Bank Soal Olimpiade Matematika SMA (III)

Summer Camp Persiapan OSN 2018

Departemen Matematika - Wardaya College

1. If it is known that the equation $x^4 + ax^3 + 2x^2 + bx + 1 = 0$ has a real root, prove the inequality $a^2 + b^2 \geq 8$.

   First solution : Let $-r$ be the real root of the equation. Then,

   $r^4 - ar^3 + 2r^2 - br + 1 = 0$ so, $r(ar^2 + b) = (r^2 + 1)^2$ and $(ar^2 + b) = \frac{(r^2+1)^2}{r}$.

   We know that $(r^4 + 1)(a^2 + b^2) \geq (ar^2 + b)^2$. Then

   $a^2 + b^2 \geq \frac{(r^2+1)^4}{r^2(r^4+1)} = \frac{(r^4+1)}{r^2} + \frac{4r^2}{(r^4+1)} + 4 \geq 4 + 4 = 8$ by the AM-GM ineq.

   Second solution : Any real quartic is the product of two real quadratics. So, it can be written $x^4 + ax^3 + 2x^2 + bx + 1 = (x^2 + px + q)(x^2 + sx + t)$ ... (1) where p, q, s, t are real. By the assumption at least one of the quadratic equations, say the second, has a real root. So the discriminant ≥ 0. Implies $s^2 \geq 4t$. Equating coefficients in (1) gives $a = p + t\,; 2 = q + t + s\,; b = pt + qs\,; 1 = qt$. So,

   $a^2 + b^2 = p^2 + s^2 + 2ps + (pt)^2 + (qs)^2 + 2ptqs = p^2(1 + t^2) + s^2(1 + q^2) + 4ps \geq p^2(1 + t^2) + 4(t + q + ps) \geq 8$.

2. Prove that if a, b, and c are integers and both $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a}$ are also integers, then $|a| = |b| = |c|$.

   Solution : Let $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = p$ and $\frac{a}{c} + \frac{c}{b} + \frac{b}{a} = q$. Then the roots of

   $x^3 - px^2 + qx - 1 = 0$ are $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Because the coefficient are integers and the roots are rational, then the possible root is ± 1. So, $|a| = |b| = |c|$.

3. The two tangents to the incircle of a right-angled triangle ABC that are perpendicular to the hypotenuse AB intersect the hypotenuse a P and $Q.$ Find angle PCQ.

   Solution : Let I be the incenter of triangle ABC, Then $\angle CBI = \angle ABI.$ Let the extension of BI cut CA at S, and drop the perpendicular ST from S to $AB.$ Then triangle CBS and TBS are congruent, so that $SC = ST.$ Since SC is tangent to the incircle, so is ST by symmetry. Hence T is in fact one of P and Q, say P. Similarly, if the tangent through Q cuts BC at R, then we have $RC = RQ$. Drop the perpendicular CD from C to AB. Then $\angle PCS = \angle CPS = \angle PCD$, and $\angle QCR = \angle CQR = \angle QCD$. Hence $\angle PCQ = \frac{1}{2}\angle SCR = 45^0$.

4. Each of 17 students talk with every other student. They all talked about three different topics. Each pair of students talked about one topic. Prove that there are three students that talked about the same topic among themselves.

   Solution : Let us call the topics T_1, T_2, T_3. Consider an arbitrary student A. By the pigeonhole principle there is a topic, say T_3, he discussed with at least 6 other students. If two of these 6 students discussed 3, then we are done. Suppose now that the 6 students discussed only T_1 and T_2 and choose one of them, say B. By the pigeonhole principle he discussed one of the topics, say T_2, with three of these students. If two of these three students also discussed T_2, then we are done. Otherwise, all the three students discussed only T_1, which completes the task.