Divisibility Theory

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1 Show that if x, y, z are positive integers, then $(xy+1)(yz+1)(zx+1)$ is a perfect square if and only if $xy+1$, $yz+1$, $zx+1$ are all perfect squares.

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2 Find infinitely many triples (a, b, c) of positive integers such that a, b, c are in arithmetic progression and such that $ab+1$, $bc+1$, and $ca+1$ are perfect squares.

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3 Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

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4 If a, b, c are positive integers such that

$$0 < a^2+b^2-abc \leq c,$$

show that a^2+b^2-abc is a perfect square.

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5 Let x and y be positive integers such that xy divides x^2+y^2+1. Show that

$$\frac{x^2+y^2+1}{xy} = 3.$$

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6 - Find infinitely many pairs of integers a and b with $1 < a < b$, so that ab exactly divides a^2+b^2-1. - With a and b as above, what are the possible values of

$$\frac{a^2+b^2-1}{ab}?$$

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7 Let n be a positive integer such that $2+2\sqrt{28n^2+1}$ is an integer. Show that $2+2\sqrt{28n^2+1}$ is the square of an integer.

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8 The integers a and b have the property that for every nonnegative integer n the number of $2^n a+b$ is the square of an integer. Show that $a=0$.

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9 Prove that among any ten consecutive positive integers at least one is relatively prime to the product of the others.

10 Let n be a positive integer with $n \geq 3$. Show that

$$n^{n^{n^n}} - n^{n^n}$$

is divisible by 1989.

11 Let a, b, c, d be integers. Show that the product

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

12 Let k, m, and n be natural numbers such that $m+k+1$ is a prime greater than $n+1$. Let $c_s = s(s+1)$. Prove that the product

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \cdots (c_{m+n} - c_k)$$

is divisible by the product $c_1 c_2 \cdots c_n$.

13 Show that for all prime numbers p,

$$Q(p) = \prod_{k=1}^{p-1} k^{2k-p-1}$$

is an integer.

14 Let n be an integer with $n \geq 2$. Show that n does not divide $2^n - 1$.

15 Suppose that $k \geq 2$ and $n_1, n_2, \cdots, n_k \geq 1$ be natural numbers having the property

$$n_2 \mid 2^{n_1} - 1, n_3 \mid 2^{n_2} - 1, \cdots, n_k \mid 2^{n_{k-1}} - 1, n_1 \mid 2^{n_k} - 1.$$

Show that $n_1 = n_2 = \cdots = n_k = 1$.

16 Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n.

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17 Let m and n be natural numbers such that

$$A = \frac{(m+3)^n + 1}{3m}$$

is an integer. Prove that A is odd.

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18 Let m and n be natural numbers and let $mn+1$ be divisible by 24. Show that $m+n$ is divisible by 24.

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19 Let $f(x) = x^3 + 17$. Prove that for each natural number $n \geq 2$, there is a natural number x for which $f(x)$ is divisible by 3^n but not 3^{n+1}.

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20 Determine all positive integers n for which there exists an integer m such that $2^n - 1$ divides $m^2 + 9$.

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21 Let n be a positive integer. Show that the product of n consecutive positive integers is divisible by $n!$

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22 Prove that the number

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$$

is not divisible by 5 for any integer $n \geq 0$.

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23 (Wolstenholme's Theorem) Prove that if

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is expressed as a fraction, where $p \geq 5$ is a prime, then p^2 divides the numerator.

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24 Let $p > 3$ is a prime number and $k = \lfloor \frac{2p}{3} \rfloor$. Prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2.

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25 Show that $\binom{2n}{n} \mid \operatorname{lcm}(1, 2, \cdots, 2n)$ for all positive integers n.

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26 Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

27 Show that the coefficients of a binomial expansion $(a+b)^n$ where n is a positive integer, are all odd, if and only if n is of the form 2^k-1 for some positive integer k.

28 Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of positive integers (m,n) with $n \geq m \geq 1$.

29 For which positive integers k, is it true that there are infinitely many pairs of positive integers (m,n) such that

$$\frac{(m+n-k)!}{m!\,n!}$$

is an integer?

30 Show that if $n \geq 6$ is composite, then n divides $(n-1)!$.

31 Show that there exist infinitely many positive integers n such that n^2+1 divides $n!$.

32 Let a and b be natural numbers such that

$$\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that a is divisible by 1979.

33 Let $a, b, x \in \mathbb{N}$ with $b > 1$ and such that $b^n - 1$ divides a. Show that in base b, the number a has at least n non-zero digits.

34 Let $p_1, p_2, \cdots, p_n$ be distinct primes greater than 3. Show that

$$2^{p_1 p_2 \cdots p_n} + 1$$

has at least 4^n divisors.

35 Let $p \geq 5$ be a prime number. Prove that there exists an integer a with $1 \leq a \leq p-2$ such that neither $a^{p-1}-1$ nor $(a+1)^{p-1}-1$ is divisible by p^2.

36 Let n and q be integers with $n \geq 5$, $2 \leq q \leq n$. Prove that $q-1$ divides $\left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

37 If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

38 Let p be a prime with $p>5$, and let $S=\{p-n^2|n \in \mathbb{N}, n^2<p\}$. Prove that S contains two elements a and b such that $a|b$ and $1<a<b$.

39 Let n be a positive integer. Prove that the following two statements are equivalent. - n is not divisible by 4 - There exist $a, b \in \mathbb{Z}$ such that a^2+b^2+1 is divisible by n.

40 Determine the greatest common divisor of the elements of the set

$$\{n^{13}-n \mid n \in \mathbb{Z}\}.$$

41 Show that there are infinitely many composite numbers n such that $3^{n-1}-2^{n-1}$ is divisible by n.

42 Suppose that 2^n+1 is an odd prime for some positive integer n. Show that n must be a power of 2.

43 Suppose that p is a prime number and is greater than 3. Prove that 7^p-6^p-1 is divisible by 43.

44 Suppose that 4^n+2^n+1 is prime for some positive integer n. Show that n must be a power of 3.

45 Let $b, m, n \in \mathbb{N}$ with $b>1$ and $m \neq n$. Suppose that b^m-1 and b^n-1 have the same set of prime divisors. Show that $b+1$ must be a power of 2.

46 Let a and b be integers. Show that a and b have the same parity if and only if there exist integers c and d such that $a^2+b^2+c^2+1=d^2$.

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47 Let n be a positive integer with $n > 1$. Prove that

$$\frac{1}{2}+\cdots+\frac{1}{n}$$

is not an integer.

48 Let n be a positive integer. Prove that

$$\frac{1}{3}+\cdots+\frac{1}{2n+1}$$

is not an integer.

49 Prove that there is no positive integer n such that, for $k = 1, 2, \cdots, 9$, the leftmost digit of $(n + k)!$ equals k.

50 Show that every integer $k > 1$ has a multiple less than k^4 whose decimal expansion has at most four distinct digits.

51 Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a+d = 2^k$ and $b+c = 2^m$ for some integers k and m, then $a = 1$.

52 Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a and b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

53 Suppose that x, y, and z are positive integers with $xy = z^2+1$. Prove that there exist integers a, b, c, and d such that $x = a^2 + b^2$, $y = c^2 + d^2$, and $z = ac + bd$.

54 A natural number n is said to have the property P, if whenever n divides $a^n - 1$ for some integer a, n^2 also necessarily divides $a^n - 1$. - Show that every prime number n has the property P. - Show that there are infinitely many composite numbers n that possess the property P.

55 Show that for every natural number n the product

$$\left(4-\frac{2}{1}\right)\left(4-\frac{2}{2}\right)\left(4-\frac{2}{3}\right)\cdots\left(4-\frac{2}{n}\right)$$

is an integer.

56 Let a, b, and c be integers such that $a+b+c$ divides $a^2+b^2+c^2$. Prove that there are infinitely many positive integers n such that $a + b + c$ divides $a^n + b^n + c^n$.

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57 Prove that for every $n \in \mathbb{N}$ the following proposition holds: $7|3^n + n^3$ if and only if $7|3^n n^3 + 1$.

58 Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq \frac{3k}{4}$. Let n be a composite integer. Prove that - if $n = 2p_k$, then n does not divide $(n-k)!$, - if $n > 2p_k$, then n divides $(n-k)!$.

59 Suppose that n has (at least) two essentially distinct representations as a sum of two squares. Specifically, let $n = s^2 + t^2 = u^2 + v^2$, where $s \geq t \geq 0$, $u \geq v \geq 0$, and $s > u$. Show that $\gcd(su - tv, n)$ is a proper divisor of n.

60 Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t, the number $at + b$ is a triangular number if and only if t is a triangular number.

61 For any positive integer $n > 1$, let $p(n)$ be the greatest prime divisor of n. Prove that there are infinitely many positive integers n with

$$p(n) < p(n+1) < p(n+2).$$

62 Let $p(n)$ be the greatest odd divisor of n. Prove that

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\frac{p(k)}{k} > \frac{2}{3}.$$

63 There is a large pile of cards. On each card one of the numbers $1, 2, \cdots, n$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

64 The last digit of the number $x^2 + xy + y^2$ is zero (where x and y are positive integers). Prove that two last digits of this numbers are zeros.

65 Clara computed the product of the first n positive integers and Valerid computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

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66 (Four Number Theorem) Let a, b, c, and d be positive integers such that $ab = cd$. Show that there exists positive integers p, q, r, s such that

$$a = pq, \quad b = rs, \quad c = ps, \quad d = qr.$$

67 Prove that $\binom{2n}{n}$ is divisible by $n + 1$.

68 Suppose that $S = \{a_1, \cdots, a_r\}$ is a set of positive integers, and let S_k denote the set of subsets of S with k elements. Show that

$$\operatorname{lcm}(a_1, \cdots, a_r) = \prod_{i=1}^{r} \prod_{s \in S_i} \gcd(s)^{\left((-1)^i\right)}.$$

69 Prove that if the odd prime p divides $a^b - 1$, where a and b are positive integers, then p appears to the same power in the prime factorization of $b(a^d - 1)$, where $d = \gcd(b, p - 1)$.

70 Suppose that $m = nq$, where n and q are positive integers. Prove that the sum of binomial coefficients

$$\sum_{k=0}^{n-1} \binom{\gcd(n,k)q}{\gcd(n,k)}$$

is divisible by m.

71 Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

72 Determine all pairs (n, p) of nonnegative integers such that - p is a prime, - $n < 2p$, - $(p - 1)^n + 1$ is divisible by n^{p-1}.

73 Determine all pairs (n, p) of positive integers such that - p is a prime, $n > 1$, - $(p - 1)^n + 1$ is divisible by n^{p-1}.

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74 Find an integer n, where $100 \leq n \leq 1997$, such that

$$\frac{2^n+2}{n}$$

is also an integer.

75 Find all triples (a, b, c) of positive integers such that $2^c - 1$ divides $2^a + 2^b + 1$.

76 Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of} \quad abc-1.$$

77 Find all positive integers, representable uniquely as

$$\frac{x^2+y}{xy+1},$$

where x and y are positive integers.

78 Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3+1}{mn-1}$$

is an integer.

79 Determine all pairs of integers (a, b) such that

$$\frac{a^2}{2ab^2-b^3+1}$$

is a positive integer.

80 Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m+a-1}{a^n+a^2-1}$$

is itself an integer.

81 Determine all triples of positive integers (a, m, n) such that $a^m + 1$ divides $(a+1)^n$.

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82 Which integers can be represented as

$$\frac{(x+y+z)^2}{xyz}$$

where x, y, and z are positive integers?

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83 Find all $n \in \mathbb{N}$ such that $\lfloor \sqrt{n} \rfloor$ divides n.

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84 Determine all $n \in \mathbb{N}$ for which - n is not the square of any integer, - $\lfloor \sqrt{n} \rfloor^3$ divides n^2.

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85 Find all $n \in \mathbb{N}$ such that 2^{n-1} divides $n!$.

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86 Find all positive integers (x, n) such that $x^n + 2^n + 1$ divides $x^{n+1} + 2^{n+1} + 1$.

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87 Find all positive integers n such that $3^n - 1$ is divisible by 2^n.

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88 Find all positive integers n such that $9^n - 1$ is divisible by 7^n.

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89 Determine all pairs (a, b) of integers for which $a^2 + b^2 + 3$ is divisible by ab.

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90 Determine all pairs (x, y) of positive integers with $y|x^2 + 1$ and $x^2|y^3 + 1$.

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91 Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

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92 Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

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93 Find the largest positive integer n such that n is divisible by all the positive integers less than $\sqrt[3]{n}$.

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94 Find all $n \in \mathbb{N}$ such that $3^n - n$ is divisible by 17.

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95 Suppose that a and b are natural numbers such that

$$p = \frac{b}{4}\sqrt{\frac{2a - b}{2a + b}}$$

is a prime number. What is the maximum possible value of p?

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96 Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2 \cdots, d_{16}$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$, $d_6 = 18$, and $d_9 - d_8 = 17$.

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97 Suppose that n is a positive integer and let

$$d_1 < d_2 < d_3 < d_4$$

be the four smallest positive integer divisors of n. Find all integers n such that

$$n = {d_1}^2 + {d_2}^2 + {d_3}^2 + {d_4}^2.$$

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98 Let n be a positive integer with $k \geq 22$ divisors $1 = d_1 < d_2 < \cdots < d_k = n$, all different. Determine all n such that

$${d_7}^2 + {d_{10}}^2 = \left(\frac{n}{d_{22}}\right)^2.$$

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99 Let $n \geq 2$ be a positive integer, with divisors

$$1 = d_1 < d_2 < \cdots < d_k = n\ .$$

Prove that

$$d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$$

is always less than n^2, and determine when it divides n^2.

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100 Find all positive integers n such that n has exactly 6 positive divisors $1 < d_1 < d_2 < d_3 < d_4 < n$ and $1 + n = 5(d_1 + d_2 + d_3 + d_4)$.

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101 Find all composite numbers n having the property that each proper divisor d of n has $n - 20 \leq d \leq n - 12$.

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102 Determine all three-digit numbers N having the property that N is divisible by 11, and $\frac{N}{11}$ is equal to the sum of the squares of the digits of N.

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103 When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

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104 A wobbly number is a positive integer whose *digits* in base 10 are alternatively non-zero and zero the units digit being non-zero. Determine all positive integers which do not divide any wobbly number.

105 Find the smallest positive integer n such that - n has exactly 144 distinct positive divisors, - there are ten consecutive integers among the positive divisors of n.

106 Determine the least possible value of the natural number n such that $n!$ ends in exactly 1987 zeros.

107 Find four positive integers, each not exceeding 70000 and each having more than 100 divisors.

108 For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples (x, y, z) of distinct positive integers satisfying - x, y, z are in arithmetic progression, - $p(xyz) \leq 3$.

109 Find all positive integers a and b such that

$$\frac{a^2+b}{b^2-a} \text{ and } \frac{b^2+a}{a^2-b}$$

are both integers.

110 For each positive integer n, write the sum $\sum_{m=1}^{n} 1/m$ in the form p_n/q_n, where p_n and q_n are relatively prime positive integers. Determine all n such that 5 does not divide q_n.

111 Find all natural numbers n such that the number $n(n+1)(n+2)(n+3)$ has exactly three different prime divisors.

112 Prove that there exist infinitely many pairs (a, b) of relatively prime positive integers such that

$$\frac{a^2-5}{b} \text{ and } \frac{b^2-5}{a}$$

are both positive integers.

113 Find all triples (l, m, n) of distinct positive integers satisfying

$$\gcd(l, m)^2 = l + m, \ \gcd(m, n)^2 = m + n, \text{ and } \gcd(n, l)^2 = n + l.$$

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114 What is the greatest common divisor of the set of numbers

$$\{16^n + 10n - 1 \mid n = 1, 2, \cdots\}?$$

115 Does there exist a 4-digit integer (in decimal form) such that no replacement of three of its digits by any other three gives a multiple of 1992?

116 What is the smallest positive integer that consists base 10 of each of the ten digits, each used exactly once, and is divisible by each of the digits 2 through 9?

117 Find the smallest positive integer n such that

$$2^{1989} \mid m^n - 1$$

for all odd positive integers $m > 1$.

118 Determine the highest power of 1980 which divides

$$\frac{(1980n)!}{(n!)^{1980}}.$$

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