalent to

$$\sqrt[3]{(a^3+b^3+c^3)(p^3+q^3+r^3)(x^3+y^3+z^3)} \geq apx+bqy+crz.$$

Verify that this proof also generalizes to the general inequality, and is similar to the one of the Cauchy-Schwarz inequality. Here are some applications:

Example 2.2.1. (IMO 2001) Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Solution. By Hölder's inequality,

$$\left(\sum \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum a(a^2+8bc)\right) \geq (a+b+c)^3.$$

Thus we need only show that

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc,$$

which is equivalent to

$$(a+b)(b+c)(c+a) \geq 8abc.$$

This is just Example 1.1.3.

Example 2.2.2. (Vasile Cirtoaje) For $a, b, c > 0$ prove that

$$\sum \frac{a}{\sqrt{a+2b}} \geq \sqrt{a+b+c} \geq \sum \frac{a}{\sqrt{2a+b}}.$$

Solution. For the left part, we have from Hölder's inequality,

$$\left(\sum \frac{a}{\sqrt{a+2b}}\right)\left(\sum \frac{a}{\sqrt{a+2b}}\right)\left(\sum a(a+2b)\right) \geq (a+b+c)^3.$$

Thus

$$\left(\sum \frac{a}{\sqrt{a+2b}}\right)^2 \geq a+b+c.$$

Now for the right part, by Cauchy-Schwarz inequality we have

$$\sum \frac{a}{\sqrt{2a+b}} \leq \sqrt{(a+b+c)\left(\sum \frac{a}{2a+b}\right)}.$$

So it remains to show that

$$\sum \frac{a}{2a+b} \leq 1,$$

which is Example 2.1.5.

Example 2.2.3. (Samin Riasat) Let a, b, c be the side lengths of a triangle. Prove that

$$\frac{1}{8abc+(a+b-c)^3} + \frac{1}{8abc+(b+c-a)^3} + \frac{1}{8abc+(c+a-b)^3} \leq \frac{1}{3abc}.$$

Solution. We have

$$\begin{aligned}
& \sum \frac{1}{8abc+(a+b-c)^3} \leq \frac{1}{3abc} \\
\Leftrightarrow\ & \sum \left(\frac{1}{8abc} - \frac{1}{8abc+(a+b-c)^3}\right) \geq \frac{3}{8abc} - \frac{1}{3abc} \\
\Leftrightarrow\ & \sum \frac{(a+b-c)^3}{8abc+(a+b-c)^3} \geq \frac{1}{3}.
\end{aligned}$$

By Hölder's inequality we obtain

$$\sum \frac{(a+b-c)^3}{8abc+(a+b-c)^3} \geq \frac{(a+b+c)^3}{3(24abc+(a+b-c)^3+(a+c-b)^3+(b+c-a)^3)} = \frac{1}{3}.$$

In this solution, the following inequality was used: for all positive real numbers a, b, c, x, y, z,

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

The proof of this is left to the reader as an exercise.

Example 2.2.4. (IMO Shortlist 2004) If a, b, c are three positive real numbers such that $ab+bc+ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a}+6b} + \sqrt[3]{\frac{1}{b}+6c} + \sqrt[3]{\frac{1}{c}+6a} \leq \frac{1}{abc}.$$

Solution. Note that $\dfrac{1}{a} + 6b = \dfrac{7ab+bc+ca}{a}$. Hence our inequality becomes

$$\sum \sqrt[3]{bc(7ab+bc+ca)} \leq \frac{1}{(abc)^{\frac{2}{3}}}.$$

From Hölder's inequality we have

$$\sum \sqrt[3]{bc(7ab+bc+ca)} \leq \sqrt[3]{\left(\sum a\right)^2 \left(9\sum bc\right)}.$$

Hence it remains to show that

$$9(a+b+c)^2(ab+bc+ca) \le \frac{1}{(abc)^2}$$
$$\Leftrightarrow [3abc(a+b+c)]^2 \le (ab+bc+ca)^4,$$

which is obviously true since

$$(ab+bc+ca)^2 \ge 3abc(a+b+c) \Leftrightarrow \sum a^2(b-c)^2 \ge 0.$$

Another formulation of Hölder's inequality is the following: for positive real numbers $a_i, b_i, p, q\,(1 \le i \le n)$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \le \left(a_1^p + a_2^p + \cdots + a_n^p\right)^{\frac{1}{p}} \left(b_1^q + b_2^q + \cdots + b_n^q\right)^{\frac{1}{q}}.$$

Exercise 2.2.1. Prove Exercise 2.1.3 using Hölder's inequality.

Exercise 2.2.2. Let a, b, x and y be positive numbers such that $1 \ge a^{11} + b^{11}$ and $1 \ge x^{11} + y^{11}$. Prove that $1 \ge a^5x^6 + b^5y^6$.

Exercise 2.2.3. Prove that for all positive real numbers a, b, c, x, y, z,

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \ge \frac{(a+b+c)^3}{3(x+y+z)}.$$

Exercise 2.2.4. Let $a, b,$ and c be positive real numbers. Prove the inequality

$$\frac{a^6}{b^2+c^2} + \frac{b^6}{a^2+c^2} + \frac{c^6}{a^2+b^2} \ge \frac{abc(a+b+c)}{2}.$$

Exercise 2.2.5. **(Kyiv 2006)** Let x, y and z be positive numbers such that $xy + xz + yz = 1$. Prove that

$$\frac{x^3}{1+9y^2xz} + \frac{y^3}{1+9z^2yx} + \frac{z^3}{1+9x^2yz} \ge \frac{(x+y+z)^3}{18}.$$

Exercise 2.2.6. Let $a, b, c > 0$. Prove that

$$\frac{ab}{\sqrt{ab+2c^2}} + \frac{bc}{\sqrt{bc+2a^2}} + \frac{ca}{\sqrt{ca+2b^2}} \ge \sqrt{ab+bc+ca}.$$

2.3 More Challenging Problems

Exercise 2.3.1. Let $a, b, c > 0$ and $k \ge 2$. Prove that

$$\frac{a}{ka+b} + \frac{b}{kb+c} + \frac{c}{kc+a} \le \frac{3}{k+1}.$$

Exercise 2.3.2. (Samin Riasat) Let a, b, c, m, n be positive real numbers. Prove that

$$\frac{a^2}{b(ma+nb)} + \frac{b^2}{c(mb+nc)} + \frac{c^2}{a(mc+na)} \geq \frac{3}{m+n}.$$

Another formulation: Let a, b, c, m, n be positive real numbers such that $abc = 1$ Prove that

$$\frac{1}{b(ma+nb)} + \frac{1}{c(mb+nc)} + \frac{1}{a(mc+na)} \geq \frac{3}{m+n}.$$

Exercise 2.3.3 (Michael Rozenberg, Samin Riasat) Let $x, y, z > 0$. Prove that

$$\sum_{cyc} \sqrt{x^2+xy+y^2} \geq \sum_{cyc} \sqrt{2x^2+xy}.$$

Exercise 2.3.4 (Vasile Cirtoaje and Samin Riasat) Let $a, b, c, k > 0$. Prove that

$$\frac{a}{\sqrt{ka+b}} + \frac{b}{\sqrt{kb+c}} + \frac{c}{\sqrt{kc+a}} < \sqrt{\frac{k+1}{k}(a+b+c)}.$$

Exercise 2.3.5. (Michael Rozenberg and Samin Riasat) Let x, y, z be positive real numbers such that $xy + yz + zx \geq 3$. Prove that

$$\frac{x}{\sqrt{4x+5y}} + \frac{y}{\sqrt{4y+5z}} + \frac{z}{\sqrt{4z+5x}} \geq 1.$$

Exercise 2.3.6. Let $a, b, c > 0$ such that $a + b + c = 1$. Prove that

$$\frac{\sqrt{a^2+abc}}{c+ab} + \frac{\sqrt{b^2+abc}}{a+bc} + \frac{\sqrt{c^2+abc}}{b+ca} \leq \frac{1}{2\sqrt{abc}}.$$

***Exercise 2.3.7. (Ji Chen, Pham Van Thuan and Samin Riasat)** Let x, y, z be positive real numbers. Prove that

$$\sqrt[4]{\frac{27(x^2+y^2+z^2)}{4}} \geq \frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \geq \sqrt[4]{\frac{27(yz+zx+xy)}{4}}.$$

Chapter 3

Rearrangement and Chebyshev's Inequalities

3.1 Rearrangement Inequality

A wonderful inequality is that called the Rearrangement inequality. The statement of the inequality is as follows:

Rearrangement Inequality. Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be sequences of positive numbers increasing or decreasing in the same direction. That is, either $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ or $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then for any permutation (c_n) of the numbers (b_n) we have the following inequalities

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n a_i c_i \geq \sum_{i=1}^n a_i b_{n-i+1}.$$

That is, the maximum of the sum occurs when the two sequences are similarly sorted, and the minimum occurs when they are oppositely sorted.

Proof. Let S denote the sum $a_1b_1 + a_2b_2 + \cdots + a_nb_n$ and S' denote the sum $a_1b_1 + a_2b_2 + \cdots + a_xb_y + \cdots + a_yb_x + \cdots + a_nb_n$. Then

$$S - S' = a_xb_x + a_yb_y - a_xb_y - a_yb_x = (a_x - a_y)(b_x - b_y) \geq 0,$$

since both of $a_x - a_y$ and $b_x - b_y$ are either positive, or negative, as the sequences are similarly sorted. Hence the sum gets smaller whenever any two of the terms alter. This implies that the maximum must occur when the sequences are sorted similarly. The other part of the inequality follows in a quite similar manner and is left to the reader.

A useful technique. Let $f(a_1, a_2, \ldots, a_n)$ be a symmetric expression in $a_1, a_2, \ldots, a_n$. That is, for any permutation $a_1', a_2', \ldots, a_n'$ we have $f(a_1, a_2, \ldots, a_n) = f(a_1', a_2', \ldots, a_n')$. Then in order to prove $f(a_1, a_2, \ldots, a_n) \geq 0$ we may assume, without loss of generality, that $a_1 \geq a_2 \geq \cdots \geq a_n$. The reason we can do so is because f remains invariant under any permutation of the a_i's. This assumption is quite useful sometimes; check out the following examples:

Example 3.1.1. Let a, b, c be real numbers. Prove that

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Solution. We may assume, WLOG[1], that $a \geq b \geq c \geq 0$, since the signs of a, b, c does not affect the left side of the inequality. Applying the Rearrangement inequality for the sequences (a, b, c) and (a, b, c) we conclude that

$$a \cdot a + b \cdot b + c \cdot c \geq a \cdot b + b \cdot c + c \cdot a$$
$$\Longrightarrow \; a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Example 3.1.2. For positive real numbers a, b, c prove that

$$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a.$$

Solution. WLOG let $a \geq b \geq c$. Applying Rearrangement inequality for (a^2, b^2, c^2) and (a, b, c) we conclude that

$$a^2 \cdot a + b^2 \cdot b + c^2 \cdot c \geq a^2 \cdot b + b^2 \cdot c + c^2 \cdot a.$$

Example 3.1.3. (Nesbitt's inequality) For positive real numbers a, b, c prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution. Since the inequality is symmetric in a, b, c we may WLOG assume that $a \geq b \geq c$. Then verify that $b + c \leq c + a \leq a + b$ i.e. $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. Now applying the Rearrangement inequality for the sequences (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ we conclude that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b},$$

and

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}.$$

Adding the above inequalities we get

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{c+a}{c+a} + \frac{a+b}{a+b} = 3.$$

which was what we wanted.

Example 3.1.4 (IMO 1975) We consider two sequences of real numbers $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$. Let $z_1, z_2, \ldots, z_n$ be a permutation of the numbers $y_1, y_2, \ldots, y_n$. Prove that

$$\sum_{i=1}^{n} (x_i - \; y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2.$$

Solution. Expanding and using the fact that $\sum y_i^2 = \sum z_i^2$, we are left to prove that

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i z_i,$$

[1]WLOG=Without loss of generality.

which is the Rearrangement inequality.

Example 3.1.5. (Rearrangement inequality in Exponential form) Let $a, b, c \geq 1$. Prove that

$$a^a b^b c^c \geq a^b b^c c^a$$

First Solution. First assume that $a \geq b \geq c \geq 1$. Then

$$\begin{aligned} & a^a b^b c^c \geq a^b b^c c^a \\ \Leftrightarrow\ & a^{a-b} b^{b-c} \geq c^{a-b} c^{b-c}, \end{aligned}$$

which is true, since $a^{a-b} \geq c^{a-b}$ and $b^{b-c} \geq c^{b-c}$.

Now let $1 \leq a \leq b \leq c$. Then

$$\begin{aligned} & a^a b^b c^c \geq a^b b^c c^a \\ \Leftrightarrow\ & c^{c-b} c^{b-a} \geq b^{c-b} a^{c-a}, \end{aligned}$$

which is also true. Hence the inequality holds in all cases.

Second Solution. Here is another useful argument: take ln on both sides

$$a \ln a + b \ln b + c \ln c \geq a \ln b + b \ln c + c \ln a.$$

It is now clear that this holds by Rearrangement since the sequences (a, b, c) and $(\ln a, \ln b, \ln c)$ are sorted similarly.

Note that the inequality holds even if $a, b, c > 0$, and in this case the first solution works but the second solution needs some treatment which is left to the reader to fix.

Example 3.1.6. Let $a, b, c \geq 0$. Prove that

$$a^3 + b^3 + c^3 \geq 3abc.$$

Solution. Applying the Rearrangement inequality for $(a, b, c), (a, b, c), (a, b, c)$ we conclude that

$$a \cdot a \cdot a + b \cdot b \cdot b + c \cdot c \cdot c \geq a \cdot b \cdot c + b \cdot c \cdot a + c \cdot a \cdot b.$$

In the same way as above, we can prove the AM-GM inequality for n variables for any $n \geq 2$. This demonstrates how strong the Rearrangement inequality is. Also check out the following example, illustrating the strength of Rearrangement inequality:

Example 3.1.7 (Samin Riasat) Let a, b, c be positive real numbers. Prove that

$$\left(\sum_{cyc} \frac{a}{b+c}\right)^2 \leq \left(\sum_{cyc} \frac{a^2}{b^2+bc}\right)\left(\sum_{\text{cyc}} \frac{a}{c+a}\right).$$

Solution. Note that the sequences $\left(\sqrt{\frac{a^3}{b+c}}, \sqrt{\frac{b^3}{c+a}}, \sqrt{\frac{c^3}{a+b}}\right)$ and $\left(\sqrt{\frac{1}{ca+ab}}, \sqrt{\frac{1}{ab+bc}}, \sqrt{\frac{1}{bc+ca}}\right)$ are oppositely sorted. Therefore by Rearrangement inequality we get

$$\sum \frac{a}{b+c} = \sum \sqrt{\frac{a^3}{b+c}} \cdot \sqrt{\frac{1}{ca+ab}} \leq \sum \sqrt{\frac{a^3}{b+c}} \cdot \sqrt{\frac{1}{ab+bc}}.$$

Now from Cauchy-Schwarz inequality

$$\sum \sqrt{\frac{a^3}{b+c}} \cdot \sqrt{\frac{1}{ab+bc}} = \sum \sqrt{\frac{a^2}{b(b+c)}} \cdot \sqrt{\frac{a}{c+a}} \le \sqrt{\left(\sum \frac{a^2}{b(b+c)}\right)\left(\sum \frac{a}{c+a}\right)},$$

which was what we wanted.

Can you generalize the above inequality?

Exercise 3.1.1. Prove Example 1.1.4 using Rearrangement inequality.

Exercise 3.1.2. For $a, b, c > 0$ prove that

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \ge a + b + c.$$

Exercise 3.1.3. Let $a, b, c > 0$. Prove that

$$\frac{a^8 + b^8 + c^8}{a^3 b^3 c^3} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Exercise 3.1.4. Prove Exercise 2.1.3 using Rearrangement inequality.

Exercise 3.1.5. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)} + \frac{b}{c(c+a)} + \frac{c}{a(a+b)} \ge \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}.$$

Exercise 3.1.6. (Yaroslavle 2006) Let $a > 0, b > 0$ and $ab = 1$. Prove that

$$\frac{a}{a^2+3} + \frac{b}{b^2+3} \le \frac{1}{2}.$$

Exercise 3.1.7. Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that

$$ab^2 + bc^2 + ca^2 \ge a + b + c.$$

Exercise 3.1.8. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{ab+c}{a+b} + \frac{ac+b}{a+c} + \frac{bc+a}{b+c} \ge 2.$$

Exercise 3.1.9. (Novosibirsk 2007) Let a and b be positive numbers, and $n \in \mathbb{N}$. Prove that

$$(n+1)(a^{n+1} + b^{n+1}) \ge (a+b)(a^n + a^{n-1}b + \cdots + b^n).$$

3.2 Chebyshev's inequality

Chebyshev's inequality is a direct consequence of the Rearangement inquality. The statement is as follows:

Chebyshev's Inequality. Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be two sequences of positive real numbers.

(i) If the sequences are similarly sorted, then

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n}. \tag{3.1}$$

(ii) If the sequences are oppositely sorted, then

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n}. \tag{3.2}$$

Proof. We will only prove (3.1). Since the sequences are similarly sorted, Rearrangement inequality implies

$$\begin{aligned}
a_1b_1 + a_2b_2 + \cdots + a_nb_n &= a_1b_1 + a_2b_2 + \cdots + a_nb_n,\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geq a_1b_2 + a_2b_3 + \cdots + a_nb_1,\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geq a_1b_3 + a_2b_4 + \cdots + a_nb_2,\\
&\vdots\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geq a_1b_n + a_2b_1 + \cdots + a_nb_{n-1}.
\end{aligned}$$

Adding the above inequalities we get

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n),$$

which was what we wanted.

Example 3.2.1. For $a, b, c > 0$ prove that

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2.$$

Solution. Applying Chebyshev's inequality for (a, b, c) and (a, b, c) we conclude that

$$3(a \cdot a + b \cdot b + c \cdot c) \geq (a + b + c)(a + b + c).$$

Example 3.2.2. Let $a, b, c > 0$. Prove that

$$\frac{a^8 + b^8 + c^8}{a^3b^3c^3} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Solution. From Chebyshev's inequality we conclude that

$$\begin{aligned}
3(a^8 + b^8 + c^8) &\geq (a^6 + b^6 + c^6)(a^2 + b^2 + c^2)\\
&\geq 3a^2b^2c^2(a^2 + b^2 + c^2)\\
&\geq 3a^2b^2c^2(ab + bc + ca),
\end{aligned}$$

hence

$$\frac{a^8+b^8+c^8}{a^3b^3c^3} \geq \frac{ab+bc+ca}{abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Example 3.2.3. Let $a \geq b \geq c \geq 0$ and $0 \leq x \leq y \leq z$. Prove that

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z} \geq \frac{a+b+c}{\sqrt[3]{xyz}} \geq 3\left(\frac{a+b+c}{x+y+z}\right). \tag{3.3}$$

Solution. Applying Chebyshev's inequality for $a \geq b \geq c$ and $\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$ we deduce that

$$3\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right) \geq (a+b+c)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq \frac{3(a+b+c)}{\sqrt[3]{xyz}} \geq 9\left(\frac{a+b+c}{x+y+z}\right),$$

which was what we wanted. Here the last two inequalities follow from AM-GM.

Example 3.2.4. Let $a, b, c > 0$. Prove that

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c.$$

Solution. WLOG assume that $a \geq b \geq c$. Then $a^3 \geq b^3 \geq c^3$ and $bc \leq ca \leq ab$. Hence using (3.3) and (3.1) we conclude that

$$\sum \frac{a^3}{bc} \geq \frac{3(a^3+b^3+c^3)}{ab+bc+ca} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca} \geq a+b+c.$$

Exercise 3.2.1. Prove the second Chebyshev's inequality (3.2).

Exercise 3.2.2. Let $a_1, a_2, \ldots, a_n \geq 0$ and $k \geq 1$. Prove that

$$\sqrt[k]{\frac{a_1^k+a_2^k+\cdots+a_n^k}{n}} \geq \frac{a_1+a_2+\cdots+a_n}{n}.$$

Exercise 3.2.3. Deduce a proof of Nesbitt's inequality from Chebyshev's inequality. (Hint: you may use Example 3.2.3)

Exercise 3.2.4. Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be positive. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \geq \frac{a_1+a_2+\cdots+a_n}{\sqrt[n]{b_1b_2\ldots b_n}} \geq \frac{n(a_1+a_2+\cdots+a_n)}{b_1+b_2+\cdots+b_n}.$$

Exercise 3.2.5. Let $a \geq c \geq 0$ and $b \geq d \geq 0$. Prove that

$$(a+b+c+d)^2 \geq 8(ad+bc).$$

Exercise 3.2.6. (Radu Titiu) Let $a, b, c > 0$ such that $a^2+b^2+c^2 \geq 3$. Show that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{3}{2}.$$

Exercise 3.2.7. Let $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{9}{2} \leq \sum_{cyc} \frac{1}{c^2(a+b)} \cdot \sum_{cyc} ab \leq 3 \sum_{cyc} \frac{ab}{c^2(a+b)}.$$

Exercise 3.2.8. Let $a, b, c > 0$. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

3.3 More Chellenging Problems

Exercise 3.3.1. (Samin Riasat) Let $a, b, c > 0$. Prove that

$$\frac{\max\{a,b,c\}}{\min\{a,b,c\}} + \frac{\min\{a,b,c\}}{\max\{a,b,c\}} \geq \frac{a+b+c}{\sqrt[3]{abc}} - 1.$$

Exercise 3.3.2. Let $a_1 \geq a_2 \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be positive.

(i) If $(c_i)_{i=1}^n$ is a permutation of $(b_i)_{i=1}^n$ prove that

$$a_1^{b_1 - c_1} a_2^{b_2 - c_2} \cdots a_n^{b_n - c_n} \geq 1.$$

(ii) Let $b = \dfrac{b_1 + b_2 + \cdots + b_n}{n}$. Prove that

$$a_1^{b_1 - b} a_2^{b_2 - b} \cdots a_n^{b_n - b} \geq 1.$$

Exercise 3.3.3. Let $x, y, z \in \mathbb{R}^+$. Prove that

$$\frac{x^3 + y^3 + z^3}{3xyz} + \frac{3\sqrt[3]{xyz}}{x+y+z} \geq 2.$$

Exercise 3.3.4. (Samin Riasat) Let a, b, c be positive real numbers. Prove that

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{2}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

***Exercise 3.3.5. (Samin Riasat)** Let a, b, c be positive real numbers and n be a positive integer. Prove that

$$\left(\sum_{cyc} \frac{a}{b+c}\right)^n \leq \left(\sum_{cyc} \frac{a^n}{b^n + b^{n-1}c}\right)\left(\sum_{cyc} \frac{a}{c+a}\right)^{n-1}.$$

Chapter 4

Other Useful Strategies

4.1 Schur's Inequality

Let a, b, c be positive real numbers, and n be positive. Then the following inequality holds:

$$a^n(a-b)(a-c) + b^n(b-c)(b-a) + c^n(c-a)(c-b) \geq 0,$$

with equality if and only if $a = b = c$ or $a = b, c = 0$ and permutations.

The above inequality is known as Schur's inequality, after Issai Schur.

Proof. Since the inequality is symmetric in a, b, c WLOG we may assume that $a \geq b \geq c$. Then the inequality is equivalent to

$$(a-b)(a^n(a-c) - b^n(b-c)) + c^n(a-c)(b-c) \geq 0,$$

which is obviously true.

4.2 Jensen's Inequality

Suppose f is a convex function in $[a, b]$. Then the inequality

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leq \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

is true for all $a_i \in [a, b]$. Similarly, if f is concave in the interval the sign of inequality turns over. This is called Jensen's inequality.

The convexity is usually determined by checking if $f''(x) \geq 0$ holds for all $x \in [a, b]$. Similarly for concavity one can check if $f''(x) \leq 0$ for all $x \in [a, b]$. Here is an example:

Example 4.2.1. Let $a, b, c > 0$. Prove that

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

Solution. Consider the function $f(x) = x \ln x$. Verify that $f''(x) = 1/x > 0$ for all $x \in \mathbb{R}^+$. Thus f is convex in $\mathbb{R}^+$ and by Jensen's inequality we conclude that

$$f(a)+f(b)+f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \Leftrightarrow \ln a^a + \ln b^b + \ln c^c \geq 3\ln\left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}},$$

which is equivalent to

$$\ln(a^a b^b c^c) \geq \ln\left(\frac{a+b+c}{3}\right)^{a+b+c},$$

which was what we wanted.

4.3 Minkowski's Inequality

Minkowski's inequality states that for positive numbers x_i, y_i and p the following inequality holds:

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}.$$

4.4 Ravi Transformation

Suppose that a, b, c are the side lengths of a triangle. Then positive real numbers x, y, z exist such that $a = x+y, b = y+z$ and $c = z+x$.

To verify this, let s be the semi-perimeter. Then denote $z = s-a, x = s-b, y = s-c$ and the conclusion is obvious since $s - a = \dfrac{b+c-a}{2} > 0$ and similarly for the others.

Geometrically, let D, E, F denote the points of tangency of BC, CA, AB, respectively, with the incircle of triangle ABC. Then $BD = BF = x, CD = CE = y$ and $AE = AF = z$ implies the conclusion.

Here are some examples of how the Ravi transformation can transform a geometric inequality into an algebraic one:

Example 4.4.1. (IMO 1964) Let a, b, c be the side lengths of a triangle. Prove that

$$a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c) \leq 3abc.$$

First Solution. Verify that the inequality can be written as

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

Let $a = x+y, b = y+z$ and $c = z+x$. Then the above inequality becomes

$$8xyz \leq (x+y)(y+z)(z+x),$$

which is Example 1.1.3.

Second Solution. The inequality is equivalent to

$$a^3 + b^3 + c^3 + 3abc \geq a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2,$$

or,

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0,$$

which is Schur's inequality.

Example 4.4.2. Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{3\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)} \geq \sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b}.$$

Solution. Let $x, y, z > 0$ such that $a = x+y, b = y+z, c = z+x$. Then our inequality is equivalent to

$$3\sum_{cyc}\sqrt{(x+y)(y+z)} \geq 2\left(\sum_{cyc}\sqrt{x}\right)^2.$$

From Cauchy-Schwarz inequality,

$$\begin{aligned} 3\sum_{cyc}\sqrt{(x+y)(y+z)} &\geq 3\sum_{cyc}(y+\sqrt{zx}) \\ &\geq 2\sum_{cyc} y + 4\sum_{cyc}\sqrt{zx} \\ &= 2\left(\sum_{cyc}\sqrt{x}\right)^2. \end{aligned}$$

4.5 Normalization

Homogeneous inequalities can be normalized, e.g. applied restrictions with homogeneous expressions in the variables. For example, in order to show that $a^3+b^3+c^3-3abc \geq 0$, one may assume, WLOG, that $abc = 1$ or $a+b+c = 1$ etc. The reason is explained below.

Suppose that $abc = k^3$. Let $a = ka', b = kb', c = kc'$. This implies $a'b'c' = 1$, and our inequality becomes $a'^3+b'^3+c'^3-3a'b'c' \geq 0$, which is the same as before. Therefore the restriction $abc = 1$ doesn't change anything of the inequality. Similarly one might also assume $a+b+c = 1$. The reader is requested to find out how it works.

4.6 Homogenization

This is the opposite of Normalization. It is often useful to substitute $a = x/y, b = y/z, c = z/x$, when the condition $abc = 1$ is given. Similarly when $a+b+c = 1$ we can substitute $a = x/x+y+z, b = y/x+y+z, c = z/x+y+z$ to homogenize the inequality. For an example of homogenization note that we can write the inequality in exercise 1.3.1 in the following form:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}.$$

On the other hand, if we substitute $a = x/y, b = y/z, c = z/x$ the inequality becomes,

$$\frac{zx}{y^2}+\frac{xy}{z^2}+\frac{yz}{x^2} \geq \frac{x}{y}+\frac{y}{z}+\frac{z}{x},$$

which clearly looks easier to deal with (Hint: Rearrangement). Many such substitutions exist, and the reader is urged to study them and find them using his/her own ideas.

Chapter 5

Supplementary Problems

Exercise 5.1.1. Let a, b, c be nonnegative reals. Prove that

$$\sqrt{\frac{ab+bc+ca}{3}} \leq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

Exercise 5.1.2. For $a, b, c > 0$ prove that

$$\frac{a}{(b+c)^4} + \frac{b}{(c+a)^4} + \frac{c}{(a+b)^4} \geq \frac{3}{2(a+b)(b+c)(c+a)}.$$

Exercise 5.1.3. Let a, b, c be real numbers. Prove that

$$2 + (abc)^2 + a^2 + b^2 + c^2 \geq 2(ab+bc+ca).$$

Exercise 5.1.4. (Michael Rozenberg) Let a, b, c be non-negative numbers such that $a+b+c=3$. Prove that

$$a\sqrt{2b+c^2} + b\sqrt{2c+a^2} + c\sqrt{2a+b^2} \leq 3\sqrt{3}.$$

Exercise 5.1.5. For any acute-angled triangle ABC show that

$$\tan A + \tan B + \tan C \geq \frac{s}{r},$$

where s and r denote the semi-perimeter and the inraduis, respectively.

Exercise 5.1.6. (Iran 2008) Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:

$$x\sqrt{y} + y\sqrt{z} + z\sqrt{x} \leq K\sqrt{(x+y)(y+z)(z+x)}.$$

Exercise 5.1.7. (USA 1997) Prove the following inequality for $a, b, c > 0$

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{a^3+c^3+abc} \leq \frac{1}{abc}.$$

Exercise 5.1.8. Let $x \geq y \geq z > 0$ be real numbers. Prove that

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2.$$

Exercise 5.1.9. (Greece 2007) Let a, b, c be sides of a triangle. Show that

$$\frac{(c+a-b)^4}{a(a+b-c)} + \frac{(a+b-c)^4}{b(b+c-a)} + \frac{(b+c-a)^4}{c(c+a-b)} \geq ab+bc+ca.$$

Exercise 5.1.10 (Samin Riasat) Let a, b, c be sides of a triangle. Show that

$$\frac{(c+a-b)^4}{a(a+b-c)} + \frac{(a+b-c)^4}{b(b+c-a)} + \frac{(b+c-a)^4}{c(c+a-b)} \geq a^2+b^2+c^2.$$

Exercise 5.1.11. (Crux Mathematicorum) If a, b and c are the sidelengths of a triangle, then prove the inequality

$$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \geq 6.$$

Exercise 5.1.12. Let a, b, c be the side-lengths of a triangle. Prove that

$$\sum_{cyc}(a+b)(b+c)\sqrt{a-b+c} \geq 4(a+b+c)\sqrt{(-a+b+c)(a-b+c)(a+b-c)}.$$

Exercise 5.1.13. Prove that if $a, b, c > 0$ then

$$\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c}) + (a+b+c)^2 \geq 4\sqrt{3abc(a+b+c)}.$$

Exercise 5.1.14. Let a, b, c be non-negative real numbers such that $a+b+c=1$. Prove that

$$ab+bc+ca \leq \frac{1}{8}\sum_{cyc}\sqrt{(1-ab)(1-bc)} \leq a^2+b^2+c^2.$$

Exercise 5.1.15. Let a, b and c be nonnegative real numbers such that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2.$$

Prove that $ab+bc+ca \leq \dfrac{3}{2}$.

Exercise 5.1.16. (Korea 1998) Let I be the incenter of triangle ABC. Prove that

$$3(IA^2+IB^2+IC^2) \geq AB^2+BC^2+CA^2.$$

Exercise 5.1.17. (Samin Riasat) Let a, b, c be positive real numbers such that $a^6+b^6+c^6=3$. Prove that

$$a^7b^2+b^7c^2+c^7a^2 \leq 3.$$

Exercise 5.1.18. (Samin Riasat) Let x, y, z be positive real numbers. Prove that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \sqrt{\frac{x+y}{2z}} + \sqrt{\frac{y+z}{2x}} + \sqrt{\frac{z+x}{2y}}.$$

Exercise 5.1.19. (Samin Riasat) Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{xy}{(x+y)(y+z)}} + \sqrt{\frac{yz}{(y+z)(z+x)}} + \sqrt{\frac{zx}{(z+x)(x+y)}} \leq \frac{3}{2}.$$

Chapter 6

Hints and Solutions to Selected Problems

1.1.2. Expand and use Example 1.1.1.
1.1.3. $a^3 + a^3 + b^3 \geq 3a^2b$.
1.3.1. Use $\frac{a}{b} + \frac{a}{b} + \frac{b}{c} \geq \frac{3a}{\sqrt[3]{abc}}$ to prove

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+b+c}{\sqrt[3]{abc}}.$$

1.3.4. See hint for 1.3.1.
1.3.5. Prove and use the following:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}.$$

2.1.2. $\frac{a}{b+c} = \frac{a^2}{ab+ca}$.
2.1.5. Use Example 2.1.5.
2.3.6. Solution: Note that $\sum \frac{\sqrt{a^2+abc}}{c+ab} = \sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}$.

Therefore our inequality is equivalent to

$$\begin{aligned}
&\sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)} \leq \frac{a+b+c}{2\sqrt{abc}} \\
\iff &\sum a(a+b)\sqrt{bc(c+a)(a+b)} \leq \frac{1}{2}(a+b+c)(a+b)(b+c)(c+a).
\end{aligned}$$

By AM-GM,

$$\begin{aligned}
\sum a(a+b)\cdot 2\sqrt{bc(c+a)(a+b)} &\leq \sum a(a+b)(b(c+a)+c(a+b)) \\
&= \sum a(a+b)(ab+2bc+ca).
\end{aligned}$$

Now

$$\begin{aligned}
\sum a(a+b)(ab+2bc+ca) &= \sum a^2(ab+bc+ca) + \sum a^2bc + \sum ab(ab+bc+ca) + \sum ab^2c \\
&= (a^2+b^2+c^2+ab+bc+ca)(ab+bc+ca) + 2abc(a+b+c) \\
&= (a+b+c)^2(ab+bc+ca) - (ab+bc+ca)^2 + 2abc(a+b+c) \\
&= (a+b+c)^2(ab+bc+ca) - (a^2b^2+b^2c^2+c^2a^2) \\
&\leq (a+b+c)^2(ab+bc+ca) - abc(a+b+c) \\
&= (a+b+c)(a+b)(b+c)(c+a),
\end{aligned}$$

which was what we wanted.

2.3.7. Solution: For the right part, from Hölder's Inequality we have

$$\left(\sum \frac{x}{\sqrt{x+y}}\right)\left(\sum \frac{x}{\sqrt{x+y}}\right)\left(\sum x(x+y)\right) \geq (x+y+z)^3$$
$$\Leftrightarrow \sum \frac{x}{\sqrt{x+y}} \geq \sqrt{\frac{(x+y+z)^3}{x^2+y^2+z^2+xy+yz+zx}}.$$

So it remains to show that

$$\left(\frac{(x+y+z)^3}{x^2+y^2+z^2+xy+yz+zx}\right)^2 \geq \frac{27}{4}(yz+zx+xy).$$

Let $x+y+z=1$ and $xy+yz+zx=t$. Thus we need to prove that

$$\left(\frac{1}{1-t}\right)^2 \geq \frac{27t}{4}$$
$$\Leftrightarrow t(1-t)^2 \leq \frac{4}{27}$$
$$\Leftrightarrow (4-3t)(1-3t)^2 \geq 0,$$

which is obvious using $t \leq 1/3$.

Now for the left part,we need to prove that

$$\sum_{cyc} \frac{x}{\sqrt{x+y}} \leq \sqrt[4]{\frac{27}{4}(x^2+y^2+z^2)}.$$

We have

$$\sum_{cyc} \frac{x}{\sqrt{x+y}} = \sum_{cyc} \left(\sqrt[4]{\frac{x}{x+y}}\right)\left(\sqrt[4]{\frac{x^3}{x+y}}\right)$$
$$\leq \sqrt{\left(\sum_{cyc} \sqrt{\frac{x}{x+y}}\right)\left(\sum_{cyc} \sqrt{\frac{x^3}{x+y}}\right)}$$
$$\leq \sqrt{\frac{3}{\sqrt{2}} \sum_{cyc} x\sqrt{\frac{x}{x+y}}}.$$

Thus we need to show that

$$\left(\frac{3}{\sqrt{2}} \sum_{cyc} x\sqrt{\frac{x}{x+y}}\right)^2 \leq \frac{27}{4}(x^2+y^2+z^2).$$

Or,

$$\left(\sum_{cyc} x\sqrt{\frac{x}{x+y}}\right)^2 \leq \frac{3}{2}(x^2+y^2+z^2).$$

But we have

$$\begin{aligned}\left(\sum_{cyc} x\sqrt{\frac{x}{x+y}}\right)^2 &= \left(\frac{\sum x\sqrt{x}\sqrt{y+z}\sqrt{z+x}}{\sqrt{(x+y)(y+z)(z+x)}}\right)^2\\ &= \frac{\left(\sum (x\sqrt{y+z})(\sqrt{x}\sqrt{z+x})\right)^2}{(x+y)(y+z)(z+x)}\\ &\le \frac{\left(\sum x^2(y+z)\right)\left(\sum x(z+x)\right)}{(x+y)(y+z)(z+x)}.\end{aligned}$$

Let $p = x+y+z, q = xy+yz+zx, r = xyz$. Then it remains to show that

$$\begin{aligned}&\frac{(pq-3r)(p^2-q)}{pq-r} \le \frac{3}{2}(p^2-2q)\\ &\Leftrightarrow 2(p^3q - pq^2 - 3p^2r + 3qr) \le 3(p^3q - p^2r - 2pq^2 + 2qr)\\ &\Leftrightarrow p^3q + 3p^2r \ge 4pq^2\\ &\Leftrightarrow p^2q + 3pr \ge 4q^2\\ &\Leftrightarrow (x+y+z)^2(xy+yz+zx) + 3xyz(x+y+z) \ge 4(xy+yz+zx)^2\\ &\Leftrightarrow (x^2+y^2+z^2)(xy+yz+zx) + 2(xy+yz+zx)^2 + 3xyz(x+y+z) \ge 4(xy+yz+zx)^2\\ &\Leftrightarrow (x^2+y^2+z^2)(xy+yz+zx) + 3xyz(x+y+z) \ge 2(xy+yz+zx)^2\\ &\Leftrightarrow (x^2+y^2+z^2)(xy+yz+zx) \ge 2(x^2y^2+y^2z^2+z^2x^2) + xyz(x+y+z)\\ &\Leftrightarrow \sum_{sym} x^3y \ge 2\sum_{sym} x^2y^2\\ &\Leftrightarrow \sum_{sym} xy(x-y)^2 \ge 0,\end{aligned}$$

which is obviously true.

3.1.5. $a \ge b \ge c$ implies $a/b+c \ge b/c+a \ge c/a+b$.
3.2.6. Use Example 3.2.3.
3.3.4. Solution: WLOG assume that $a \ge b \ge c$. This implies $a+b \ge c+a \ge b+c$. Therefore

$$\frac{a}{b+c} \ge \frac{b}{c+a} \ge \frac{c}{a+b}$$

On the other hand, we have $ab \ge ca \ge bc$. Therefore $a(b+c) \ge b(c+a) \ge c(a+b)$. Applying Chebyshev's inequality for the similarly sorted sequences $\left(\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}\right)$ and $(a(b+c), b(c+a), c(a+b))$ we get

$$\begin{aligned}&3\sum \frac{a}{b+c}\cdot a(b+c) \ge \left(\sum \frac{a}{b+c}\right)\left(\sum a(b+c)\right)\\ &\Leftrightarrow 3(a^2+b^2+c^2) \ge 2(ab+bc+ca)\sum \frac{a}{b+c}\end{aligned}$$

which was what we wanted.

3.3.5. Solution: Let $x = \frac{a^n}{b+c}, y = \frac{b^n}{c+a}, z = \frac{c^n}{a+b}$. Then

$$\begin{aligned}\sum \frac{a}{b+c} &= \sum \sqrt[n]{\frac{a^n}{(b+c)^n}} \\ &= \sum \sqrt[n]{\frac{x}{(b+c)^{n-1}}} \\ &= \sum \sqrt[n]{\frac{a^{n-1}x}{(ca+ab)^{n-1}}}.\end{aligned}$$

But the sequences $(\sqrt[n]{a^{n-1}x}, \sqrt[n]{b^{n-1}y}, \sqrt[n]{c^{n-1}z})$ and $\left(\sqrt[n]{\frac{1}{(ca+ab)^{n-1}}}, \sqrt[n]{\frac{1}{(ab+bc)^{n-1}}}, \sqrt[n]{\frac{1}{(bc+ca)^{n-1}}}\right)$ are oppositely sorted, since the sequences (a, b, c) and (x, y, z) are similarly sorted. Hence by Rearrangement inequality we get

$$\sum \sqrt[n]{\frac{a^{n-1}x}{(ca+ab)^{n-1}}} \le \sum \sqrt[n]{\frac{a^{n-1}x}{(ab+bc)^{n-1}}} = \sum \sqrt[n]{\frac{xa^{n-1}}{b^{n-1}(c+a)^{n-1}}}.$$

Finally using Hölder's inequality

$$\sum \frac{a}{b+c} \le \sum \sqrt[n]{\frac{xa^{n-1}}{b^{n-1}(c+a)^{n-1}}} \le \sqrt[n]{\left(\sum \frac{x}{b^{n-1}}\right)\left(\sum \frac{a}{c+a}\right)^{n-1}},$$

which was what we wanted.

5.1.2. The following stronger inequality holds:

$$\frac{a}{(b+c)^4} + \frac{b}{(c+a)^4} + \frac{c}{(a+b)^4} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)(a+b)(b+c)(c+a)}.$$

You may use Chebyshev's inequality to prove it.

5.1.3. First Solution: Consider the numbers a^2-1, b^2-1, c^2-1. Two of them must be of the same sign i.e. either positive or negative. WLOG suppose that a^2-1 and b^2-1 are of the same sign. Then $(a^2-1)(b^2-1) \ge 0 \Rightarrow a^2b^2+1 \ge a^2+b^2 \ge \frac{(a+b)^2}{2}$.

Now the inequality can be written as

$$c^2(a^2b^2+1) - 2c(a+b) + 2 + (a-b)^2 \ge 0.$$

Using the above argument, we'll be done if we can show that

$$\frac{c^2(a+b)^2}{2} - 2c(a+b) + 2 + (a-b)^2 \ge 0.$$

Or,

$$\frac{1}{2}(ca+bc-2)^2 + (a-b)^2 \ge 0,$$

which is obviously true.

Second Solution: WLOG we may assume that a, b, c are positive. First we have the following inequality

$$a^2+b^2+c^2+3(abc)^{\frac{2}{3}} \ge 2(ab+bc+ca).$$

This follows from Schur and AM-GM

$$(a^{\frac{2}{3}})^3+(b^{\frac{2}{3}})^3+(c^{\frac{2}{3}})^3+3(abc)^{\frac{2}{3}} \geq \sum\left(\left(a^{\frac{2}{3}}\right)^2 b^{\frac{2}{3}}+a^{\frac{2}{3}}\left(b^{\frac{2}{3}}\right)^2\right) \geq \sum 2ab.$$

So we'll be done if we can show that

$$2+(abc)^2 \geq 3(abc)^{\frac{2}{3}}.$$

Let $(abc)^{\frac{2}{3}}=t$. We need to show that

$$2+t^3 \geq 3t,$$

or,

$$(t-1)^2(t+2) \geq 0,$$

which is obviously true.

5.1.7. Solution: The inequality is equivalent to

$$\sum \frac{a^3+b^3}{a^3+b^3+abc} \geq 2$$

Verify that

$$\begin{aligned} \frac{a^3+b^3}{a^3+b^3+abc} &\geq \frac{a+b}{a+b+c} \\ \Leftrightarrow\ c(a^2+b^2-ab) &\geq abc \\ \Leftrightarrow\ c(a-b)^2 &\geq 0 \end{aligned}$$

which is obviously true. Hence we conclude that

$$\sum \frac{a^3+b^3}{a^3+b^3+abc} \geq \sum \frac{a+b}{a+b+c}=2.$$

5.1.12. Solution: The inequality is equivalent to

$$\sum \frac{(a+b)(b+c)}{\sqrt{(b+c-a)(c+a-b)}} \geq 4(a+b+c).$$

From AM-GM we get

$$\frac{(a+b)(b+c)}{\sqrt{(b+c-a)(c+a-b)}} \geq \frac{2(a+b)(b+c)}{b+c-a+c+a-b}=\frac{(a+b)(b+c)}{c}.$$

Therefore it remains to show that

$$\sum \frac{(a+b)(b+c)}{c} \geq 4(a+b+c).$$

Since the sequences $\{\frac{1}{a},\frac{1}{b},\frac{1}{c}\}$ and $\{(c+a)(a+b),(a+b)(b+c),(b+c)(c+a)\}$ are oppositely sorted, from Rearrangement we get

$$\sum \frac{(a+b)(b+c)}{c} \geq \sum \frac{(a+b)(b+c)}{b}=a+b+c+\frac{ca}{b}.$$

Therefore it remains to show that

$$\sum \frac{ca}{b} \geq a+b+c,$$

which follows from Rearrangement

$$\sum \frac{ca}{b} \geq \sum \frac{ca}{c} = a + b + c.$$

5.1.15. Solution: Let $x = \frac{1}{2a^2+2}, y = \frac{1}{2b^2+2}, z = \frac{1}{2c^2+2}$. Thus $a = \sqrt{\frac{1-2x}{2x}}, b = \sqrt{\frac{1-2y}{2y}}, c = \sqrt{\frac{1-2z}{2z}}$. Then from the given condition $x + y + z = 1$ and we need to prove that

$$\sum_{cyc} \sqrt{\frac{(1-2x)(1-2y)}{2x \cdot 2y}} \leq \frac{3}{2},$$

or

$$\sum_{cyc} \sqrt{\frac{(y+z-x)(z+x-y)}{xy}} \leq 3.$$

But we have

$$\sum_{cyc} 2\sqrt{\frac{(y+z-x)(z+x-y)}{xy}} \leq \sum_{cyc} \left(\frac{y+z-x}{y} + \frac{z+x-y}{x} \right) = 6.$$

Hence we are done.

Remark: The inequality holds even if a, b, c are real numbers.

References

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