Length Bashing in Olympiad Geometry

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1 Introduction

Geometry is an important part of Mathematics Olympiad and Competitions. In many problems you need to compute some of the lengths or angles. Angle chasing is simple enough for a lot of us. But sometimes finding the lengths is a boring and hard job to do; especially if you don't like calculations in geometry. Here are some important theorems, lemmas and important triangle lengths that will appear a lot in bashing geometry problems. In the last section I've provided a bunch of computational problems for you. At the end I have to thank my dear friend Matin Yousefi for helping me by gathering many of those wonderful problems and Amir Hossein Parvardi who helped me a lot in writing this article and improving it. I hope you like it and Happy bashing!

2 Notation and Definition

In triangle $\triangle ABC$, we have:

- A, B, C, as the vertices of the triangle.
- a, b, c, as the sides BC, AC, AB, respectively.
- r as the inraduis of the triangle.
- R as the circumradius of the triangle.
- $[ABC]$ as the area of the triangle.
- p as the semiperimeter of the triangle; In other words:

$$p = \frac{a+b+c}{2}$$

3 Useful Trigonometry Formulas

In this section, there are some useful trigonometry formulas that can help you while bashing. Sum and difference formulas are the most important ones:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
\end{aligned}
$$

Using the equations above, you will have double and half angle formulas:

$$
\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
\cos(2\alpha) &= 2\cos^2\alpha - 1 \\
\cos(2\alpha) &= 1 - 2\sin^2\alpha \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
\tan\left(\frac{\alpha}{2}\right) &= \frac{1-\cos\alpha}{\sin\alpha} \\
\tan\left(\frac{\alpha}{2}\right) &= \frac{\sin\alpha}{1+\cos\alpha}
\end{aligned}
$$

Factorization may help you during the bashing as well:

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta))$$
$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))$$
$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$$
$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$
$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha+\cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

4 Some Important Theorems

Here are some theorems which are the bases of computational geometry and will be used in this article a lot.

Theorem 4.1. *(Law of Sines) In the triangle $\triangle ABC$ we have:*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Theorem 4.2. *(Law of Cosines) In the triangle $\triangle ABC$ we have:*

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

We have the exact same formula for b and c.

Corollary 4.3. *We can calculate the cosine of an angle in a triangle directly using the Law of Cosines:*

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$\cos B$ and $\cos C$ can be written in the exact same way.

As an exercise, try to prove that the equations below are true:

$$\cos\left(\frac{A}{2}\right) = \sqrt{\frac{p(p-a)}{bc}}$$
$$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(p-b)(p-c)}{bc}}$$

Theorem 4.4. *(Stewart's Theorem) In the triangle $\triangle ABC$, if a line passing through A meets BC at X, we have:*

$$b^2 \cdot BX + c^2 \cdot CX = a(AX^2 + BX \cdot CX).$$

Theorem 4.5. *(Angle Bisector Theorem) In the triangle $\triangle ABC$, we have:*

$$\frac{b}{c} = \frac{CD_a}{BD_a} = \frac{\sin B}{\sin C}$$

If and only if AD_a is an angle bisector.

Theorem 4.6. *(Medians Theorem) In the triangle $\triangle ABC$, we have:*

$$\frac{b}{c} = \frac{\sin B}{\sin C} = \frac{\sin (BAM_a)}{\sin (CAM_a)}$$

If and only if AM_a bisects BC.

Theorem 4.7. *(Ptolemy's Inequality) If the vertices of the quadrilateral are A, B, C, and D in order, then the theorem states that:*

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

And the equality holds if and only if $ABCD$ is cyclic.

Theorem 4.8. *(Ceva's Theorem) We call a line drawn from one of the vertices of triangle meeting the opposite side, a Cevian. In triangle $\triangle ABC$, three Cevians AA', BB', and CC' are concurrent if and only if we have:*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1.$$

Theorem 4.9. *(Trigonometric Ceva) In triangle $\triangle ABC$, three Cevians AA', BB', and CC' are concurrent if and only if we have:*

$$\frac{\sin CAA'}{\sin A'AB} \cdot \frac{\sin ABB'}{\sin B'BC} \cdot \frac{\sin BCC'}{\sin C'CA} = 1.$$

Theorem 4.10. *(Menelaus's Theorem) Given a triangle $\triangle ABC$, and a transversal line ℓ that meets BC, AC, and AB at points D, E, and F respectively, then we will have:*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

The inverse of this theorem is also true.

Theorem 4.11. *(Area of the Triangle) The area of a triangle can be calculated using following formulas which can be proved easily.*
Let h_a be the altitude from vertex A, then the area of $\triangle ABC$ can be written as these equations:

$$[ABC] = \frac{a \cdot h_a}{2}$$

$$[ABC] = \frac{bc \cdot \sin A}{2}$$

Let R, r and p be the circumradius, inradius and semiperimeter of $\triangle ABC$ respectively, we will have the following equations as well:

$$[ABC] = pr$$

$$[ABC] = \frac{abc}{4R}$$

$$[ABC] = 2R^2 \sin A \sin B \sin C$$

$$[ABC] = \sqrt{p(p-a)(p-b)(p-c)}$$

Theorem 4.12. *(Euler Line) In the triangle $\triangle ABC$, let O be the circumcenter, G be the centroid, N_9 be the center of Nine-Point Circle, and H be the orthocenter.*
Then, O, G, N_9 and H are colinear and they share the ratio:

$$OG : GN_9 : N_9H = 2 : 1 : 3.$$

Theorem 4.13. *(Nagel Line) In the triangle $\triangle ABC$, let I be the incenter, G be the centroid, and N be the Nagel Point.*
Then, I, G, and N are colinear and they share the ratio:

$$IG : GN = 1 : 2.$$

Theorem 4.14. *In every triangle $\triangle ABC$, with orthocenter H and circumcenter O, if M is the midpoint of BC, we have:*

$$AH : OM = 2 : 1.$$

Theorem 4.15. *In every triangle $\triangle ABC$, with Centroid G, if M is the midpoint of BC, we have:*

$$AG : GM = 2 : 1.$$

5 Some Lemmas for Problem Solving

Here are some important lemmas (which are not always considered trivial and in many situations you need to mention and prove them) that will be useful for real Olympiad-based problem solving. Using these lemmas you can tackle many hard problems.

Lemma 5.1. *(Lemma of Ratio) In the triangle $\triangle ABC$, if a line passing through A meet BC at X, we have:*

$$\frac{BX}{CX} = \frac{c \cdot \sin(BAX)}{b \cdot \sin(CAX)}.$$

Corollary 5.2. *In the quadrilateral $ABCD$, we have:*

$$\frac{\sin BAD}{\sin CAD} = \frac{\sin B}{\sin C} \cdot \frac{BD}{CD}.$$

Lemma 5.3. *(The Lemma of Perpendicularly) For a line BC, Point A on the plane and point H on BC are assumed. For a point P that lies on AH, We have:*

$$AB^2 - AC^2 = PB^2 - PC^2.$$

If and only if $AH \perp BC$.

Lemma 5.4. *For four angles α , α' , β , β', if we know that:*

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{\sin\beta}{\sin\beta'} \quad \textit{and} \quad \alpha + \alpha' = \beta + \beta'$$

Then we can conclude that: $\alpha = \beta$ and $\alpha' = \beta'$.

Lemma 5.5. *For four angles α , α' , β , β', if we know that:*

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{\sin\beta}{\sin\beta'} \quad \textit{and} \quad \alpha - \alpha' = \beta - \beta'$$

Then we can conclude that: $\{\alpha, \beta\} = \{\alpha', \beta'\}$.

Note that $\sin\theta = \sin\gamma$, doesn't give us $\theta = \gamma$ right away. It has two cases: One of the cases is that θ and γ are equal angles, the other case is that θ and γ are supplementary (their sum is equal to 180^o).

Lemma 5.6. *(Trigonometric Ptolemy) Let A, B, C, D, be four points on the plane. $ABCD$ is a cyclic quadrilateral of this order, if and only if we have:*

$$AB \cdot \sin BAC + AD \cdot \sin CAD = AC \cdot \sin BAD.$$

Lemma 5.7. *(Trigonometric Colinearity) Let A, B, C, D, be four points on the plane. B, C, D are colinear if and only if we have:*

$$\frac{\sin BAC}{AD} + \frac{\sin CAD}{AB} = \frac{\sin BAD}{AC}.$$

Lemma 5.8. *(Isogonal Lines Lemma) In the triangle $\triangle ABC$, P is a point on segment a and Q is a point on arc $\widehat{BC}$, if we know that $\angle PAB = \angle CAQ$, we will have:*

$$AP \cdot AQ = bc.$$

Proof. Since we know that $\angle PAB = \angle CAQ$, it's obvious that $\angle ABP = \angle AQC = \angle B$, so we get:

$$\triangle ABP \sim \triangle AQC \Rightarrow \frac{b}{AP} = \frac{AQ}{c}$$
$$\implies AP \cdot AQ = bc.$$

□

Corollary 5.9. *Let I be the incenter of $\triangle ABC$ and I_a be the center of the excircle relative to the vertex A, then this lemma can be written for I and I_a as well.*

In the triangle $\triangle ABC$, we have:

$$AI \cdot AI_a = bc.$$

Proof. We know that $\angle BAI_a = \angle IAC = \angle \frac{A}{2}$; with some angle chasing we can see that $\angle ABI_a = \angle AIC = \angle B + \angle \frac{A+C}{2}$. So we have:

$$\triangle ABI_a \sim \triangle AIC \Rightarrow \frac{c}{AI_a} = \frac{AI}{b}$$
$$\implies AI \cdot AI_a = bc.$$

□

6 Geometric Calculations

6.1 Median

In the triangle $\triangle ABC$, we have:

- M_a , M_b , M_c , as the midpoints of sides a, b, c respectively.
- m_a , m_b , m_c , as the medians from midpoints of a, b, c, respectively.

6.1.1 Length of m_a

Using the Law of Cosines, we have:

$$m_a^2 = c^2 + \left(\frac{a}{2}\right)^2 - 2c\left(\frac{a}{2}\right)\cos B = c^2 + \frac{a^2}{4} - ac\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$m_a^2 = c^2 + \frac{a^2}{4} - \frac{a^2}{2} - \frac{c^2}{2} + \frac{b^2}{2}$$
$$m_a^2 = \frac{c^2}{2} + \frac{b^2}{2} - \frac{a^2}{4}$$
$$\implies m_a = \sqrt{\frac{2b^2+2c^2-a^2}{4}}.$$

6.2 Altitude

In the triangle $\triangle ABC$, we have:

- H as the Orthocenter of triangle.
- h_a , h_b , h_c , as the altitudes from A, B, C, respectively.
- H_a , H_b , H_c are where the altitudes h_a , h_b , h_c , touch a, b, c, respectively.

6.2.1 Length of h_a

$i)$ Using the Law of Sines, we have:

$$\frac{\sin B}{h_a} = \frac{\sin 90}{c}$$

$$\Longrightarrow h_a = c \cdot \sin B.$$

$ii)$ Using the formulas of the area, we will get:

$$\frac{a \cdot h_a}{2} = \frac{bc \cdot \sin A}{2}$$

$$\Rightarrow h_a = \frac{bc \cdot \sin A}{a}.$$

By the Law of Sines we know that $\frac{a}{\sin A} = 2R$, using that, we get:

$$\Longrightarrow h_a = \frac{bc}{2R}.$$

6.2.2 Length of BH_a

In the triangle $\triangle ABH_a$, we have:

$$\frac{\sin(90 - B)}{BH_a} = \frac{\sin B}{c}.$$

Since we know that $\sin(90 - B) = \cos B$, we will get:

$$\Longrightarrow BH_a = c \cdot cosB.$$

We can also change it this way:

$$BH_a = c\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$\Longrightarrow BH_a = \frac{a^2 + c^2 - b^2}{2a}$$

6.2.3 Length of AH

We know that HH_bCH_a is a cyclic quadrilateral; So:

$$AH \cdot AH_a = AH_b \cdot AC.$$

Knowing that $AH_a = b \cdot \sin C$ and $AH_b = c \cdot \cos A$, we get:

$$AH \cdot b \cdot \sin C = c \cdot \cos A \cdot b$$

$$\implies AH = a \cdot \cot A.$$

Using $\frac{a}{\sin A} = 2R$, it can be written like:

$$\implies AH = 2R \cdot \cos A.$$

6.2.4 Length of HH_a

Since we know that HH_bCH_a is a cyclic quadrilateral; So we get $\angle BHH_a = \angle C$. Using the law of sines in triangle $\triangle BHH_a$ and $BH = b \cdot \cot B$, we have:

$$\implies HH_a = b \cdot \cot B \cdot \cos C.$$

Which can be transformed as:

$$\implies HH_a = 2R \cdot \cos B \cdot \cos C.$$

6.2.5 Length of H_bH_c

With some angle chasing we will know that $\angle AH_bH_c = \angle ABC$ and $\angle AH_cH_b = \angle ACB$, this will lead us to the similarity $\triangle AH_bH_c \sim \triangle ABC$; so:

$$\frac{H_bH_c}{a} = \frac{AH_b}{c} = \cos A$$

$$\implies H_bH_c = a \cdot \cos A.$$

6.3 Internal and Exterior Angle Bisectors

In the triangle $\triangle ABC$, we have:

- I as the incenter of the triangle.
- d_a , d_b , d_c , as the angle bisectors of $\angle A$, $\angle B$, $\angle C$.
- D_a , D_b , D_c are where the bisectors d_a , d_b , d_c , meet a, b, c, respectively.
- E_a , E_b , E_c are where the incircle touch a, b, c, respectively.
- d'_a , d'_b , d'_c as the exterior angle bisectors of $\angle A$, $\angle B$, $\angle C$, respectively.
- D'_a , D'_b , D'_c are where d'_a , d'_b , d'_c meet the extension of BC, AC, AB, respectively.

6.3.1 Length of d_a

Using the Stewart's Theorem, we will have:

$$a\left(d_a^2+\frac{a^2bc}{(b+c)^2}\right)+b^2\left(\frac{ac}{b+c}\right)+c^2\left(\frac{ab}{b+c}\right)$$

$$\Rightarrow d_a^2=bc-BD_a\cdot CD_a$$

$$d_a^2=bc\left(1-\frac{a^2}{(b+c)^2}\right)=\frac{2b^2c^2}{(b+c)^2}\left(\frac{(b^2+c^2-a^2)+2bc}{2bc}\right)$$

$$d_a^2=\frac{4b^2c^2}{(b+c)^2}\cdot\cos^2\left(\frac{A}{2}\right)$$

$$\Longrightarrow d_a=\frac{2bc}{b+c}\cdot\cos\left(\frac{A}{2}\right).$$

6.3.2 d'_a and D'_a

We can calculate the length of d'_a , just like d_a with the Stewart's Theorem. The results are quite similar:

$$\Rightarrow (d'_a)^2=BD'_a\cdot CD'_a-bc$$

$$\Longrightarrow d'_a=\frac{2bc}{|b-c|}\cdot\sin\left(\frac{A}{2}\right)$$

6.3.3 Length of BD_a

Using the Angle Bisector Theorem, we have:

$$\frac{BD_a}{CD_a} = \frac{c}{b}$$

$$\Rightarrow \frac{BD_a}{BD_a + CD_a} = \frac{c}{b+c}$$

$$\frac{BD_a}{a} = \frac{c}{b+c}$$

$$\Longrightarrow BD_a = \frac{a \cdot c}{b+c}.$$

6.3.4 Length of AI

$i)$ Using the Angle Bisector Theorem in the triangle $\triangle ABD_a$, we will get:

$$\frac{AI}{ID_a} = \frac{c}{BD}$$

$$\frac{AI}{ID_a} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a}$$

$$\frac{AI}{AI + ID_a} = \frac{b+c}{a+b+c}$$

$$\Rightarrow \frac{AI}{AD_a} = \frac{b+c}{2p}$$

$$AI = AD_a \cdot \frac{b+c}{2p} = \frac{2bc}{b+c} \cdot \cos\left(\frac{A}{2}\right) \cdot \frac{b+c}{2p}$$

$$\Longrightarrow AI = \frac{bc}{p} \cdot \cos\left(\frac{A}{2}\right).$$

$ii)$ If we connect I to E_c , it's obvious that $IE_c \perp AB$ and $IE_c = r$; in the triangle $\triangle AIE_c$, we will get:

$$\Rightarrow AI = \frac{r}{\sin\left(\frac{A}{2}\right)}.$$

$iii)$ Having E_c and the incircle again, it's easy to prove that $AE_c = p - a$; in the triangle $\triangle AIE_c$, we will get:

$$\Longrightarrow AI = \frac{p-a}{\cos\left(\frac{A}{2}\right)}.$$

6.4 Mixtilinear Incircles

A Mixtilinear Incircle is a circle tangent to two of a triangle's segments, and the circumcircle of that triangle. There are three Mixtilinear Incircles in a triangle, we show Mixtilinear Incircle opposite to vertex A (the circle that is tangent to AB, AC and circumcircle of $\triangle ABC$) with Γ_a.
Γ_b and Γ_c are defined is the same way.
Let Γ_a touch AB and AC at E and F respectively; it can be proved that I lies on EF (which is actually an special case of Sawayama Thebault's Theorem). Using that, since $AI \perp EF$ we will have:

$$AE = AF = \frac{AI}{\cos\left(\frac{A}{2}\right)}.$$

Using the length of AI directly, we get:

$$\Longrightarrow AE = AF = \frac{bc}{p}.$$

Lengths of BE and CF are easy to find as well:

$$BE = c - \frac{bc}{p}$$

$$\Longrightarrow BE = \frac{c(p-b)}{p}$$

$$CF = b - \frac{bc}{p}$$

$$\Longrightarrow CF = \frac{b(p-c)}{p}$$

6.5 T and T'

We defined T and T' as the points where the internal and exterior bisectors of $\angle A$ meet the circumcircle of the triangle $\triangle ABC$; or we can say that T is the midpoint of arc $\widehat{BC}$ not containing A, and T' is the midpoint of arc $\widehat{BC}$ containing A. We use the same notations as the previous parts in this section:

- d_a , d_b , d_c , as the angle bisectors of $\angle A$, $\angle B$, $\angle C$.
- D_a , D_b , D_c are where the bisectors d_a , d_b , d_c , meet a, b, c, respectively.
- E_a , E_b , E_c are where the incircle touch a, b, c, respectively.
- d'_a , d'_b , d'_c as the exterior angle bisectors of $\angle A$, $\angle B$, $\angle C$, respectively.
- D'_a , D'_b , D'_c are where d'_a , d'_b , d'_c meet the extension of BC, AC, AB, respectively.

6.5.1 Length of AT

An special case of the Isogonal Lines Lemma that we proved, is when A, P, Q are collinear; And that is when $P \equiv D_a$ and $Q \equiv T$.
Using the lemma, we get:

$$AD_a \cdot AT = bc.$$

We know that $AD_a = \frac{2bc}{b+c} \cdot \cos\left(\frac{A}{2}\right)$, by putting that in our equation we have:

$$AT \cdot \frac{2bc}{b+c} \cdot \cos\left(\frac{A}{2}\right) = bc$$

$$\implies AT = \frac{b+c}{2\cos\left(\frac{A}{2}\right)}.$$

6.5.2 Length of AT'

Here, we need to use D'_a again. it's obvious that T', A and D'_a are collinear; with angle chasing we know that $\angle ABD'_a = \angle AT'C$ and $\angle D'_aAB = \angle CAT'$, so:

$$\triangle ABD'_a \sim \triangle AT'C \Rightarrow \frac{c}{AT'} = \frac{d'_a}{b}$$

Just like the length of AT, by putting $d'_a = \frac{2bc}{|b-c|} \cdot \sin\left(\frac{A}{2}\right)$ in the equation, we get:

$$\implies AT' = \frac{|b-c|}{2\sin\left(\frac{A}{2}\right)}.$$

6.5.3 Length of BT, CT and IT

Reading the title, you might have guessed the point of this part: BT, CT and IT are all equal. with some angle chasing, it's simple to show that $\angle TBI = \angle TIB = \angle\frac{A+B}{2}$ and $\angle TCI = \angle TIC = \angle\frac{A+C}{2}$; so $BT = CT = IT$ (and it's not hard to prove they are all equal to TI_a as well).
Adding TM_a to the structure, where M_a is the midpoint of side BC, we will have:

$$TB \cdot \cos\left(\frac{A}{2}\right) = \frac{a}{2}$$

$$\implies TB = TC = TI = \frac{a}{2\cos\left(\frac{A}{2}\right)}.$$

6.5.4 T and T' images on AC

In this part, without loss of generality, we assume that $AC > AB$.
S and S' will be images of T and T' on the segment AC.
By the definition, $T'S' \perp AC$ and $TS \perp AC$, using the Law of Sines, we will have the following results:

$$\Longrightarrow AS = CS' = \frac{b+c}{2}$$

$$\Longrightarrow AS' = CS = \frac{|b-c|}{2}$$

6.6 Excircles and Excenters

Just like Mixtilinear Incircles, there are three Excircles in a triangle. We define the Excircle opposite to vertex A with ω_a.
ω_b and ω_c are defined in the same way. In this section, in triangle $\triangle ABC$, we have:

- I_a , I_b , I_c , as the centers of ω_a, ω_b, ω_c , respectively.
- X_a , X_b , X_c , are where ω_a, ω_b, ω_c , touch a, b, c, respectively.
- r_a , r_b , r_c , as inradii of excircles ω_a, ω_b, ω_c , respectively.
- E_a , E_b , E_c are where the incircle touch a, b, c, respectively.
- I as the incenter of the triangle.

6.6.1 Length of AIa

$i)$ Using the Isogonal Lines Lemma again, and having $AI = \frac{bc}{p} \cdot \cos\left(\frac{A}{2}\right)$, we get:

$$AI_a \cdot \frac{bc}{p} \cdot \cos\left(\frac{A}{2}\right) = bc.$$

With some calculations, we have:

$$\Longrightarrow AI_a = \frac{bc}{p-a} \cdot \cos\left(\frac{A}{2}\right).$$

$ii)$ Let ω_a touch AB at Z_c.
It's obvious that and $I_aZ_c \perp AZ_c$ and $I_aZ_c = r_a$; using that in the triangle $\triangle AZ_cI_a$, we get:

$$\Longrightarrow AI_a = \frac{r_a}{\sin\left(\frac{A}{2}\right)}.$$

iii) Having Z_c again; it's easy to prove that $AZ_c = p$; using that in the triangle $\triangle AZ_cI_a$, we get:

$$\Longrightarrow AI_a = \frac{p}{\cos\left(\frac{A}{2}\right)}.$$

6.6.2 Length of AI_b

If we connect I_b to X_b , it's obvious that $I_bX_b \perp AC$ and $I_bX_b = r_b$; using that, we will have:

$$AI_b = \frac{r_b}{\sin\left(90 - \frac{A}{2}\right)}$$

$$\Longrightarrow AI_b = \frac{r_b}{\cos\left(\frac{A}{2}\right)}.$$

6.6.3 Length of r_a

i) Calculating the length of r_a is quite like proving $r = \frac{[ABC]}{p}$; we use the area of $\triangle ABC$ to calculate it.
Consider r_a ; we know that:

$$[ABC] = [AI_aB] + [AI_aC] - [CI_aB].$$

$$\Rightarrow [ABC] = \frac{p \cdot r_a}{2} + \frac{p \cdot r_a}{2} - \frac{a \cdot r_a}{2}$$

With a little calculations, we get:

$$[ABC] = (p - a) \cdot r_a$$

$$\Longrightarrow r_a = \frac{[ABC]}{p - a}.$$

In the exact same way, we can prove that $r_b = \frac{[ABC]}{p-b}$ and $r_c = \frac{[ABC]}{p-c}$.

ii) Let ω_a touch AB at Z_c. It's obvious that and $I_aZ_c \perp AZ_c$; If we connect I to E_c , we know that $IE_c \perp AB$ as well. It is well known that A, I and I_a are colinear, so we can prove that $\triangle AIE_c \sim \triangle AI_aZ_c$.
Since in triangle $\triangle AIE_c$ we have $AE_c = p - a$, $IE_c = r$ and in triangle $\triangle AI_aZ_c$ we have $AZ_c = p$, $I_aZ_c = r_a$, by using the similarity:

$$\frac{AE_c}{AZ_c} = \frac{IE_c}{IZ_c}$$

$$\frac{p - a}{p} = \frac{r}{r_a}$$

By putting $r = \frac{[ABC]}{p}$, we get the same result:

$$\Longrightarrow r_a = \frac{[ABC]}{p - a}$$

7 Training Problems

Here are some problems gathered from Mathematics Olympiads all over the world that can be solved by length bashing for some exercise. You may solve some of these in much easier ways; But please don't! There's much more benefit in solving them by calculations than other ways.

1. (Blanchet's Theorem) In an acute triangle $\triangle ABC$, F is the foot of the perpendicular from C to AB and P is a point on CF. Let the lines AP and BP meet BC and AC at D and E, respectively. Show that the angles $\angle EFC$ and $\angle DFC$ are equal.

2. Quadrilateral $XABY$ is inscribed in the semicircle ω with diameter XY. Segments AY and BX meet at P. Point Z is the foot of the perpendicular from P to line XY. Point C lies on ω such that line XC is perpendicular to line AZ. Let Q be the intersection of segments AY and XC. Prove that:

$$\frac{BY}{XP} + \frac{CY}{XQ} = \frac{AY}{AX}.$$

3. Let B_1 , C_1 be the midpoints of sides AC, AB of a triangle $\triangle ABC$. The rays CC_1 , BB_1 meet the tangents to the circumcircle at B and C at K and L respectively. Prove that the angles $\angle BAK$ and $\angle CAL$ are equal.

4. Equal circles ω_1 and ω_2 are passing through each others centers. The triangle $\triangle ABC$ is inscribed into the circle ω_1 such that lines AC and BC are tangent to the circle ω_2. Prove that: $\cos A + \cos B = 1$.

5. In a right angled-triangle $\triangle ABC$, $\angle ACB = 90^o$. Its incircle Γ meets BC, AC, AB at D,E,F respectively. AD cuts Γ at P. If $\angle BPC = 90^o$, Prove that: $AE + AP = PD$.

6. In a triangle $\triangle ABC$, let M_a, M_b, M_c be the midpoints of the sides BC, CA, AB, respectively, and T_a, T_b, T_c be the midpoints of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$ of the circumcircle of $\triangle ABC$, not containing the vertices A, B, C, respectively. For $i \in \{a, b, c\}$, let w_i be the circle with M_iT_i as diameter. Let p_i be the common external common tangent to the circles w_j and w_k (for all $\{i, j, k\} = \{a, b, c\}$) such that w_i lies on the opposite side of p_i than w_j and w_k do. Prove that the lines p_a, p_b, p_c form a triangle similar to $\triangle ABC$ and find the ratio of similitude.

7. Let $\triangle ABC$ be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of: $\frac{KP}{R^3}$

8. Bisector of side BC intersects circumcircle of triangle $\triangle ABC$ in points P and Q. Points A and P lie on the same side of line BC. Point R is an orthogonal projection of point P on line AC. Point S is middle of line segment AQ. Show that points A, B, R, S lie on one circle.

9. Let D, E, F be points on the sides BC, CA, AB respectively of a triangle $\triangle ABC$ such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Show that $\triangle ABC$ is equilateral.

10. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD, BE, and CF. Let H be the orthocentre, that is, the point where the altitudes meet. Prove that:

$$\frac{AB \cdot AC + BC \cdot CA + CA \cdot CB}{AH \cdot AD + BH \cdot BE + CH \cdot CF} \leq 2.$$

11. Let $\triangle ABC$ be an acute angled triangle. Let D, E, F be points on BC, CA, AB such that AD is the median, BE is the internal bisector and CF is the altitude. Suppose that $\angle FDE = \angle C, \angle DEF = \angle A$ and $\angle EFD = \angle B$. Show that $\triangle ABC$ is equilateral.

12. The incircle of a triangle $\triangle ABC$ touches the side AB and AC at respectively at X and Y. Let K be the midpoint of the arc $\widehat{AB}$ on the circumcircle of $\triangle ABC$. Assume that XY bisects the segment AK. What are the possible measures of angle $\angle BAC$?

13. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

14. Bisector of angle $\angle BAC$ of triangle $\triangle ABC$ intersects circumcircle of this triangle in point $D \neq A$. Points K and L are orthogonal projections on line AD of points B and C, respectively. Prove that $AD \geq BK + CL$.

15. Let N be a point on the longest side AC of a triangle $\triangle ABC$. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle $\triangle KBM$

16. Let $\triangle ABC$ be an acute angled triangle. D is the midpoint of BC. E, F are on sides AC, AB such that $\angle EDF = \angle A$ and $DE = DF$. Prove that:

$$DE \geq \frac{AB + AC}{4}.$$

17. Let $ABCD$ be a cyclic quadrilateral inscribed in a circle. Let E, F, G, H be the midpoints of arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, $\widehat{AD}$ of the circle respectively. Suppose that $AC \cdot BD = EG \cdot FH$. Show that AC, BD, EG, FH are all concurrent.

18. In a scalene triangle $\triangle ABC$ the altitudes AA_1 and CC_1 intersect at H, O is the circumcenter, and B_0 the midpoint of side AC. The line BO intersects side AC at P, while the lines BH and A_1C_1 meet at Q. Prove that the lines HB_0 and PQ are parallel.

19. Triangle $\triangle ABC$ has integer-length sides. Line ℓ is tangent to the circumcircle of $\triangle ABC$ at A. Let S be the intersection of ℓ and BC. if AS is an integer as well, Prove that: $gcd(AB, AC) > 1$.

20. Let AD be an angle bisector of triangle $\triangle ABC$. Perpendiculars from D to the other sides of $\triangle ABC$, meet AB at X and AC at Y, Let P be the intersection of segments BY and CX. Prove that: $AK \perp BC$.

21. Let I and G be the incenter and centroid of triangle $\triangle ABC$ respectively. Prove that IG is perpendicular to BC if and only if we have: $AB + BC = 3BC$.

22. (Droz-Farny Theorem) Let H be the orthocenter of the triangle $\triangle ABC$. Let L_1 and L_2 be any two mutually perpendicular lines through H. Let A_1 , B_1 , and C_1 be the points where L_1 intersects the side lines BC, CA, and AB, respectively. Similarly, let Let A_2 , B_2 , and C_2 be the points where L_2 intersects those side lines. Prove that the midpoints of the three segments A_1A_2, B_1B_2, and C_1C_2 are collinear.

23. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Lines AC and BD meet at P. H_1 and H_2 are the orthocenters of triangles $\triangle APD$ and $\triangle BPC$ respectively. Let M be the midpoint of H_1H_2 . Prove that: $MP \perp CD$.

24. Let P , Q be two points on the side BC of triangle $\triangle ABC$ such that $2PQ = BC$ and P is closer to B than Q. Perpendiculars at P and Q to BC, meet AB and AC at X and Y respectively. of O is the circumcenter of $\triangle ABC$, Prove that $AXOY$ is cyclic.

25. In a quadrilateral $ABCD$, Lines AC and BD meet at P. Let H_1 and H_2 be the orthocenters of $\triangle ABP$ and $\triangle CDP$ respectively. Let G_1 and G_2 be the centroids of $\triangle ADP$ and $\triangle BCP$ respectively. Prove that: $H_1H_2 \perp G_1G_2$.

26. Let ω be the incircle of triangle $\triangle ABC$ with center I. Circle ω touches AB and AC at E and F respectively. Line BI and CI intersect EF at X and Y respectively. If M is the midpoint of BC, Prove that $\triangle MXY$ is equilateral if and only if $\angle A = 60^o$.

27. Let Ω be the circumcircle of triangle $\triangle ABC$ and G be its centroid. Lines AG, BG and CG meet Ω at A', B' and C' respectively. Prove that:

$$\sum A'G \geq \sum AG.$$

28. $\triangle ABC$ is a triangle with incenter I, Midpoints of AB and AC are M and N. Line CI meets MN at K. We call the line perpendicular to MN at K, ℓ_1 and the line parallel to BI passing through N, ℓ_2 . Let P be the intersection of ℓ_1 and ℓ_2 . Prove that: $PI \perp AC$.

29. If I is the incenter of triangle $\triangle ABC$ and a circle passing through A and I meets AB and AC at E and F, And M is the midpoint of EF, Show that: $\angle BMC > 90^o$

30. The incircle of triangle $\triangle ABC$ touches the sides AB, BC and CA at D, E and F respectively. Perpendicular line from E to DF, meets DF at X. Prove that two angles $\angle BXE$ and $\angle CXE$ are equal.

31. BB' and CC' are two altitudes of triangle $\triangle ABC$, Angle bisector of $\angle A$ meets $B'C'$ and BC at E and F. Let ℓ_1 be the perpendicular line to $B'C'$ at E and ℓ_2 be the perpendicular line to BC at F. The intersection of ℓ_1 and ℓ_2 is X. Prove that AX is a median of $\triangle ABC$.

32. (Leibniz Theorem) Let G be the centroid of triangle $\triangle ABC$. Prove that, for any point P on the plane, we have:

$$\sum PA^2 = \frac{1}{3}\sum AB^2 + 3PG.$$

33. Circumcircle of triangle $\triangle ABC$ is Ω. Let K be the midpoint of arc $\widehat{BAC}$. Prove that: $KB + KC \geq AB + AC$.

34. Points P and Q are inside the triangle $\triangle ABC$ such that $\angle PBA = \angle QBC$, $\angle PCB = \angle QCA$ and $\angle PAB = \angle QAC$. Prove that:

$$AP \cdot \sin BPC = AQ \cdot \sin BQC.$$

35. AA', BB' and CC' a altitudes of triangle $\triangle ABC$. We extent lines AA', CC' from A', C' and extend BB' from B. Let X, Y and Z be points on lines AA', BB' and CC' such that $AA' = A'X$, $CC' = C'Z$ and $BY = 3BB'$. Prove that angles $\angle ABC$ and $\angle XYZ$ are equal.

36. Let M be the midpoint of side BC in triangle $\triangle ABC$. Points X, Y are outside the triangle and $\angle AXB = \angle AYC = 90^o$ and $\angle XBA = \angle YCA$.
a) Prove that: $XM = YM$.
b) Prove that: $\angle XMB = 2\angle XBA$.

37. Let r be the radius of the circle inscribed in triangle $\triangle ABC$ and Ω_a be the excircle relative to vertex A and I_a be its center. Ω_a touches lines BC, AB and AC at D, E and F respectively. EF meets I_aB and I_aC at X and Y. The intersection of XC and YB is K. Prove that the distance of K and BC is equal to r.

38. A circle is inscribed in Trapezoid $ABCD$ with $AB \parallel CD$. Lines AC and BD meet at P. Prove that: $\angle CPD > 90^o$.

39. Line ℓ is passing through triangle $\triangle ABC$. AX, BY and CZ are perpendiculars to ℓ. Prove that the centroid of $\triangle ABC$ lies on ℓ if and only if we have: $AX = BY + CZ$.

40. A Line passing through the centroid of triangle $\triangle ABC$, meets AB at X and AC at Y. Prove that:

$$\frac{AX}{BX} \cdot \frac{AY}{CY} \geq 4.$$

41. In triangle $\triangle ABC$, Points X, Y are chosen on AB, AC such that $BX = CY$. If M is the midpoint of BC and N is the midpoint of XY, Prove that MN is parallel to the angle bisector of $\angle A$.

42. Let Ω be the circumcircle of triangle $\triangle ABC$. A line passing through the centroid of $\triangle ABC$ meets arcs $\widehat{AB}$ and $\widehat{AC}$ of Ω at X and Y. Prove that:

$$AX \cdot AY = BX \cdot BY + CX \cdot CY.$$

43. Let AH, AD and AM be an altitude, an angle bisector and a median through A in triangle $\triangle ABC$ respectively. If we know that $DH < DM$, Prove that: $BC < 2AD$

44. $ABCD$ is a quadrilateral and points E, F are on BC, AD such that $\frac{BC}{BE} = \frac{AD}{AF} = n$ for some real number n. Prove that:

$$[ABEF] \leq \frac{AD \cdot BC + n(n-1) \cdot AB^2 + n \cdot AB \cdot DC}{2n^2}.$$

(Here $[ABCD]$ represents the area of quadrilateral $ABCD$).

45. In the triangle $\triangle ABC$, points X, X' are on BC, Y, Y' are on CA and Z, Z' are on AB such that $BX = X'C$, $CY = Y'A$ and $AZ = Z'B$. Prove that: $[XYZ] = [X'Y'Z']$.
(Here $[ABC]$ represents the area of triangle $\triangle ABC$).

46. The incircle of triangle $\triangle ABC$ with center I touches AB, BC and CA at D, E and F. ℓ is a line passing through A parallel to DF. Lines ED and EF meet ℓ at K and J. Show that: $\angle JIK < 90^o$.

47. The incircle of triangle $\triangle ABC$ with center I touches AB, BC and CA at D, E and F. ℓ is a line passing through A parallel to BC. Lines ED and EF meet ℓ at K and J. Show that: $\angle JIK < 90^o$.

48. Let h_a, h_b and h_c be the altitudes of triangle $\triangle ABC$ with circumradius R and area S, Prove that:

$$\frac{1}{3}\sum \frac{1}{h_a} \geq \sqrt[3]{\frac{R}{2S^2}}.$$

49. In the triangle $\triangle ABC$, Point A' is symmetry of A with respect to BC. B' and C' are defined in the same way. Prove that: $[A'B'C'] \leq 4[ABC]$
. (Here $[ABC]$ represents the area of triangle $\triangle ABC$).

50. If G is the centroid of triangle $\triangle ABC$ and angles $\angle BAG$ and $\angle ACG$ are equal. Prove that:

$$\sin CAG + \sin CBG \leq \frac{2}{\sqrt{3}}.$$

51. Triangle $\triangle ABC$ has circumcenter O and circumradius R. Circumcenter of triangle $\triangle BOC$ is Ω_a. The line AO meets Ω_a for the second time at A'. B' and C' are defined the same way. Prove that:

$$OA' \cdot OB' \cdot OC' \geq 8R^2.$$

52. Let h_a, h_b and h_c be the altitudes of triangle $\triangle ABC$ with semiperimeter P . Prove that:

$$\sum h_a \leq \sqrt{3}P.$$

53. Let h_a , h_b , h_c , be the altitudes of $\triangle ABC$ from A, B, C and r_a , r_b , r_c be inradii of excircles relative to vertices A, B, C. Prove that:

$$\sum \frac{r_a}{h_a} \geq 3.$$

54. Triangle $\triangle ABC$ with circumradius R and inradius r is given. Prove that:

$$\sum \left(\sin\left(\frac{A}{2}\right) \cdot \sin\left(\frac{B}{2}\right) \right) \leq \frac{5}{8} + \frac{r}{4R}.$$

55. Circumcircle of triangle $\triangle ABC$ is Ω. Let the angle bisector of $\angle A$ meet BC and Ω at D and T. We choose an arbitrary point X on BT. The intersection point of XD and TC is Y. Show that the angle $\angle XAY$ is greater that $\angle BAC$.

56. Circle ω with radius R is inscribed inside trapezoid $ABCD$ with $AB \parallel CD$. Show that: $AB \cdot CD \geq 4R^2$.

57. Triangle $\triangle ABC$ has integer-length sides and $AC = 2007$. The Internal bisector of $\angle A$ meets BC at D. Given that $AB = CD$. Determine AB and BC.

58. (Casey's Theorem) Assume the line ℓ is the radical axis of two circle C_1 and C_2 with centers O_1 and O_2. For an arbitrary point A on the plane, H is the projection of A on ℓ. Prove that:

$$|P^K_{C_1} - P^K_{C_2}| = 2AH \cdot O_1O_2.$$

59. A triangle $\triangle ABC$ is given with $AB > AC$. Line ℓ is tangent to the circumcircle of $\triangle ABC$ at A. A circle centered in A with radius $|AC|$ cuts AB in the point D and the line ℓ in points E, F (such that C and E are in the same halfplane with respect to AB). Prove that the line DE passes through the incenter of $\triangle ABC$.

60. Let P be a moving point on side BC of triangle $\triangle ABC$. Line ℓ_1 is drawn from P such that ℓ_1 and PB make a specific angle α. Line ℓ_2 is also drawn from P the same way such that ℓ_2 and PC make a specific angle β. Lines ℓ_1 and ℓ_2 meet AB and AC at K and L. Prove that the circumcircle of triangle $\triangle AKL$ passes trough a fixed point.

61. Triangle $\triangle ABC$ with circumcircle Γ is given. Let M be the midpoint of arc $\widehat{ABC}$ and N be the midpoint of arc $\widehat{BAC}$. Prove that the incenter of $\triangle ABC$ lies on line MN if and only if we have: $AC + BC = 3AB$.

62. Circle ω is internally tangent to circle Ω at K. Let A be a point on Ω and chords AB and AC of Ω be tangent to ω at M and N. AK meets MN at S. Prove that the angles $\angle MSB$ and $\angle NSC$ are equal.

63. $ABCD$ is a convex quadrilateral with $\angle BAC = 30^\circ$, $\angle CAD = 20^\circ$, $\angle ABD = 50^\circ$, $\angle DBC = 30^\circ$. If the diagonals intersect at P, show that: $PC = PD$.

64. Let H be the orthocenter of an acute-angled triangle $\triangle ABC$. Show that the triangles $\triangle ABH, \triangle BCH$ and $\triangle CAH$ have the same perimeter if and only if the triangle $\triangle ABC$ is equilateral.

65. In the rectangle $ABCD$, Points X, Y are chosen on sides DC, BC such that $CX \neq CY$. Let P be the intersection of DY and BX. If $AP \perp XY$, show that $DX = BY$.

66. Let P, Q and R be the midpoints of sides AB, AC and BC in triangle $\triangle ABC$. Points X and Y are on PR and QR such that $BX \perp PR$ and $CY \perp QR$. if O' is the circumcenter of PQR, prove that RO' bisects XY.

67. In the isosceles triangle $\triangle ABC$ with $AB = AC$, Point K is on segment BC such that $BK = 2KC$ and point P is chosen on AK such that angles $\angle BPK$and $\angle BAC$ are equal. Prove that: $2\angle CPK = \angle BAC$.

68. Circle ω is inscribed in triangle $\triangle ABC$ and touches AC at B_1 . Points E and F are chosen on ω such that $\angle AEB_1 = 90^\circ$ and $\angle CFB_1 = 90^\circ$. Let M be the midpoint of $\widehat{AC}$. Prove that lines DM, CF and AE are concurrent.

69. In an acute-angled triangle $\triangle ABC$ with $AB \neq AC$ and $A = 60^\circ$, O and H are circumcircle and orthocenter respectively. if OH meets AB and AC (not their extension) at P and Q and we show the area of triangle $\triangle APQ$ with S and area of quadrilateral $BPQC$ with R, find all possible values of $\frac{S}{R}$.

70. Circle ω is internally tangent to circle Ω at S. Let A be a point on Ω and chords AB and AC of Ω be tangent to ω at E and F. Let T be the midpoint of arc $\widehat{BC}$. Show that lines BC, EF and TS are concurrent.

71. Let $\triangle ABC$ be a triangle with incenter I, incircle ω and circumcircle Γ. Let M, N, P be the midpoints of sides BC, CA, AB and let E, F be the tangency points of ω with CA and AB, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc $\widehat{BAC}$ of Γ.
a) Prove that I lies on ray CV.
b) Prove that line XI bisects UV.

72. Circle ω is tangent to line ℓ at M. Let A be the point on ω such that AM is the diameter. Points X and Y are moving on ℓ such that always $MX \cdot MY = k$ for some real k and M is between X, Y. Lines AX and AY meet ω at E and F. Prove that by Moving X and Y, EF passes trough a fixed point.

73. In an equilateral triangle $\triangle ABC$, circle ω meets side AB at D, E; side BC at F, G; and side CA at H, I internally (D, E, F, G, H and I are on ω in order). Show that:

$$AD + BF + CH = BE + GH + IA.$$

74. Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle $\triangle ABC$, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.

75. Chord AB of circle Γ is assumed. P is an arbitrary point on AB. Circles ω_1 with radius r_1 and ω_2 with radius r_2 are on two sides of AB such that they're tangent to AB at P and internally tangent to Γ. Show that with changing the position of P, the ratio $\frac{r_1}{r_2}$ won't change.

76. The lengths of the sides of a convex hexagon $ABCDEF$ satisfy $AB = BC$, $CD = DE$, $EF = FA$. Prove that:

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

77. Given a triangle $\triangle ABC$ satisfying $AC + BC = 3AB$. The incircle of triangle $\triangle ABC$ has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.

78. In an acute-angled and not isosceles triangle $\triangle ABC$. we draw the median AM and the height AH. Points Q and P are marked on the lines AB and AC, respectively, so that the $QM \perp AC$ and $PM \perp AB$. The circumcircle of $\triangle PMQ$ intersects the line BC for second time at point X. Prove that $BH = CX$.