HMMT Through the Years

Edited by

The HMMT

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Preface

This is a compilation of selected nice problems and solutions from the past few years of the Harvard-MIT Math Tournament.

1 Algebra

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1. Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

   Solution: $\boxed{\dfrac{1}{6}}$

   We have $2x^2y^2 = (x^2+y^2)^2 - (x^4+y^4) = \frac{1}{18}$, so $xy = \frac{1}{6}$.

2. Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

   Solution: $\boxed{242}$

   This is $[2^1, 2^2) \cup [2^4, 2^8) \cup [2^{16}, 2^{32}) \ldots$, and $2^8 < 2008 < 2^{16}$ so we have exactly the first two intervals.

3. Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

   Solution: $\boxed{1, 2}$

   Let $f(x) = x^2 + 2ax + 3a$. Note that $f(-3/2) = 9/4$, so the graph of f is a parabola that goes through $(-3/2, 9/4)$. Then, the condition that $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution means that the parabola has exactly one point in the strip $-1 \leq y \leq 1$, which is possible if and only if the parabola is tangent to $y = 1$. That is, $x^2 + 2ax + 3a = 2$ has exactly one solution. Then, the discriminant $\Delta = 4a^2 - 4(3a-2) = 4a^2 - 12a + 8$ must be zero. Solving the equation yields $a = 1, 2$.

4. Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
   (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

   Solution: $\boxed{2006}$

   We have
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor = \left\lfloor \frac{\left(2007 \cdot 2006 + \frac{1}{2005}\right) \cdot 2005!}{(2006+1) \cdot 2005!} \right\rfloor = \left\lfloor \frac{2007 \cdot 2006 + \frac{1}{2005}}{2007} \right\rfloor = \left\lfloor 2006 + \frac{1}{2005 \cdot 2007} \right\rfloor.$$

5. Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy.

   Solution: $\boxed{-3}$

   We have $28 = x^3 - y^3 = (x-y)(x^2+xy+y^2) = (x-y)((x-y)^2 + 3xy) = 4 \cdot (16 + 3xy)$, from which $xy = -3$.

6. Three real numbers x, y, and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y.

   Solution: $\boxed{\dfrac{1}{2}}$

   Because the first and third fractions are equal, adding their numerators and denominators produces another fraction equal to the others: $((x+4)+(x+5))/(2+(z-5)) = (2x+9)/(z-3)$. Then $y + 9 = 2x + 9$, etc.

7. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

Solution: $\boxed{14{:}24}$

Let the distance between Harvard and MIT be 1, and let c, s, r denote the speeds of the current and Larry's swimming and rowing, respectively. Then we are given

$$s + c = \frac{1}{40} = \frac{9}{360}, \qquad s - c = \frac{1}{45} = \frac{8}{360}, \qquad r - c = \frac{1}{15} = \frac{24}{360},$$

so

$$r + c = (s + c) - (s - c) + (r - c) = \frac{9 - 8 + 24}{360} = \frac{25}{360},$$

and it takes Larry $360/25 = 14.4$ minutes, or 14:24, to row from Harvard to MIT.

8. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

Solution: $\boxed{(3,3),\ (-4,-4),\ \left(\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}\right),\ \left(\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2}\right)}$

We have $x^2 + y = y^2 + x$ which can be written as $(x - y)(x + y - 1) = 0$. The case $x = y$ yields $x^2 + x - 12 = 0$, hence $(x, y) = (3, 3)$ or $(-4, -4)$. The case $y = 1 - x$ yields $x^2 + 1 - x - 12 = x^2 - x - 11 = 0$ which has solutions $x = \frac{1\pm\sqrt{1+44}}{2} = \frac{1\pm3\sqrt{5}}{2}$. The other two solutions follow.

9. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

Solution: $\boxed{\frac{13}{48}}$

Suppose we fix the time at which Train B arrives at Intersection X; then call the interval during which Train A could arrive (given its schedule) and collide with Train B the "disaster window."

We consider two cases:

(i) *Train B enters Intersection X between 9:30 and 9:45.* If Train B arrives at 9:30, the disaster window is from 9:00 to 10:15, an interval of $1\frac{1}{4}$ hours. If Train B arrives at 9:45, the disaster window is $1\frac{1}{2}$ hours long. Thus, the disaster window has an average length of $(1\frac{1}{4}+1\frac{1}{2}) \div 2 = \frac{11}{8}$. From 9:00 to 2:30 is $5\frac{1}{2}$ hours. The probability of a collision is thus $\frac{11}{8} \div 5\frac{1}{2} = \frac{1}{4}$.

(ii) *Train B enters Intersection X between 9:45 and 12:30.* Here the disaster window is always $1\frac{1}{2}$ hours long, so the probability of a collision is $1\frac{1}{2} \div 5\frac{1}{2} = \frac{3}{11}$.

From 9:30 to 12:30 is 3 hours. Now case (i) occurs with probability $\frac{1}{4} \div 3 = \frac{1}{12}$, and case (ii) occurs with probability $\frac{11}{12}$. The overall probability of a collision is therefore $\frac{1}{12} \cdot \frac{1}{4} + \frac{11}{12} \cdot \frac{3}{11} = \frac{1}{48} + \frac{1}{4} = \frac{13}{48}$.

10. How many real numbers x are solutions to the following equation?

$$|x - 1| = |x - 2| + |x - 3|$$

Solution: $\boxed{2}$

If $x < 1$, the equation becomes $(1-x) = (2-x)+(3-x)$ which simplifies to $x = 4$, contradicting the assumption $x < 1$. If $1 \leq x \leq 2$, we get $(x-1) = (2-x)+(3-x)$, which gives $x = 2$. If $2 \leq x \leq 3$, we get $(x-1) = (x-2)+(3-x)$, which again gives $x = 2$. If $x \geq 3$, we get $(x-1) = (x-2)+(x-3)$, or $x = 4$. So 2 and 4 are the only solutions, and the answer is 2.

11. How many real numbers x are solutions to the following equation?

$$2003^x + 2004^x = 2005^x$$

Solution: $\boxed{1}$

Rewrite the equation as $(2003/2005)^x + (2004/2005)^x = 1$. The left side is strictly decreasing in x, so there cannot be more than one solution. On the other hand, the left side equals $2 > 1$ when $x = 0$ and goes to 0 when x is very large, so it must equal 1 somewhere in between. Therefore there is one solution.

12. Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

Solution: $\boxed{-\frac{1}{2}}$

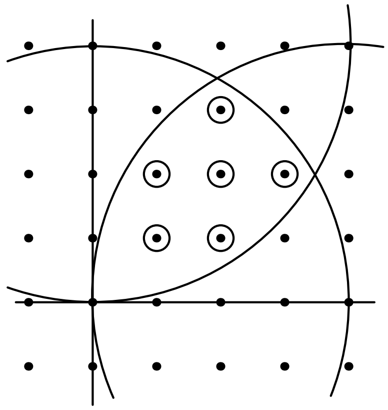
Note that $0 = (x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$. Rearranging, we get that $xy+yz+zx = -\frac{1}{2}(x^2+y^2+z^2)$, so that in fact the quantity is always equal to $-1/2$.

13. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2 + b^2 &< 16 \\ a^2 + b^2 &< 8a \\ a^2 + b^2 &< 8b \end{aligned}$$

Solution: $\boxed{6}$

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0,0), (4,0), (0,4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



14. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

    Solution: $\boxed{6}$

    For positive integers a, b, we have

    $$a! \mid b! \quad \Leftrightarrow \quad a! \leq b! \quad \Leftrightarrow \quad a \leq b.$$

    Thus,
    $$((n!)!)! \mid (2004!)! \quad \Leftrightarrow \quad (n!)! \leq 2004! \quad \Leftrightarrow \quad n! \leq 2004 \quad \Leftrightarrow \quad n \leq 6.$$

15. Compute:
    $$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor.$$

    Solution: $\boxed{8}$

    Let $x = 2004$. Then the expression inside the floor brackets is

    $$\frac{(x+1)^3}{(x-1)x} - \frac{(x-1)^3}{x(x+1)} = \frac{(x+1)^4 - (x-1)^4}{(x-1)x(x+1)} = \frac{8x^3 + 8x}{x^3 - x} = 8 + \frac{16x}{x^3 - x}.$$

    Since x is certainly large enough that $0 < 16x/(x^3 - x) < 1$, the answer is 8.

16. Find the smallest value for x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a.

    Solution: $\boxed{49}$

    We want to find the maximum possible value of $14\sqrt{a} - a = 49 - (\sqrt{a} - 7)^2$, which is clearly 49, achieved when $a = 49$.

17. Compute $\frac{\tan^2(20°) - \sin^2(20°)}{\tan^2(20°)\sin^2(20°)}$.

    Solution: $\boxed{1}$

    If we multiply top and bottom by $\cos^2(20°)$, the numerator becomes $\sin^2(20°) \cdot (1 - \cos^2 20°) = \sin^4(20°)$, while the denominator becomes $\sin^4(20°)$ also. So they are equal, and the ratio is 1.

18. Find the smallest n such that $n!$ ends in 290 zeroes.

    Solution: $\boxed{1170}$

Each 0 represents a multiple of 10, factored into $2 \cdot 5$. Thus, we wish to find the smallest factorial that contains at least 290 2's and 290 5's in its prime factorization. Let this number be $n!$, so $n! = 2^p \cdot 5^q \cdots$ where

$$p = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \ldots \text{ and } q = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \ldots$$

(this takes into account one factor for each single multiple of 2 or 5, an additional factor for each square, and so on). Naturally, p will be larger because 2 is smaller than 5. Thus, we want to bring q as low to 290 as possible. If $q = \left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \cdots$, we form a rough geometric sequence (by taking away the floor function) whose sum is represented by $290 = \frac{\frac{n}{5}}{1-\frac{1}{5}}$. Hence $n = 1160$, and $q = 288$. Adding 10 to the value of n gives the necessary two additional factors of 5, and so the answer is 1170.

19. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

    Solution: $\boxed{\frac{4}{3}}$

    $\frac{24}{6 \cdot 3} = \frac{x^4 - y^4}{(x^2+y^2)(x+y)} = \frac{(x^2+y^2)(x+y)(x-y)}{(x^2+y^2)(x+y)} = x - y = \frac{4}{3}$.

20. Find $(x+1)(x^2+1)(x^4+1)(x^8+1)\cdots$, where $|x| < 1$.

    Solution: $\boxed{\frac{1}{1-x}}$

    Let $S = (x+1)(x^2+1)(x^4+1)(x^8+1)\cdots = 1 + x + x^2 + x^3 + \cdots$. Since $xS = x + x^2 + x^3 + x^4 + \cdots$, we have $(1-x)S = 1$, so $S = \frac{1}{1-x}$.

21. How many times does 24 divide into 100! (factorial)?

    Solution: $\boxed{32}$

    We first determine the number of times 2 and 3 divide into $100! = 1 \cdot 2 \cdot 3 \cdots 100$. Let $\langle N \rangle_n$ be the number of times n divides into N (i.e. we want to find $\langle 100! \rangle_{24}$). Since 2 only divides into even integers, $\langle 100! \rangle_2 = \langle 2 \cdot 4 \cdot 6 \cdots 100 \rangle$. Factoring out 2 once from each of these multiples, we get that $\langle 100! \rangle_2 = \left\langle 2^{50} \cdot 1 \cdot 2 \cdot 3 \cdots 50 \right\rangle_2$. Repeating this process, we find that $\langle 100! \rangle_2 = \left\langle 20^{50+25+12+6+3+1} \cdot 1 \right\rangle_2 = 97$. Similarly, $\langle 100! \rangle_3 = \left\langle 3^{33+11+3+1} \right\rangle_3 = 48$. Now $24 = 2^3 \cdot 3$, so for each factor of 24 in 100! there needs to be three multiples of 2 and one multiple of 3 in 100!. Thus $\langle 100! \rangle_{24} = ([\langle 100! \rangle_2 /3] + \langle 100! \rangle_3) = \boxed{32}$, where $[N]$ is the greatest integer less than or equal to N.

22. Which is greater: $(3^5)^{(5^3)}$ or $(5^3)^{(3^5)}$?

    Solution: $\boxed{(5^3)^{(3^5)}}$

    Notice that $3^6 = 729$ while $5^4 = 625$, and since $\ln 5 > \ln 3$, it follows that $5^4 \ln 3 < 3^6 \ln 5$, so $5^3 * 5 \ln 3 < 3^5 * 3 \ln 5$, and so by laws of logarithms, $5^3 \ln 3^5 < 3^5 \ln 5^3$. Again applying laws of logarithms, it follows that $\ln{(3^5)^{(5^3)}} < \ln{(5^3)^{(3^5)}}$. So, since $\ln x$ is an increasing function, it follows that $(3^5)^{(5^3)} < (5^3)^{(3^5)}$, and so it follows that $\mathbf{(5^3)^{(3^5)}}$ **is greater**.

23. If $a = 2b + c$, $b = 2c + d$, $2c = d + a - 1$, $d = a - c$, what is b?

    Solution: $\boxed{\frac{2}{9}}$

    Since $d = a - c$, substitute into the equation $b = 2c + d$ to get $b = 2c + a - c = a + c$. Also, substitute into $2c = d + a - 1$ to get $2c = a - c + a - 1$, or $3c = 2a - 1$. Now, since $b = a + c$, we can substitute into $a = 2b + c$ to get $a = 2a + 2c + c$, or $a = -3c$. Since we know $3c = 2a - 1$ from above, we substitute in to get $3c = -6c - 1$, or $c = -\frac{1}{9}$. Thus, we find that $a = \frac{1}{3}$, $d = \frac{4}{9}$, and $\boldsymbol{b = \frac{2}{9}}$.

24. Bobo the clown was juggling his spherical cows again when he realized that when he drops a cow is related to how many cows he started off juggling. If he juggles 1, he drops it after 64 seconds. When juggling 2, he drops one after 55 seconds, and the other 55 seconds later. In fact, he was able to create the following table:

cows	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
seconds	64	55	47	40	33	27	22	18	14	13	12	11	10	9	8	7	6	5	4	3	2	1

He can only juggle up to 22 cows. To juggle the cows the longest, what number of cows should he start off juggling? How long (in minutes) can he juggle for?

Solution: $\boxed{5 \text{ cows}, 2\frac{3}{4} \text{ minutes}}$

The time that Bobo can juggle is the number of cows times seconds. So, we get the following table:

cows started	total time
1	64
2	110
3	141
4	160
5	165
6	162
7	154
8	144
9	126
10	130
11	121
12	121
13	130
14	126
15	120
16	112
17	102
18	90
19	76
20	60
21	42
22	22

Thus, we see that the maximum occurs with **5 cows**, and the total time is 165 seconds = $\mathbf{2\frac{3}{4}}$ **minutes**.

25. If $a@b = \frac{a^3-b^3}{a-b}$, for how many real values of a does $a@1 = 0$?

Solution: $\boxed{0}$

If $\frac{a^3-1}{a-1} = 0$, then $a^3 - 1 = 0$, or $(a-1)(a^2+a+1) = 0$. Thus $a = 1$, which is an extraneous solution since that makes the denominator of the original expression 0, or a is a root of $a^2 + a + 1$. But this quadratic has no real roots, in particular its roots are $\frac{-1\pm\sqrt{-3}}{2}$. Therefore there are no such real values of a, so the answer is **0**.

26. For what single digit n does 91 divide the 9-digit number $12345n789$?

    Solution: $\boxed{7}$

    123450789 leaves a remainder of 7 when divided by 91, and 1000 leaves a remainder of 90, or -1, so adding **7** multiples of 1000 will give us a multiple of 91.

    Alternate Solution: For those who don't like long division, there is a quicker way. First notice that $91 = 7 \cdot 13$, and $7 \cdot 11 \cdot 13 = 1001$. Observe that $12345n789 = 123 \cdot 1001000 + 45n \cdot 1001 - 123 \cdot 1001 + 123 - 45n + 789$ It follows that 91 will divide $12345n789$ iff 91 divides $123 - 45n + 789 = 462 - n$. The number 462 is divisible by 7 and leaves a remainder of **7** when divided by 13.

27. Alex is stuck on a platform floating over an abyss at 1 ft/s. An evil physicist has arranged for the platform to fall in (taking Alex with it) after traveling 100ft. One minute after the platform was launched, Edward arrives with a second platform capable of floating all the way across the abyss. He calculates for 5 seconds, then launches the second platform in such a way as to maximize the time that one end of Alex's platform is between the two ends of the new platform, thus giving Alex as much time as possible to switch. If both platforms are 5 ft long and move with constant velocity once launched, what is the speed of the second platform (in ft/s)?

    Solution: $\boxed{3}$

    The slower the second platform is moving, the longer it will stay next to the first platform. However, it needs to be moving fast enough to reach the first platform before it's too late. Let v be the velocity of the second platform. It starts 65 feet behind the first platform, so it reaches the back of the first platform at $\frac{60}{v-1}$ seconds, and passes the front at $\frac{70}{v-1}$ seconds, so the time to switch is $\frac{10}{v-1}$. Hence we want v to be as small as possible while still allowing the switch before the first platform falls. Therefore the time to switch will be maximized if the back of the second platform lines up with the front of the first platform at the instant that the first platform has travelled 100ft, which occurs after 100 seconds. Since the second platform is launched 65 seconds later and has to travel 105 feet, its speed is $105/35 = \mathbf{3}$ft/s.

2 Calculus

2.1 Calculus Problems

2.2 Calculus Solutions

1. Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.

   Solution: $\boxed{5050}$

   Note that $f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$, so $f'(1) = 1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050$.

2. Let ℓ be the line through $(0, 0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ.

   Solution: $\boxed{13}$

   Let the point of tangency be $(t, t^3 + t + 16)$, then the slope of ℓ is $(t^3 + t + 16)/t$. On the other hand, since $dy/dx = 3x^2 + 1$, the slope of ℓ is $3t^2 + 1$. Therefore,

   $$\frac{t^3 + t + 16}{t} = 3t^2 + 1.$$

   Simplifying, we get $t^3 = 8$, so $t = 2$. It follows that the slope is $3(2)^2 + 1 = 13$.

3. Find all $y > 1$ satisfying $\displaystyle\int_1^y x \ln x \, dx = \frac{1}{4}$.

   Solution: $\boxed{\sqrt{e}}$

   Applying integration by parts with $u = \ln x$ and $v = \frac{1}{2}x^2$, we get

   $$\int_1^y x \ln x \, dx = \frac{1}{2}x^2 \ln x\Big|_1^y - \frac{1}{2}\int_1^y x \, dx = \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + \frac{1}{4}.$$

   So $y^2 \ln y = \frac{1}{2}y^2$. Since $y > 1$, we obtain $\ln y = \frac{1}{2}$, and thus $y = \sqrt{e}$.

4. Compute:

   $$\lim_{x \to 0} \quad \frac{x^2}{1 - \cos(x)}$$

   Solution: $\boxed{2}$

   Since $\sin^2(x) = 1 - \cos^2(x)$, we multiply the numerator and denominator by $1 + \cos(x)$ and use the fact that $x/\sin(x) \to 1$, obtaining

   $$\lim_{x \to 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \to 0} \frac{x^2(1 + \cos(x))}{1 - \cos^2(x)} = \lim_{x \to 0} \left(\frac{x}{\sin(x)}\right)^2 \cdot 2 = 2$$

   Remark: Another solution, using *L'Hôpital's rule*, is possible: $\lim_{x \to 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \to 0} \frac{2x}{\sin(x)} = 2$.

5. Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

   Solution: $\boxed{1}$

   Being a relative minimum, we have $0 = f'(a) = 4a^3 - 3a^2 - 2a + a = a(4a + 1)(a - 1)$. Then $a = 0, 1, -1/4$ are the only possibilities. However, it is easily seen that $a = 1$ is the only value satisfying $f(a) = a$.

6. Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{\mathrm{d}x}{x+\sqrt{x}}$$

achieves its smallest possible value.

Solution: $\boxed{3-2\sqrt{2}}$

Let $F(a)$ denote the given definite integral. Then

$$F'(a) = \frac{\mathrm{d}}{\mathrm{d}a} \int_a^{a^2} \frac{\mathrm{d}x}{x+\sqrt{x}} = 2a \cdot \frac{1}{a^2+\sqrt{a^2}} - \frac{1}{a+\sqrt{a}}.$$

Setting $F'(a) = 0$, we find that $2a + 2\sqrt{a} = a + 1$ or $(\sqrt{a}+1)^2 = 2$. We find $\sqrt{a} = \pm\sqrt{2} - 1$, and because $\sqrt{a} > 0$, $a = (\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$.

7. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

Solution: $\boxed{\frac{1}{18}}$

Suppose that the leading term of $f(x)$ is cx^n, where $c \neq 0$. Then the leading terms of $f'(x)$ and of $f''(x)$ are cnx^{n-1} and $cn(n-1)x^{n-2}$, respectively, so $cx^n = cnx^{n-1} \cdot cn(n-1)x^{n-2}$, which implies that $n = (n-1) + (n-2)$, or $n = 3$, and $c = cn \cdot cn(n-1) = 18c^2$, or $c = \frac{1}{18}$.

8. Compute $\displaystyle\lim_{x\to 0} \frac{e^{x\cos x} - 1 - x}{\sin(x^2)}$.

Solution: $\boxed{\frac{1}{2}}$

Let's compute all the relevant Maclaurin series expansions, up to the quadratic terms:

$$x\cos x = x + \dots, \qquad e^{x\cos x} = 1 + x + \frac{1}{2}x^2 + \dots, \qquad \sin(x^2) = x^2 + \dots,$$

so

$$\lim_{x\to 0} \frac{e^{x\cos x} - 1 - x}{\sin(x^2)} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 + \dots}{x^2 + \dots} = \frac{1}{2}.$$

9. At time 0, an ant is at $(1, 0)$ and a spider is at $(-1, 0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

Solution: $\boxed{\frac{\sqrt{14}}{4}}$

Picture an instant in time where the ant and spider have x-coordinates a and s, respectively. If $1 \le s \le 3$, then $a \le 0$, and the distance between the bugs is at least 1. If $s > 3$, then, needless to say the distance between the bugs is at least 2. If $-1 \le s \le 1$, then $s = 1 - 2a$, and the distance between the bugs is

$$\sqrt{(a-(1-2a))^2 + (1-a^2)} = \sqrt{8a^2 - 6a + 2} = \sqrt{\frac{(8a-3)^2 + 7}{8}},$$

which attains the minimum value of $\sqrt{7/8}$ when $a = 3/8$.

10. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

Solution: $\boxed{1}$

Since $f'(x) = 3x^2 + a$, we must have $3a^2 + a = 3b^2 + a$. Then $a^2 = b^2$, and since $a \neq b$, $a = -b$. Thus $f(1) = 1 + a + b = 1$.

11. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u}\, du$ and $y(t) = \int_t^\infty \frac{\sin u}{u}\, du$ for $1 \leq t \leq 2$. What is the length of the curve?

Solution: $\boxed{\ln 2}$

By the Second Fundamental Theorem of Calculus, $\frac{dx}{dt} = -\frac{\cos t}{t}$ and $\frac{dy}{dt} = -\frac{\sin t}{t}$. Therefore, the length of the curve is

$$\int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\, dt = \int_1^2 \frac{1}{t}\, dt = [\ln t]_1^2 = \ln 2.$$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

Solution: $\boxed{486/5}$

$$0 = \int_0^1 f(x)f'(x)dx = \int_{f(0)}^{f(1)} u\, du = \frac{1}{2}(f(1)^2 - f(0)^2), \text{ and}$$

$$18 = \int_0^1 f(x)^2 f'(x)dx = \int_{f(0)}^{f(1)} u^2 du = \frac{1}{3}(f(1)^3 - f(0)^3).$$

The first equation implies $f(0) = \pm f(1)$. The second equation shows that $f(0) \neq f(1)$, and in fact $54 = f(1)^3 - f(0)^3 = 2f(1)^3$, so $f(1) = 3$ and $f(0) = -3$. Then

$$\int_0^1 f(x)^4 f'(x)dx = \int_{f(0)}^{f(1)} u^4 du = \frac{1}{5}(f(1)^5 - f(0)^5) = \frac{1}{5}(243 + 243) = \frac{486}{5}.$$

13. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h \to 0} \frac{f(x+h) - f(h)}{x}$ at $x = \pi$.

Solution: $\boxed{0}$

The expression $\frac{f(x+h)-f(h)}{x}$ is continuous at $h = 0$, so the limit is just $\frac{f(x)-f(0)}{x}$. Letting $x = \pi$ yields $\frac{\sin(\sin \pi) - \sin(\sin 0)}{\pi} = 0$.

14. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

Solution: $\boxed{19}$

Let $g(x) = f(x) - f(2x)$. Then we want the derivative of

$$f(x) - f(4x) = (f(x) - f(2x)) + (f(2x) - f(4x)) = g(x) + g(2x)$$

at $x = 1$. This is $g'(x) + 2g'(2x)$ at $x = 1$, or $5 + 2 \cdot 7 = 19$.

15. Find $\lim_{x\to\infty}(\sqrt[3]{x^3+x^2}-\sqrt[3]{x^3-x^2})$.

Solution: $\boxed{2/3}$

Observe that

$$\lim_{x\to\infty}\left[(x+1/3)-\sqrt[3]{x^3+x^2}\right]=\lim_{x\to\infty}\frac{x/3+1/27}{(\sqrt[3]{x^3+x^2})^2+(\sqrt[3]{x^3+x^2})(x+1/3)+(x+1/3)^2},$$

by factoring the numerator as a difference of cubes. The numerator is linear in x, while the denominator is at least $3x^2$, so the limit as $x\to\infty$ is 0. By similar arguments, $\lim_{x\to\infty}[(x-1/3)-\sqrt[3]{x^3-x^2}]=0$. So, the desired limit equals

$$2/3+\lim_{x\to\infty}[(x-1/3)-\sqrt[3]{x^3-x^2}]-\lim_{x\to\infty}[(x+1/3)-\sqrt[3]{x^3+x^2}]=2/3.$$

16. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

Solution: $\boxed{\frac{2}{3}}$

The probability of the point falling between a distance r and $r+dr$ from the center is $\frac{2\pi r\,dr}{\pi}=2r\,dr$. Then the expected distance is $\int_0^1 2r^2\,dr=\frac{2}{3}$.

17. A particle moves along the x-axis in such a way that its velocity at position x is given by the formula $v(x)=2+\sin x$. What is its acceleration at $x=\frac{\pi}{6}$?

Solution: $\boxed{\frac{5\sqrt{3}}{4}}$

$a=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}\cdot v=\cos x\cdot(2+\sin x)=\frac{5\sqrt{3}}{4}$.

18. What is the area of the region bounded by the curves $y=x^{2003}$ and $y=x^{1/2003}$ and lying above the x-axis?

Solution: $\boxed{\frac{1001}{1002}}$

We have

$$\int_0^1 x^{1/2003}-x^{2003}\,dx=\left[\frac{x^{\frac{2004}{2003}}}{\frac{2004}{2003}}-\frac{x^{2004}}{2004}\right]_0^1=\frac{1001}{1002}.$$

19. A sequence of ants walk from $(0,0)$ to $(1,0)$ in the plane. The nth ant walks along n semicircles of radius $\frac{1}{n}$ with diameters lying along the line from $(0,0)$ to $(1,0)$. Let L_n be the length of the path walked by the nth ant. Compute $\lim\limits_{n\to\infty} L_n$.

Solution: $\boxed{\pi}$

A semicircle of radius $\frac{1}{n}$ has length $\frac{1}{2}\pi\left(\frac{2}{n}\right)=\frac{\pi}{n}$, so n such semicircles have total length $\boxed{\pi}$.

20. The polynomial $3x^5-250x^3+735x$ is interesting because it has the maximum possible number of relative extrema and points of inflection at integer lattice points for a quintic polynomial. What is the sum of the x-coordinates of these points?

Solution: $\boxed{75}$

The first derivative is $15x^4-750x^2+735$, whose roots (which give the relative extrema) sum to $750/15=50$. The second derivative is $60x^3-1500x$, whose roots (which give the points of inflection) sum to $1500/60=25$, for a grand total of 75.

21. A balloon that blows up in the shape of a perfect cube is being blown up at a rate such that at time t fortnights, it has surface area $6t$ square furlongs. At how many cubic furlongs per fortnight is the air being pumped in when the surface area is 144 square furlongs?

    Solution: $\boxed{3\sqrt{6}}$

    The surface area at time t is $6t$, so the volume is $t^{3/2}$. Hence the air is being pumped in at a rate of $\frac{3}{2}\sqrt{t}$. When the surface area is 144, $t = 24$, so the answer is $3\sqrt{6}$.

22. A rectangle of length $\frac{1}{4}\pi$ and height 4 is bisected by the x-axis and is in the first and fourth quadrants. The graph of $y = sin(x) + C$ divides the area of the square in half. What is C?

    Solution: $\boxed{\frac{2\sqrt{2}+4}{\pi}}$

    $\int_0^{\frac{1}{4}\pi} \sin x + C = 0$, from the statement of the problem. So, $[-\cos x + Cx]|_0^{\frac{\pi}{4}} = 0$. Thus, $\cos\frac{\pi}{4} + \frac{\pi}{4}C + 1 = 0$. So, $-\frac{\sqrt{2}}{2} + \frac{\pi}{4}C + 1 = 0$, and solving for C, we find that $C = \frac{2\sqrt{2}+4}{\pi}$.

23. For what value of x $(0 < x < \frac{\pi}{2})$ does $\tan x + \cot x$ achieve its minimum?

    Solution: $\boxed{\frac{\pi}{4}}$

    Let $y = \tan x$. So, we want to find the minimum of $y + \frac{1}{y}$, where $0 \leq y \leq \infty$. Taking the derivative and minimizing, we find that the minimum occurs at $y = 1$, so the minimum of the given function occurs at $\arctan 1 = \frac{\pi}{4}$.

24. What is the average value of $\sin^2\theta$ over its period?

    Solution: $\boxed{\frac{1}{2}}$

    By Mean Value Theorem (??), the average value of $\sin^2\theta = \frac{\int_0^\pi \sin^2\theta d\theta}{\pi - 0} = \frac{1}{2\pi}\int_0^\pi (1 - \cos 2\theta)d\theta = \frac{1}{2\pi}[\theta - \frac{2\theta}{2}]|_0^\pi = \frac{1}{2\pi}[\pi - 0 - (-0 - 0)] = \frac{1}{2}$.

25. Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

    Solution: $\boxed{80x + 19}$

    Since $f''(x) = 0$ we must have $f(x) = ax + b$ for some real numbers a, b. Thus $f(0) = b = 19$ and $f(1) = a + 19 = 99$, so $a = 80$. Therefore $f(x) = f80x + 19$.

26. A rectangle has sides of length $\sin x$ and $\cos x$ for some x. What is the largest possible area of such a rectangle?

    Solution: $\boxed{\frac{1}{2}}$

    We wish to maximize $\sin x \cdot \cos x = \frac{1}{2}\sin 2x$. But $\sin 2x \leq 1$, with equality holding for $x = \pi/4$, so the maximum is $\frac{1}{2}$.

27. Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} (\frac{\sin x}{1 + x^4} + 1)dx.$$

    Solution: $\boxed{8\pi\sqrt{2}}$

    The function $\frac{\sin x}{1+x^4}$ is odd, so its integral over this interval is 0. Thus we get the same answer if we just integrate dx, namely, $8\pi\sqrt{2}$.

3 Combinatorics

3.1 Combinatorics Problems

3.2 Combinatorics Solutions

1. A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

   Solution: $\boxed{60}$

   There are 6 such slices, and each slice gives 10 valid pairs (with no overcounting). Therefore, there are 60 such pairs.

2. Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S.

   Solution: $\boxed{\dfrac{2009}{2}}$

   For any subset A, we can define the "reflected subset" $A' = \{i \mid 2009 - i \in A\}$. Then $m(A) = 2009 - m(A')$. Note that as A is taken over all nonempty subsets of S, A' goes through all the nonempty subsets of S as well. Thus, the average of $m(A)$ is equal to the average of $\frac{m(A)+m(A')}{2}$, which is the constant $\frac{2009}{2}$.

3. Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

   Solution: $\boxed{100800}$

   Since there are 9 cow and pigs combined and 7 horses, there must be a pair with 1 cow and 1 pig, and all the other pairs must contain a horse. There are 4×5 ways of selecting the cow-pig pair, and $7!$ ways to select the partners for the horses. It follows that the answer is $4 \times 5 \times 7! = 100800$.

4. A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

   Solution: $\boxed{41}$

   If Biff and Jacob are on the committee, there are $\binom{7}{3} = 35$ ways for the other members to be chosen. Amongst these 35 possibilities, we reject the $\binom{5}{1} = 5$ choices where both Alice and Jane are also serving. If Biff and Jacob are not serving, then there are $\binom{7}{5} = 21$ ways to choose the remaining 5 members. Again, we reject the $\binom{5}{3} = 10$ instances where Alice and Jane are chosen, so the total is $(35 - 5) + (21 - 10) = 41$.

5. How many 5-digit numbers $\overline{abcde}$ exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

   Solution: $\boxed{330}$

   Note that $a > 0$, so that $b > c$, and $e \geq 0$ so that $d \geq c$. Conversely, for each choice of (b, c, d) with $b > c$ and $d \geq c$, there exists a unique pair (a, e) such that $\overline{abcde}$ is a number having the desired property. Thus, we compute

$$\sum_{c=0}^{9}(9-c)(10-c) = \sum_{c=0}^{9} c^2 - 19c + 90 = 330.$$

6. Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

   Solution: $\boxed{\dfrac{12}{37}}$

   The desired probability is the relative probability that Jill draws the spade. In the first round, Jack, Jill, and John draw a spade with probability $1/4, 3/4 \cdot 1/4$, and $(3/4)^2 \cdot 1/4$ respectively. Thus, the probability that Jill draws the spade is

$$\frac{3/4 \cdot 1/4}{1/4 + 3/4 \cdot 1/4 + (3/4)^2 \cdot 1/4} = \frac{12}{37}.$$

7. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

   Solution: $\boxed{11}$

   Suppose that n seniors play all three sports and that $2n$ are on the lacrosse team. Then, by the principle of inclusion-exclusion, $85 = (74 + 26 + 2n) - (17 + 18 + 13) + (n) = 100 + 2n - 48 + n = 52 + 3n$. It is easily seen that $n = 11$.

8. Compute

$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

   Solution: $\boxed{1953}$

   The given sum counts the number of non-decreasing 61-tuples of integers $(n_0, \ldots, n_{60})$ from the set $\{0, 1, 2\}$. Such 61-tuples are in one-to-one correspondence with strictly increasing 61-tuples of integers $(m_0, \ldots, m_{60})$ from the set $\{0, 1, 2, \ldots, 62\}$: simply let $m_k = n_k + k$. But the number of such $(m_0, \ldots, m_{60})$ is almost by definition $\binom{63}{61} = \binom{63}{2} = 1953$.

9. A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

   Solution: $\boxed{48}$

   Let X, Y, Z be the three directions in which the moth can intially go. We can symbolize the trajectory of the moth by a sequence of stuff from Xs, Ys, and Zs in the obvious way: whenever the moth takes a step in a direction parallel or opposite to X, we write down X, and so on.

   The moth can reach B in either exactly 3 or exactly 5 steps. A path of length 3 must be symbolized by XYZ in some order. There are $3! = 6$ such orders. A trajectory of length 5 must by symbolized by $XYZXX$, $XYZYY$, or $XYZZZ$, in some order, There are $3 \cdot \frac{5!}{3!1!1!} = 3 \cdot 20 = 60$ possibilities here. However, we must remember to subtract out those trajectories that already arrive at B by the 3rd step: there are $3 \cdot 6 = 18$ of those. The answer is thus $60 - 18 + 6 = 48$.

10. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

    Solution: $\boxed{22}$

    Suppose that either nine or ten of the questions have the same answer. Then no matter which five questions we pick to have this answer, we will be right at least four times. Conversely, suppose that there are at least two questions with each answer; we will show that we can get a score less than four. By symmetry, assume there are at least five questions whose answer is true. Then if we label five of these false, not only will we get these five wrong, but we will also have answered all the false questions with true, for a total of at least seven incorrect. There are 2 ways for all the questions to have the same answer, and $2 \cdot 10 = 20$ ways for one question to have a different answer from the others, for a total of 22 ways.

11. How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

    Solution: $\boxed{1365}$

    If a is the smallest element of such a set, then $13 - a$ is the largest element, and for the remaining elements we may choose any (or none) of the $12-2a$ elements $a+1, a+2, \ldots, (13-a)-1$. Thus there are 2^{12-2a} such sets whose smallest element is a. Also, $13 - a \geq a$ clearly implies $a < 7$. Summing over all $a = 1, 2, \ldots, 6$, we get a total of

    $$2^{10} + 2^8 + 2^6 + \cdots + 2^0 = 4^5 + 4^4 + \cdots + 4^0 = (4^6 - 1)/(4 - 1) = 4095/3 = 1365$$

    possible sets.

12. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

    Solution: $\boxed{\frac{1}{2}}$

    Note that if we imagine that the series always continues to seven games even after one team has won four, this will never change the winner of the series. Notice also that the probability that the Red Sox will win Game n is precisely the probability that the Yankees will win Game $8 - n$. Therefore, the probability that the Yankees win at least four games is the same as the probability that the Red Sox win at least four games, namely $1/2$.

13. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

    Solution: $\boxed{31}$

    We can prove by induction on n that the following pattern holds for $0 \leq n \leq 499$: after $2n$ minutes, the first room contains $1000 - 2n$ people and the next n rooms each contain 2 people, and after $2n+1$

minutes, the first room contains $1000-(2n+1)$ people, the next n rooms each contain 2 people, and the next room after that contains 1 person. So, after 60 minutes, we have one room with 940 people and 30 rooms with 2 people each.

14. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

    Solution: $\boxed{21600}$

    In the top row, you can mark any of the 6 squares that is not a corner. In the bottom row, you can then mark any of the 5 squares that is not a corner and not in the same column as the square just marked. Then, in the second row, you have 6 choices for a square not in the same column as either of the two squares already marked; then there are 5 choices remaining for the third row, and so on down to 1 for the seventh row, in which you make the last mark. Thus, altogether, there are $6 \cdot 5 \cdot (6 \cdot 5 \cdots 1) = 30 \cdot 6! = 30 \cdot 720 = 21600$ possible sets of squares.

15. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

    Solution: $\boxed{6}$

    Suppose the last student solved n problems, and the total number of problems on the test was p. Then the total number of correct solutions written was $7p$ (seven per problem), and also equal to $36+n$ (the sum of the students' scores), so $p = (36+n)/7$. The smallest $n \geq 0$ for which this is an integer is $n = 6$. But we also must have $n \leq p$, so $7n \leq 36+n$, and solving gives $n \leq 6$. Thus $n = 6$ is the answer.

16. You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, with both possibilities equally likely. You perform the following operation: for each switch S, if the two switches next to S are initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?

    Solution: $\boxed{\dfrac{1}{2^{2002}}}$

    In general, there are 2^n equally likely starting configurations. In this case (or more generally for any odd n), all switches end up ON if and only if switches $1, 3, 5, 7, \ldots, 2003,\ 2, 4, \ldots, 2002$ — i.e. all 2003 of them — are initially in the same position. This initial position can be ON or OFF, so this situation occurs with probability $2/2^{2003} = 1/2^{2002}$.

17. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

    Solution: $\boxed{36}$

    Since each row has only two squares, it is impossible for two marked squares to be in the same row. Therefore, exactly nine of the ten rows contain marked squares. Consider two cases:

    Case 1: The first or last row is empty. These two cases are symmetrical, so assume without loss of generality that the first row is empty. There are two possibilities for the second row: either the square in the first column is marked, or the square in the second column is marked. Since the third row must contain a marked square, and it cannot be in the same column as the marked square in the second row, the third row is determined by the second. Similarly, all the remaining rows are determined. This leaves two possibilities if the first row is empty. Thus, there are four possibilities if the first or last row is empty.

Case 2: The empty row is not the first or last. Then, there are two sets of (one of more) consecutive rows of marked squares. As above, the configuration of the rows in each of the two sets is determined by the position of the marked square in the first of its rows. That makes $2 \times 2 = 4$ possible configurations. There are eight possibilities for the empty row, making a total of 32 possibilities in this case.

Together, there are 36 possible configurations of marked squares.

18. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

    Solution: $\boxed{\dfrac{9}{13}}$

    Consider the situation after two points. Daniel has a 9/25 chance of winning, Scott, 4/25, and there is a 12/25 chance that the players will be tied. In the latter case, we revert to the original situation. Hence the probability that Daniel eventually wins the game is $(9/25)/(9/25 + 4/25) = 9/13$.

19. One of the receipts for a math tournament showed that 72 identical trophies were purchased for \$-99.9-, where the first and last digits were illegible. How much did each trophy cost?

    Solution: $\boxed{\$11.11}$

    The price must be divisible by 8 and 9. Thus the last 3 digits must be divisible by 8, so the price ends with 992, and the first digit must be 7 to make the total divisible by 9. $\$799.92/72 = \$11.11$.

20. Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending \$1024. When she comes out of the last shop she has no money left. What is the minimum possible value of d?

    Solution: $\boxed{1023}$

    Work backwards. Before going into the last shop she had \$1024, before the lottery she had \$512, then \$1536, \$768, ... We can easily prove by induction that if she ran out of money after n shops, $0 \leq n \leq 10$, she must have started with $1024 - 2^{10-n}$ dollars. Therefore d is 1023.

21. An unfair coin has the property that when flipped four times, it has the same probability of turning up 2 heads and 2 tails (in any order) as 3 heads and 1 tail (in any order). What is the probability of getting a head in any one flip?

    Solution: $\boxed{\dfrac{3}{5}}$

    Solution: Let p be the probability of getting a head in one flip. There are 6 ways to get 2 heads and 2 tails, each with probability $p^2(1-p)^2$, and 4 ways to get 3 heads and 1 tail, each with probability $p^3(1-p)$. We are given that $6p^2(1-p)^2 = 4p^3(1-p)$. Clearly p is not 0 or 1, so we can divide by $p^2(1-p)$ to get $6(1-p) = 4p$. Therefore p is $\frac{3}{5}$.

4 Geometry

4.1 Geometry Problems

4.2 Geometry Solutions

1. How many different values can $\angle ABC$ take, where A,B,C are distinct vertices of a cube?

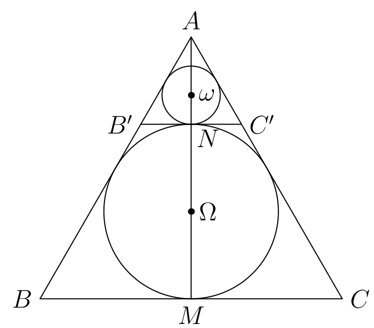
   Solution: $\boxed{5}$

   In a unit cube, there are 3 types of triangles, with side lengths $(1, 1, \sqrt{2})$, $(1, \sqrt{2}, \sqrt{3})$ and $(\sqrt{2}, \sqrt{2}, \sqrt{2})$. Together they generate 5 different angle values.

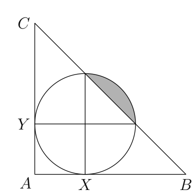
2. Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

   Solution: $\boxed{3}$

   Label the diagram as shown below, where Ω and ω also denote the center of the corresponding circles. Note that AM is a median and Ω is the centroid of the equilateral triangle. So $AM = 3M\Omega$. Since $M\Omega = N\Omega$, it follows that $AM/AN = 3$, and triangle ABC is the image of triangle $AB'C'$ after a scaling by a factor of 3, and so the two incircles must also be related by a scale factor of 3.



3. Let ABC be a triangle with $\angle BAC = 90°$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



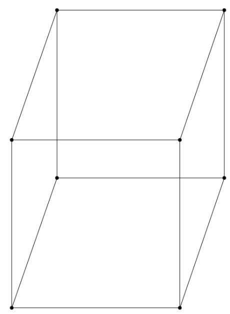
   Solution: $\boxed{\pi - 2}$

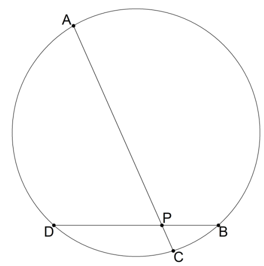
   Let O be the center of the circle, and r its radius, and let X' and Y' be the points diametrically opposite X and Y, respectively. We have $OX' = OY' = r$, and $\angle X'OY' = 90°$. Since triangles $X'OY'$ and BAC are similar, we see that $AB = AC$. Let X'' be the projection of Y' onto AB. Since $X''BY'$ is similar to ABC, and $X''Y' = r$, we have $X''B = r$. It follows that $AB = 3r$, so $r = 2$.

4. A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

Solution: $\boxed{1,\ 5}$

The volume of the cube is s^3 and its surface area is $6s^2$, so we have $6s^2 = s^3 + 5s$, or $0 = s^3 - 6s^2 + 5s = s(s-1)(s-5)$.





5. A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

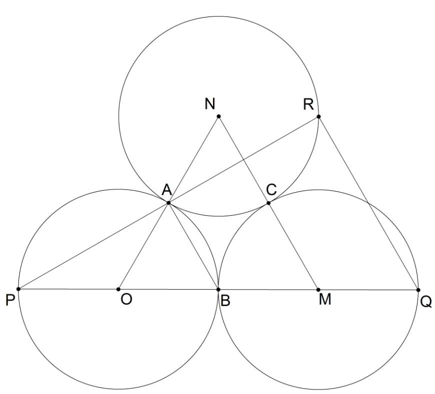
Solution: $\boxed{2}$

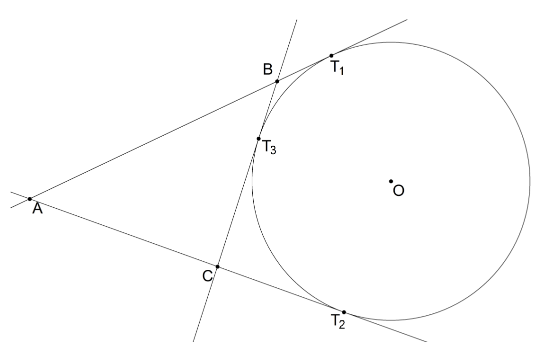
Writing $BP = x$ and $PD = 6 - x$, we have that $BP < 3$. Power of a point at P gives $AP \cdot PC = BP \cdot PD$ or $8 = x(6 - x)$. This can be solved for $x = 2$ and $x = 4$, and we discard the latter.

6. Circles ω_1, ω_2, and ω_3 are centered at M, N, and O, respectively. The points of tangency between ω_2 and ω_3, ω_3 and ω_1, and ω_1 and ω_2 are tangent at A, B, and C, respectively. Line MO intersects ω_3 and ω_1 again at P and Q respectively, and line AP intersects ω_2 again at R. Given that ABC is an equilateral triangle of side length 1, compute the area of PQR.

Solution: $\boxed{2\sqrt{3}}$

Note that ONM is an equilateral triangle of side length 2, so $m\angle BPA = m\angle BOA/2 = \pi/6$. Now BPA is a 30-60-90 triangle with short side length 1, so $AP = \sqrt{3}$. Now A and B are the midpoints of segments PR and PQ, so $[PQR] = \frac{PR}{PA} \cdot \frac{PQ}{PB}[PBA] = 2 \cdot 2[PBA] = 2\sqrt{3}$.





7. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

Solution: $\boxed{20+\sqrt{2}}$

Extend sides AB, CD, EF, GH to form a rectangle: let X be the intersection of lines GH and AB; Y that of AB and CD; Z that of CD and EF; and W that of EF and GH.

As $BC = 2$, we have $BY = YC = \sqrt{2}$. As $DE = 4$, we have $DZ = ZE = 2\sqrt{2}$. As $FG = 2$, we have $FW = WG = \sqrt{2}$.

We can compute the dimensions of the rectangle: $WX = YZ = YC + CD + DZ = 3 + 3\sqrt{2}$, and $XY = ZW = ZE + EF + FW = 2 + 3\sqrt{2}$. Thus, $HX = XA = XY - AB - BY = 1 + 2\sqrt{2}$, and so $AH = \sqrt{2}HX = 4 + \sqrt{2}$, and $GH = WX - WG - HX = 2$. The perimeter of the octagon can now be computed by adding up all its sides.

8. Suppose ABC is a scalene right triangle, and P is the point on hypotenuse $\overline{AC}$ such that $\angle ABP = 45°$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC.

   Solution: $\boxed{\frac{9}{5} \text{ or } 1\frac{4}{5}}$

   Notice that $\overline{BP}$ bisects the right angle at B. Thus, we write $AB = 2x$, $BC = x$. By the Pythagorean theorem, $5x^2 = 9$, from which the area $\frac{1}{2}(x)(2x) = x^2 = \frac{9}{5}$.

9. Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

   Solution: $\boxed{31}$

   Suppose X, P, Q, Y lie in that order. Let $PX = x$ and $QY = y$. By power of a point from P, $x \cdot (27 + y) = 30$, and by power of a point from Q, $y \cdot (27 + x) = 84$. Subtracting the first from the second, $27 \cdot (y - x) = 54$, so $y = x + 2$. Now, $x \cdot (29 + x) = 30$, and we find $x = 1, -30$. Since -30 makes no sense, we take $x = 1$ and obtain $XY = 1 + 27 + 3 = 31$.

10. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

    Solution: $\boxed{6\sqrt{3}}$

    If the side length of the cube is s inches, then the condition implies $s^3 + 3 \cdot 12s = 2 \cdot 6s^2$, or $s(s^2 - 12s + 36) = s(s-6)^2 = 0$. Therefore $s = 6$, and the long diagonal has length $s\sqrt{3} = 6\sqrt{3}$.

11. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

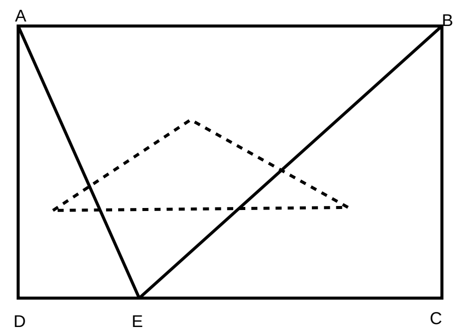
    Solution: $\boxed{1+2\sqrt{3}}$

    The plane intersects each face of the tetrahedron in a midline of the face; by symmetry it follows that the intersection of the plane with the tetrahedron is a square of side length 1. The surface area of each piece is half the total surface area of the tetrahedron plus the area of the square, that is, $\frac{1}{2} \cdot 4 \cdot \frac{2^2\sqrt{3}}{4} + 1 = 1 + 2\sqrt{3}$.

12. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD. What is the area of the triangle formed by the centroids of triangles ABE, BCE, and ADE?

    Solution: $\boxed{1/9}$

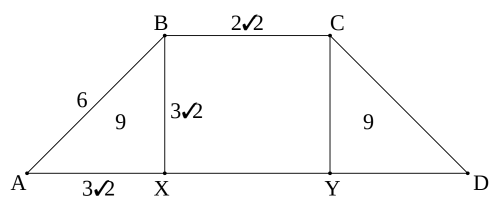
Let the centroids of ABE, BCE, and ADE be denoted by X, Y, and Z, respectively. Let $d(P, QR)$ denote the distance from P to line QR. Since the centroid lies two-thirds of the distance from each vertex to the midpoint of the opposite edge, $d(X, AB) = d(Y, CD) = d(Z, CD) = \frac{1}{3}BC$, so YZ is parallel to CD and $d(X, YZ) = BC - \frac{2}{3}BC = \frac{1}{3}BC$. Likewise, $d(Z, AD) = \frac{1}{3}DE$ and $d(Y, BC) = \frac{1}{3}CE$, so that since YZ is perpendicular to AD and BC, we have that $YZ = CD - \frac{1}{3}(DE + CE) = \frac{2}{3}CD$. Therefore, the area of XYZ is $\frac{1}{2}(\frac{1}{3}BC)(\frac{2}{3}CD) = \frac{1}{9}BC \cdot CD = \frac{1}{9}$.



13. In trapezoid $ABCD$, AD is parallel to BC. $\angle A = \angle D = 45°$, while $\angle B = \angle C = 135°$. If $AB = 6$ and the area of $ABCD$ is 30, find BC.

Solution: $\boxed{2\sqrt{2}}$

Draw altitudes from B and C to AD and label the points of intersection X and Y, respectively. Then ABX and CDY are $45° - 45° - 90°$ triangles with $BX = CY = 3\sqrt{2}$. So, the area of ABX and the area of CDY are each 9, meaning that the area of rectangle $BCYX$ is 12. Since $BX = 3\sqrt{2}$, $BC = 12/(3\sqrt{2}) = 2\sqrt{2}$.



14. A parallelogram has 3 of its vertices at (1, 2), (3,8), and (4, 1). Compute the sum of the possible x-coordinates for the 4th vertex.

Solution: $\boxed{8}$

There are 3 possible locations for the 4th vertex. Let (a, b) be its coordinates. If it is opposite to vertex $(1, 2)$, then since the midpoints of the diagonals of a parallelogram coincide, we get $(\frac{a+1}{2}, \frac{b+2}{2}) = (\frac{3+4}{2}, \frac{8+1}{2})$. Thus $(a, b) = (6, 7)$. By similar reasoning for the other possible choices of opposite vertex, the other possible positions for the fourth vertex are $(0, 9)$ and $(2, -5)$, and all of these choices do give parallelograms. So the answer is $6 + 0 + 2 = 8$.

15. A swimming pool is in the shape of a circle with diameter 60 ft. The depth varies linearly along the east-west direction from 3 ft at the shallow end in the east to 15 ft at the diving end in the west (this is so that divers look impressive against the sunset) but does not vary at all along the north-south direction. What is the volume of the pool, in ft^3?

Solution: $\boxed{8100\pi}$

Take another copy of the pool, turn it upside-down, and put the two together to form a cylinder. It has height 18 ft and radius 30 ft, so the volume is $\pi(30 \text{ ft})^2 \cdot 18 \text{ ft} = 16200\pi \text{ ft}^3$. Since our pool is only half of that, the answer is $8100\pi \text{ ft}^3$.

16. A circle of radius 3 crosses the center of a square of side length 2. Find the positive difference between the areas of the nonoverlapping portions of the figures.

Solution: $\boxed{9\pi^2 - 4}$

Call the area of the square s, the area of the circle c, and the area of the overlapping portion x. The area of the circle not overlapped by the square is $c - x$ and the are of the square not overlapped by the circle is $s - x$, so the difference between these two is $(c - x) - (s - x) = c - s = 9\pi^2 - 4$.

17. Call three sides of an opaque cube adjacent if someone can see them all at once. Draw a plane through the centers of each triple of adjacent sides of a cube with edge length 1. Find the volume of the closed figure bounded by the resulting planes.

Solution: $\boxed{\frac{1}{2}}$

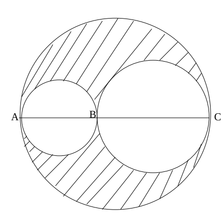
The volume of the figure is half the volume of the cube (which can be seen by cutting the cube into 8 equal cubes and realizing that the planes cut each of these cubes in half), namely $\frac{1}{2}$.

18. Square $ABCD$ is drawn. Isosceles Triangle CDE is drawn with E a right angle. Square $DEFG$ is drawn. Isosceles triangle FGH is drawn with H a right angle. This process is repeated infinitely so that no two figures overlap each other. If square $ABCD$ has area 1, compute the area of the entire figure.

Solution: $\boxed{\frac{5}{2}}$

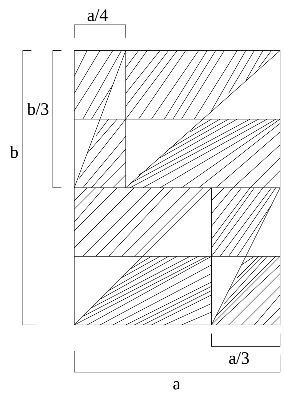
Let the area of the nth square drawn be S_n and the area of the nth triangle be T_n. Since the hypotenuse of the nth triangle is of length $\sqrt{S_n}$, its legs are of length $l = \sqrt{\frac{S_n}{2}}$, so $S_{n+1} = l^2 = \frac{S_n}{2}$ and $T_n = \frac{l^2}{2} = \frac{S_n}{4}$. Using the recursion relations, $S_n = \frac{1}{2^{n-1}}$ and $T_n = \frac{1}{2^{n+1}}$, so $S_n + T_n = \frac{1}{2^{n-1}} + \frac{1}{2^{n+1}} = \left(\frac{1}{2} + 2\right)\frac{1}{2^n} = \frac{5}{2}\frac{1}{2^n}$. Thus the total area of the figure is $\sum_{n=1}^{\infty} S_n + T_n = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{5}{2}$.

19. Given: AC has length 5, AB has radius 1, BC has diameter 3. What percent of the big circle is shaded?



Solution: $\boxed{48\%}$

The area of the big circle is $\left(\frac{5}{2}\right)^2\pi = \frac{25}{4}\pi$. The area of the circle with diameter AB is π, and the area of the circle with diameter BC is $\frac{9}{4}\pi$. Thus, the percentage of the big circle that is shaded is $\frac{\frac{25}{4}\pi - \pi - \frac{9}{4}\pi}{\frac{25}{4}\pi} = \frac{25-4-9}{25} = \frac{12}{25} = 48\%$.



20. Find the total area of the non-triangle regions in the figure below (the shaded area).

Solution: $\boxed{\frac{3}{4}ab}$

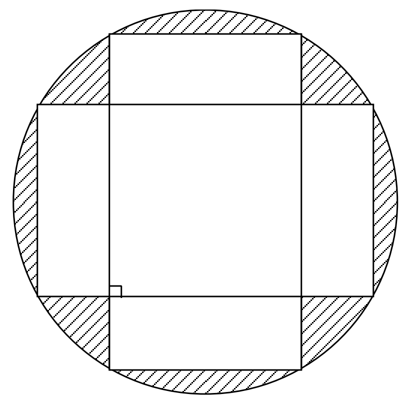
Notice that in general, when there is a rectangle of side length x and y, the area of the non-triangle regions (created by drawing a line connecting the midpoint of two opposite lines and a line connecting two opposite corners - see diagram for examples) is simply $\frac{3}{4}$ of the original area of the box, since the area of the excluded triangles are $\frac{1}{2}\frac{y}{2}\frac{x}{2}+\frac{1}{2}\frac{y}{2}\frac{x}{2}=2\frac{xy}{2}=\frac{xy}{4}$, so the desired area is simply $ab-\frac{ab}{4}=\frac{3}{4}ab$, as desired. So, if we consider the four rectangles making up the large rectangle that are divided this way (the one with dimensions $\frac{a}{4}x\frac{b}{3}$, etc.), we can say that the total shaded area is $\frac{3}{4}$ of the total area of the rectangle - that is, $\frac{3}{4}ab$.

21. A sphere is inscribed inside a pyramid with a square as a base whose height is $\frac{\sqrt{6}}{2}$ times the length of one edge of the base. A cube is inscribed inside the sphere. What is the ratio of the volume of the pyramid to the volume of the cube?

Solution: $\boxed{\frac{9\sqrt{3}}{4}}$

Let t be the length of a vertical edge of the pyramid, and set $t=s\sqrt{2}$. Thus, the radius of the sphere is $\frac{t}{2\sqrt{3}}=\frac{s}{\sqrt{6}}$. Notice that the diameter of the cube is the diameter of the sphere, so let l be the length of a side of the cube, so the diameter of the cube is $l\sqrt{3}=\frac{2s}{\sqrt{6}}$, so $l=s\frac{\sqrt{2}}{3}$. Then, the volume of the pyramid is $\frac{s^3\sqrt{6}}{6}$, and the volume of the cube is $l^3=\frac{2\sqrt{2}s^3}{27}$, so the ratio is $\frac{\frac{s^3\sqrt{6}}{6}}{\frac{2\sqrt{2}s^3}{27}}=\frac{9\sqrt{3}}{4}$.

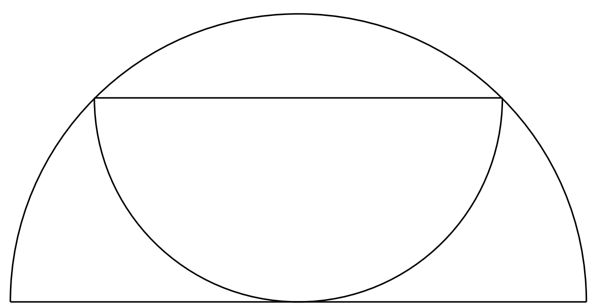
22. Two 10×24 rectangles are inscribed in a circle as shown. Find the shaded area.



Solution: $\boxed{169\pi - 380}$

The rectangles are 10×24, so their diagonals, which are diameters of the circle, have length 26. Therefore the area of the circle is $\pi 13^2$, and the overlap is a 10×10 square, so the shaded area is $\pi 13^2 - 2 \cdot 10 \cdot 24 + 10^2 = \mathbf{169}\pi - \mathbf{380}$.

23. A semicircle is inscribed in a semicircle of radius 2 as shown. Find the radius of the smaller semicircle.



Solution: $\boxed{\sqrt{2}}$

Draw a line from the center of the smaller semicircle to the center of the larger one, and a line from the center of the larger semicircle to one of the other points of intersection of the two semicircles. We now have a right triangle whose legs are both the radius of the smaller semicircle and whose hypotenuse is 2, therefore the radius of the smaller semicircle is $\sqrt{2}$.

24. In a cube with side length 6, what is the volume of the tetrahedron formed by any vertex and the three vertices connected to that vertex by edges of the cube?

Solution: $\boxed{36}$

We have a tetrahedron whose base is half a face of the cube and whose height is the side length of the cube, so its volume is $\frac{1}{3} \cdot (\frac{1}{2} \cdot 6^2) \cdot 6 = 36$.

5 Appendix

5.1 About the HMMT

The Harvard-MIT Mathematics Tournament (HMMT) is an annual math tournament for high school students, held at MIT and at Harvard in alternate years. It is run exclusively by MIT and Harvard students.