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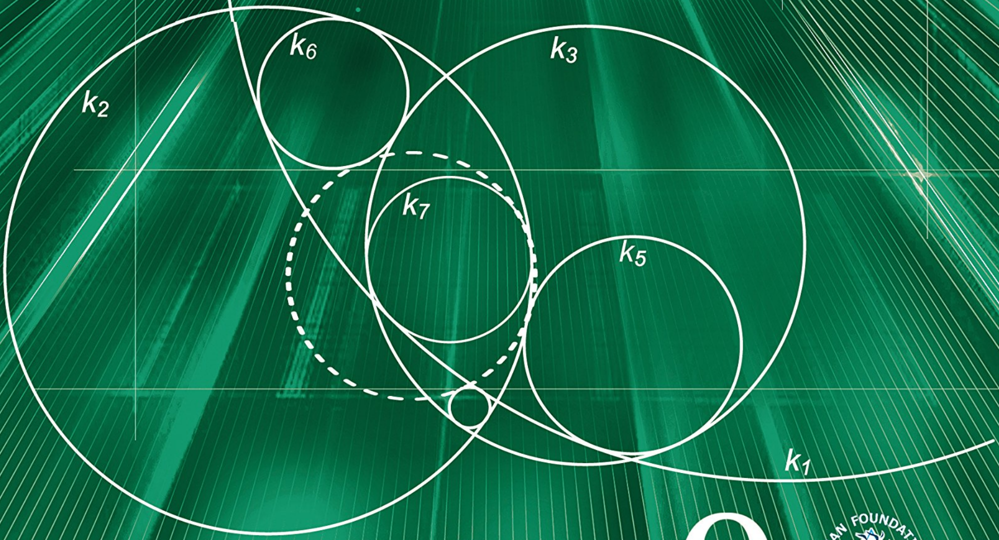
555 GEOMETRY PROBLEMS

SOLUTIONS BASED ON "GEOMETRY IN FIGURES" BY A. V. AKOPYAN



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UNIMAT SMB
Union of Bulgarian Mathematicians
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555 Geometry Problems

Solutions based on "Geometry in Figures" by A. V. Akopyan

This book contains the solutions of one of the best collections of elementary geometry problems – "Geometry in Figures" by the Russian author Arseny V. Akopyan. The purpose of this book is to serve as a resource for teachers and students interested in Olympiad Geometry. Math enthusiasts, explorers and researchers could also benefit from it. We believe this is an excellent material for preparation for Mathematical Olympiads, from ones on a local scale up to the International Mathematical Olympiad.

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June 2017
ISBN: 978-954-8880-48-0

Introduction to the Bulgarian edition

The facts in this book, collected by Akopyan, provide a large scope of classical and original constructions. The problem difficulty varies – some problems are easy and could be useful at school, while others have difficulty comparable to the one of shortlisted problems for the IMO.

The organization of the problems and their numbers are equivalent to those in "Geometry in Figures". This means that they are not ordered by difficulty, but by theme. This grouping would allow the reader to easily make connections between the constructions, which can further facilitate additional exploration.

This edition includes a diversity of methods for solving Olympiad problems. Provisionally, they are classified as either synthetic or computational. Among the computational methods, one can find trigonometry, complex numbers, dot product of vectors, metric relations and barycentric coordinates. The synthetic methods include similar triangles, (oriented) areas, angle chasing, uniqueness of construction, radical axes, harmonic division and quadrilaterals, cross-ratios, isogonal conjugation, poles and polars, inversion, homothety, uniform motion and projective transformations.

We use a number of different methods, hence the reader could possibly be unfamiliar with some of them. For this reason, we offer a list of related literature. We recommend "Advanced Euclidean Geometry" by Roger A. Johnson, since it captures many fundamental topics.

We have added some classical theorems which do not appear in Akopyan's book, such as those of Bretschneider, Stewart, Fontene, Pascal and Brianchon (the last two appear in chapter 11 of "Geometry in Figures", where the general case for an ellipse is illustrated).

We have covered chapters 2-10 from Akopyan's book. Chapter 1 contains easy and basic problems, and we assume that the reader is familiar with most of them. Throughout the book, we use these results without proof.

Problem 10.16 (also known as the nine circles theorem) is the only one to which we could not find a satisfactory elementary solution, and consequently, it is not presented in the book. In some solutions, only a certain configuration of the points is considered but analogous arguments apply for any other configuration.

A collection of GeoGebra applets for all problems is available on the website of Viva Cognita (`www.vivacognita.org/555geometry.html`). You are welcome to comment and share your thoughts, insights and solutions.

We do not guarantee that the solutions presented here are the easiest or the most concise ones. Instead, our goal is to make use of as many methods as possible, so that the reader could get familiar with their applications.

We would appreciate your comments and questions. Contact us by email at schobanov95@gmail.com (Stanislav Chobanov), lubem@abv.bg (Lyuben Lichev), stennli.smg@gmail.com (Stanislav Dimitrov) and miroslav.marinov@stcatz.ox.ac.uk (Miroslav Marinov).

The figures were made using the free interactive software "GeoGebra" and the vector graphics language Asymptote.

We would like to express our gratitude towards everyone who supported us throughout all this time so that we could turn this project into reality. We thank Prof. Peter Boyvalenkov, Todor Branzov, Mihaela Nikolova, Mark Andonov, Nikolay Beluhov, Stoyan Boev, Ivo Dilov, Assel Ismoldayeva, Yordan Yordanov, Teofil Todorov and Alexandar Cherganski.

Introduction to the English edition

The huge success of the Bulgarian edition of this book motivated us to publish it in English. We believe that this act would open the doors to elementary geometry for many students and teachers around the globe. This is the time to express our deepest gratitude to Miroslav Marinov. He became a vital part of our team by spending countless hours helping us with the translation and the editing process of the entire book. We would wish to expand our work by translating it into other languages in the future. We hope that the book would provide an intellectual satisfaction to the geometry enthusiasts, who have had the courage to read the introduction to its very end, so good luck!

The authors

Glossary of symbols

Unless stated otherwise, we will use the following notations.

$\triangle ABC$	Triangle ABC
c	AB
a	BC
b	CA
γ	$\angle ACB$
α	$\angle BAC$
β	$\angle ABC$
R	The circumradius of the triangle ABC
r	The inradius of the triangle ABC
H	The orthocenter of the triangle ABC
O	The circumcenter of the triangle ABC
I	The incenter of the triangle ABC
s	The semiperimeter of the triangle ABC
r_a, r_b, r_c	The radii of the excircles opposite to A, B, and C, respectively
I_a, I_b, I_c	The excenters opposite to A, B, and C, respectively
$k(O)$	A circle k with center O
$k(O, R)$	The circle k with center O and radius R
(ABC)	The circumcircle of triangle ABC
$t(A, k)$	The tangent line to the circle k that passes through the point $A \in k$. Also denoted by $t(A)$ if the circle is implied
S_{ABC}	The area of triangle ABC
$[ABC]$	The oriented area of triangle ABC
S_{AB}	The perpendicular bisector of the segment AB
π_A	The polar line of the point A with respect to the pole-polar transformation π
π_{AB}	The pole of the line AB with respect to the pole-polar transformation π
$\varphi(O, R)$	The inversion φ with center O and radius R
$h(O, x)$	The homothety h with center O and ratio x
(A, B, C, D)	The cross-ratio of the four collinear points A, B, C and D. It is defined as $\dfrac{AD}{DC} \cdot \dfrac{CB}{BA}$

$(A, B, C, D) = 1$	The points A, B, C and D are in harmonic division, or their cross-ratio equals one. Also, a cyclic quadrilateral is called harmonic if $\frac{AD}{DC} \cdot \frac{CB}{BA} = 1$
(XA, XB, XC, XD)	The cross-ratio of the lines XA, XB, XC and XD in a pencil
$pow(A, k)$	The power of the point A a with respect to the circle k
$dist(A, l)$	The distance from the point A to the line l
$\rho(k_1, k_2)$	The radical axis of the non-concentric circles k_1 and k_2. It is defined as the locus of points that have the same power with respect to the two circles.

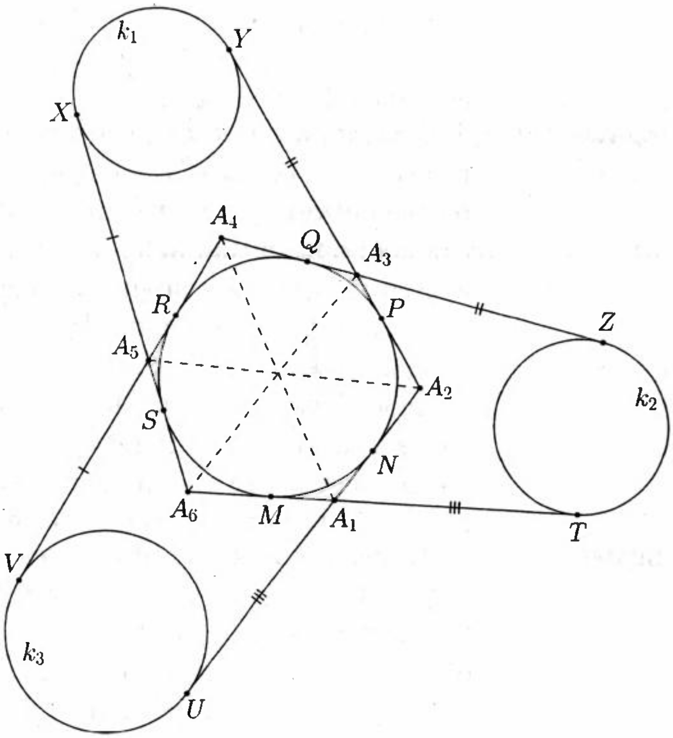
In chapters 2 and 3, the notations are as they appear in Akopyan's "Geometry in Figures".

Glossary of terms

Mixtilinear incircle	The circle that is tangent to two of the sides of a triangle and that touches its circumcircle internally
Mixtilinear excircle	The circle that is tangent to two of the sides of a triangle and that touches its circumcircle externally
Isotomic conjugates	Two points are isotomic conjugates with respect to a vertex of a triangle if they lie on the opposing side and are equidistant to the midpoint of that side. The same term is used for the lines passing through the vertex and the two points. Two points are isotomic conjugates with respect to a triangle if each of the three pairs of lines connecting them to the vertices of the triangle are isotomic conjugates with respect to the corresponding vertex
Isogonal conjugates	Two lines are isogonal conjugates with respect to a vertex of a triangle if they pass through the vertex and the reflection of one of them with respect to the angle bisector at that vertex is the other line. The same term is used for the points where the two lines meet the opposing side of the triangle. Two points are isogonal conjugates with respect to a triangle if each of the three pairs of lines connecting them to the vertices of the triangle are isogonal conjugates with respect to the corresponding vertex
The cotangent technique	Given a $\triangle A_1A_2A_3$, the following equality holds $\cot\angle A_3A_1A_2 = \dfrac{\dfrac{A_3A_1}{A_3A_2} - \cos\angle A_1A_3A_2}{\sin\angle A_1A_3A_2}$. A powerful method is based on this relation.

Useful Theorems

Brianchon's Theorem. Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon all of whose sides are tangent to a circle ω. Then the lines A_1A_4, A_2A_5 and A_3A_6 are concurrent.



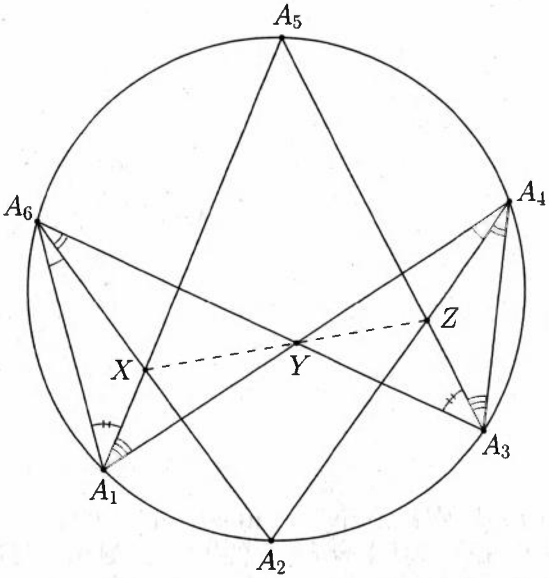
Proof. Let M, N, P, Q, R and S be the points of tangency of ω and the sides of the hexagon, as shown in the figure. Points X, Y, Z, T, U and V are chosen such that $SX = PY = QZ = MT = NU = RV$, as shown in the figure.

Clearly, there exists a circle k_1 that touches XS and YP at the points X and Y. Similarly, we can define circles k_2 and k_3. We have $YP = ZQ$ and $QA_3 = PA_3$ and thus $YA_3 = ZA_3$. Therefore, A_3 lies on the radical axis of k_1 and k_2. Moreover, as $MT = SX$ and $MA_6 = SA_6$, it follows that $TA_6 = XA_6$. Hence, A_6 also lies on the radical axis of circles k_1 and k_2.

It follows that A_3A_6 is the radical axis of k_1 and k_2. Analogously, A_1A_4 and A_2A_5 are the radical axes of the pairs (k_2, k_3) and (k_3, k_1), respectively.

Applying the radical axes theorem (Problem 6.5.1) to k_1, k_2 and k_3, we get that the lines A_1A_4, A_2A_5 and A_3A_6 are concurrent.

Pascal's Theorem. Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed in a circle k. (The points may not lie in this order along the circle.) Let $X = A_1A_5 \cap A_2A_6$, $Y = A_1A_4 \cap A_3A_6$ and $Z = A_2A_4 \cap A_3A_5$. Then X, Y and Z are collinear.



Proof. We have

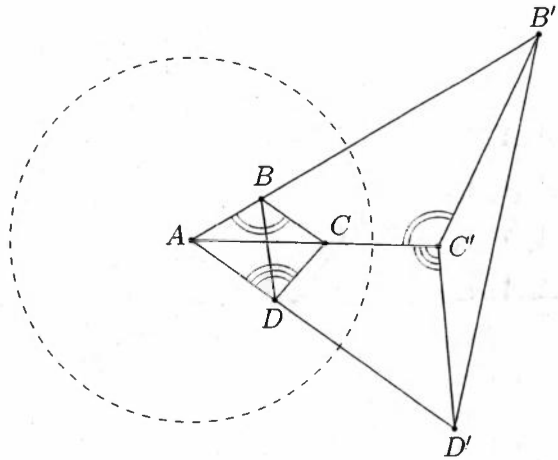
$$\angle A_1A_6A_2 = \angle A_1A_4A_2,$$
$$\angle A_5A_1A_6 = \angle A_5A_3A_6,$$
$$\angle A_2A_6A_3 = \angle A_2A_4A_3,$$
$$\angle A_5A_1A_4 = \angle A_5A_3A_4.$$

On the other hand, we have that $\triangle A_6YA_1 \sim \triangle A_4YA_3$. Now, if X' is the isogonal conjugate of X in $\triangle A_1YA_6$, then it is easy to show that X' and Z correspond to each other in the similar triangles. Therefore, $\angle XYA_1 = \angle X'YA_6 = \angle ZYA_4$ and thus the points X, Y and Z are collinear.

Note. Pascal's Theorem is the dual of Brianchon's Theorem. If we take one of them and consider a pole-polar transformation with respect to the circle k or ω, respectively, the result will be the other theorem. This fact means that we can prove either theorem by only using the other one.

Bretschneider's Theorem. Let $ABCD$ be a quadrilateral (it may be concave). Then the following equality holds

$$AC^2.BD^2=AB^2.CD^2+BC^2.AD^2-2AB.BC.CD.DA.\cos(\angle BAD+\angle BCD).$$



Proof. We apply an inversion φ with center A and an arbitrary radius $R>0$. Let $\varphi(B)=B'$, $\varphi(C)=C'$ and $\varphi(D)=D'$.

It follows that $\angle ABC=\angle AC'B'$ and $\angle ADC=\angle AC'D'$, so $\angle B'C'D'=\angle BAD+\angle BCD$. We have $\triangle ADC\sim\triangle AC'D'$, $\triangle ABC\sim\triangle AC'B'$ and $\triangle ABD\sim\triangle AD'B'$.

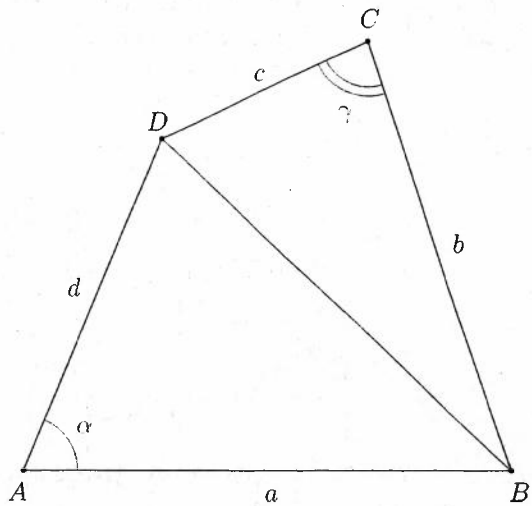
Thus, $\dfrac{C'D'}{CD}=\dfrac{AC'}{AD}=\dfrac{R^2}{AC.AD}$, and therefore $C'D'=\dfrac{CD.R^2}{AC.AD}$. Analogously, $B'C'=\dfrac{BC.R^2}{AC.AB}$ and $B'D'=\dfrac{BD.R^2}{AD.AB}$.

The law of cosines in $\triangle B'C'D'$ yields

$$B'C'^2+D'C'^2-2B'C'.D'C'.\cos(\angle BAD+\angle BCD)=B'D'^2.$$

Plugging $C'D'$, $B'C'$ and $B'D'$ from above and simplifying, we get the desired result.

Bretschneider's Formula. Let $ABCD$ be a quadrilateral such that $AB = a$, $BC = b$, $CD = c$, $DA = d$, $\angle BAD = \alpha$ and $\angle BCD = \gamma$. Then $S = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(\cos(\alpha+\gamma)+1)}$, where S and s are the area and the semiperimeter of the quadrilateral, respectively.



Proof. Note that $S = S_{\triangle ABD} + S_{\triangle BCD}$. Moreover, $2S_{\triangle ABD} = ad\sin\alpha$ and $2S_{\triangle BCD} = bc\sin\gamma$. Therefore, $2S = ad\sin\alpha + bc\sin\gamma$. Squaring both sides yields

$$4S^2 = (ad)^2\sin^2\alpha + (bc)^2\sin\gamma + 2abcd\sin\alpha\sin\gamma.$$

We also have that $BD^2 = a^2 + d^2 - 2ad\cos\alpha = b^2 + c^2 - 2bc\cos\gamma$ and hence $\dfrac{a^2+d^2-b^2-c^2}{2} = ad\cos\alpha - bc\cos\gamma$. We square both sides and get

$$\left(\frac{a^2+d^2-b^2-c^2}{2}\right)^2 = (ad)^2\cos^2\alpha + (bc)^2\cos^2\gamma - 2abcd\cos\alpha\cos\gamma$$

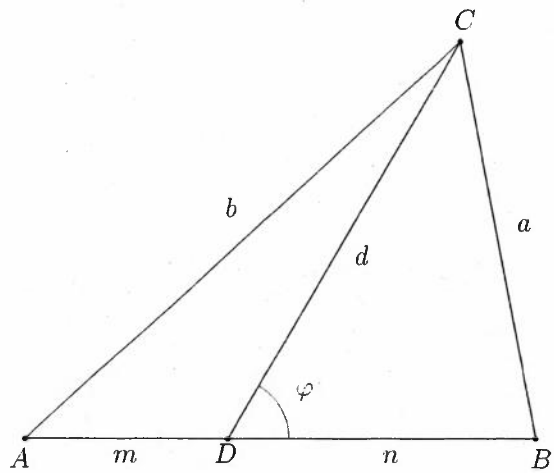
$$\Rightarrow 4S^2 + \left(\frac{a^2+d^2-b^2-c^2}{2}\right)^2 = (ad)^2 + (bc)^2 - 2abcd\cos(\alpha+\gamma),$$

which simplifies to

$$16S^2 = 16(s-a)(s-b)(s-c)(s-d) - 8abcd(1+\cos(\alpha+\gamma)).$$

To get the desired result, divide by 16 and take the square root.

Stewart's Theorem. Given a $\triangle ABC$, let D be a point on the side AB. Denote $BC = a$, $AC = b$, $CD = d$, $AD = m$ and $BD = n$. Then $d^2 = \dfrac{ma^2 + nb^2}{m+n} - mn$.



First Proof. Denote $\angle ACB = \gamma$. We apply Bretschneider's Theorem to the degenerate quadrilateral $CADB$.

This yields $(am)^2 + (bn)^2 - 2abmn\cos(180° + \gamma) = d^2(m+n)^2$, or equivalently $(am)^2 + (bn)^2 + 2abmn\cos\gamma = d^2(m+n)^2$.

The law of cosines in $\triangle ABC$ gives $2ab\cos\gamma = a^2 + b^2 - c^2$. After substituting $c = m+n$, we get $(am)^2+(bn)^2+mn(c^2-a^2-b^2) = d^2(m+n)^2$, which yields the desired result.

Second Proof. Denote $\angle BDC = \varphi$. Then $\angle ADC = 180° - \varphi$.

The law of cosines in $\triangle BDC$ yields

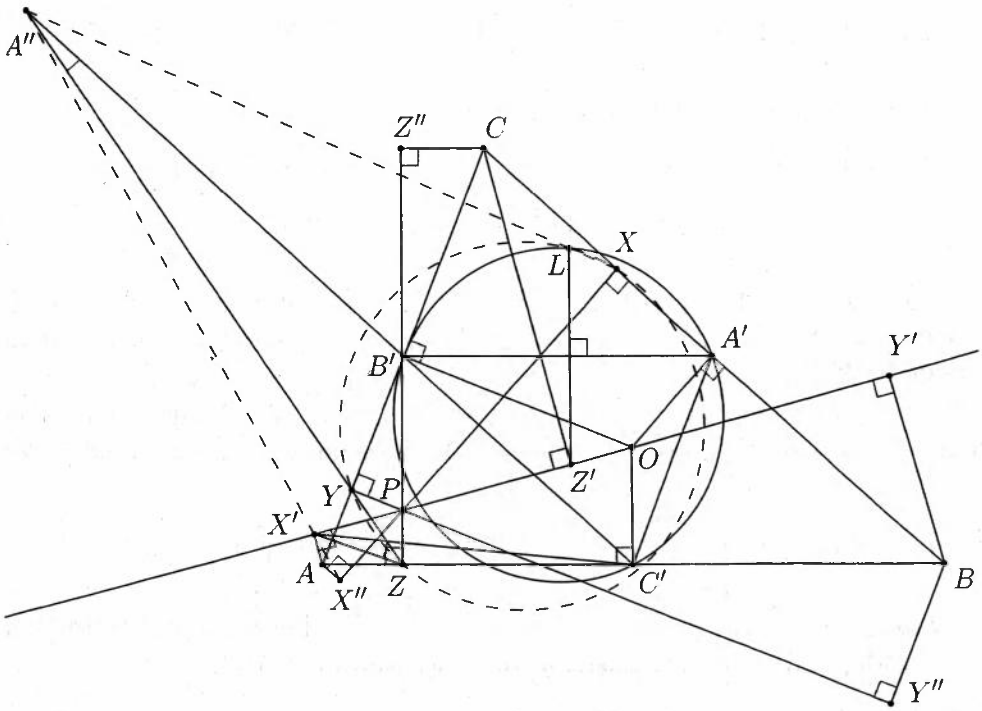
$$d^2 + n^2 - a^2 = 2dn\cos\varphi.$$

The law of cosines in $\triangle ADC$ yields

$$d^2 + m^2 - b^2 = 2md\cos(180° - \varphi) = -2md\cos\varphi.$$

Therefore, $d^2m + mn^2 - a^2m + d^2n + nm^2 - b^2n = 0$, which yields the desired result.

First and Second Fontene Theorems. Let ABC be a triangle with circumcenter O. Let A', B' and C' be the midpoints of the sides BC, CA and AB, respectively. Let P be an arbitrary point in the plane. Let X, Y and Z be the projections of P onto the lines BC, CA and AB, respectively. Let $YZ \cap B'C' = A''$, $XZ \cap A'C' = B''$ and $XY \cap A'B' = C''$. Then the lines $A''X$, $B''Y$ and $C''Z$ all pass through a single point, which lies on the circumcircles of $\triangle A'B'C'$ and $\triangle XYZ$, and this point is independent of the position of P on the line OP.



Proof. Obviously, point O is the orthocenter of $\triangle A'B'C'$. We apply Problem 4.2.3 to this triangle and the line OP. It shows the existence of a point on the circumcircle of $\triangle A'B'C'$ that lies on the reflections of the line OP with respect to the sides of $\triangle A'B'C'$.

This point is called the Anti-Steiner Point and we will denote it by L. From the definition of L it follows that its reflections in $B'C'$, $C'A'$ and $A'B'$ – points X', Y' and Z', lie on OP.

Moreover, Problem 4.8.1 tells us that the circumcircle of $\triangle A'B'C'$ and the circumcircle of $\triangle A'B'C$ are symmetric with respect to $A'B'$. Therefore, the point Z' lies on the circumcircle of $\triangle A'B'C$. The analogous statements about the points X' and Y' are also true.

Now we have $\angle OZ'C = \angle OB'C = 90°$. Thus, X', Y' and Z' are the projections of A, B and C onto OP, respectively. The point L is also called the orthopole (see the note below) of $\triangle ABC$ with respect to OP. We saw that it lies on the nine-point circle of $\triangle ABC$.

Let points X'', Y'' and Z'' be the reflections of X, Y and Z with respect to $B'C'$, $C'A'$ and $A'B'$, respectively. Now Z'' lies on the circle with diameter CP. The analogous statements for the points X'' and Y'' are also true. We have

$$\angle ZX'C' = \angle AX'C' - \angle AX'Z = \angle AB'C' - \angle AYZ = \angle ZA''C'.$$

Therefore, the quadrilateral $ZX'A''C'$ is cyclic. Thus,

$$\begin{aligned}\angle A''X'X'' &= \angle A''X'Z + \angle ZX'X'' = \angle B'C'B + \angle ZPX'' \\ &= 180° - \angle ZBX + \angle ZBX = 180°.\end{aligned}$$

It follows that the points A'', X' and X'' are collinear. Moreover, L is symmetric to X' with respect to $B'C'$ and X is symmetric to X'' with respect to $B'C'$.

Therefore, the points A'', L and X are collinear. Analogously, the points B'', L and Y, as well as the points C'', L and Z, are collinear. We have

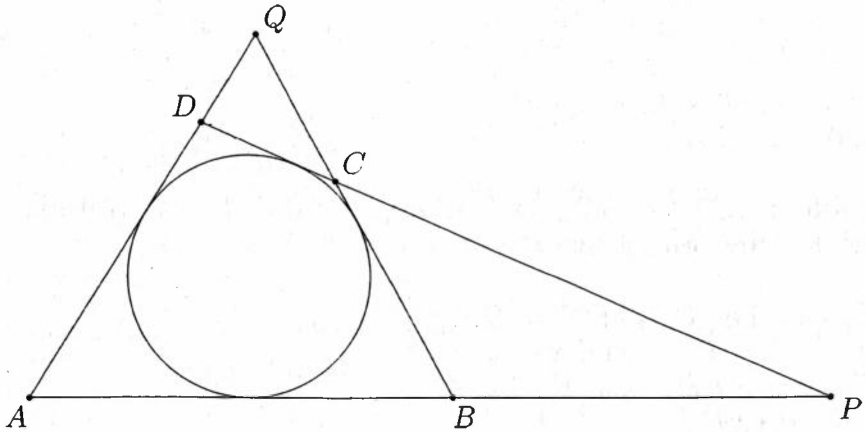
$$A''L = A''X',\ A''X = A''X'' \Rightarrow A''L.A''X = A''X'.A''X'' = A''Y.A''Z.$$

Thus, L lies on the circumcircle of $\triangle XYZ$. Furthermore, the definition of L is independent of the position of P on the line OP.

Note. We define the orthopole of a line l with respect to $\triangle ABC$ as follows. Construct the feet of the perpendiculars from each vertex of the triangle to the line l. From each foot, we draw a perpendicular to the corresponding side of the triangle (if the foot came from A, we draw the perpendicular to BC and so on). These three perpendiculars are concurrent at a point called the orthopole of the line l with respect to $\triangle ABC$.

Given a convex quadrilateral $ABCD$, such that the rays AB and DC intersect at the point P and the rays BC and AD intersect at the point Q, we will use the three necessary and sufficient conditions for the quadrilateral to be circumscribed without proof. These conditions are

$$AB + CD = BC + AD,$$
$$AP + CQ = AQ + CP,$$
$$QB + BP = QD + DP.$$



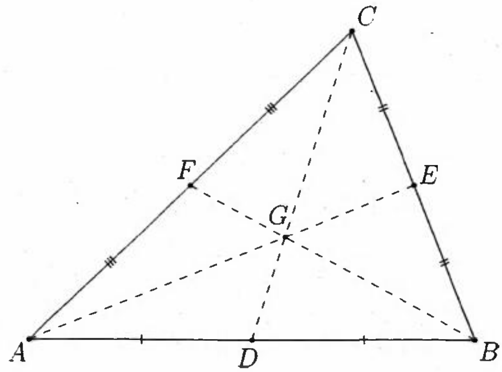
Also, if the circle touches the lines BC and CD on the opposite side, analogous necessary and sufficient conditions hold, and we will use them without proof as well. These conditions are

$$AB + BC = CD + DA,$$
$$AP + PC = AQ + QC,$$
$$DP + PB = DQ + QB.$$

See Problem 6.9 in V. Prasolov's edition of "Problems in Plane and Solid Geometry" – `http://students.imsa.edu/~tliu/Math/planegeo.pdf`.

Problems

Problem 2.1. Let ABC be a triangle. Prove that its medians CD, AE and BF are concurrent.

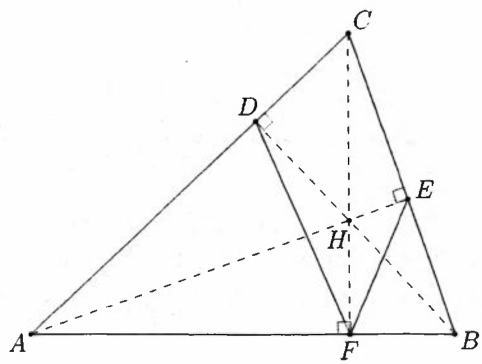


Solution. Let $AE \cap BF = G$. Let $CG \cap AB = D'$. We have

$$\frac{S_{\triangle AGC}}{S_{\triangle BGC}} = \frac{S_{\triangle AGC}}{S_{\triangle AGB}} \cdot \frac{S_{\triangle AGB}}{S_{\triangle BGC}} = \frac{CE}{EB} \cdot \frac{AF}{FC} = \frac{1}{1} \cdot \frac{1}{1} = 1.$$

Therefore, D' is the midpoint of AB, and so $D \equiv D'$.

Problem 2.2. Let ABC be a triangle. Prove that its altitudes AE, CF and BD are concurrent.

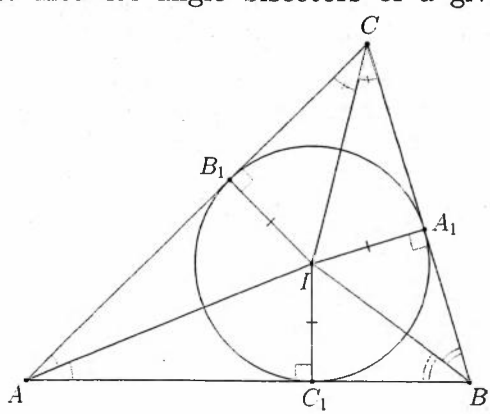


Solution. Let $BD \cap CF = H$. Let $AH \cap BC = E'$. Therefore, the quadrilaterals $AFHD$ and $BFDC$ are cyclic. Hence,

$$\angle DAH = \angle DFC = \angle CBD.$$

Thus, the quadrilateral $ADE'B$ is cyclic. It follows that $\angle AE'B = 90°$, and so $E \equiv E'$.

Problem 2.3. Prove that the internal angle bisectors of a given $\triangle ABC$ are concurrent.



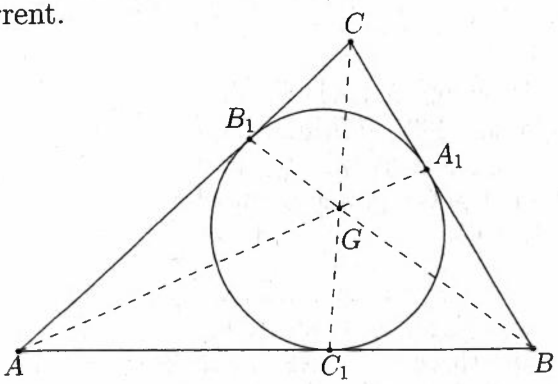
Solution. Let the bisectors of $\angle BAC$ and $\angle ABC$ intersect at I. Then $dist(I, AC) = dist(I, AB)$ and $dist(I, AB) = dist(I, BC)$.

We get that

$$dist(I, BC) = dist(I, AC),$$

and thus the point I also lies on the bisector of $\angle ACB$.

Problem 2.4. Let ABC be a triangle. Its incircle touches AB, BC and CA at the points C_1, A_1 and B_1, respectively. Prove that the lines CC_1, BB_1 and AA_1 are concurrent.



Solution. We use the properties of tangent lines to the incircle to get that $AB_1 = AC_1$, $BC_1 = BA_1$ and $CA_1 = CB_1$. Thus,

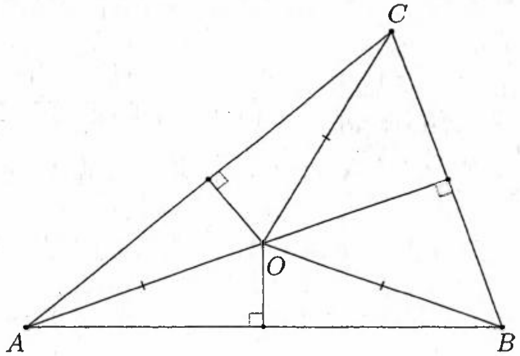
$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$$

and Ceva's Theorem (Problem 4.9.15), applied to $\triangle ABC$ and the lines CC_1, AA_1 and BB_1, yields the desired result.

Note. The intersection point of AA_1, BB_1 and CC_1 is called the Gergonne point of $\triangle ABC$.

Problem 2.5. Prove that the perpendicular bisectors of the sides of a given $\triangle ABC$ are concurrent.

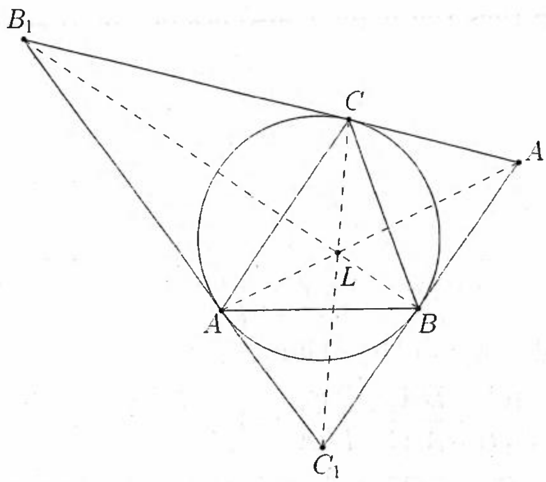
Solution. Let $S_{AB} \cap S_{AC} = O$. Since O lies on both perpendicular bisectors, $AO = BO$ and $AO = CO$. Hence, $BO = CO$. Hence, the point O lies on the perpendicular bisector of BC. Thus, O is the intersection point of the three perpendicular bisectors of the sides of $\triangle ABC$.



Problem 2.6. Let ABC be a triangle with circumcircle k. Let l_A, l_B and l_C be the tangent lines to k at the points A, B and C, respectively. If $l_A \cap l_B = C_1$, $l_B \cap l_C = A_1$ and $l_C \cap l_A = B_1$, prove that the lines AA_1, BB_1 and CC_1 are concurrent.

Solution. Note that k is inscribed in $\triangle A_1B_1C_1$, and also note that A, B and C are the points of tangency of k and B_1C_1, C_1A_1 and A_1B_1, respectively.

Problem 2.4 yields that the lines AA_1, BB_1 and CC_1 are concurrent at L, and this is the Gergonne point of $\triangle A_1B_1C_1$.



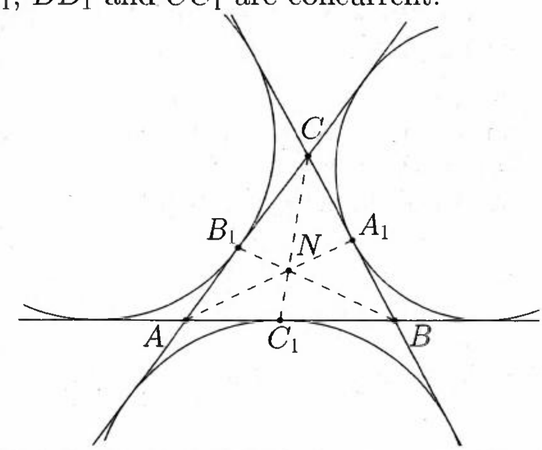
Note. Problem 4.4.1 tells us that AA_1, BB_1 and CC_1 are the three symmedians of $\triangle ABC$. The point L is called the Lemoine point of $\triangle ABC$.

Problem 2.7. Let ABC be a triangle. Let A_1, B_1 and C_1 be the points of tangency of the segments BC, CA and AB and the excircles of $\triangle ABC$. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.

Solution. We have $AC_1 = A_1C$, $B_1A = BA_1$ and $C_1B = CB_1$. Thus,

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

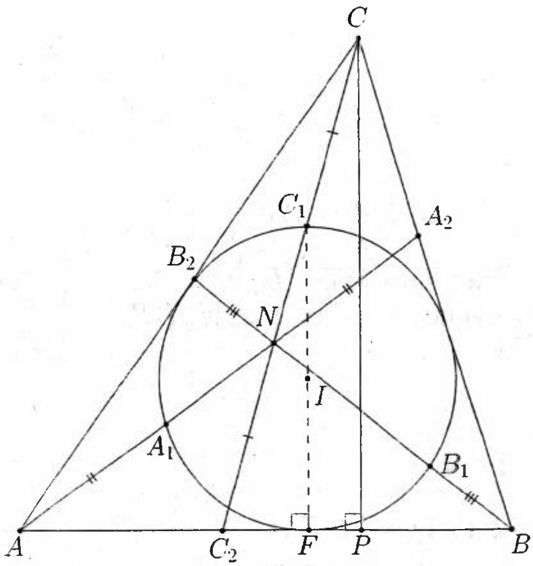
Ceva's Theorem (Problem 4.9.15), applied to $\triangle ABC$ and the lines AA_1, BB_1 and CC_1, yields the desired result.



Note. The intersection point of AA_1, BB_1 and CC_1 is called the Nagel point of $\triangle ABC$.

Problem 2.8. Let ABC be a triangle and let N be its Nagel point. Let AN, BN and CN intersect the incircle of $\triangle ABC$ at the points A_1, B_1 and C_1 (as shown in the figure), and the sides BC, CA and AB at the points A_2, B_2 and C_2, respectively. Prove that $AA_1 = NA_2$, $BB_1 = NB_2$ and $CC_1 = NC_2$.

Solution. We will prove that $CC_1 = NC_2$, and the remaining two statements will follow analogously. Let P be the foot of the altitude of $\triangle ABC$ through C, and let I be the incenter of $\triangle ABC$. Let the incircle of $\triangle ABC$ touch AB at the point F.



Let C_3 be the diametrically opposite point to F with respect to the incircle of $\triangle ABC$.

Consider a homothety that sends the triangle, formed by the lines CA, CB and the tangent line to the incircle at the point C_3, to $\triangle ABC$. It shows that the points C_2, C_3 and C are collinear and so $C_1 \equiv C_3$.

Now the intercept theorem yields $\dfrac{C_2C_1}{C_1C} = \dfrac{2r}{CP - 2r}$. We apply Menelaus' Theorem to $\triangle AC_2C$ and the line B_2NB to get

$$\frac{CN}{NC_2} = \frac{CB_2}{B_2A} \cdot \frac{AB}{BC_2} = \frac{s-a}{s-c} \cdot \frac{c}{s-a} = \frac{c}{s-c}.$$

But the equality that we want to prove is equivalent to $\dfrac{C_2C_1}{C_1C} = \dfrac{CN}{NC_2}$, i. e. $\dfrac{2r}{CP-2r} = \dfrac{c}{s-c}$, which can be simplified to $2rs = c.CP$. The last equality holds because $2rs = 2S_{ABC} = c.CP$.

Problem 2.9. Let ABC be a triangle. The equilateral triangles $\triangle ABC_1$, $\triangle AB_1C$ and $\triangle A_1BC$ are constructed externally to its sides. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.

Solution. Let the circumcircles of $\triangle ABC_1$ and $\triangle A_1BC$ intersect at the point T_1. Then $\angle AT_1B = 120° = \angle BT_1C$ and hence

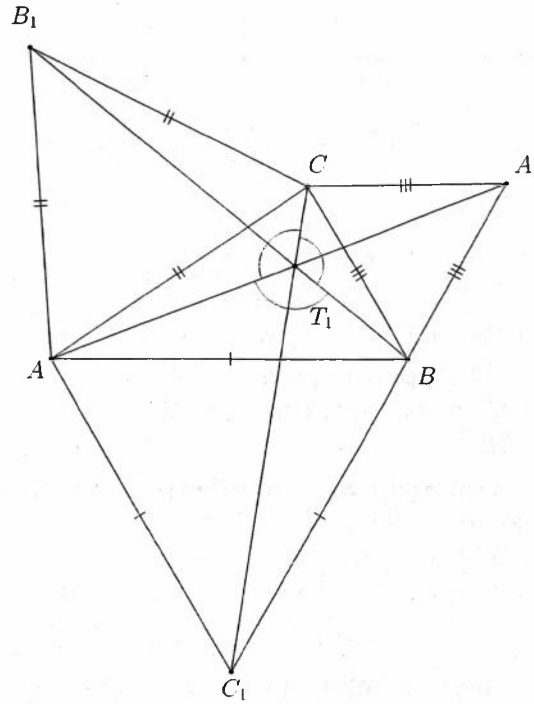
$$\angle CT_1A = 120°.$$

Therefore, AT_1CB_1 is cyclic. The cyclic quadrilateral AC_1BT_1 yields

$$\angle AT_1C_1 = 60° = \angle BT_1C_1.$$

Analogously,

$$\begin{aligned}\angle BT_1A_1 &= \angle CT_1A_1\\ &= \angle CT_1B_1\\ &= \angle AT_1B_1 = 60°.\end{aligned}$$



We have that $\angle AT_1A_1 = 180°$, and so the points A, T_1 and A_1 are collinear. Analogously, it follows that T_1 lies on the lines BB_1 and CC_1.

Note. The point T_1 is called the first Fermat-Torricelli point.

Problem 2.10. Let ABC be a triangle. The equilateral triangles $\triangle ABC_1$, $\triangle AB_1C$ and $\triangle A_1BC$ are constructed internally to its sides. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.

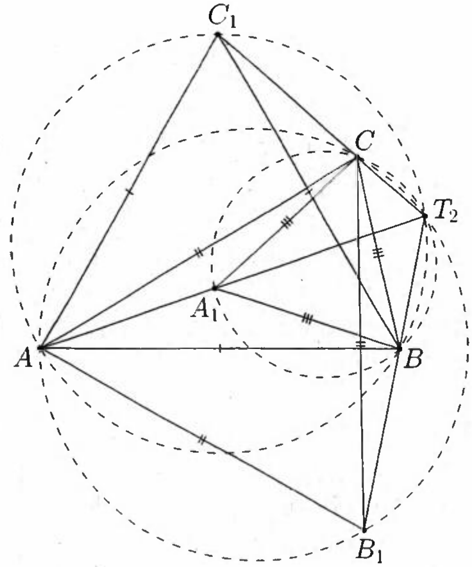
Solution. Let the circumcircles of $\triangle ABC_1$ and $\triangle A_1BC$ intersect at the point T_2. Then

$$\begin{aligned}\angle AT_2C &= \angle BT_2C - \angle AT_2B\\ &= 120° - 60° = \angle AB_1C.\end{aligned}$$

Therefore, the quadrilateral AB_1T_2C is cyclic. Now we have that

$$\angle AT_2B = 60° = \angle AT_2B_1,$$

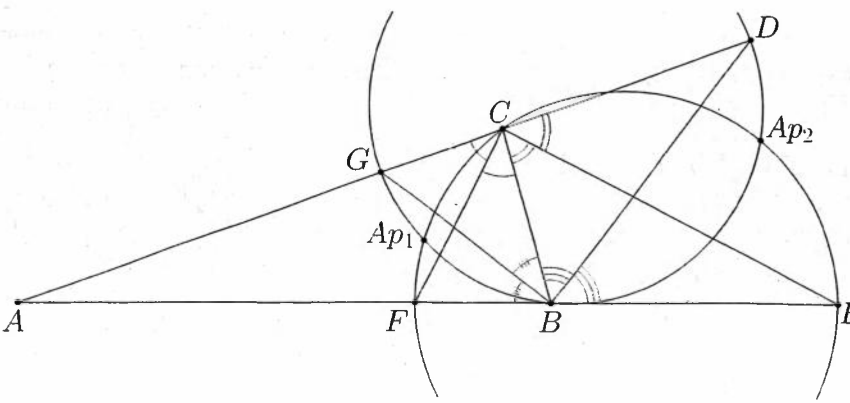
and hence the points T_2, B and B_1 are collinear. Analogously, we get that the points T_2, A and A_1 are collinear and that the points T_2, C and C_1 are collinear.



Note. The point T_2 is called the second Fermat-Torricelli point.

Problems 2.11 and 2.12. Prove that for a given $\triangle ABC$, there exist exactly two points X such that the equality $AX.BC = BX.AC = CX.AB$ holds.

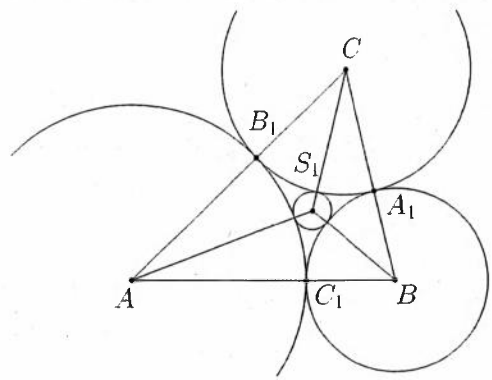
Solution. We have $\dfrac{AX}{BX} = \dfrac{AC}{BC}$. It follows that X lies on the Apollonius circle of $\triangle ABC$ with respect to C (see Problem 2.26).



Analogously, the point X lies on the Apollonius circle of $\triangle ABC$ with respect to B and on the Apollonius circle of $\triangle ABC$ with respect to A. Since these three circles have exactly two common points (see Problem 2.26), X must be one of them.

Note. These points are called the first and the second isodynamic points.

Problem 2.13. Let ABC be a triangle. Prove that there exists a unique point S such that the equality $BC + AS = CA + BS = AB + CS$ holds.



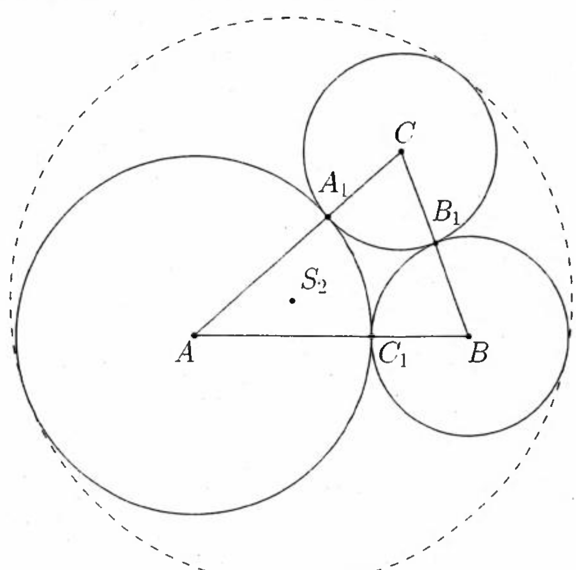
Solution. Let ω be the incircle of $\triangle ABC$ and let ω touch the sides AB, BC and CA at the points C_1, A_1 and B_1, respectively.

Consider the circles $k_1(A, AC_1)$, $k_2(B, BA_1)$ and $k_3(C, CB_1)$. It is clear that each one of these circles touches the other two. Let us construct a circle k_4 with center S_1 and radius r_1, which touches externally k_1, k_2 and k_3.

Then $AS_1 = AC_1 + r_1 = s - a + r_1$ and it follows that $AS_1 + BC = s + r_1$. Analogously, we get that $BS_1 + CA$ and $CS_1 + AB$ both equal $s + r_1$.

Note. The point S_1 is called the first Soddy center.

Problem 2.14. Let ABC be a triangle. Prove that there exists a unique point S such that the equality $BC - AS = CA - BS = AB - CS$ holds.



Solution. Let ω be the incircle of $\triangle ABC$ and let ω touch the sides AB, BC and CA at the points C_1, A_1 and B_1, respectively.

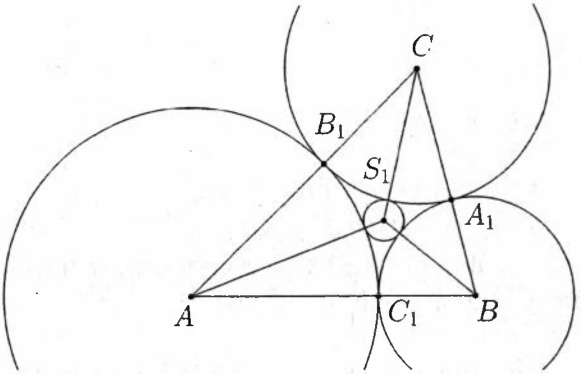
We construct the circles $k_1(A, AC_1)$, $k_2(B, BA_1)$ and $k_3(C, CB_1)$. It is clear that each one of these circles touches the other two. Let us construct a circle k_4 such that k_1, k_2 and k_3 touch it internally. Let S_2 be its center, and let r_2 be its radius.

Then $AS_2 = r_2 - (s - a) = r_2 + a - s$ and it follows that $BC - AS_2 = a - r_2 - a + s = s - r_2$. Analogously, we get that $BS_1 + CA$ and $CS_1 + AB$ are both equal to $s - r_2$.

Note. The point S_2 is called the second Soddy center.

Problem 2.15. Three circles $k_1(A)$, $k_2(B)$ and $k_3(C)$ are given, and they all touch each other externally. Let C_1 and B_1 be the points of tangency of k_1 and k_2, and of k_1 and k_3, respectively. Let A_1 be the point of tangency of k_2 and k_3. The circle k_4 touches the other three circles externally. Prove that the first Soddy center of $\triangle ABC$ coincides with the center of k_4.

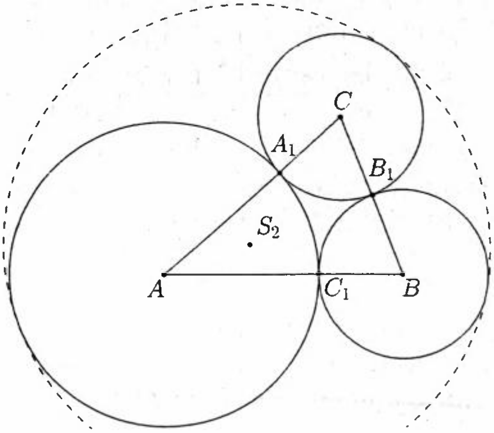
Solution. The points A_1, B_1 and C_1 coincide with the points of tangency of the incircle of $\triangle ABC$ and the sides of the triangle. Thus, the statement follows from Problem 2.13, in which we use k_4 to prove the existence of the first Soddy center. We will refer to k_4 as the first Soddy circle of $\triangle ABC$.



Problem 2.16. Three circles $k_1(A)$, $k_2(B)$ and $k_3(C)$ are given, and they all touch each other externally. Let C_1 and B_1 be the points of tangency of k_1 and k_2, and of k_1 and k_3, respectively. Let A_1 be the point of tangency of k_2 and k_3. The circle k_4 touches the other three circles internally. Prove that the second Soddy center of $\triangle ABC$ coincides with the center of k_4.

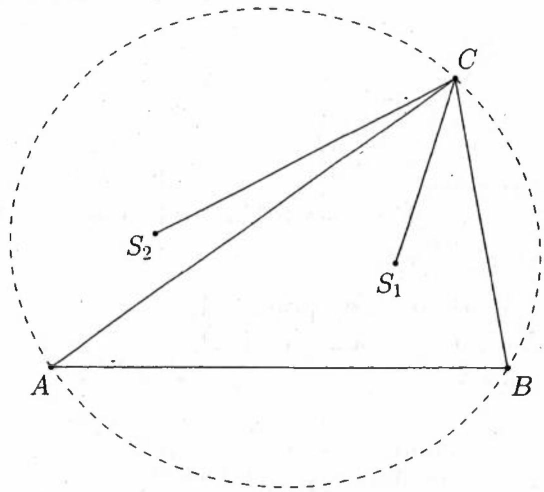
Solution. The points A_1, B_1 and C_1 are the points of tangency of the incircle of $\triangle ABC$ and the sides of the triangle.

Thus, the statement follows from Problem 2.14, in which we use k_4 to prove the existence of the second Soddy center. We will refer to k_4 as the second Soddy circle of $\triangle ABC$.



Problem 2.17. Let ABC be a triangle and let S_1 and S_2 be its first and second Soddy centers, respectively. Prove that the points A, B and C lie on an ellipse with foci S_1 and S_2.

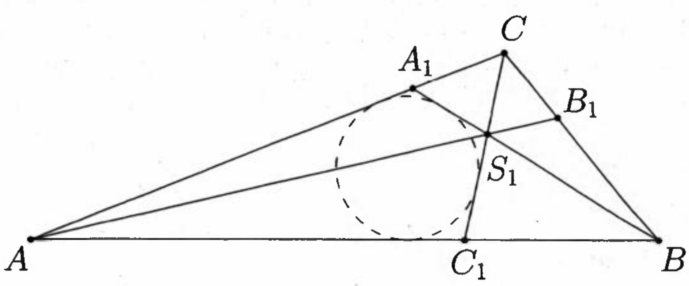
Solution. The definition of Soddy centers yields $S_1A + S_2A =$



$$\begin{aligned} &= (S_1A + BC) + (S_2A - BC) \\ &= (S_1B + AC) + (S_2B - AC) \\ &= S_1B + S_2B \\ &= (S_1C + AB) + (S_2C - AB) \\ &= S_1C + S_2C. \end{aligned}$$

An ellipse with foci S_1 and S_2 is the locus of all points X for which $S_1X + S_2X = const$. From here it follows that the points A, B and C lie on such an ellipse.

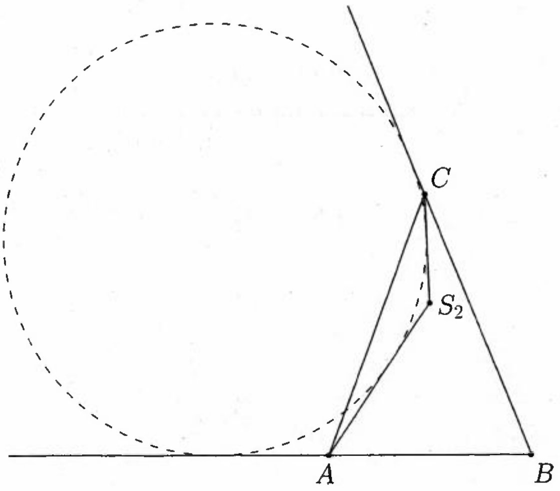
Problem 2.18. Let ABC be a triangle and let S_1 be its first Soddy center. Prove that there exists a circle, inscribed in the convex quadrilateral, formed by the lines CS_1, BS_1, AC and AB.



Solution. We use the notations from the figure to prove that the quadrilateral $AA_1S_1C_1$ is circumscribed. Problem 2.13 yields $BS_1 + AC = CS_1 + AB$, and the result follows.

Problem 2.19. Let ABC be a triangle and let S_2 be its second Soddy center. Prove that there exists a circle that touches the lines BA and BC, and the segments AS_2 and CS_2.

Solution. Problem 2.14 yields $CS_2 + BC = AS_2 + AB$, and the result follows from the concave quadrilateral $ABCS_2$.



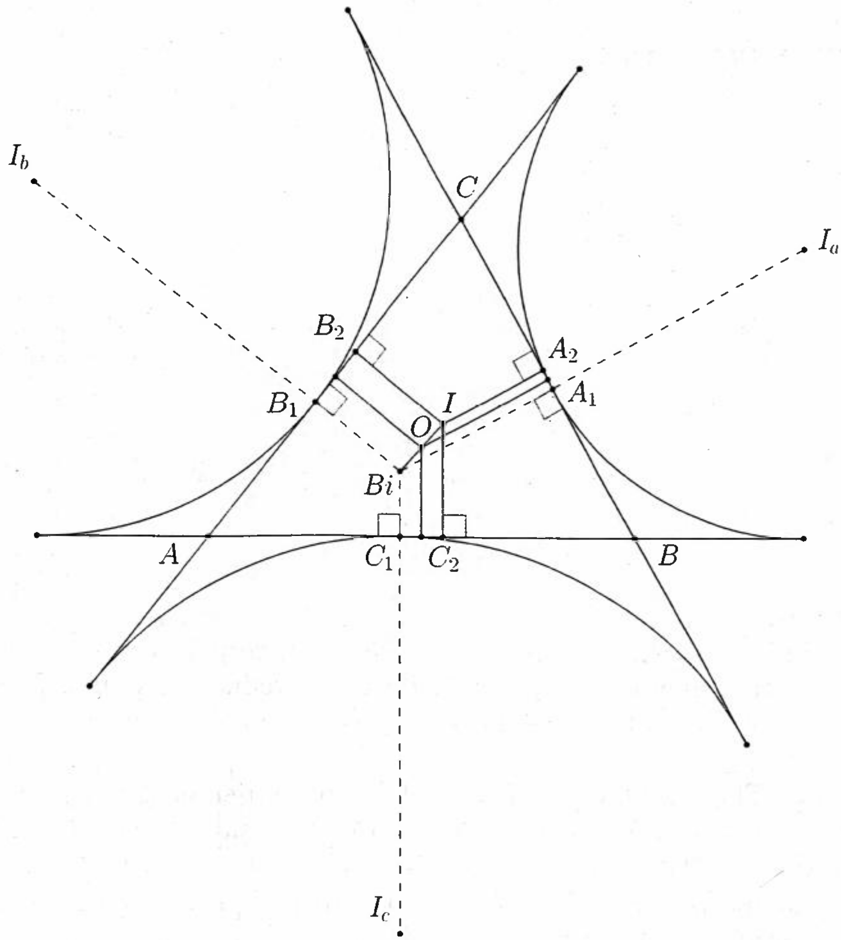
Problem 2.20. Let ABC be a triangle with excircles ω_a, ω_b and ω_c. Let I_a, I_b and I_c be the centers of ω_a, ω_b and ω_c, respectively. Let A_1 be the point of tangency of ω_a and the side BC. Define the points B_1 and C_1 analogously. Prove that the lines C_1I_c, B_1I_b and A_1I_a are concurrent.

Solution. The lines C_1I_c, B_1I_b and A_1I_a are perpendicular to BA, AC and CB, respectively. We will prove that the perpendiculars, drawn from A_1, B_1 and C_1 to the respective sides of $\triangle ABC$, are concurrent.

Let I be the incenter of $\triangle ABC$. Denote the projections of I onto the sides AB, BC and CA by C_2, A_2 and B_2, respectively. Then A_1 and A_2 are symmetric with respect to the midpoint of BC. Analogously, the points B_1 and B_2 are symmetric with respect to the midpoint of CA, and the points C_1 and C_2 are symmetric with respect to the midpoint of AB.

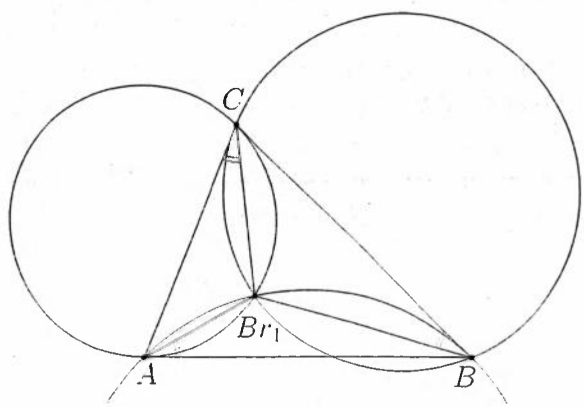
But we know that the perpendiculars, drawn from A_2, B_2 and C_2 to the respective sides, intersect at the point I.

We also know that the perpendiculars, drawn from the midpoints of the sides to the respective sides, intersect at the circumcenter O. Therefore, the lines C_1I_c, B_1I_b and A_1I_a intersect at the symmetric point of I with respect to O. This point is called the Bevan point of $\triangle ABC$. Note that here we also solved Problem 3.5.



Problem 2.21. Let ABC be a triangle. The first Brocard point Br_1 is defined as the point for which $\angle BABr_1 = \angle ACBr_1 = \angle CBBr_1$. Prove that it always exists.

Solution. Construct the circle through A and C which touches AB, and construct the circle through B and C which touches AC. Let their second intersection point be Br_1.



The alternate segments theorem yields

$$\angle BABr_1 = \angle ACBr_1 = \angle CBBr_1$$

and the problem is solved.

Note that the circle that passes through A and B and touches BC also passes through Br_1. Let us calculate Brocard's angle $\angle BABr_1 = \varphi$. We will apply the modified law of cosines three times:

$$Br_1B^2 = Br_1A^2 + AB^2 - 4S_{\triangle ABBr_1} \cot\varphi,$$

$$Br_1A^2 = Br_1C^2 + CA^2 - 4S_{\triangle CABr_1} \cot\varphi,$$

$$Br_1C^2 = Br_1B^2 + BC^2 - 4S_{\triangle BCBr_1} \cot\varphi.$$

Since $S = S_{\triangle ABC} = S_{\triangle ABBr_1} + S_{\triangle CABr_1} + S_{\triangle BCBr_1}$, it follows that

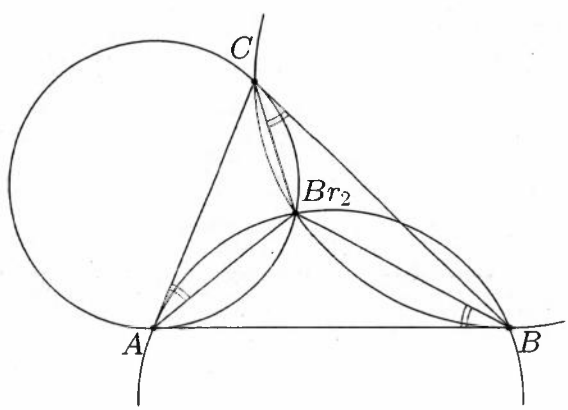
$$\cot\varphi = \frac{AB^2 + CA^2 + BC^2}{4S}.$$

It turns out that the angle defined by the second Brocard point is the same.

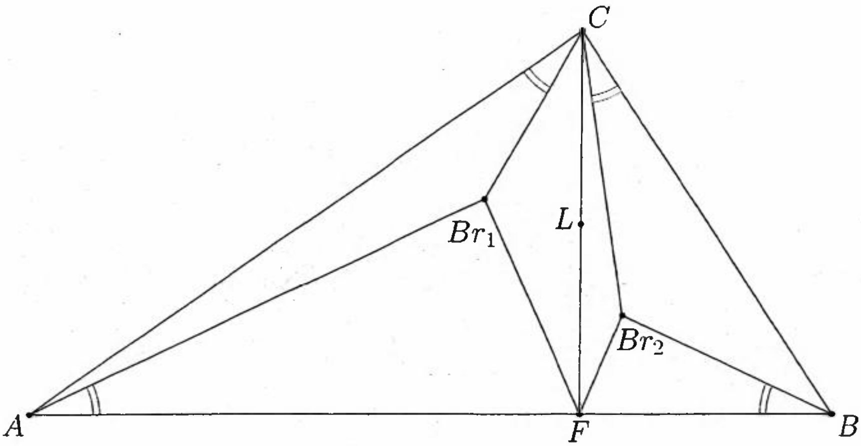
Problem 2.22. Let ABC be a triangle. The second Brocard point Br_1 is defined as the point for which $\angle ABBr_2 = \angle CABr_2 = \angle BCBr_2$. Prove that it always exists.

Solution. Construct the circle through A and C, which touches CB, and construct the circle through B and C which touches AB. Let their second intersection point be Br_2. The alternate segments theorem yields

$$\angle ABBr_2 = \angle BCBr_2 = \angle CABr_2.$$



Problem 2.23. Let ABC be a triangle. Let L be its Lemoine point and let Br_1 and Br_2 be its first and second Brocard points, respectively. Let $CL \cap AB = F$. Prove that $\angle AFBr_1 = \angle BFBr_2$.



Solution. The definition of the Brocard points and Problems 2.21 and 2.22 yield

$$\angle FABr_1 = \angle ACBr_1 = \angle Br_2CB = \angle Br_2BF = \varphi.$$

Since CL is isogonally conjugate to the median through C in $\triangle ABC$ (see Problem 4.4.1), we have that $\dfrac{\sin \angle ACL}{\sin \angle LCB} = \dfrac{\sin \beta}{\sin \alpha}$. Therefore,

$$\frac{AF}{BF} = \frac{b}{a} \cdot \frac{\sin \angle ACL}{\sin \angle LCB} = \frac{b^2}{a^2}$$

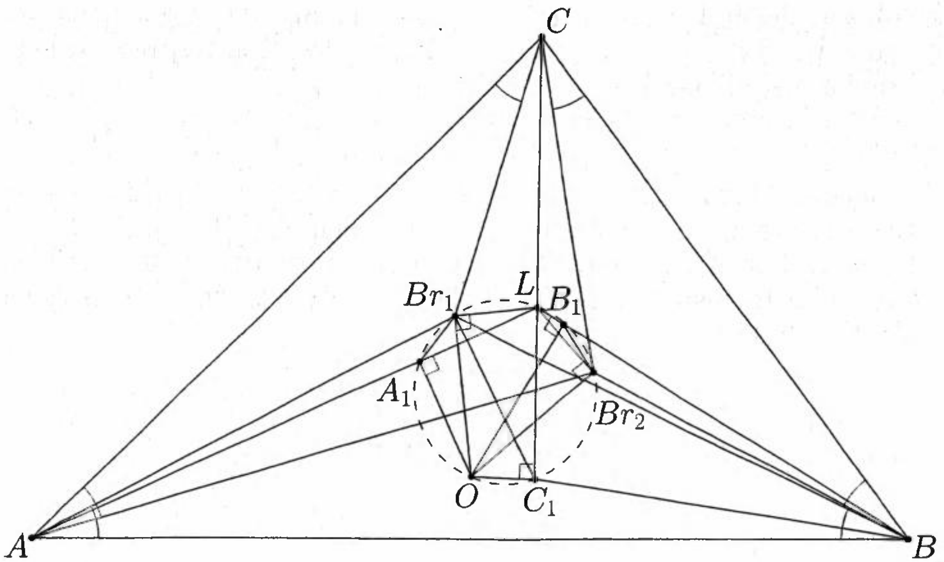
and thus $AF = \dfrac{b^2 c}{a^2 + b^2}$ and $BF = \dfrac{a^2 c}{a^2 + b^2}$.

We have that $\angle BBr_2C = 180° - \beta$, and so $BBr_2 = \dfrac{a \sin \varphi}{\sin \beta}$ and similarly $ABr_1 = \dfrac{b \sin \varphi}{\sin \alpha}$. Now we get

$$\frac{ABr_1}{AF} = \frac{\sin \varphi (a^2 + b^2)}{bc \sin \alpha} = \frac{\sin \varphi (a^2 + b^2)}{2S} = \frac{\sin \varphi (a^2 + b^2)}{ac \sin \beta} = \frac{BBr_2}{BF}.$$

Hence, $\triangle AFBr_1 \sim \triangle BFBr_2$, and thus $\angle AFBr_1 = \angle BFBr_2$.

Problem 2.24. Let ABC be a triangle with circumcenter O. Let L be its Lemoine point and let Br_1 and Br_2 be its first and second Brocard points, respectively. Prove that the equalities $\angle OBr_1L = \angle OBr_2L = 90°$ and $Br_1L = Br_2L$ hold.



Solution. Let us denote the projections of O onto AL, BL and CL by A_1, B_1 and C_1. We will prove that the quadrilateral ABC_1O is cyclic. Let the tangent lines to the circumcircle of $\triangle ABC$ at the points A and B intersect at T.

Then the points A, B and C_1 lie on the circle with diameter TO because C, L, C_1 and T are collinear. Moreover, $\angle TC_1B = \angle TC_1A = \gamma$ and thus $\angle AC_1C = \angle BC_1C = 180° - \gamma$.

It follows that $\angle C_1BC = \angle C_1CA$ and $\angle C_1CB = \angle C_1AC$. Therefore, the points C_1 and Br_2 lie on the circle that passes through A and C and touches BC. Furthermore, C_1 and Br_1 lie on the circle that passes through B and C and touches AC.

Analogously, we get that the points B_1 and Br_2 lie on the circle that passes through B and C and touches AB, that the points B_1 and Br_1 lie on the circle that passes through A and B and touches BC, that the points A_1 and Br_2 lie on the circle that passes through A and B and touches AC, and that the points A_1 and Br_1 lie on the circle that passes through A and C and touches AB. Now we have that $\angle Br_1A_1L = \angle Br_1CA = \angle Br_1BC = \angle Br_1C_1L$.

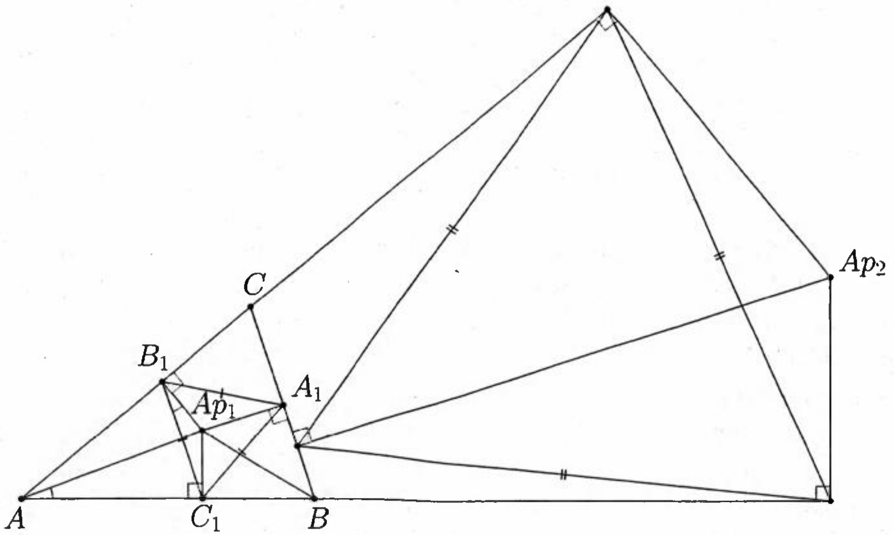
Thus, Br_1 lies on the circle with diameter OL. Analogously, we obtain that Br_2 also lies there. Now we have

$$\begin{aligned}\angle Br_1A_1L &= \angle ACBr_1 = \angle Br_2CB \\ &= \angle Br_2B_1B = \angle LBr_1Br_2 \Rightarrow LBr_1 = LBr_2.\end{aligned}$$

Problem 2.25. Let ABC be a triangle. Let Ap_1 and Ap_2 be the two isodynamic points. Prove that the pedal triangles with respect to these two points are equilateral.

Solution. Let X be any one of Ap_1 and Ap_2.

Problems 2.11 and 2.12 yield $AX.BC = BX.CA = CX.AB$. Therefore, there exists a real constant $k > 0$ such that $AX.BC = BX.CA = CX.AB = k$. Let A_1, B_1 and C_1 be the projections of X onto the lines BC, CA and AB, respectively. Then the quadrilateral AC_1XB_1 is cyclic and $\angle B_1XC_1 = 180° - \alpha$.



Denote $\angle XAC_1 = x$. Thus, $\angle C_1B_1X = x$.

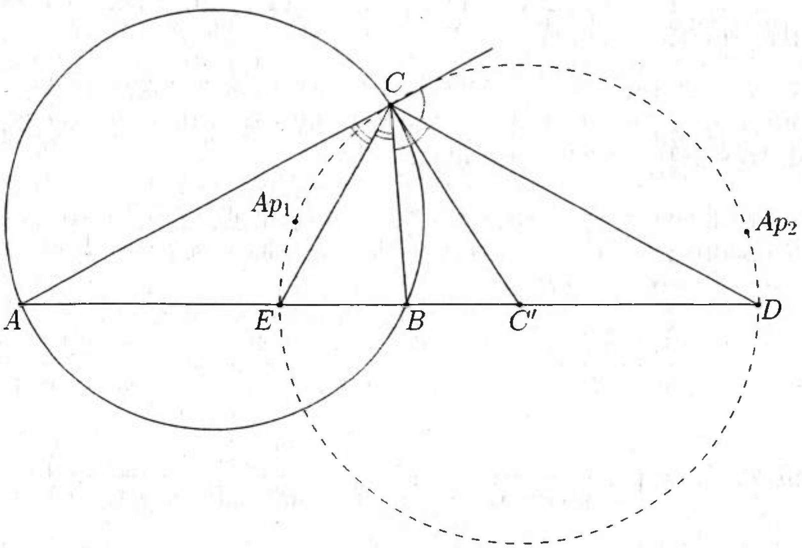
From the right-angled $\triangle AC_1X$ we get $\dfrac{C_1X}{\sin x} = AX$, and the law of sines in $\triangle B_1XC_1$ yields $\dfrac{C_1X}{\sin x} = \dfrac{B_1C_1}{\sin(180° - \alpha)}$.

Therefore, $B_1C_1 = AX.\sin\alpha = \dfrac{k}{BC}\sin\alpha = \dfrac{k}{2R}$. Similarly, we prove that $A_1B_1 = C_1A_1 = \dfrac{k}{2R}$. Then $A_1B_1 = B_1C_1 = C_1A_1$, and so $\triangle A_1B_1C_1$ is equilateral.

Problem 2.26. Let ABC be a triangle and let E and D be the feet of the internal and external bisectors at C, respectively. Prove that the two isodynamic points of $\triangle ABC$ (defined as in Problems 2.11 and 2.12) lie on the circle with diameter ED.

Solution. We will prove that the circle with diameter ED is the locus of points X for which the foot of the bisector of $\angle AXB$ is the point E. Since $\dfrac{AE}{EB} = \dfrac{AC}{CB} = \dfrac{AD}{DB}$, then D is the foot of the external bisector of $\angle AXB$. Thus, $\angle EXD = 90°$.

If $\angle EXD = 90°$, then from $(A, E, B, D) = 1$ we get that XE and XD are the internal and external bisectors of $\angle AXB$, respectively.



Let C' be the midpoint of ED. Then $\angle C'CB = \angle ECC' - \angle ECB = \angle BAC$, and so $\dfrac{BC'}{AC'} = \dfrac{BC'}{C'C} \cdot \dfrac{CC'}{C'A} = \dfrac{\sin^2 \alpha}{\sin^2 \beta}$.

We define the points B' and A' analogously. Menelaus' Theorem, applied to $\triangle ABC$, yields that the points A', B' and C' are collinear. It is easy to see that each two of the Apollonius circles intersect each other.

But if O is the circumcenter of $\triangle ABC$, then OC is tangent to the circle with center C' and radius CC', OB is tangent to the circle with center B' and radius BB', and OA is tangent to the circle with center A' and radius AA'.

Therefore, O lies on the radical axes of the three circles. These axes are parallel to each other because the centers are collinear. Thus, these radical axes coincide. It follows that the three Apollonius circles intersect at two common points, and these points are the two isodynamic points.

Problem 2.27. Let ABC be a triangle and let T_1 be its first Fermat-Torricelli point. Prove that $\angle AT_1B = \angle BT_1C = 120°$.

Solution. See Problem 2.9.

Problem 2.28. Let ABC be a triangle and let T_2 be its second Fermat-Torricelli point. Prove that exactly one of the equalities $\angle AT_2B = \angle AT_2C = 60°$, $\angle BT_2A = \angle BT_2C = 60°$ and $\angle CT_2B = \angle CT_2A = 60°$ holds.

Solution. See Problem 2.10. The position of T_2 determines which one of the three equations holds.

Now we are going to derive the barycentric coordinates of the isodynamic points and the Fermat-Torricelli points because we are going to need them to solve the next problems.

Given a $\triangle ABC$, let T_1 be its first Fermat-Torricelli point. Problem 2.9 (see its figure) yields that if $\triangle ABC_1$ is equilateral and external to $\triangle ABC$, then $T_1 \in CC_1$. We have

$$\frac{S_{AT_1C}}{S_{BT_1C}} = \frac{b}{a} \cdot \frac{\sin \angle ACC_1}{\sin \angle BCC_1} = \frac{b}{a} \cdot \frac{AC_1 . \sin \angle CAC_1}{BC_1 . \sin \angle CBC_1} = \frac{b . \sin(60° + \alpha)}{a . \sin(60° + \beta)}$$

Therefore, $T_1 = \left(\dfrac{a}{\sin(\alpha + 60°)}, \dfrac{b}{\sin(\beta + 60°)}, \dfrac{c}{\sin(\gamma + 60°)} \right)$.

Analogously, we find $T_2 = \left(\dfrac{a}{\sin(\alpha - 60°)}, \dfrac{b}{\sin(\beta - 60°)}, \dfrac{c}{\sin(\gamma - 60°)} \right)$.

For the first isodynamic point Ap_1 let $m = AAp_1.a = BAp_1.b = CAp_1.c$ and $\angle AAp_1B = x$. We apply Bretschneider's Theorem to the quadrilateral $CBAp_1A$ and obtain

$$(AAp_1.a)^2 + (BAp_1.b)^2 - 2AAp_1.a.BAp_1.b\cos(x - \gamma) = (CAp_1.c)^2$$

Therefore, $m^2 + m^2 - 2m^2 \cos(x - \gamma) = m^2$, and thus $\cos(x - \gamma) = \dfrac{1}{2}$. Since $x - \gamma < 180°$, then $x = \gamma + 60°$. We obtain $\angle AAp_1B = \gamma + 60°$, $\angle BAp_1C = 60° + \alpha$ and $\angle CAp_1A = 60° + \beta$. Now for the barycentric coordinates of Ap_1 we have

$$Ap_1 = \left(\frac{S_{BCAp_1}}{AAp_1.BAp_1.CAp_1}, \frac{S_{ACAp_1}}{AAp_1.BAp_1.CAp_1}, \frac{S_{ABAp_1}}{AAp_1.BAp_1.CAp_1}\right)$$

or $Ap_1 = \left(\dfrac{\sin(60^\circ+\alpha)}{AAp_1}, \dfrac{\sin(60^\circ+\beta)}{BAp_1}, \dfrac{\sin(60^\circ+\gamma)}{CAp_1}\right).$

We multiply the three coordinates by m and we get

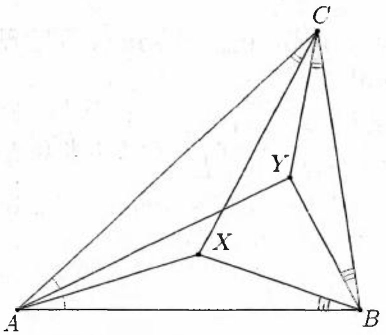
$$Ap_1 = (a.\sin(60^\circ+\alpha), b.\sin(60^\circ+\beta), c.\sin(60^\circ+\gamma)).$$

Analogously, we find $Ap_2 = (a.\sin(\alpha-60^\circ), b.\sin(\beta-60^\circ), c.\sin(\gamma-60^\circ))$

Problems 2.29 and 2.30. Let ABC be a triangle. Prove that the first isodynamic point and the first Fermat-Torricelli point are isogonal conjugates with respect to $\triangle ABC$, as well as that the second isodynamic point and the second Fermat-Torricelli point are isogonal conjugates with respect to $\triangle ABC$.

Solution. First we are going to prove a lemma for the barycentric coordinates of isogonal conjugates with respect to a given triangle.

Lemma. Given a $\triangle ABC$, let X have barycentric coordinates $X = (u, v, w)$. Let Y and X be isogonal conjugates with respect to $\triangle ABC$. Then the barycentric coordinates of Y are $\left(\dfrac{a^2}{u}, \dfrac{b^2}{v}, \dfrac{c^2}{w}\right)$.



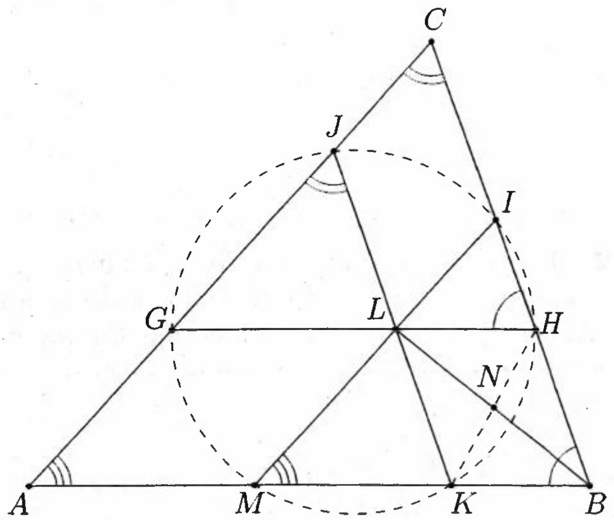
Proof. Let $Y = (u_1, v_1, w_1)$. Then

$$\frac{v}{u} = \frac{S_{AXC}}{S_{BXC}} = \frac{b.\sin\angle ACX}{a.\sin\angle BCX} = \frac{b^2}{a^2}\cdot\frac{a.\sin\angle BCY}{b.\sin\angle ACY} = \frac{b^2}{a^2}\cdot\frac{S_{BCY}}{S_{ACY}} = \frac{b^2}{a^2}\cdot\frac{u_1}{v_1}$$

Therefore, $u_1 : v_1 = \dfrac{a^2}{u} : \dfrac{b^2}{v}$, and so $Y = (u_1, v_1, w_1) = \left(\dfrac{a^2}{u}, \dfrac{b^2}{v}, \dfrac{c^2}{w}\right)$.

Now we use the lemma, and the barycentric coordinates of the isodynamic points and the Fermat-Torricelli points to get the desired result.

Problem 2.31. Let ABC be a triangle and let L be its Lemoine point. The points M, $K \in AB$, H, $I \in BC$ and J, $G \in AC$ are chosen such that $MI \parallel AC$, $GH \parallel AB$, $KJ \parallel BC$ and $MI \cap KJ \cap GH = L$. Prove that the points M, K, H, I, J and G lie on a circle.



Solution. We will prove that the quadrilaterals $MKHI$, $HIJG$ and $IJGM$ are cyclic. We have that BL is a symmedian in $\triangle MBI$ because of the homothety with center B that sends CA to IM. Let N be the midpoint of KH.

Then B, N and L are collinear because $KBHL$ is a parallelogram. Now the law of sines yields

$$\frac{BI}{BM} = \frac{\sin \angle IBL}{\sin \angle MBL} = \frac{BK}{BH} \Rightarrow BI.BH = BM.BK.$$

Therefore, the quadrilateral $KHIM$ is cyclic. Analogously, the quadrilaterals $HGJI$ and $JGMK$ are also cyclic. If we assume that these three circles are distinct from each other, then their pairwise radical axes will be JG, IH and MK. These lines are not concurrent, a contradiction.

Thus, two of these three circles coincide, and so the points M, K, H, I, J and G lie on a circle.

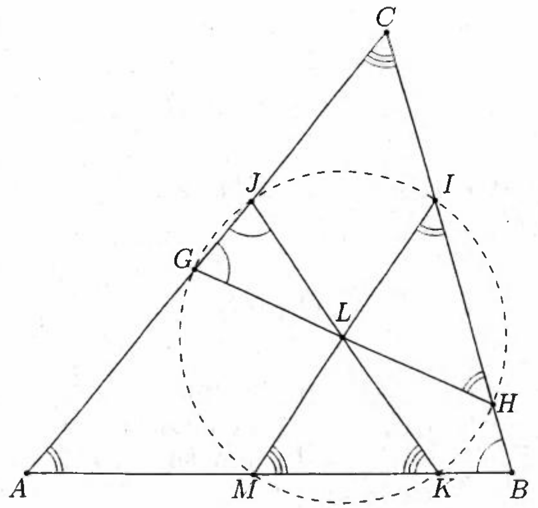
Problem 2.32. Let ABC be a triangle and let L be its Lemoine point. The points M, $K \in AB$, H, $I \in BC$ and J, $G \in AC$ are chosen such that the quadrilaterals $MICA$, $GHBA$ and $KJCB$ are cyclic and $MI \cap KJ \cap GH = L$. Prove that the points M, K, H, I, J and G lie on a circle with center L.

Solution. Since $\triangle MBI \sim \triangle CBA$, then BL is a median in $\triangle MBI$.

Therefore, $ML = LI$ and analogously, $HL = LG$ and $JL = LK$.

But simple angle chasing leads us to the fact that $\triangle MLK$, $\triangle LHI$ and $\triangle LJG$ are isosceles.

It is now easy to obtain that all the desired segments with an endpoint in L are equal to each other.

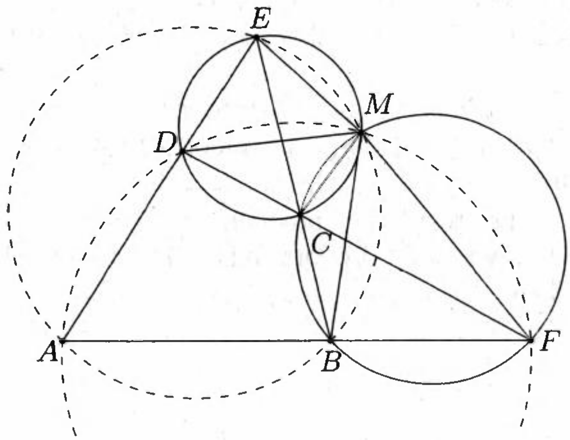


Problem 2.33. Let $ABCD$ be a convex quadrilateral such that $AB \cap CD = F$ and $AD \cap BC = E$. Prove that the circumcircles of $\triangle BFC$, $\triangle AFD$, $\triangle DCE$ and $\triangle ABE$ pass through a single point.

Solution. Let the circumcircles of $\triangle BFC$ and $\triangle CDE$ intersect each other at the points C and M. We have

$$\begin{aligned}\angle DMF &= \angle DMC + \angle CMF \\ &= \angle DEC + \angle CBA \\ &= 180^\circ - \angle BAE,\end{aligned}$$

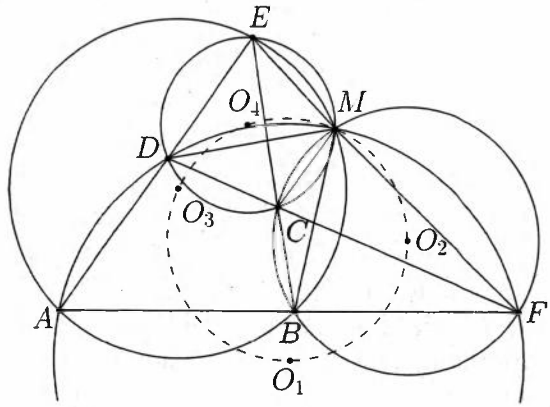
hence the quadrilateral $AFMD$ is cyclic and analogously, the quadrilateral $ABME$ is also cyclic.



Problem 2.34. The construction of Problem 2.33 is given. Prove that point M and the respective centers O_1, O_2, O_3 and O_4 of the circumcircles of $\triangle AFD$, $\triangle BFC$, $\triangle ABE$ and $\triangle DCE$ lie on a circle.

Solution. We have that

$$\angle O_4O_3O_2 = 180° - \angle EMB,$$



because $O_3O_4 \perp EM$ and $O_2O_3 \perp BM$. But

$$\angle EMB = \angle O_4MO_2,$$

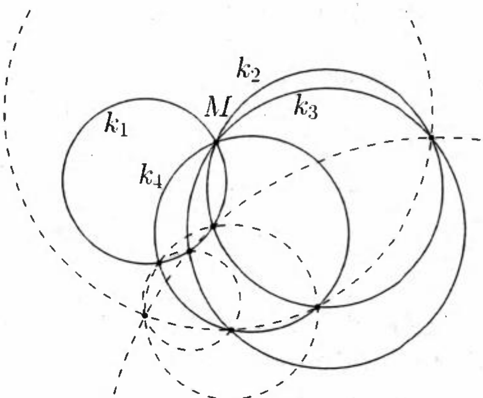
because of the spiral homothety with center M that sends $\triangle EMB$ to $\triangle DMF$, and that sends O_4 to O_2.

Then the quadrilateral $O_3O_4MO_2$ is cyclic. Analogously, the quadrilateral $O_1O_4MO_2$ is also cyclic.

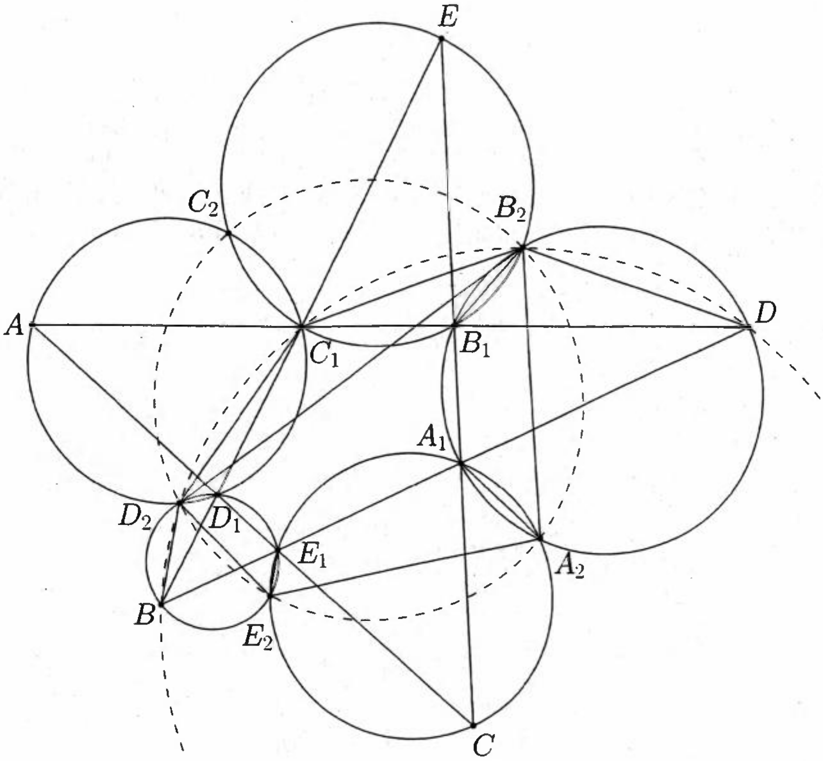
Problem 2.35. The circles k_1, k_2, k_3 and k_4 are given such that they pass through a single point M. Prove that the circles through the intersection points of (k_1, k_2, k_3), (k_1, k_2, k_4), (k_1, k_3, k_4) and (k_2, k_3, k_4), different from M, also pass through a single point.

Solution. Consider an inversion with center M and an arbitrary radius.

The resulting problem is exactly Miquel's Theorem (Problem 2.33).



Problem 2.36. Let $ABCDE$ be a convex pentagon such that $AC \cap BE = D_1$, $BD \cap AC = E_1$, $BD \cap EC = A_1$, $EC \cap AD = B_1$ and $AD \cap BE = C_1$. Let (XYZ) denote the circumcircle of $\triangle XYZ$. Let $(AD_1C_1) \cap (B_1C_1E) = \{C_1, C_2\}$. Let $(B_1C_1E) \cap (A_1B_1D) = \{B_1, B_2\}$. Let $(A_1B_1D) \cap (A_1E_1C) = \{A_1, A_2\}$. Let $(A_1E_1C) \cap (E_1D_1B) = \{E_1, E_2\}$ and let $(E_1D_1B) \cap (C_1D_1A) = \{D_1, D_2\}$. Prove that the points A_2, B_2, C_2, D_2 and E_2 lie on a circle.



Solution. We will prove that the points A_2, B_2, D_2 and E_2 lie on a circle. By analogous reasoning it will follow that the point C_2 lies on the same circle. First, we will prove that BC_1B_2D is cyclic. We have

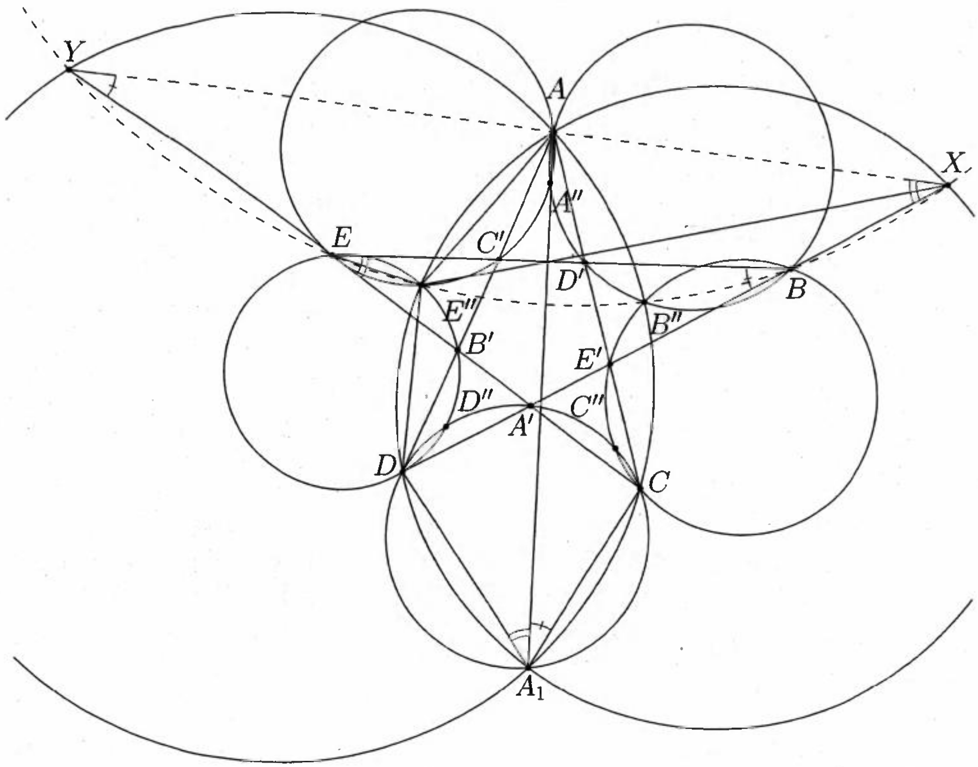
$$\begin{aligned}\angle C_1BD = \angle EBA_1 &= \angle DA_1B_1 - \angle C_1EB_1 \\ &= 180^\circ - \angle B_1B_2D - \angle C_1B_2B_1 = 180^\circ - \angle C_1B_2D,\end{aligned}$$

and thus BC_1B_2D is cyclic. Analogously, D_2 also lies on this circle. Now we have

$$\begin{aligned}\angle B_2D_2E_2 = \angle B_2D_2B - \angle E_2D_2B &= 180^\circ - \angle B_2DA_1 - \angle BE_1E_2 \\ &= 180^\circ - \angle B_2A_2A_1 - \angle A_1A_2E_2 = 180^\circ - \angle B_2A_2E_2,\end{aligned}$$

hence the points A_2, B_2, D_2 and E_2 lie on a circle.

Problem 2.37. Let $ABCDE$ be a convex pentagon such that $AC \cap BE = D'$, $BD \cap AC = E'$, $BD \cap EC = A'$, $EC \cap AD = B'$ and $AD \cap BE = C'$. Let (XYZ) denote the circumcircle of $\triangle XYZ$. Let $(AD'B) \cap (BE'C) = \{B, B''\}$. Let $(BE'C) \cap (CA'D) = \{C, C''\}$. Let $(CA'D) \cap (DB'E) = \{D, D''\}$. Let $(DB'E) \cap (AC'E) = \{E, E''\}$ and let $(AC'E) \cap (AD'B) = \{A, A''\}$. Prove that the lines AA'', BB'', CC'', DD'' and EE'' are concurrent.



Solution. Due to the radical axes theorem, it is enough to show that the quadrilateral $BB''E''E$ is cyclic. Let $(ADE'') \cap (ACB'') = A_1$.

We have

$$\begin{aligned}\angle BEC &= 180^\circ - \angle EB'C' - \angle EC'B' = \angle EC'A + \angle EB'D - 180^\circ \\ &= \angle AE''E + \angle DE''E - 180^\circ = 180^\circ - \angle AE''D = \angle AA_1D.\end{aligned}$$

Analogously, we obtain $\angle EBD = \angle AA_1C$. But

$$\angle BA'C = \angle EBD + \angle BEC = \angle DA_1C.$$

Therefore, the quadrilateral DA_1CA' is cyclic. Let $DB \cap (ADE'') = \{D, X\}$ and let $CE \cap (CB''A) = \{C, Y\}$. Now we have

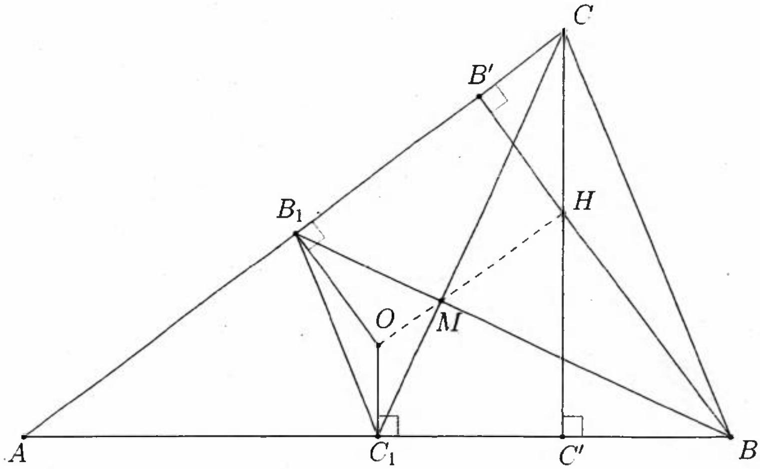
$$\angle XAY = \angle XAA_1 + \angle YAA_1 = \angle XDA_1 + \angle YCA_1 = 180^\circ.$$

Thus, the points X, A and Y are collinear. Therefore, $\angle XYE = \angle AA_1C = \angle EBA'$, and so the quadrilateral $XYEB$ is cyclic. Furthermore,

$$\angle XE''E = \angle XE''A + \angle AE''E = \angle XDA + \angle BC'D = 180^\circ - \angle EBD = \angle XBE.$$

Thus, the quadrilateral $XBE''E$ is cyclic, and analogously, the quadrilateral $YEB''B$ is cyclic, but the quadrilateral $XYEB$ is also cyclic. It follows that $XYEE''B''B$ is a cyclic hexagon.

Problem 3.1. Let ABC be a triangle and let O be its circumcenter. Let M be its centroid and let H be its orthocenter. Prove that the points H, O and M are collinear.



Solution. Let B' and C' be the feet of the altitudes from B and C, and let B_1 and C_1 be the midpoints of AC and AB, respectively. Then we have that $OB_1 \parallel BH$ and $OC_1 \parallel CH$. But $B_1C_1 \parallel BC$ as a midsegment of $\triangle ABC$ and hence $\triangle HBC \sim \triangle OB_1C_1$.

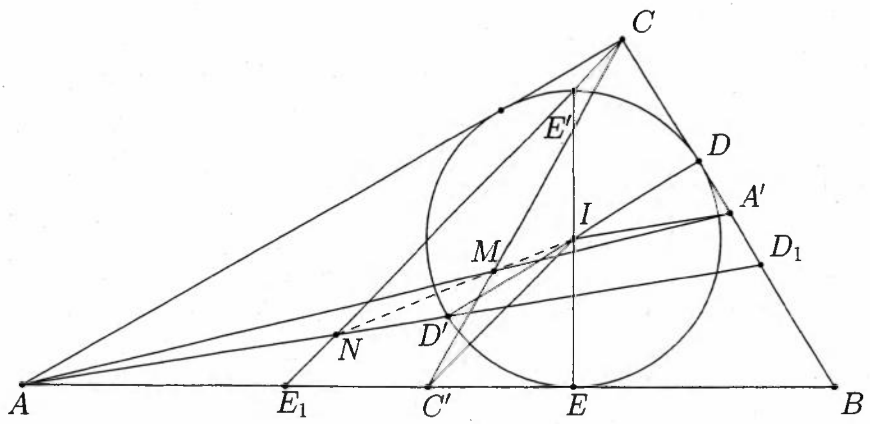
But $B_1C_1 = \dfrac{BC}{2}$ and thus $\dfrac{CH}{OC_1} = \dfrac{BC}{B_1C_1} = 2 = \dfrac{CM}{MC_1}$.

Therefore, $\triangle CMH \sim \triangle C_1MO$, and so $\angle CMH = \angle C_1MO$. Thus, the points H, O and M are collinear.

Problem 3.2. Let ABC be a triangle and let N be its Nagel point. Let M be its centroid and let I be its incenter. Prove that the points N, I and M are collinear.

Solution. The Nagel point is defined as the intersection point of the lines that connect each vertex of the triangle and the point of tangency of the excircle and the corresponding side of the triangle (see Problem 2.7). Let the incircle touch the sides BC and AB at the points D and E, respectively, and let D' and E' be their symmetric points with respect to I, respectively. Let C' and A' be the midpoints of AB and BC.

Let D_1 and E_1 be the points of tangency of the excircles and the segments BC and AB, respectively. The tangent line to the incircle that passes through D' is parallel to BC. The homothety that sends the incircle to the A-excircle shows that the points A, N, D' and D_1 are collinear.



Analogously, we get that the points C, E', N and E_1 are collinear. If we consider the homothety φ with center M and coefficient $-1/2$, it follows that $\varphi(C) = C'$ and $\varphi(A) = A'$. The midsegment in $\triangle EE_1E'$ shows that $C'I \parallel E'E_1$. Analogously, $A'I \parallel D_1D'$.

Therefore, $\varphi(C'I) = CE_1$ and $\varphi(A'I) = AD_1$, and so $I = C'I \cap A'I$ is sent to $N = CE_1 \cap AD_1$. The desired result follows immediately.

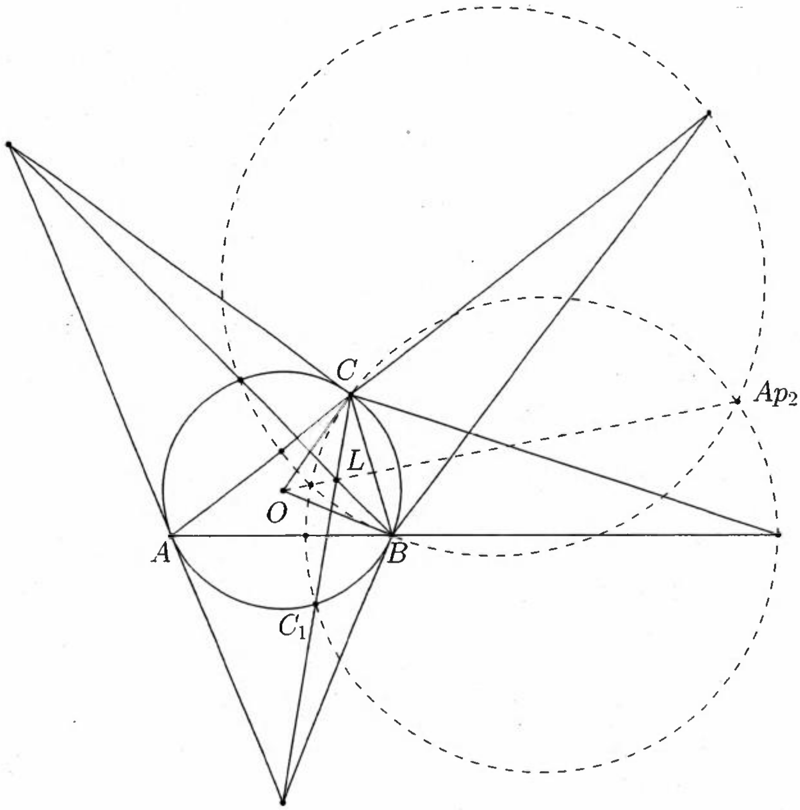
Problem 3.3. Let ABC be a triangle. Prove that the lines, formed by the first and second Fermat-Torricelli points, and the first and second isodynamic points, intersect at the Lemoine point L. Also, prove that the circumcenter of $\triangle ABC$ lies on the line, formed by the isodynamic points.

Solution. First, we will prove that the points O, Ap_1, L and Ap_2 are collinear. Since Ap_1Ap_2 is the radical axis of the three Apollonius circles, it is enough to show that the powers of O and L with respect to two of them are equal.

Problem 4.3.3. yields that the line OC is tangent to the Apollonius circle k_c that passes through C.

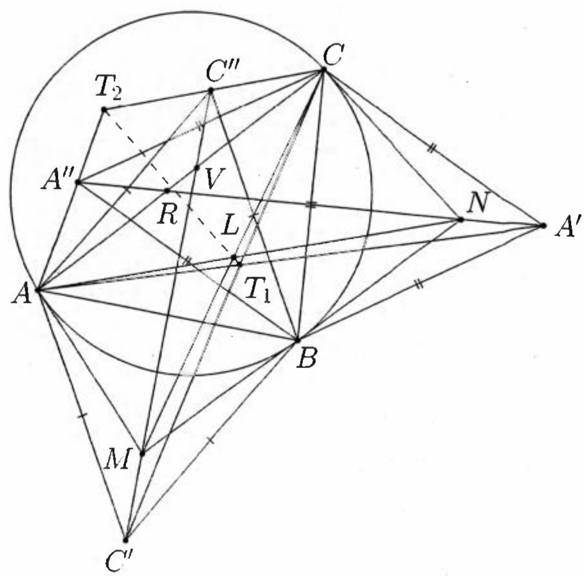
Therefore, the power of O with respect to k_c is R^2. Analogously, its power with respect to k_b and k_a is also R^2. Problem 4.4.5 yields that if $CL \cap k = C_1$, then the quadrilateral CBC_1A is harmonic. Thus, $\dfrac{CA}{CB} = \dfrac{AC_1}{C_1B}$ and hence the point C_1 lies on k_c. It follows that L lies on the radical axis of k and k_c.

Analogously, L lies on the radical axis of k and k_b and hence L also lies on the radical axis of k_c and k_b, which is Ap_1Ap_2.



We are left to prove that the points T_1, L and T_2 are collinear.

Let A' and A'' be the vertices of the equilateral triangles with base BC such that A'' and A lie in the same half-plane with respect to the line BC. Analogously, let C' and C'' be the vertices of the equilateral triangles with base AB such that C'' and C lie in the same half-plane with respect to the line AB.

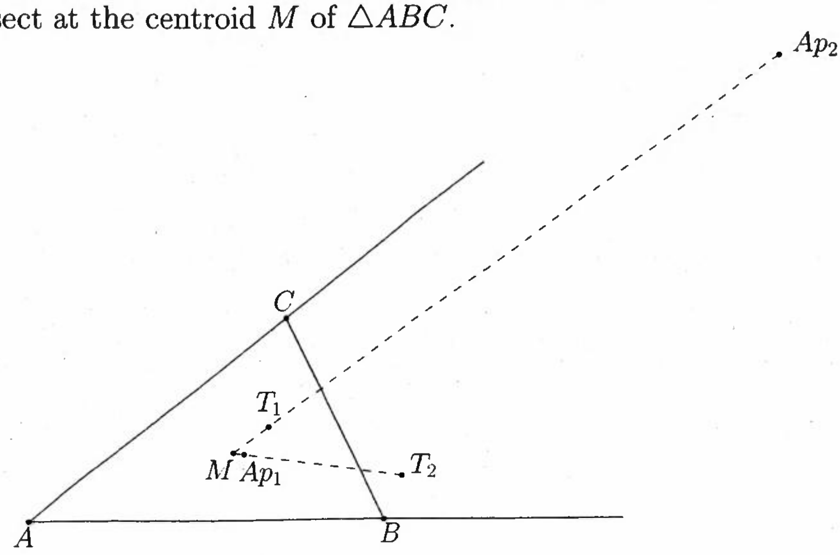


Let $t(A) \cap t(B) = M$ and $t(B) \cap t(C) = N$ and let the perpendicular bisector of AB intersect AC at V and let the perpendicular bisector of BC intersect AC at R. It suffices to show that $(AT_2, AC, AL, AT_1) = (CT_2, CA, CL, CT_1)$. (Prove that it is impossible that the cross-ratios are equal but the points are not collinear.) We have

$$\begin{aligned}(AT_2, AC, AL, AT_1) = A(A'', R, N, A') &= \frac{A''R}{RN} \cdot \frac{NA'}{A'A''} \\ &= \frac{A''R}{RC} \cdot \frac{RC}{RN} \cdot \frac{NA'}{A'C} \cdot \frac{A'C}{A'A''} \\ &= \frac{\sin(60^\circ - \gamma)}{\sin 30^\circ} \cdot \frac{\sin(90^\circ - \alpha)}{\sin(\gamma + \alpha)} \cdot \frac{\sin(60^\circ - \alpha)}{\sin(90^\circ - \alpha)} \cdot \frac{\sin 30^\circ}{\sin 120^\circ} \\ &= \frac{\sin(60^\circ - \gamma)}{\sin 120^\circ} \cdot \frac{\sin(60^\circ - \alpha)}{\sin\beta}.\end{aligned}$$

Analogously, $(CT_2, CA, CL, CT_1) = \dfrac{\sin(60^\circ - \gamma)}{\sin 120^\circ} \cdot \dfrac{\sin(60^\circ - \alpha)}{\sin\beta}$.

Problem 3.4. Let ABC be a triangle. Prove that Ap_2T_1 and Ap_1T_2 intersect at the centroid M of $\triangle ABC$.



Solution. Consider the following barycentric coordinates

$$T_1 = \left(\frac{\sin\alpha}{\sin(\alpha + 60^\circ)}, \frac{\sin\beta}{\sin(\beta + 60^\circ)}, \frac{\sin\gamma}{\sin(\gamma + 60^\circ)}\right),$$

$$T_2 = \left(\frac{\sin\alpha}{\sin(\alpha - 60^\circ)}, \frac{\sin\beta}{\sin(\beta - 60^\circ)}, \frac{\sin\gamma}{\sin(\gamma - 60^\circ)}\right),$$

$$Ap_1 = (\sin\alpha\sin(\alpha+60^\circ), \sin\beta\sin(\beta+60^\circ), \sin\gamma\sin(\gamma+60^\circ)),$$
$$Ap_2 = (\sin\alpha\sin(\alpha-60^\circ), \sin\beta\sin(\beta-60^\circ), \sin\gamma\sin(\gamma-60^\circ)).$$

In order to prove that the points M, T_1 and Ap_2 are collinear, it is enough to show that

$$\begin{vmatrix} \dfrac{\sin\alpha}{\sin(\alpha+60^\circ)} & \dfrac{\sin\beta}{\sin(\beta+60^\circ)} & \dfrac{\sin\gamma}{\sin(\gamma+60^\circ)} \\ \sin\alpha\sin(\alpha-60^\circ) & \sin\beta\sin(\beta-60^\circ) & \sin\gamma\sin(\gamma-60^\circ) \\ \dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} \end{vmatrix} = 0$$

$$\Leftrightarrow \sum_{cyc} \sin\alpha\sin(\alpha-60^\circ)\left(\frac{\sin\gamma}{\sin(\gamma+60^\circ)} - \frac{\sin\beta}{\sin(\beta+60^\circ)}\right) = 0$$

$$\Leftrightarrow \sum_{cyc} \sin\alpha\sin(\alpha-60^\circ)\sin(\alpha+60^\circ)(\sin\beta\sin(\gamma+60^\circ) - \sin\gamma\sin(\beta+60^\circ)) = 0$$

$$\Leftrightarrow \sum_{cyc} \sin\alpha(\cos 120^\circ - \cos 2\alpha)\cos 60^\circ(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} \cos 120^\circ\cos 60^\circ(\sin\alpha\sin\gamma\cos\beta - \sin\alpha\sin\beta\cos\gamma)$$
$$- \sum_{cyc} \sin\alpha\cos 2\alpha\cos 60^\circ(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} \sin\alpha\cos(2\alpha)(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} (\sin 3\alpha + \sin(-\alpha))(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} \sin 3\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma)$$
$$- \sum_{cyc} \sin\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} \sin 3\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$

$$\Leftrightarrow \sum_{cyc} 4\sin\alpha\cos^2\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma)$$

$$-\sum_{cyc} \sin\alpha(\sin\gamma\cos\beta - \sin\beta\cos\gamma) = 0$$
$$\Leftrightarrow \sum_{cyc} 4\cos^2\alpha \sin\alpha \sin(\gamma - \beta) = 0$$
$$\Leftrightarrow \sum_{cyc} \cos^2\alpha(\cos(\alpha + \gamma - \beta) - \cos(\alpha + \beta - \gamma) = 0$$
$$\Leftrightarrow \sum_{cyc} \cos^2\alpha(\cos 2\gamma - \cos 2\beta) = 0$$
$$\Leftrightarrow \sum_{cyc} \cos^2\alpha(\cos^2\gamma - \cos^2\beta) = 0,$$

which obviously holds.

We proved that the points M, T_1 and Ap_2 are collinear. In order to prove that the points M, Ap_1 and T_2 are collinear, we have to show that

$$\begin{vmatrix} \dfrac{\sin\alpha}{\sin(\alpha - 60^\circ)} & \dfrac{\sin\beta}{\sin(\beta - 60^\circ)} & \dfrac{\sin\gamma}{\sin(\gamma - 60^\circ)} \\ \sin\alpha\sin(\alpha + 60^\circ) & \sin\beta\sin(\beta + 60^\circ) & \sin\gamma\sin(\gamma + 60^\circ) \\ \dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} \end{vmatrix} = 0$$
$$\Leftrightarrow \sum_{cyc} \sin\alpha\sin(\alpha + 60^\circ)\left(\frac{\sin\gamma}{\sin(\gamma - 60^\circ)} - \frac{\sin\beta}{\sin(\beta - 60^\circ)}\right) = 0$$
$$\Leftrightarrow \sum_{cyc} \sin\alpha\sin(\alpha + 60^\circ)\sin(\alpha - 60^\circ)(\sin\beta\sin(\gamma - 60^\circ) - \sin\gamma\sin(\beta - 60^\circ)) = 0$$
$$\Leftrightarrow \sum_{cyc} \sin\alpha(\cos 120^\circ - \cos 2\alpha)\cos 60^\circ(\sin\beta\cos\gamma - \sin\gamma\cos\beta) = 0,$$

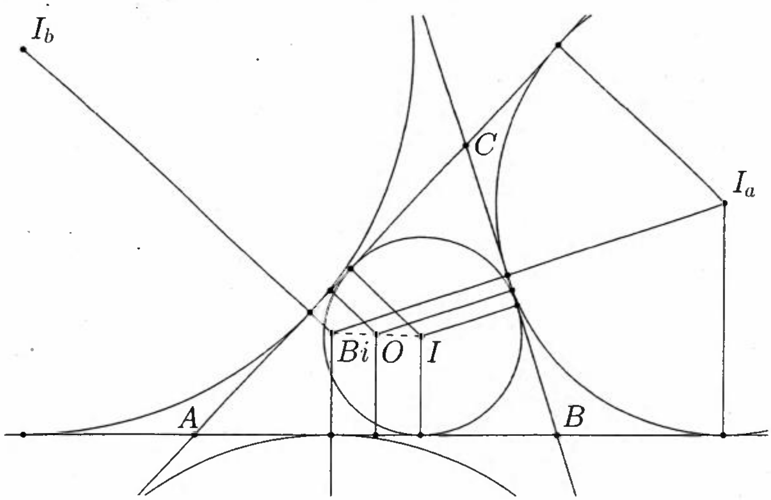
which we have already showed is true.

Note. The barycentric coordinates method is very helpful if we want to prove collinearity. The reader can exercise it by applying it to the first three problems of this chapter.

Problem 3.5. Let ABC be a triangle. Let I be its incenter and let O be its circumcenter. Let Bi be the Bevan point of $\triangle ABC$ (see Problem 2.20). Prove that the points I, O and Bi are collinear.

Solution. Let Bi_1 be the symmetric point of I with respect to O. Then the projections of the points Bi_1 and I onto the sides of the triangle are equidistant from the respective midpoints of the sides. Indeed, these midpoints are the projections of the midpoint of the segment Bi_1I onto the sides of $\triangle ABC$.

Therefore, the point Bi_1 is the intersection point of the perpendiculars, drawn at the points of tangency of the excircles to the respective sides. This is the definition of the Bevan point, thus $Bi_1 \equiv Bi$.



Problem 3.6. Let ABC be a triangle. Let I be its incenter, let G be its Gergonne point, and let S_1 and S_2 be its first and second Soddy centers, respectively. Prove that the points I, G, S_1 and S_2 are collinear.

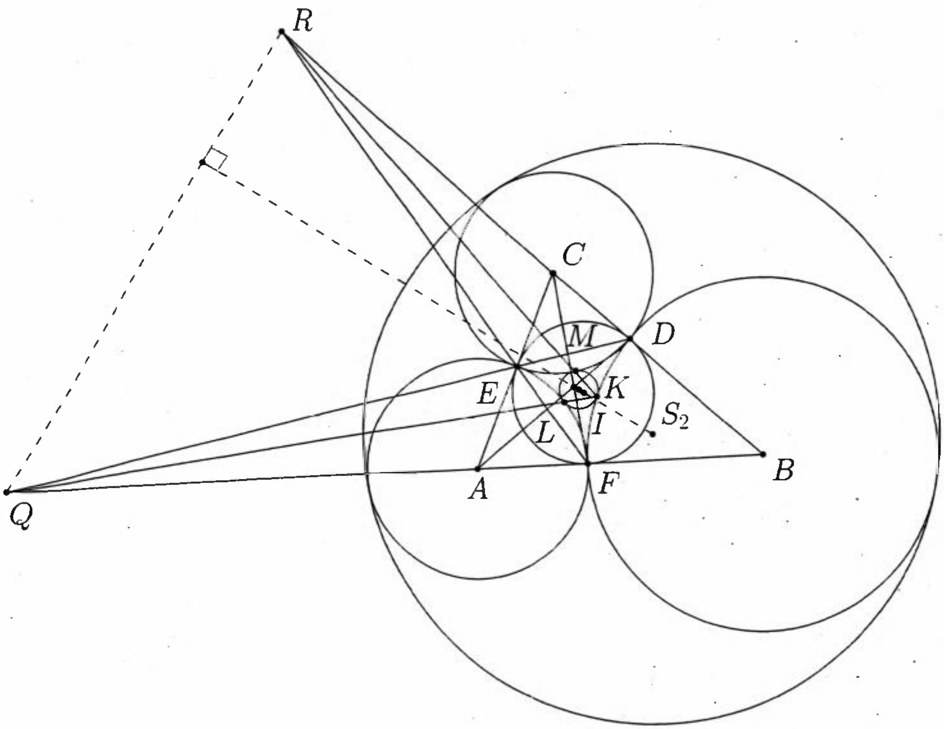
Solution. Consider an inversion with center I and radius r.

It sends the circles $k_1(A, s-a)$, $k_2(B, s-b)$ and $k_3(C, s-c)$ to themselves, and it sends the first Soddy circle (with center S_1) to the second Soddy circle.

Therefore, the points S_1, S_2 and I are collinear. We are left to prove that the points G, I and S_1 are collinear.

Denote the points of tangency of the incircle and BC, CA and AB by D, E and F, respectively. Denote the points of tangency of the first Soddy circle and k_1, k_2 and k_3 by L, K and M, respectively. Moreover, let $EF \cap BC = R$ and $ED \cap AB = Q$.

The polar lines of R and Q with respect to the incircle of $\triangle ABC$ are AD and CF, respectively. They intersect each other at G and hence QR is the polar line of G with respect to the incircle, and so $IG \perp QR$.



Now we apply Menelaus' Theorem to $\triangle ABC$ and the line DEQ and we get $\dfrac{AQ}{QB} = \dfrac{AE}{EC} \cdot \dfrac{CD}{DB} = \dfrac{AF}{FB}$ and hence Q is the external center of homothety of k_1 and k_2.

Monge's Theorem, applied to the first Soddy circle, k_1 and k_2 yields that the points K, L and Q are collinear.

Consider an inversion with center Q and radius QF. It sends L to K, thus $QL.QK = QF^2 = QE.QD$. Therefore, the quadrilateral $EDKL$ is cyclic and so Q lies on the radical axis of the first Soddy circle and the incircle.

Analogously, R also lies on this radical axis, thus $S_1 I \perp QR$ and hence the points G, I and S_1 are collinear.

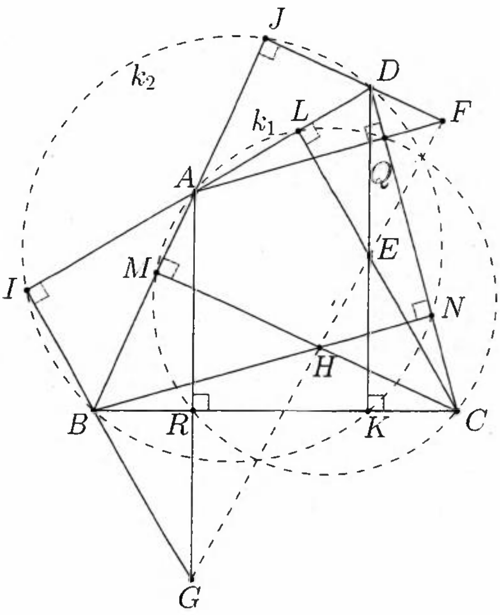
Problem 3.7. Let $ABCD$ be a quadrilateral. Let the feet of the perpendiculars from A to BC and CD be R and Q, respectively. Let the feet of the perpendiculars from B to CD and DA be N and I, respectively. Let the feet of the perpendiculars from C to DA and AB be L and M, respectively. Let the feet of the perpendiculars from D to AB and BC be J and K, respectively. Let $AR \cap BI = G$, $BN \cap CM = H$, $CL \cap DK = E$ and $AQ \cap DJ = F$. Prove that the points E, F, G and H are collinear.

Solution. Consider the circle k_1 with diameter AC and the circle k_2 with diameter BD.

We have $GR.GA = GB.GI$ because the quadrilateral $BRAI$ is cyclic and we have $HM.HC = HN.HB$ because the quadrilateral $BCNM$ is cyclic.

We have $LE.EC = DE.EK$ because the quadrilateral $KCDL$ is cyclic and we have $FQ.FA = FD.FJ$ because the quadrilateral $AQDJ$ is cyclic.

Therefore, the points G, H, E and F lie on the radical axis of k_1 and k_2.

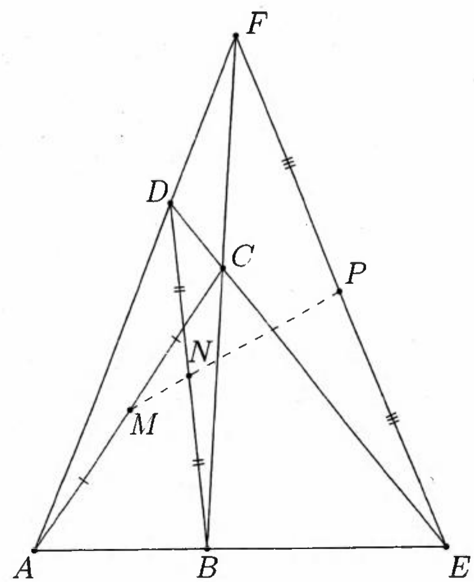


Problem 3.8. Let $ABCD$ be a quadrilateral such that $AB \cap CD = E$ and $AD \cap BC = F$. Prove that the midpoints M, N and P of the segments AC, BD and EF, respectively, are collinear.

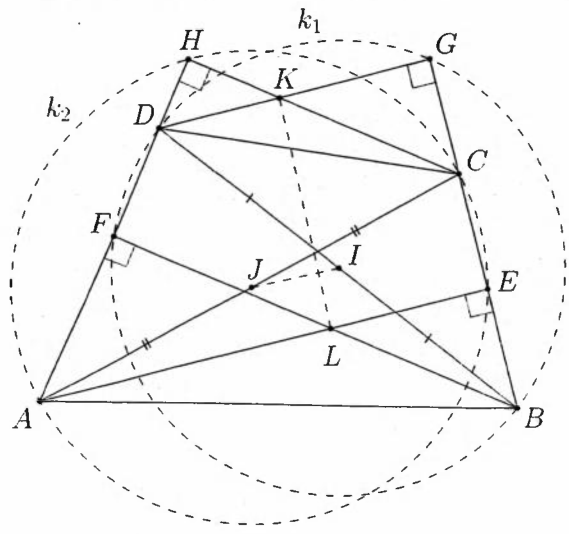
Solution. We will use oriented areas.

$$\begin{aligned}
[MNP] &= \frac{[MNE] + [MNF]}{2} \\
&= \frac{\frac{[MBE] + [MDE]}{2} + \frac{[MBF] + [MDF]}{2}}{2} \\
&= \frac{1}{8}([CBE] + [ADE] + [ABF] + [CDF]) \\
&= \frac{[ABCD] + [ADCB]}{8} = 0.
\end{aligned}$$

Therefore, the area of MNP is 0, and hence the points M, N and P are collinear.



Problem 3.9. Let $ABCD$ be a quadrilateral. Let J and I be the midpoints of the diagonals AC and BD, respectively. Let the perpendicular DG to BC ($G \in BC$) intersect the perpendicular CH to AD ($H \in AD$) at the point K. The perpendicular BF to AD ($F \in AD$) intersects the perpendicular AE to BC ($E \in BC$) at the point L. Prove that $KL \perp JI$.



Solution. Consider the circle k_1 with diameter AC and the circle k_2 with diameter BD.

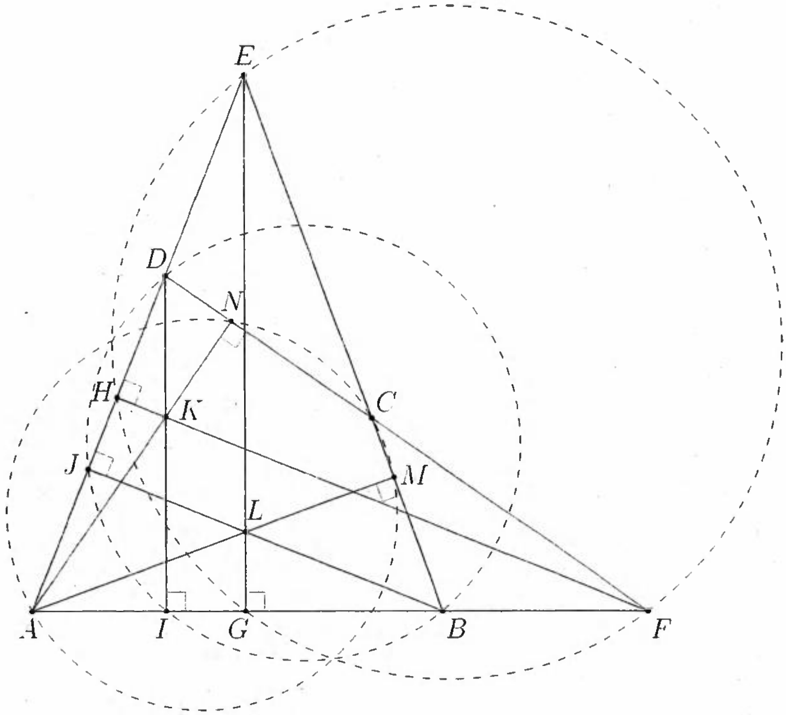
Their centers are I and J, respectively. We have that $KG.KD = KC.HK$ because the quadrilateral $DCGH$ is cyclic. Therefore, the point K lies on the radical axis of k_1 and k_2.

But the same statement is also true for L because the quadrilateral $ABEF$ is cyclic, leading to $LB.LF = LA.LE$.

We know that the radical axis of two circles is always perpendicular to the line that connects their centers, and thus $KL \perp JI$.

Note. We proved that the Aubert line (KL) and the Newton-Gauss line (JI) are perpendicular.

Problem 3.10. Let $ABCD$ be a quadrilateral such that $AB \cap DC = E$ and $AD \cap BC = F$. The circles k_1, k_2 and k_3 have AC, BD and EF as diameters, respectively. Prove that they share a common radical axis.



Solution. Let the altitudes FH, AN and DI intersect each other at the orthocenter K of $\triangle AFD$, and let the altitudes AM, BJ and EG intersect each other at the orthocenter L of $\triangle ABE$.

Then $KI.KD = KH.KF = KN.KA$ and $LM.LA = LB.LJ = LG.LE$, and hence K and L are radical centers of the three circles. If we assume that their pairwise radical axes do not coincide, then they must intersect at exactly one point, a contradiction.

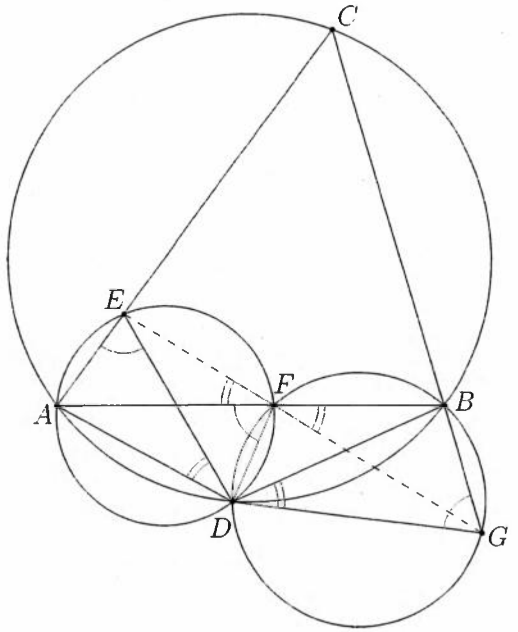
Problems 3.11 and 3.12. *(Simson line)* Let ABC be a triangle with circumcircle k. An arbitrary point D is chosen on the arc $\widehat{AB}$ of k which does not contain C. The points E, F and G lie on CA, AB and BC, respectively, and are chosen such that $\angle AED = \angle AFD = \angle BGD = \varphi$. Prove that the points E, F and G are collinear.

Solution. Using the equal angles, it is easy to show that the quadrilaterals $ADBC$, $ADFE$, $BFDG$ and $EDGC$ are cyclic.

Now we have

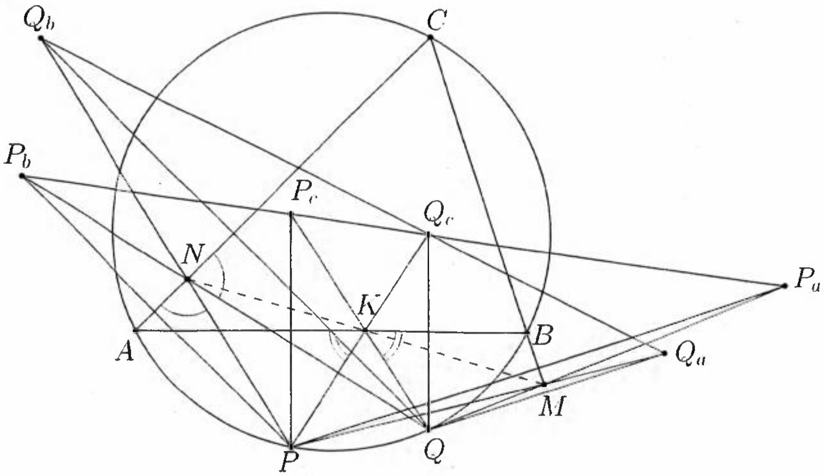
$$\begin{aligned}\angle EFG =& \angle EFD + \angle DFG\\ =& 180^\circ - \angle DAE + \angle DBG\\ =& \angle DBC + \angle DBG\\ =& 180^\circ.\end{aligned}$$

Therefore, the points E, F and G are collinear.



Note. When $\varphi = 90^\circ$, the construction represents the classic Simson line. We proved the generalization of 3.11, which is 3.12.

Problem 3.13. Let ABC be a triangle with circumcircle k. Two arbitrary points P and Q are chosen on the arc $\widehat{AB}$ which does not contain C. Points M, N and K are chosen on BC, CA and AB, respectively, such that $\angle(PM, BC) = \angle(QM, CB)$, $\angle(PN, AC) = \angle(QN, CA)$ and $\angle(QK, AB) = \angle(PK, BA)$. Prove that the points M, N and K are collinear.



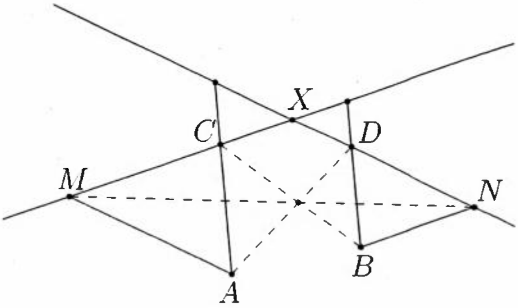
Solution. Consider the reflections of P and Q with respect to the sides of $\triangle ABC$. Denote them by P_a, P_b, P_c, Q_a, Q_b and Q_c, as shown in the figure. Problem 3.11 yields that they lie on the Simson lines after a homothety with coefficient 2 is applied. The points M, N and K are the intersection points of the diagonals of the three isosceles trapezoids that have a common leg PQ. We are going to prove the following lemma.

Lemma. Given two lines a and b, let X be their intersection point, and let A and B be two points not lying on the lines. Two parallel lines through A and B intersect a and b at the points C and D, respectively. Then, when the direction of the two parallel lines changes, the intersection point of AD and BC moves along a line.

Proof.

Let $AM \parallel b$, $M \in a$, and $BN \parallel a$, $N \in b$.

Desargues' Theorem (Problem 7.1), applied to to $\triangle ACM$ and $\triangle DBN$, yields that the intersection point of AD and BC moves along the line MN which is fixed.



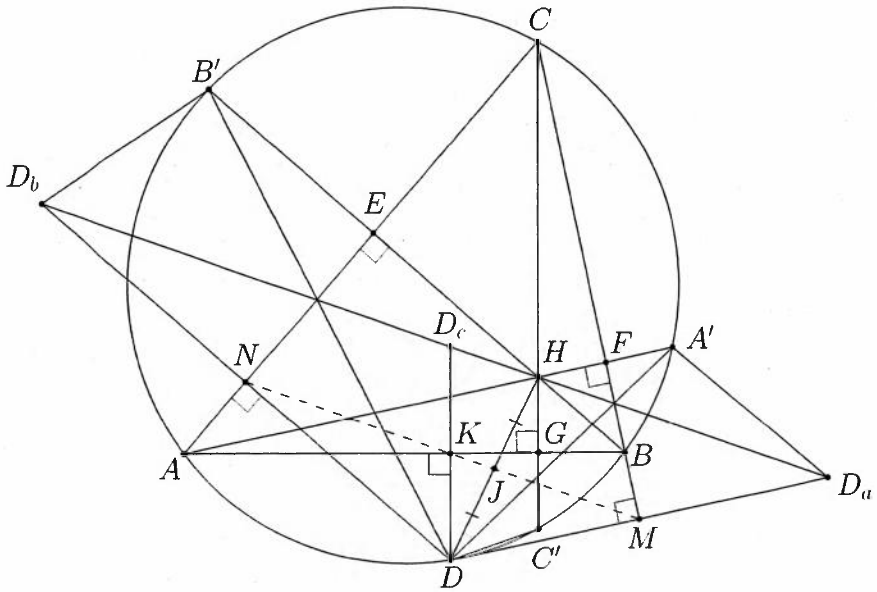
Now it is easy to see that MNK is the line from the lemma, where the two initial lines are Q_aQ_b and P_aP_b.

Problem 3.14. Let ABC be a triangle and let point D lie on its circumcircle. Prove that the midpoint J of the segment DH lies on the Simson line of $\triangle ABC$ and the point D.

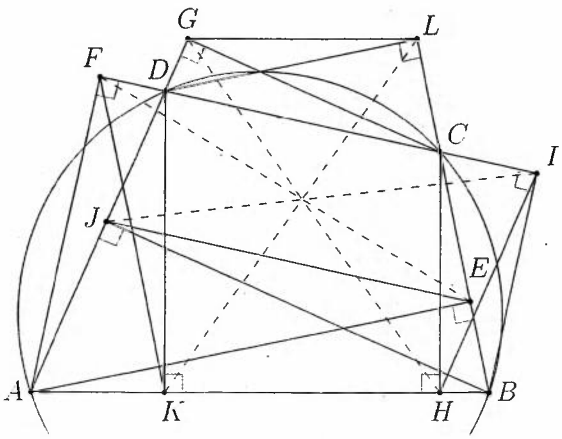
Solution. Let M, N and K be the feet of the perpendiculars drawn from D to the sides BC, CA and AB, respectively. Let D_a, D_b and D_c be the reflections of D with respect to the lines BC, CA and AB. Let F, E and G be the feet of the altitudes from A, B and C of $\triangle ABC$. Let A', B' and C'' be the reflections of H with respect to the sides of $\triangle ABC$.

Then it suffices to prove that D_b, D_c, D_a and H are collinear. Obviously, D_bDHB', $DC''HD_c$ and $DD_aA'H$ are isosceles trapezoids.

We have $\angle D_bHD = \angle B'HD - \angle B'HD_b = 180^\circ - \angle DB'B - \angle NDH = \angle ACB + \angle HDM - \angle DB'B = \angle HDM + \angle ACD = \angle HDM + \angle AA'D = 180^\circ - \angle DHD_a$ and hence the points D_b, H and D_a are collinear. Similarly, the points D_c, H and D_a are also collinear.



Problem 3.15. Let $ABCD$ be a cyclic quadrilateral. The feet of the perpendiculars from A to BC and CD are E and F, respectively. The feet of the perpendiculars from B to CD and DA are I and J, respectively. The feet of the perpendiculars from C to DA and AB are G and H, respectively. The feet of the perpendiculars from D to AB and BC are K and L, respectively. Prove that the lines JI, EF, GH and KL are concurrent.



Solution. We have that $\angle JEL = \angle BAJ = \angle DCL$ and it follows that $JE \parallel FI$. Analogously, $GL \parallel KH$, $HI \parallel JG$ and $KF \parallel EL$.

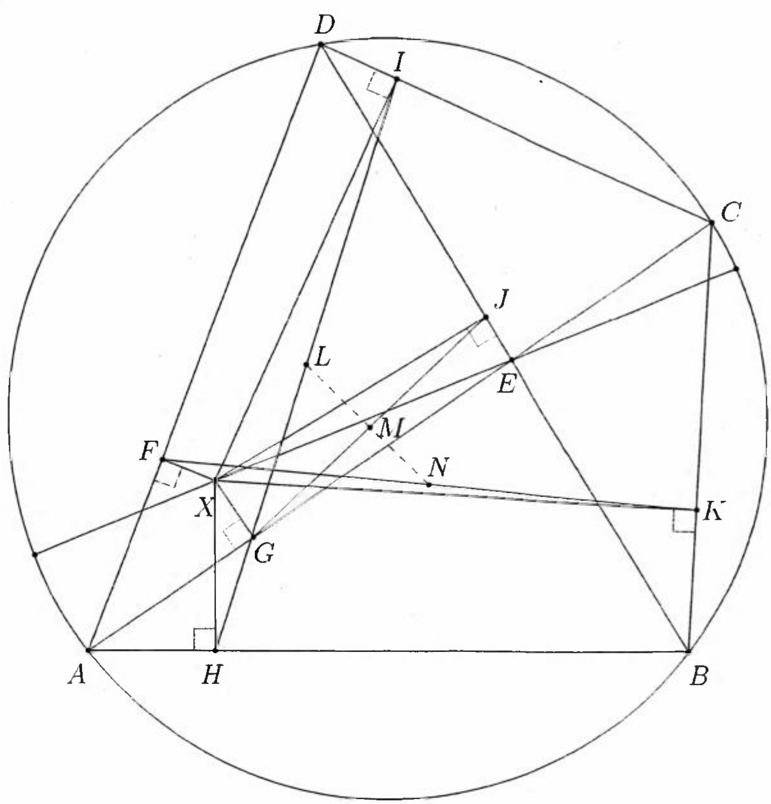
We have $\dfrac{KH}{GL} = \dfrac{KH}{DC} \cdot \dfrac{DC}{GL} = \dfrac{\cos\angle(AB, DC)}{\cos\angle(AD, BC)}$.

Similarly, $\dfrac{FI}{JE} = \dfrac{KF}{EL} = \dfrac{HI}{GJ} = \dfrac{\cos\angle(AB, DC)}{\cos\angle(AD, BC)}$.

Therefore, the quadrilaterals $JELG$ and $IFKH$ are similar and they have parallel sides, and hence they are homothetic. Then the lines JI, EF, GH and KL intersect at the center of this homothety.

Problem 3.16. Let ABC be a cyclic quadrilateral and let X be an arbitrary point. The feet of the perpendiculars from X to AB and CD are H and I, respectively. The feet of the perpendiculars from X to BC and DA are K and F, respectively. The feet of the perpendiculars from X to AC and BD are G and J, respectively. The midpoints of HI, GJ and KF are L, M and N, respectively. Prove that the points M, N and L are collinear.

Solution. Let $AC \cap BD = E$. Let us move the point X uniformly along a line through E. Then J and G are moving uniformly along BD and AC, and the line JG is moving uniformly, keeping the same direction. Therefore, M is moving uniformly along a line through E.



The points L and N are moving uniformly in a straight line because they are the midpoints of segments whose endpoints are moving uniformly in a straight line.

We will use the following property of the geometry of uniform motion. Given three points, each moving uniformly in a straight line, if they are collinear in three distinct points in time, then they are always collinear. One can prove this property by using vectors or other analytical methods. The proof with vectors can be sketched as follows:

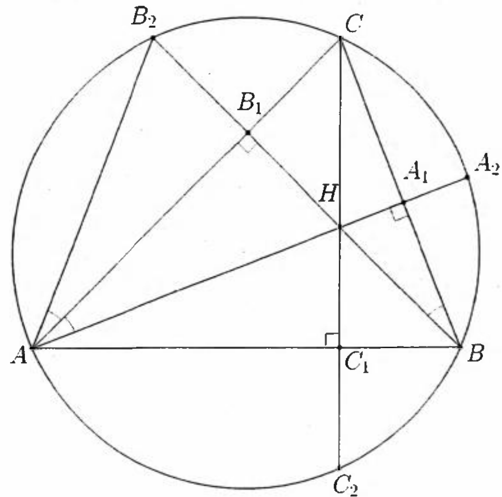
Let $\overrightarrow{x}$, $\overrightarrow{y}$ and $\overrightarrow{z}$ be the vectors of motion of the three points, i.e. directed towards the direction of motion, having length that corresponds to the point's speed. The three collinearities give two vector equations. These equations yield that the vectors $\overrightarrow{x} - \overrightarrow{y}$ and $\overrightarrow{y} - \overrightarrow{z}$ are collinear. From here, one can choose an arbitrary position in time and plug $\overrightarrow{x} - \overrightarrow{y} = k(\overrightarrow{y} - \overrightarrow{z})$. The conclusion follows.

Now, it is enough to prove the statement for three distinct positions of the point X along the line XE.

Position 1.) The points X and E coincide. Then Problem 5.7.7. yields that the quadrilateral $HKIF$ is circumscribed and its inscribed circle has X as a center. Now we apply Newton's Theorem to the quadrilateral $HKIF$.

Positions 2. and 3.) The point X coincides with one of the two distinct intersection points of EX and the circumcircle of $ABCD$. Then Problem 3.11 yields that GHK, JIK, JFH and GFI are the Simson lines of $\triangle ABC$, $\triangle BCD$, $\triangle ABD$, $\triangle ACD$ and the point X, respectively. It follows that $FJHIGK$ is a complete quadrilateral, and the line we are trying to prove corresponds to the Newton-Gauss line.

Problem 4.1.1. Let ABC be a triangle. The circumcircle of $\triangle ABC$ is k and its orthocenter is H. The altitude through B intersects AC and k at the points B_1 and B_2, respectively. Prove that the points H and B_2 are symmetric with respect to B_1.

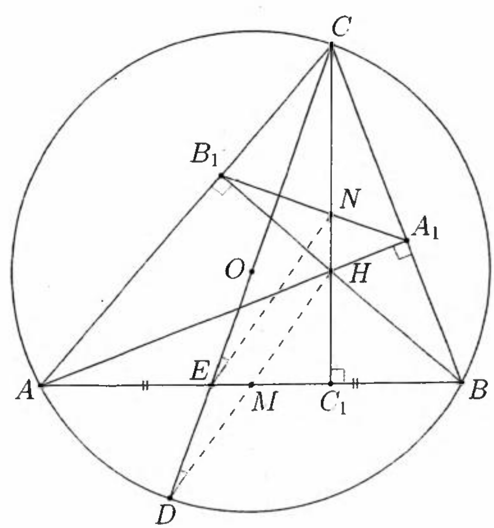


Solution. It is clear that

$$\angle B_1AB_2 = \angle CBB_2 = \angle B_1AH.$$

Then in $\triangle B_2AH$, the segment AB_1 is both an angle bisector and an altitude. Hence, it is also a median, which means that H and B_2 are symmetric with respect to B_1.

Problem 4.1.2. Let O be the circumcenter of $\triangle ABC$ with altitudes AA_1, BB_1 and CC_1. The lines CC_1 and A_1B_1 intersect at the point N and the lines CO and AB intersect at the point E. Prove that $HM \parallel EN$, where M is the midpoint of AB.



Solution. Problem 4.2.1 tells us that the lines HM and CO intersect on the circumcircle at a point that is symmetric to H with respect to M. Denote this point by D.

We know that $\angle EAC = \angle NA_1C$ and $\angle A_1CN = \angle ACE$.

Hence, the triangles CEA and CNA_1 are similar with a ratio of $\dfrac{AC}{A_1C} = \dfrac{CE}{CN}$. On the other hand, $\triangle DAC \sim \triangle HA_1C$ with a ratio of $\dfrac{AC}{A_1C} = \dfrac{CD}{CH}$.

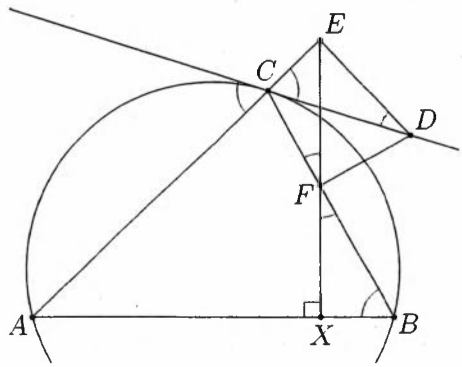
Note that $\dfrac{CD}{CH} = \dfrac{AC}{A_1C} = \dfrac{CE}{CN}$. Now, by the intercept theorem, applied to $\triangle DCH$ and the segment EN, we conclude that $EN \parallel DH$.

Problem 4.1.3. Let ABC be a triangle with circumcircle k. Let D be an arbitrary point on the tangent line to k at C. The points E and F are its projections onto AC and BC, respectively. Prove that $EF \perp AB$.

Solution. The quadrilateral $CEDF$ is cyclic. If $\angle ABC = \beta$, then $\angle DCE = \beta$ too. Hence, we have

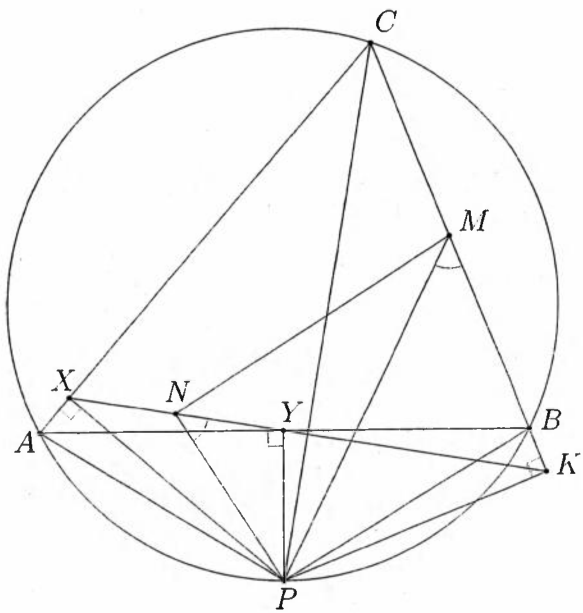
$$\begin{aligned}\angle CDE &= \angle CFE \\ &= \angle BFX = 90^\circ - \beta.\end{aligned}$$

Therefore, $\angle BXF = 90^\circ$.



Problem 4.1.4. Let ABC be a triangle. Let P be an arbitrary point on the smaller arc $\widehat{AB}$ of the circumcircle of $\triangle ABC$. The projections of P onto AC and AB are X and Y, respectively. The points M and N are the midpoints of BC and XY, respectively. Prove that $\angle PNM = 90^\circ$.

Solution. Let K be the projection of P onto BC. Then the points X, Y and K lie on the Simson line of P (Problem 3.11). It suffices to prove that the quadrilateral $PNMK$ is cyclic.



We show that $\triangle PYX \sim \triangle PBC$. We have $\angle PXY = \angle PAY = \angle PCB$ and $\angle XPY = \angle XAY = \angle CPB$.

Now the similarity follows and hence $\angle PNK = \angle PMK$ because PN and PM are respective medians in these triangles.

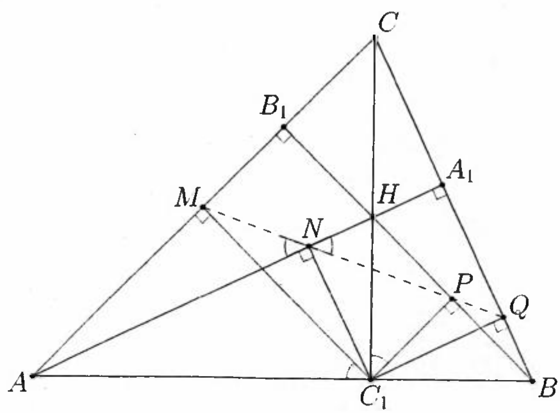
Therefore, the quadrilateral $PNMK$ is cyclic.

Problem 4.1.5. Let ABC be a triangle with altitudes AA_1, BB_1 and CC_1. The points M, N, P and Q are the projections of C_1 onto the lines AC, AA_1, BB_1 and BC, respectively. Prove that the points M, N, P and Q are collinear.

Solution. It is clear that $\angle MC_1P = 90°$. Hence, $\angle AC_1M = \angle PC_1H$.

The quadrilaterals AC_1NM and HNC_1P are cyclic. Hence, $\angle AC_1M = \angle ANM$ and $\angle PNH = \angle PC_1H$.

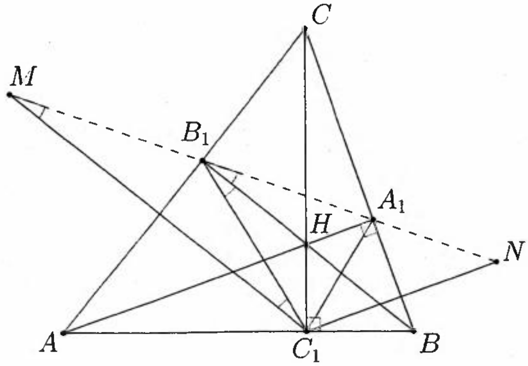
Now the equality $\angle ANM = \angle PNH$ yields that the points M, N and P are collinear. Analogously, the points N, P and Q are collinear. Therefore, the points M, N, P and Q are collinear.



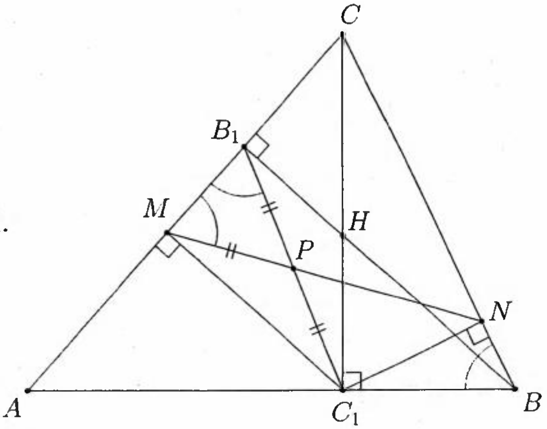
Problem 4.1.6. Let ABC be a triangle with altitudes AA_1, BB_1 and CC_1. Denote the reflections of C_1 with respect to the sides AC and BC by M and N, respectively. Prove that the points M, B_1, A_1 and N are collinear.

Solution. It is clear that $C_1M \parallel BB_1$. Hence, $\angle MC_1B_1 = \angle C_1B_1B$. On the other hand, $\angle C_1B_1B = \angle C_1AA_1 = \angle A_1B_1B$.

Also, $\angle MC_1B_1 = \angle C_1MB_1$, hence we derive $\angle MB_1C_1 + \angle C_1B_1A_1 = 180°$, which implies that the points M, B_1 and A_1 are collinear. Analogously, we deduce that N lies on this line.



Problem 4.1.7. Let ABC be a triangle with altitudes AA_1, BB_1 and CC_1. The points M and N are the projections from C_1 onto the sides AC and BC, respectively. Let $P = MN \cap B_1C_1$. Prove that P is the midpoint of B_1C_1.



Solution. We have $\angle ABC = \angle NC_1C = \angle NMC$ because NC_1MC is a cyclic quadrilateral.

Moreover, $\angle ABC = \angle C_1B_1A$. Hence, $\angle PB_1M = \angle PMB_1$. Therefore, $PM = PB_1$.

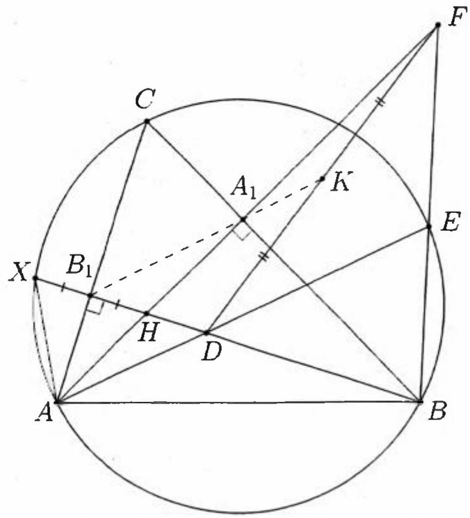
Also, $\angle PMC_1 = 90° - \angle PMB_1 = \angle PC_1M$, therefore $PB_1 = PM = PC_1$.

Problem 4.1.8. Let ABC be a triangle. Let k be its circumcircle and let H be its orthocenter. Let AA_1 and BB_1 be altitudes in the triangle. Let D be an arbitrary point on the segment BH. The line AD intersects k for the second time at the point E. Let $BE \cap AA_1 = F$ and let K be the midpoint of FD. Prove that the points A_1, B_1 and K are collinear.

Solution. Let the line BH intersect k for the second time at the point X. We have $\angle XAB_1 = \angle HBA_1 = \angle B_1AH$. Hence, $B_1H = B_1X$. Also, we have $\angle B_1AD = \angle CBE = \angle A_1BF$. This implies that $\triangle ADX \sim \triangle BFH$ with AB_1 and BA_1 being respective altitudes in these triangles.

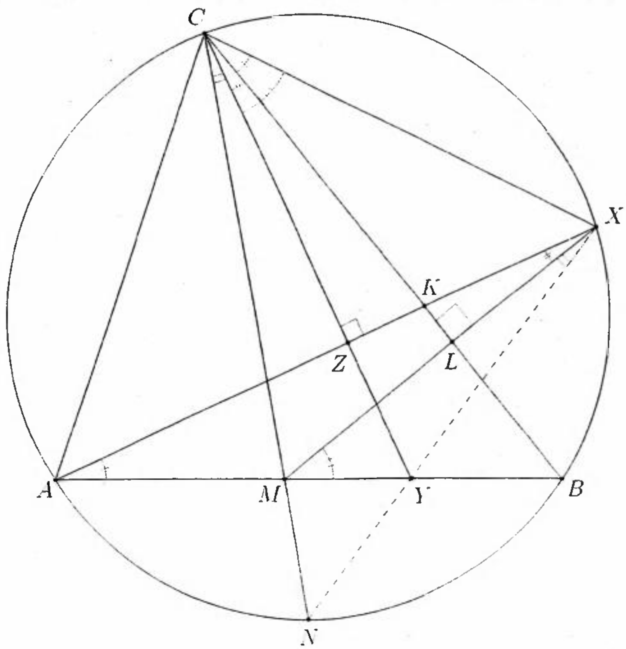
Therefore,

$$\frac{DB_1}{B_1X} = \frac{FA_1}{A_1H} \Rightarrow \frac{DB_1}{B_1H} = \frac{FA_1}{A_1H} \Rightarrow \frac{DB_1}{B_1H} \cdot \frac{A_1H}{FA_1} \cdot \frac{FK}{KD} = 1.$$



Now the statement follows by Menelaus' Theorem applied to $\triangle HDF$.

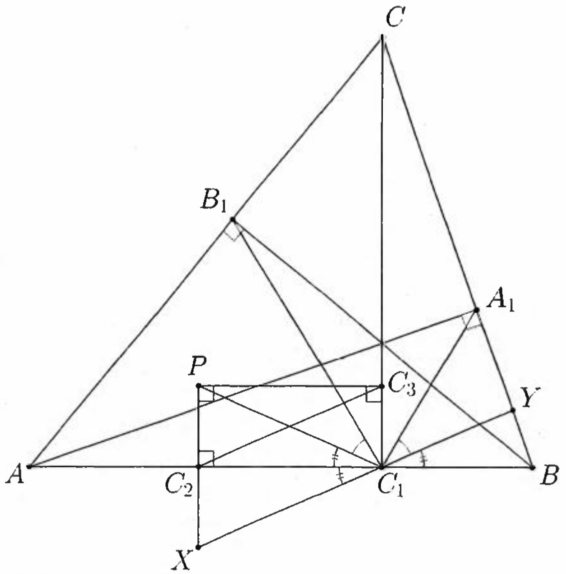
Problem 4.1.9. Let ABC be a triangle with circumcircle k. The line CM ($M \in AB$) is the angle bisector of $\angle ACB$, and it intersects k at the point N. The line through M, perpendicular to BC, intersects BC and the smaller arc $\widehat{BC}$ from k at the points L and X, respectively. The line through C, perpendicular to AX, intersects AX and AB at the points Z and Y, respectively. Prove that the points X, Y and N are collinear.



Solution. We will use the notations on the figure. We have $\angle YMX = \angle BML = 90° - \angle ABC = 90° - \angle AXC = \angle YCX$. Hence, the quadrilateral $YMCX$ is cyclic. Also, the quadrilateral $CZLX$ is cyclic as well.

Hence, $\angle MXY = \angle MCY = \angle MCB - \angle YCB = \angle ACN - \angle AXM = \angle AXN - \angle AXM = \angle MXN$, which proves that the points X, Y and N are collinear.

Problem 4.1.10. Let AA_1, BB_1 and CC_1 be the altitudes of a given triangle ABC. Let P be an arbitrary point inside of the triangle. The points C_2 and C_3 are the projections of P onto AB and CC_1, respectively The points $A_2 \in BC$, $A_3 \in AA_1$, $B_2 \in AC$ and $B_3 \in BB_1$ are defined analogously. Prove that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent.



Solution. First, consider a homothety with center P and coefficient 2. The image of the line C_2C_3 is a line, passing through C_1 and parallel to C_2C_3. Let X be the image of C_2 and denote the intersection point of BC and the image of C_2C_3 by Y.

Then $\angle XC_1A = \angle PC_1A$ and $\angle XC_1A = \angle BC_1Y$. Hence, $\angle AC_1P = \angle BC_1Y$.

On the other hand, since C_1C bisects $\angle A_1C_1B_1$, we have $\angle AC_1B_1 = \angle BC_1A_1$. Thus, $\angle PC_1B_1 = \angle YC_1A_1$.

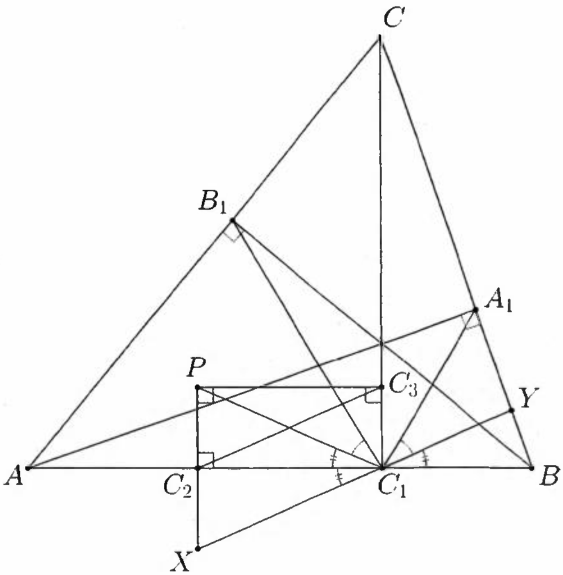
Therefore, the lines C_1P and XC_1 are isogonal conjugates in $\triangle A_1B_1C_1$. Since the lines C_1P, A_1P and B_1P are concurrent at P, then their respective isogonal conjugate lines are concurrent as well. Note that XC_1 is the image of C_2C_3 under the homothety.

We have shown that these isogonal conjugate lines are the images of A_2A_3, B_2B_3 and C_2C_3 under the homothety and that they are concurrent. Hence, the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent as well.

Solution. We will use the notations on the figure. We have $\angle YMX = \angle BML = 90^\circ - \angle ABC = 90^\circ - \angle AXC = \angle YCX$. Hence, the quadrilateral $YMCX$ is cyclic. Also, the quadrilateral $CZLX$ is cyclic as well.

Hence, $\angle MXY = \angle MCY = \angle MCB - \angle YCB = \angle ACN - \angle AXM = \angle AXN - \angle AXM = \angle MXN$, which proves that the points X, Y and N are collinear.

Problem 4.1.10. Let AA_1, BB_1 and CC_1 be the altitudes of a given triangle ABC. Let P be an arbitrary point inside of the triangle. The points C_2 and C_3 are the projections of P onto AB and CC_1, respectively The points $A_2 \in BC$, $A_3 \in AA_1$, $B_2 \in AC$ and $B_3 \in BB_1$ are defined analogously. Prove that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent.



Solution. First, consider a homothety with center P and coefficient 2. The image of the line C_2C_3 is a line, passing through C_1 and parallel to C_2C_3. Let X be the image of C_2 and denote the intersection point of BC and the image of C_2C_3 by Y.

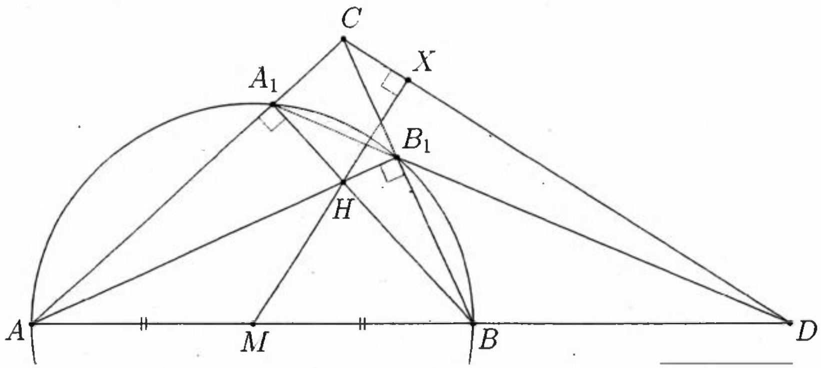
Then $\angle XC_1A = \angle PC_1A$ and $\angle XC_1A = \angle BC_1Y$. Hence, $\angle AC_1P = \angle BC_1Y$.

On the other hand, since C_1C bisects $\angle A_1C_1B_1$, we have $\angle AC_1B_1 = \angle BC_1A_1$. Thus, $\angle PC_1B_1 = \angle YC_1A_1$.

Therefore, the lines C_1P and XC_1 are isogonal conjugates in $\triangle A_1B_1C_1$. Since the lines C_1P, A_1P and B_1P are concurrent at P, then their respective isogonal conjugate lines are concurrent as well. Note that XC_1 is the image of C_2C_3 under the homothety.

We have shown that these isogonal conjugate lines are the images of A_2A_3, B_2B_3 and C_2C_3 under the homothety and that they are concurrent. Hence, the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent as well.

Problem 4.1.11. Let ABC be a triangle with altitudes AB_1 and BA_1 which intersect at the point H. The lines A_1B_1 and AB intersect at the point D and M is the midpoint of AB. Prove that $MH \perp DC$.



Solution. We have that the quadrilateral ABB_1A_1 is inscribed in a circle with center M. Consider the pole-polar transformation π with respect to this circle. By one of the polar properties, $H \in \pi_C$, hence (by La Hire's Theorem) $C \in \pi_H$.

Also, $H \in \pi_D$ and therefore $D \in \pi_H$. Then the line DC is the polar line of H, which yields $MH \perp DC$.

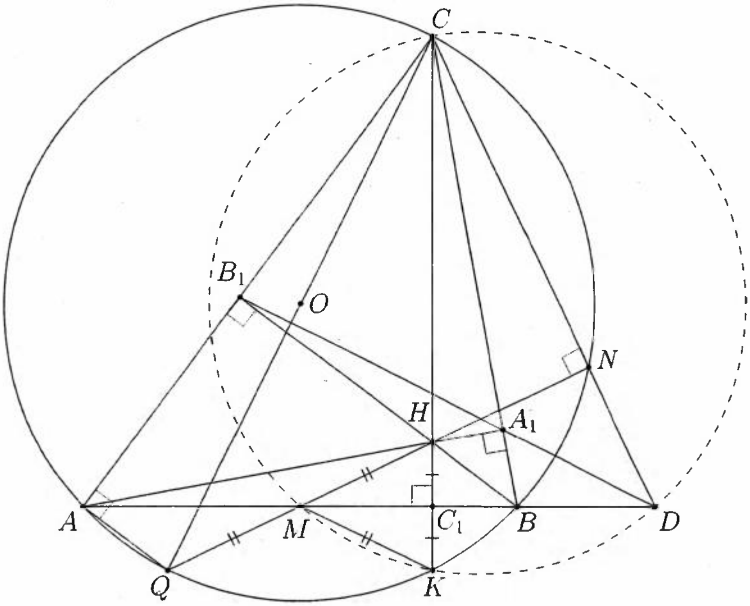
Note. In fact, this is Brocard's Theorem, applied to the quadrilateral ABB_1A_1 (see Problem 4.8.25).

Problem 4.1.12. Let ABC be an acute triangle. The point H is its orthocenter and the point M is the midpoint of AB. Let AA_1 and BB_1 be altitudes in the triangle and let $AB \cap A_1B_1 = D$. The line CH intersects the circumcircle of $\triangle ABC$ at the points C and K. Prove that the points K, M, C and D are concyclic.

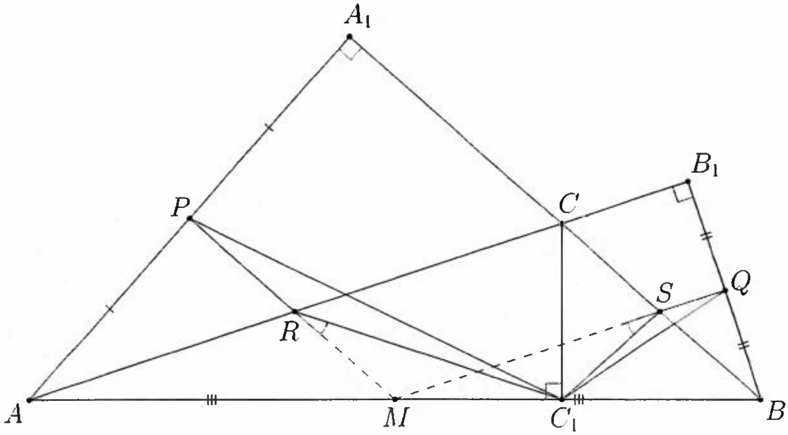
Solution. Let $CC_1 \perp AB$ $(C_1 \in AB)$. Denote the symmetric point of H with respect to M by Q. Since $\angle AQB = \angle AHB = 180° - \angle ACB$, we have that Q lies on the circumcircle of $\triangle ABC$.

Also, $BH \perp AC$ and $QA \parallel BH$, thus $QA \perp AC$, which yields $O \in CQ$. Let $QH \cap CD = N$. Brocard's Theorem (Problem 4.8.25), applied to the quadrilateral ABA_1B_1, yields that H is the orthocenter of $\triangle MCD$ and consequently $MH \perp CD$ and $\angle CNQ = 90°$.

Now we have $\angle MKC = \angle MHK = \angle MDC$. Therefore, the points K, M, C and D are concyclic.



Problem 4.1.13. Let ABC be a triangle with $\angle ACB > 90°$. Let AA_1, BB_1 and CC_1 be its altitudes. The point M is the midpoint of the side AB. Prove that the midpoints of AA_1 and BB_1, and the points M and C_1 are concyclic.



Solution. Denote the midpoints of AA_1, BB_1, AC and BC by P, Q, R and S, respectively.

Then $PR \parallel CA_1$ and $MR \parallel BC$, as midsegments in $\triangle ACA_1$ and $\triangle ABC$, respectively. Hence, the points P, R and M are collinear. Analogously, the points M, S and Q are collinear.

Note that the points R, M, C_1 and S lie on the Euler circle (Problem

4.8.1) of $\triangle ABC$, which gives $\angle MRC_1 = \angle MSC_1$ and thus $\angle C_1RP = \angle C_1SQ$. On the other hand, it is easy to see that $\triangle APR \sim \triangle BQS$ which yields $\dfrac{PR}{QS} = \dfrac{AR}{BS}$.

Since C_1R is a median to the hypotenuse in $\triangle ACC_1$, we have $C_1R = AR$. Analogously, $BS = C_1S$, so $\dfrac{C_1R}{C_1S} = \dfrac{AR}{BS} = \dfrac{PR}{QS}$.

Therefore, $\triangle PRC_1 \sim \triangle QSC_1$ and consequently, $\angle MPC_1 = \angle MQC_1$.

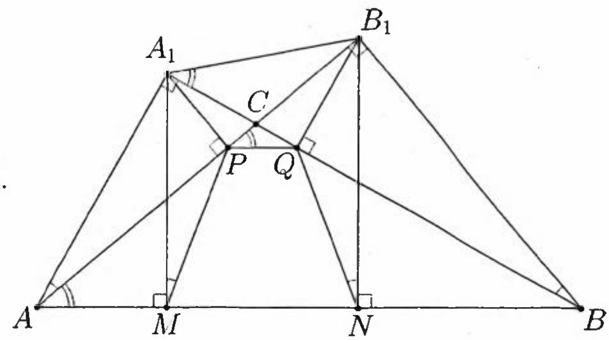
Note. The statement is true for non-obtuse triangles as well.

Problem 4.1.14. Let ABC be a triangle with $\angle ACB > 90°$ and altitudes AA_1 and BB_1. The points P and M are the projections of A_1 onto AC and AB, respectively, and Q and N are the projections of B_1 onto BC and AB, respectively. Prove that $PM = QN$.

Solution. The quadrilaterals ABB_1A_1, $AMPA_1$ and $BNQB_1$ are cyclic. We have

$$\angle B_1PQ = \angle B_1AQ = \angle CAB.$$

Hence, $PQ \parallel AB$. Moreover, we have



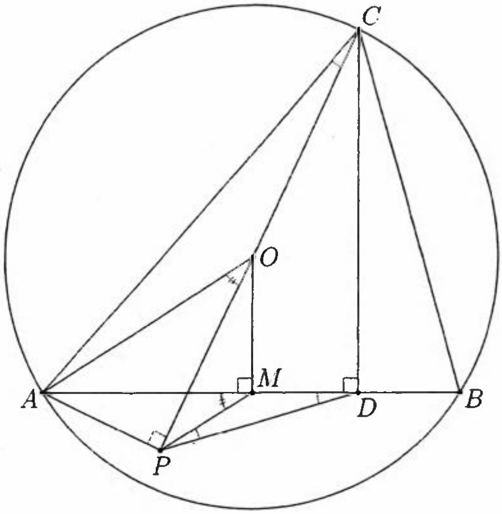
$$\angle PMA_1 = \angle PAA_1 = \angle A_1BB_1 = \angle B_1NQ.$$

Thus, $\angle PMN = \angle QNM$, which implies that the quadrilateral $MNQP$ is an isosceles trapezoid. Therefore, $MP = NQ$.

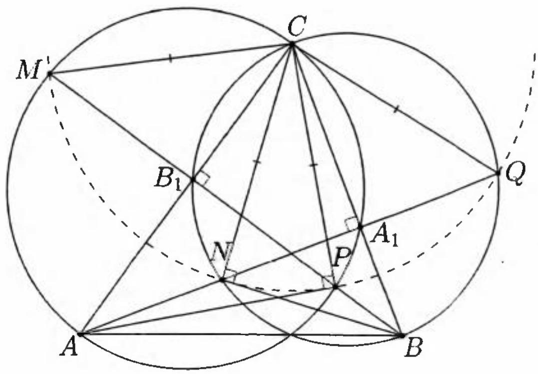
Problem 4.1.15. Let $\triangle ABC$ be a triangle with an altitude CD and circumcenter O. Let M be the midpoint of the side AB. Denote the projection of A onto CO by P. Prove that $DM = PM$.

Solution. Note that the quadrilaterals $APDC$ and $AOMP$ are cyclic.

We have $\angle AOP = \angle AMP$ and $\angle ADP = \angle ACP$. Hence, $\triangle PMD \sim \triangle AOC$. But $\triangle AOC$ is isosceles, which means that $DM = PM$.



Problem 4.1.16. Let ABC be a triangle with altitudes AA_1 and BB_1. The circle with diameter AC intersects the line BB_1 at the points P and M, so that P is between B and M. The circle with diameter BC intersects AA_1 at the points N and Q, so that N is between A and Q. Prove that the quadrilateral $MNPQ$ is cyclic.



Solution. Note that $\triangle BNC$ is right-angled and has NA_1 as an altitude. Hence,

$$CN^2 = CA_1.CB.$$

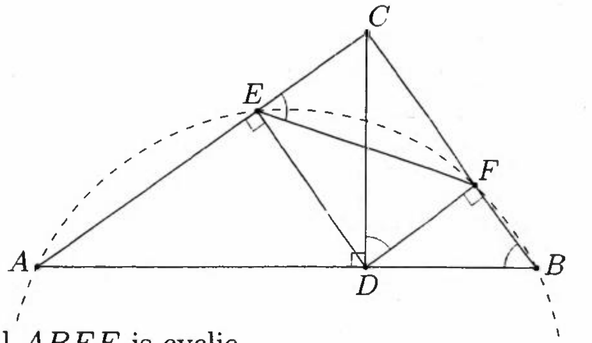
Analogously, we derive $CP^2 = CB_1.CA$. But $CB_1.CA = CA_1.CB$, due to ABA_1B_1 being a cyclic quadrilateral. Hence, $CN = CP$.

Moreover, the points N and Q are symmetric with respect to the line BC, thus $CN = CQ$. Analogously, we deduce that $CM = CP$. Therefore, the points M, N, P and Q lie on a circle centered at C.

Problem 4.1.17. Let ABC be a triangle with an altitude CD. The points E and F are the projections of D onto AC and BC, respectively. Prove that the quadrilateral $ABFE$ is cyclic.

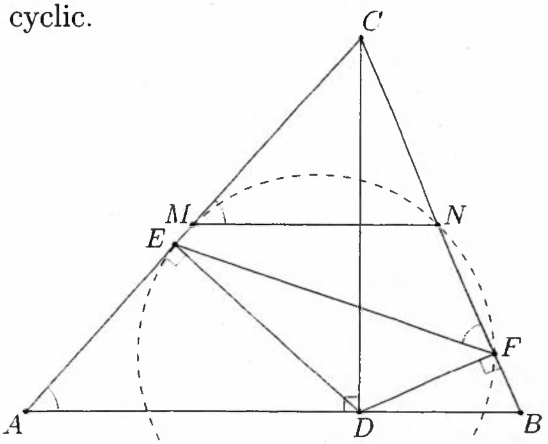
Solution. The quadrilateral $EDFC$ is cyclic, hence

$$\angle CEF = \angle CDF = \\ = 90^\circ - \angle FDB \\ = \angle ABF.$$



Therefore, the quadrilateral $ABFE$ is cyclic.

Problem 4.1.18. Let ABC be a triangle with an altitude CD. The points E and F are the projections of D onto AC and BC, respectively. The points M and N are the midpoints of AC and BC, respectively. Prove that the quadrilateral $EFNM$ is cyclic.



Solutoin. Note that Problem 4.1.17 implies $\angle BAC = \angle EFC$.

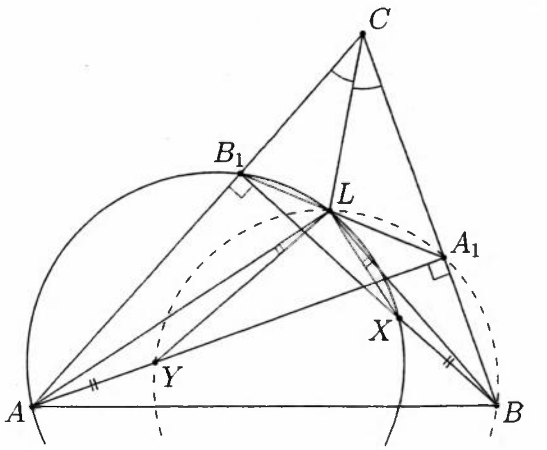
Since MN is a midsegment in $\triangle ABC$, we have $MN \parallel AB$, hence $\angle NMC = \angle BAC$.

Therefore, $\angle NMC = \angle EFC$ which yields that the quadrilateral $EFNM$ is cyclic.

Problem 4.1.19. Let ABC be a triangle. The segments AA_1 and BB_1 are altitudes, and the bisector of $\angle ACB$ intersects the segment A_1B_1 at the point L. The circumcircle of $\triangle AB_1L$ intersects BB_1 for the second time at X. Let $Y \in AA_1$ be a point such that $AY = BX$. Prove that the quadrilateral BA_1LY is cyclic.

Solution. Let the circumcircle of $\triangle BA_1L$ intersect AA_1 for the second time at the point Y'. We will show that $Y \equiv Y'$. It suffices to prove the equality $BX = AY'$.

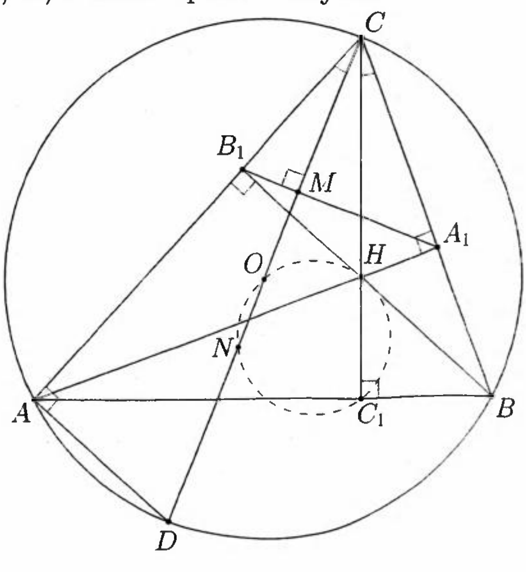
We have $\angle Y'LB = \angle Y'A_1B = \angle XB_1A = \angle XLA$. This yields $\angle ALY' = \angle BLX$.



Now the law of sines, applied to $\triangle XBL$, $\triangle Y'AL$, $\triangle ABL$, $\triangle ALC$ and $\triangle BLC$, yields

$$\begin{aligned}\frac{BX}{AY'} &= \frac{\sin\angle BLX}{\sin\angle ALY'}\cdot\frac{BL}{AL}\cdot\frac{\sin\angle LY'A_1}{\sin\angle LXB_1}\\ &= \frac{BL}{AL}\cdot\frac{\sin\angle LBC}{\sin\angle LAC} = \frac{BL}{AL}\cdot\frac{AL}{BL}\cdot\frac{\sin\angle LCB}{\sin\angle LCA} = 1.\end{aligned}$$

Problem 4.1.20. Let ABC be a triangle. Its circumcenter is O, its orthocenter is H and its altitudes are AA_1, BB_1 and CC_1. The point M is the projection of C onto A_1B_1 and N is the reflection of C with respect to A_1B_1. Prove that the points H, O, N and C_1 are concyclic.



Solution. It is clear that

$$\angle ACM = \angle ACN = \angle BCH.$$

But $\angle BCH = \angle ACO$, hence the points C, M, O and N are collinear.

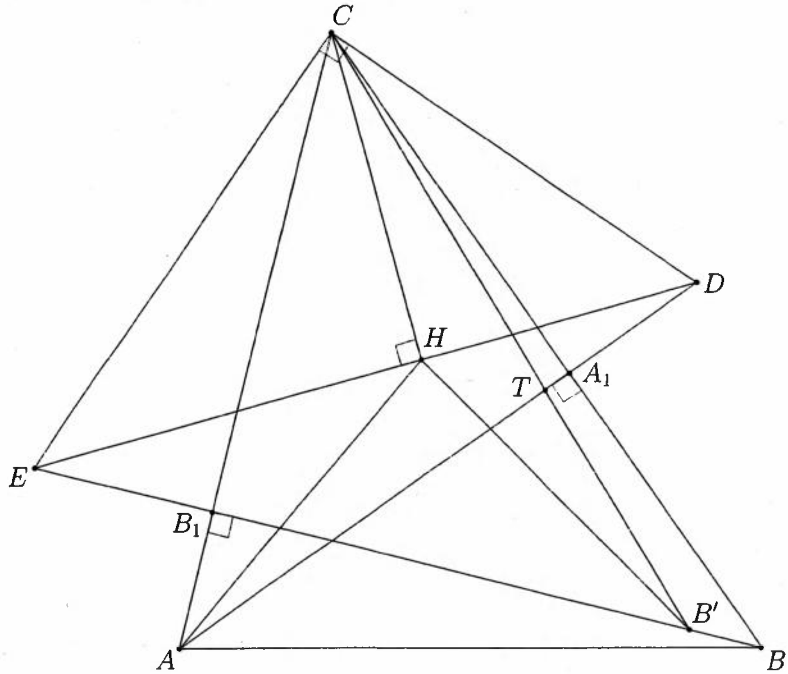
Let R be the circumradius of $\triangle ABC$.

Then $CH = 2R\cos\gamma$.

Now the similarity $\triangle ABC \sim \triangle A_1B_1C$ yields $\dfrac{CM}{CC_1} = \dfrac{CA_1}{CA} = \cos\gamma$.

Hence, $\dfrac{CH}{CO} = 2\cos\gamma = \dfrac{2CM}{CC_1} = \dfrac{CN}{CC_1}$, which is equivalent to $CO.CN = CH.CC_1$. Therefore, the quadrilateral $NOHC_1$ is cyclic.

Problem 4.1.21. Let ABC be a triangle with altitudes AA_1 and BB_1. A point D is chosen on the extension of AA_1 beyond A_1. A point E is chosen on the extension of BB_1 beyond B_1, satisfying $\angle DCE = 90°$. Let H be the foot of the perpendicular from C to ED. Prove that $\angle AHB = 90°$.



Solution. Let $B' \in BB_1$ be a point such that $B'H \perp AH$. It suffices to show that $B \equiv B'$. We have that the quadrilaterals EB_1HC and AB_1HB' are cyclic.

Hence, $\angle ACH = \angle HEB'$ and $\angle HAB_1 = \angle HB'E$, thus $\triangle AHC \sim \triangle B'HE$. But also, $\triangle EHC \sim \triangle CHD$. Therefore,

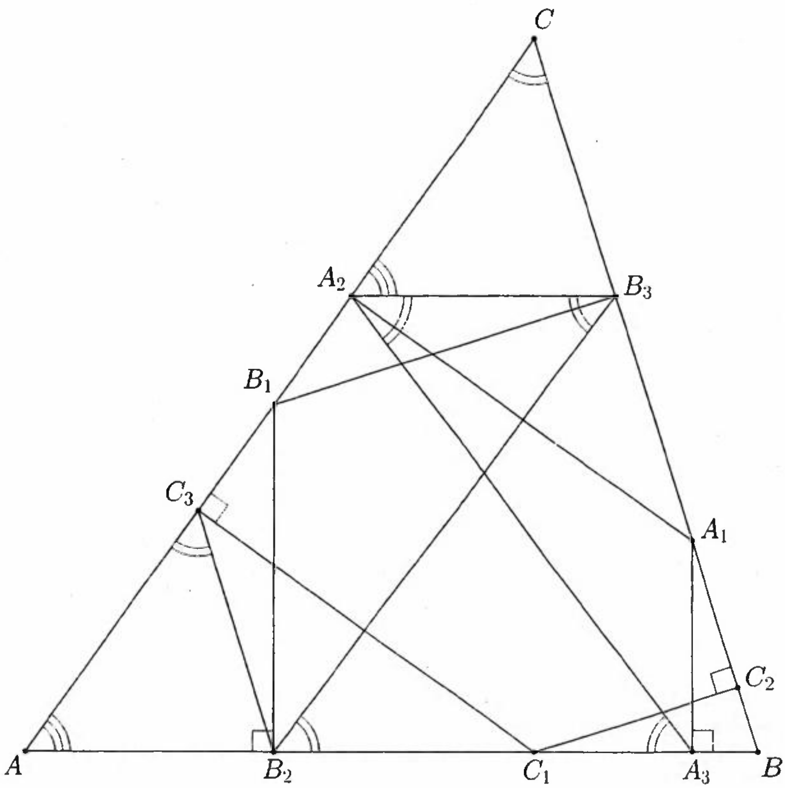
$$\frac{CH}{HD} = \frac{EH}{HC} = \frac{HB'}{AH} \Rightarrow \frac{CH}{HB'} = \frac{HD}{HA},$$

but $\angle CHB' = 90° + \angle DHB' = \angle AHD$. Hence, we also have $\triangle AHD \sim \triangle B'HC$.

We deduce that $\angle HCB' = \angle HDA$. Let $CB' \cap AD = T$. Then the quadrilateral $HCDT$ is cyclic and

$$\angle CTD = \angle CHD = 90° \Rightarrow T \equiv A_1 \Rightarrow B' \equiv B.$$

Problem 4.1.22. Let ABC be a triangle. The segments AA_1, BB_1 and CC_1 are its altitudes. The points A_2 and A_3 are the projections of A_1 onto AC and AB, respectively. The points B_2, B_3, C_2 and C_3 are defined analogously. Prove that the points A_2, A_3, B_2, B_3, C_2 and C_3 are concyclic.



Solution. The quadrilaterals BAB_1A_1 and $A_1B_1A_2B_3$ are cyclic, hence $\angle BAC = \angle B_1A_1C = \angle B_3A_2C$. Thus, $A_3B_2A_2B_3$ is a trapezoid. Problem 4.1.17 yields that the quadrilaterals BA_3A_2C and AB_2B_3C are cyclic.

Hence, $\angle B_2A_3A_2 = \angle ACB = \angle A_3B_2B_3$, so the trapezoid $A_3B_2A_2B_3$ is isosceles, and in particular, cyclic. We also have that the quadrilateral $C_1A_3C_2A_1$ is cyclic.

Hence, $\angle A_3C_2B = \angle A_1C_1B = \angle ACB = \angle B_3B_2A_3$. Therefore, the point C_2 lies on the circumcircle of $A_3B_2A_2B_3$. Analogously, the point C_3 lies on the same circle.

Problem 4.1.23. Let ABC $(AC = BC)$ be an isosceles triangle. The segment CC_1 is an altitude and M is its midpoint. Let P be the projection of C_1 onto BM. Prove that $\angle APC = 90°$.

Solution. Note that $\triangle BC_1M$ is right-angled with C_1P as its altitude. Then

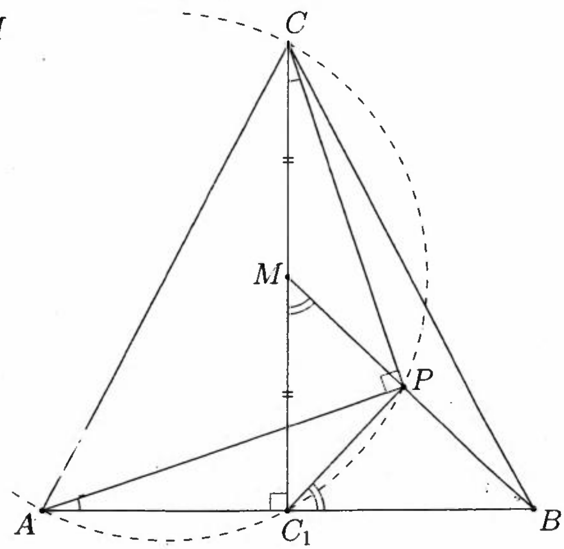
$$\triangle MPC_1 \sim \triangle C_1PB,$$

which implies

$$\frac{MC_1}{C_1B} = \frac{MP}{PC_1}.$$

But $MC_1 = MC$ and $C_1B = AC_1$, hence

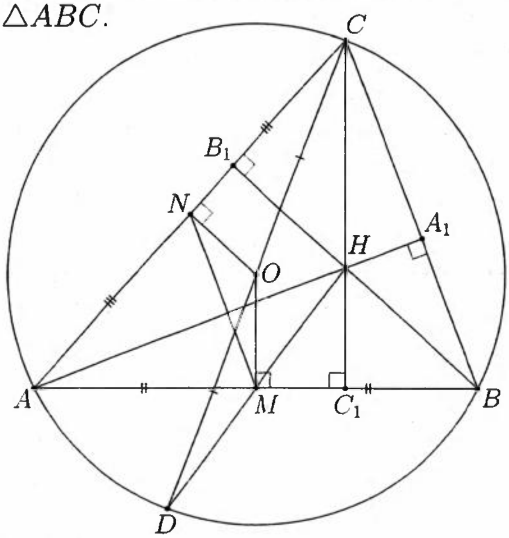
$$\frac{MC}{AC_1} = \frac{MP}{PC_1}.$$



On the other hand, $\angle AC_1P = \angle CMP$, thus $\triangle AC_1P \sim \triangle CMP$.

Then $\angle C_1AP = \angle C_1CP$. Therefore, the quadrilateral AC_1PC is cyclic, which yields that $\angle APC = \angle AC_1C = 90°$.

Problem 4.2.1. Let ABC be a triangle with orthocenter H and let M be the midpoint of the side AB. Prove that the symmetric point of H with respect to M coincides with the diametrically opposite point of C with respect to the circumcircle of $\triangle ABC$.

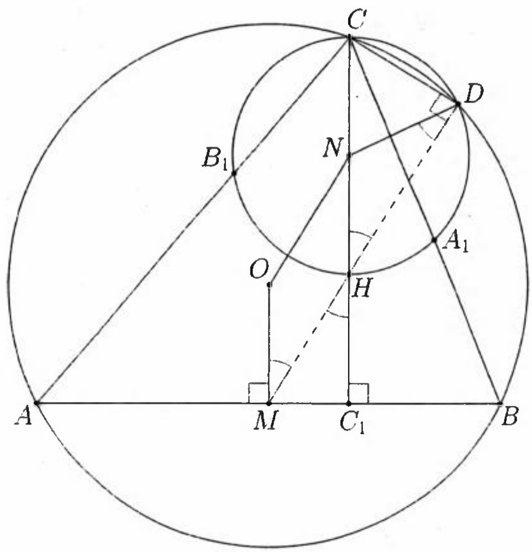


Solution. Let O be the circumcenter of $\triangle ABC$. Let N be the projection of O onto AC. We have $OM \perp AB$, $ON \perp AC$ and $MN \parallel BC$.

It follows that $\triangle MON$ and $\triangle CHB$ are similar with ratio $\dfrac{MN}{BC}$. But MN is a midsegment in $\triangle ABC$, thus $\dfrac{CH}{OM} = \dfrac{2}{1}$.

Now if D is the symmetric point of H with respect to M, we have that MO is a midsegment in $\triangle CHD$. Therefore, CD is a diameter in the circumcircle of $\triangle ABC$.

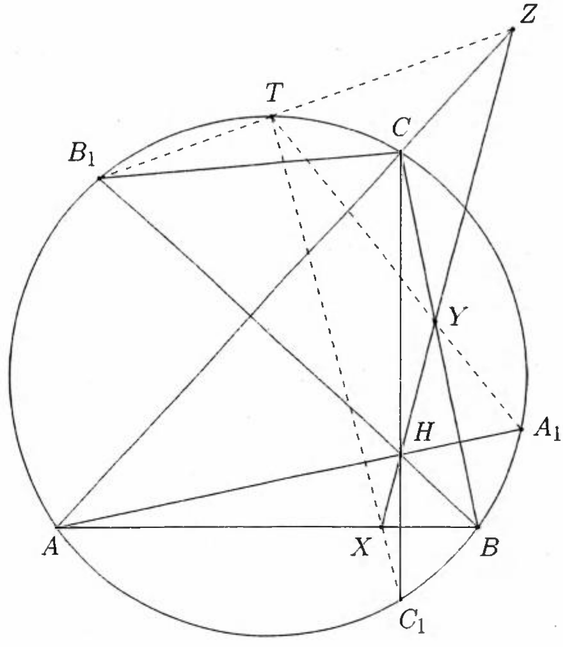
Problem 4.2.2. Let ABC be a triangle with altitudes AA_1 and BB_1 intersecting at the point H. Let D be the second intersection point of the circumcircles of $\triangle ABC$ and $\triangle A_1B_1C$, and let M be the midpoint of AB. Prove that the points D, H and M are collinear.



Solution. Let O be the circumcenter of ABC and let N be the midpoint of CH. It follows that N is the circumcenter of the cyclic pentagon CB_1HA_1D. Now Problem 4.2.1 implies $OM = NH = CN = ND$, which yields $OM = NH$ and $OM \parallel HN$.

Hence, the quadrilateral $MHNO$ is a parallelogram and $ON \parallel MH$. Note that $OC = OD$ and $NC = ND$, hence $ON \perp CD$. But $CD \perp DH$, and so $ON \parallel HD$. This means that the points D, H and M are collinear.

Problem 4.2.3. Let ABC be a triangle with circumcircle k. Let H be its orthocenter. Let l be an arbitrary line through H. Prove that the reflections of l with respect to AB, BC and CA concur at a point on k.



Solution. Let A_1, B_1 and C_1 be the second intersection points of the respective altitudes with the circumcircle of $\triangle ABC$. Let X, Y and Z be the intersection points of the line through H with the sides of $\triangle ABC$, as shown in the figure. Let $T = C_1X \cap (ABC), T \neq C_1$. Note that the points H and C_1 are symmetric with respect to AB, the points H and B_1 are symmetric with respect to AC, and the points H and A_1 are symmetric with respect to BC.

Denote $\angle YHC = \varphi$. We have

$$\angle TA_1C = \angle TC_1C = \angle XHC_1 = \varphi.$$

On the other hand, $\angle YA_1C = \angle YHC = \varphi$. Hence, $\angle TA_1C = \angle YA_1C$, thus the points T, Y and A_1 are collinear.

Moreover,

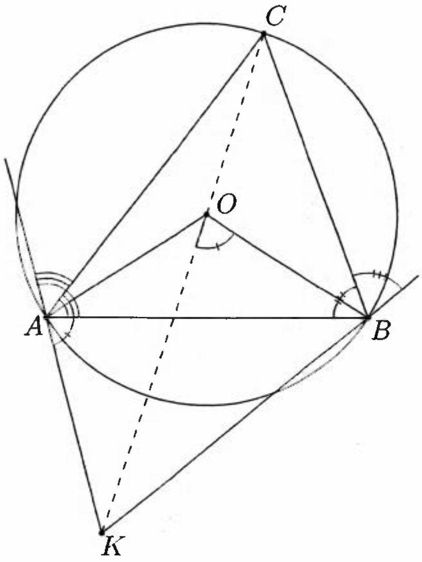
$$\angle TB_1C = \angle TC_1C = \varphi$$

and $\angle ZB_1C = \angle ZHC = \varphi$. Hence, $\angle TB_1C = \angle ZB_1C$. Thus, the points T, Z and B_1 are collinear.

Problem 4.2.4. Let ABC be a triangle with circumcenter O. The reflections of AB with respect to the lines AC and BC intersect at the point K. Prove that the points C, O and K are collinear.

Solution. Note that C is an excenter in $\triangle ABK$. Therefore, $\angle AOB = 2\angle ACB = 180° - \angle AKB$.

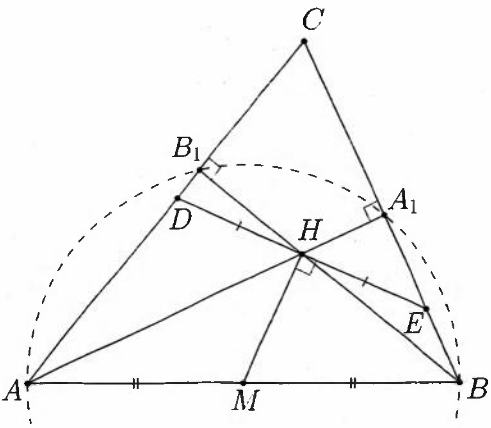
Hence, the quadrilateral $AKBO$ is cyclic. But we have $AO = BO$. Thus, KO is the angle bisector of $\angle AKB$. Therefore, the line KO passes through the point C.



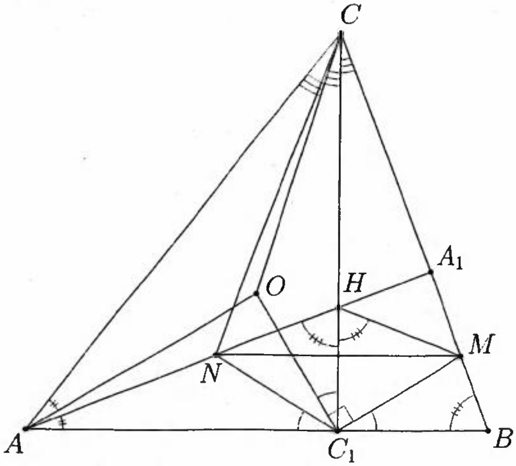
Problem 4.2.5. Let ABC be a triangle with orthocenter H, and let M be the midpoint of the side AB. The line through H, perpendicular to HM, intersects the sides AC and BC at the points D and E, respectively. Prove that $DH = EH$.

Solution. Let AA_1 and BB_1 be altitudes in $\triangle ABC$. Notice that the quadrilateral ABA_1B_1 is inscribed in a circle k_1 with center M and H is the intersection point of its diagonals.

Now we apply the butterfly theorem (Problem 6.4.3) to the circle k_1 and the quadrilateral ABA_1B_1, and we get $DH = EH$.



Problem 4.2.6. Let ABC be a triangle with orthocenter H and circumcenter O. The point M lies on the side BC and $\angle OC_1M = 90°$. Prove that $\angle ABC = \angle MHC_1$.



Solution. Let CC_1 be an altitude in $\triangle ABC$. We have $\angle BC_1C = \angle MC_1O$.

Hence, $\angle OC_1H = \angle BC_1M$. Let N be the reflection of M with respect to CC_1.

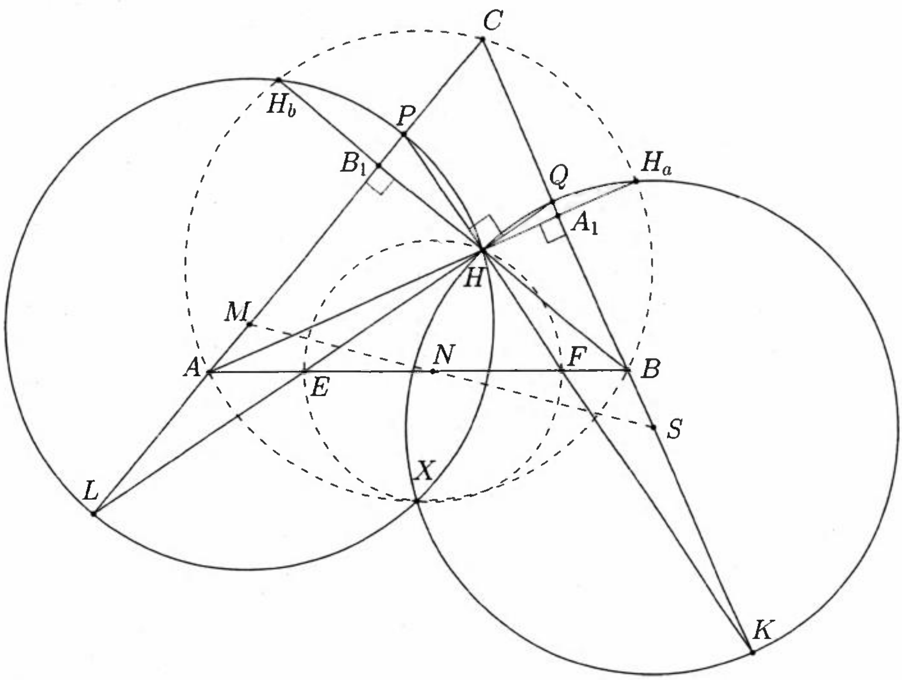
Now, the fact that O and H are isogonal conjugates in $\triangle ABC$ gives

$$\angle NCC_1 = \angle BCC_1 = \angle BCH = \angle ACO = 90° - \beta.$$

Moreover, we have $\angle AC_1N = \angle BC_1M = \angle OC_1C$. Therefore, O and N are isogonal conjugates in $\triangle ACC_1$, which implies $\angle C_1AN = \angle CAO$.

But we know that $\angle CAO = \angle C_1AH$, thus the points A, N and H are collinear. Now we have $\angle AHC_1 = \angle ABC$ and $\angle AHC_1 = \angle MHC_1$, which yields $\angle ABC = \angle MHC_1$.

Problem 4.2.7. Let ABC be a triangle. The circumcircle of $\triangle ABC$ is k and its orthocenter is H. Consider two lines through H that are perpendicular to each other. One of them intersects AB, BC and CA at the points F, K and P, respectively, and the other one – at the points E, Q and L, respectively. Prove that the midpoints S, N and M of the segments QK, EF and LP, respectively, are collinear.



Solution. Let H_a, H_b and H_c be the reflections of H with respect to BC, AC and AB, respectively. Since $\angle BH_aC = \angle BHC = 180° - \angle BAC$, we get that H_a lies on k, and the same is true for H_b and H_c.

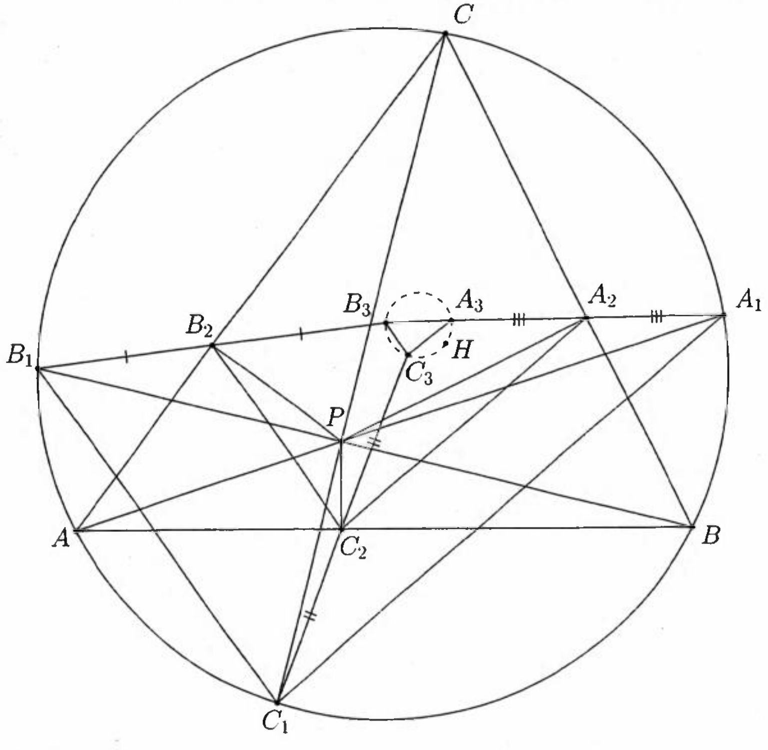
Let k_1, k_2 and k_3 be the circles with diameters KQ, EF and LP, respectively. Their respective centers are S, N and M, hence if we prove that these circles are coaxial (have a common radical axis), this will imply that S, N and M are collinear.

It is clear that H and H_a lie on k_1 and that H and H_b lie on k_3. Let $k_1 \cap k_3 = \{H, X\}$. Then we have

$$\begin{aligned}\angle H_bXH_a &= \angle H_bXH + \angle HXH_a = \angle H_bLH + \angle H_aKH \\ &= 2\angle CLH + 2\angle CKH = 2(90° - \angle ACB) = 2\angle CAH_a = \angle H_bAH_a.\end{aligned}$$

Thus, X lies on k. So the intersection point of k_1 and k, distinct from H_a, coincides with that of k_3 and k, distinct from H_b. Analogously, it coincides with that of k_2 and k, distinct from H_c. Therefore, the points H and X lie on all three of k_1, k_2 and k_3, which means that the three circles are coaxial. The desired result follows.

Problem 4.2.8. Let ABC be a triangle. The circumcircle of $\triangle ABC$ is k and its orthocenter is H. Let P be an arbitrary point from the inside of $\triangle ABC$. The lines AP, BP and CP intersect k for the second time at the points A_1, B_1 and C_1, respectively. The points A_2, B_2 and C_2 are the projections of P onto BC, CA and AB, respectively. The points A_3, B_3 and C_3 are the reflections of A_1, B_1 and C_1 with respect to A_2, B_2 and C_2, respectively. Prove that the points H, A_3, B_3 and C_3 are concyclic.

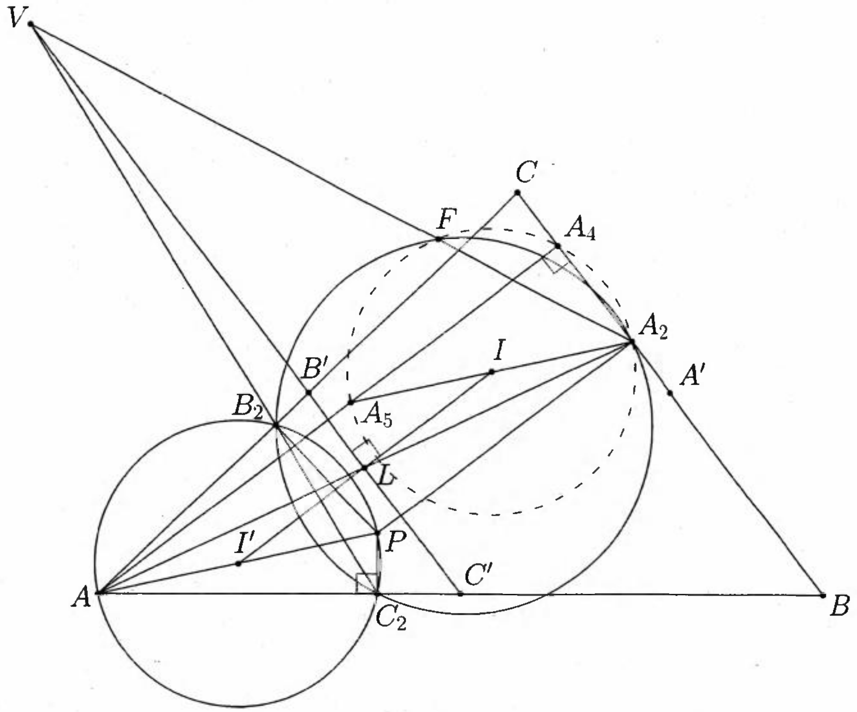


Solution. We have

$$\begin{aligned}\angle B_1C_1A_1 &= \angle B_1C_1C + \angle CC_1A_1 = \angle B_1BC + \angle A_1AC \\ &= \angle PC_2A_2 + \angle PC_2B_2 = \angle B_2C_2A_2.\end{aligned}$$

Analogously, $\angle B_1A_1C_1 = \angle B_2A_2C_2$, hence $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Moreover, these two triangles have the same orientation. Now, by Problem 4.12.9, we have that if C_3' is a point such that $\triangle A_1B_1C_1 \sim \triangle A_3B_3C_3'$, then the midpoint of C_1C_3' coincides with C_2, and hence $C_3' \equiv C_3$. This implies that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \sim \triangle A_3B_3C_3$. Now we will prove the following lemma.

Lemma. Denote the midpoints of BC, CA and AB by A', B' and C', respectively. Let $B'C' \cap B_2C_2 = V$. The line VA_2 intersects the circumcircle of $\triangle C_2B_2A_2$ for the second time at the point F. Let AA_4, BB_4 and CC_4 be the altitudes in $\triangle ABC$. The point $A_5 \in AA_4$ is chosen so that $AP \parallel A_2A_5$. Then the quadrilateral $A_2A_4FA_5$ is cyclic.

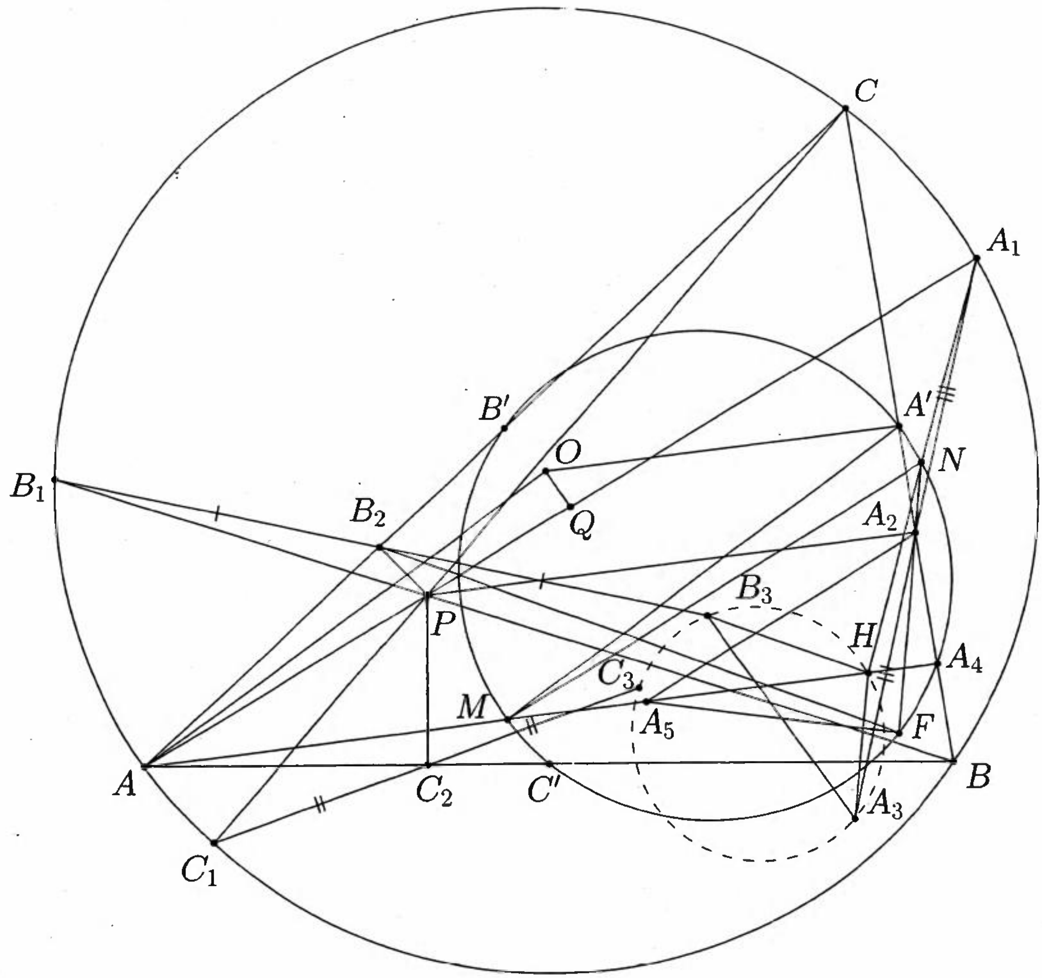


Proof. Using Fontene's Theorems, we get that F lies on the circumcircle of $\triangle A'B'C'$.

Let I', I and L be the midpoints of AP, A_2A_5 and AA_2, respectively. Then it is clear that $L \in B'C'$. Moreover, the points I', I and L lie on a line perpendicular to BC, due to APA_2A_5 being a parallelogram. This implies that I' is the reflection of I with respect to $B'C'$.

Therefore, the circles with diameters AP and A_2A_5 are symmetric with respect to $B'C'$, so $B'C'$ is the radical axis of these two circles.

Now consider the circumcircle of $\triangle A_2B_2C_2$. The above arguments show that V lies on the radical axis of the circle with diameter A_2A_5 and the circumcircle of $\triangle A_2B_2C_2$. In conclusion, the point F lies on the circle with diameter A_2A_5, which proves the lemma.



Let FA_2 intersect the circumcircle of $\triangle C'B'A'$ for the second time at the point N. Let M be the midpoint of AH and let Q be the midpoint of AA_1. Let O be the center of k.

We have $OA' = R\cos\gamma = AM$ and $AM \parallel OA'$. Therefore, $AMA'O$ is a parallelogram and in particular AO is parallel and equal to $A'M$.

The lemma yields that F lies on the line A_2A_5. Also, Problem 4.8.1 implies that the points A_4, B_4 and C_4 lie on the circumcircle of $\triangle A'B'C'$. Thus, we have

$$\angle A_5A_2F = \angle A_5A_4F = \angle MNF \Rightarrow MN \parallel A_2A_5 \parallel AP.$$

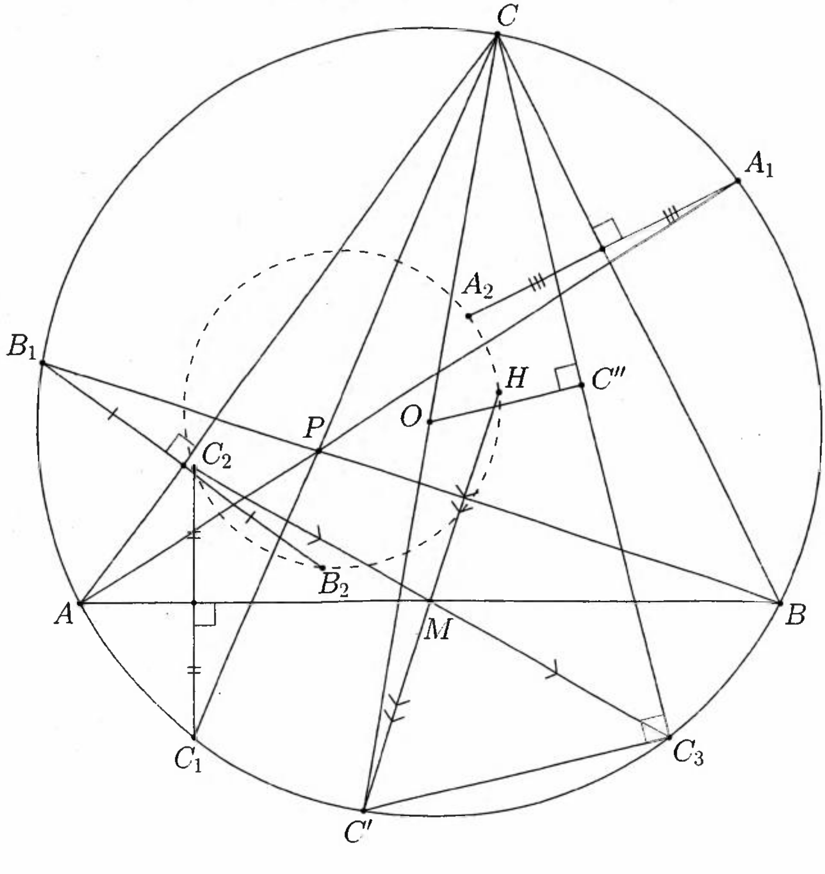
Now $AO \parallel A'M$ and $AO = A'M$ yield $\angle OAQ = \angle A'MN$. Moreover, $\angle A'NM = \angle A'A_4M = 90^\circ = \angle OQA$, hence $\triangle A'MN \cong \triangle OAQ$.

Thus, $MN = AQ = \dfrac{1}{2}AA_1$, which means that MN is a midsegment in $\triangle AHA_1$, so N is the midpoint of A_1H. On the other hand, A_2 is the midpoint of A_1A_3. Thus, A_2N is a midsegment in $\triangle A_1A_3H$.

Therefore, $HA_3 \parallel A_2N$, hence $HA_3 \parallel A_2F$ and analogously, $HB_3 \parallel B_2F$. Now using that F lies on the circumcircle of $\triangle A_2B_2C_2$ and the fact that $\triangle A_2B_2C_2 \sim \triangle A_3B_3C_3$, we conclude that

$$\angle A_3HB_3 = 180^\circ - \angle A_2FB_2 = 180^\circ - \angle A_2C_2B_2 = 180^\circ - \angle A_3C_3B_3.$$

Problem 4.2.9. Let ABC be a triangle with orthocenter H and let P be an arbitrary point inside of the triangle. The lines AP, BP and CP intersect the circumcircle k of $\triangle ABC$ at the points A_1, B_1 and C_1, respectively. The points A_2, B_2 and C_2 are the reflections of A_1, B_1 and C_1 with respect to the lines BC, AC and AB, respectively. Prove that the quadrilateral $HA_2B_2C_2$ is cyclic.



Let M, N and P be the midpoints of AB, BC and CA, respectively. Let C_3, A_3 and B_3 be the reflections of C_2, A_2 and B_2 with respect to M, N and P, respectively. We have $\angle AC_2B = \angle AC_1B = \angle AC_3B$, hence $C_3 \in k$. Analogously, $A_3, B_3 \in k$.

Moreover, $\angle C_3CB = \angle C_3AB = \angle C_2BA = \angle ABC_1 = \angle ACC_1$, which means that CC_1 and CC_3 are isogonal conjugates. Analogously, BB_1 and BB_3 are isogonal conjugates, and AA_1 and AA_3 are isogonal conjugates. Therefore, the lines CC_3, BB_3 and AA_3 concur at a point – denote it by P'. It is exactly the point which is isogonally conjugate to P in $\triangle ABC$.

Let C', A' and B' be the reflections of H with respect to M, N and P, respectively. Problem 4.2.1 implies that C', A' and B' lie on k and that CC', AA' and BB' concur at O – the center of k.

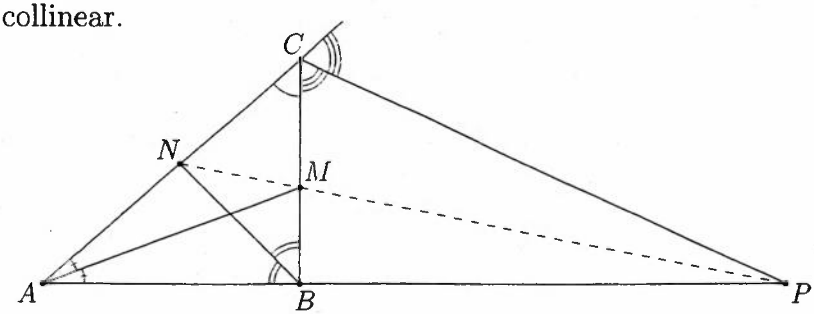
Let $C'' \in CC_3$ be a point such that $OC'' \perp CC_3$. Define the points A'' and B'' analogously. Then OC'' is a midsegment in $\triangle CC'C_3$ and

$$\overrightarrow{OC''} = \frac{1}{2}\overrightarrow{C'C_3} = \frac{1}{2}\overrightarrow{C_2H}.$$

Analogously, $\overrightarrow{OB''} = \frac{1}{2}\overrightarrow{B_2H}$ and $\overrightarrow{OA''} = \frac{1}{2}\overrightarrow{A_2H}$.

Therefore, the points H, A_2, B_2 and C_2 are concyclic if and only if the points O, A'', B'' and C'' are concyclic. Finally, note that O, A'', B'' and C'' lie on the circle with diameter OP'.

Problem 4.3.1. Let ABC be a triangle and let M and N be the feet of the angle bisectors through A and B, respectively. The point P is the foot of the external bisector through C. Prove that the points N, M and P are collinear.



Solution. The internal bisector theorem yields

$$\frac{BM}{MC} = \frac{AB}{AC}$$

and

$$\frac{CN}{NA} = \frac{CB}{AB}.$$

The external bisector theorem yields $\dfrac{AP}{PB} = \dfrac{AC}{CB}$.

Therefore, $\dfrac{AP}{PB} \cdot \dfrac{BM}{MC} \cdot \dfrac{CN}{NA} = \dfrac{AC}{CB} \cdot \dfrac{AB}{AC} \cdot \dfrac{CB}{AB} = 1$. Finally, Menelaus' Theorem gives the desired result.

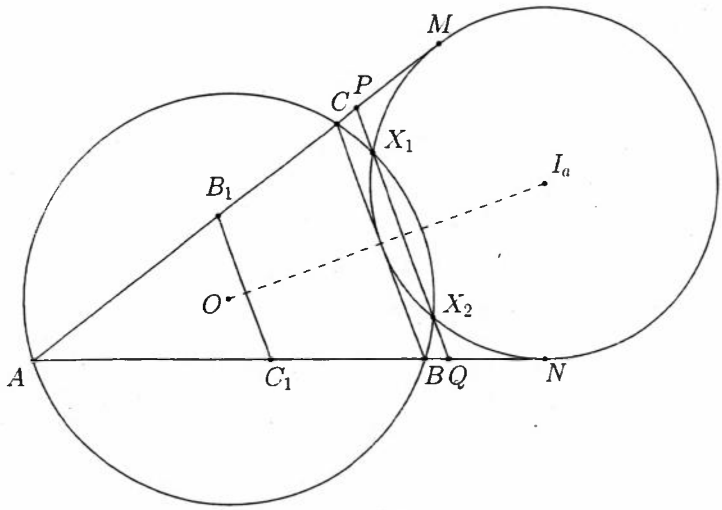
Problem 4.3.2. Let ABC be a triangle and let C_1 and B_1 be the feet of the bisectors through C and B, respectively. Let O be the circumcenter of $\triangle ABC$ and let I_a be the A-excenter. Prove that $OI_a \perp B_1C_1$.

Solution. Let X_1 and X_2 be the intersection points of the circumcircle k and the A-excircle ω_a of $\triangle ABC$. Since X_1X_2 is the radical axis of these circles, we have $OI_a \perp X_1X_2$. Let $X_1X_2 \cap AC = P$ and $X_1X_2 \cap AB = Q$.

Since $OI_a \perp PQ$, it suffices to prove that $B_1C_1 \parallel PQ$.

Let M and N be the points of tangency of ω_a and the lines AC and AB, respectively. Let $PM = x$. Then $PA = s - x$ and $PC = s - b - x$.

Since P lies on the radical axis of ω_a and k, we have $PC.PA = PM^2$, or equivalently $(s-x)(s-b-x) = x^2$ and so $s^2 - (2s-b)x - sb = 0$. Hence, $x = \dfrac{s(s-b)}{a+c}$. Now $AP = s - x = \dfrac{s(2s-b) - s(s-b)}{a+c} = \dfrac{s^2}{a+c}$.

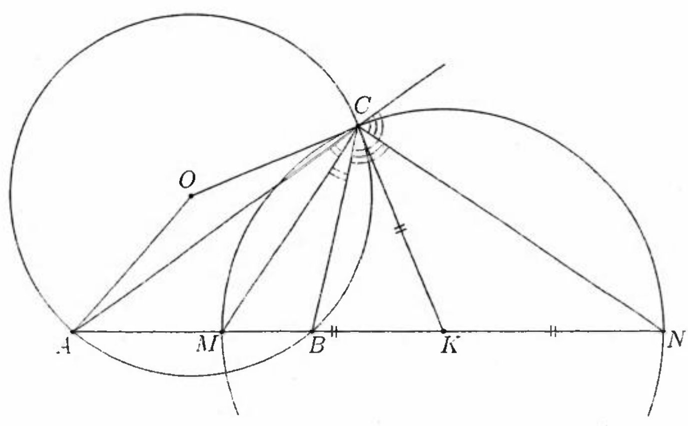


Using that $AB_1 = \dfrac{bc}{a+c}$, we get $\dfrac{AB_1}{AP} = \dfrac{bc}{s^2}$.

Analogously, $\dfrac{AC_1}{AQ} = \dfrac{bc}{s^2}$, which yields that $B_1C_1 \parallel PQ$.

Problem 4.3.3. Let ABC be a triangle ($\angle ABC > 90°$) and let CM and CN be the internal and the external bisectors of $\angle ACB$, respectively. Prove that the circles (ACB) and (MNC) are orthogonal, i.e. the tangent lines through one of their intersection points are perpendicular.

Solution. We have $\angle AMC = \beta + \dfrac{\gamma}{2}$. Note that $\angle AOC = 360° - 2\beta$ and $\angle ACO = \beta - 90°$. On the other hand, ΔMNC is right-angled with $\angle MCN = 90°$. Let K be the midpoint of MN. Then CK is a median to the hypotenuse in ΔMNC, hence $\angle MCK = \angle KMC = \alpha + \dfrac{\gamma}{2}$.



Now we have

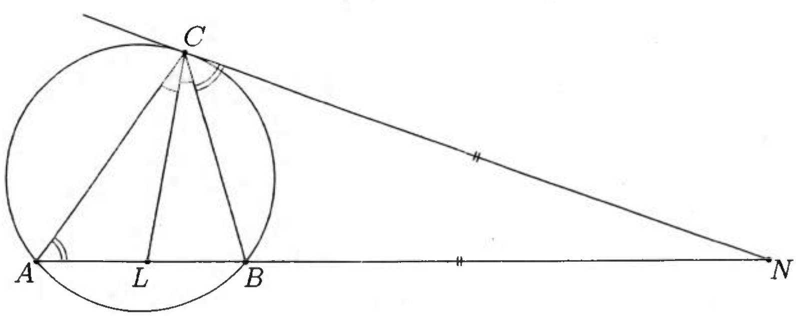
$$\angle OCK = \angle OCA + \angle ACM + \angle MCK = \beta - 90^\circ + \frac{\gamma}{2} + \alpha + \frac{\gamma}{2} = 90^\circ.$$

The equality $\angle OCK = 90^\circ$ implies that the tangent lines through C are perpendicular.

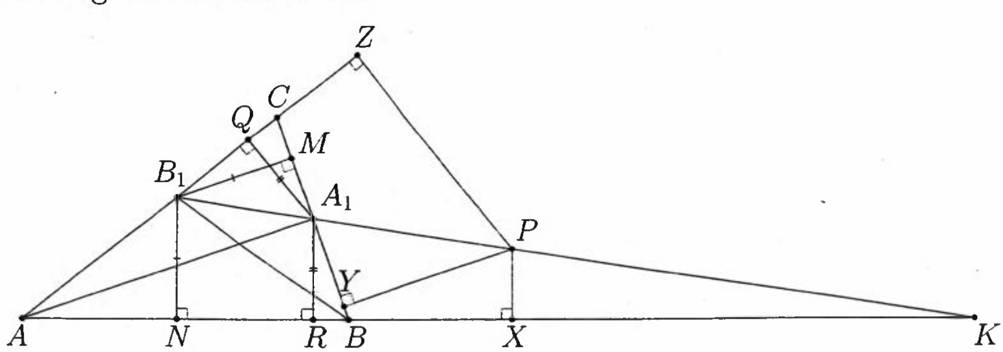
Note. The statement of the problem is true for non-obtuse triangles as well.

Problem 4.3.4. Let ABC be a triangle with circumcircle k. The line CL ($L \in AB$) is the angle bisector of $\angle ACB$. Denote the intersection point of the tangent line to k at C and the line AB by N. Prove that $NC = NL$.

Solution. We have $\angle CLN = \angle CAL + \angle ACL = \angle BCN + \angle BCL = \angle LCN$, and therefore $NC = NL$.



Problem 4.3.5. Let ABC be a triangle with bisectors AA_1, BB_1 and CC_1. Let P be an arbitrary point on the line A_1B_1 and let X, Y and Z be its projections onto the lines AB, BC and CA, respectively. Prove that the sum of the lengths of two of the segments PX, PY and PZ equals the length of the third one.



Solution. We will use the notations on the figure. Without loss of generality, we assume that $AC > BC$ and A_1 is between P and B_1. We will show that $PZ = PY + PX$. Note that $\triangle PYA_1 \sim \triangle B_1MA_1$, which implies $\frac{PY}{B_1M} = \frac{PA_1}{B_1A_1}$, hence $PY = B_1M\frac{PA_1}{A_1B_1}$.

The similarity $\triangle B_1A_1Q \sim \triangle B_1PZ$ gives $PZ = QA_1\frac{B_1P}{A_1B_1}$.

Let $AB \cap A_1B_1 = K$. Denote $B_1N = x$, $A_1R = y$, $PX = z$, $A_1B_1 = c$, $A_1P = b$ and $PK = a$. We will give an expression for z in terms of x, y, c and b.

The intercept theorem, applied to $\triangle B_1KN$ and the segment A_1R, gives $\frac{y}{x} = \frac{a+b}{a+b+c}$, hence $a = \frac{yb+yc-xb}{x-y}$. The same theorem, applied to $\triangle A_1RK$ and the segment PX, gives $\frac{z}{y} = \frac{a}{a+b}$. Thus, $PX = z = \frac{yb+yc-bx}{c}$.

Note that $B_1M = B_1N = x, A_1R = A_1Q = y$ and $A_1B_1 = c$. Then $PY = \frac{bx}{c}$ and $PZ = \frac{y(b+c)}{c}$.

Now we have $PX + PY = \frac{yb+yc-bx+bx}{c} = \frac{y(b+c)}{c} = PZ$.

Problem 4.3.6. Let ABC be a triangle with angle bisectors AA_1 and BB_1. Let E be the intersection point of the line A_1B_1 and the circumcircle of $\triangle ABC$, such that E and A lie in the same half-plane with respect to BC. Prove that $\frac{1}{EA} = \frac{1}{EB} + \frac{1}{EC}$.

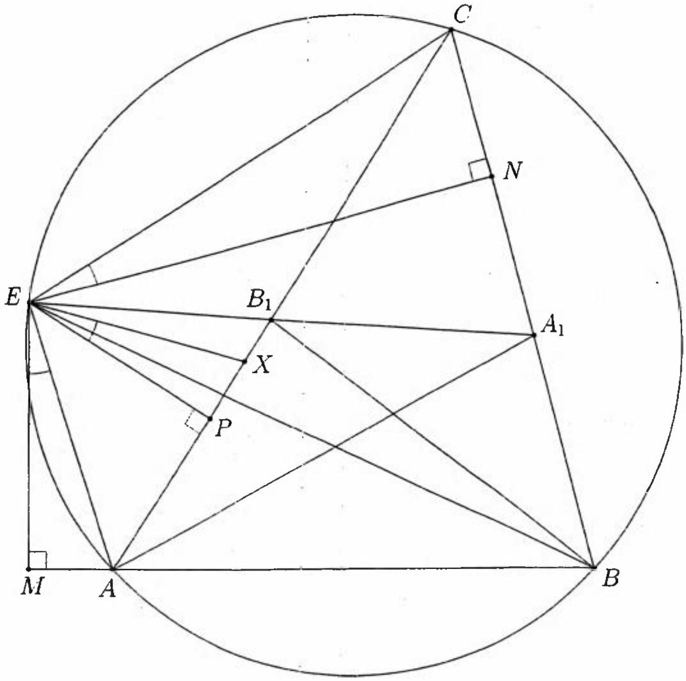
Solution. Denote the projections of E onto AB, BC and CA by M, N and P, respectively. Let $X \in AC$ and $\angle XEP = \angle CEN$. We have

$$\angle CEN = 90^\circ - \angle ECN = 90^\circ - \angle EAM = \angle MEA$$

$$\Rightarrow \triangle CEN \sim \triangle XEP \sim \triangle AEM.$$

Moreover, $\angle XEA = \angle PEM = \angle CAB = \angle CEB$.

Problem 4.3.5 yields $EM + EP = EN$ which, due to the similarities, implies $EA + EX = EC$.



Now we have

$$\begin{aligned} EB.EA + EC.EA - EB.EC &= EB.EA + EC.EA - EB(EA + EX) \\ &= EC.EA - EB.EX. \end{aligned}$$

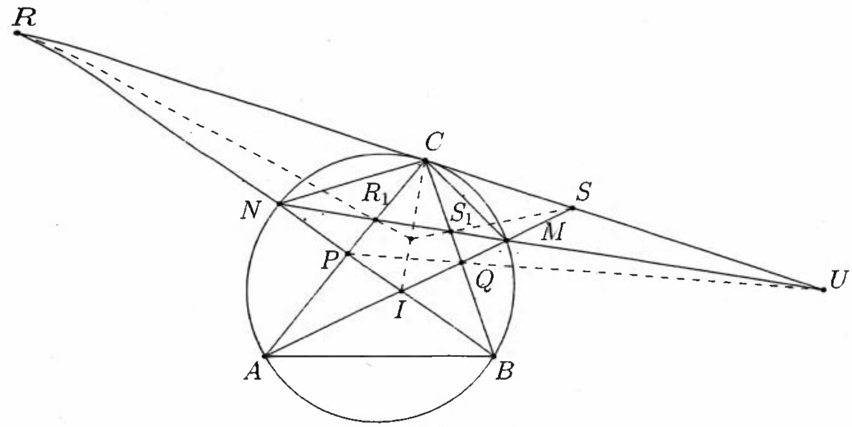
From $\angle XEA = \angle CEB$ it follows that $\angle CEX = \angle BEA$. Also, $\angle EBA = \angle ECX$. Hence, $\triangle ABE \sim \triangle XCE$, $\dfrac{EC}{EX} = \dfrac{BE}{EA}$ and thus $EC.EA - EB.EX = 0$.

Therefore, $EB.EA + EC.EA - EB.EC = 0$ or $\dfrac{1}{EA} = \dfrac{1}{EB} + \dfrac{1}{EC}$.

Problem 4.3.7. Let ABC be a triangle with angle bisectors AQ and BP and circumcircle k. The lines AQ and BP intersect k at the points M and N, respectively. Prove that the lines PQ, MN and the tangent line to k at C are concurrent.

Solution. Let l be the tangent line to k at C. We denote $U = l \cap MN$ and it suffices to prove that P, Q and U are collinear.

Let $R = BN \cap l$, $R_1 = AC \cap NM$, $S = AQ \cap l$ and $S_1 = BC \cap NM$. If we prove that the lines IC, RR_1 and SS_1 are concurrent, Desargues' Theorem (Problem 7.1), applied to $\triangle IRS$ and $\triangle CR_1S_1$, will imply that the points $P = IR \cap CR_1$, $Q = IS \cap CS_1$ and $U = RS \cap R_1S_1$ are collinear.



We will use the trigonometric form of Ceva's Theorem. The law of sines, applied to $\triangle PRR_1$ and $\triangle CRR_1$, gives

$$\frac{\sin\angle IRR_1}{\sin\angle R_1RC} = \frac{PR_1}{R_1C}\cdot\frac{\sin\angle RPC}{\sin\angle PCR} = \frac{PR_1}{R_1C}\cdot\frac{\sin\angle NPC}{\sin\beta}.$$

Also, by the trillium theorem (Problem 4.6.1), we have $NC = NI$ and $MI = MC$, which implies $\triangle NIM \cong \triangle NCM$. Thus, $\angle R_1NP = \angle R_1NC$ and the angle bisector theorem gives $\dfrac{PR_1}{R_1C} = \dfrac{NP}{NC}$.

But $\angle NCP = \dfrac{\beta}{2}$, hence the law of sines in $\triangle NPC$ gives $\dfrac{NP}{NC} = \dfrac{\sin\frac{\beta}{2}}{\sin\angle NPC}$. Then $\dfrac{\sin\angle IRR_1}{\sin\angle R_1RC} = \dfrac{\sin\frac{\beta}{2}}{\sin\angle NPC}\cdot\dfrac{\sin\angle NPC}{\sin\beta} = \dfrac{1}{2\cos\frac{\beta}{2}}$.

Analogously, we derive $\dfrac{\sin\angle RSS_1}{\sin\angle ISS_1} = 2\cos\dfrac{\alpha}{2}$.

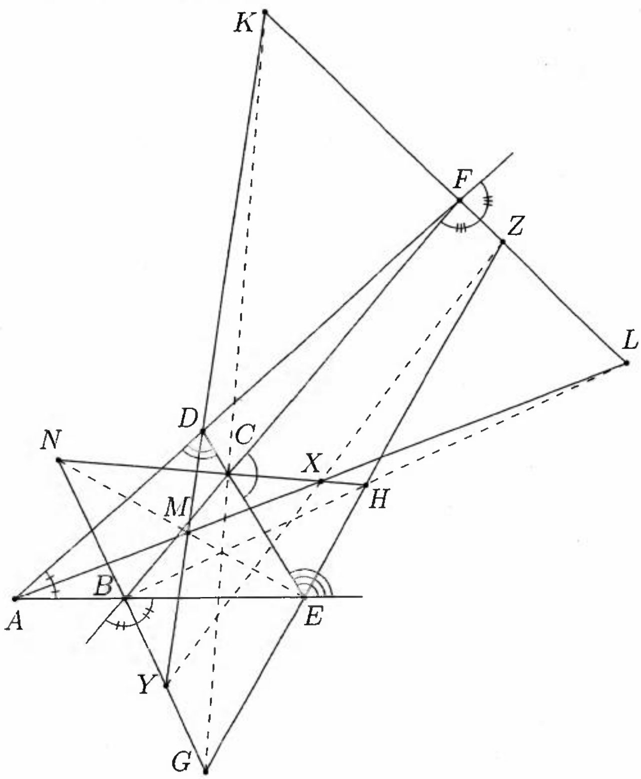
On the other hand, $\angle PIC = 90^\circ - \dfrac{\gamma}{2}$ and $\angle CIS = 90^\circ - \dfrac{\beta}{2}$. Therefore,

$$\frac{\sin\angle IRR_1}{\sin\angle R_1RU}\cdot\frac{\sin\angle RSS_1}{\sin\angle S_1SI}\cdot\frac{\sin\angle SIC}{\sin\angle CIR} = \frac{1}{2\cos\frac{\beta}{2}}\cdot\frac{2\cos\frac{\alpha}{2}}{1}\cdot\frac{\sin\left(90^\circ-\frac{\beta}{2}\right)}{\sin\left(90^\circ-\frac{\alpha}{2}\right)}$$

$$= \frac{\cos\frac{\alpha}{2}}{\cos\frac{\beta}{2}}\cdot\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = 1.$$

We conclude that the lines IC, RR_1 and SS_1 are concurrent.

Problem 4.3.8. Let $ABCD$ be a quadrilateral. The rays AB and DC intersect at the point E and the rays AD and BC intersect at the point F. The internal angle bisectors of $\angle EAF$ and $\angle ECF$ intersect at X. The internal angle bisector of $\angle ADE$ intersects the external angle bisector of $\angle EBC$ at Y. The external angle bisectors of $\angle AFB$ and $\angle AEC$ intersect at Z. Prove that the points X, Y and Z are collinear.



Solution. Denote
$YD \cap ZF = K$,
$BY \cap ZE = G$,
$CX \cap ZE = H$,
$AX \cap FZ = L$,
$DY \cap AX = M$,
$CX \cap BY = N$.

Now we have that N is the E-excenter of $\triangle BCE$, and that M is the incenter of $\triangle AED$.

Hence, M, N and E lie on the angle bisector of $\angle BEC$. We also have that H and L are the B-excenter and the A-excenter of $\triangle BEC$ and $\triangle AFB$, respectively.

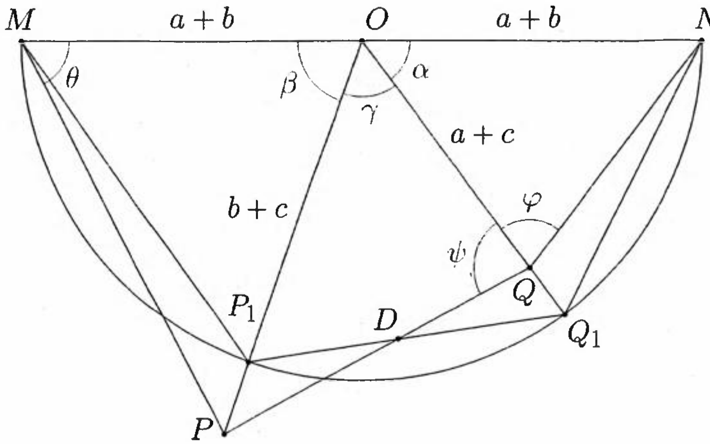
Also, note that K and G are the C-excenters of $\triangle DCF$ and $\triangle BCE$. Hence, the points M, N and E lie on the angle bisector of $\angle BEC$.

Moreover, the points B, H and L lie on the angle bisector of $\angle EBC$, and the points G, C and K lie on the angle bisector of $\angle BCE$. Therefore, the lines GK, HL and MN are concurrent.

Finally, Desargues' Theorem (Problem 7.1), applied to $\triangle NGH$ and $\triangle MKL$, yields that the points $X = ML \cap NH$, $Y = NG \cap MK$ and $Z = GH \cap KL$ are collinear.

Problem 4.3.9. Let ABC be a triangle such that $AC > AB > BC$. Let AA_1, BB_1 and CC_1 be its angle bisectors. The circumcircle of $\triangle A_1B_1C_1$ intersects AB, AC and BC for the second time at the points C_2, B_2 and A_2, respectively. Prove that $C_1C_2 = A_1A_2 + B_1B_2$.

Solution. Set $AB = c$, $AC = b$ and $BC = a$. Then $b > c > a$ and thus $b + c > b + a > c + a$. We will prove that C_2 lies between C_1 and B. Analogously, it follows that B_2 lies between B_1 and C, and A_2 lies between A_1 and B. The angle bisector theorem gives $AC_1 = \dfrac{bc}{a+b}$, $BC_1 = \dfrac{ac}{a+b}$, $BA_1 = \dfrac{ac}{b+c}$, $CA_1 = \dfrac{ab}{b+c}$, $CB_1 = \dfrac{ab}{a+c}$ and $AB_1 = \dfrac{bc}{a+c}$. Since $\angle B_1C_2A = \angle C_1A_1B_1$, it suffices to prove that $\angle B_1C_1A > \angle C_1A_1B_1$. Denote $\angle AC_1B_1 = \varphi$, $\angle B_1A_1C = \psi$ and $\angle C_1A_1B = \theta$. We will prove that $\varphi + \psi + \theta > 180°$.



Let $k(O, 2(a+b))$ be a circle with diameter MN. The points P and Q are chosen in the same half-plane with respect to MN, such that $\angle POM = \beta$, $OP = b + c$, $\angle NOQ = \alpha$ and $OQ = a + c$. Then $\triangle MOP \sim \triangle A_1BC_1$, $\triangle POQ \sim \triangle B_1CA_1$ and $\triangle QON \sim \triangle C_1AB_1$. Thus, $\angle OQN = \varphi$, $\angle PQO = \psi$ and $\angle PMO = \theta$. Let k intersect the rays OP and OQ at the points P_1 and Q_1, respectively. Then P_1 does not belong to the segment OP, while Q_1 belongs to the segment OQ. Let the intersection point of the line segments PQ and P_1Q_1 be D.

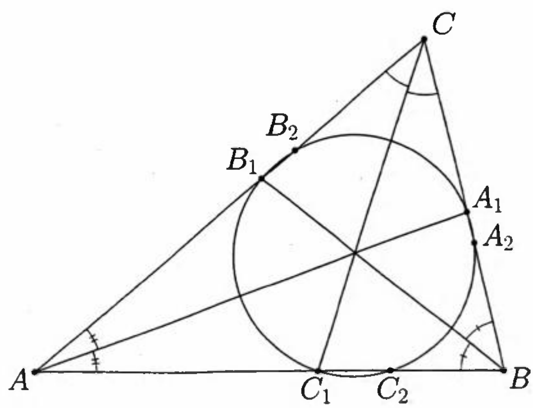
We have that $\theta > \angle P_1MN$ and

$$\varphi + \psi = \angle DQ_1N + \angle QDQ_1 + \angle QNQ_1 > \angle P_1Q_1N = 180° - \angle P_1MN.$$

Therefore, $\varphi + \psi + \theta > 180°$, as desired.

Let $A_1A_2 = x$, $B_1B_2 = y$ and $C_1C_2 = z$.

Now, using the power of the points A, B and C with respect to the circle, we derive the following system of equations:



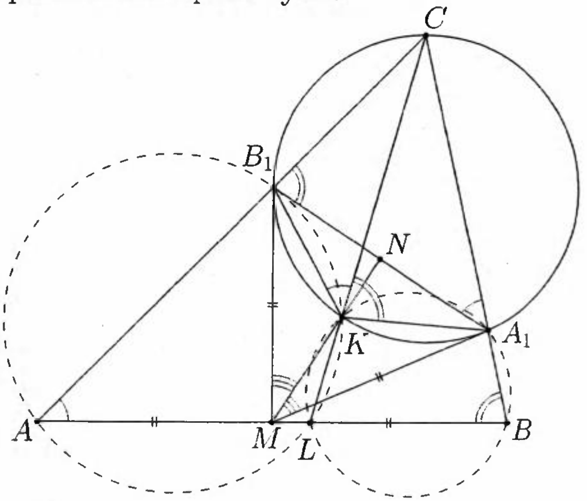
$$\left|\begin{array}{l} \dfrac{ac}{a+b}\left(\dfrac{ac}{a+b}-z\right)=\dfrac{ac}{b+c}\left(\dfrac{ac}{b+c}-x\right) \\[2ex] \dfrac{bc}{a+b}\left(\dfrac{bc}{a+b}+z\right)=\dfrac{bc}{a+c}\left(\dfrac{bc}{a+c}+y\right) \\[2ex] \dfrac{ab}{a+c}\left(\dfrac{ab}{a+c}-y\right)=\dfrac{ab}{b+c}\left(\dfrac{ab}{b+c}+x\right) \end{array}\right.$$

Solving the system with respect to x, y and z gives

$$\left|\begin{array}{l} y=\dfrac{a+c}{2}\left(\dfrac{c}{a+b}+\dfrac{b(a-c)}{(a+c)^2}-\dfrac{a}{b+c}\right) \\[2ex] x=\dfrac{b+c}{2}\left(\dfrac{b}{a+c}+\dfrac{a(c-b)}{(b+c)^2}-\dfrac{c}{a+b}\right) \\[2ex] z=\dfrac{a+b}{2}\left(\dfrac{b}{a+c}+\dfrac{c(a-b)}{(a+b)^2}-\dfrac{a}{b+c}\right) \end{array}\right.$$

We can see that $z = x + y$ by considering the fractions with the same denominators separately. For example, $\dfrac{b(a-c)}{2(a+c)} + \dfrac{(b+c)b}{2(a+c)} = \dfrac{(a+b)b}{2(a+c)}$.

Problem 4.3.10. Let ABC be a triangle with angle bisector $CL (L \in AB)$. The circle with diameter AB, centered at M, intersects the sides AC and BC for the second time at the points B_1 and A_1, respectively. The angle bisector of $\angle A_1MB_1$ intersects the line CL at the point K. Prove that the quadrilaterals $ALKB_1$ and $BLKA_1$ are cyclic.

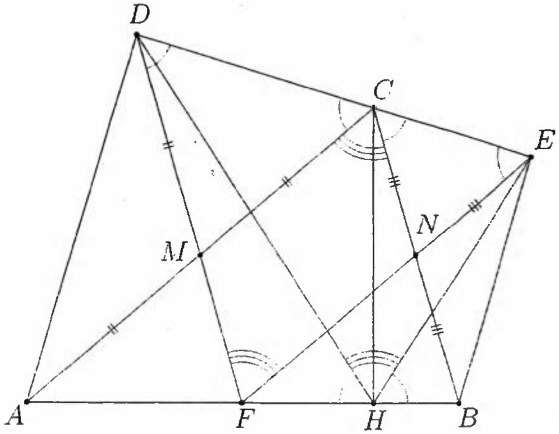


Solution. Denote the intersection point of the lines MK and A_1B_1 by N.

Now in $\triangle A_1MB_1$ we have that $MA_1 = MB_1$ and we have that MK is an angle bisector of $\angle A_1MB_1$. Hence, the line MK is the perpendicular bisector of A_1B_1. Thus, K is the midpoint of the arc $\widehat{A_1B_1}$ of the circumcircle of $\triangle A_1B_1C$ because it lies on the angle bisector of $\angle A_1CB_1$ and on the perpendicular bisector of A_1B_1.

Hence, the quadrilateral A_1CB_1K is cyclic, which implies $\angle ABC = \angle A_1B_1C = \angle A_1KC$. Therefore, the quadrilateral $BLKA_1$ is cyclic too. Analogously, the quadrilateral $ALKB_1$ is cyclic.

Problem 4.3.11. Let ABC be a triangle and let l be its external bisector through C. The points D and E are the projections of A and B onto l. The line CH $(H \in AB)$ is the altitude from C to AB and the point F is the midpoint of AB. Prove that the quadrilateral $DEHF$ is cyclic.



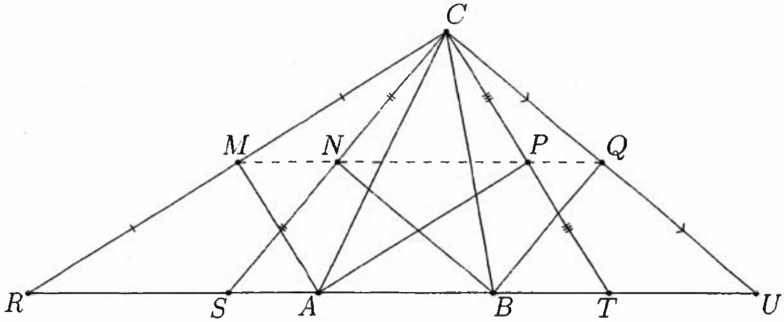
Solution. Let M and N be the midpoints of AC and BC, respectively. Then $FM \parallel BC$ and $FN \parallel AC$.

On the other hand, DM is a median to the hypotenuse in the right-angled $\triangle CDA$. Thus, $\angle MDE = \angle MCD = \angle BCE$.

Therefore, $DM \parallel BC$, hence the points D, M and F are collinear. Analogously, the points E, N and F are collinear. Also, $CMFN$ is a parallelogram, so we have $\angle EFD = \angle ACB$.

The quadrilaterals $AHCD$ and $BHCE$ are cyclic, so $\angle AHD = \angle ACD = \angle BCE = \angle BHE$. These give $\angle DHE = \angle ACB = \angle EFD$. Therefore, the quadrilateral $DEHF$ is cyclic.

Problem 4.3.12. Let ABC be a triangle. The points M, N, P and Q are the feet of the perpendiculars from C to the internal and external angle bisectors of $\angle BAC$ and $\angle ABC$, as shown in the figure. Prove that the points M, N, P and Q are collinear.



Solution. Denote the intersection points of CM, CN, CP and CQ, and the line AB by R, S, T and U, respectively. In $\triangle RAC$, the segment AM is both an altitude an angle bisector. Thus, M is the midpoint of RC.

Analogously, the points N, P and Q are the midpoints of the segments CS, CT and CU, respectively. We conclude that the points M, N, P and Q lie on the midsegment of $\triangle ABC$, parallel to AB.

Problem 4.3.13. Let ABC be a triangle with orthocenter H. The points L and P are the projections of H onto the internal and external angle bisectors of $\angle ACB$, respectively. Prove that the points M, L and P are collinear, where M is the midpoint of AB.

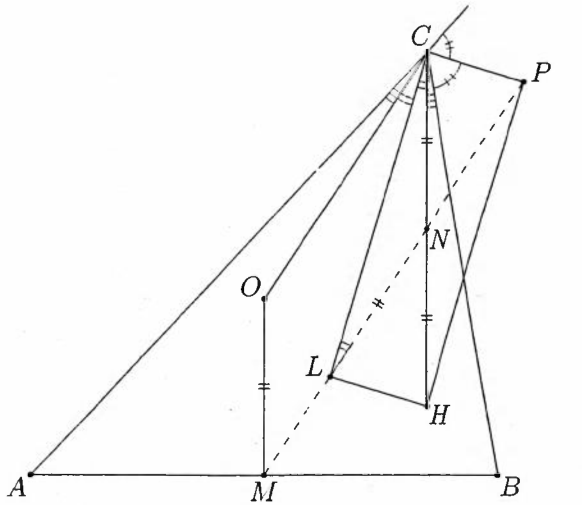
Solution. Since $\angle PCL = 90°$, the quadrilateral $LHPC$ is a rectangle. Let N be the intersection point of its diagonals. Then $LN = CN = NH$ and $\angle CLN = \angle LCN$.

However, since O and H are isogonal conjugates, we have $\angle ACO = \angle BCH$. Since CL is a bisector, we have

$\angle OCL = \angle HCL = \angle CLN.$

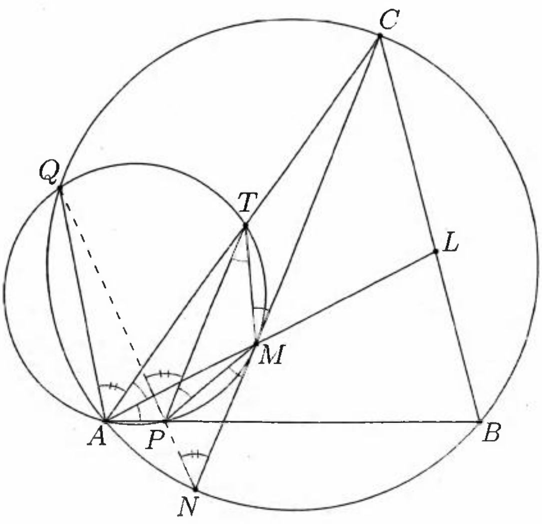
Hence, the lines CO and LN are parallel. By Problem 4.2.1, we have that CN is parallel and equal to OM. This implies that $CONM$ is a parallelogram and $OC \parallel MN$.

Therefore, $MN \parallel LN$ and we conclude that the points M, L and N lie on a line passing through the point P.



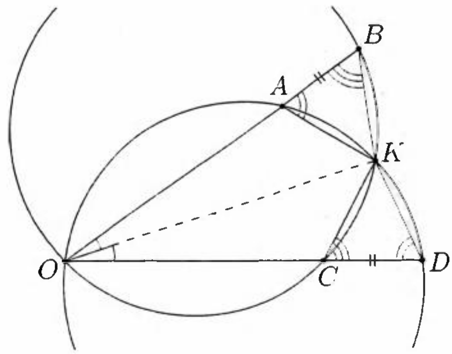
Problem 4.3.14. Let ABC be a triangle. A line through C intersects the angle bisector through A and the circumcircle of $\triangle ABC$ at the points M and N, respectively. A circle k_1 passes through A and touches CM at the point M. Let k_1 intersect AB and the circumcircle of $\triangle ABC$ at the points P and Q, respectively. Prove that the points N, P and Q are collinear.

Solution. Let $k_1 \cap AC = \{A, T\}$. The quadrilateral $APMT$ is cyclic, hence $\angle PMN = \angle PAM = \angle MAT = \angle MPT$. Thus, $TP \parallel CN$. Moreover, $\angle CNQ = \angle CAQ = \angle TPQ$, which means that the points N, P and Q are collinear.



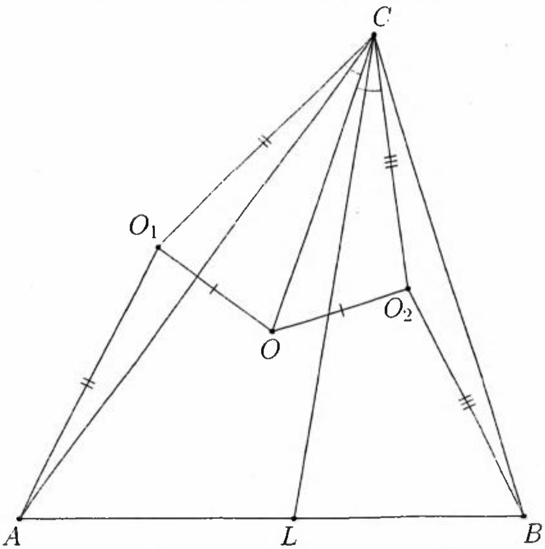
Problem 4.3.15. Let $\angle AOC$ be an angle. The point B lies on the ray OA and the point D lies on the ray OC. Moreover, $AB = CD$, A lies between O and B, and C lies between O and D. The circumcircles of $\triangle ADO$ and $\triangle BOC$ intersect again at the point K. Prove that OK is the angle bisector of $\angle AOC$.

Solution. Since the quadrilaterals $AKDO$ and $BKCO$ are cyclic, we have $\angle DCK = \angle OBK$ and $\angle ODK = \angle KAB$. On the other hand, $CD = AB$, thus $\triangle KBA \cong \triangle KCD$.



Therefore, the distances from K to the arms of $\angle AOC$ are equal, as respective altitudes in these congruent triangles. We conclude that OK is the angle bisector of $\angle AOC$.

Problem 4.3.16. Let ABC $(AC > BC)$ be a triangle, let CL $(L \in AB)$ be its angle bisector, and let O be its circumcenter. Denote the circumcenters of $\triangle ALC$ and $\triangle BLC$ by O_1 and O_2, respectively. Prove that $OO_1 = OO_2$.



Solution. We have

$$\angle ALC = \beta + \frac{\gamma}{2}$$

which gives

$$\angle AO_1C = 360^\circ - 2\beta - \gamma.$$

and $\angle O_1CA = \beta + \dfrac{\gamma}{2} - 90^\circ$.

Then

$$\angle O_1CO = \angle ACO + \angle ACO_1 = \beta + \frac{\gamma}{2} - 90^\circ + 90^\circ - \beta = \frac{\gamma}{2}$$

Analogously, we have $\angle O_2CO = \dfrac{\gamma}{2}$. Let R_1 and R_2 be the circumradii of $\triangle ALC$ and $\triangle BLC$, respectively. The law of sines, applied to $\triangle ABC$.

$\triangle ACL$ and $\triangle BCL$, yields

$$R_1 + R_2 = \frac{AL}{2\sin\frac{\gamma}{2}} + \frac{BL}{2\sin\frac{\gamma}{2}} = \frac{c}{2\sin\frac{\gamma}{2}} = \frac{c.\cos\frac{\gamma}{2}}{\sin\gamma} = 2R\cos\frac{\gamma}{2}.$$

Now we multiply both sides of the equality $R_1 + R_2 = 2R.\cos\frac{\gamma}{2}$ by $R_1 - R_2$ and obtain

$$R_1^2 - 2RR_1.\cos\frac{\gamma}{2} = R_2^2 - 2RR_2.\cos\frac{\gamma}{2}$$
$$\Leftrightarrow R_1^2 + R^2 - 2RR_1.\cos\frac{\gamma}{2} = R_2^2 + R^2 - 2RR_2.\cos\frac{\gamma}{2} \Leftrightarrow OO_1^2 = OO_2^2.$$

(We applied the law of cosines in $\triangle OO_1C$ and $\triangle OO_2C$.)

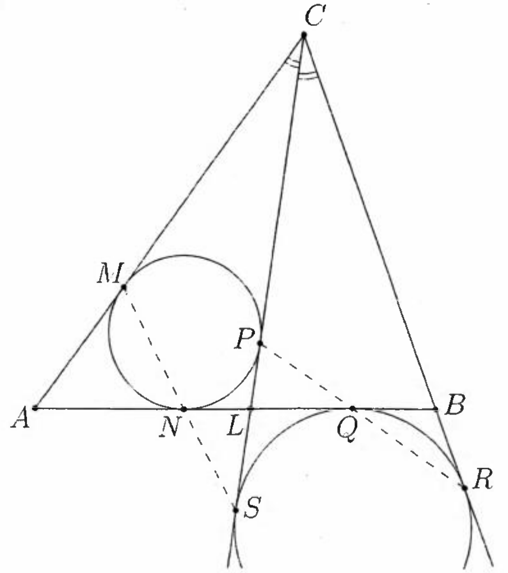
Note. The condition $AC > BC$ implies that O_1 is outside and O_2 is inside $\triangle ABC$.

Problem 4.3.17. Let ABC be a triangle and let CL ($L \in AB$) be its angle bisector. The incircle of $\triangle ALC$ touches AC, AL and CL at the points M, N and P, respectively, and the C-excircle of $\triangle BCL$ touches the lines BL, CB and CL at the points Q, R and S, respectively. Prove that each of the triples of points M, N, S and P, Q, R are collinear.

Solution. Let $\angle BLC = x$. Denote the semiperimeters of $\triangle BLC$ and $\triangle ALC$ by s_1 and s_2, respectively. We have $CR = s_1$, $QL = s_1 - BC$, $LP = s_2 - AC$ and $PC = s_2 - AL$.

Now we give expressions for these four segments, where R_1 and R_2 are the circumradii of $\triangle BLC$ and $\triangle ALC$.

$$CR = 4R_1\cos\frac{\alpha}{2}\cos\frac{x}{2}\cos\frac{\gamma}{4},$$
$$QL = 4R_1\cos\frac{\alpha}{2}\sin\frac{x}{2}\sin\frac{\gamma}{4},$$



$$LP = 4R_2 \cos\left(90 - \frac{x}{2}\right) \sin\frac{\gamma}{4} \sin\frac{\beta}{2},$$

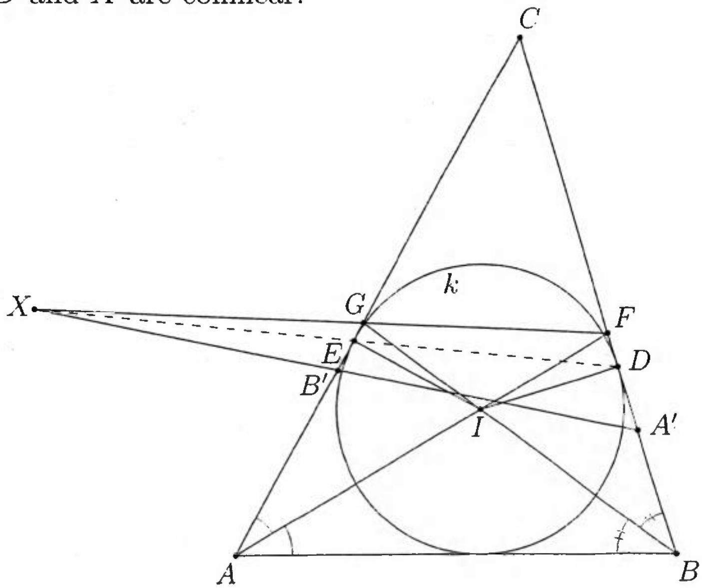
$$PC = 4R_2 \cos\frac{\gamma}{4} \sin\left(90 - \frac{x}{2}\right) \sin\frac{\beta}{2}.$$

Hence, $\dfrac{CR}{QL} = \cot\dfrac{x}{2}\cot\dfrac{\gamma}{4}$ and $\dfrac{LP}{PC} = \operatorname{tg}\dfrac{x}{2}\operatorname{tg}\dfrac{\gamma}{4}$, which implies that $\dfrac{CR}{QL}\cdot\dfrac{LP}{PC} = 1$.

On the other hand, $BR = QB$, thus $\dfrac{CR}{RB}\cdot\dfrac{BQ}{QL}\cdot\dfrac{LP}{PC} = 1$.

Now we apply Menelaus' Theorem to $\triangle BLC$ to get that the points P, Q and R are collinear. Analogously, we prove that the points M, N and S are collinear.

Problem 4.3.18. Let ABC be a triangle with altitudes AA' and BB' and incircle k centered at I. The lines AI and BI intersect BC and AC at the points F and G, respectively. The circle k touches BC and CA at the points D and E, respectively. If $A'B' \cap GF = X$, prove that the points E, D and X are collinear.

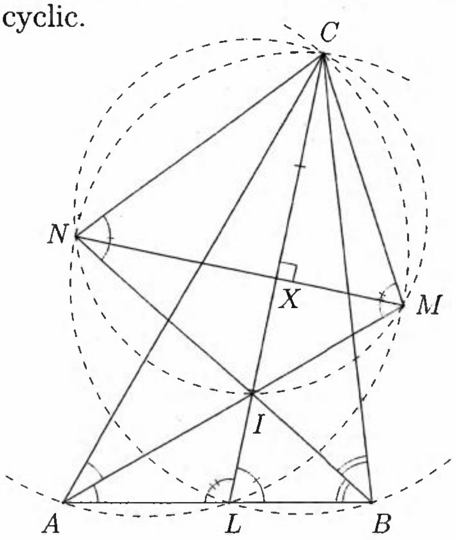


Solution. It follows by the cross-ratio properties, that it suffices to prove that $\dfrac{B'E.CG}{EG.B'C} = \dfrac{A'D.FC}{DF.CA'}$. But the intercept theorem and the angle bisector theorem give

$$\frac{B'E}{EG} = \frac{BI}{IG} = \frac{BC}{CG}, \frac{A'D}{DF} = \frac{AI}{IF} = \frac{AC}{CF}.$$

Hence, the assertion follows from $\frac{CB}{CB'} = \frac{CA}{CA'}$, which is true because $\triangle ABC \sim \triangle A'B'C$.

Problem 4.3.19. Let ABC be a triangle with angle bisector CL and incenter I. The perpendicular bisector of the segment CL intersects the angle bisectors of $\angle BAC$ and $\angle ABC$ at the points M and N, respectively. Prove that the quadrilateral $MINC$ is cyclic.



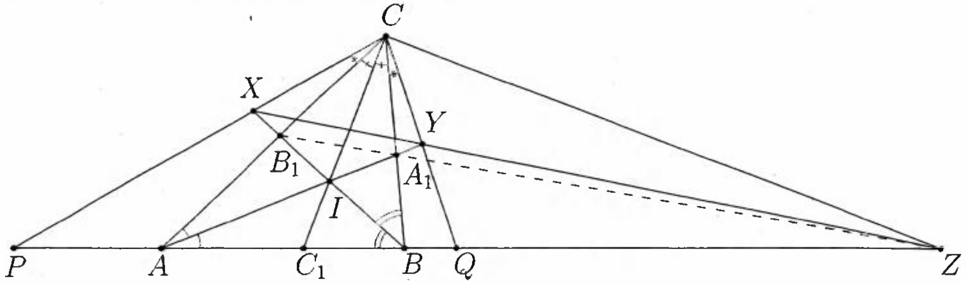
Solution. The points M and N always exist because $AC > AL$ and $BC > BL$. Since M lies on both the angle bisector of $\angle LAC$ and the perpendicular bisector of CL, the quadrilateral $ALMC$ is cyclic. Analogously, $BLNC$ is also cyclic.

Thus,

$$\angle MCI = \angle MAL = \frac{\alpha}{2},$$
$$\angle NCI = \angle NBA = \frac{\beta}{2}.$$

Hence, $\angle NCM = \frac{\alpha+\beta}{2}$, but $\angle MIN = \angle AIB = 180^\circ - \frac{\alpha+\beta}{2}$, which means that the quadrilateral $MINC$ is cyclic.

Problem 4.3.20. Let ABC be a triangle with incenter I. The points $Y \in AI$ and $X \in BI$ are chosen such that $\angle XCA = \angle YCB$. Prove that the lines AX, CI and BY are concurrent.



Solution. By applying Desargues' Theorem (Problem 7.1) to $\triangle ABC$ and $\triangle XYI$, we see that it suffices to show that the points $B_1 = AC \cap XI$, $A_1 = BC \cap YI$ and $Z = AB \cap XY$ are collinear.

We will use the notations on the figure. By Problem 4.3.1, applied to $\triangle ABC$, we see that it suffices to show that CZ is an external angle bisector of $\angle ACB$. The latter is equivalent to $\angle C_1CZ = 90°$. If we prove that the lines PY, QX and CC_1 are concurrent, we will derive $(P, C_1, Q, Z) = 1$ and consequently $\angle C_1CZ = 90°$.

Note that $\dfrac{PB}{AQ} = \dfrac{S_{\triangle BPC}}{S_{\triangle QAC}} = \dfrac{PC.CB}{AC.CQ}$. Hence,

$$\frac{BC}{BP} \cdot \frac{PC}{CQ} \cdot \frac{AQ}{AC} = 1.$$

By the angle bisector theorem we get

$$\frac{CX}{XP} \cdot \frac{PC_1}{C_1Q} \cdot \frac{QY}{YC} = \frac{BC}{BP} \cdot \frac{PC}{CQ} \cdot \frac{AQ}{AC} = 1,$$

and the statement follows by Ceva's Theorem.

Note. We can solve the problem by applying Ceva's Theorem in trigonometric form to $\triangle ABC$ and the points I, X and Y. Indeed, we need to prove that $\dfrac{\sin\angle BAX}{\sin\angle XAB_1} = \dfrac{\sin\angle ABY}{\sin\angle YBA_1}$. We manipulate the first ratio by applying the law of sines in $\triangle ABX$, $\triangle AB_1X$, $\triangle BXC$ and $\triangle B_1XC$. Analogously, we manipulate the second ratio by applying the law of sines in $\triangle ABY$, $\triangle A_1BY$, $\triangle AYC$ and $\triangle A_1YC$.

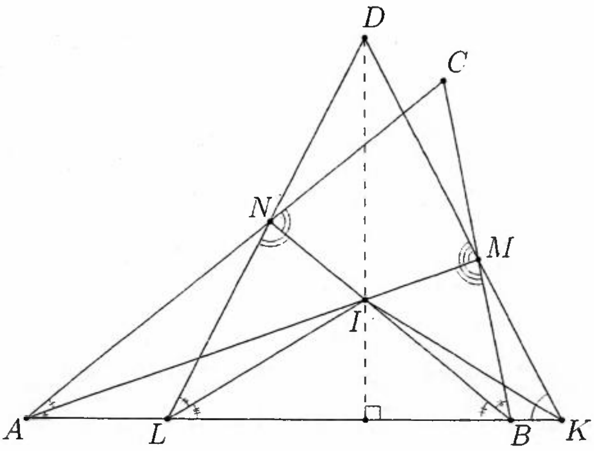
Problem 4.3.21. Let ABC be a triangle with angle bisectors AM and BN, intersecting at the point I. Points L and K are chosen on the line AB, such that LN and CN are symmetric with respect to BN, and such that CM and KM are symmetric with respect to AM. Let $D = LN \cap KM$. Prove that $DI \perp AB$.

Solution. The symmetry with respect to BN gives $\angle BLN = \angle BCN$, and the symmetry with respect to AM gives $\angle AKM = \angle ACM$. Therefore, $\triangle LDK$ is isosceles.

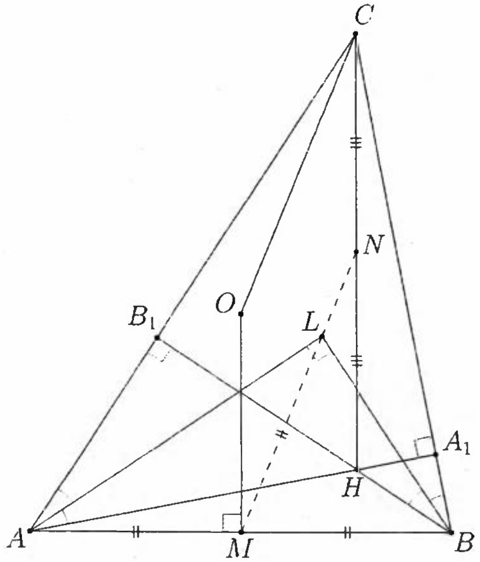
Since I is the K-excenter of $\triangle BKM$, we deduce that KI is the angle bisector of $\angle LKD$.

Analogously, LI is the angle bisector of $\angle KLD$.

Therefore, I is the incenter of $\triangle LKD$, which implies that $DI \perp AB$.



Problem 4.3.22. Let ABC $(AC > BC)$ be a triangle with altitudes AA_1 and BB_1, intersecting at the point H. The angle bisectors of $\angle HAC$ and $\angle HBC$ intersect at the point L. Let M and N be the midpoints of AB and CH, respectively. Prove that the points M, L and N are collinear.



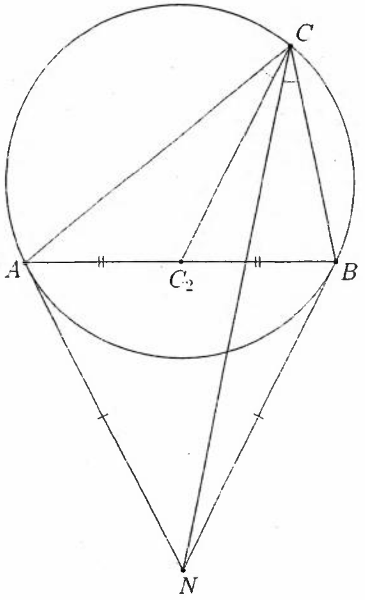
Solution. Let the point O be the circumcenter of $\triangle ABC$. By Problem 4.2.1, we have that OM is parallel and equal to NC, so $OMNC$ is a parallelogram. Thus, $\angle OMN = \angle OCN = \beta - \alpha$.

Our goal is to prove that $\angle OML = \angle OCN$. We have that $\angle ALB = \angle LAC + \angle LBC + \angle ACB = \angle A_1AC + \angle ACB = 90^\circ$. Hence, LM is the median to the hypotenuse in the right-angled $\triangle ABL$. Thus,

$$\begin{aligned}\angle OML &= 90^\circ - \angle BML = 90^\circ - (180^\circ - 2\angle ABL)\\ &= 2\angle ABL - 90^\circ = 2\beta - (90^\circ - \gamma) - 90^\circ\\ &= \beta - \alpha = \angle OCN = \angle OMN.\end{aligned}$$

Therefore, the points M, L and N are collinear.

Problem 4.4.1. Let ABC be a triangle and let C_2 be the midpoint of AB. The tangent lines to the circumcircle of $\triangle ABC$ at A and B intersect each other at the point N. Prove that $\angle ACC_2 = \angle BCN$.

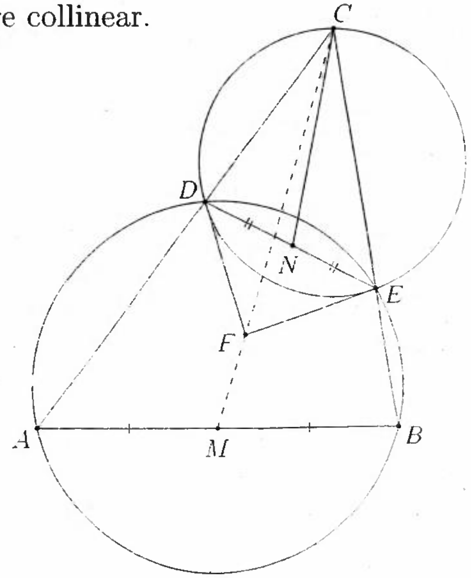


Solution. The law of sines, applied to $\triangle AC_2C$, $\triangle BC_2C$, $\triangle ANC$ and $\triangle BNC$, gives

$$\frac{\sin\angle ACC_2}{\sin\angle BCC_2} = \frac{AC_2}{BC_2}\cdot\frac{\sin\alpha}{\sin\beta} = \frac{\sin(\beta+\gamma)}{\sin(\alpha+\gamma)}$$
$$= \frac{BN}{CN}\cdot\frac{\sin(\beta+\gamma)}{\sin(\alpha+\gamma)} = \frac{\sin\angle BCN}{\sin\angle ACN}.$$

We also have $\angle ACC_2 + \angle BCC_2 = \angle BCN + \angle ACN$. We use that if $x + y = z + t$ and $\dfrac{\sin x}{\sin y} = \dfrac{\sin z}{\sin t}$, then $\cot x = \cot z$ and $\cot y = \cot t$. We get $\cot\angle ACC_2 = \cot\angle BCN$, and so $\angle ACC_2 = \angle BCN$.

Problem 4.4.2. Let ABC be a triangle. The points D and E are chosen on the sides AC and BC, such that the quadrilateral $ABED$ is cyclic. Let M be the midpoint of AB and let F be the intersection point of the tangent lines to the circumcircle of $\triangle DEC$ at the points D and E. Prove that the points M, F and C are collinear.

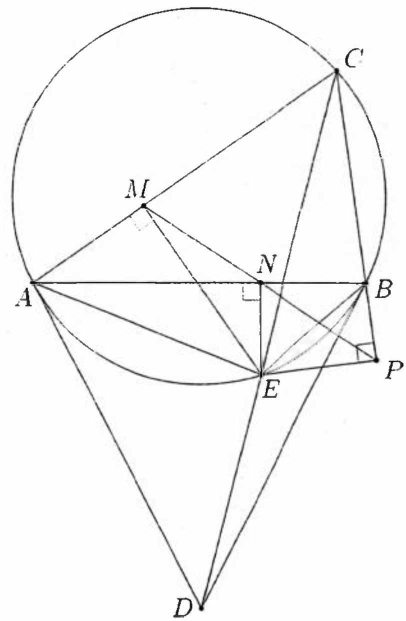


Solution. Let N be the midpoint of DE. Problem 4.4.1 yields that CF and CN are isogonal conjugates with respect to the vertex C in $\triangle EDC$. Also, note that $\triangle ABC \sim \triangle EDC$. The segments CM and CN are their respective medians.

Therefore, CM and CN are isogonal conjugates with respect to the vertex C in $\triangle ABC$. Hence, $\angle FCD = \angle NCE = \angle MCA$. This yields that the points M, F and C are collinear.

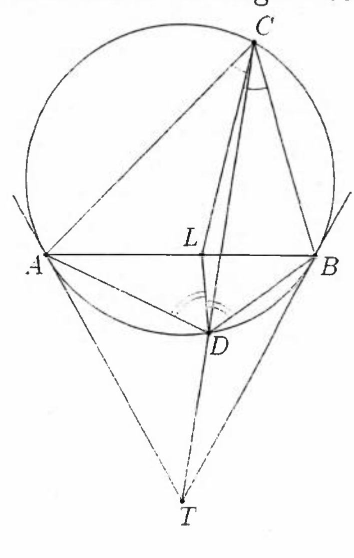
Problem 4.4.3. Let ABC be a triangle with circumcircle k. The tangent lines to k at A and B intersect at the point D. Let CD intersect k for the second time at the point E. The projections of E onto AB, BC and CA are N, P and M, respectively. Prove that N is the midpoint of PM.

Solution. By Problem 3.11, we have that the points M, N and P are collinear (they lie on the Simson line). Since $PBNE$ is inscribed in the circle with diameter BE and $AENM$ is inscribed in the circle with diameter AE, we have



$$\begin{aligned}\frac{PN}{NM} &= \frac{BE\sin\beta}{AE\sin\alpha}\\ &= \frac{\sin\angle BCD\sin\beta}{\sin\angle ACD\sin\alpha}\\ &= \frac{BD\sin\angle CBD\sin\beta}{AD\sin\angle CAD\sin\alpha}\\ &= \frac{\sin(\beta+\gamma)\sin\beta}{\sin(\alpha+\gamma)\sin\alpha} = 1.\end{aligned}$$

Problem 4.4.4. Let ABC be a triangle. The tangent lines to its circumcircle at the points A and B intersect at T. The line CT intersects the circumcircle of $\triangle ABC$ for the second time at the point D. Let $CL (L \in AB)$ be the angle bisector of $\angle ACB$. Prove that DL is the angle bisector of $\angle ADB$.



Solution. Problem 4.4.1 yields that CT is a symmedian in $\triangle ABC$. Thus, the quadrilateral $ADBC$ is harmonic. Hence,

$$AC.BD = AD.BC \Rightarrow \frac{AC}{BC} = \frac{AD}{BD}.$$

The angle bisector theorem applied to $\triangle ABC$ gives

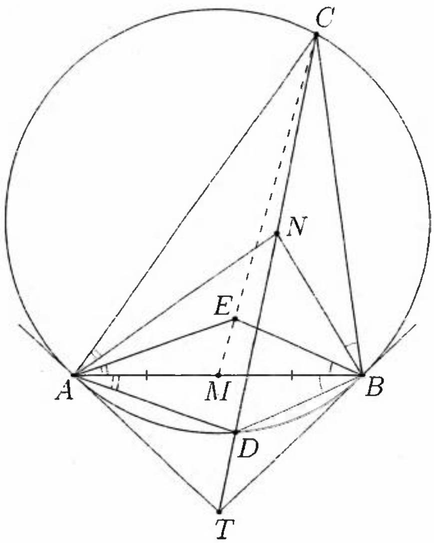
$$\frac{AC}{BC} = \frac{AL}{BL} \Rightarrow \frac{AD}{BD} = \frac{AL}{BL}.$$

Hence, DL is the angle bisector of $\angle ADB$.

Problem 4.4.5. Let ABC be a triangle. The tangent lines to its circumcircle at the points A and B intersect at T. The line CT intersects the circumcircle of $\triangle ABC$ for the second time at the point D. Let E be the reflection of D with respect to AB and let M be the midpoint of AB. Prove that the points C, E and M are collinear.

Solution. Let N be the midpoint of CD. As in Problem 4.4.6 we deduce that $\angle CBN = \angle DBA$. Moreover, the symmetry gives $\angle DBA = \angle ABE$.

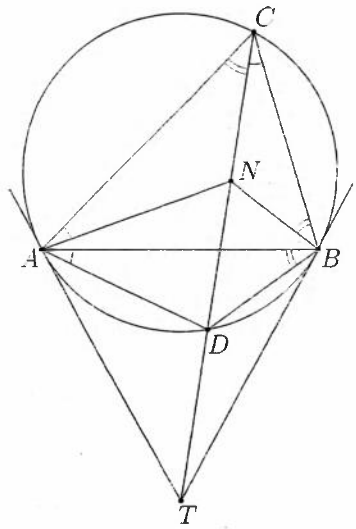
Hence, the lines BE and BN are isogonal conjugates with respect to the vertex B in $\triangle ABC$. Analogously, AE and AN are isogonal conjugates with respect to A in the same triangle.



Now Problem 4.9.1 yields that E and N are isogonal conjugates in $\triangle ABC$. Thus, $\angle ACE = \angle BCN$. But Problem 4.4.1 yields that $\angle ACM = \angle BCN$. Therefore, $\angle ACE = \angle ACM$ and the statement follows.

Problem 4.4.6. Let ABC be a triangle. The tangent lines to its circumcircle at the points A and B intersect at T. The line CT intersects the circumcircle of $\triangle ABC$ for the second time at the point D. Let $N \in CD$ be a point such that $\angle NBC = \angle ACN$. Prove that $\angle BCN = \angle CAN$.

Solution. By Problem 4.4.1, we have that CT is a symmedian in $\triangle ABC$. Thus, the quadrilateral $ADBC$ is harmonic. On the other hand, we have that $\angle ABD = \angle ACD$.



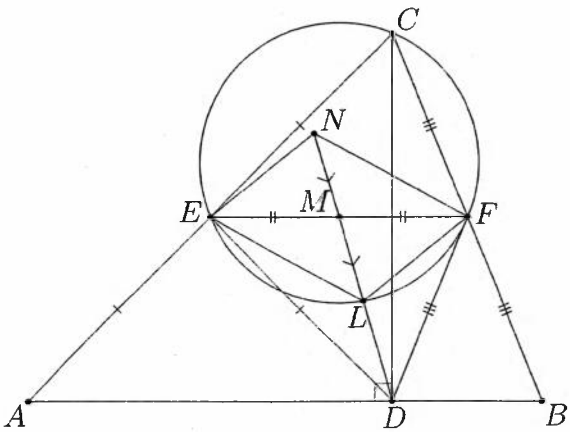
Then $\angle NBC = \angle DBA$, and hence the lines BA and BN are isogonal conjugates with respect to $\angle DBC$.

Since BA is a symmedian in $\triangle BDC$, the line BN is a median in the same triangle. Hence, $CN = ND$. Note that $\angle BCD = \angle BAD$.

We have that AB is a symmedian in $\triangle DAC$. But AN is a median in the same triangle. Therefore, $\angle CAN = \angle BAD = \angle NCB$.

Problem 4.4.7. Let ABC be a triangle. Let E, F and M be the midpoints of AC, BC and EF, respectively. The segment CD is an altitude in $\triangle ABC$. Prove that the circumcircles of $\triangle ECF$, $\triangle BDF$ and $\triangle ADE$ are concurrent at a point on the line DM.

Solution. Let the circumcircle of $\triangle EFC$ intersect the segment DM at the point L. Let N be the symmetric point to L with respect to M. Then $ELFN$ is a parallelogram. We have



$$AE = EC = ED$$

and

$$BF = FC = FD.$$

Hence, $\triangle EFC \cong \triangle EFD$. Therefore,

$$\angle ENF = \angle ELF = 180^\circ - \angle ECF = 180^\circ - \angle EDF.$$

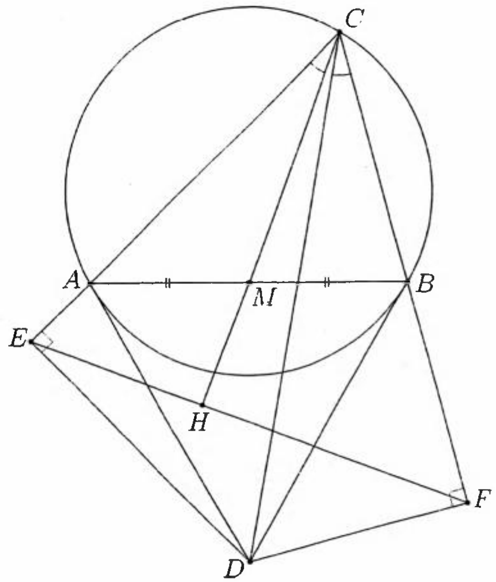
This implies that $ENFD$ is a cyclic quadrilateral. In particular,

$$\angle DBF = \angle EFC = \angle EFD = \angle END = \angle FLN,$$

which implies that $BDLF$ is a cyclic quadrilateral too.

Analogously, we deduce that the quadrilateral $ADLE$ is cyclic.

Problem 4.4.8. Let ABC be a triangle. Let M be the midpoint of AB and let D be the intersection point of the tangent lines to the circumcircle of $\triangle ABC$ at the points A and B. The projections of D onto the lines CA and CB are E and F, respectively. Prove that $CM \perp EF$.



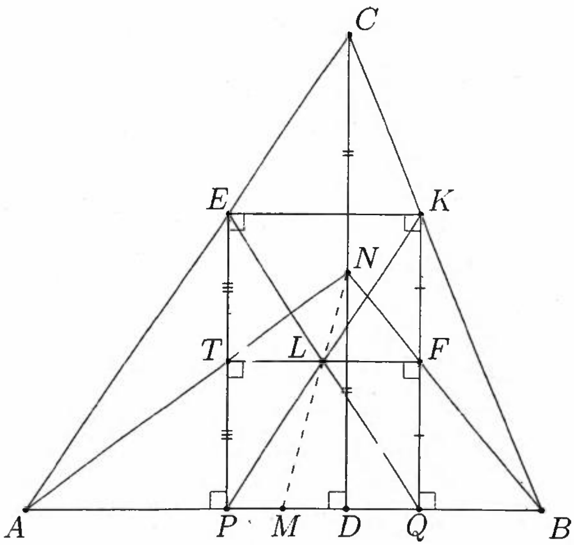
Solution. Let $CM \cap EF = H$. Note that Problem 4.4.1 yields $\angle ACM = \angle DCB$. It is clear that $EDFC$ is a cyclic quadrilateral.

Therefore, $\angle ECH = \angle DCB = \angle DEF = 90° - \angle HEC$, which means that $\angle EHC$ is right.

Problem 4.4.9. Let ABC be a triangle. Let N be the midpoint of the altitude CD and let M be the midpoint of AB. Let L be the Lemoine point of $\triangle ABC$. Prove that the points N, L and M are collinear.

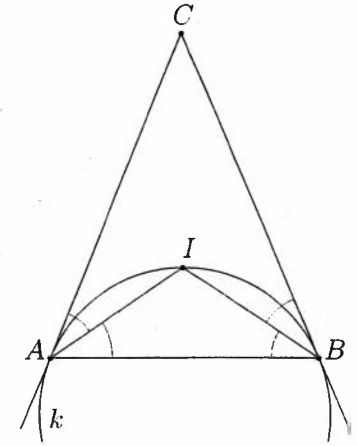
Solution. By Problem 2.32, we have that the Lemoine point is the center of the second Lemoine circle. Hence, there exists a rectangle $PQKE$, whose diagonals intersect at L and such that the points P and Q lie on AB, and the points E and K lie on AC and BC, respectively.

Let F be the midpoint of KQ and let T be the midpoint of EP.



Steiner's Theorem (Problem 5.2.1), applied to the trapezoids $DQKC$, $PDCE$ and $ABFT$, yields that the triples B, F, N; A, T, N and N, L, M are collinear.

Problem 4.5.1. Let k be a circle. A point C is chosen outside of k. Let the tangent lines from C to k touch the circle at the points A and B. Prove that the incenter of $\triangle ABC$ lies on k.



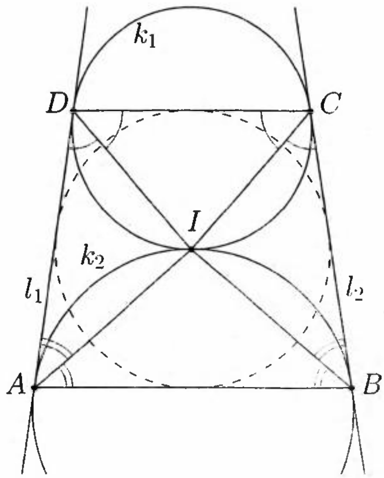
Solution. Let I be the midpoint of the smaller arc $\widehat{AB}$ of k.

Then $\angle CAI = \dfrac{\widehat{AI}}{2}$ and $\angle BAI = \dfrac{\widehat{BI}}{2}$. Thus, $\angle CAI = \angle BAI$.

Hence, AI is the bisector of $\angle BAC$. Analogously, BI is the bisector of $\angle ABC$.

Therefore, I is the incenter of $\triangle ABC$.

Problem 4.5.2. The circles k_1 and k_2 touch each other externally at the point I. Let l_1 and l_2 be the common external tangent lines of the two circles. Let the points of tangency of l_1 and l_2 to k_2 be A and B, respectively. Let the points of tangency of l_1 and l_2 to k_1 be D and C, respectively. Prove that the quadrilateral $ABCD$ is circumscribed.



Solution. Observe that l_1 and l_2 are symmetric with respect to the line that connects the centers of k_1 and k_2, and note that I lies on that line.

Hence, I is the midpoint of the arc $\widehat{CD}$ of k_1, and it is also the midpoint of the arc $\widehat{AB}$ of k_2. Thus, $\angle ADI = \angle CDI$.

It follows that DI is the bisector of $\angle ADC$. Analogously, AI, BI and CI are bisectors of $\angle BAD$, $\angle ABC$ and $\angle BCD$, respectively. Hence, the quadrilateral $ABCD$ is circumscribed.

Problem 4.5.3. Let ABC be a triangle with incenter I. The incircle of $\triangle ABC$ touches AB at N. The C-excircle of $\triangle ABC$ touches AB at M. Let P and Q be the orthogonal projections of A and B onto the line CI, respectively. Prove that the points P, M, Q and N lie on a circle with diameter MN.

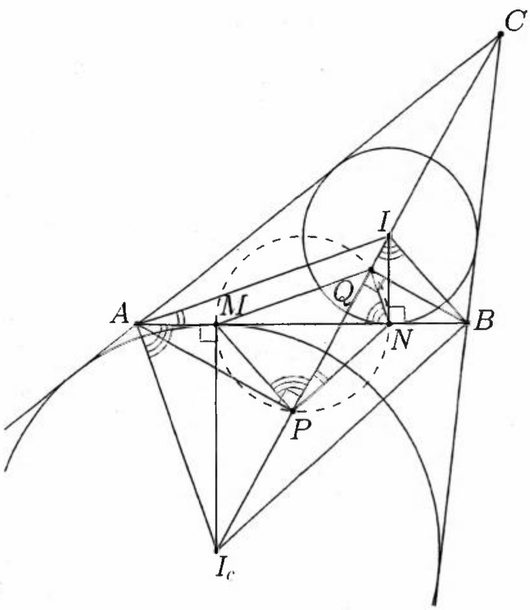
Solution. Let I_c be the C-excenter of $\triangle ABC$. Then C, I and I_c are collinear.

The quadrilateral I_cAIB is cyclic and hence

$$\angle I_cAB = \angle BII_c.$$

Observe that AI_cPM is also cyclic, and thus $\angle I_cAB = \angle MPQ$. Note that $BIQN$ is cyclic too. Thus, $\angle BIQ = \angle QNM$.

We have that $\angle MPQ = \angle MNQ$ and so $MPNQ$ is cyclic. But $PNIA$ is also cyclic, which means that $\angle IPN = \angle IAN$. Hence,



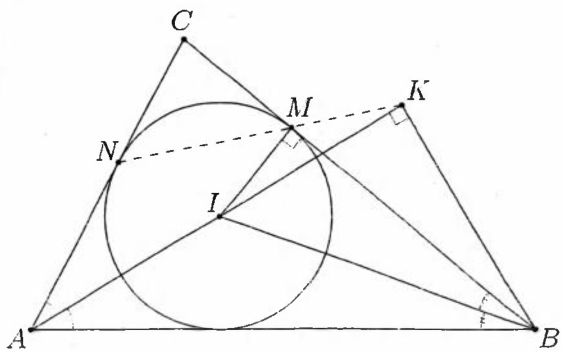
$$\angle MPN = \angle MPQ + \angle QPN = \angle MAI_c + \angle IAN = \angle I_cAI = 90^\circ$$

because AI and AI_c are an internal and an external bisector, respectively.

Problem 4.5.4. Let ABC be a triangle with incircle ω and with incenter I. Denote the orthogonal projection of B onto the line AI by K. Let N and M be the points of tangency of AC and BC to ω, respectively. Prove that the points M, N and K are collinear.

Solution. Using $\angle IMB = \angle IKB = 90^\circ$, we obtain that $IBKM$ is cyclic. Hence,

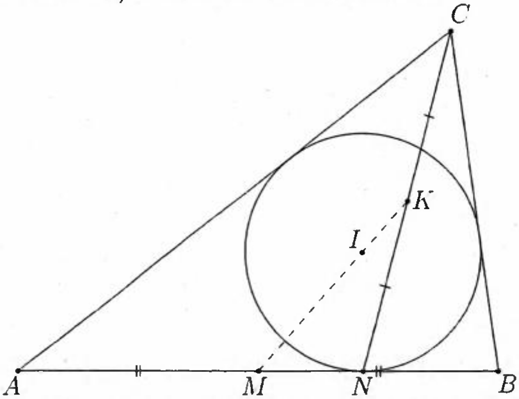
$$\begin{aligned}\angle KMB &= \angle KIB \\ &= 90^\circ - \frac{\angle ACB}{2} \\ &= \angle CMN\end{aligned}$$



and from here we conclude that M, N and K are collinear.

Problem 4.5.5. Let ABC be a triangle with incircle ω and with incenter I. Let AB touch ω at the point N. Let K and M be the midpoints of CN and AB, respectively. Prove that K, I and M are collinear.

Solution. Apply Newton's Theorem (Problem 5.4.6) to the degenerate quadrilateral $ANBC$, in which $\angle ANB = 180°$. The midpoints of its diagonals AB and CN, and its incenter I are collinear.



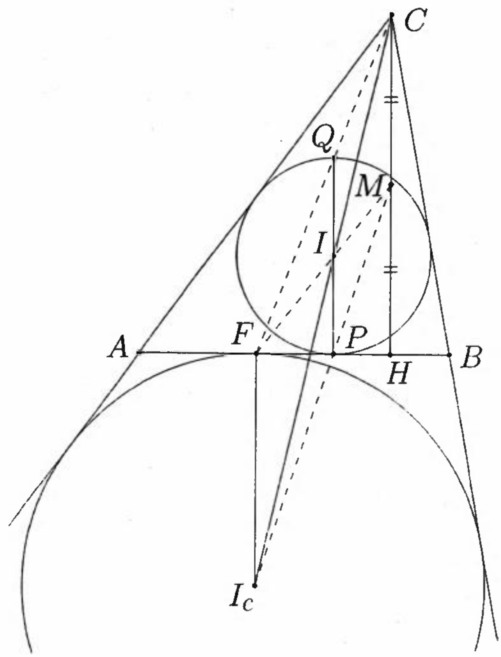
Problem 4.5.6. Let ABC be a triangle with incircle k and incenter I. Let AB touch k at the point P. Let CH ($H \in AB$) be an altitude in $\triangle ABC$. Denote the midpoint of CH by M. Let k_c (I_c) be the C-excircle of $\triangle ABC$, and let it touch AB at F. Prove that the points M, I and F are collinear, and prove that the points M, P and I_c are collinear.

Solution. Let Q be the reflection of P with respect to I. Let h be the homothety with center C that sends k to k_c. Then $h(Q) = F$. Hence, the points C, Q and F are collinear and $\dfrac{CQ}{CF} = \dfrac{IQ}{I_cF}$.

Observe that $PHCQ$ is a trapezoid. Steiner's Theorem (Problem 5.2.1), applied to that trapezoid, yields that the points F, I and M are collinear.

By the intercept theorem, we obtain

$$\frac{MI}{MF} = \frac{CQ}{CF} = \frac{IQ}{I_cF} = \frac{IP}{I_cF}.$$



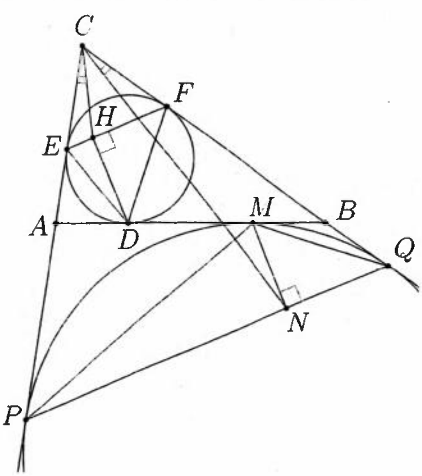
Hence, the points M, P and I_c are collinear.

Note. This problem is directly connected to 4.5.16. We are going to show the connection.

Consider the circle ω with diameter II_c. That circle passes trough A and B, so AB is the radical axis of ω and (ABC). Observe that the radical axis of ω and k coincides with the polar line of I_c with respect to k.

We conclude that there is a point X that lies on the line AB, on the polar line of I_c with respect to k, and on the radical axis of k and (ABC). By the polar property, we get that I_c lies on the polar line of X with respect to k, and that polar line passes through P. Hence, MP is its polar line, which means that the second intersection point of MP and k is the point of tangency of the circle, passing through A and B and touching k. If we know the result of the Problem 4.5.16, then we can easily prove Problem 4.5.6.

Problem 4.5.7. Let ABC be a triangle. Let its incircle touch the sides BC, CA and AB at the points F, E and D, respectively. Let its C-excircle touch the lines BC, CA and AB at the points Q, P and M, respectively. Let MN ($N \in PQ$) be an altitude in $\triangle PMQ$, and let DH ($D \in EF$) be an altitude in $\triangle EDF$. Prove that $\angle ACN = \angle BCH$.



Solution. It is enough to prove that

$$\frac{\sin\angle PCN}{\sin\angle NCQ} = \frac{\sin\angle FCH}{\sin\angle HCE}.$$

We have that

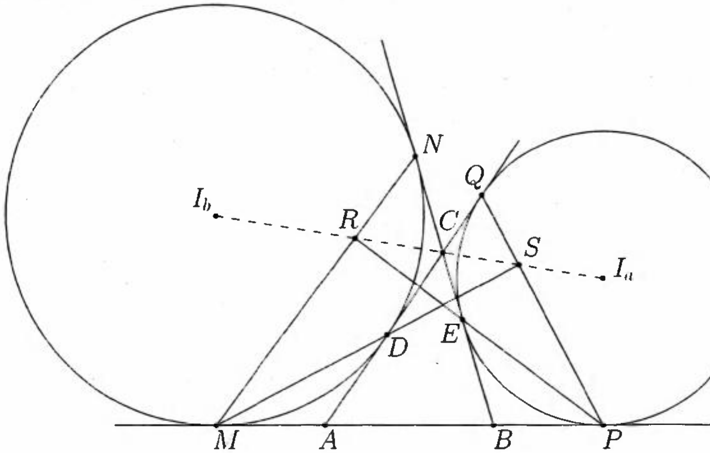
$$\angle DEF = 90^\circ - \frac{\beta}{2},\ \angle DFE = 90^\circ - \frac{\alpha}{2}.$$

Hence, $\angle EDH = \dfrac{\beta}{2}$ and $\angle HDF = \dfrac{\alpha}{2}$.

Observe that $\angle MPN = \dfrac{\beta}{2}$ and $\angle MQN = \dfrac{\alpha}{2}$. Hence, $\angle PMN = 90^\circ - \dfrac{\beta}{2}$, $\angle NMQ = 90^\circ - \dfrac{\alpha}{2}$ and by the law of sines we get

$$\frac{\sin\angle PCN}{\sin\angle NCQ} = \frac{PN}{NQ}\cdot\frac{CQ}{CP} = \frac{PN}{NM}\cdot\frac{MN}{NQ}$$
$$= \cot\frac{\beta}{2}\operatorname{tg}\frac{\alpha}{2} = \frac{DH}{HE}\cdot\frac{HF}{DH} = \frac{HF}{HE}\cdot\frac{EC}{FC} = \frac{\sin\angle FCH}{\sin\angle HCE}.$$

Problem 4.5.8. Let ABC be a triangle with an A-excircle ω_a (I_a) and a B-excircle ω_b (I_b). These circles touch AB, BC and AC as shown in the figure. Let $PE \cap MN = R$ and $MD \cap PQ = S$. Prove that the points I_b, R, C, S and I_a are collinear.



Solution. Observe that I_b, C and I_a lie on the external bisector of $\angle ACB$. We will prove that $\angle QCS = 90^\circ - \dfrac{\gamma}{2}$, and then it will follow that $S \in I_aI_b$. Analogously, we will get that $R \in I_aI_b$.

We know that $CD = s - a$, $CQ = s - b$, and so $DQ = c$. Obviously, $\angle AMD = \angle ADM = \angle QDS = \dfrac{\alpha}{2}$ and $\angle DQS = 90^\circ - \dfrac{\alpha}{2}$. Hence, $\angle DSQ = 90^\circ$ and $QS = DQ \sin\dfrac{\alpha}{2} = c\sin\dfrac{\alpha}{2}$. Let $\angle QCS = \psi$.

The law of sines, applied to $\triangle QCS$, yields

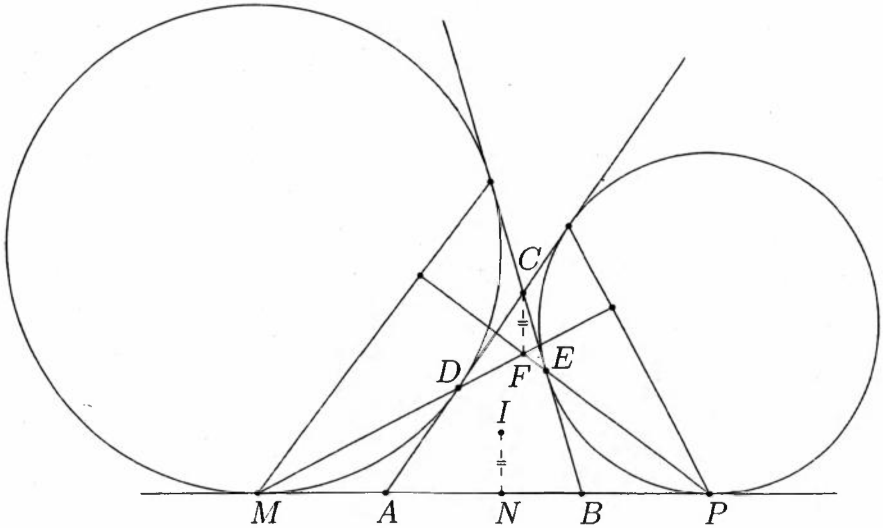
$$\frac{QC}{QS} = \frac{\cos\left(\dfrac{\alpha}{2} - \psi\right)}{\sin\psi} \Rightarrow \frac{s-b}{c\sin\dfrac{\alpha}{2}} = \cos\frac{\alpha}{2}\cot\psi + \sin\frac{\alpha}{2}$$

$$\Rightarrow \frac{4R\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\cos\dfrac{\beta}{2}}{4R\sin\dfrac{\gamma}{2}\cos\dfrac{\gamma}{2}\sin\dfrac{\alpha}{2}} = \cos\frac{\alpha}{2}\cot\psi + \sin\frac{\alpha}{2}$$

$$\Rightarrow \frac{\cos\dfrac{\beta}{2}}{\cos\dfrac{\gamma}{2}} = \cos\frac{\alpha}{2}\cot\psi + \sin\frac{\alpha}{2}$$

$$\Rightarrow \cot\psi = \frac{\cos\frac{\beta}{2} - \cos\frac{\gamma}{2}\sin\frac{\alpha}{2}}{\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}} = \frac{\sin\left(\frac{\alpha+\gamma}{2}\right) - \cos\frac{\gamma}{2}\sin\frac{\alpha}{2}}{\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}}{\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}}$$
$$= \cot\left(90^\circ - \frac{\gamma}{2}\right) \Rightarrow \psi = 90^\circ - \frac{\gamma}{2}.$$

Problem 4.5.9. Let ABC be a triangle with an A-excircle ω_a and a B-excircle ω_b. These circles touch AB, BC and AC as shown in the figure. Let $PE \cap MD = F$. Prove that $CF = r$, where r is the inradius of $\triangle ABC$.

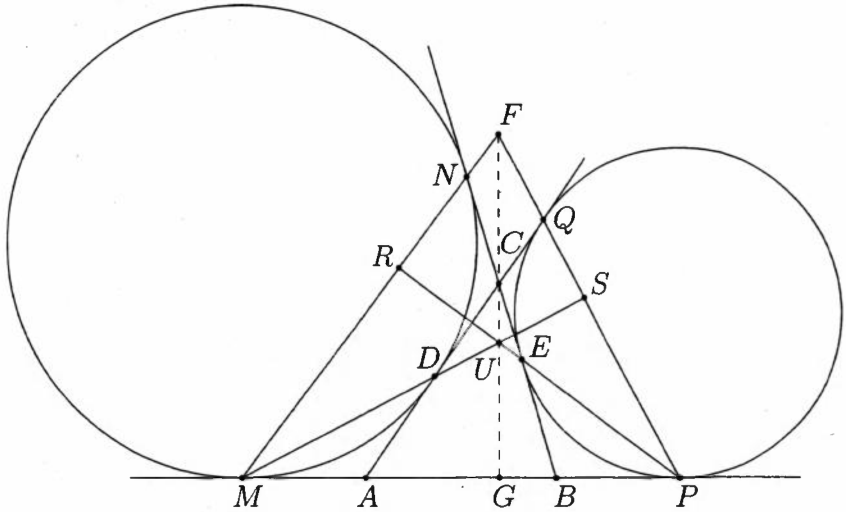


Solution. Problem 4.5.10 gives $CF \perp AB$. But we know that $\angle BPE = \angle BEP = \angle FEC = \frac{\beta}{2}$, which means that $\angle EFC = 90^\circ + \frac{\beta}{2}$.

The law of sines, applied to $\triangle CEF$, yields

$$CF = \frac{CE\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}} = \frac{(s-b)\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}} = \frac{4R\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$$
$$= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = r.$$

Problem 4.5.10. Let ABC be a triangle with an A-excircle ω_a and a B-excircle ω_b. These circles touch AB, BC and AC as shown in the figure. Let $PE \cap MD = U$ and $PQ \cap MN = F$. Prove that $CF \perp AB$ and $CU \perp AB$.



Solution. We have $\angle CQF = 90^\circ + \dfrac{\alpha}{2}$, $\angle CNF = 90^\circ + \dfrac{\beta}{2}$, $CQ = s - b$ and $CN = s - a$.

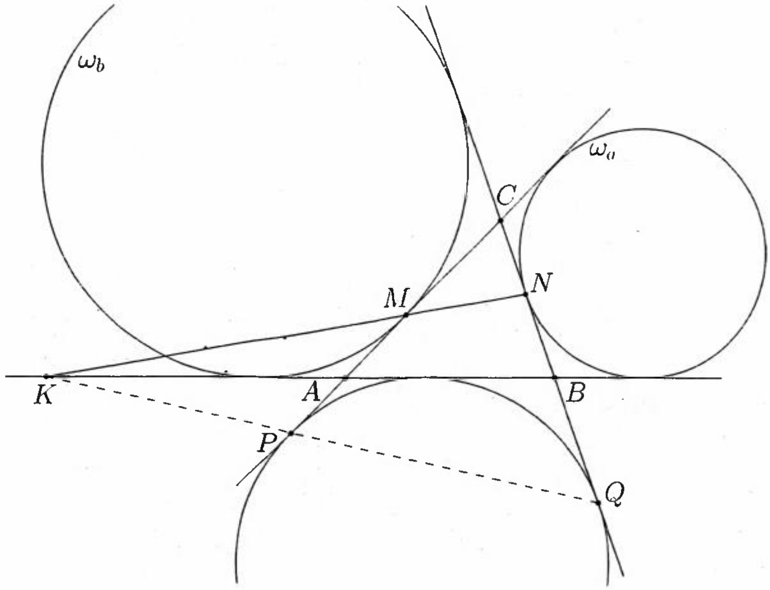
The law of sines, applied to $\triangle CFN$ and $\triangle CFQ$, yields

$$\frac{\sin\angle NFC}{\sin\angle CFQ} = \frac{(s-a)\cos\frac{\beta}{2}}{(s-b)\cos\frac{\alpha}{2}} = \frac{4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{4R\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}}.$$

Note that $\angle NFC + \angle CFQ = \dfrac{\alpha}{2} + \dfrac{\beta}{2}$ and thus $\cot\angle CFQ = \cot\dfrac{\alpha}{2}$, i.e. $\angle CFQ = \dfrac{\alpha}{2}$. Hence, $CF \perp AB$. We get that $\angle DQS = 90^\circ - \dfrac{\alpha}{2}$ and $\angle SDQ = \dfrac{\alpha}{2}$.

Hence, MS is an altitude in $\triangle MPF$. Analogously, PR is an altitude in the same triangle. Hence, $UF \perp AB$ and finally, $CU \perp AB$.

Problem 4.5.11. Let ABC be a triangle and let its excircles touch AB, BC and AC as shown in the figure. Prove that the lines PQ, MN and AB are concurrent.



Solution. Let $MN \cap AB = K$. We know that $AM = s-c$, $CM = s-a$, $CN = s-b$, $NB = s-c$, $PA = s-b$ and $BQ = s-a$. By Menelaus' Theorem, applied to $\triangle ABC$ and the points K, M and N, it follows that

$$\frac{BK}{KA} \cdot \frac{AM}{MC} \cdot \frac{CN}{NB} = 1 \Rightarrow \frac{BK}{KA} \cdot \frac{s-c}{s-a} \cdot \frac{s-b}{s-c} = 1 \Rightarrow \frac{BK}{KA} = \frac{s-a}{s-b}.$$

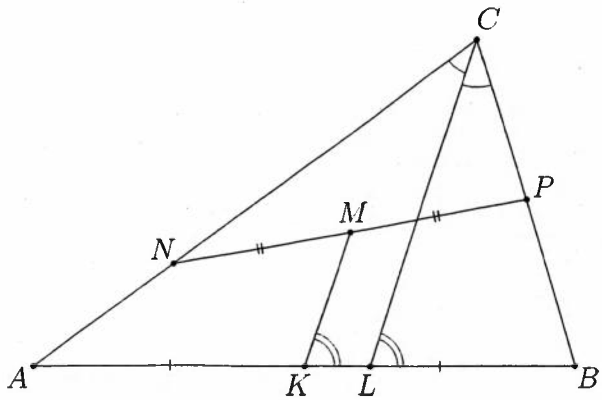
We get that $\dfrac{BK}{KA} \cdot \dfrac{AP}{PC} \cdot \dfrac{CQ}{QB} = \dfrac{s-a}{s-b} \cdot \dfrac{s-b}{s} \cdot \dfrac{s}{s-a} = 1.$

Menelaus' Theorem, applied to $\triangle ABC$ and the points K, P and Q, yields that these points are collinear.

Problem 4.5.12. Let ABC be a triangle. Let K be the midpoint of AB. The line CL ($L \in AB$) is the angle bisector of $\angle ACB$. The B-excircle of $\triangle ABC$ touches AC at the point N. The A-excircle of $\triangle ABC$ touches BC at the point P. Let M be the midpoint of NP. Prove that $KM \parallel CL$.

Solution. We know that $CN = s-a$ and $CP = s-b$. Define $\overrightarrow{CA} = \overrightarrow{x}$ and $\overrightarrow{BC} = \overrightarrow{y}$. We will express all necessary vectors as functions of $\overrightarrow{x}$, $\overrightarrow{y}$, a, b and c.

Observe that $\overrightarrow{CN} = \dfrac{s-a}{b}\overrightarrow{x}$ and $\overrightarrow{PC} = \dfrac{s-b}{a}\overrightarrow{y}$.



Obviously, $\overrightarrow{PN} = \overrightarrow{PC} + \overrightarrow{CN}$ and $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{PN}$. Also, $\overrightarrow{NA} = \frac{s-c}{b}\overrightarrow{x}$.

We know that $\overrightarrow{AK} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{CB}) = -\frac{1}{2}(\overrightarrow{x} + \overrightarrow{y})$.

Then $\overrightarrow{MK} = \overrightarrow{MN} + \overrightarrow{NA} + \overrightarrow{AK} = \frac{s-c}{2b}\overrightarrow{x} - \frac{a+b-p}{2a}\overrightarrow{y}$.

Furthermore, $\overrightarrow{BL} = \frac{a}{a+b}\overrightarrow{BA} = \frac{a}{a+b}(\overrightarrow{x} + \overrightarrow{y})$.

Thus, $\overrightarrow{CL} = \overrightarrow{BL} - \overrightarrow{BC} = \frac{a}{a+b}\overrightarrow{x} - \frac{b}{a+b}\overrightarrow{y}$.

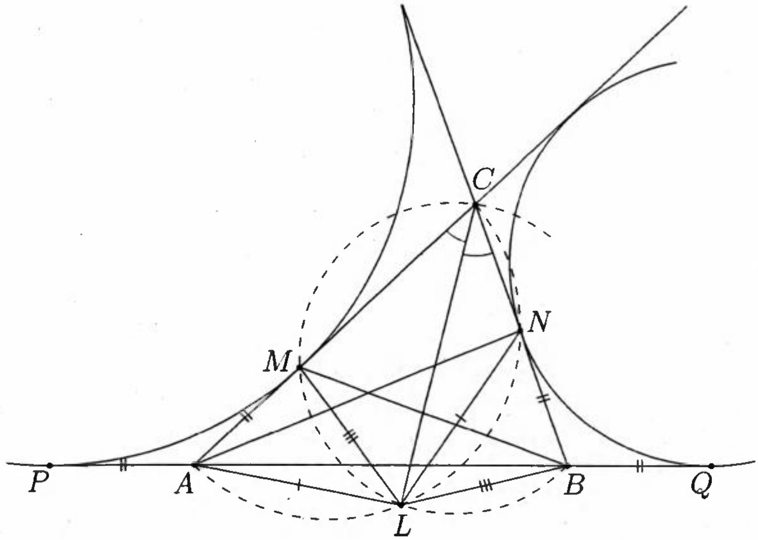
Since

$$\frac{\frac{s-c}{2b}}{\frac{a}{a+b}} = \frac{\frac{a+b-s}{2a}}{\frac{b}{a+b}} \Leftrightarrow \frac{s-c}{2ab} = \frac{a+b-s}{2ab} \Leftrightarrow a+b+c = 2p,$$

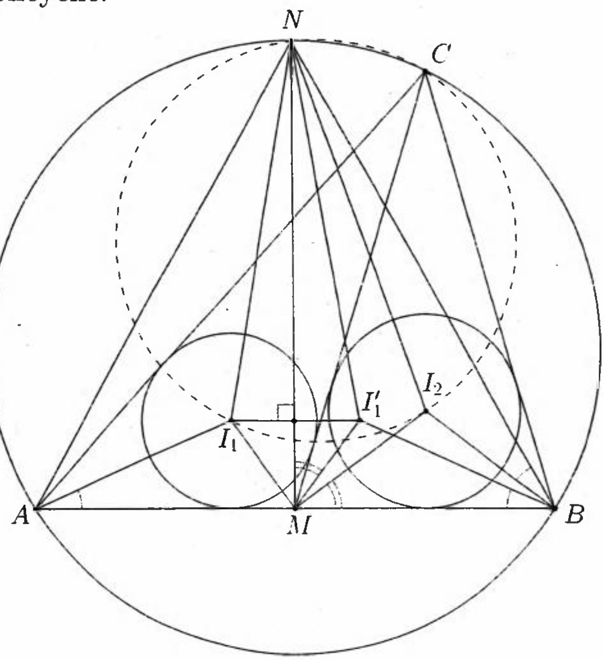
it follows that $\overrightarrow{CL}$ and $\overrightarrow{MK}$ are collinear.

Problem 4.5.13. Let ABC be a triangle with an A-excircle ω_a and a B-excircle ω_b. These circles touch AB, BC and AC as shown in the figure. Prove that the perpendicular bisectors of AN and BM, and the bisector of $\angle ACB$ are concurrent.

Solution. Let L be the intersection point of the two given perpendicular bisectors. Then $AL = LN$ and $BL = LM$. Because of the excircles, we have that $AM = BN$. Hence, $\triangle LAM \cong \triangle LNB$. Thus, the point L is equidistant from AM and BN. Hence, it lies on the bisector of $\angle ACB$.



Problem 4.5.14. Let ABC be a triangle with circumcircle k. Let M be the midpoint of AB, and let N be the midpoint of $\widehat{ACB}$. Let I_1 and I_2 be the incenters of $\triangle ACM$ and $\triangle BCM$, respectively. Prove that the points N, C, I_1 and I_2 are concyclic.



Solution. Note that

$$\angle I_1CI_2 = \angle MNB = \frac{\gamma}{2}.$$

So it is enough to prove that

$$\angle I_1NM = \angle I_2NB.$$

Let I_1' be the reflection of I_1 with respect to MN. Hence,

$$\angle I_1NM = \angle I_1'NM.$$

We will prove that I_1' and I_2 are isogonal conjugates in $\triangle MBN$. We have

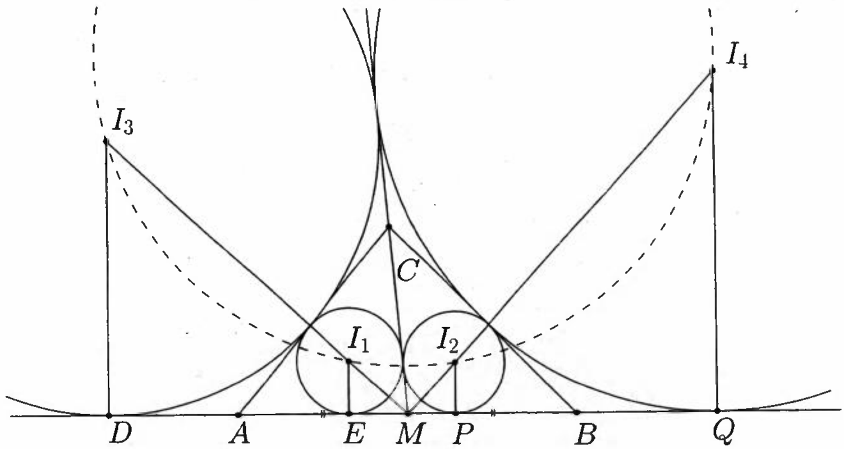
$$\angle I_1MI_2 = \angle NMB = 90^\circ \Rightarrow \angle I_1'MN = \angle NMI_1 = \angle I_2MB.$$

The symmetry with respect to MN yields

$$\angle I_1'BM = \angle I_1AM = \frac{\alpha}{2} = \left(90^\circ - \frac{\gamma}{2}\right) - \frac{\beta}{2} = \angle NBM - \angle I_2BM = \angle NBI_2.$$

Therefore, I_1' and I_2 are isogonal conjugates in $\triangle MBN$, and thus $\angle I_1NM = \angle I_2NB$.

Problem 4.5.15. Let ABC be a triangle with median CM. Let $k_1(I_1)$ and $k_2(I_2)$ be the incircles of $\triangle ACM$ and $\triangle BCM$, respectively. Let $k_3(I_3)$ and $k_4(I_4)$ be the M-excircles of $\triangle ACM$ and $\triangle BCM$, respectively. Prove that the points I_1, I_2, I_3 and I_4 are concyclic.



Solution. Let the circles k_1, k_2, k_3 and k_4 touch the line AB at the points E, P, D and Q, respectively. We have

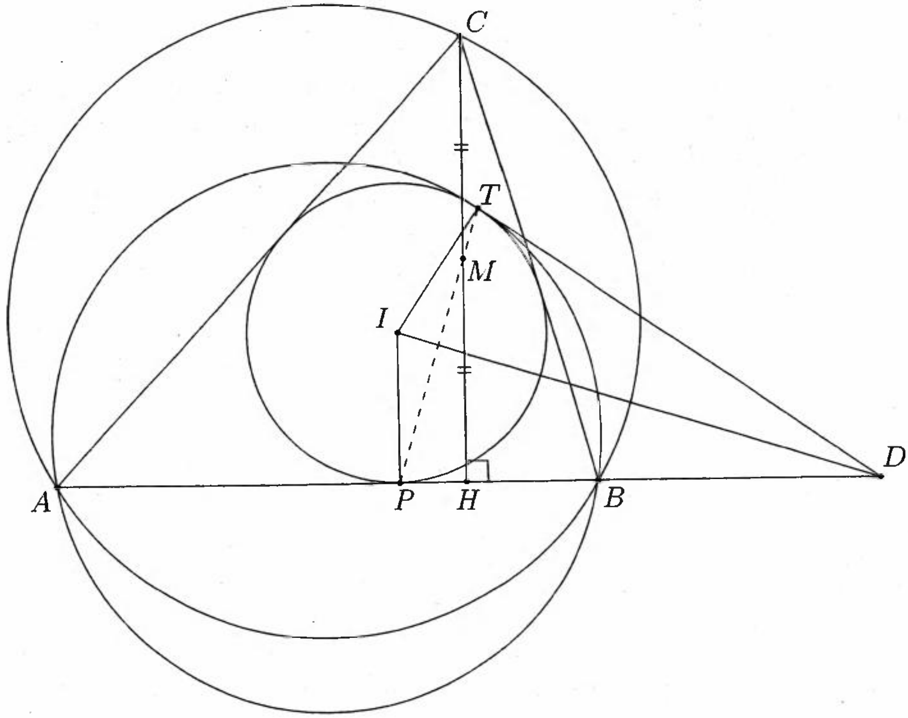
$$\angle I_3MI_4 = 90^\circ \Rightarrow \angle I_4MQ = \angle MI_1E \Rightarrow \triangle MI_1E \sim \triangle I_4MQ \Rightarrow \frac{MI_1}{MI_4} = \frac{EI_1}{MQ}.$$

Analogously, $\dfrac{MI_2}{MI_3} = \dfrac{PI_2}{MD}$. The triangles $\triangle ACM$ and $\triangle BCM$ have the same area. By the formula $S = pr$, we get

$$MD.I_1E = S_{\triangle ACM} = S_{\triangle BCM} = MQ.I_2P$$
$$\Rightarrow \frac{MI_1}{MI_4} = \frac{EI_1}{MQ} = \frac{I_2P}{MD} = \frac{MI_2}{MI_3} \Rightarrow MI_1.MI_3 = MI_2.MI_4.$$

Hence, the points I_1, I_2, I_3 and I_4 are concyclic.

Problem 4.5.16. Let ABC be a triangle with incircle k and incenter I. Let k_1 be the circle that passes through A and B, and touches k at the point T. Let CH be an altitude in $\triangle ABC$, and let M be the midpoint of CH. Let P be the point of tangency of AB and k. Prove that the points P, M and T are collinear.



Solution. (See also Problem 4.5.6) Without loss of generality, let $AC > BC$. Denote the circle (ABC) by k_2. Let the tangent line to k at T intersect AB at D.

By the radical axes theorem, applied to the circles k, k_1 and k_2, it follows that D lies on the radical axis of k and k_2. Hence, we have $DP^2 = DB.DA$.

Denote $DP = x$. Now,

$$(x - (s-b))(x + (s-a)) = x^2 \Rightarrow x = \frac{(s-b)(s-a)}{b-a}.$$

We have

$$\frac{IP}{PD}\cdot\frac{MH}{HP} = \frac{rh_c(b-a)}{2(s-b)(s-a)(s-b-a\cos\beta)}$$

$$= \frac{\sin\frac{\beta}{2}\sin\left(\frac{\beta-\alpha}{2}\right)}{\sin\frac{\gamma}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)}$$

$$= \frac{\sin\frac{\beta}{2}\sin\left(\frac{\beta-\alpha}{2}\right)}{\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\sin^2\frac{\beta}{2}\cos\frac{\alpha}{2}}$$

$$= \frac{\sin\frac{\beta}{2}\sin\left(\frac{\beta-\alpha}{2}\right)}{\sin\frac{\beta}{2}\left(\sin\frac{\beta}{2}\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\sin\frac{\alpha}{2}\right)} = \frac{\sin\frac{\beta}{2}\sin\left(\frac{\beta-\alpha}{2}\right)}{\sin\frac{\beta}{2}\sin\left(\frac{\beta-\alpha}{2}\right)} = 1.$$

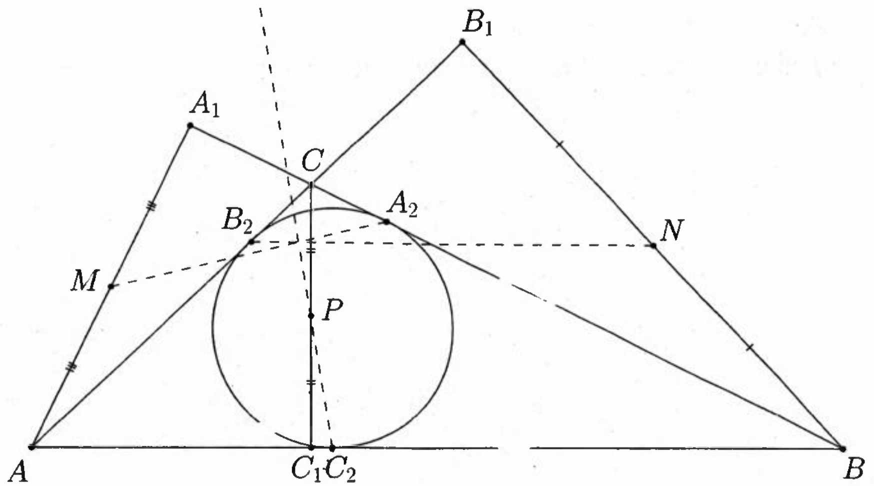
Hence,

$$\frac{IP}{PD} = \frac{HP}{HM} \Rightarrow \Delta IPD \sim \Delta PHM$$
$$\Rightarrow \angle MPH = \angle PID = \angle TPH.$$

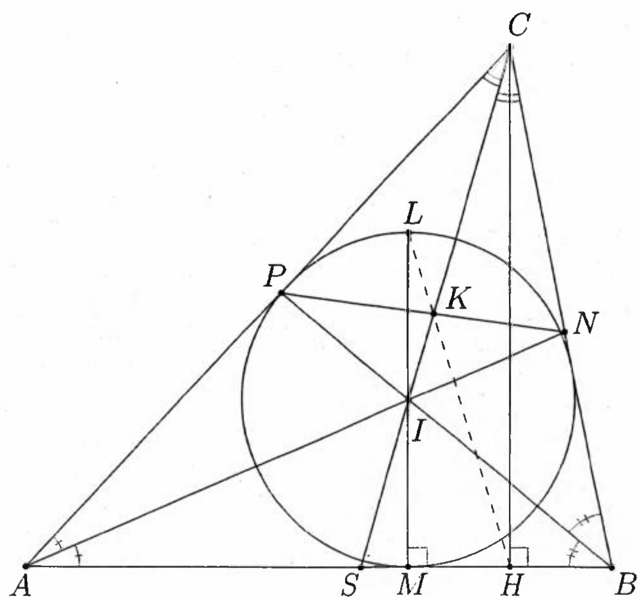
Thus, the points P, M and T are collinear.

Problem 4.5.17. Let ABC be a triangle with altitudes AA_1, BB_1 and CC_1. Let the incircle of $\triangle ABC$ touch AB, BC and CA at C_2, A_2 and B_2, respectively. Denote the midpoints of AA_1, BB_1 and CC_1 by M, N and P, respectively. Prove that the lines MA_2, NB_2 and PC_2 are concurrent.

Solution. Problem 4.5.16 yields that the poles of the lines MA_2, NB_2 and PC_2 with respect to the incircle of $\triangle ABC$ lie on the radical axis of the incircle and the circumcircle of $\triangle ABC$. Thus, these three lines are concurrent. (If we use the notations of Problem 4.5.16, then $DP^2 = DT^2 = DA.DB$)



Problem 4.5.18. Let ABC be a triangle with incircle ω (I) that touches AB at M. Let L be the the reflection of M with respect to I. Let $AI \cap BC = N$, $BI \cap AC = P$, and $PN \cap IC = K$. Let H be the foot of the perpendicular from C to AB. Prove that the points L, K and H are collinear.



Solution. Menelaus' Theorem, applied to $\triangle CSI$ and the line AB, to $\triangle AIC$ and the line PN, and to $\triangle ANC$ and the line BP, yields

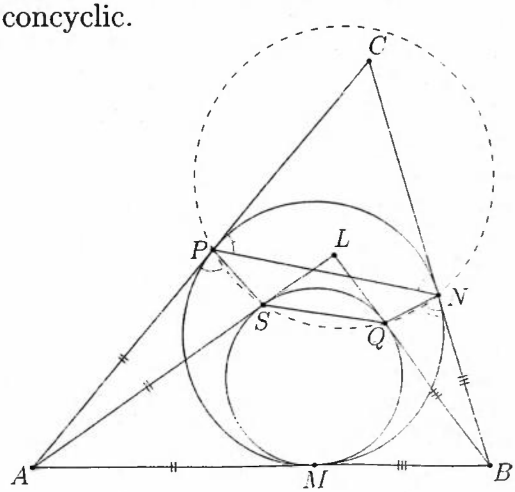
$$\frac{CS}{SI} \cdot \frac{IA}{AN} \cdot \frac{NB}{BC} = 1,$$
$$\frac{AN}{NI} \cdot \frac{IK}{KC} \cdot \frac{CP}{PA} = 1,$$
$$\frac{CB}{BN} \cdot \frac{NI}{IA} \cdot \frac{AP}{PC} = 1.$$

Multiplying these equalities, we conclude that $\dfrac{CS}{SI} \cdot \dfrac{IK}{KC} = 1$. Thus, the points C, K, I and S are in a harmonic division. Now the intercept theorem yields

$$\frac{CK}{KI} = \frac{CS}{SI} = \frac{CH}{IM} = \frac{CH}{IL}.$$

Hence, $\triangle CHK \sim \triangle ILK$ and finally, the points H, K and L are collinear.

Problem 4.5.19. Let ABC be a triangle with incircle ω that touches the sides AB, BC and CA at the points M, N and P, respectively. A circle k_1 is constructed such that it touches AB at M. Let L be the intersection point of the tangent lines from A and B to k_1, different from AB. Let k_1 touch LA and LB at the points S and Q, respectively. Prove that the points P, N, Q and S are concyclic.



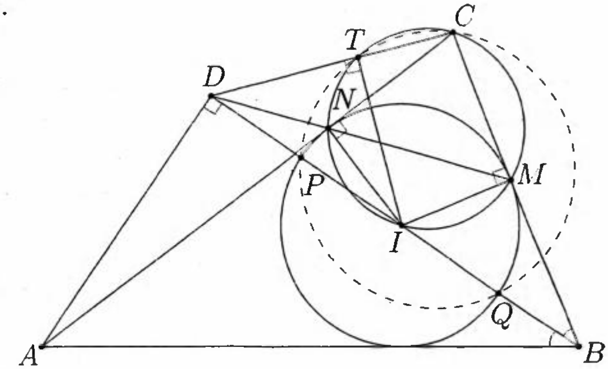
Solution. Let $\angle PAS = 2\varphi$ and $\angle NBQ = 2\psi$. Then $BN = BM = BQ$ and $AP = AM = AS$. Hence, $\angle NQL = 90^\circ + \psi$ and $\angle APS = 90^\circ - \varphi$.

Now, $\angle CPN = 90^\circ - \dfrac{\gamma}{2}$ and so $\angle SPN = \varphi + \dfrac{\gamma}{2}$.

Thus, $\angle ALB = \angle ACB + \angle CAL + \angle CBL = \gamma + 2\varphi + 2\psi$. Hence, $\angle LQS = 90^\circ - \dfrac{\gamma}{2} - \varphi - \psi$.

Then $\angle SPN + \angle SQN = \varphi + \dfrac{\gamma}{2} + 90^\circ - \dfrac{\gamma}{2} - \varphi - \psi + 90^\circ + \psi = 180^\circ$.

Problem 4.5.20. Let ABC be a triangle with incircle ω (I) that touches BC and CA at the points M and N, respectively. Let D be the projection of A onto BI, and let T be the projection of I onto CD. Let P and Q be the intersection points of ω and BD. Prove that the points P, Q, C and T are concyclic.

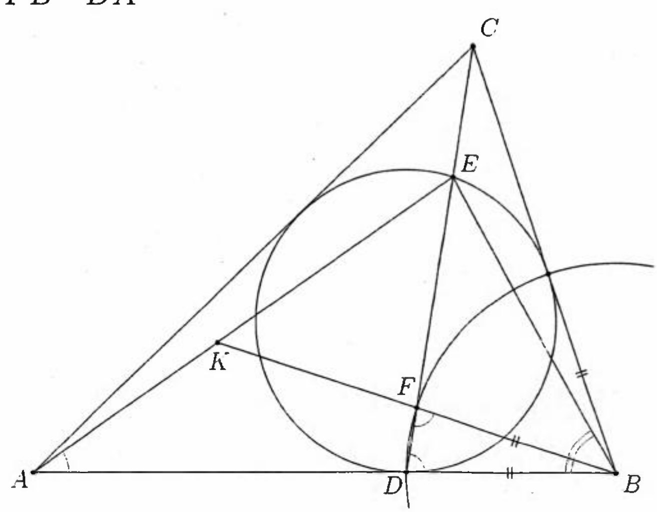


Solution. Without loss of generality, let Q be located between B and P. The pentagon $IMCTN$ is iscribed in the circle k_1 with diameter CI. Problem 4.5.4 yields that the points D, N and M are collinear.

Considering the powers of D with respect to k_1 and ω, we get that $DT.DC = DN.DM$ and $DN.DM = DP.DQ$. Therefore, $DT.DC = DP.DQ$. Hence, the points P, Q, C and T are concyclic.

Problem 4.5.21. Let ABC be a triangle with incircle ω that touches AB at D. Let $CD \cap \omega = E \neq D$. The circle with center B and radius BD intersects CD at the point $F \neq D$. Let $BF \cap AE = K$. Prove that $KF = FB$.

Solution. Menelaus' Theorem, applied to $\triangle AKB$ and the line ED, yields $\dfrac{AE}{EK} \cdot \dfrac{KF}{FB} \cdot \dfrac{BD}{DA} = 1$.



It follows that $\dfrac{AE}{EK}\cdot\dfrac{KF}{DA}=1$. We are left to prove that $\dfrac{AE}{EK}=\dfrac{s-a}{s-b}$.

Define $\angle BDF=\angle BFD=\varphi$, $\angle DAE=\psi$ and $\angle ABE=\psi_1$. The law of sines yields

$$\begin{aligned}\frac{AE}{EK}&=\frac{\sin\psi_1}{\sin(\psi_1-(180^\circ-2\varphi))}\cdot\frac{\sin(180^\circ-2\varphi+\psi)}{\sin\psi}\\&=\frac{\sin(-2\varphi+\psi)}{\sin\psi}\cdot\frac{\sin\psi_1}{\sin(2\varphi+\psi_1)}\\&=\frac{\cos 2\varphi-\cot\psi\sin 2\varphi}{\sin 2\varphi\cot\psi_1+\cos 2\varphi}.\end{aligned}$$

Observe that $ED=2r\sin\varphi$. The cotangent technique, applied to $\triangle ADE$ yields

$$\begin{aligned}\cot\psi=\frac{\dfrac{s-a}{2r\sin\varphi}+\cos\varphi}{\sin\varphi}&\Rightarrow\cot\psi\sin 2\varphi=\cot\frac{\alpha}{2}\cot\varphi+2\cos^2\varphi\\&\Rightarrow\cos 2\varphi-\cot\psi\sin 2\varphi=-1-\cot\frac{\alpha}{2}\cot\varphi.\end{aligned}$$

The same technique, applied to $\triangle BDE$, yields

$$\cot\psi_1=\frac{\dfrac{s-b}{2r\sin\varphi}-\cos\varphi}{\sin\varphi}\Rightarrow\sin 2\varphi\cot\psi_1+\cos 2\varphi=\cot\frac{\beta}{2}\cot\varphi-1.$$

Now we apply the cotangent technique to $\triangle BDC$ and get

$$\cot\varphi=\frac{\dfrac{s-b}{a}-\cos\beta}{\sin\beta}.$$

$$\begin{aligned}\Rightarrow\frac{AE}{EK}&=\frac{1+\cot\dfrac{\alpha}{2}\cot\varphi}{1-\cot\dfrac{\beta}{2}\cot\varphi}=\frac{1+\cot\dfrac{\alpha}{2}\left(\dfrac{s-b-a\cos\beta}{a\sin\beta}\right)}{1-\cot\dfrac{\beta}{2}\left(\dfrac{s-b-a\cos\beta}{a\sin\beta}\right)}\\&=\frac{a\sin\beta+\cot\dfrac{\alpha}{2}(s-b)-a\cot\dfrac{\alpha}{2}\cos\beta}{a\sin\beta-\cot\dfrac{\beta}{2}(s-b)+a\cot\dfrac{\beta}{2}\cos\beta}=\end{aligned}$$

$$= \frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\beta + \cos\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2} - \cos^2\frac{\alpha}{2}\cos\beta}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sin\beta - \dfrac{\cos^2\frac{\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\sin\frac{\beta}{2}} + \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\dfrac{\cos\frac{\beta}{2}}{\sin\frac{\beta}{2}}\cos\beta}$$

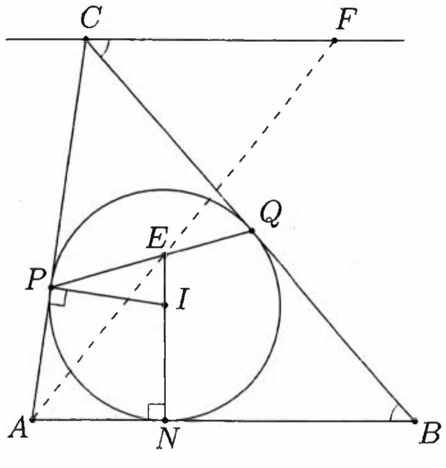
$$= \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} \cdot \frac{\sin\frac{\alpha}{2}\sin\beta + \sin\frac{\gamma}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\cos\beta}{\cos\frac{\alpha}{2}\sin\beta - \dfrac{\cos^2\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\beta}{2}} + \cos\frac{\alpha}{2}\dfrac{\cos\frac{\beta}{2}}{\sin\frac{\beta}{2}}\cos\beta}$$

$$= \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} \cdot \frac{\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}} \cdot \frac{\sin\frac{\alpha}{2}\sin\beta + \sin\frac{\gamma}{2}\cos\frac{\beta}{2} - \cos\frac{\alpha}{2}\cos\beta}{2\cos\frac{\alpha}{2}\sin^2\frac{\beta}{2} - \cos\frac{\beta}{2}\sin\frac{\gamma}{2} + \cos\frac{\alpha}{2}\cos\beta}$$

$$= \cot\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2} \cdot \frac{\cos\frac{\gamma}{2}}{\cos\frac{\gamma}{2}} = \cot\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2} = \frac{s-a}{s-b}.$$

Therefore, $\dfrac{AE}{EK} = \dfrac{s-a}{s-b}$.

Problem 4.5.22. Let ABC be a triangle with incircle $\omega\ (I)$ that touches the sides AB, BC and CA at the points N, Q and P, respectively. Let $NI \cap PQ = E$. Let F be a point such that $CF = CP$, $CF \parallel AB$, and F and A lie in different half-planes, with respect to BC. Prove that the points A, E and F are collinear.



Solution. It suffices to prove that

$$\cot\angle CAE = \cot\angle CAF.$$

We have $\angle ACF = 180^\circ - \alpha$,

$$CF = CP = s - c$$
$$= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2},$$
$$CA = b = 4R\sin\frac{\beta}{2}\cos\frac{\beta}{2},$$
$$\angle EPI = \frac{\gamma}{2}, \angle EIP = \alpha.$$

Also, $\angle IEP = 90^\circ - \left(\dfrac{\alpha-\beta}{2}\right)$, $PA = s - a = r\cot\dfrac{\alpha}{2}$ and $\angle APE = 90^\circ + \dfrac{\gamma}{2}$. Hence,

$$EP = \frac{r\sin\alpha}{\cos\left(\dfrac{\alpha-\beta}{2}\right)} \Rightarrow \frac{AP}{PE} = \frac{r\cot\dfrac{\alpha}{2}\cos\left(\dfrac{\alpha-\beta}{2}\right)}{r\sin\alpha} = \frac{\cos\left(\dfrac{\alpha-\beta}{2}\right)}{2\sin^2\dfrac{\alpha}{2}}.$$

The cotangent technique, applied to $\triangle APE$, yields

$$\cot\angle PAE = \frac{\dfrac{AP}{PE} - \cos\left(90^\circ + \dfrac{\gamma}{2}\right)}{\sin\left(90^\circ + \dfrac{\gamma}{2}\right)} = \frac{\dfrac{\cos\left(\dfrac{\alpha-\beta}{2}\right)}{2\sin^2\dfrac{\alpha}{2}} + \sin\dfrac{\gamma}{2}}{\cos\dfrac{\gamma}{2}}$$

$$= \frac{\cos\left(\dfrac{\alpha-\beta}{2}\right) + 2\sin^2\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}}{2\sin^2\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}}.$$

The same technique, applied to $\triangle ACF$, yields

$$\cot\angle CAF = \frac{\dfrac{CA}{CF} - \cos(180^\circ - \alpha)}{\sin(180^\circ - \alpha)} = \frac{\dfrac{4R\sin\dfrac{\beta}{2}\cos\dfrac{\beta}{2}}{4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}} + \cos\alpha}{\sin\alpha}$$

$$= \frac{\dfrac{\cos\dfrac{\beta}{2}}{\sin\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}} + \cos\alpha}{\sin\alpha} = \frac{\cos\dfrac{\beta}{2} + \cos\alpha\sin\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}}{2\sin^2\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}\cos\dfrac{\gamma}{2}}.$$

We are left to show that

$$\cos\frac{\beta}{2} + \cos\alpha\sin\frac{\alpha}{2}\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\left(\cos\left(\frac{\alpha-\beta}{2}\right) + 2\sin^2\frac{\alpha}{2}\sin\frac{\gamma}{2}\right).$$

We consecutively have

$$\cos\frac{\beta}{2}+\cos\alpha\sin\frac{\alpha}{2}\cos\frac{\gamma}{2}-\cos\frac{\alpha}{2}\cos\left(\frac{\alpha-\beta}{2}\right)-2\sin^2\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}$$

$$=\cos\frac{\beta}{2}\left(\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\right)+\cos\alpha\sin\frac{\alpha}{2}\cos\frac{\gamma}{2}$$

$$-\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)-2\sin^2\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}$$

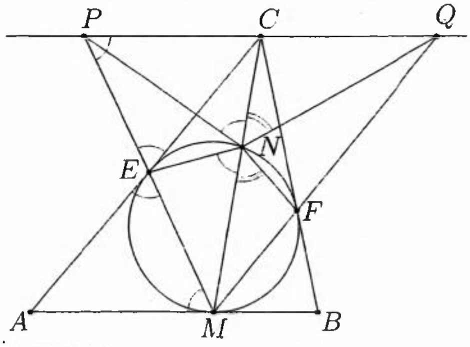
$$=\cos\frac{\beta}{2}\sin^2\frac{\alpha}{2}+\cos\alpha\sin\frac{\alpha}{2}\cos\frac{\gamma}{2}-\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}-2\sin^2\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}$$

$$=\sin\frac{\alpha}{2}\left(\cos\frac{\beta}{2}\sin\frac{\alpha}{2}+\cos\alpha\cos\frac{\gamma}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}-\sin\alpha\sin\frac{\gamma}{2}\right)$$

$$=\sin\frac{\alpha}{2}\left(\sin\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\alpha+\frac{\gamma}{2}\right)\right)$$

$$=\sin\frac{\alpha}{2}\left(\sin\left(\frac{\alpha-\beta}{2}\right)+\sin\left(\frac{\beta-\alpha}{2}\right)\right)=0.$$

Problem 4.5.23. Let ABC be a triangle with incircle ω that touches the sides AB, BC and CA at the points M, F and E, respectively. Let $CM\cap\omega=N\neq M$. Let $l\parallel AB$ be a line through C. The points P and Q are the intersection points of l and the lines ME and MF, respectively. Prove that $\angle ENF=\angle PNQ$.



Solution. The parallel lines and the alternate segments theorem give

$$\angle ENM=\angle EMA=\angle EPC$$

and

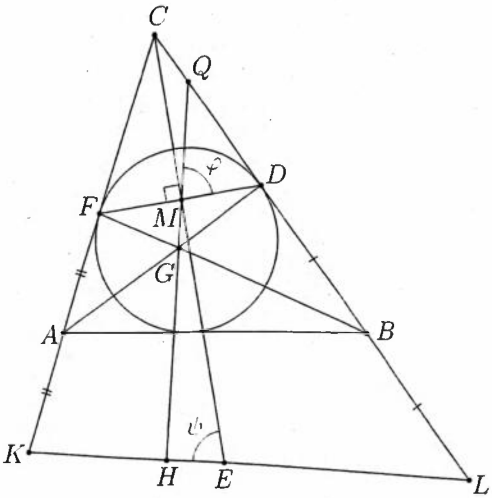
$$\angle MNF=\angle FMB=\angle FQC.$$

Hence, the quadrilaterals $ENCP$ and $FNCQ$ are cyclic. Therefore,

$$\angle ENF=\angle ENM+\angle MNF=\angle AEM+\angle BFM$$
$$=\angle PEC+\angle CFQ=\angle PNC+\angle CNQ=\angle PNQ.$$

Problem 4.5.24. Let ABC be a triangle with incircle ω that touches the sides BC and CA at the points D and F, respectively. Let $AD\cap BF=G$. Denote the midpoint of FD by M. Let K and L be the reflections of F and D, with respect to A and B, respectively. Prove that $MG\perp KL$.

Solution. Let $MG \cap KL = H$, $MG \cap BC = Q$ and $CM \cap KL = P$. Denote $DQ = x$. We know that $MC \perp FD$, thus we are left to prove that $\angle MPH = \angle QMD$. We will show that $\cot\angle MPH = \cot\angle QMD$.



Let $\angle MPH = \psi$ and $\angle QMD = \varphi$. We have

$$\cot\varphi = \frac{\sin\angle CMQ}{\sin\angle QMD} = \frac{CQ}{QD}\operatorname{tg}\frac{\gamma}{2} = \frac{s-c-x}{x}\operatorname{tg}\frac{\gamma}{2}.$$

Menelaus' Theorem yields

$$1 = \frac{BQ}{QD}\cdot\frac{DM}{MF}\cdot\frac{FG}{GB} = \frac{s-b+x}{x}\cdot\frac{FC}{CA}\cdot\frac{AE}{EB} = \frac{s-b+x}{x}\cdot\frac{s-c}{b}\cdot\frac{s-a}{s-b}.$$

Hence, $x = \dfrac{(s-a)(s-b)(s-c)}{b(s-b)-(s-a)(s-c)}$ and thus $\dfrac{s-c-x}{x} =$

$$= \frac{b(s-b)-a(s-a)}{(s-b)(s-a)}$$

$$= \frac{4R\sin\frac{\beta}{2}\cos\frac{\beta}{2}4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} - 4R\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$$

$$= \frac{\cos^2\frac{\beta}{2} - \cos^2\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}} = \frac{\cos\beta - \cos\alpha}{2\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)\cos\frac{\gamma}{2}}{\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}}.$$

Therefore, $\cot\varphi = \dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)\cos\frac{\gamma}{2}}{\cos\frac{\beta}{2}\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}}\operatorname{tg}\dfrac{\gamma}{2} = \dfrac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\beta}{2}\cos\frac{\alpha}{2}}.$

The law of sines, applied to $\triangle KPC$ and $\triangle LPC$, yields

$$\frac{\sin\left(\psi+\frac{\gamma}{2}\right)}{\sin\left(\psi-\frac{\gamma}{2}\right)} = \frac{a+s-b}{b+s-a} \Rightarrow \frac{\cos\frac{\gamma}{2}+\cot\psi\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}-\cot\psi\sin\frac{\gamma}{2}} = \frac{a+s-b}{b+s-a}.$$

Hence, $\dfrac{\cos\dfrac{\gamma}{2}}{\cos\dfrac{\gamma}{2}-\cot\psi\sin\dfrac{\gamma}{2}}=\dfrac{s}{b+s-a}$ and so

$$cot\psi=\frac{(a-b)\cos\dfrac{\gamma}{2}}{s\sin\dfrac{\gamma}{2}}=\frac{2R(\sin\alpha-\sin\beta)\cos\dfrac{\gamma}{2}}{s\sin\dfrac{\gamma}{2}}$$

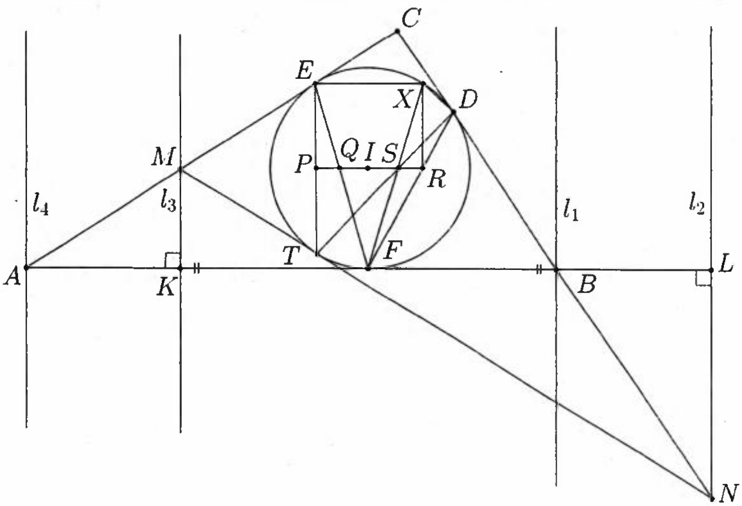
$$=\frac{4R\sin\left(\dfrac{\alpha-\beta}{2}\right)\sin\dfrac{\gamma}{2}\cos\dfrac{\gamma}{2}}{4R\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}\sin\dfrac{\gamma}{2}}=\frac{\sin\left(\dfrac{\alpha-\beta}{2}\right)}{\cos\dfrac{\beta}{2}\cos\dfrac{\alpha}{2}}=\cot\varphi.$$

Problem 4.5.25. Let ABC ($AC \geq BC$) be a triangle with incircle k that touches the sides BC, CA and AB at the points D, E and F, respectively. Let L and K be the respective reflections of A and B with respect to F. The lines through L and K, perpendicular to AB, intersect CB and CA at N and M, respectively. Prove that MN touches k.

Solution. Let the secont tangent line from N to k touch k at T, and let it intersect AC at M'. We will prove that $M \equiv M'$.

Let $K' \in AB$ be a point such that $M'K' \perp AB$. We are left to show that $K'F = FB$. Denote the lines through B, L, K' and A that are perpendicular to AB by l_1, l_2, l_3 and l_4, respectively.

Let I be the center of k. We are going to show that the poles of l_1 and l_3 with respect to k are equidistant from I. Let those poles be R and P, respectively. Let S and Q be the poles of l_2 and l_4, with respect to k.

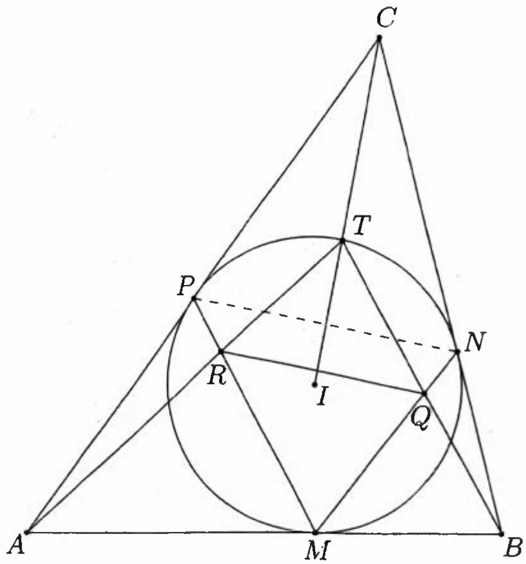


We know that $AF = FL$, so $QI = IS$ and also, the points P, Q, I, S and R lie on a line that is parallel to AB. Since $N \in l_2$, $B \in l_1$, $M \in l_3$ and $A \in l_4$, we get that $S \in TD$, $R \in FD$, $P \in ET$ and $Q \in EF$.

Let FS intersect k for the second time at X. We will prove that $PQ = SR$. Using the equality $QI = IS$, we obtain $EX \parallel QS$. Hence, $\angle XSR = \angle SXE = \angle FEX = 180° - \angle RDX$, which means that the quadrilateral $SRDX$ is cyclic.

Hence, $\angle SXR = \angle SDR = \angle PEQ$. We also have that $EQ = XS$ and $\angle XSR = \angle EQP$. Therefore, $\triangle SXR \cong \triangle QEP$ and hence $PQ = SR$, which means that the poles of l_1 and l_3 are equidistant from I. Thus, $K \equiv K'$ and $M \equiv M'$.

Problem 4.5.26. Let ABC be a triangle with incircle ω that touches the sides AB, BC and CA at the points M, N and P, respectively. Let T be the intersection of the segment CI and ω. Let $MP \cap AT = R$ and $MN \cap BT = Q$. Prove that $RQ \perp IC$.



Solution. We have $CI \perp PN$. We will show that $PN \parallel RQ$. This is equivalent to proving that $\dfrac{PR}{RM} = \dfrac{NQ}{MQ}$.

Now,

$$\frac{S_{\triangle ARP}}{S_{\triangle AMR}} = \frac{\sin\angle RAP}{\sin\angle RAM} = \frac{PR}{RM},$$

$$\frac{S_{\triangle BNQ}}{S_{\triangle BMQ}} = \frac{\sin\angle NBQ}{\sin\angle MBQ} = \frac{NQ}{MQ}.$$

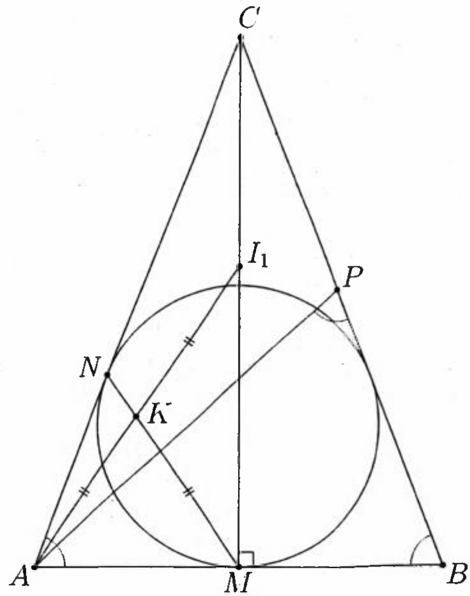
We will apply the trigonometric form of Ceva's Theorem to $\triangle ABC$ and the point T.

$$\frac{\sin\angle RAP}{\sin\angle RAM} \cdot \frac{\sin\angle MBQ}{\sin\angle NBQ} \cdot \frac{\sin\angle NCT}{\sin\angle PCT} = 1$$

$$\Rightarrow \frac{\sin\angle RAP}{\sin\angle RAM} \cdot \frac{\sin\angle MBQ}{\sin\angle NBQ} = 1.$$

It follows that $\dfrac{\sin\angle RAP}{\sin\angle RAM} = \dfrac{\sin\angle NBQ}{\sin\angle MBQ}$ and thus $\dfrac{PR}{RM} = \dfrac{NQ}{MQ}$.

Problem 4.5.27. Let ABC be an isosceles triangle ($AC = BC$) and let $AC > AB$. Its incircle ω touches AB and CA at M and N, respectively. Let P be a point of the side BC such that $AP = AB$. Let I_1 be the incenter of $\triangle APC$. The lines AI_1 and MN intersect at K. Prove that K is the midpoint of AI_1.



Solution. The points C, I_1 and M lie on the bisector of $\angle ACB$, which is perpendicular to AB, and hence $\angle I_1MA = 90°$. Denote $\angle BAC = \alpha = \angle ABC$.

We have that $AM = AN$, so

$$\angle AMK = \angle AMN = 90° - \frac{\alpha}{2}$$

Also,

$$\angle PAB = 180° - 2\alpha.$$

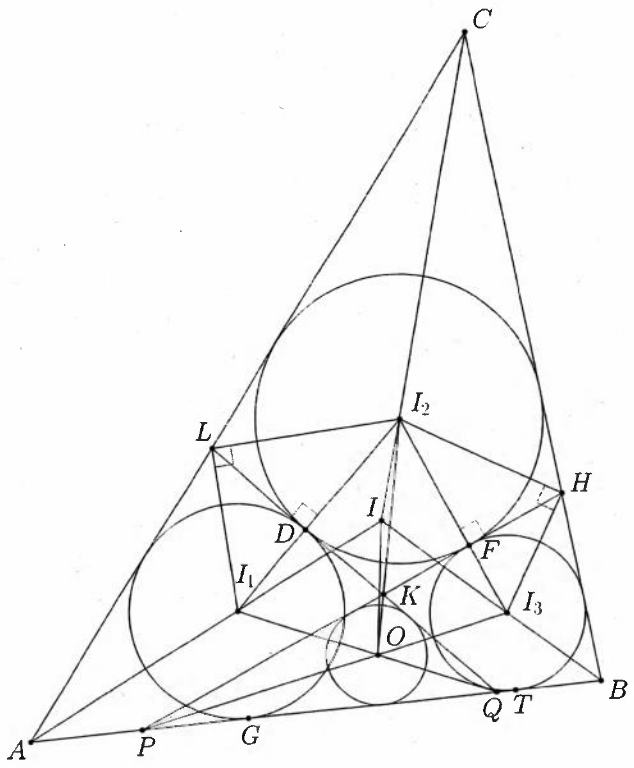
Thus, $\angle CAP = 3\alpha - 180°$ and $\angle I_1AP = \dfrac{3\alpha}{2} - 90°$.

We get that $\angle MAK = \angle BAP + \angle I_1AP = \dfrac{3\alpha}{2} - 90° + 180° - 2\alpha = 90° - \dfrac{\alpha}{2}$. Finally, $\angle MAK = \angle AMK$, which means that $AK = KM = KI_1$.

Problem 4.5.28. Let ABC be a triangle with incenter I. Let k_2 (I_2) be a circle that touches the segments AC and BC. Let k_1 (I_1) be a circle that touches AB at G, AC, and k_2 at D. Let $k_3(I_3)$ be a circle that touches AB at T, BC, and k_2 at F. Prove that I lies on the perpendicular bisector of the segment, formed by the intersection points of the two segments AI and BI, and the circumcircle of $\triangle GTF$.

Solution. Let $L \in AC$ lie on the common internal tangent line of k_1 and k_2. Let $H \in BC$ lie on the common internal tangent line of k_3 and k_2. Let $HF \cap AB = P$, $LD \cap AB = Q$ and $HF \cap LD = K$.

The perpendicular bisectors of DG, FQ and DF intersect at O, which is the circumcenter of $\triangle PQK$. Hence, the quadrilateral $GTFD$ is cyclic.



We will show that $\angle OII_3 = \angle OII_1$, and the desired property will follow because of congruent triangles. The law of sines yields

$$\begin{aligned}\frac{\sin\angle OII_1}{\sin\angle OII_3} &= \frac{OI_1}{OI_3}\cdot\frac{\sin\angle OI_1O}{\sin\angle OI_3O} = \frac{\sin\angle I_2LK}{\sin\angle I_2HK}\cdot\frac{\sin\angle I_1I_2O}{\sin\angle OI_2I_3}\cdot\frac{\sin\angle OI_3I_2}{\sin\angle OI_1I_2}\\ &= \frac{\sin\angle LKI_2}{\sin\angle HKI_2}\cdot\frac{I_2H}{I_2L}\cdot\frac{\cos\angle OPK}{\cos\angle OQK} = \frac{\sqrt{I_2F.I_2I_3}}{\sqrt{I_2D.I_2I_1}}\cdot\frac{\cos\angle OPK}{\cos\angle OQK}\\ &= \frac{\sqrt{I_2I_3}}{\sqrt{I_2I_1}}\cdot\frac{\cos\angle OPK}{\cos\angle OQK}.\end{aligned}$$

However,

$$\frac{I_2I_3}{I_2I_1} = \frac{\sin\angle I_2OI_3}{\sin\angle I_2OI_1}\cdot\frac{\sin\angle I_2I_1O}{\sin\angle I_2I_3O} = \frac{\cos\angle OQK}{\cos\angle OPK}\cdot\frac{\cos\angle OQK}{\cos\angle OPK} = \frac{\cos^2\angle OQK}{\cos^2\angle OPK}.$$

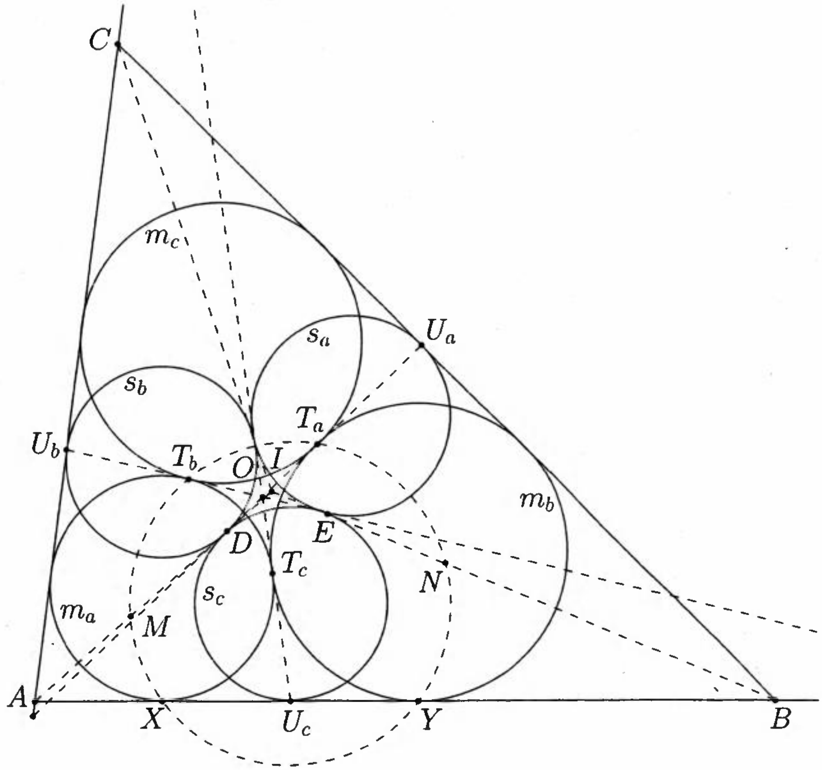
Hence, $\dfrac{\sin\angle OII_1}{\sin\angle OII_3} = 1$, and so $\angle OII_3 = \angle OII_1$.

Problem 4.5.29. See Problems 6.2.7 and 6.3.3.

Problem 4.5.30. Let m_a, m_b and m_c be the Malfatti circles of a triangle ABC with incenter I. Let $m_a \cap m_b = T_c$, $m_b \cap m_c = T_a$ and $m_a \cap m_c = T_b$. The circles m_a and m_b touch AB at X and Y, respectively. Prove that the quadrilateral XYT_aT_b is inscribed in a circle, and let this circle intersect AI and BI at the points M and N, respectively. Also prove that $AM = AX$ and $BN = BY$.

Solution. Let O be the radical center of m_a, m_b and m_c.

Lemma 1. Let s_c be the circle that touches the ray $T_aO^{\rightarrow}$ beyond O, the ray $T_bO^{\rightarrow}$ beyond O, and the segment AB, such that s_c and C lie in the same half-plane, with respect to AB. Denote the last point of tangency by U_c. Analogously, we define s_a and s_b, as well as U_a and U_b. Then U_c lies on OT_c.



Proof. The perpendicular bisectors of YT_a, XT_b and T_aT_b intersect at the center of s_c and thus the quadrilateral XYT_aT_b is cyclic. Hence, U_c is the midpoint of XY. The radical axis of m_a and m_b bisects XY, and thus U_c lies on OT_c.

Lemma 2. The second common internal tangent line of s_b and s_c passes through A.

Proof. One of the common external tangent lines of s_b and m_a, one of the common external tangent lines of s_c and m_a, and one of the common internal tangent lines of s_b and s_c are concurrent at the point O. Problem 5.4.16 yields that the second tangent lines of these pairs of circles are also concurrent. But their common point is A.

Lemma 3. The length of the tangent segment from U_b to s_c, and the length of the tangent segment from U_c to s_b are equal.

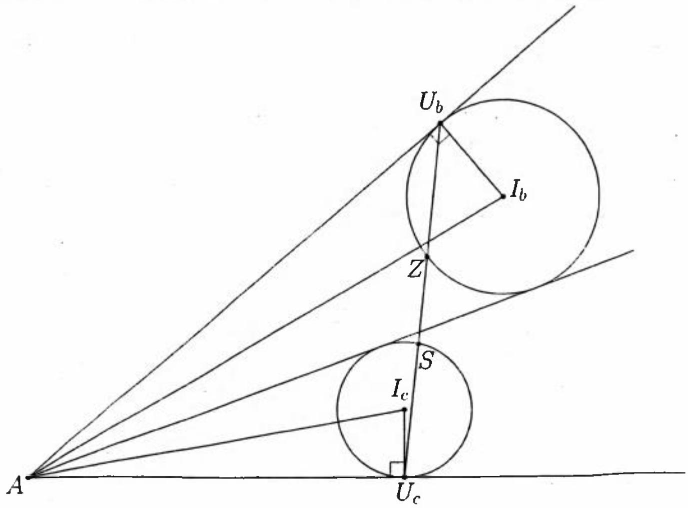
Proof. Let $t(X,k)$ denote the length of the tangent segment from X to the circle k. We have

$$\begin{aligned} t(U_b,s_c)-t(U_c,s_b) &= t(O,s_c)-t(O,s_b)+U_bO-U_cO \\ &= t(O,s_c)-t(O,s_b)+AU_b-AU_c \\ &= t(O,s_c)-t(O,s_b)+t(A,s_b)-t(A,s_c)=0. \end{aligned}$$

We used that AU_cOU_b is a circumscribed quadrilateral.

Lemma 4. The line AI is the second common internal tangent line of s_b and s_c.

Proof. Let U_bU_c intersect s_b and s_c at Z and S for the second time, respectively. Denote the centers of s_b and s_c by I_b and I_c, respectively. Let r_b and r_c be the radii of those circles, respectively.



Then

$$\frac{\operatorname{tg}\angle I_cAU_c}{\operatorname{tg}\angle I_bAU_b} = \frac{r_c}{r_b}\cdot\frac{AU_b}{AU_c} = \frac{SU_c}{2\sin\angle SU_cA}\cdot\frac{2\sin\angle ZU_bA}{ZU_b}\cdot\frac{\sin\angle AU_cU_b}{\sin\angle AU_bU_c} = \frac{SU_c}{ZU_b}.$$

Lemma 3 yields that the power of U_c with respect to s_b equals the power of U_b with respect to s_c. Hence, $\dfrac{SU_b}{ZU_c} = \dfrac{SU_b.U_bU_c}{ZU_c.U_bU_c} = 1.$

Thus,

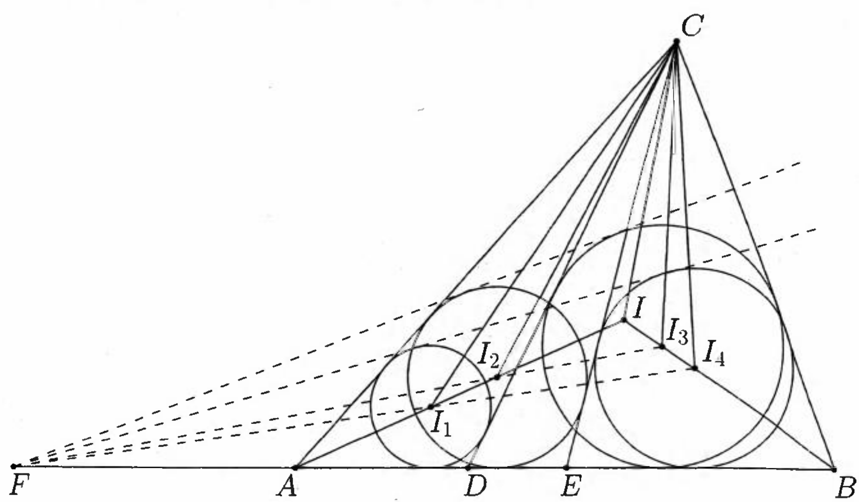
$$SU_b = ZU_c \Rightarrow ZU_b = SU_c \Rightarrow \operatorname{tg}\angle I_cAU_c = \operatorname{tg}\angle I_bAU_b \Rightarrow \angle I_cAU_c = \angle I_bAU_b.$$

Therefore, AI is the second common internal tangent line of s_b and s_c.

As we noted in Lemma 1, XYT_aT_b is a cyclic quadrilateral and its circumcircle and s_c are concentric. But Lemma 4 yields that s_c coincides with the incircle of $\triangle AIB$. Thus, the points M, N, X and Y lie on a circle, and this circle and s_c are concentric. So those four points have the same powers with respect to s_c.

Let s_c touch AI and BI at D and E, respectively. Lemma 1 yields that s_c touches AB at U_c. Hence, $BN = BE - EN = BU_c - U_cY = BY$ and analogously, $AM = AD - DM = AU_c - U_cX = AX$.

Problem 4.5.32. Let ABC be a triangle. Points D and E lie on the segment AB (D lies between A and E). Let k_1, k_2, k_3 and k_4 be the incircles of $\triangle ADC$, $\triangle AEC$, $\triangle BDC$ and $\triangle BEC$, respectively. Prove that AB and the external tangent lines (different from AB) of the pairs of circles (k_1, k_4) and (k_2, k_3) are concurrent.



Solution. Let I be the incenter of $\triangle ABC$. It is clear that $I_1I_2 \cap I_3I_4 = I$. It suffices to prove that I_1I_4, I_2I_3 and AB are concurrent. By Menelaus' Theorem, this is equivalent to

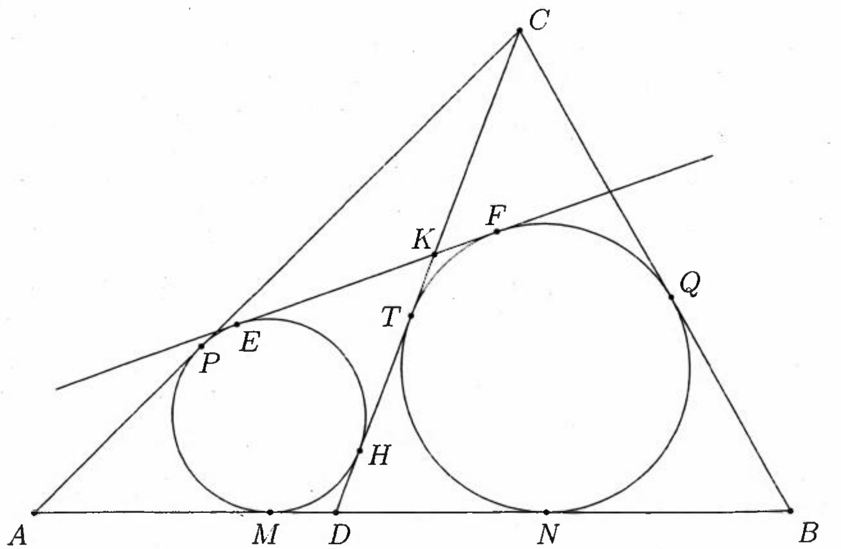
$$\frac{AI_1}{I_1I} \cdot \frac{II_4}{I_4B} = \frac{AI_2}{I_2I} \cdot \frac{II_3}{I_3B}.$$

Define $\angle ACI_1 = \varphi$, $\angle BCI_4 = \psi$ and $\angle ACB = 2\gamma$. Hence, $\angle ICI_3 = \varphi$, $\angle I_2CI = \psi$, $\angle BCI_3 = \gamma - \phi$ and $\angle ACI_2 = \gamma - \psi$.

The law of sines, applied to $\triangle AI_1C$, $\triangle I_1IC$, $\triangle II_4C$, $\triangle I_4BC$, $\triangle AI_2C$, $\triangle I_2IC$, $\triangle I_2I_3C$ and $\triangle I_3BC$ yields

$$\begin{aligned}\frac{AI_1}{I_1I} \cdot \frac{II_4}{I_4B} &= \left(\frac{\sin\varphi}{\sin(\gamma-\varphi)} \cdot \frac{AC}{CI}\right)\left(\frac{\sin(\gamma-\psi)}{\sin\psi} \cdot \frac{CI}{CB}\right) \\ &= \left(\frac{\sin(\gamma-\psi)}{\sin\psi} \cdot \frac{AC}{CI}\right)\left(\frac{\sin\varphi}{\sin(\gamma-\varphi)} \cdot \frac{CI}{CB}\right) = \frac{AI_2}{I_2I} \cdot \frac{II_3}{I_3B}.\end{aligned}$$

Problem 4.5.33. Let ABC be a triangle. Let D be an arbitrary point on the segment AB. Let k_1 and k_2 be the incircles of $\triangle ADC$ and $\triangle BDC$, respectively. Let the common external tangent line (different from AB) of k_1 and k_2 intersect CD at K. Prove that $CK = \dfrac{AC + BC - AB}{2}$.



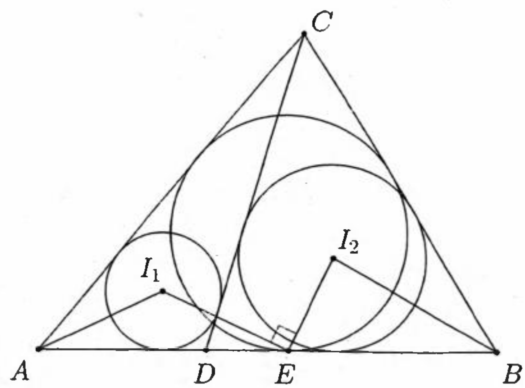
Solution. Denote the points of tangency of k_1 and k_2 as shown in the figure.

Then we consecutively have

$$\begin{aligned}CK &= CT - TK = CQ - KF = CB - BQ - EF + EK\\ &= CB - BN - MN + KH = CB - BM + CH - CK\\ &= BC - AB + AM + CP - CK = BC - AB + AP + CP - CK\\ &= BC - AB + AC - CK.\end{aligned}$$

Therefore, $CK = \dfrac{BC + AC - AB}{2}$.

Problem 4.5.34. Let ABC be a triangle with incircle k that touches AB at E. A point D lies on the segment AB. Let $k_1(I_1)$ and $k_2(I_2)$ be the incircles of $\triangle ADC$ and $\triangle BDC$, respectively. Prove that $\angle I_1EI_2 = 90°$.



Solution. Let $\angle BDC = \varphi$. Then $BE = s - b$ and $AE = s - a$.

Let R_1 and R_2 be the radii of the circumcircles of $\triangle ADC$ and $\triangle BDC$. Hence,

$$BI_2 = 4R_2 \sin\frac{\varphi}{2}\sin\frac{180° - \beta - \varphi}{2}.$$

The cotangent technique, applied to $\triangle BEI_2$, yields

$$\cot\angle I_2EB = \frac{\dfrac{BE}{BI_2} - \cos\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}} = \frac{\dfrac{4R\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\cos\dfrac{\beta}{2}}{4R_2\sin\dfrac{\varphi}{2}\cos\left(\dfrac{\beta+\varphi}{2}\right)} - \cos\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}}$$

$$= \frac{\dfrac{4R\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\cos\dfrac{\beta}{2}}{2\dfrac{2R\sin\alpha}{\sin\varphi}\sin\dfrac{\varphi}{2}\cos\left(\dfrac{\beta+\varphi}{2}\right)} - \cos\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}} = \frac{\dfrac{\sin\dfrac{\gamma}{2}\cos\dfrac{\beta}{2}}{\dfrac{\cos\dfrac{\alpha}{2}}{\cos\dfrac{\varphi}{2}}\cos\left(\dfrac{\beta+\varphi}{2}\right)} - \cos\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}} =$$

$$= \frac{\cos\frac{\beta}{2}\left(\sin\frac{\gamma}{2}\cos\frac{\varphi}{2} - \cos\frac{\alpha}{2}\cos\left(\frac{\beta+\varphi}{2}\right)\right)}{\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\left(\frac{\beta+\varphi}{2}\right)}.$$

Analogously,

$$AI_1 = 4R_1\cos\frac{\varphi}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)$$

$$= 2\frac{2R\sin\beta}{\sin\varphi}\cos\frac{\varphi}{2}\sin\left(\frac{\varphi-\alpha}{2}\right) = \frac{4R\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)}{\sin\frac{\varphi}{2}}.$$

The cotangent technique, applied to $\triangle AEI_1$, yields

$$\cot\angle I_1EA = \frac{\dfrac{AE}{AI_1} - \cos\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} = \frac{\dfrac{\dfrac{4R\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}\sin\frac{\beta}{2}}{4R\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)}}{\sin\frac{\varphi}{2}} - \cos\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} =$$

$$= \frac{\dfrac{\cos\frac{\alpha}{2}\sin\frac{\gamma}{2}\sin\frac{\varphi}{2}}{\cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)} - \cos\dfrac{\alpha}{2}}{\sin\dfrac{\alpha}{2}} = \frac{\cos\frac{\alpha}{2}\left(\sin\frac{\gamma}{2}\sin\frac{\varphi}{2} - \cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)\right)}{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)}.$$

Then $\cot\angle I_1EA\cot\angle I_2EB =$

$$= \frac{\cos\frac{\alpha}{2}\left(\sin\frac{\gamma}{2}\sin\frac{\varphi}{2} - \cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)\right)}{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)} \cdot \frac{\cos\frac{\beta}{2}\left(\sin\frac{\gamma}{2}\cos\frac{\varphi}{2} - \cos\frac{\alpha}{2}\cos\left(\frac{\beta+\varphi}{2}\right)\right)}{\sin\frac{\beta}{2}\cos\frac{\alpha}{2}\cos\left(\frac{\beta+\varphi}{2}\right)}$$

$$= \frac{\cos\frac{\beta}{2}\sin\frac{\alpha}{2}\cos\frac{\varphi}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\varphi}{2}}{\sin\frac{\alpha}{2}\sin\left(\frac{\varphi-\alpha}{2}\right)} \cdot \frac{\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\varphi}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\varphi}{2}}{\sin\frac{\beta}{2}\cos\left(\frac{\beta+\varphi}{2}\right)}$$

$$= \frac{-\sin\frac{\beta}{2}\sin\frac{\varphi}{2} + \cos\frac{\beta}{2}\cos\frac{\varphi}{2}}{\cos\left(\frac{\beta+\varphi}{2}\right)} \cdot \frac{-\sin\frac{\alpha}{2}\cos\frac{\varphi}{2} + \cos\frac{\alpha}{2}\sin\frac{\varphi}{2}}{\sin\left(\frac{\varphi-\alpha}{2}\right)} = 1.$$

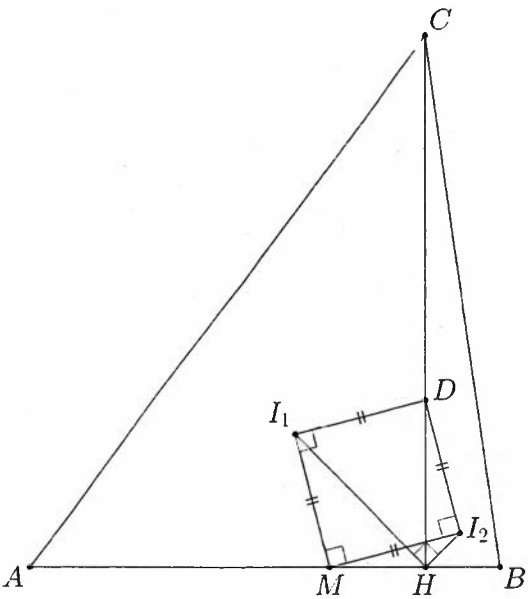
Hence, $\cot\angle I_1EA = \operatorname{tg}\angle I_2EB$. Finally, $\angle I_1EI_2 = 90°$.

Problem 4.5.35. Let ABC be a triangle with altitude CH and with incircle that touches AB at M. Denote the incenters of $\triangle AHC$ and $\triangle BHC$ by I_1 and I_2, respectively. A point D_1 is chosen such that $MI_2D_1I_1$ is a rectangle. Prove that this rectangle is a square, and prove that D_1 lies on CH.

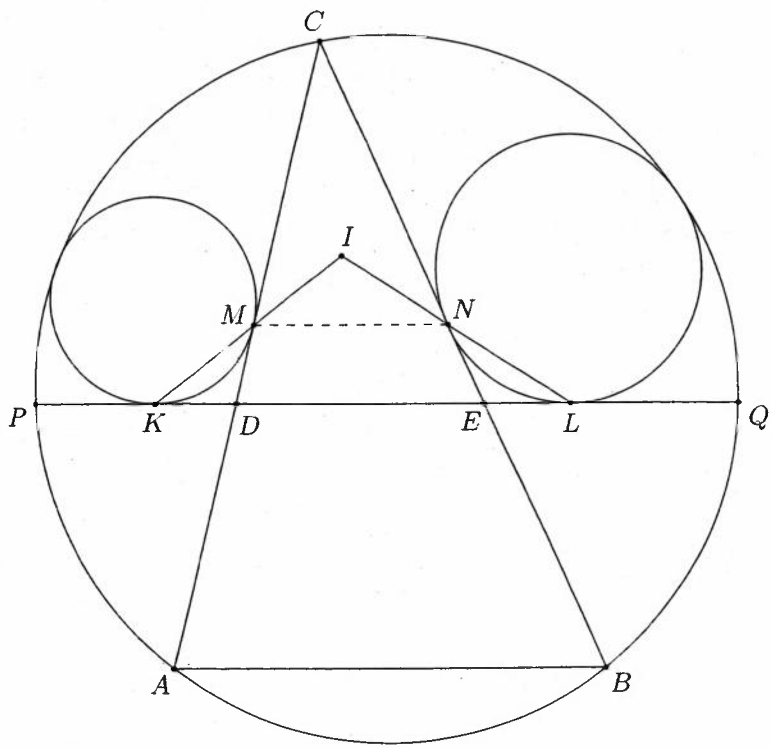
Solution. Problem 4.5.34 yields that $\angle I_1MI_2 = 90°$. Also, $\angle I_2HI_1 = 90°$. Let $D \in CH$ be a point such that $\angle MI_1D = 90°$. The pentagon MI_1DI_2H is cyclic.

Observe that HI_1 is the angle bisector of $\angle MHC$, so $MI_1 = DI_1$.

The quadrilateral I_1MI_2D is cyclic, which means that it is a rectangle because $\angle I_1MI_2 = 90°$ and $\angle MI_1D = 90°$. Using the fact that $MI_1 = DI_1$, we get that I_1MI_2D is a square. Hence, $D \equiv D_1$ and $D_1 \in CH$.



Problem 4.5.36. Let ABC be a triangle with circumcircle k. Let the line $l \parallel AB$ intersect AC and BC at the points D and E, respectively. Let DE intersect k at the points P and Q, respectively (D lies between P and E). Let k_1 be the circle that touches the segments PD, DC, and the smaller arc $\widehat{PC}$ of k. Let k_2 be the circle that touches the segments EQ, EC, and the smaller arc $\widehat{QC}$ of k. Let k_1 touch DC at M. Let k_2 touch EC at N. Prove that $MN \parallel l$.



Solution. Let k_1 and k_2 touch l at K and L, respectively.

Sawayama's Lemma (Problem 4.7.13) yields that KM and NL intersect at I – the incenter of $\triangle PQC$.

It is clear that $DK = DM$ and $EN = EL$. Hence,

$$\angle MIN = 180^\circ - \angle MKD - \angle NLE = 180^\circ - \frac{\angle CDE}{2} - \frac{\angle CED}{2}$$
$$= 180^\circ - \frac{180^\circ - \angle MCN}{2} = 90^\circ + \frac{\angle MCN}{2}.$$

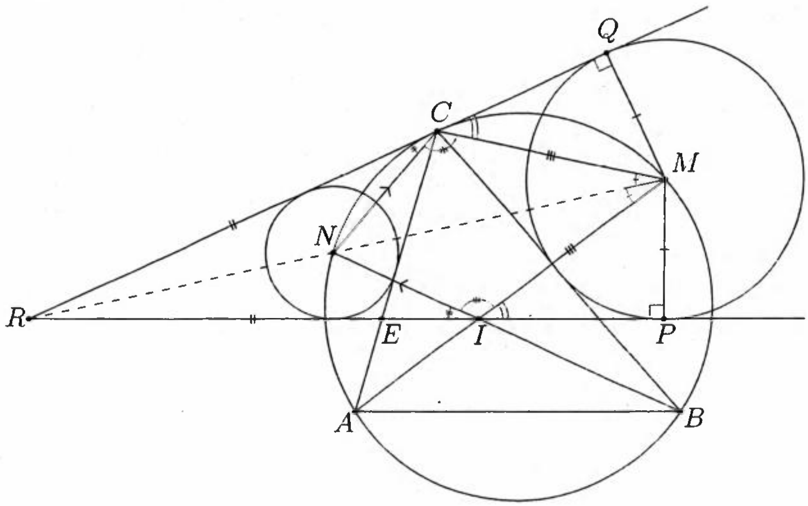
Also, CI is the bisector of $\angle PCQ$, and the arcs $\widehat{PA}$ and $\widehat{QB}$ are equal because $AB \parallel l$. Hence, $\angle PCA = \angle QCB$, which means that CI is the bisector of $\angle MCN$.

From $\angle MIN = 90^\circ + \dfrac{\angle MCN}{2}$ it follows that I is the incenter of $\triangle MNC$. Finally,

$$\angle NMI = \angle IMC = \angle KMD = \angle IKD,$$

which means that $MN \parallel l$.

Problem 4.5.37. Let ABC be a triangle with circumcircle k. The midpoints of the smaller arcs $\widehat{BC}$ and $\widehat{AC}$ are M and N, respectively. The lines AM and BN intersect at I. Let k_1 (N) be the circle that touches AC, and let k_2 (M) be the circle that touches BC. Prove that one of the common external tangent lines of k_1 and k_2 passes through C, and the other one passes through I.



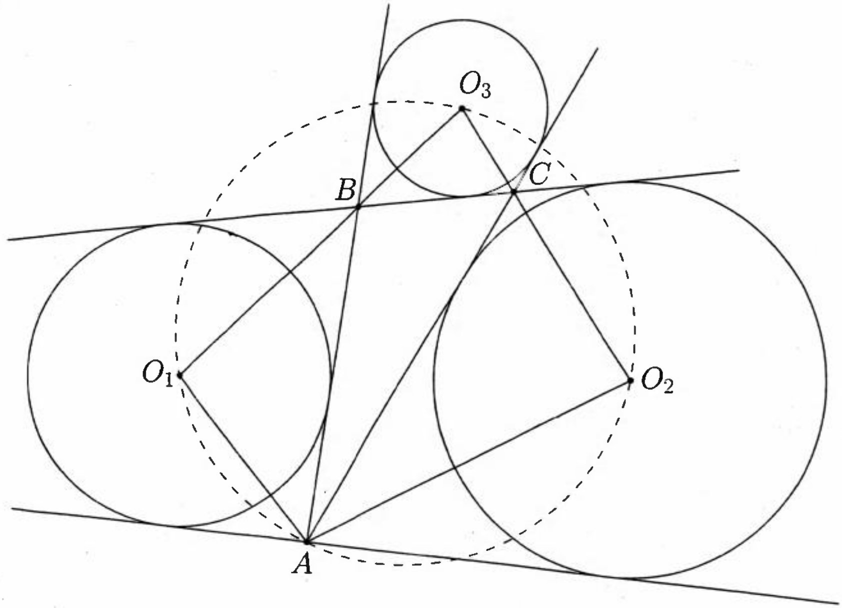
Solution. It is clear that AM and BN are the bisectors of $\angle BAC$ and $\angle ABC$, respectively, and so I is the incenter of $\triangle ABC$.

Let l be the tangent line to k at C.

Let $l \cap MN = R$. Then $\angle RCN = \angle CBN$ and $\angle NCA = \angle NBA$. Hence, $\angle RCN = \angle ACN$. It follows that N lies on the bisector of $\angle ACR$, which means that it is equidistant from AC and RC. Thus, k_1 touches l and analogously, k_2 touches l. Hence, l is one of the common external tangent lines of k_1 and k_2, and it passes through C. The trillium theorem (Problem 4.6.1) yields that $CN = IN$ and $CM = IM$. Thus, $\triangle CNM \cong \triangle INM$.

Hence, C and I are symmetric with respect to the line that connects the centers of k_1 and k_2. Then I lies on the other common external tangent line of k_1 and k_2.

Problem 4.5.38. Let ABC be a triangle and let l be a line passing through A, which does not intersect the segment BC. Let $k_1(O_3)$, $k_2(O_2)$ and $k_3(O_3)$ be circles that touch the lines AB, BC, CA and l, as shown in the figure. Prove that the quadrilateral $O_1O_3O_2A$ is cyclic.



Solution. Note that $\angle BO_3C = 90° - \dfrac{\alpha}{2}$ and

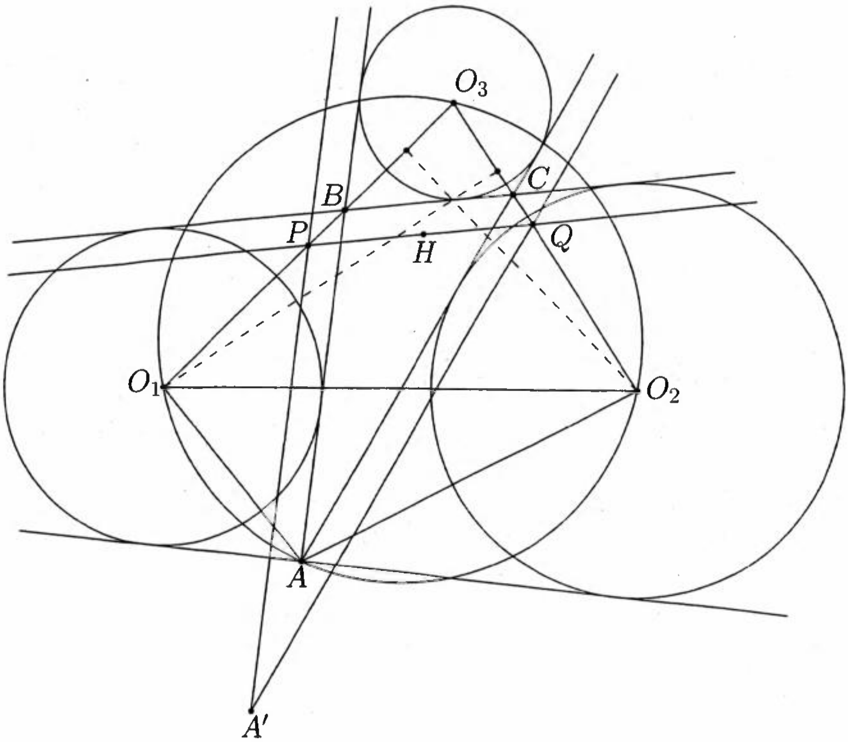
$$\angle O_1AO_2 = \alpha + (\angle O_1AB + \angle O_2AC) = \alpha + \left(\frac{180° - \alpha}{2}\right) = 90° + \frac{\alpha}{2}.$$

Then $\angle O_1AO_2 + \angle O_1O_3O_2 = 180°$. Hence, the quadrilateral $O_1O_3O_2A$ is cyclic.

Problem 4.5.39. Given the construction of Problem 4.5.38, prove that the orthocenter of $\Delta A_1O_2O_3$ lies on BC.

Solution. We have that the quadrilateral $O_1AO_2O_3$ is cyclic. Let H be the orthocenter of $\Delta O_1O_2O_3$.

If $H \in BC$, then AB and AC are the reflections of BC with respect to two of the sides of $\Delta O_1O_2O_3$. Problem 4.2.3 yields that the three reflections have a common point which lies on the circumcircle of $\Delta O_1O_2O_3$. So in this particular case, A is that intersection point. Assume that H does not lie on BC. Let us construct the line passing trough H and parallel to BC, and let P and Q be the intersection points of that line with O_1O_3 and O_2O_3, respectively.

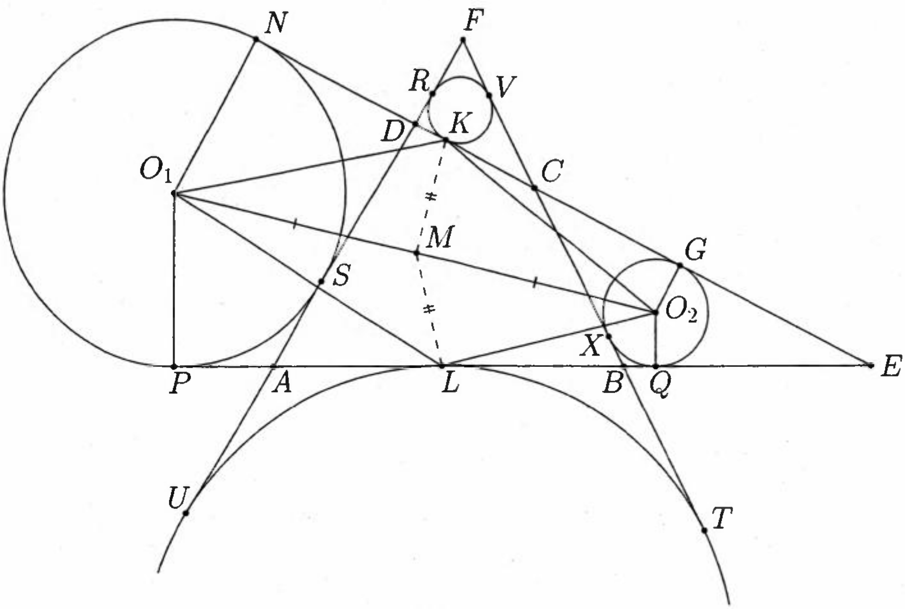


Consider the reflections of that line with respect to O_1O_3 and O_2O_3. Let A' be the intersection point of those reflections. Problem 4.2.3 yields that A' lies on the circumcircle of $O_1AO_2O_3$ from Problem 4.5.38. Note that $QA' \parallel CA$ and $PA' \parallel BA$.

Therefore, if we start moving the line PQ uniformly, without changing its slope, then P and Q would move uniformly in a straight line along O_1O_3 and O_2O_3, respectively. This means that the lines PA' and QA' would move uniformly, without changing their slope. It follows that point A' would move uniformly in a straight line. If $P \equiv B$, then $A' \equiv A$, and if $P \equiv O_3$, then $A' \equiv O_3$.

Hence, the line along which A' is moving intersects the circumcircle of $O_1AO_2O_3$ at the points A and O_3. As we saw, A' lies on that circle. Therefore, the only possibility is $H \in BC$ and $A \equiv A'$.

Problem 4.5.40. Let $ABCD$ be a convex quadrilateral. The four excircles of that quadrilateral are constructed (see the figure). Let O_1 and O_2 be the centers of the circles that touch the segments AD and BC. The other two circles touch AB and DC at L and K, respectively. Let M be the midpoint of O_1O_2. Prove that $MK = ML$.



Solution. We will use the notations from the figure.

Let $PQ = NG = x$. We have $RU = VT$, which means that $RS+SU = VX + XT$. Hence, $NK + PL = KG + LQ$ and so $x - KG + PL = KG + x - PL$. It follows that $KG = PL$ and $KN = LQ$.

We also have that $O_1P = O_1N$ and $O_2G = O_2Q$. Using the formula for the length of the median, and the Pythagorean Theorem, we consecutively have

$$\begin{aligned}
&KM^2 - LM^2 = \\
&= \frac{1}{4}(2O_1K^2 + 2O_2K^2 - O_1O_2^2 - (2O_1L^2 + 2O_2L^2 - O_1O_2^2)) \\
&= \frac{1}{2}(O_1K^2 + O_2K^2 - O_1L^2 - O_2L^2) \\
&= \frac{1}{2}(O_1N^2 + NK^2 + O_2G^2 + GK^2 - O_1P^2 - PL^2 - O_2Q^2 - QL^2) = 0.
\end{aligned}$$

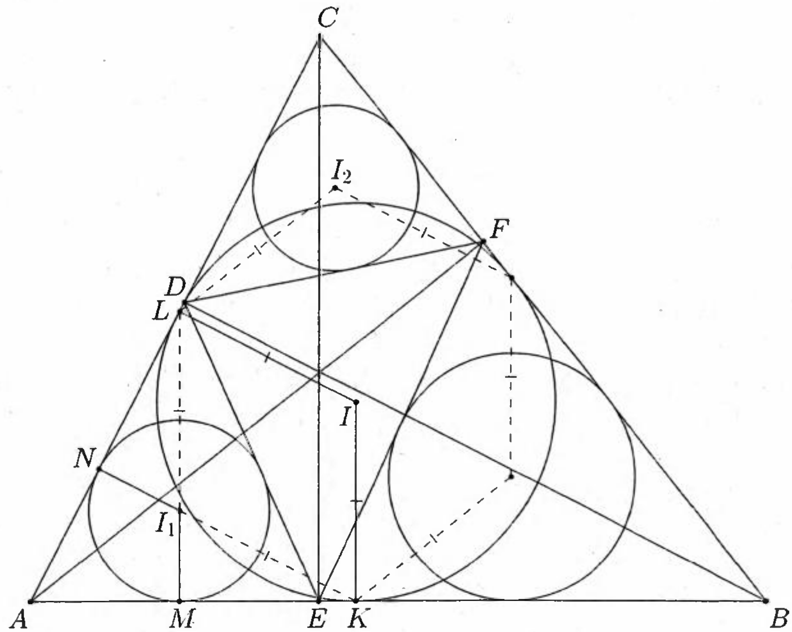
Problem 4.5.41. Let ABC be a triangle with altitudes AF, BD and CE. Let its incircle touch AB and AC at the points K and L, respectively. Let I_1 and I_2 be the incenters of $\triangle AED$ and of $\triangle CDF$, respectively. Prove that $KI_1 = LI_1 = LI_2$.

Solution. We will prove that $KI_1 = LI_1 = r$, where r is the inradius of $\triangle ABC$.

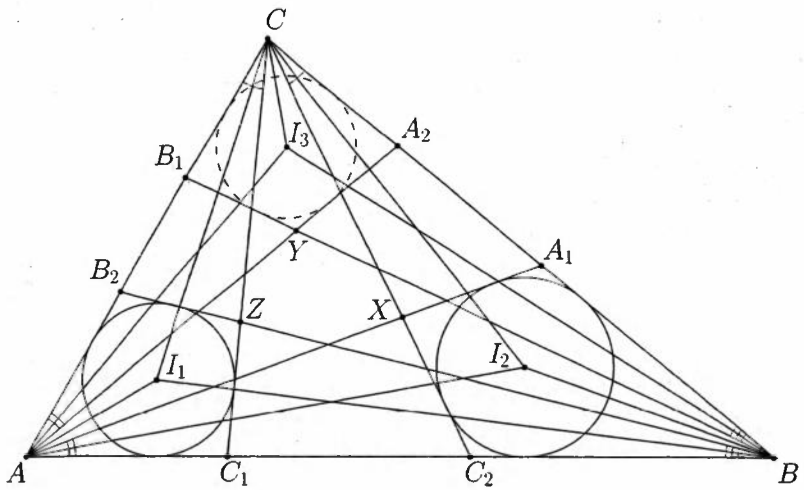
Let the incircle of $\triangle AED$ touch AB and AC at the points M and N, respectively. Note that $\triangle AED \sim \triangle ACB$ with ratio $\frac{AE}{AC} = \cos\alpha$.

Hence, $\frac{AM}{AL} = \cos\alpha$. But $\angle MAL = \alpha$, and so $LM \perp AM$. Thus, the points L, I_1 and M are collinear. Analogously, the points N, I_1 and K are also collinear.

Hence, $IL \parallel KI_1$ and $IK \parallel LI_1$. Therefore, the quadrilateral IKI_1L is a rhombus, and so $KI_1 = LI_1 = r$.



Problem 4.5.42. Let ABC be a triangle, and let CC_1 and CC_2, AA_1 and AA_2, BB_1 and BB_2 be three pairs of isogonally conjugate lines with respect to $\triangle ABC$. Also, we assume that $C_1 \in AB$, $C_2 \in AB$, $B_1 \in AC$, $B_2 \in AC$, $A_1 \in BC$ and $A_2 \in BC$. Let $AA_1 \cap CC_2 = X$, $BB_1 \cap AA_2 = Y$ and $CC_1 \cap BB_2 = Z$. Prove that if the quadrilaterals AC_1ZB_2 and BA_1XC_2 are circumscribed, then the quadrilateral B_1YA_2C is also circumscribed.



Solution. Denote the incenters of AC_1ZB_2 and BA_1XC_2 by I_1 and I_2, respectively. Let the bisectors of $\angle B_1BC$ and $\angle A_2AC$ intersect at I_3. Denote $\angle CAI_3 = \angle I_3AA_2 = \angle A_1AI_2 = \angle I_2AB = \varphi$, $\angle ABI_1 = \angle I_1BB_2 = \angle B_1BI_3 = \angle I_3BC = \psi$ and $\angle ACI_1 = \angle I_1CC_1 = \angle C_2CI_2 = \angle I_2CB = \delta$.

We will apply the trigonometric form of Ceva's Theorem to $\triangle ABC$ and the points I_1 and I_2. We get

$$\frac{1}{1} \cdot \frac{\sin\psi}{\sin(\beta-\psi)} \cdot \frac{\sin(\gamma-\delta)}{\sin\delta} = 1,$$
$$\frac{1}{1} \cdot \frac{\sin\delta}{\sin(\gamma-\delta)} \cdot \frac{\sin(\alpha-\varphi)}{\sin\varphi} = 1.$$

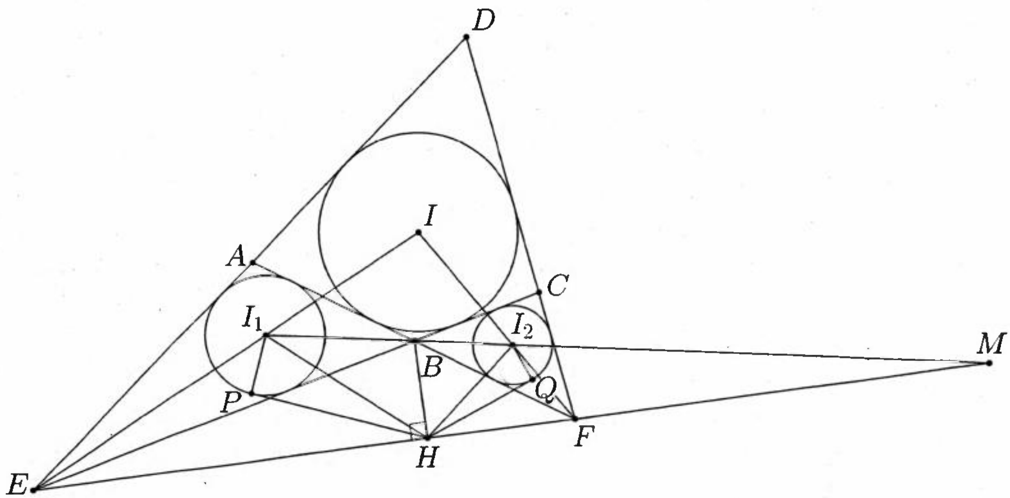
Multiplying these equalities, we get $\dfrac{\sin\psi}{\sin(\beta-\psi)} \cdot \dfrac{\sin(\alpha-\varphi)}{\sin\varphi} = 1.$

We will apply the trigonometric form of Ceva's Theorem to $\triangle ABC$ and the point I_3. We get

$$\frac{\sin\psi}{\sin(\beta-\psi)} \cdot \frac{\sin(\alpha-\varphi)}{\sin\varphi} \cdot \frac{\sin\angle ACI_3}{\sin\angle I_3CB} = 1 \Rightarrow \frac{\sin\angle ACI_3}{\sin\angle I_3CB} = 1.$$

Hence, I_3 is the incenter of both $\triangle AA_2C$ and $\triangle BB_1C$. Thus, the quadrilateral B_1YA_2C is circumscribed.

Problem 4.5.43. Let $ABCD$ be a circumscribed quadrilateral with incircle k. Let the rays DA and CB intersect at the point E, and let the rays DC and AB intersect at the point F. Let H be the projection of B onto the line EF. Let k_1 and k_2 be the incircles of $\triangle ABE$ and $\triangle BCF$, respectively. Let I_1 and I_2 be their respective centers. Let HP and HQ be the tangent lines from H respectively to k_1 and k_2 ($P \in k_1$ and $Q \in k_2$). Prove that $\angle I_1HP = \angle I_2HQ$.



Solution. Let $I_1I_2 \cap EF = M$. Monge's Theorem (Problem 6.2.3), applied to the circles k, k_1 and k_2, yields that M is the external center of homothety of k_1 and k_2.

Note that B is the internal center of homothety of k_1 and k_2. Hence, $\dfrac{MI_1}{MI_2} = \dfrac{r_1}{r_2} = \dfrac{I_1B}{BI_2}$, which means that the points I_1, B, I_2 and M are in a harmonic division.

Using the harmonic division properties, we obtain that HB is the bisector of $\angle I_1HI_2$ because $\angle MHB = 90°$. The angle bisector theorem yields

$$\frac{HI_1}{HI_2} = \frac{I_1B}{I_2B} = \frac{r_1}{r_2} = \frac{I_1P}{I_2Q},$$

implying that $\triangle I_1HP \sim \triangle I_2HQ$.

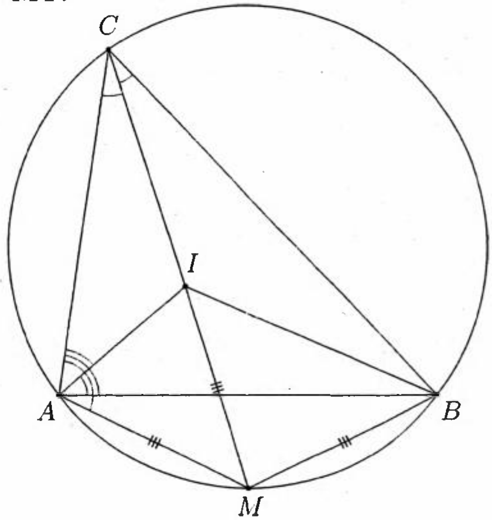
Problem 4.6.1. *(Trillium theorem)* Let ABC be a triangle. Its incenter is I and its circumcircle is k. Let M be the second intersection point of CI and k. Prove that $MA = MB = MI$.

Solution: We have

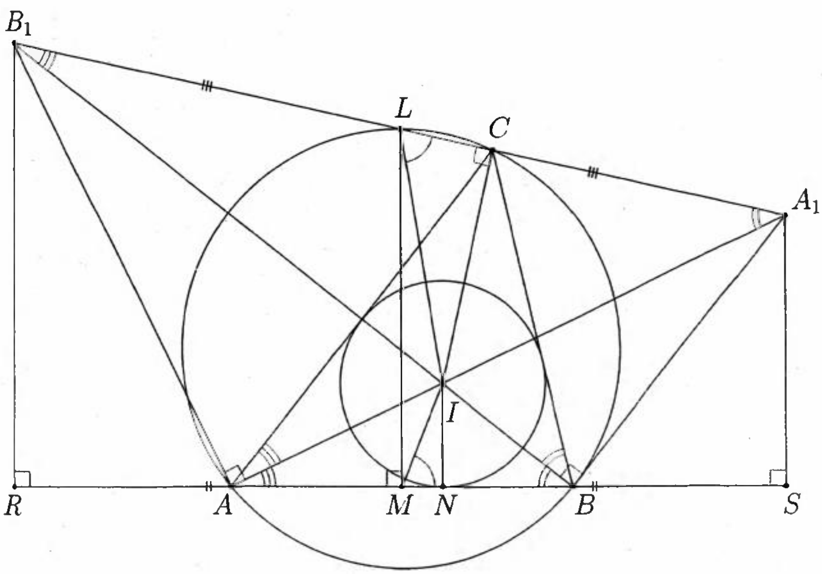
$$\begin{aligned}\angle AIM &= \angle IAC + \angle ICA \\ &= \angle IAB + \angle ICB \\ &= \angle IAB + \angle MAB \\ &= \angle MAI.\end{aligned}$$

Therefore, $AM = MI$.

Analogously, we deduce that $MI = IB$.



Problem 4.6.2. Let ABC be a triangle with incenter I. Let M be the midpoint of AB, and let L be the midpoint of $\widehat{ACB}$. Prove that $\angle ILC = \angle IMB$.



First solution. Let A_1 and B_1 be the intersection points of CL and the lines AI and BI, respectively. The given conditions imply that L lies on the external bisector of $\angle ACB$.

Therefore, A_1 and B_1 are the excenters of $\triangle ABC$, opposite to the vertices A and B, respectively. Thus, the quadrilaterals $AICB_1$ and $BICA_1$ are cyclic. Now we have $\angle IB_1A_1 = \angle IAC = \angle IAB$ and $\angle IA_1B_1 = \angle IBC = \angle IBA$.

Hence, $\triangle ABI \sim \triangle B_1IA_1$. If we prove that L is the midpoint of A_1B_1, this will yield $\triangle ILA_1 \sim \triangle IMB$ and the desired statement will follow.

Let R and S be the points of tangency of the line AB and the excircles, as shown in the figure. Let s be the semiperimeter of $\triangle ABC$. The fact that $BR = AS = s$ yields $MR = MS$. Moreover, ML is the midsegment in the trapezoid RSA_1B_1, which means that L is the midpoint of A_1B_1.

Second solution. Without loss of generality, let $AC > BC$. The line CL is the external bisector of $\angle ACB$, hence $\angle LCI = 90°$. We have $CI = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$ and $\angle CBL = \frac{\beta-\alpha}{2}$, thus $CL = 2R\sin\left(\frac{\beta-\alpha}{2}\right)$. Then

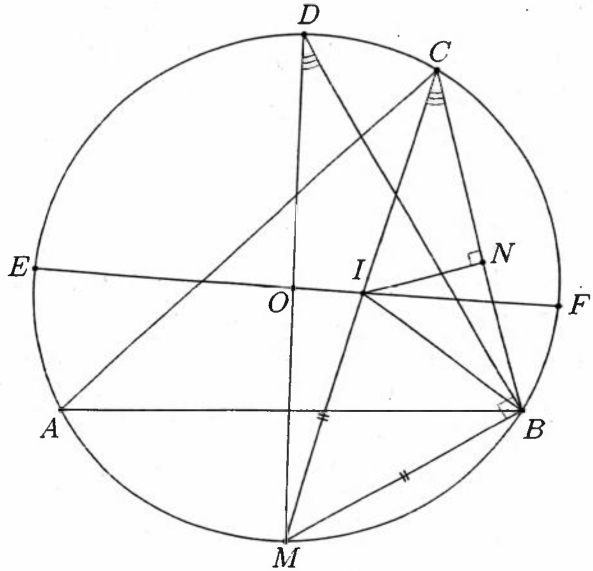
$$\cot\angle ILC = \frac{LC}{CI} = \frac{2R\sin\left(\frac{\beta-\alpha}{2}\right)}{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} = \frac{\sin\left(\frac{\beta-\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}.$$

We have $\angle MBI = \frac{\beta}{2}$, $MB = 2R\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$ and $BI = 4R\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}$. The cotangent technique, applied to $\triangle MBI$, yields

$$\cot\angle IMB = \frac{\frac{MB}{BI} - \cos\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \frac{\frac{2R\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{4R\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}} - \cos\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \frac{\frac{\cos\frac{\gamma}{2}}{2\sin\frac{\alpha}{2}} - \cos\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$
$$= \frac{\sin\left(\frac{\alpha+\beta}{2}\right) - 2\cos\frac{\beta}{2}\sin\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} = \frac{\sin\left(\frac{\beta-\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} = \cot\angle ILC$$

and the statement follows.

Problem 4.6.3. *(Euler's formula)* For a given triangle ABC, prove that $OI^2 = R^2 - 2Rr$, where O and I are the circumcenter and the incenter, respectively, and R and r are the circumradius and the inradius, respectively.



Solution. Let M be the second intersection point of CI and k, where k is the circumcircle of $\triangle ABC$.

Let D be the diametrically opposite point to M in k. Denote the projection of I onto the line BC by N.

We have that $\angle MDB = \angle MCB$ and $\angle MBD = 90°$. Then $\triangle MBD \sim \triangle INC$, which implies $\dfrac{MD}{IC} = \dfrac{MB}{IN}$.

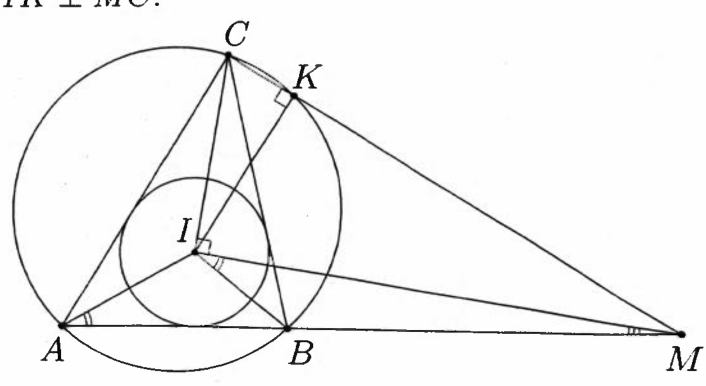
Hence, $CI.MB = DM.IN$.

The trillium theorem (Problem 4.6.1) yields $MB = MI$, thus $CI.MB = CI.IM = MD.IN$.

But $IN = r$ and $DM = 2R$, so $CI.IM = 2Rr$.

Now, by considering the power of I with respect to k, we get $CI.IM = R^2 - OI^2$, which gives $R^2 - 2Rr = OI^2$.

Problem 4.6.4. Let ABC be a triangle with circumcircle k, incircle ω, and incenter I. The line through I, perpendicular to CI, intersects the line AB at M. Let K be the second intersection point of k and MC. Prove that $IK \perp MC$.



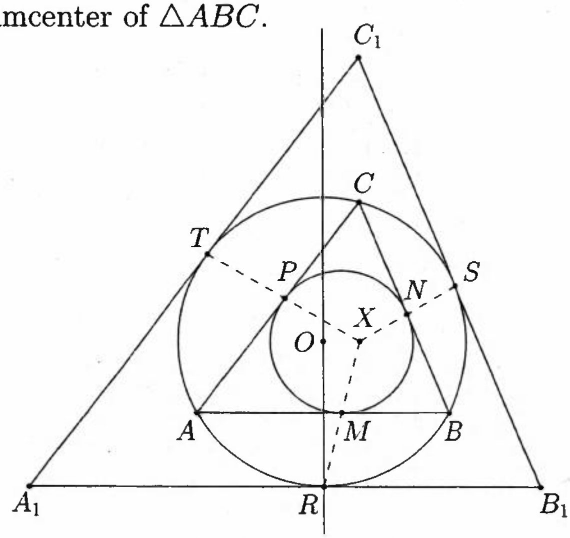
Solution. We have $\angle CIB = 90^\circ + \dfrac{\angle CAB}{2} = 90^\circ + \angle IAB$ and hence $\angle MIB = \angle BAI$.

Thus, $\triangle AMI \sim \triangle IMB$ and $MB.MA = MI^2$.

Considering the power of M with respect to k, we get $MK.MC = MB.MA = MI^2$. Therefore, $\triangle MIC \sim \triangle MKI$ and $\angle MKI = 90^\circ$.

Problem 4.6.5. Let ABC be a triangle with incircle ω and circumcircle k. The circle ω touches AB, BC and CA at the points M, N and P, respectively. The points R, S and T are the midpoints of the smaller arcs $\widehat{AB}$, $\widehat{BC}$ and $\widehat{CA}$, respectively. Prove that the lines MR, SN and PT are concurrent.

Solution. Let O be the circumcenter of $\triangle ABC$.

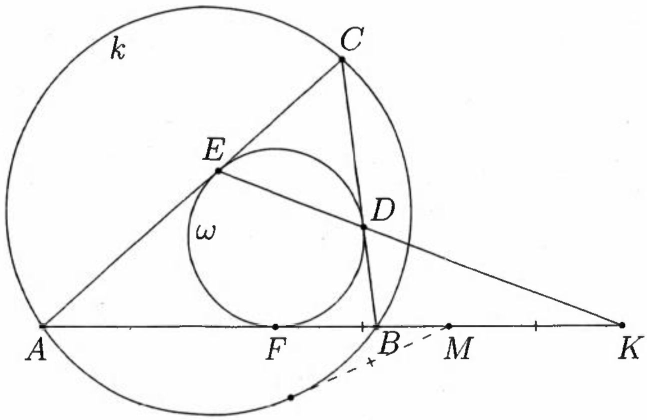


Let the tangent lines to k at the points R, S and T form the triangle $A_1B_1C_1$, as shown in the figure. It is clear that there exists a homothety that maps $\triangle ABC$ to $\triangle A_1B_1C_1$.

On the one hand, in $\triangle A_1B_1C_1$, the points R, S and T are the points of tangency of the incircle k and its sides.

On the other hand, in $\triangle ABC$, the points M, N and P are the points of tangency of the incircle ω and its sides. Thus, the points R, S and T correspond to the points M, N and P in the similar triangles $A_1B_1C_1$ and $\triangle ABC$. Therefore, the lines RM, SN and TP are concurrent at the center of the homothety mentioned above.

Problem 4.6.6. Let ABC be a triangle with incircle ω and circumcircle k. The circle ω touches BC, CA and AB at the points D, E and F, respectively. Let $ED \cap AB = K$. Prove that the midpoint M of KF lies on the radical axis of k and ω.



Solution. It suffices to prove that $MF^2 = MB.MA$. Applying Menelaus' Theorem to $\triangle ABC$ and the points E, D and K yields

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BK}{KA} = 1.$$

After performing simple manipulations, we get

$$\frac{BK}{KA} = \frac{BD}{EA} = \frac{s-b}{s-a},$$

or $\dfrac{AB}{BK} = \dfrac{b-a}{s-b}$, thus $BK = \dfrac{c(s-b)}{b-a}$. Hence,

$$FK = (s-b) + \frac{c(s-b)}{b-a} = \frac{2(s-a)(s-b)}{b-a}.$$

We have

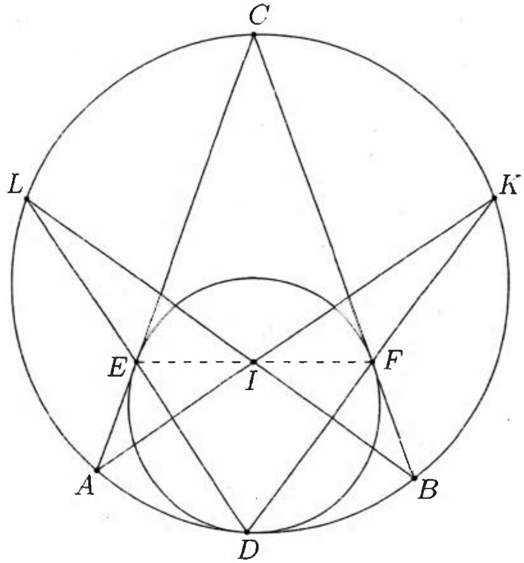
$$MF = \frac{(s-a)(s-b)}{b-a},$$

$$MB = (s-b)\left(\frac{s-a}{b-a} - 1\right) = \frac{(s-b)^2}{b-a},$$

$$MA = (s-a)\left(\frac{s-b}{b-a} + 1\right) = \frac{(s-a)^2}{b-a},$$

and the statement follows.

Problem 4.7.1. *(Verrièr's Lemma)* Let ABC be a triangle and let k be its circumcircle. Its C-mixtilinear incircle ω touches AC, BC and k at the points E, D and F, respectively. Prove that the incenter of $\triangle ABC$ is the midpoint of the segment EF.



Solution. Let the lines DF and DE intersect k again at the points K and L, respectively.

Problem 6.1.1 yields that L is the midpoint of the smaller arc $\widehat{AC}$, and K is the midpoint of the smaller arc $\widehat{BC}$.

Let $I = AK \cap BL$. Then I is the incenter of $\triangle ABC$. Pascal's Theorem, applied to the hexagon $ADBKCL$, yields that the points E, I and F are collinear. Finally, I is the midpoint of EF because $\triangle EFC$ is isosceles.

Problem 4.7.2. Let ABC be a triangle with circumcircle k. Its C-mixtilinear incircle ω touches AC, BC and k at the points E, F and X, respectively. The point D is the midpoint of the arc $\widehat{ACB}$. Prove that the points X, D and the midpoint of EF are collinear.

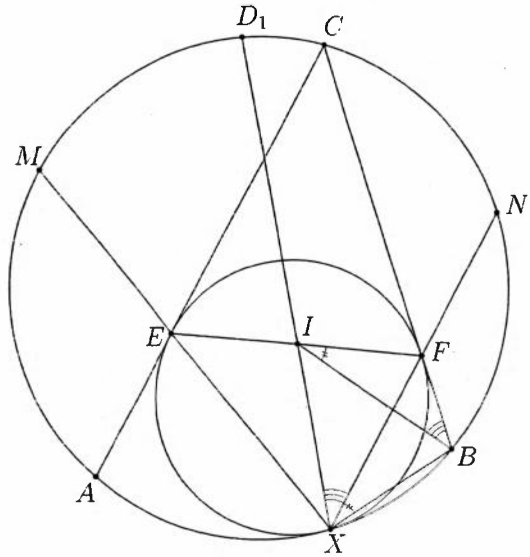
Solution. Problem 4.7.1 gives that the midpoint of EF is the incenter I of $\triangle ABC$.

Let the line IX intersect k again at the point D_1. Let M and N be the second intersection points of XE and XF with k, respectively. Problem 6.1.1 gives that M is the midpoint of the smaller arc $\widehat{AC}$ and N is the midpoint of the smaller arc $\widehat{BC}$ of k.

Hence, $\angle FXB = \angle NAB = \dfrac{\alpha}{2}$.

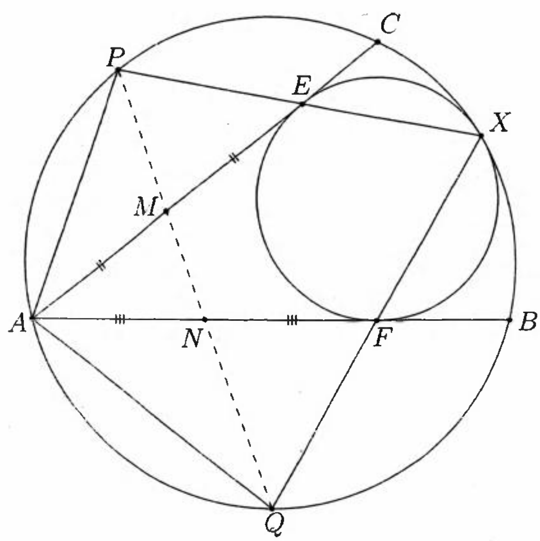
Moreover, $\angle FIB = \angle EFC - \angle IBF = 90^\circ - \dfrac{\gamma}{2} - \dfrac{\beta}{2} = \dfrac{\alpha}{2}$.

Therefore, the quadrilateral $BXIF$ is cyclic. Analogously, we get that the quadrilateral $AXIE$ is cyclic. In conclusion, $\angle BXD_1 = \angle EFC = \angle FEC = \angle AXD_1$ and hence $D \equiv D_1$.



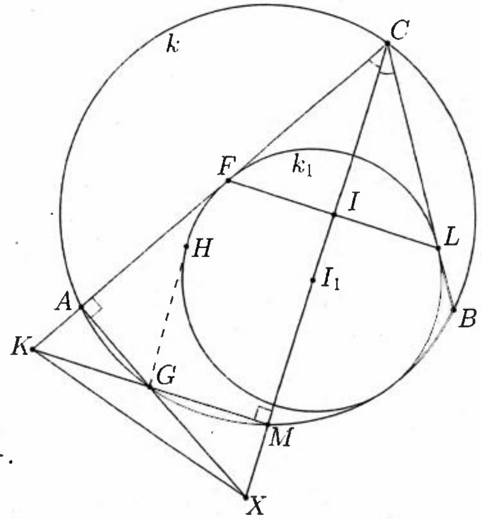
Problem 4.7.3. Let ABC be a triangle with circumcircle k. The A-mixtilinear incircle ω touches AC, AB and k at the points E, F and X, respectively. The points P and Q are the midpoints of the smaller arcs $\widehat{AC}$ and $\widehat{AB}$, respectively. The points M and N are the midpoints of AE and AF, respectively. Prove that the points P, M, N and Q are collinear.

Solution. Problem 6.1.1 gives that the points X, E and P, as well as the points X, F and Q are collinear. We have that $\angle PXC = \angle PAC = \angle PCA$ and hence $\triangle PCE \sim \triangle PXC$.



We deduce that $PA^2 = PC^2 = PE.PX$. Thus, the point P lies on the radical axis of the circle ω and the point A. Analogously, the point Q lies on the same radical axis. The points M and N also lie on it because $AM = ME$ and $AN = NF$.

Problem 4.7.4. Let ABC be a triangle with circumcircle k. Let M be the midpoint of the smaller arc $\widehat{AB}$ of k. Let the point G be diametrically opposite to C with respect to k, and let $AG \cap CM = X$. Let k_1 be the C-mixtilinear incircle of $\triangle ABC$. The segment GH ($H \in k_1$) is tangent to k_1. Prove that $GH = GX$.



Solution. It suffices to show that G lies on the radical axis of the point X and the circle k_1.

We will use the notations on the diagram. Since $\angle GMC = \angle GAC = 90°$, it suffices to prove that K lies on the radical axis of the point X and the circle k_1. This is equivalent to $KX = KF$.

We have $KX = \dfrac{AM}{\cos\frac{\gamma}{2}} = \dfrac{2R\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}}$.

Verrièr's Lemma (Problem 4.7.1) gives that the incenter I of $\triangle ABC$ is the midpoint of FL. The law of sines, applied to $\triangle AIC$, gives $CI = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$, and so $CF = \dfrac{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$.

Also, $CK = \dfrac{CM}{\cos\frac{\gamma}{2}} = \dfrac{2R\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}}$ and therefore

$$KF = KC - CF = \frac{2R\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}} - \frac{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}} =$$

$$= \frac{2R(\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2})}{\cos\frac{\gamma}{2}} = \frac{2R\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}} = KX.$$

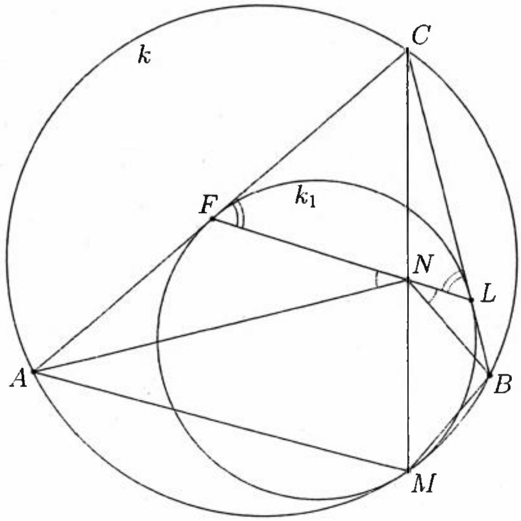
Problem 4.7.5. Let ABC be a triangle with circumcircle k. The C-mixtilinear incircle k_1 touches k at the point M, and the lines AC and BC at the points F and L, respectively. Let $FL \cap CM = N$. Prove that $\angle ANF = \angle BNL$.

Solution. It suffices to show that $\triangle ANF \sim \triangle BNL$. Since $\angle CFL = \angle CLF$, it remains to show that $\dfrac{AF}{BL} = \dfrac{FN}{NL}$.

We will use Lemma 1 from the proof of Casey's Theorem (Problem 6.1.10). We have

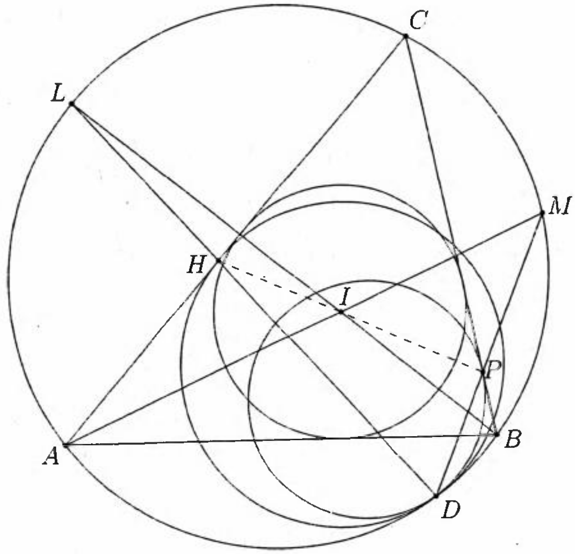
$$AF = AM\sqrt{\frac{R-R_1}{R}}, \quad BL = BM\sqrt{\frac{R-R_1}{R}}.$$

Therefore, $\dfrac{AF}{BL} = \dfrac{AM}{BM} = \dfrac{\sin\angle ACM}{\sin\angle BCM} = \dfrac{FN}{NL}$.



Problem 4.7.6. Let ABC be a triangle with circumcircle k. Let D be an arbitrary point on the smaller arc $\widehat{AB}$ of k. The circle k_1 touches k internally at the point D and the line BC at the point P. The circle k_2 touches k internally at the point D and the line AC at the point H. The point I is the incenter of $\triangle ABC$. Prove that the points H, I and P are collinear.

Solution. Let L and M be the midpoints of the smaller arcs $\widehat{AC}$ and $\widehat{BC}$ of k, respectively. Problem 6.1.1 implies that the points D, H and L, as well as the points D, P and M are collinear. Also, the points A, I and M, as well as B, I and L are collinear.



Finally, Pascal's Theorem, applied to the points A, D, B, M, C and L, yields that the points H, I and P are collinear.

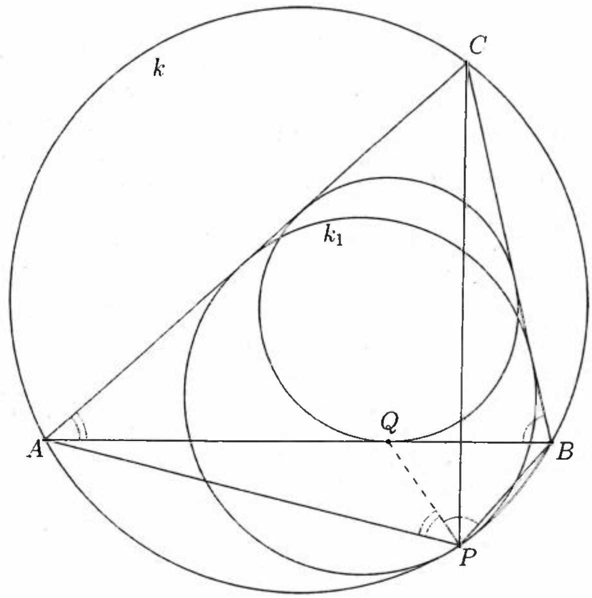
Problem 4.7.7. Let ABC be a triangle with circumcircle k and C-mixtilinear incircle k_1. Its incircle k_2 touches AB at the point Q. Prove that $\angle APQ = \angle BAC$ and $\angle BPQ = \angle ABC$.

Solution. Note that $\dfrac{\sin\angle APQ}{\sin\angle BPQ} = \dfrac{AQ}{QB} \cdot \dfrac{BP}{PA} = \dfrac{s-a}{s-b} \cdot \dfrac{\sin\angle BCP}{\sin\angle ACP}$.

Problem 4.7.11 implies that

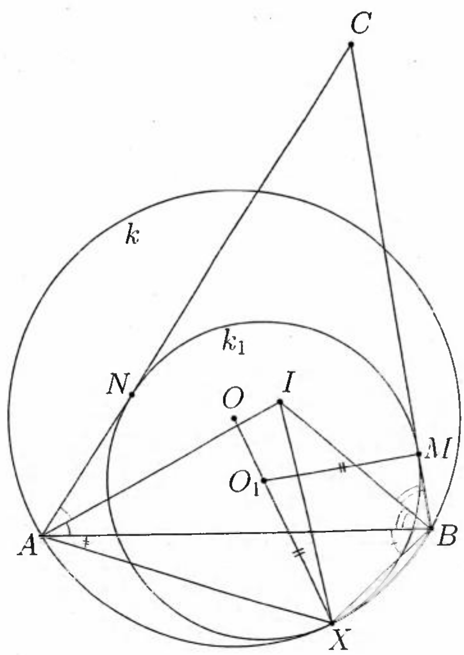
$$\frac{\sin\angle BCP}{\sin\angle ACP} = \frac{s-b}{s-a} \cdot \frac{a}{b}$$
$$\Rightarrow \frac{\sin\angle APQ}{\sin\angle BPQ} = \frac{s-a}{s-b} \cdot \frac{s-b}{s-a} \cdot \frac{a}{b} = \frac{\sin\alpha}{\sin\beta}.$$

But $\angle APQ + \angle BPQ = \alpha + \beta$ and hence $\cot\angle BPQ = \cot\beta$ and $\cot\angle APQ = \cot\alpha$, i.e. $\angle APQ = \angle BAC$ and $\angle BPQ = \angle ABC$.



Problem 4.7.8. Let ABC be a triangle with incenter I. Let k be an arbitrary circle through A and B. The circle k_1 touches AC and BC at the points N and M, respectively. It also touches the smaller arc $\widehat{AB}$ of k internally at the point X. Prove that $\angle AXI = \angle IXB$.

Solution. Let O and R be the center and the radius of k, respectively. Let O_1 and r_1 be the center and the radius of k_1, respectively. Lemma 1 from the proof of Casey's Theorem (Problem 6.1.10) yields



$$BM = \frac{BX\sqrt{R(R-r_1)}}{R},$$

whence $\dfrac{BM}{BX} = \sqrt{\dfrac{R-r_1}{R}}$.

Analogously,

$$\frac{AN}{AX} = \sqrt{\frac{R-r_1}{R}},$$

whence $\dfrac{BM}{BX} = \dfrac{AN}{AX}$.

Let $\angle BAX = \alpha$, $\angle ABX = \beta$, $\angle AXB = \gamma$, $\angle IAB = \angle CAI = \varphi$ and $\angle IBA = \angle CBI = \psi$. Clearly, the points O, O_1 and X are collinear and k is the circumcircle of $\triangle ABX$. We have that $\angle O_1XB = 90^\circ - \alpha$, $\angle O_1XA = 90^\circ - \beta$ and $\angle O_1MB = 90^\circ$. Therefore,

$$\angle XO_1M = 360^\circ - 90^\circ - \beta - 2\psi - 90^\circ + \alpha = 180^\circ - 2\psi - \beta + \alpha,$$

$$O_1M = O_1X = r_1 \Rightarrow \angle O_1MX = \psi + \frac{\beta - \alpha}{2}$$

$$\Rightarrow \angle XMB = 90^\circ - \psi + \frac{\alpha - \beta}{2}, \ \angle MXB = 90^\circ - \psi - \frac{\alpha + \beta}{2}.$$

The law of sines, applied to $\triangle XBM$, yields $\dfrac{BM}{BX} = \dfrac{\cos\left(\psi + \frac{\alpha+\beta}{2}\right)}{\cos\left(\psi + \frac{\beta-\alpha}{2}\right)}$.

Analogously, $\dfrac{AN}{AX} = \dfrac{\cos\left(\varphi + \frac{\alpha+\beta}{2}\right)}{\cos\left(\varphi + \frac{\alpha-\beta}{2}\right)}$. Using $\dfrac{BM}{BX} = \dfrac{AN}{AX}$, we get

$$\frac{\cos\left(\psi + \frac{\alpha+\beta}{2}\right)}{\cos\left(\psi + \frac{\beta-\alpha}{2}\right)} = \frac{\cos\left(\varphi + \frac{\alpha+\beta}{2}\right)}{\cos\left(\varphi + \frac{\alpha-\beta}{2}\right)}.$$

Now we have

$$\cos\left(\psi + \frac{\alpha+\beta}{2}\right)\cos\left(\varphi + \frac{\alpha-\beta}{2}\right) = \cos\left(\psi + \frac{\beta-\alpha}{2}\right)\cos\left(\varphi + \frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow \frac{\cos(\varphi+\psi+\alpha) + \cos(\psi-\varphi+\beta)}{2} = \frac{\cos(\varphi+\psi+\beta) + \cos(\varphi-\psi+\alpha)}{2}$$

$$\Rightarrow \cos(\varphi+\psi+\alpha) - \cos(\varphi-\psi+\alpha) = \cos(\varphi+\psi+\beta) - \cos(\psi-\varphi+\beta)$$

$$\Rightarrow 2\sin(\varphi+\alpha)\sin(-\psi) = 2\sin(\psi+\beta)\sin(-\varphi)$$

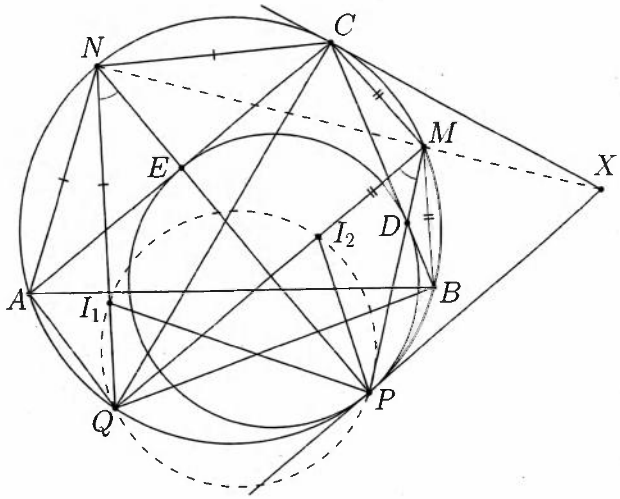
$$\Rightarrow \sin(\varphi+\alpha)\sin\psi = \sin(\psi+\beta)\sin\varphi.$$

The law of sines, applied to $\triangle AXI$, $\triangle BXI$ and $\triangle ABI$ gives

$$\frac{\sin\angle AXI}{\sin\angle BXI} = \frac{AI}{BI}\cdot\frac{XI}{XI}\cdot\frac{\sin\angle IAX}{\sin\angle IBX} = \frac{\sin\psi}{\sin\varphi}\cdot\frac{\sin(\alpha+\varphi)}{\sin(\beta+\psi)} = 1.$$

Hence, $\sin\angle AXI = \sin\angle IXB$. Finally, since $\angle AXI + \angle IXB < 180^\circ$, we conclude that $\angle AXI = \angle IXB$.

Problem 4.7.9. Let ABC be a triangle with circumcircle k. The C-mixtilinear circle k_1 touches k at the point P. Let Q be an arbitrary point on the arc $\widehat{AB}$ of k, not containing C. The points I_1 and I_2 are the incenters of $\triangle AQC$ and $\triangle QBC$, respectively. Prove that the quadrilateral QPI_2I_1 is cyclic.



Solution. Let k_1 touch AC and BC at the points E and D, respectively. The midpoints of the smaller arcs $\widehat{AC}$ and $\widehat{BC}$ are N and M, respectively. Let the lines tangent to k at the points C and P intersect at the point X.

Clearly, the points Q, I_1 and N and the points Q, I_2 and M are collinear. Moreover, by Problem 6.1.1, we have that the points P, D and M are collinear and that the points P, E and N are collinear.

Hence, $\angle QNP = \angle QMP$. Problem 4.6.1 gives that $I_2M = MB = MC$ and $I_1N = NA = NC$.

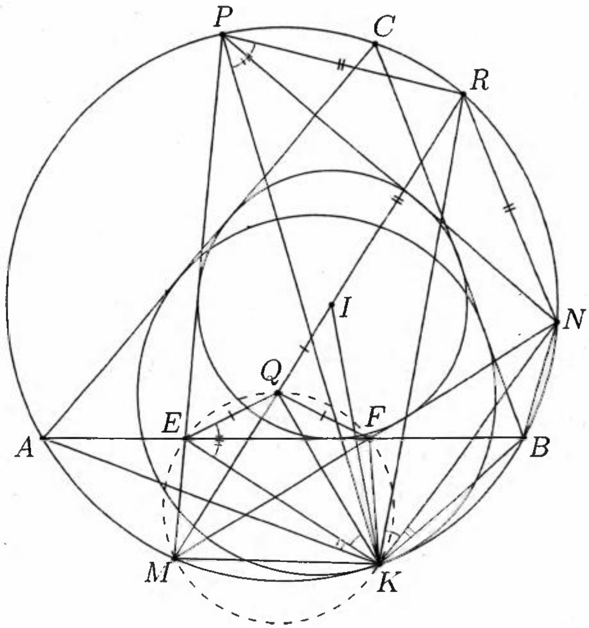
Clearly, X is the radical center of the point C and the circles k and k_1. Hence, X lies on the radical axis of C and k_1, which, by Problem 4.7.3, is the line MN. Thus, the points M, N and X are collinear. This implies that the quadrilateral $CNPM$ is harmonic. Therefore,

$$\frac{MP}{MI_2} = \frac{MP}{MC} = \frac{NP}{NC} = \frac{NP}{NI_1}.$$

This, together with $\angle QNP = \angle QMP$, gives $\triangle PI_2M \sim \triangle PI_1N$, so $\angle PI_2Q = \angle PI_1Q$ and we conclude that the quadrilateral QPI_2I_1 is cyclic.

Problem 4.7.10. Let ABC be a triangle with circumcircle k and incircle ω. The C-mixtilinear incircle k_1 touches k at the point K. The point $M \in \widehat{AKB}$ is arbitrary. The tangent lines through M to ω intersect AB at the points E and F. Prove that the quadrilateral $MEFK$ is cyclic.

Solution. Let K' be the second intersection point of the circumcircle of $\triangle MEF$ and k. We will show that $K' \equiv K$.



Problems 4.7.2 and 4.7.1 yield that it suffices to show that $K'I$ bisects $\angle AK'B$. Indeed, then K' will be the other intersection point of k and the line through I and the midpoint of the arc $\widehat{ACB}$.

Let $ME \cap k = \{M, P\}$ and $MF \cap k = \{M, N\}$.

Problem 10.17 gives that PN touches ω. Let $MI \cap k = \{M, R\}$ and let MI intersect the circumcircle of $\triangle MEF$ again at the point Q. Now, the points Q and R are the midpoints of the arcs $\widehat{EF}$ and $\widehat{PN}$. We have

$$\angle AK'E = \angle K'EF - \angle K'AB = \angle K'MN - \angle K'MB = \angle BMN = \angle BK'N$$

and $\angle EK'Q = \angle PMR = \angle RMN = \angle RK'N$. From here, it suffices to show that $K'I$ bisects $\angle QK'R$. We have that $\angle EK'Q = \angle PMR = \angle PK'R$ and $\angle QEK' = \angle RMK' = \angle RPK'$.

These imply that $\triangle QEK' \sim \triangle RPK'$ and hence $\dfrac{RK'}{K'Q} = \dfrac{RP}{QE} = \dfrac{RI}{IQ}$.

We used Problem 4.6.1. Finally, applying the angle bisector theorem, we get that $K'I$ is the angle bisector of $\angle QK'R$.

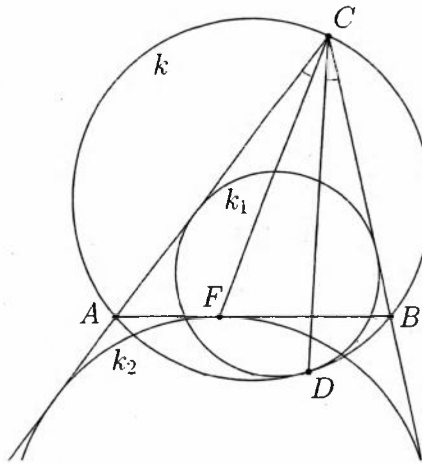
Problem 4.7.11. Let ABC be a triangle with circumcircle k. Its C-excircle k_2 touches AB at the point F. Its C-mixtilinear incircle k_1 touches k at the point D. Prove that $\angle ACF = \angle BCD$.

Solution. Consider the composition φ of an inversion with center C and radius $\sqrt{CA.CB}$ and a reflection with respect to the angle bisector of $\angle ACB$.

Clearly, $\varphi(A) = B$, $\varphi(B) = A$, $\varphi(k) = AB$ and $\varphi(AB) = k$.

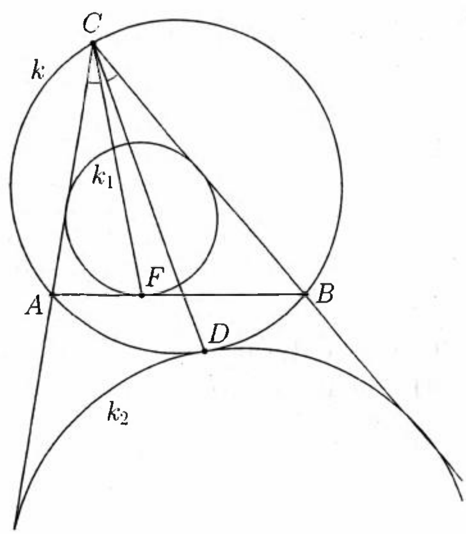
Hence, $\varphi(k_1) = k_2$ and $\varphi(k_2) = k_1$, so $\varphi(F) = D$ and $\varphi(D) = F$.

Therefore, the lines CF and CD are symmetric with respect to the angle bisector of $\angle ACB$, which means that $\angle ACF = \angle BCD$.



Problem 4.7.12. Let ABC be a triangle with circumcircle k. Its incircle k_1 touches the segment AB at the point F. Its C-mixtilinear excircle k_2 touches k at the point D. Prove that $\angle ACF = \angle BCD$.

Solution. As in Problem 4.7.11, the statement follows from the fact that CF is the image of CD under the same composition of inversion and reflection.

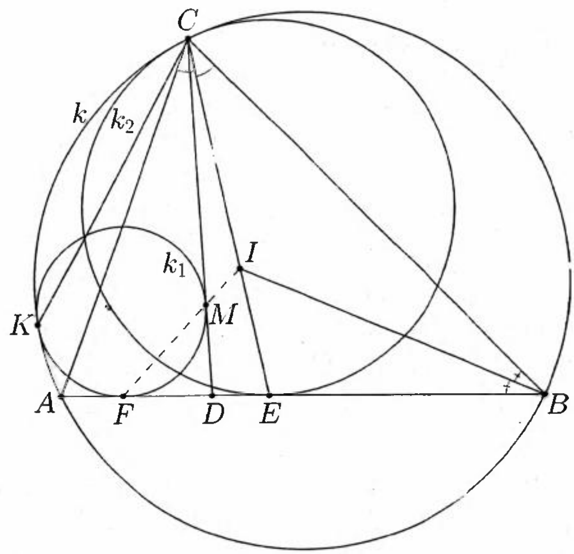


Problem 4.7.13. *(Sawayama's Lemma)* Let ABC be a triangle with circumcircle k. Let D be an arbitrary point on AB. The circle k_1 touches the segments AD and CD, as well as k internally at a point on the arc $\overparen{AC}$. Let k_1 touch AB at F and CD at M. The point I is the incenter of $\triangle ABC$. Prove that the points F, M and I are collinear.

Solution. Let the lines CI and AB intersect at E and let k_1 touch k at the point K.

Problem 6.1.1 yields that the circle k_2, which touches the segment AB and k internally at the point C, touches AB exactly at the point E.

Let the radii of k, k_1 and k_2 be R, R_1 and R_2, respectively.



Lemma 1 from the proof of Casey's Theorem (Problem 6.1.10) and the angle bisector theorem yield

$$\frac{IE}{IC} = \frac{BE}{BC} = \sqrt{\frac{R-R_2}{R}}, \frac{EF}{CK} = \frac{\sqrt{(R-R_1)(R-R_2)}}{R}, \frac{CM}{CK} = \sqrt{\frac{R-R_1}{R}}.$$

Therefore, $\dfrac{EF}{FD} \cdot \dfrac{DM}{MC} \cdot \dfrac{CI}{IE} = \dfrac{EF}{CK} \cdot \dfrac{CK}{MC} \cdot \dfrac{CI}{IE} =$

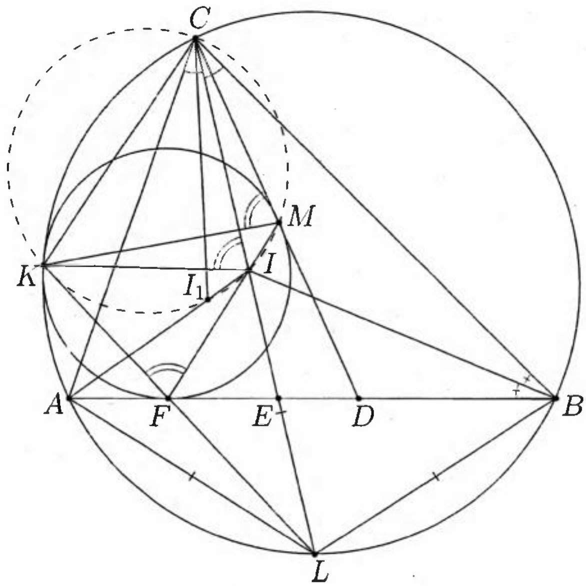
$$= \frac{\sqrt{(R-R_1)(R-R_2)}}{R}\sqrt{\frac{R}{R-R_1}}\sqrt{\frac{R}{R-R_2}} = 1.$$

The desired statement follows by Menelaus' Theorem, applied to $\triangle DEC$ and the points F, M and I.

Problem 4.7.14. Let ABC be a triangle with circumcircle k. The point D lies on the segment AB. The circle k_1 touches AB at the point F, touches CD at the point M and touches k internally at the point K on the smaller arc $\widehat{AC}$. The point I is the incenter of $\triangle ABC$, and the point I_1 is the incenter of $\triangle ADC$. Prove that the pentagon KI_1IMC is cyclic.

Solution. Let L be the second intersection point of CI and k. Problem 4.6.1 gives that $LA = LB = LI$.

Problem 6.1.1 yields that the points K, F and L are collinear. Problem 4.7.13 yields that the points F, I and M are collinear. Also, the points A, I_1 and I are collinear. Now we have



$$\begin{aligned}\angle AIF &= \angle IFD - \angle IAF \\ &= 90^\circ - \frac{\angle FDM}{2} \\ &- \frac{\angle CAD}{2} = \frac{\angle ACD}{2} \\ &= \angle I_1CM.\end{aligned}$$

Hence, the points I_1, I, M and C are concyclic.

We have that $\angle AKL = \angle ABL = \angle FAL$ and thus $\triangle ALF \sim \triangle KLA$. We deduce that $IL^2 = AL^2 = LF.LK$, and so LI is the tangent line through I to the circumcircle of $\triangle FIK$. This gives

$$\angle CMK = \angle MFK = \angle CIK.$$

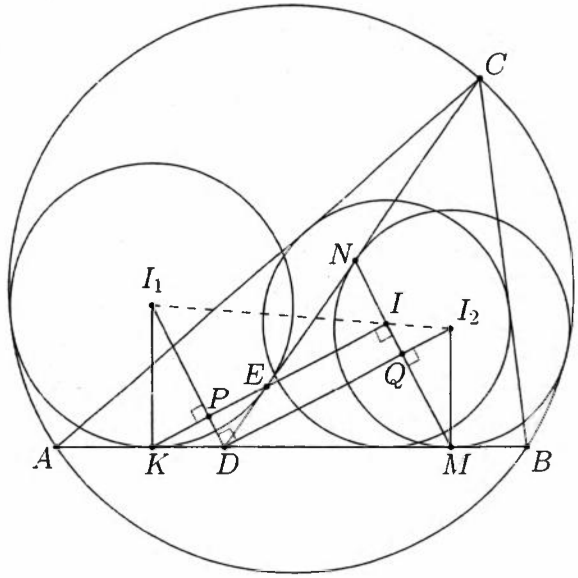
Therefore, the points K, I, M and C are concyclic, and so the pentagon KI_1IMC is cyclic.

Problem 4.7.15. *(Thébault's Theorem)* Let ABC be a triangle with circumcircle k. The point D lies on the segment AB. The circle k_1 touches the segment AD at the point K, the segment CD at the point E and the circle k internally at a point on the arc $\widehat{AC}$. The circle k_2 touches the segment BD at the point M, the segment CD at the point N and the circle k internally at a point of the arc $\widehat{BC}$. The point I is the incenter of $\triangle ABC$ and the points I_1 and I_2 are the centers of k_1 and k_2, respectively. Prove that the points I, I_1 and I_2 are collinear.

Solution. By Sawayama's Lemma (Problem 4.7.13), the points K, E and I are collinear, and the points M, N and I are collinear.

We have that $DI_2 \perp DI_1$. We also have $DI_2 \perp MN$ and $DI_1 \perp KE$. Hence, the quadrilateral $DPIQ$ is a rectangle.

Now it is easy to see that $\triangle KDI_1 \sim \triangle MI_2D$, with KP and MQ being respective altitudes in these triangles. Hence,



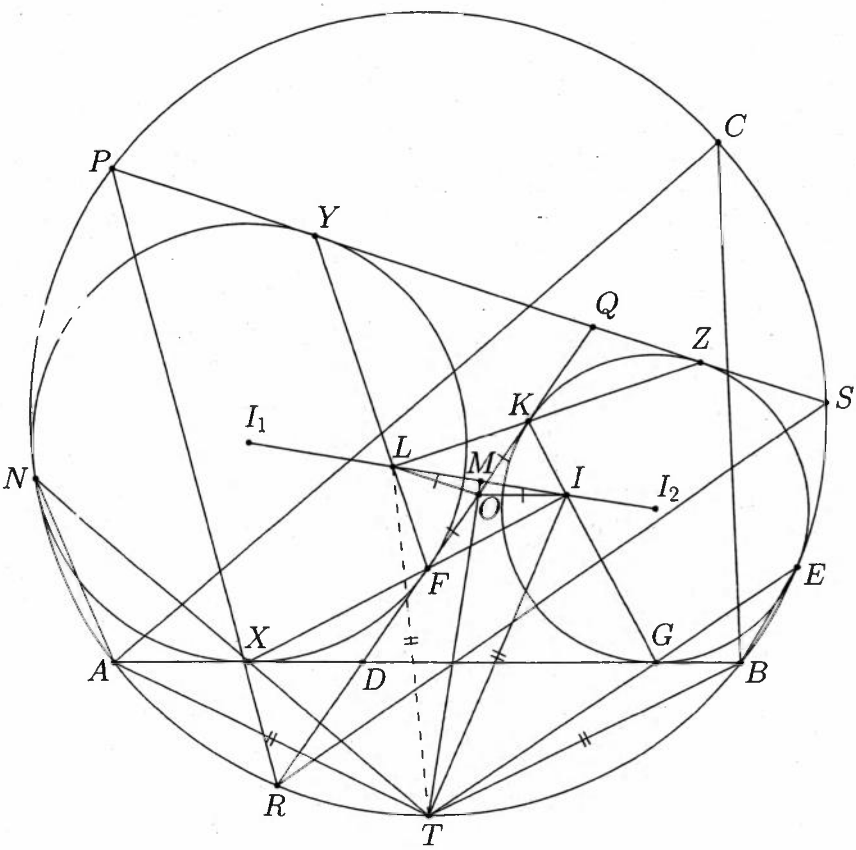
$$\frac{DP}{PI_1} = \frac{I_2Q}{QD} \Rightarrow \frac{IQ}{PI_1} = \frac{I_2Q}{IP},$$

whence $\triangle I_1PI \sim \triangle IQI_2$ and finally $\angle I_1IP + \angle I_2IQ = 90°$. Therefore, the points I, I_1 and I_2 are collinear.

Problem 4.7.16 Let ABC be a triangle with circumcircle k. The point D lies on the segment AB. The circle k_1 touches the circle k internally at a point on the arc $\widehat{AC}$, touches the segment AD and touches the segment CD at the point F. The circle k_2 touches the circle k internally at a point on the arc $\widehat{BC}$, touches the segment BD and touches the segment CD at the point K. The point I is the incenter of $\triangle ABC$. The segment YZ is the second common external tangent line of k_1 and k_2, such that $Y \in k_1$ and $Z \in k_2$. Let $ZK \cap FY = L$. Prove that the points A, B, I and L are concyclic.

Solution. Problem 4.6.1 yields that the midpoint T of the smaller arc $\widehat{AB}$ is the circumcenter of $\triangle ABI$. Therefore, it suffices to prove that the point T lies on the perpendicular bisector of IL. Let $YZ \cap k = \{P, S\}$ and $CD \cap k = \{C, R\}$.

Let also $CD \cap PS = Q$. Denote the centers of k_1 and k_2 by I_1 and I_2, respectively. Let k_1 touch AB at the point X and let k_2 touch AB at the point G. Let also $k_1 \cap k = N$ and $k_2 \cap k = E$. By Sawayama's Lemma (Problem 4.7.13), we have that the points K, G and I are collinear and that the points X, F and I are collinear.



We have $DI_2 \perp DI_1$. Moreover, $DI_2 \perp KG$ and $DI_1 \perp XF$ and thus $XI \perp IG$. Thébault's Theorem (Problem 4.7.15) yields that I lies on the line I_1I_2. Problem 6.1.1 gives that the points T, X and N are collinear, and that the points T, G and E are collinear. Now note that

$$\angle ANT = \angle ABT = \angle BAT = \angle BET \Rightarrow \triangle ATX \sim \triangle NTA,$$
$$\triangle BTG \sim \triangle ETB \Rightarrow TA^2 = TB^2 = TG.TE = TX.TN.$$

Hence, the point T lies on the radical axis of k_1 and k_2.

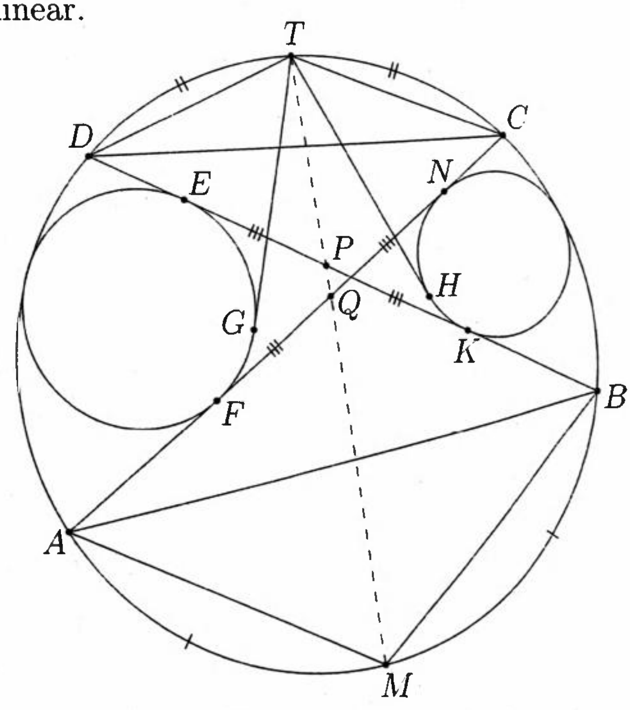
Now consider $\triangle RSP$ and Q as an arbitrary point on PS. Sawayama's Lemma (Problem 4.7.13) gives that the lines ZK and FY intersect at the incenter of $\triangle RSP$. Hence, the point L is exactly this center.

We have that QI_2 and QI_1 are the internal and external angle bisectors of $\angle SQR$. Thus, $QI_2 \perp QI_1$. Also, $QI_2 \perp KZ$ and $QI_1 \perp YF$, so $YL \perp LZ$.

Thébault's Theorem (Problem 4.7.15) gives that the point L lies on the line I_1I_2. Therefore, the points I and L lie on the circle with diameter FK.

Let M be the midpoint of IL and let O be the midpoint of FK. Then OM is the perpendicular bisector of LI, which implies that $OM \perp I_1I_2$. The equality $OK = OF$ gives that O lies on the radical axis of the circles k_1 and k_2. Therefore, OM is this radical axis. But as we saw, the point T also lies on this radical axis and hence T lies on the perpendicular bisector of IL. In conclusion, the points A, B, I, L lie on a circle with center T.

Problem 4.7.17. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The circle k_1 touches the circle k internally at a point on the smaller arc $\widehat{AD}$, touches the segment AC at the point F, and touches the segment BD at the point E. The circle k_2 touches the circle k internally at a point on the smaller arc $\widehat{BC}$, touches the segment AC at the point N and touches the segment BD at the point K. The points P and Q are the midpoints of EK and NF, respectively. The points T and S are the midpoints of the smaller arcs $\widehat{CD}$ and $\widehat{AB}$, respectively. Prove that the points T, P, Q and M are collinear.



Solution. Clearly, the points P and Q lie on the radical axis l of k_1 and k_2. Therefore, it suffices to prove that T and M also lie on l.

Let TG and TH be the tangent lines through T to k_1 and k_2, respectively, so that $G \in k_1$ and $H \in k_2$. We will show that $TG = TH$.

Casey's Theorem (Problem 6.1.10), applied to the circle k_1 and the points D, T and C, yields

$$TG = \frac{TC.DE + TD.CF}{DC} = \frac{TC(DE+CF)}{DC} = \frac{TC(DK+CN)}{DC}.$$

The same theorem, applied to the circle k_2 and the points D, T and C, gives

$$TH = \frac{TC.DK + TD.CN}{DC} = \frac{TC(DK+CN)}{DC} = TG.$$

Analogously, $M \in l$.

Problem 4.7.18. Let $CYBED$ be a cyclic pentagon with circumcircle k. The circle k_1 touches the line CB at the point N, touches the line EY and touches the circle k internally at a point on the smaller arc $\widehat{CE}$. The circle k_2 touches the line CB at the point M, touches the line DY and touches the circle k internally at a point on the smaller arc $\widehat{BD}$. The circle k_3 touches the lines CD and BE and the circle k internally at the point X on the smaller arc $\widehat{CB}$. Prove that the quadrilateral $XYNM$ is cyclic.

Solution. Firstly, we will prove the following lemma.

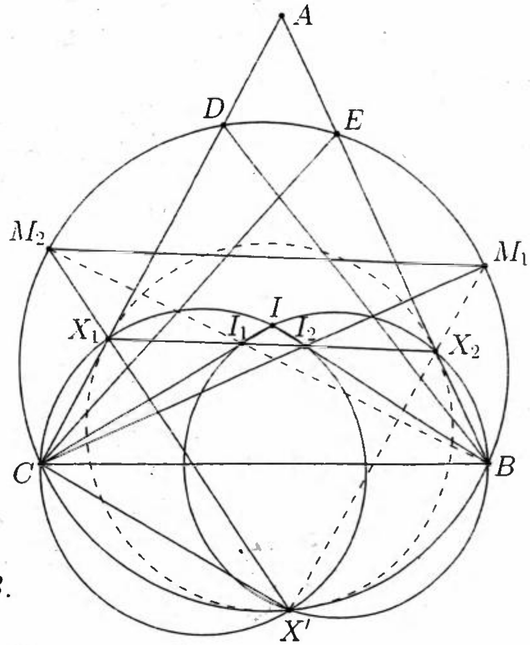
Lemma. Using the notations of the problem, let $CD \cap BE = A$. Let I_1, I_2 and I be the incenters of $\triangle CBD$, $\triangle CBE$ and $\triangle CBA$, respectively. Then the quadrilaterals CXI_2I and BXI_1I are cyclic.

Proof. Let the circumcircle of $\triangle CI_1I_2$ intersect k for the second time at the point X'. We will show that $X \equiv X'$. Denote

$$I_1I_2 \cap CD = X_1,$$
$$I_1I_2 \cap BE = X_2,$$
$$X'X_1 \cap k = \{X', M_2\},$$
$$CI_2 \cap k = \{C, M_1\}.$$

We have

$$\angle CI_1B = 90^\circ + \frac{\angle CDB}{2} = 90^\circ + \frac{\angle CEB}{2} = \angle CI_2B.$$



Therefore, the quadrilateral CI_2I_1B is cyclic. It follows that $\angle II_2X_1 = \angle I_1CB = \angle ICX_1$ and thus the pentagon $CX'I_2IX_1$ is cyclic. Hence,

$$\angle M_2BC = \angle M_2M_1C = \angle M_2X'C = \angle CI_2X_1 = \angle I_1BC.$$

Consequently, the points M_2, I_1 and B are collinear, i.e. M_2 is the midpoint of $\widehat{CD}$. Also, we have shown that $M_1M_2 \parallel X_1X_2$.

The quadrilateral BI_1IX_2 is cyclic because

$$\angle II_1X_2 = \angle I_2BC = \angle IBX_2.$$

Moreover,

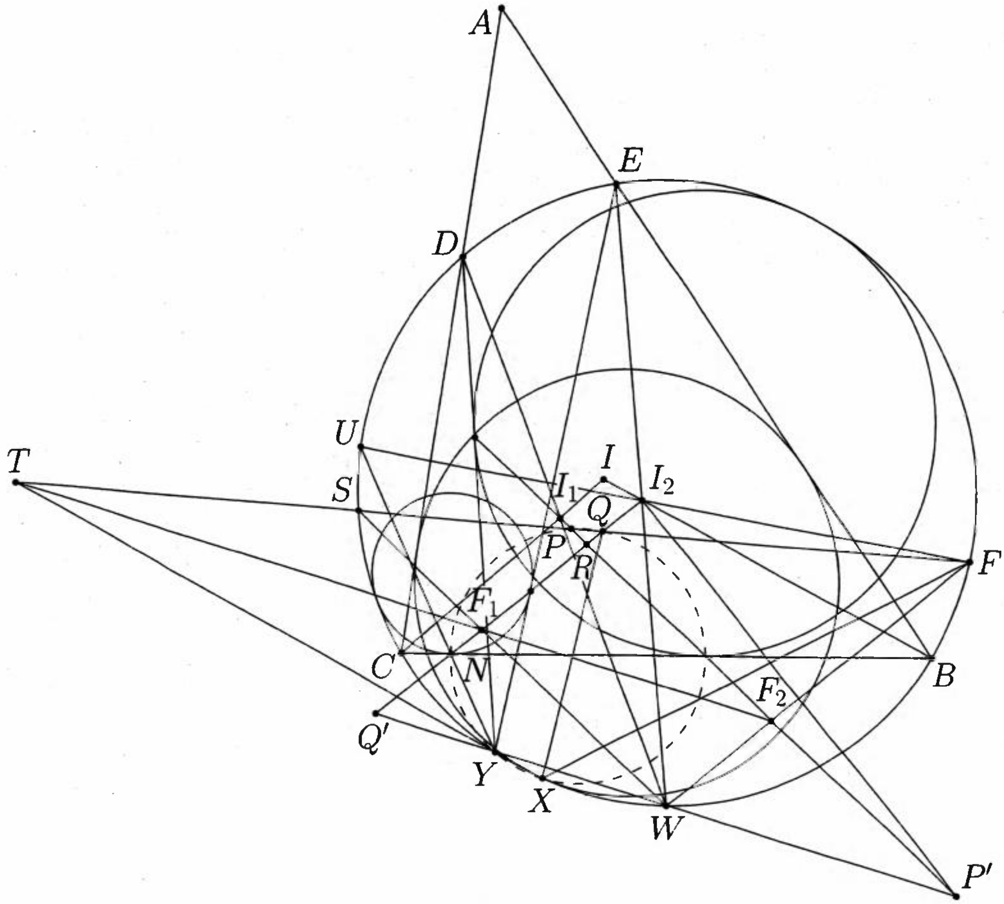
$$\begin{aligned}\angle IX'B &= \angle CX'B - \angle CX'I = 180^\circ - \angle CEB - \angle CI_2I \\ &= 180^\circ - \angle CEB - 90^\circ + \frac{\angle CEB}{2} \\ &= 90^\circ - \frac{\angle CEB}{2} = 90^\circ - \frac{\angle CDB}{2} = \angle II_1B.\end{aligned}$$

Therefore, the quadrilateral $BX'I_1IX_2$ is cyclic. Analogously, we deduce that the points X', X_2 and M_1 are collinear. Now consider a homothety with center C that maps X_2 to M_1. Since $M_1M_2 \parallel X_1X_2$, the image of X_1 is M_2. Hence, the image of the circumcircle of $\triangle X'X_1X_2$ is k, and since M_1 and M_2 are the midpoints of the arcs $\widehat{BE}$ and $\widehat{CD}$, respectively, it follows that the circumcircle of $\triangle X'X_1X_2$ coincides with k_3. In conclusion, $X \equiv X'$, and the lemma is proven.

We will use the notations in the lemma. Let W be the midpoint of $\widehat{BYC}$, let F be the midpoint of $\widehat{YBE}$ and let S be the midpoint of $\widehat{YCD}$.

Sawayama's Lemma (Problem 4.7.13) yields that the line NI_2 intersects BC and EY at the same angle. The same property holds for the line WF because W and F are the midpoints of the arcs $\widehat{BYC}$ and $\widehat{YBE}$, respectively. Hence, $WF \parallel NI_2$ and analogously, $WS \parallel MI_1$. Let $FS \cap MI_1 = P$ and $FS \cap NI_2 = Q$. Problem 4.6.1 yields that the point W is the circumcenter of the quadrilateral CI_1I_2B.

Let this circle intersect WY at the points Q' and P', so that Y lies between Q' and W. Since W is the midpoint of $\widehat{BYC}$, the line WF is an external angle bisector of $\triangle YWE$, and so it is perpendicular to the internal angle bisector of $\angle YWE$. Also, $WF \parallel NI_2$, and so NI_2 is perpendicular to the internal angle bisector of $\angle YWE$. Now $WQ' = WI_2$ implies that the points Q', N and I_2 are collinear.



Analogously, the points P', M and I_1 lie on a line, parallel to WS. Let $QQ' \cap PP' = R$ and let F_1 and F_2 be the midpoints of $Q'R$ and $P'R$, respectively. Considering the midsegments in $\triangle P'Q'R$, we get that $F_2 \in WF$ and $F_1 \in WS$. Let $F_1F_2 \cap FS = T$. The intercept theorem gives

$$\frac{TF}{TQ} = \frac{TF_2}{TF_1} = \frac{TP}{TS} \Rightarrow TF.TS = TP.TQ.$$

Also, the lemma gives

$$\begin{aligned}\angle I_1XI_2 = \angle I_1XI + \angle IXI_2 &= \angle I_1BI + \angle I_2CI \\ &= \frac{\angle CBE}{2} - \frac{\angle CBD}{2} + \frac{\angle BCD}{2} - \frac{\angle BCE}{2} \\ &= \frac{\angle DBE}{2} + \frac{\angle DCE}{2} = \angle I_1WI_2.\end{aligned}$$

Therefore, the quadrilateral I_1I_2WX is cyclic.

We have that $WQ' = WI_2 = WI_1 = WP'$, and so $\angle Q'I_2P' = \angle Q'I_1P' = 90°$. Hence, the point X lies on the Euler circle (Problem 4.8.1) of $\triangle P'Q'R$, which means that the quadrilateral F_1F_2WX is cyclic.

Therefore, X is the Miquel point (Problem 2.33) of the quadrilateral SFF_2F_1. Consequently, the quadrilateral FF_2XT is cyclic. Now we have that $\angle FXT = \angle FF_2T = \angle FWY = \angle FXY$.

Consequently, the points X, Y and T are collinear. This implies that

$$TP.TQ = TF.TS = TX.TY.$$

Hence, the quadrilateral $XYPQ$ is cyclic. It now suffices to show that N (and analogously, M) lies on its circle. We consecutively have

$$\begin{aligned}\angle I_2XF &= \angle I_2XW - \angle FXW = \angle I_2I_1W - \angle FSW\\ &= 90° - \frac{\angle I_1WI_2}{2} - \angle QPR\\ &= 90° - \angle I_1P'I_2 - 180° + \angle I_2QF + \angle Q'RP'\\ &= \angle I_2RP' - 180° + \angle I_2QF + \angle Q'RP' = \angle I_2QF.\end{aligned}$$

We deduce that the quadrilateral XFI_2Q is cyclic. Let the line YN intersect k for the second time at the point U.

Then since the quadrilaterals $Q'CI_2B$ and $YCUB$ are cyclic, we get $NU.NY = NC.NB = NQ'.NI_2$, and so the quadrilateral $Q'YI_2U$ is cyclic.

We deduce that $\angle Q'I_2U = \angle Q'YU = \angle UFW$. But $WF \parallel Q'I_2$, and so the points U, I_2 and F are collinear. Now we have

$$\angle NQX = \angle XFI_2 = \angle XFU = 180° - \angle XYU = 180° - \angle NYX.$$

Hence, the quadrilateral $NYXQ$ is cyclic. Thus, the pentagon $NYXQP$ is cyclic, and so the hexagon $NYXMQP$ is cyclic. In conclusion, the quadrilateral $NYXM$ is cyclic.

Note. Actually, the proof of the lemma gives us a synthetic proof of Sawayama's Lemma (Problem 4.7.13) in the case when the arbitrary point is external to the segment (in this case the point A does not belong to the segment BE).

Problem 4.7.19. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The circle k_1 touches the lines AC and BD, and the circle k internally at the point E on the smaller arc $\widehat{AD}$. The circle k_2 touches the lines AC and BD, and the circle k internally at the point Z on the smaller arc $\widehat{BC}$. Let $X \in k_2$ and $K \in k_1$ be points, such that XK is a common external tangent line of k_1 and k_2, and C and X lie in the same half-plane with respect to EZ. The circle k_3 touches k_1 and k_2 externally. Let $k_3 \cap BD = M$, so that M is the intersection point closer to D, and let $k_3 \cap AC = T$, so that T is the intersection point closer to C. Prove that $MT \parallel XK \parallel CD$.

Solution. Let $KX \cap k = \{F, Y\}$, so that K lies between X and F.

Also, let $k_1 \cap k_3 = N$, $k_2 \cap k_3 = S$, $XZ \cap EK = Q$ and $XS \cap KN = P$. Problem 6.1.1 yields that Q is the midpoint of $\widehat{YF}$. Moreover, Problem 6.1.2 yields that the point Q lies on the radical axis of k_1 and k_2.

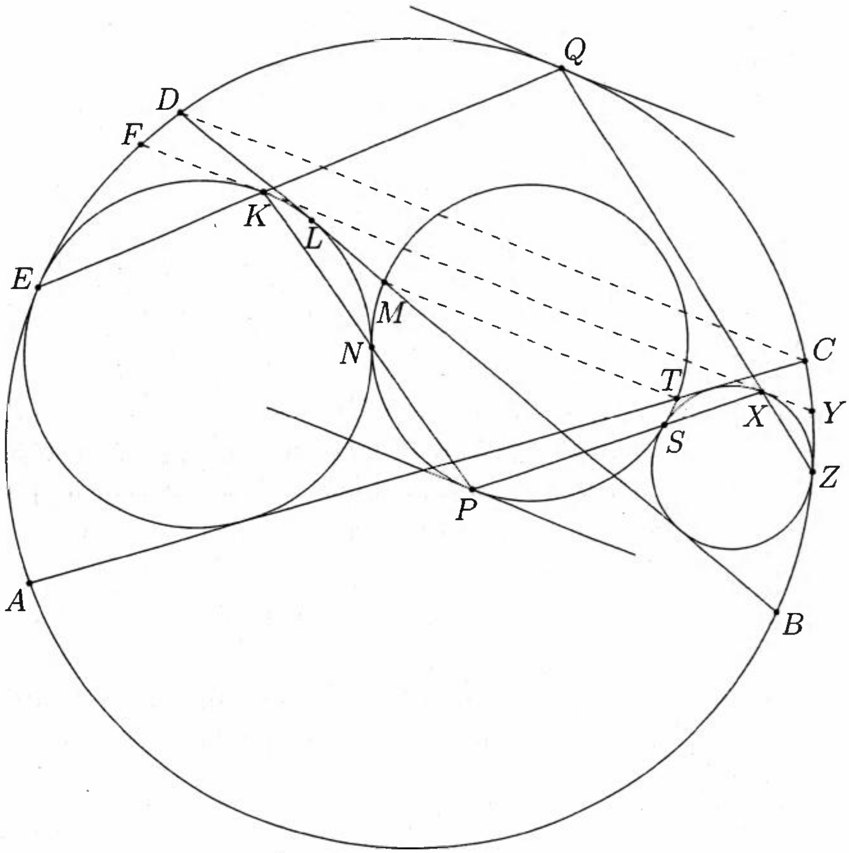
Problem 4.7.17 gives that Q is the midpoint of $\widehat{CD}$. This implies that $XK \parallel CD$, which lines are parallel to the tangent line to k through Q.

Problem 6.1.3 yields that the point P lies on k_3, and that the tangent line to k_3 through P is parallel to KX. The arcs $\widehat{PS}$ and $\widehat{SX}$ are equal because the homothety with center S that maps k_3 to k_2 sends $\widehat{PS}$ to $\widehat{SX}$.

We have that

$$\angle SNP = \frac{\widehat{PS}}{2} = \frac{\widehat{SX}}{2} = \angle KXS.$$

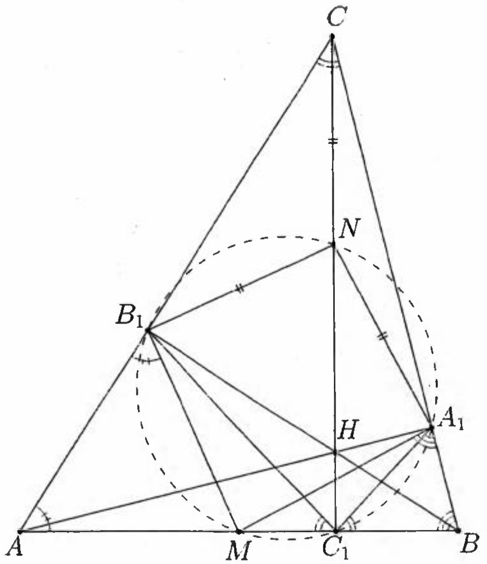
Therefore, the quadrilateral $XSNK$ is cyclic.



We deduce that $PS.PX = PN.PK$, i.e. the point P lies on the radical axis of k_1 and k_2. On the other hand, the equivalent problem of 4.7.17 for externally tangent circles implies that P is the midpoint of the arc $\widehat{MT}$. Indeed, we consider the points where the common internal tangent lines of k_1 and k_2 intersect the circle k_3, to which k_1 and k_2 are externally tangent. These points are M and T. Thus, the midpoint of $\widehat{MPT}$ lies on the radical axis of k_1 and k_2.

Then the tangent line to k_3 through P is parallel to MT, which gives $MT \parallel XK$. In conclusion, $MT \parallel XK \parallel CD$.

Problem 4.8.1. *(Euler Circle)* Let ABC be a triangle with orthocenter H. Prove that the midpoints of its sides, the feet of its altitudes and the midpoints of the segments AH, BH and CH are concyclic.



Solution. Let the feet of the respective altitudes be A_1, B_1 and C_1, and let M and N be the midpoints of AB and CH, respectively. We have that $\angle A_1C_1B_1 = 180° - 2\gamma$.

Since N is the circumcenter of the quadrilateral B_1HA_1C, we have $\angle B_1NA_1 = 2\angle ACB = 2\gamma$. Hence, the point N lies on the circumcircle of $\triangle A_1B_1C_1$.

Also, since M is the circumcenter of the quadrilateral AB_1A_1B, we have $\angle B_1MA_1 = 2\angle CAA_1 = 180° - 2\gamma$. Hence, the point M lies on the circumcircle of $\triangle A_1B_1C_1$.

Analogously, the other midpoints also lie on this circle.

Remark. The radius of the Euler circle is half of the circumradius of $\triangle ABC$.

Problem 4.8.2. Let ABC be a triangle. Let k, k_a, k_b and k_c be the incircle and the three excircles, respectively. Let A_1, B_1 and C_1 be the midpoints of BC, CA and AB, respectively. Prove that the circumcircle ω of $\triangle A_1B_1C_1$ is tangent to all circles.

Solution. Without loss of generality, let $b \geq a$. We will show that ω is tangent to k and k_c. Let k touch AB at the point J_3 and let k_c touch AB at the point J_c.

Clearly, $J_cC_1 = J_3C_1 = \dfrac{b-a}{2}$. Let t be the second common internal tangent line of k and k_c. Denote $t \cap AC = B'$ and $t \cap BC = A'$.

Consider the inversion with center C_1 and radius $\dfrac{b-a}{2}$. Since k and k_c are orthogonal to the circle of inversion, we have $k \to k$ and $k_c \to k_c$. Let $C_1B_1 \cap t = B_2$ and $C_1A_1 \cap t = A_2$.

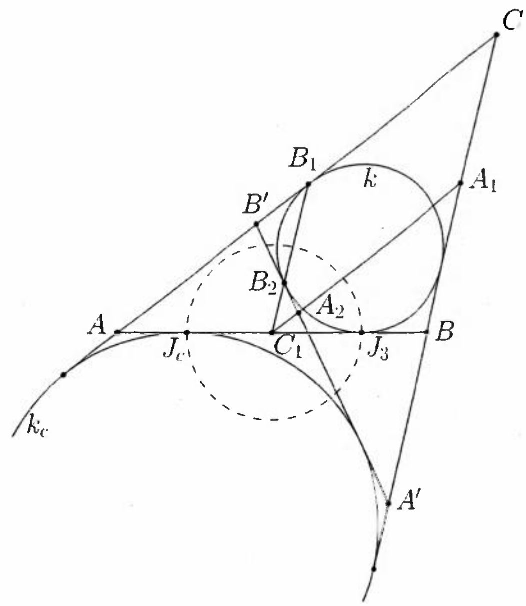
We will show that $A_1 \to A_2$ and $B_1 \to B_2$. Indeed, as a consequence $\omega \to t$, but since t touches k and k_c, we will conclude that ω touches k and k_c as well. We will prove that $C_1A_2.C_1A_1 = \dfrac{(b-a)^2}{4}$.

Note that $C_1A_1 = \dfrac{b}{2}$. It remains to find an expression for C_1A_2. Since the construction is symmetric with respect to the angle bisector of $\angle ACB$, it follows that

$$\angle A_2A'A_1 = \angle BAC = \alpha$$

and

$$\angle A_2A_1A' = \angle ACB = \gamma.$$



Therefore, $\triangle ABC \sim \triangle A'A_2A_1$. Now we deduce that

$$A_1A' = CA' - CA_1 = CA - \frac{a}{2} = b - \frac{a}{2}$$

$$\Rightarrow A_1A_2 = \frac{A_1A'.BC}{AC} = \frac{(2b-a)a}{2b}.$$

Hence, $C_1A_2 = C_1A_1 - A_1A_2 = \dfrac{b}{2} - \dfrac{(2b-a)a}{2b} = \dfrac{(b-a)^2}{2b}$.

We get $C_1A_2.C_1A_1 = \dfrac{(b-a)^2}{2b} \cdot \dfrac{b}{2} = \dfrac{(b-a)^2}{4}$.

Analogously, $C_1B_2.C_1B_1 = \dfrac{(b-a)^2}{4}$, thus $A_1 \to A_2$ and $B_1 \to B_2$.

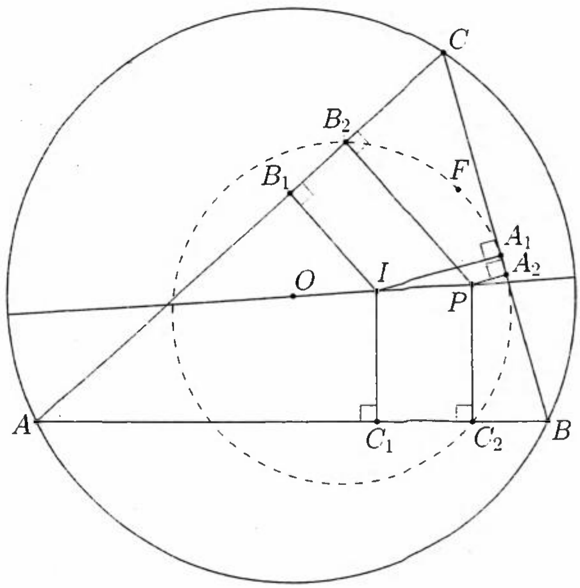
Note. The point of tangency of ω and k is called the Feuerbach point.

Problem 4.8.3. Let ABC be a triangle with incenter I and circumcenter O. Let P be an arbitrary point on the line OI. Denote the feet of the perpendiculars from P to AB, BC and CA by C_2, A_2 and B_2, respectively. Prove that the Feuerbach point F lies on the circumcircle of $\triangle A_2B_2C_2$.

Solution. Let the feet of the perpendiculars from I to AB, BC and CA be C_1, A_1 and B_1, respectively.

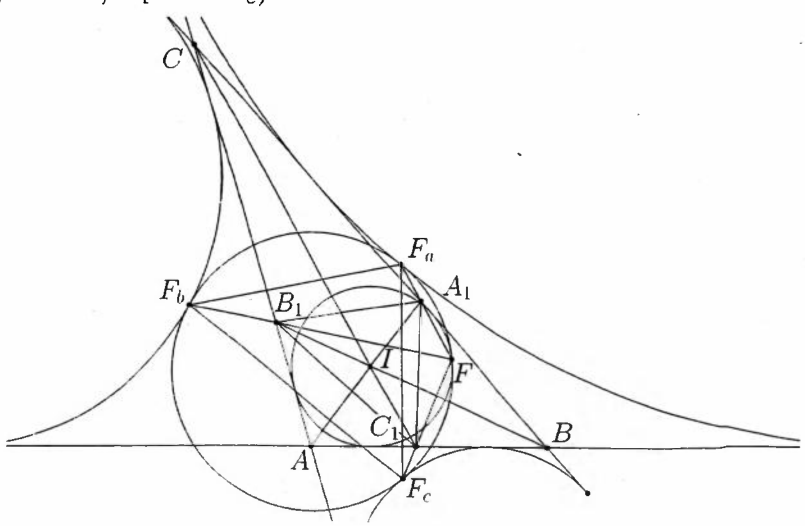
The first and second Fontene Theorems, applied to the point I, yield that there exists a fixed point lying on both the circumcircle of $\triangle A_1B_1C_1$ and the Euler circle. But by Feuerbach's Theorem (Problem 4.8.2), we have that the incircle and the Euler circle are tangent at F – the Feuerbach point.

The Fontene Theorems, applied to the point P, give that F lies on the circumcircle of $\triangle A_2B_2C_2$.



Problem 4.8.4. Let ABC be a triangle with angle bisectors AA_1, BB_1 and CC_1. Let F be the Feuerbach point of $\triangle ABC$. Prove that the quadrilateral $A_1B_1C_1F$ is cyclic.

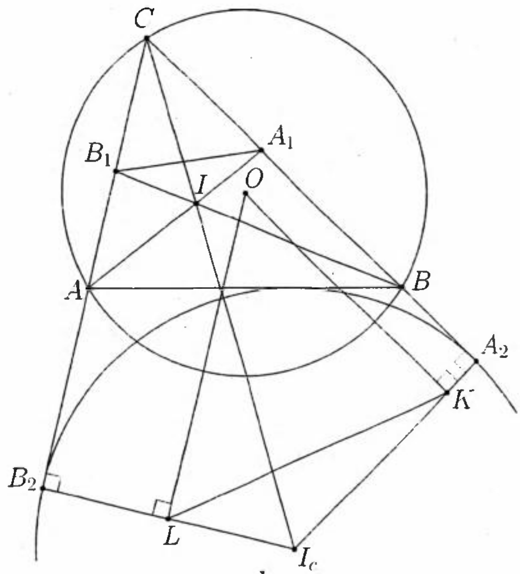
Solution. Let F_a, F_b and F_c be the three external Feuerbach points of $\triangle ABC$. Monge's Theorem (in the form of Problem 6.2.4), applied to the incircle, to the Euler circle and to one of the excircles, yields that the points F, A_1 and F_a are collinear (the same holds for the triples F, B_1 and F_b, and F, C_1 and F_c).



We will show that $\triangle A_1B_1C_1 \sim \triangle F_aF_bF_c$. Consequently, we would get

$$\angle A_1B_1C_1 = \angle F_aF_bF_c = 180^\circ - \angle F_aFF_c = 180^\circ - \angle A_1FC_1,$$

which would mean that the quadrilateral $A_1B_1C_1F$ is cyclic.



It suffices to show that

$$\frac{A_1B_1}{F_aF_b} = \frac{B_1C_1}{F_bF_c} = \frac{C_1A_1}{F_cF_a}.$$

Let O be circumcenter of $\triangle ABC$ and let A_2 and B_2 be the feet of the perpendiculars from I_c to the lines BC and AC, respectively, where I_c is the C-excenter of $\triangle ABC$.

Let the projections of O onto I_cA_2 and I_cB_2 are K and L, respectively. We have $\angle KOL = \angle ACB = \gamma$. Also, $OL = p - \frac{b}{2}$ and $OK = p - \frac{a}{2}$. Thus, we have

$$\frac{OL}{OK} = \frac{a+c}{b+c} = \frac{ab}{b+c} \cdot \frac{a+c}{ab} = \frac{CA_1}{CB_1} \Rightarrow \triangle OKL \sim \triangle CB_1A_1,$$

which implies

$$A_1B_1 = LK \cdot \frac{CA_1}{OL} = LK \cdot \frac{ab}{(a+c)(b+c)} = OI_c \cdot \frac{ab\sin\gamma}{(a+c)(b+c)}.$$

By an analogue of Euler's Formula (see Problem 4.6.3), we have that $OI_c = \sqrt{R(R+2r_c)}$.

We deduce that $A_1B_1 = \dfrac{ab\sin\gamma\sqrt{R(R+2r_c)}}{(a+c)(b+c)}$.

Now we apply Lemma 1 from the proof of Casey's Theorem (Problem 6.1.10) to get

$$F_aF_b = \frac{R(a+b)}{\sqrt{\left(\frac{R}{2}+r_a\right)\left(\frac{R}{2}+r_b\right)}} = \frac{2R(a+b)}{\sqrt{(R+2r_a)(R+2r_b)}}.$$

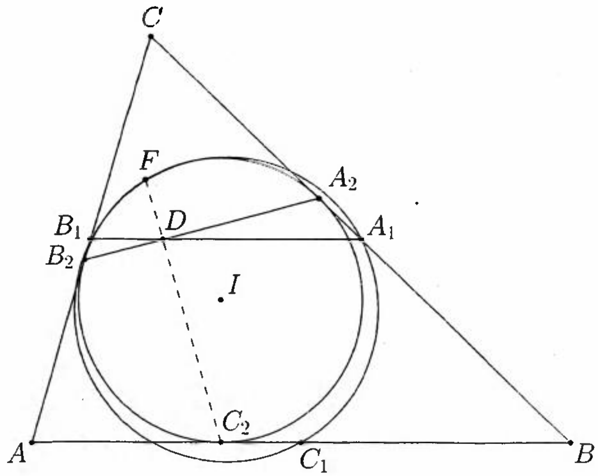
Therefore,

$$\frac{A_1B_1}{F_aF_b} = \frac{ab\sin\gamma\sqrt{R(R+2r_c)}}{(a+c)(b+c)} \cdot \frac{\sqrt{(R+2r_a)(R+2r_b)}}{2R(a+b)}$$
$$= \frac{abc\sqrt{R(R+2r_a)(R+2r_b)(R+2r_c)}}{4R^2(a+b)(b+c)(c+a)} = \frac{A_1C_1}{F_aF_c} = \frac{C_1B_1}{F_cF_b}.$$

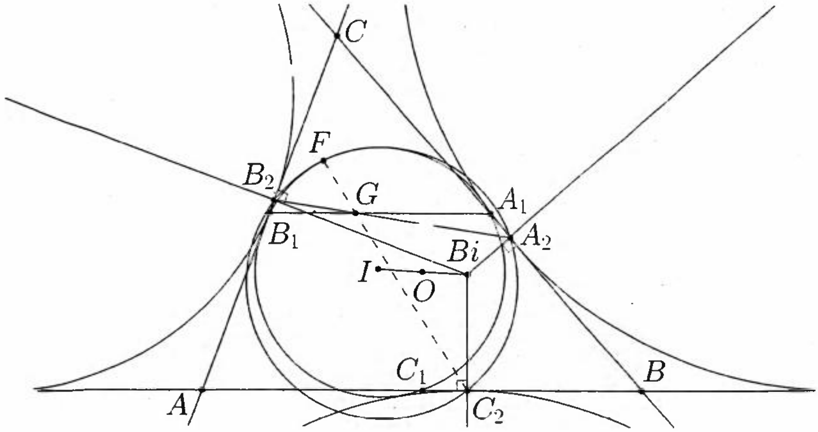
Problems 4.8.5 and 4.8.6. Let ABC be a triangle with incenter I. Its incircle touches AB, BC and AC at the points C_2, A_2 and B_2, respectively. The midpoints of AB, BC and CA are C_1, A_1 and B_1, respectively. Let $A_1B_1 \cap A_2B_2 = D$. Prove that the Feuerbach point F lies on the line DC_2.

Solution. This problem is a corollary of Problem 4.8.2 and Fontene's Theorems, where we have chosen I as the arbitrary point.

Indeed, the line C_2D intersects the Euler circle at a point on the pedal circle of I. But this pedal circle is exactly the incircle, which has exactly one common point with the Euler circle – the Feuerbach point.



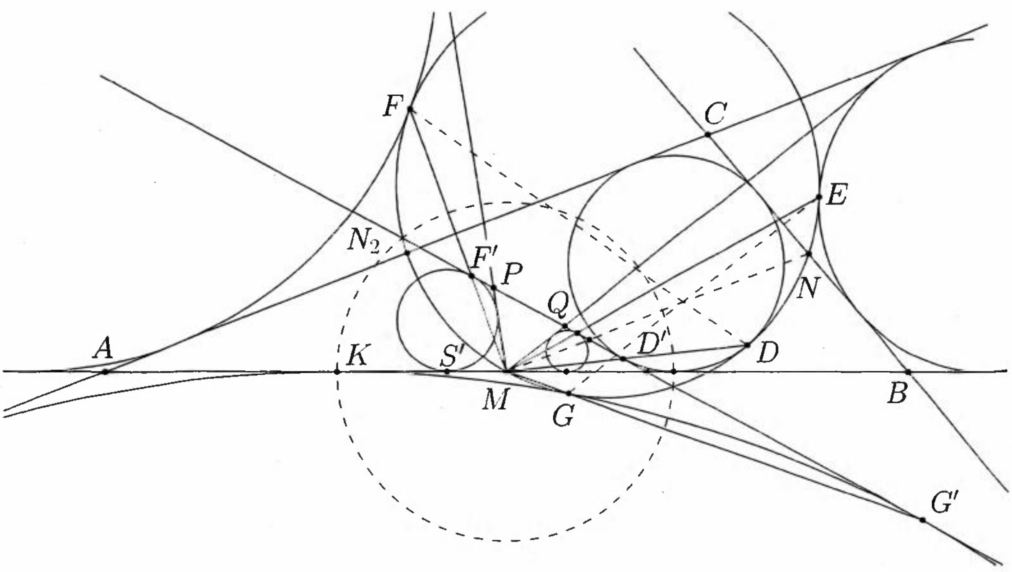
Problem 4.8.7. Let ABC be a triangle. Denote the midpoints of AB, BC and CA by C_1, A_1 and B_1, respectively. Denote the points of tangency of the excircles and the segments AB, BC and CA by C_2, A_2 and B_2, respectively. Let $A_1B_1 \cap A_2B_2 = G$. Prove that the Feuerbach point F lies on the line GC_2.



Solution. Define the Bevan point Bi as in Problem 2.20. Problem 3.5 yields that Bi lies on the line OI, where O and I are the circumcenter and the incenter of $\triangle ABC$, respectively.

By Problem 4.8.3, we have that F lies on the circumcircle of $\triangle A_2B_2C_2$. Also, F lies on the Euler circle. By Fontene's Theorems, we conclude that F lies on the line C_2G.

Problem 4.8.8. Let ABC be a triangle. The points M and N are the midpoints of AB and BC, respectively. Denote the incircle, the Euler circle and the excircles of $\triangle ABC$ by k, k_1, k_a, k_b and k_c, respectively. Let $k \cap k_1 = D$, $k_a \cap k_1 = E$, $k_b \cap k_1 = F$ and $k_c \cap k_1 = G$. Prove that the lines MN, DF and EG are concurrent.



Solution. Without loss of generality, let $b \geq a$. Denote $k \cap AB = H$ and $k_c \cap AB = K$.

We have that $AK = BH = s - b$. Thus, M is the midpoint of KH. Let the angle bisector of $\angle ACB$ intersect AB at the point L, and let k_1' be the second common internal tangent line of k and k_c. Clearly, $L \in k_1'$.

Consider the inversion φ with respect to the circle with diameter KH. Then $\varphi(k) = k$ and $\varphi(k_c) = k_c$ because they are orthogonal to the circle of inversion. Moreover, $M \in k_1$, so the image of k_1 is a line tangent to both k and k_c, i.e. $\varphi(k_1) = k_1'$. Let $k_1' \cap k_c = G'$ and $k_1' \cap k = D'$. Hence, $\varphi(G) = G'$ and $\varphi(D) = D'$. Clearly, $\varphi(MN) = MN$ and $\varphi(AB) = AB$.

It remains to find the images of F and E. Since the radius of the inversion R_1 equals $\dfrac{b-a}{2}$, we have that the image of k_b is a circle k_b', which is tangent to AB and k_1', and which is the image of k_b under a homothety with center M. The same is true for $\varphi(k_a) = k_a'$. So if $k_b' \cap k_1' = F'$ and $k_a' \cap k_1' = E'$, we have $\varphi(F) = F'$ and $\varphi(E) = E'$.

Let l_a and l_b be the second tangent lines from M to k_a and k_b, respectively. Let $l_b \cap k_1' = P$ and $l_a \cap k_1' = Q$. Hence, k_b' is an excircle of $\triangle LMP$ and k_a' is the incircle of $\triangle LMQ$. Let $k_a' \cap AB = T'$, $k_b' \cap AB = S'$, $k_a \cap AB = T$ and $k_b \cap AB = S$.

Then $\varphi(T) = T'$ and $\varphi(S) = S'$. We deduce that

$$MT.MT' = R_1^2 = \frac{(b-a)^2}{4}$$

and

$$MT = s - \frac{c}{2} = \frac{a+b}{2} \Rightarrow MT' = \frac{(b-a)^2}{2(a+b)}.$$

Analogously, $MS' = \dfrac{(b-a)^2}{2(a+b)} = MT'$. The angle bisector property gives

$$AL = \frac{bc}{a+b} \Rightarrow ML = \frac{bc}{a+b} - \frac{c}{2} = \frac{c(b-a)}{2(a+b)}.$$

It follows that

$$E'L = T'L = ML - MT' = \frac{c(b-a)}{2(a+b)} - \frac{(b-a)^2}{2(a+b)} = \frac{(b-a)(c+a-b)}{2(a+b)}.$$

Analogously,

$$F'L = S'L = ML + MS' = \frac{c(b-a)}{2(a+b)} + \frac{(b-a)^2}{2(a+b)} = \frac{(b-a)(c+b-a)}{2(a+b)}.$$

We have $LD' = LH = AH - AL = s - a - \dfrac{bc}{a+b} = \dfrac{(b-a)(b+a-c)}{2(a+b)}$.
Also, $LG' = LK = AL - AK = \dfrac{bc}{a+b} - (s-b) = \dfrac{(b-a)(a+b+c)}{2(a+b)}$.

Denote $MN \cap k_1' = X$. Then $\angle MLX = \beta - \alpha$ and $\angle HMX = \alpha$. It follows that $\angle MXL = 180^\circ - \beta$ and thus

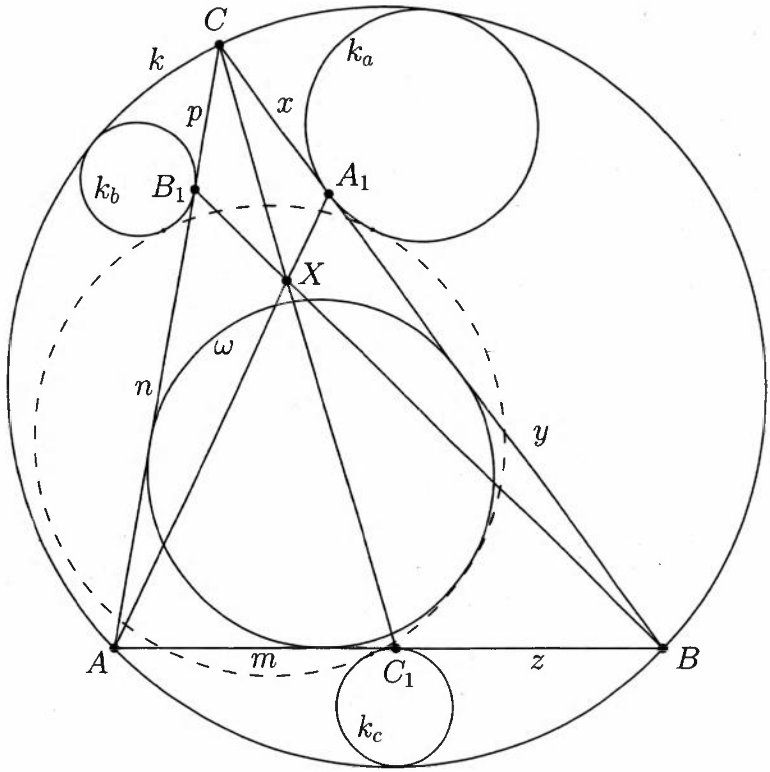
$$LX = \frac{LM \sin\alpha}{\sin\beta} = \frac{LM.a}{b} = \frac{ca(b-a)}{2b(a+b)}.$$

The statement that the lines MN, DF and EG are concurrent is equivalent to the statement that the circumcircles of $\triangle MD'F'$ and $\triangle ME'G'$ have MN as a radical axis. So it suffices to show that X lies on their radical axis. We consecutively have

$$\begin{aligned}
XD'.XF' - XE'.XG' &= (XL - LD')(F'L - XL) \\
&\quad - (E'L - LX)(XL + G'L) \\
&= XL(F'L + D'L + G'L - E'L) \\
&\quad - F'L.D'L - E'L.G'L \\
&= \frac{ca}{2b}\left(\frac{c+b}{2} + \frac{b-c}{2} + \frac{b+c}{2} - \frac{c-b}{2}\right) \\
&\quad - \frac{c+b-a}{2}\cdot\frac{b+a-c}{2} + \frac{a+c-b}{2}\cdot\frac{b+a+c}{2} \\
&= ca - \frac{b^2 - a^2 - c^2 + 2ac}{4} + \frac{a^2 + c^2 + 2ac - b^2}{4} \\
&= ca - ca = 0,
\end{aligned}$$

whence X lies on the radical axis of the circumcircles of $\triangle MD'F'$ and $\triangle ME'G'$. Since $X \in MN$, it follows that MN is that radical axis.

Problem 4.8.9. Let ABC be a triangle with circumcircle k. Let X be an arbitrary point inside $\triangle ABC$. Denote $AX \cap BC = A_1$, $BX \cap AC = B_1$ and $CX \cap AB = C_1$. The circle k_a is external to $\triangle ABC$, touches k internally and also touches the segment BC at the point A_1. The circles k_b and k_c are defined analogously. Denote the incircle of $\triangle ABC$ by ω. Prove that there exists a circle which is tangent to k_a, k_b and k_c externally and to ω internally.



Solution. Denote $CA_1 = x$, $BA_1 = y$, $BC_1 = z$, $AC_1 = m$, $AB_1 = n$ and $CB_1 = p$. Let t_{ij} be the length of the common external tangent segment of k_i and k_j. We will treat the points A, B and C as circles k_A, k_B and $k_{C'}$ with zero radii. Let t_{Oi} be the length of the common internal tangent segment of k and k_i. Casey's Theorem (Problem 6.1.10), applied to (A, B, C, k_a), (A, B, C, k_b), (A, B, C, k_c), (A, B, k_a, k_b), (C, B, k_c, k_b) and (A, C, k_a, k_c), gives

$$a.t_{Aa} = by + cx \Rightarrow t_{Aa} = \frac{by + cx}{a},$$

$$b.t_{Bb} = cp + an \Rightarrow t_{Bb} = \frac{cp + an}{b},$$

$$c.t_{Cc} = am + bz \Rightarrow t_{Cc} = \frac{am + bz}{c}.$$

Therefore, $t_{Aa}t_{Bb} = c.t_{ab} + ny$, whence $t_{ab} = \dfrac{t_{Aa}t_{Bb} - ny}{c} =$

$$= \frac{abt_{Aa}t_{Bb} - abny}{abc} = \frac{(by + cx)(cp + an) - abny}{abc} = \frac{bcyp + c^2xp + acxn}{abc}$$

Analogously, we get that $t_{ac} = \dfrac{abym + b^2yz + bcxz}{abc}$ and also $t_{bc} = \dfrac{acpm + a^2nm + abnz}{abc}$. Set $s = \dfrac{a+b+c}{2}$.

We have $x + y = a$, $m + z = c$ and $n + p = b$. Moreover,

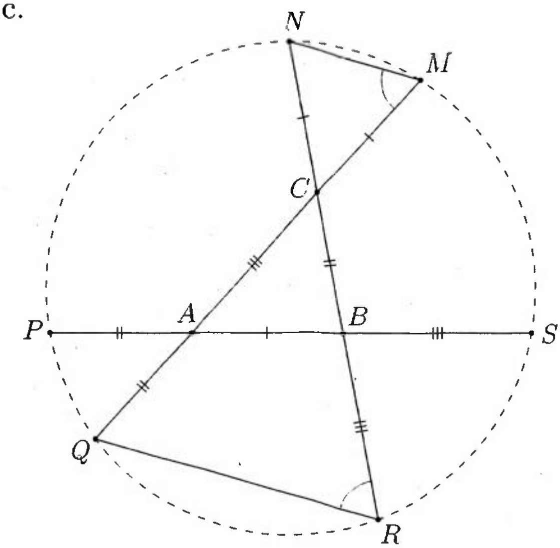
$$t_{Oa} = y - (s-b) = \frac{2y - a - c + b}{2},$$
$$t_{Ob} = (s-c) - p = \frac{a + b - c - 2p}{2},$$
$$t_{Oc} = m - (s-a) = \frac{2m - b - c + a}{2}.$$

Ceva's Theorem gives $xzn = ymp$. Considering

$$\begin{aligned}
t_{ab}t_{Oc} + t_{bc}t_{Oa} - t_{ac}t_{Ob} &= \frac{1}{2abc}((bcyp + c^2xp + acxn)(2m - b - c + a) \\
&+ (acpm + a^2nm + abnz)(2y - a - c + b) \\
&+ (abym + b^2yz + bcxz)(2p - a - b + c)) \\
&= \frac{1}{2abc}(2bcypm + 2c^2xpm + 2acxnm \\
&- b^2yp(m+z) - bcxp(m+z) - abxn(m+z) \\
&- bcyp(m+z) - c^2xp(m+z) - acxn(m+z) \\
&+ abyp(m+z) + acxp(m+z) + a^2xn(m+z) \\
&+ 2acpmy + 2a^2nmy + 2abnzy - acpm(x+y) \\
&- a^2nm(x+y) - abnz(x+y) - c^2pm(x+y) \\
&- acnm(x+y) - bcnz(x+y) + bcpm(x+y) \\
&+ abnm(x+y) + b^2nz(x+y) + 2abpym + 2b^2yzp \\
&+ 2bcxzp - a^2ym(n+p) - abyz(n+p) - acxz(n+p) \\
&- abym(n+p) - b^2yz(n+p) - bcxz(n+p) \\
&+ acym(n+p) + bcyz(n+p) + c^2xz(n+p)) = 0
\end{aligned}$$

and using the converse of Casey's Theorem, we conclude that the desired circle exists.

Problem 4.8.10. *(Conway circle)* Let ABC be a triangle. The points P and S lie on the line AB, such that the points P, A, B and S are in this order, $AP = BC$ and $BS = AC$. The points R and N lie on the line BC, such that the points R, B, C and N are in this order, $CN = AB$ and $BR = AC$. The points M and Q lie on the line CA, such that the points Q, A, C and M are in this order, $CM = AB$ and $AQ = BC$. Prove that the hexagon $MNPQRS$ is cyclic.



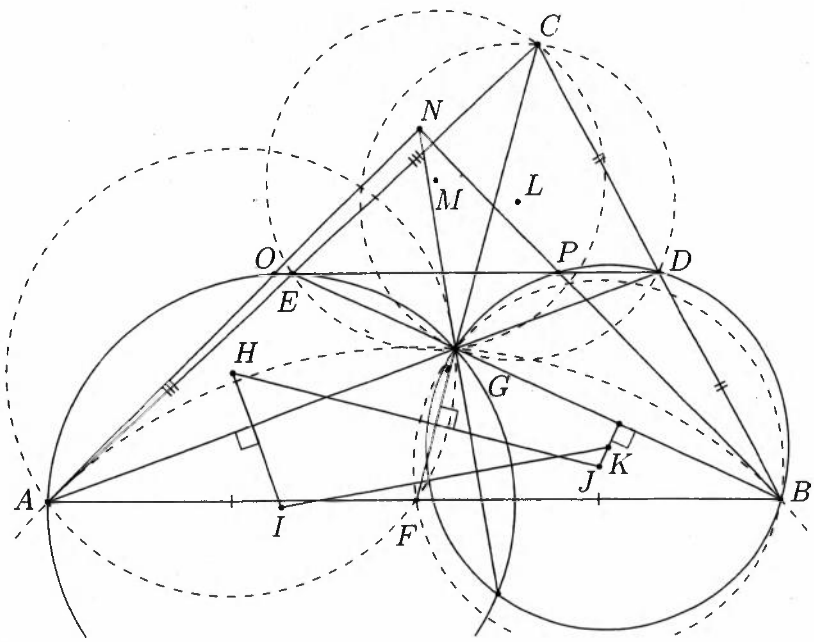
Solution. We have $CN = CM$ and $CQ = CR$. Hence, $\angle QRC = \angle NMC$, which means that the quadrilateral $QRMN$ is cyclic. Analogously, the quadrilaterals $PQSM$ and $NPRS$ are cyclic.

Assume the circumcircles of the quadrilaterals $QRMN$, $PQSM$ and $NPRS$ are distinct. Denote them by k_1, k_2 and k_3, respectively.

From $CM.CQ = CN.CR$ it follows that C lies on the radical axis of k_2 and k_3 (which is the line PS). But $C \notin PS$, a contradiction. Hence, at least two of k_1, k_2 and k_3 coincide. Then all three circles coincide, so the hexagon $MNPQRS$ is cyclic.

Problem 4.8.11. Let ABC be a triangle. The points D, E and F are the midpoints of the sides BC, CA and AB, respectively. The point G is the centroid of $\triangle ABC$. Prove that the circumcenters of $\triangle AFG$, $\triangle AEG$, $\triangle GDB$, $\triangle FGB$, $\triangle GDC$ and $\triangle EGC$ are concyclic.

Solution. Denote the circumcenters by H, I, J, K, L and M, respectively. We will show that the quadrilateral $HIJK$ is cyclic (the same will hold analogously for $HILM$ and $JKLM$). Then the desired statement will follow from radical axes, analogously to how we prove the lemma in Problem 5.4.8, but applied to the triangle formed by the perpendicular bisectors of AG, BG and CG.



We have $HI \perp AG$, $HJ \perp FG$ and $JK \perp GB$. Hence, $\angle IHJ = \angle AGF$ and since this is the angle at the median in $\triangle ABG$, it suffices to show that IK is perpendicular to the G-symmedian of $\triangle ABG$.

If the tangent lines through A and B to (ABG) intersect at N, we will show that NG is the radical axis of (AEG) and (FDB).

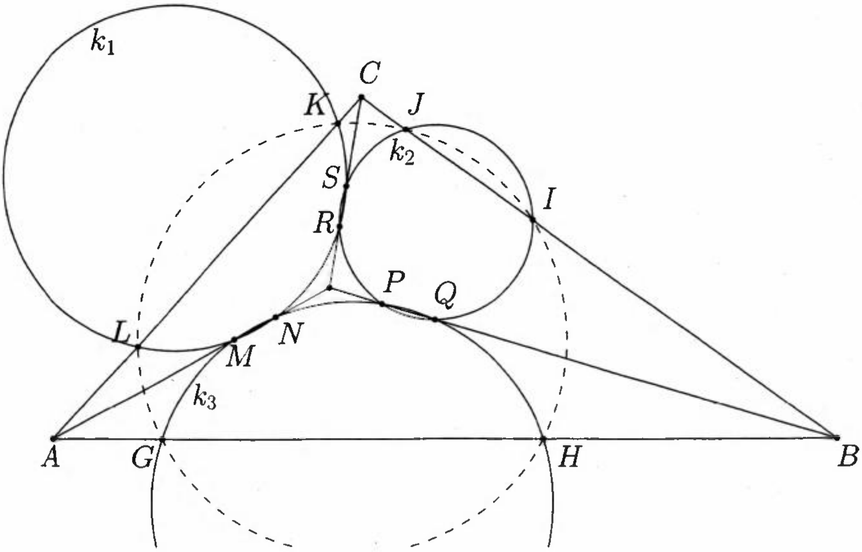
Let $AN \cap k(AGE) = \{A, O\}$ and $BN \cap k(BDG) = \{B, P\}$. Then

$$\angle OEG = 180^\circ - \angle OAG = 180^\circ - \angle ABG,$$

which implies that $OE \parallel AB$. Analogously, $PD \parallel AB$. But $ED \parallel AB$ as a midsegment in $\triangle ABC$ and hence the points O, E, P and D are collinear.

Therefore, $ABPO$ is an isosceles trapezoid, and so it is cyclic. The desired statement follows because the radical axes of the circumcircles of $\triangle AEG$, $\triangle GDB$ and $ABPO$ are concurrent.

Problem 4.8.12. The circles k_1, k_2 and k_3 are chosen such that $k_1 \cap k_2 = \{R, S\}$, $k_2 \cap k_3 = \{P, Q\}$ and $k_3 \cap k_1 = \{M, N\}$. Denote the radical center of k_1, k_2 and k_3 by X. The points A, B and C are chosen on the rays XM, XQ and XS beyond M, Q and P, respectively. Let $AB \cap k_3 = \{G, H\}$, $BC \cap k_2 = \{I, J\}$ and $CA \cap k_1 = \{K, L\}$. Prove that the points H, I, J, K, L and G are concyclic.



Solution. The point A lies on the radical axis of k_1 and k_3 Hence, $AG.AH = AM.AN = AL.AK$ and thus the quadrilateral $GHKL$ is cyclic. Analogously, the quadrilaterals $GHIJ$ and $IJKL$ are cyclic. The desired statement follows by observing that the radical axes of the circumcircles of $GHKL$, $GHIJ$ and $IJKL$ are not concurrent, assuming the circles are distinct.

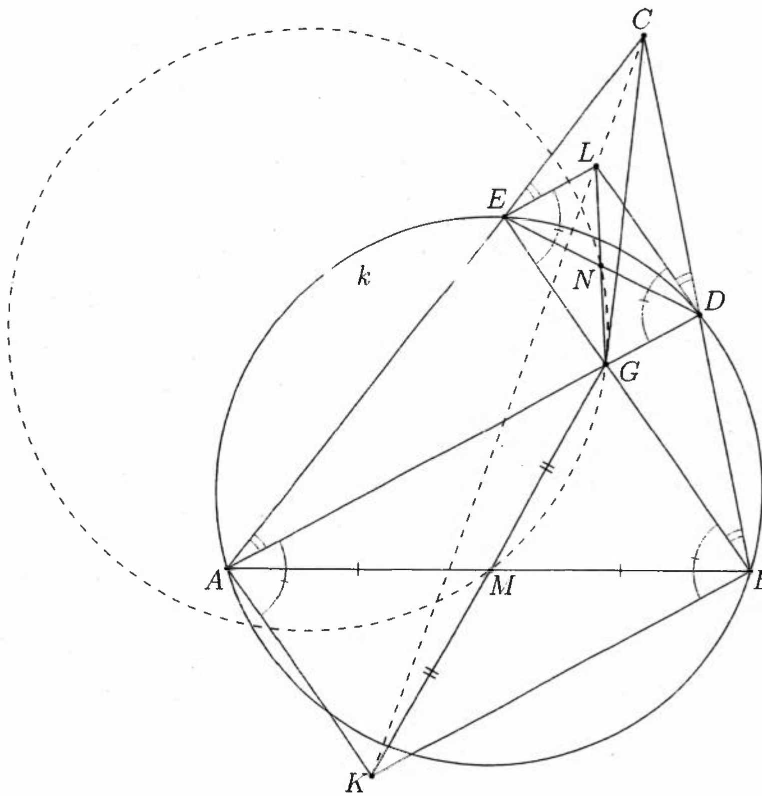
Problem 4.8.13. Let ABC be a triangle. A circle k through A and B intersects the sides AC and BC at the points E and D, respectively. Let $AD \cap BE = G$. Denote the midpoints of AB and DE by M and N, respectively. Prove that the line CG is tangent to the circumcircle of $\triangle MGN$.

Solution. Denote the reflections of G with respect to M and N by K and L, respectively. We have that $AKBG$ and $EGDL$ are parallelograms. Then $\angle CEL = \angle CAD = \angle EBC$ and $\dfrac{EL}{GB} = \dfrac{GD}{GB} = \dfrac{AB}{ED} = \dfrac{CB}{CE}$.

Therefore, $\triangle ELC \sim \triangle GBC$. Analogously, $\triangle AKC \sim \triangle DGC$ and thus $\angle LCE = \angle GCD = \angle KCA$. We deduce that the points C, L and K are collinear.

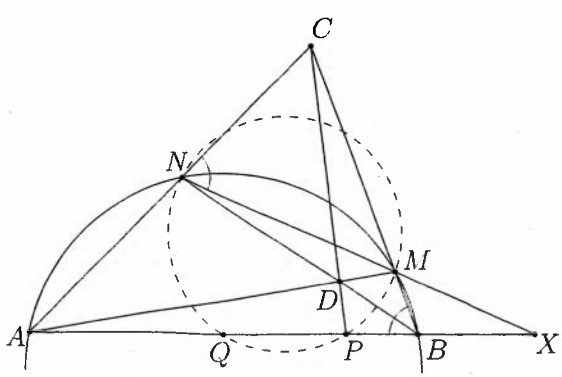
We also have that $\dfrac{CL}{CG} = \dfrac{CE}{CB} = \dfrac{CD}{CA} = \dfrac{CG}{CK}$, which implies $CL.CK = CG^2$. Hence, the circumcircle of $\triangle KLG$ is tangent to CG.

Now use the homothety with center G and coefficient $\dfrac{1}{2}$.



Problem 4.8.14 Let ABC be a triangle. The points N and M are chosen on the sides AC and BC, respectively, so that $ABMN$ is cyclic. Let $AM \cap BN = D$ and $CD \cap AB = P$. Denote the midpoint of the segment AB by Q. Prove that the quadrilateral $MNQP$ is cyclic.

Solution. Without loss of generality, let $AC > BC$. Denote $AB \cap MN = X$. Then B is between X and A.



Menelaus' Theorem, applied to $\triangle ABC$ and the points N, M and X, gives

$$\frac{AX}{BX} = \frac{AN}{NC} \cdot \frac{CM}{MB}.$$

Ceva's Theorem, applied to $\triangle ABC$ and the points P, M and N, gives $\dfrac{AP}{BP} = \dfrac{AN}{NC} \cdot \dfrac{CM}{MB} = \dfrac{AX}{BX}$.

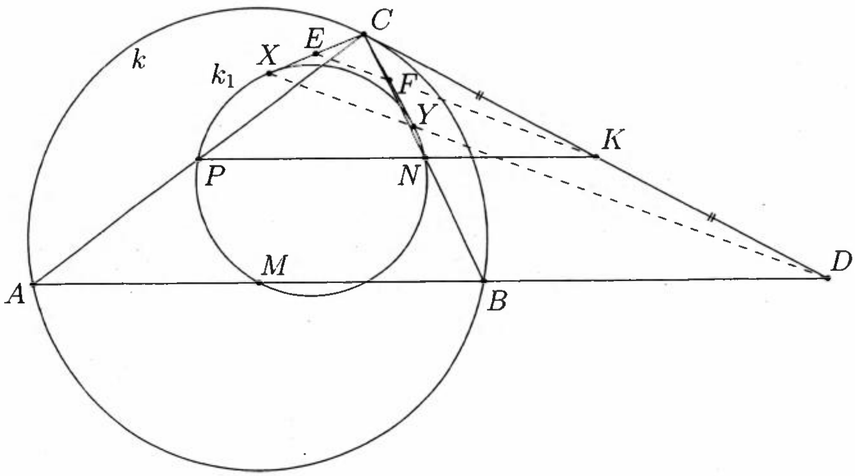
Note that since $\dfrac{AX}{BX} > 1$, the inequality $\dfrac{AP}{BP} > 1$ holds, and so P is between Q and B.

We have that $XA.XB = XM.XN$, and so it suffices to show that $XA.XB = XP.XQ$.

Let $AQ = a + b$, $QP = a$, $PB = b$ and $XB = c$. Then we have $\dfrac{2a+b}{b} = \dfrac{2a+2b+c}{c}$, which implies $ac = ab + b^2$.

Equivalently, $(b+c)(a+b+c) = c(2a+2b+c)$. Therefore, $XP.XQ = XA.XB$.

Problem 4.8.15. Let ABC be a triangle with circumcircle k. Let M, N and P be the midpoints of AB, BC and CA, respectively. Let $D \in AB$ be a point such that CD is tangent to k. Denote the circumcircle of $\triangle MNP$ by k_1. Prove that the point C lies on the polar line of D with respect to k_1.



Solution. By La Hire's Theorem, it suffices to show that the point D lies on the polar line of C with respect to k_1.

Let CX and CY be the tangent lines to k_1. We will show that the points X, Y and D are collinear. Denote the midpoints of CX, CY and CD by E, F and K, respectively. Now it suffices to show that the points E, F and K are collinear.

Midsegments give that the points P, N and K are collinear. A homothety with center C and coefficient 2 implies that the circumcircle of $\triangle CPN$ is tangent to k at the point C. Therefore, $KN.KP = KC^2$ and thus the point K lies on the radical axis of the point C and the circle k_1.

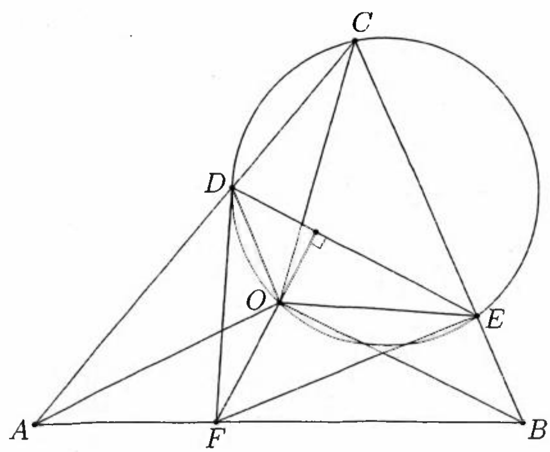
This means that the points E, F and K are collinear.

Problem 4.8.16. Let ABC be a triangle with circumcenter O. Let k be a circle through the points C and O. Denote $k \cap AC = \{C, D\}$ and $k \cap BC = \{C, E\}$. Prove that the orthocenter of $\triangle DOE$ lies on AB.

Solution. Let $F \in AB$ be a point such that $OF \perp DE$. We will show that F is the orthocenter of $\triangle DOE$. We have

$$\begin{aligned}\angle FOE &= 90^\circ + \angle OED = 90^\circ + \angle OCD = 90^\circ + 90^\circ - \beta \\ &= 180^\circ - \beta = 180^\circ - \angle FBE.\end{aligned}$$

Hence, the quadrilateral $BEOF$ is cyclic. Analogously, the quadrilateral $AFOD$ is cyclic.



Therefore,

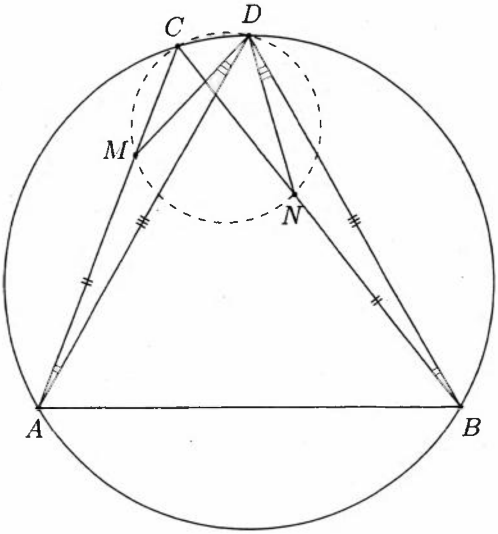
$$\begin{aligned}\angle DEO = \angle FDE &= \angle DCO + \angle FDO + \angle ODE \\ &= \angle DCO + \angle FAO + \angle OCE = \angle ACB + \angle BAO = 90^\circ.\end{aligned}$$

This means that $EO \perp DF$ and in conclusion, F is the orthocenter of $\triangle DOE$.

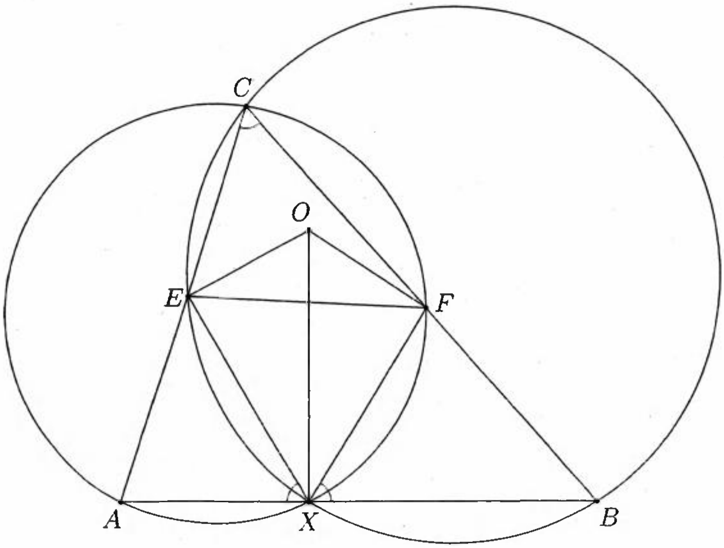
Problem 4.8.17. Let ABC be a triangle with circumcircle k. The point D is the midpoint of $\widehat{ACB}$. An arbitrary point M is chosen on the side AC. The point $N \in BC$ satisfies $AM = BN$. Prove that the points C, D, N and M are concyclic.

Solution. The point D lies on the perpendicular bisector of the side AB and hence $AD = BD$.

On the other hand, we have $\angle CAD = \angle CBD$ and $AM = BN$. Therefore, we obtain $\triangle MAD \cong \triangle NBD$, whence we deduce that $\angle DMC = \angle DNC$, which completes the proof.



Problem 4.8.18. Let ABC be a triangle. Let X be an arbitrary point on AB. The points E and F lie on the sides AC and BC and are chosen such that the quadrilaterals $AXFC$ and $BCEX$ are cyclic. Let O be the circumcenter of $\triangle EFC$. Prove that $OX \perp AB$.



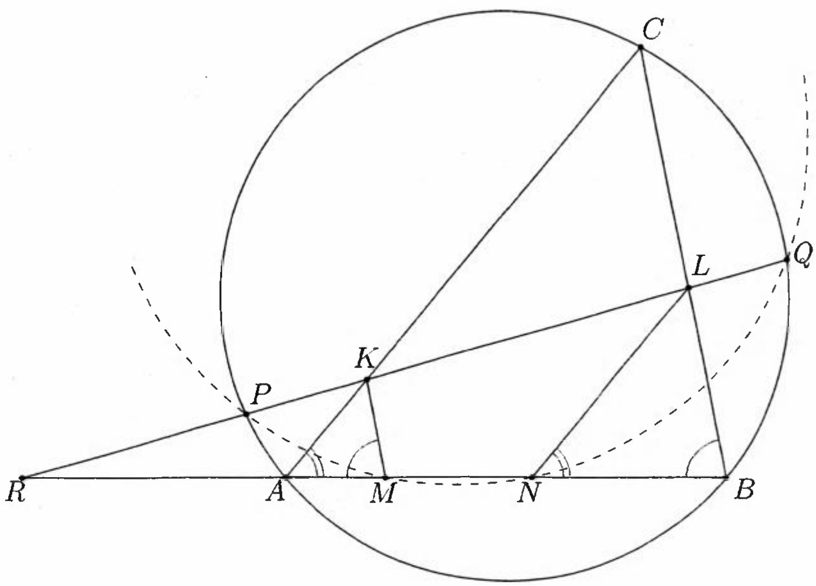
Solution. Cyclic quadrilaterals give

$$\angle EXA = \angle ACB = \angle FXB.$$

On the other hand, we have $\angle EOF = 2\angle ACB$. Thus, the quadrilateral $OEXF$ is cyclic and finally,

$$90^\circ - \angle ACB = \angle OEF = \angle OXF \Rightarrow \angle OXB = 90^\circ.$$

Problem 4.8.19. Let ABC be a triangle with circumcircle k. Let M and N be arbitrary points on the side AB, so that M lies between A and N. The points $K \in AC$ and $L \in BC$ are chosen such that $MK \parallel BC$ and $NL \parallel AC$. The line LK intersects k at the points P and Q. Prove that the quadrilateral $MNQP$ is cyclic.



Solution. Let $PQ \cap AB = R$. Without loss of generality, we will assume that A lies between B and R. Denote $AM = x$, $BN = y$, $AR = z$ and $AB = c$.

The intercept theorem gives that $\dfrac{AK}{KC} = \dfrac{AM}{MB} = \dfrac{x}{c-x}$. Analogously, $\dfrac{CL}{LB} = \dfrac{c-y}{y}$. Now applying Menelaus' Theorem to $\triangle ABC$ and the points L, K and R, we get $\dfrac{BR}{RA} \cdot \dfrac{AK}{KC} \cdot \dfrac{CL}{LB} = 1$.

Hence, $\dfrac{c+z}{z} \cdot \dfrac{x}{c-x} \cdot \dfrac{c-y}{y} = 1$ and thus $z(z+c) = (z+x)(z+c-y)$, i.e. $RA.RB = RM.RN$. On the other hand, we have $RA.RB = RP.RQ$. In conclusion, $RP.RQ = RM.RN$.

Remark. If $PQ \parallel AB$, the statement can be proven as follows. From $\dfrac{NB}{BA} = \dfrac{BL}{BC} = \dfrac{AK}{AC} = \dfrac{AM}{AC}$ we deduce that $AM = BN$.

The quadrilateral $ABPQ$ is an isosceles trapezoid, so $AP = BQ$ and

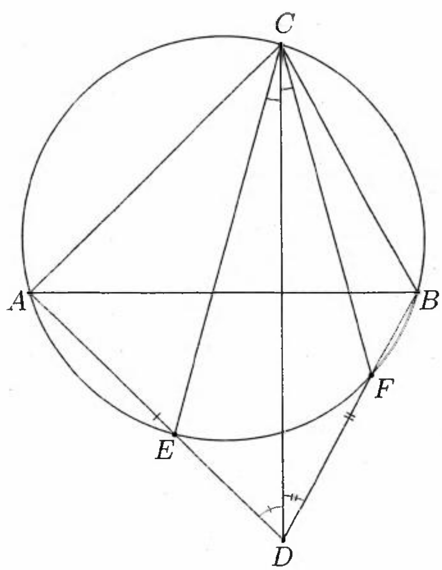
$\angle NBQ = \angle MAP$. Therefore, $\triangle MAP \cong \triangle NBQ$, which means that $PMNQ$ is an isosceles trapezoid.

Problem 4.8.20. Let $ABCD$ be a convex quadrilateral, such that $AC = AD$ and $BC = BD$. The circumcircle of $\triangle ABC$ intersects AD and BD at the points E and F, respectively. Prove that $\angle ECD = \angle FCD$.

Solution. Since AB is the perpendicular bisector of CD, we have $AB \perp CD$.

Therefore,

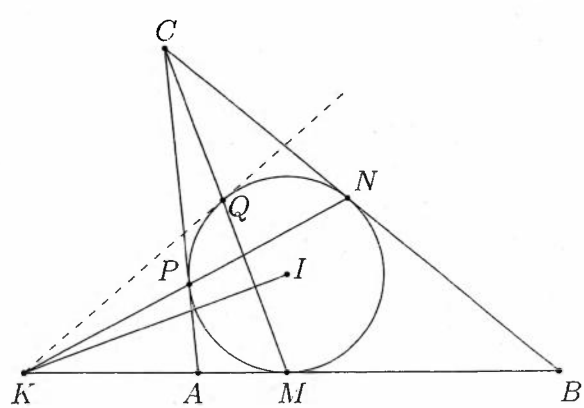
$$\begin{aligned}\angle FCD &= \angle CFB - \angle CDB\\ &= \angle CAB - \angle DCB\\ &= \angle CAB + \angle ABC - 90^\circ\\ &= \angle ABC + \angle CAB - 90^\circ\\ &= \angle ABC - \angle DCA\\ &= \angle AEC - \angle ADC = \angle ECD.\end{aligned}$$



Problem 4.8.21. Let ABC be a triangle. Its incircle is ω and its incenter is I. Let ω touch AB, BC and CA at the points M, N and P, respectively. Let $AB \cap PN = K$. Prove that $CM \perp KI$.

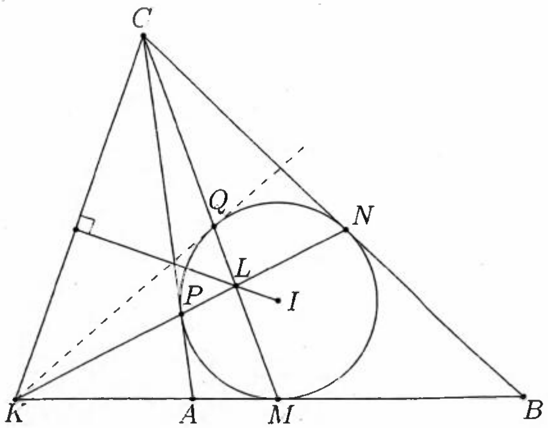
Solution. Let π be the pole-polar transformation with respect to ω. Clearly, NP is the polar line of C. By La Hire's Theorem and $K \in \pi_C$, we get $C \in \pi_K$.

On the other hand, we have $M \in \pi_K$. Therefore, $\pi_K \equiv CM$, which means that $CM \perp KI$.



Remark. The statement also follows from Problem 4.8.23, where the point X coincides with I.

Problem 4.8.22. Let ABC be a triangle. Its incircle is ω and its incenter is I. Let ω touch AB, BC and CA at the points M, N and P, respectively. Let $AB \cap PN = K$ and $CM \cap KN = L$. Prove that $IL \perp KC$.

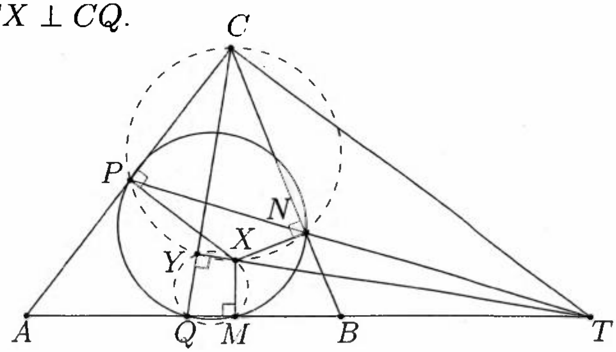


Solution. Consider the pole-polar transformation π with respect to ω. Let KQ be the second tangent line to ω through K, so that $Q \in \omega$.

Problem 4.8.21 yields that the points C, Q and M lie on π_K. But since $L \in CM$, it follows that $L \in \pi_K$. Consequently, $K \in \pi_L$.

We have that $L \in PN$ and that PN is the polar line of the point C, which implies that $C \in \pi_L$. Since $K \in \pi_L$ and $C \in \pi_L$, it follows that $CK \equiv \pi_L$. Therefore, $IL \perp CK$.

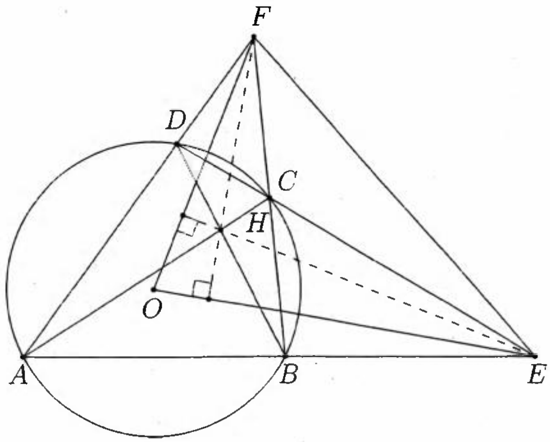
Problem 4.8.23. Let ABC be a triangle and let X be a point inside of it. The points M, N and P are the feet of the perpendiculars from X to AB, BC and CA, respectively. Let Q be the second intersection point of the circumcircle of $\triangle MNP$ and AB, and let the lines AB and PN intersect at T. Prove that $TX \perp CQ$.



Solution. Let Y be a point, such that $Y \in CQ$ and $XY \perp CQ$. Then the pentagon $PYXNC$ is inscribed in the circle with diameter CX. Clearly, the quadrilateral $QMXY$ is cyclic.

We apply the radical axes theorem (Problem 6.5.2) to the circumcircles of $QMNP$, $MXYQ$ and $CPYXN$. We deduce that the lines PN, QM and XY are concurrent. But $PN \cap XY = T$ and thus $T \in XY$. Therefore, $TX \perp CQ$.

Problems 4.8.24 and 4.8.25. *(Brocard's Theorem)* A non-trapezoid quadrilateral $ABCD$ is inscribed in a circle with center O. Let $AC \cap BD = H$, $AB \cap CD = E$ and $AD \cap BC = F$. Prove that H is the orthocenter of $\triangle OEF$.

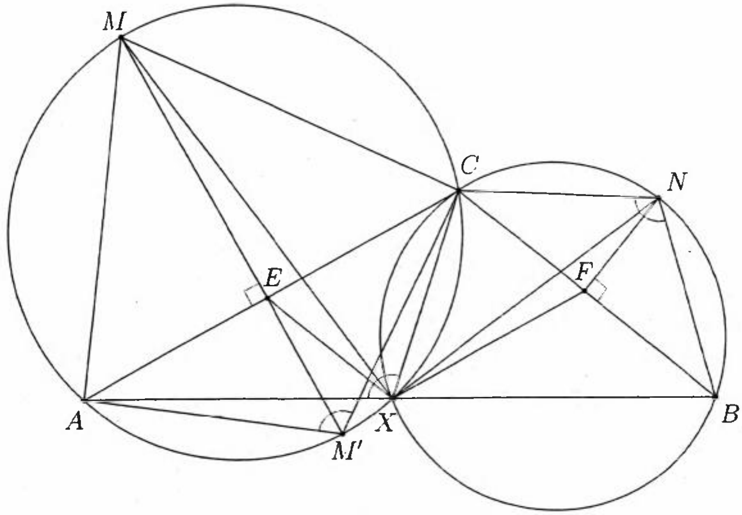


Solution. Consider the pole-polar transformation with respect to the circumcircle of $ABCD$. By one of the polar properties, it follows that HE and HF are the polar lines of F and E, respectively. We deduce that $HE \perp FO$ and $HF \perp EO$. Therefore, H is the orthocenter of $\triangle OEF$.

Problem 4.8.26. Let ABC be a triangle. Let X be an arbitrary point on AB. Denote the circumcircles of $\triangle AXC$ and $\triangle BXC$ by k_1 and k_2, respectively. The points $E \in AC$ and $F \in BC$ are chosen such that $EX \parallel BC$ and $FX \parallel AC$. The point $M \in k_1$ is chosen such that $ME \perp AC$, and such that M and X lie in different half-planes with respect to AC. The point $N \in k_2$ is chosen such that $NF \perp BC$, and such that N and X lie in different half-planes with respect to BC. Prove that $\angle MXN = 90°$.

Solution. Let the line ME intersect k_1 for the second time at the point M'. Then we have $\angle AM'C = \angle AXC = \angle CNB$. Moreover, the intercept theorem gives $\dfrac{AE}{EC} = \dfrac{AX}{XB} = \dfrac{CF}{FB}$.

We will now show that $\triangle AM'C$ and $\triangle CNB$ are similar. Let M'' be a point such that $\triangle AM''C \sim \triangle CNB$, with M'' and M' being in the same half-plane with respect to the line AC. Let $M''E''$ be an altitude in $\triangle AM''C$.



Now we have $\dfrac{AE''}{E''C} = \dfrac{CF}{FB} = \dfrac{AE}{EC}$ and therefore $E'' \equiv E$.

Hence, $\angle AM''C = \angle AM'C$ and we deduce that $M' \equiv M''$. Consequently, $\triangle AM'C \sim \triangle CNB$. Finally,

$$\angle MXN = 180^\circ - \angle MXA - \angle NXB = 180^\circ - \angle MM'A - \angle NCB$$
$$= 180^\circ - \angle CNF - \angle NCF = 90^\circ.$$

Problem 4.8.27. Let ABC be a triangle ($AC > BC$). Let $D \in AB$, such that DC touches the circumcircle of $\triangle ABC$. The point B' is the reflection of B with respect to the line CD. The point C' is the reflection of C with respect to the point D. Prove that the quadrilateral $AC'B'C$ is cyclic.

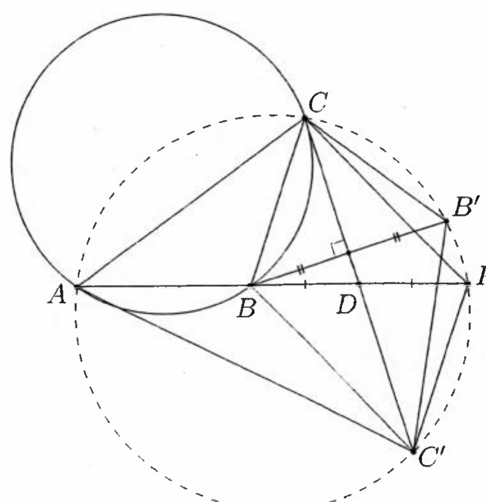
Solution. Denote the reflection of B with respect to the point D by K. Then $CBC'K$ is a parallelogram. Hence,

$$\angle CAK = \angle BCC' = \angle CC'K.$$

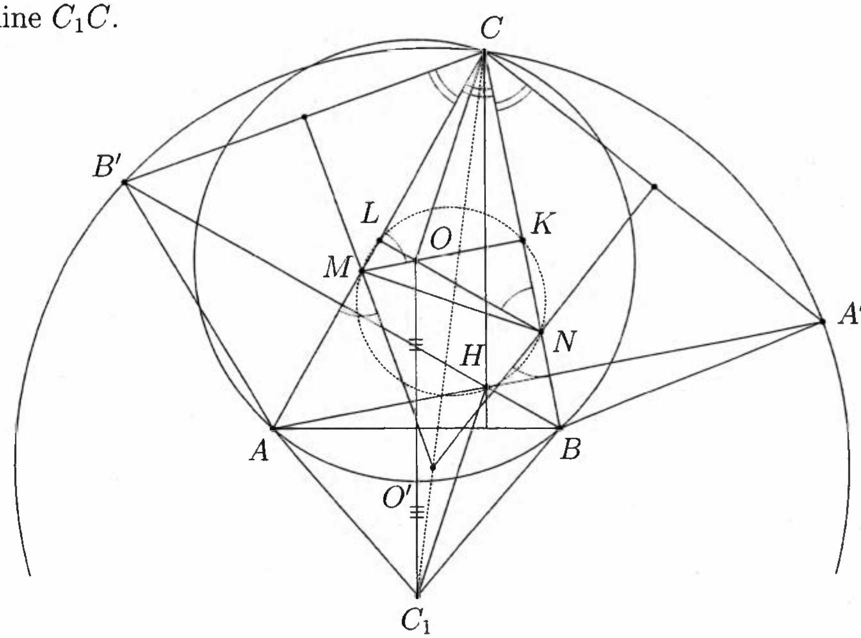
Thus, the quadrilateral $AC'KC$ is cyclic. Due to

$$\angle CB'C' = \angle CBC' = \angle CKC',$$

the point B' lies on the same circle.



Problem 4.8.28 Let $\triangle ABC$ be a triangle with $\angle ACB < 60°$ and circumcenter O. The points A' and B' are the reflections of A and B with respect to BC and AC, respectively. The point C_1 is the reflection of O with respect to AB. Prove that the circumcenter O' of $\triangle B'CA'$ lies on the line C_1C.



Solution. Let the perpendicular bisector of $B'C$ intersect AC at the point M and let the perpendicular bisector of $A'C$ intersect BC at the point N. Using symmetry, we get that the points O, N and the midpoint L of AC are collinear and that the points O, M and the midpoint K of BC are collinear. Due to $\angle MKN = \angle NLM = 90°$, the quadrilateral $MLKN$ is cyclic.

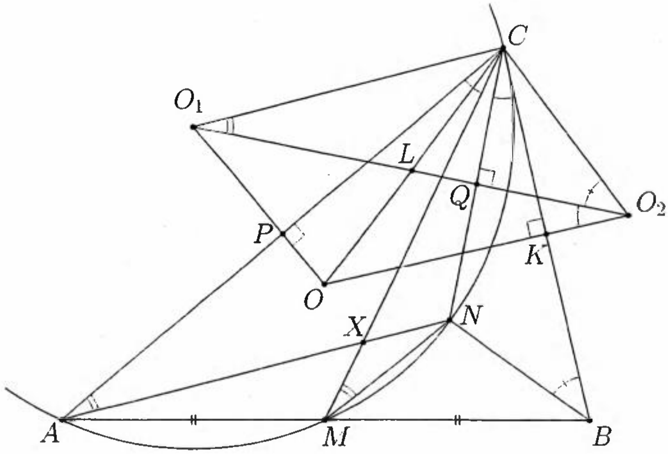
Hence, $\angle O'NB = \angle LNC = \angle KMC = \angle O'MA$. Moreover, $\angle MNC = \angle KLC = \angle BAC$. Therefore, $\triangle ABC \sim \triangle NMC$.

Define the point X such that $AX \parallel MO'$ and $BX \parallel NO'$. Clearly, the quadrilaterals $AXBC$ and $NO'MC$ are similar. Therefore, the lines CX and CO' are isogonal conjugates with respect to $\angle ACB$. It suffices to show that CX and CC_1 are isogonal conjugates with respect to $\angle ACB$.

We have that $\angle CAX + \angle BAC_1 = 90° + \angle ACB + \angle OAB = 180°$ and thus the lines AX and AC_1 are isogonal conjugates with respect to $\angle BAC$. Similarly, the lines BX and BC_1 are isogonal conjugates with respect to $\angle ABC$.

In conclusion, the points X and C_1 are isogonal conjugates in $\triangle ABC$, which implies that $\angle ACX = \angle BCC_1$.

Problem 4.8.29. Let ABC be a triangle with median CM. The symmedian of $\triangle ABC$ at C intersects the circumcircle of $\triangle AMC$ at the point N. Let O, O_1 and O_2 be the circumcenters of $\triangle ABC$, $\triangle AMC$ and $\triangle BNC$, respectively. Prove that OC bisects the segment O_1O_2.



Solution. We know that OO_1, O_1O_2 and OO_2 are the perpendicular bisectors of AC, CN and CB, respectively. Let $CM \cap AN = X$. Set $\angle ACB = \gamma$, $\angle ACM = \varphi$, $\angle NAC = \delta$ and $\angle NBC = \psi$. Denote the midpoints of AC, CN and CB by P, Q and K, respectively, and let $CO \cap O_1O_2 = L$.

The quadrilateral $CQKO_2$ is cyclic, so $\angle O_1O_2O = \varphi$. The quadrilateral $CPOK$ is cyclic, so $\angle O_1OO_2 = 180° - \gamma$. Therefore, $\angle O_2O_1O = \gamma - \varphi$.

Also, $\angle CO_2Q = \dfrac{\angle CO_2N}{2} = \angle CBN = \psi$. Moreover, we have that $\angle CO_1O_2 = \dfrac{\angle CO_1N}{2} = \angle CAN = \delta$.

The quadrilateral $CAMN$ is cyclic, which implies that $\angle CMN = \angle CAN = \delta$ and $\angle NMB = \angle ACN = \gamma - \varphi$. The law of sines, applied to $\triangle BMC$, $\triangle BNC$ and $\triangle AXC$, gives

$$\frac{\sin(\gamma - \varphi)}{\sin(\gamma - \varphi + \delta)} = \frac{MB}{CB} = \frac{AM}{CB},$$

$$\frac{\sin(\varphi + \psi)}{\sin\psi} = \frac{CB}{CN},$$

$$\frac{\sin\delta}{\sin\varphi} = \frac{CX}{AX} = \frac{CN}{AM}.$$

Therefore,

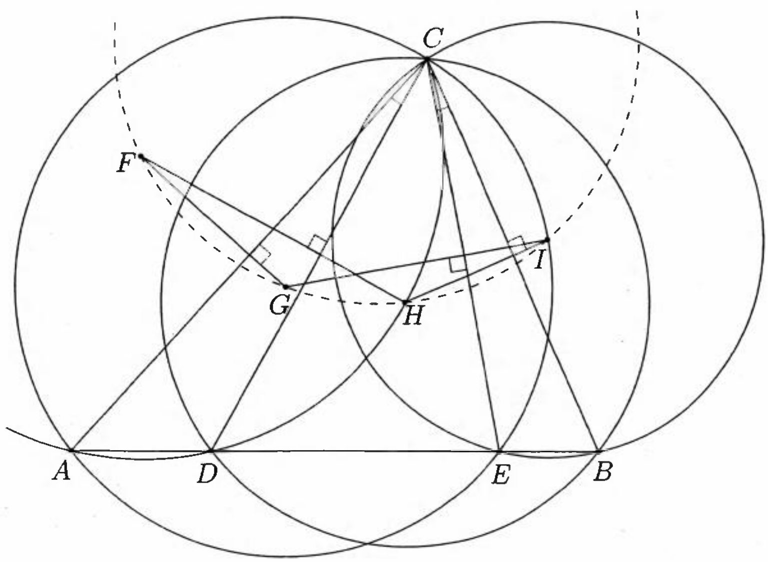
$$\frac{\sin(\gamma-\varphi)}{\sin(\gamma-\varphi+\delta)}\cdot\frac{\sin(\varphi+\psi)}{\sin\psi}\cdot\frac{\sin\delta}{\sin\varphi}=\frac{AM}{CB}\cdot\frac{CB}{CN}\cdot\frac{CN}{AM}=1.$$

But we have

$$\frac{\sin(\gamma+\delta-\varphi)}{\sin(\varphi+\psi)}=\frac{\sin\angle O_1OL}{\sin\angle O_2OL}\cdot\frac{CO_2}{CO_1}=\frac{\sin\angle O_1OL}{\sin\angle O_2OL}\cdot\frac{\sin\delta}{\sin\psi}.$$

Therefore, $\dfrac{\sin\angle O_1OL}{\sin\angle O_2OL}=\dfrac{\sin(\gamma-\varphi)}{\sin\varphi}$ and so $O_1L=O_2L$.

Problem 4.8.30. Let ABC be a triangle. The points D and E from the segment AB are such that $\angle ACD=\angle BCE$. Let F, G, H, I be the circumcenters of $\triangle ADC$, $\triangle AEC$, $\triangle BDC$ and $\triangle BEC$, respectively. Prove that the quadrilateral $EGHI$ is cyclic.



Solution. Clearly, the lines FG, GI, IH and HF are the perpendicular bisectors of AC, CE, CB and CD, respectively. We have

$$\angle HIG=\angle(HI,IG)=\angle(BC,EC)=\angle BCE,$$
$$\angle HFG=\angle(HF,FG)=\angle(DC,CA)=\angle ACD.$$

But $\angle ACD=\angle BCE$ and so $\angle HIG=\angle HFG$ and the desired statement follows.

Problem 4.8.31. Let ABC be a triangle. Let l be a line through C that lies inside of $\angle ACB$. The lines l_1 and l are isogonally conjugate with respect to $\angle ACB$. The points K and M are the projections of A and B onto l_1, and the points L and N are the projections of A and B onto l, respectively. Prove that the points K, L, M and N are concyclic.

Solution. Clearly, we have $\triangle ALC \sim \triangle BMC$ and $\triangle AKC \sim \triangle BNC$.

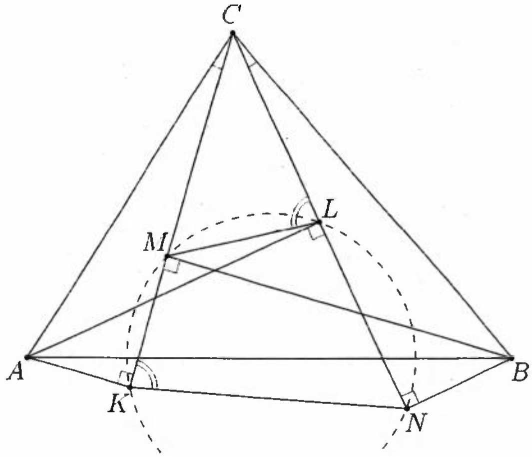
Therefore,

$$\frac{CM}{CL} = \frac{CB}{CA} = \frac{CN}{CK}.$$

We deduce that

$$CM.CK = CL.CN.$$

In conclusion, the quadrilateral $KMLN$ is cyclic.



Problem 4.8.32. Let k be a circle with center A and let C be a point outside of k. The lines CD and CE are tangent to k, such that $D, C \in k$. The point G is the projection of C onto an arbitrary line l through A, and the point H is an arbitrary point on the ray CG beyond G. The circumcircle of $\triangle HGD$ intersects l at the points G and F. The circumcircle of $\triangle HGE$ intersects l at the points G and I. Prove that $FA = IA$.

Solution. First, the pentagon $AGDCE$ is inscribed in the circle with diameter AC.

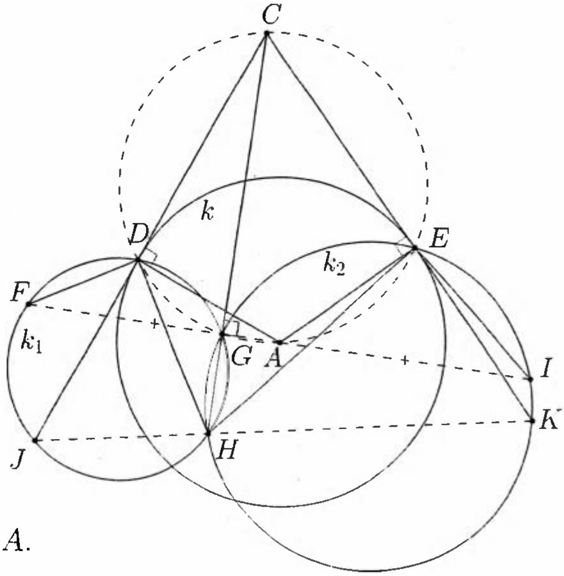
Then $\angle DAG = \angle DCG$ and $\angle DFG = \angle DHG$, which implies that $\triangle DFA \sim \triangle DHC$.

Therefore,

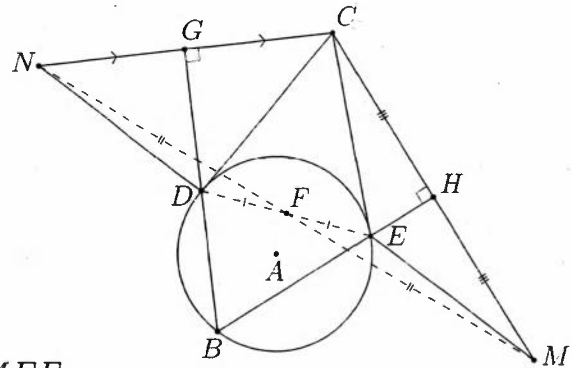
$$FA = \frac{AD.HC}{DC}.$$

Analogously,

$$IA = \frac{AE.HC}{EC} = \frac{AD.HC}{DC} = FA.$$



Problem 4.8.33. Let k be a circle with center A and let C be a point outside of k. The lines CD and CE are tangent to k, such that $D, E \in k$. The point B lies on k, such that the points B and C lie in different half-planes with respect to ED. Let G and H be the projections of C onto BD and BE, respectively. The points N and M are the reflections of C with respect to G and H, respectively. Prove that the segments MN and DE bisect each other.



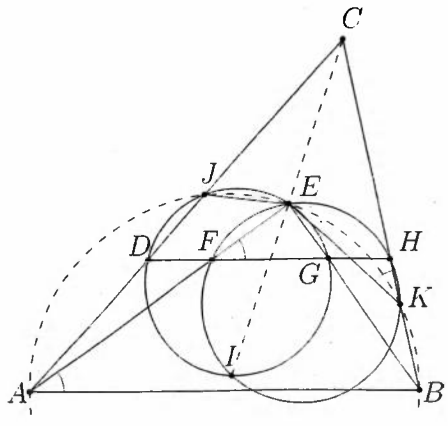
Solution. First, note that $DN = DC = CE = EM$.

Let $\angle EBD = \gamma$ and $\angle ECM = \varphi$. Then

$$\angle DCE = 180^\circ - 2\gamma,$$
$$\angle GCH = 180^\circ - \gamma,$$
$$\angle NCD = \gamma - \varphi,$$
$$\angle EDN = 180^\circ - \gamma + 2\varphi = \angle MEF.$$

It follows that the segments ND and EM are parallel and of equal length. Therefore, the quadrilateral $DMEN$ is a parallelogram, and the desired statement follows.

Problem 4.8.34. Let ABC be a triangle. Let $DH \parallel AB$, where $D \in AC$ and $H \in BC$. Let E be an arbitrary point inside of $\triangle DHC$. Denote $G = EB \cap DH$ and $F = AE \cap DH$. Prove that CE passes through the second intersection point I of the circumcircles of $\triangle DEG$ and $\triangle FHE$.



Solution. Let the circumcircle of $\triangle DGE$ intersect AC at the points D and J. Let the circumcircle of $\triangle FEH$ intersect BC at the points H and K.

We have

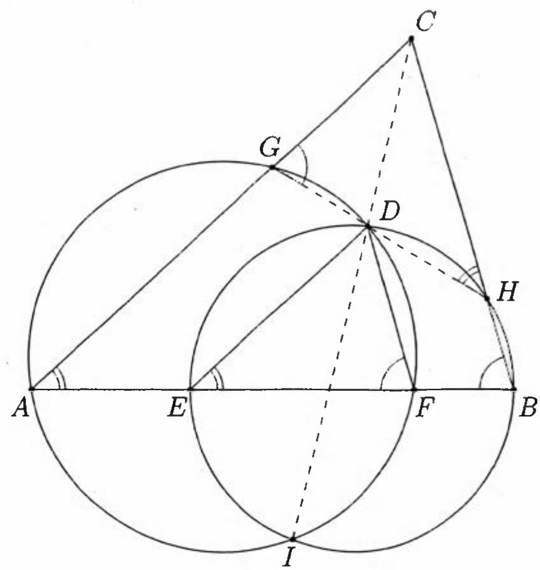
$$\angle ABE = \angle DGE = \angle EJC$$

and $\angle BAE = \angle GFE = \angle EKH$, which implies that the points J and K lie on the circumcircle of $\triangle ABE$.

Then $\angle JDH = \angle JAB = \angle JKH$. Hence, the quadrilateral $DKHJ$ is cyclic.

The radical axes theorem (Problem 6.5.2), applied to the circumcircles of the quadrilaterals $DJEI$, $IKHE$ and $DKHJ$, yields that the point C lies on the line EI.

Problem 4.8.35. Let ABC be a triangle and let D be an arbitrary point inside of it. The points E and F are chosen such that $DE \parallel AC, E \in AB$ and $DF \parallel BC, F \in AB$. Prove that the line CD passes through the second intersection point I of the circumcircles of $\triangle EDB$ and $\triangle AFG$.



Solution. Let the circumcircle of $\triangle AFD$ intersect AC at the points A and G, and let the circumcircle of $\triangle BED$ intersect BC at the points B and H. We have

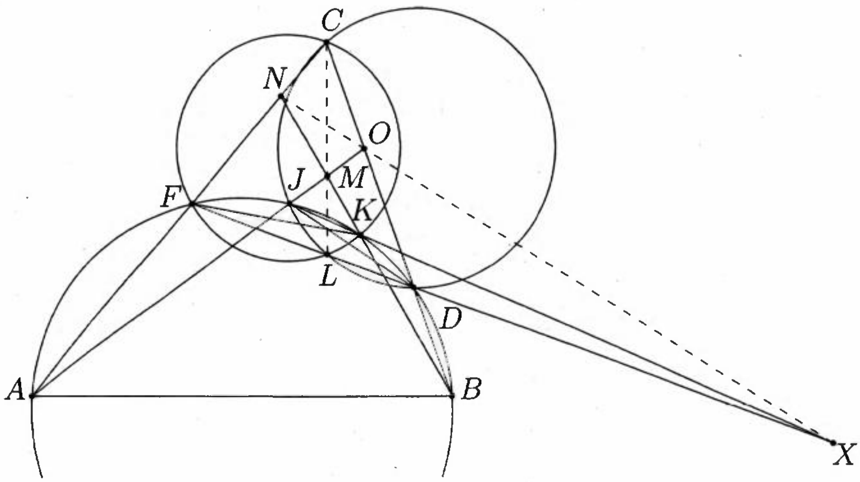
$$\angle DHC = \angle DEB = \angle CAB$$

and $\angle DGC = \angle GFA = \angle CBA$. Now considering the sum of the angles of the quadrilateral $GCHD$, we get that $\angle GDH = 180°$. Thus, the points G, D and H are collinear. This means that the quadrilateral $ABHG$ is cyclic.

The radical axis theorem (Problem 6.5.2), applied to the circumcircles of the quadrilaterals $AIDG$, $BIDH$ and $ABHG$, yields that the point C lies on the line DI.

Problem 4.8.36. Let ABC be a triangle. The circle k contains the points A and B and intersects BC and CA at the points D and F, respectively. Let $J, K \in k$ and let L be the second intersection point of the circumcircles of $\triangle FKC$ and $\triangle DJC$. Prove that $M = AJ \cap BK$ lies on LC.

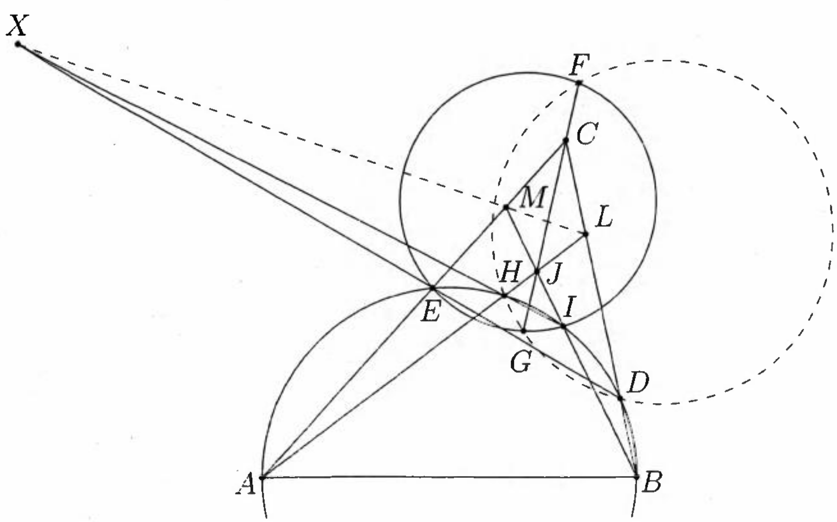
Solution. Denote $N = AF \cap BK$, $O = AJ \cap BD$ and $X = FD \cap JK$. Pascal's Theorem, applied to the hexagon $AJKBDF$, gives that the points N, O and X are collinear.



Desargues' Theorem (Problem 7.1), applied to the triangles $\triangle JKM$ and $\triangle DFC$, gives that the lines FK, DJ and CM are concurrent. On the other hand, the radical axes theorem (Problem 6.5.1) gives that the lines FK, DJ and CL are concurrent.

Therefore, the points C, M and L are collinear.

Problem 4.8.37. Let ABC be a triangle and let J be a point inside of it. A circle k passes through A and B and intersects CA and CB for the second time at the points E and D, respectively. Let $JA \cap k = H$ and $JB \cap k = I$. The point F lies on the ray JC beyond C. The circumcircle of $\triangle EIF$ intersects the line FJ at the points F and G. Prove that the quadrilateral $DGHF$ is cyclic.



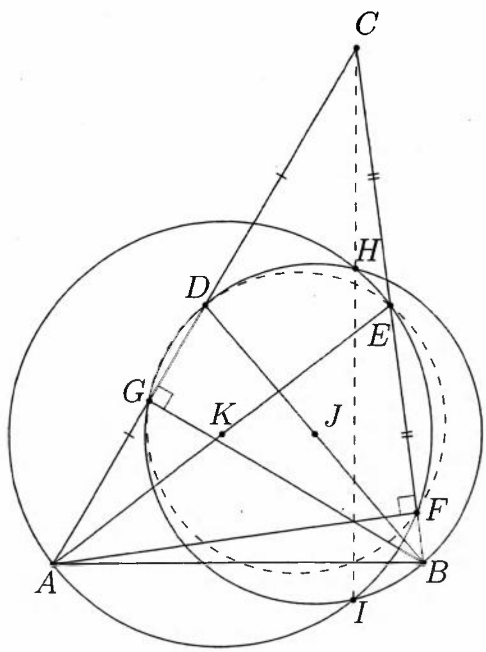
Solution. Let the circumcircles of $\triangle DHF$ and $\triangle EGF$ intersect for the second time at the point G'. We will show that $G \equiv G'$. Denote $M = AC \cap BJ$, $L = AJ \cap BC$ and $X = ED \cap HI$. Pascal's Theorem, applied to the hexagon $EAHIBD$, gives that the points L, M and X are collinear.

Moreover, applying Desargues' Theorem (Problem 7.1) to $\triangle HIJ$ and $\triangle DEF$ gives that the lines EI, DH and JF are concurrent. On the other hand, the radical axes theorem (Problem 6.5.1) gives that the lines EI, DH and FG' are concurrent. Therefore, $G' \in JF$ and $G \equiv G'$.

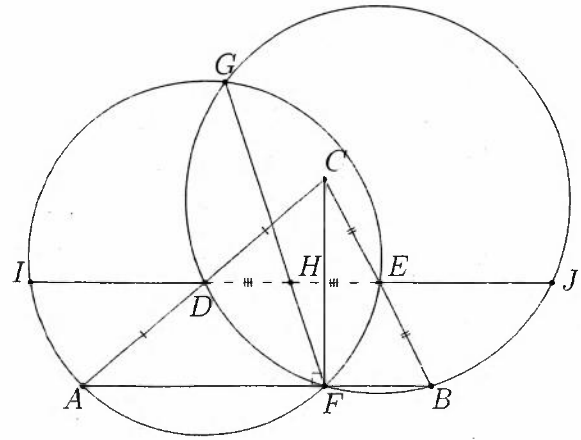
Problem 4.8.38. Let ABC be a triangle. Let D and E be the midpoints of the sides AC and BC, respectively. The circles with diameters BD and AE intersect each other at the points H and I. Prove that the points H, I and C are collinear.

Solution. We will show that if F and G are the feet of the altitudes from A and B in $\triangle ABC$, then the quadrilateral $GDEF$ is cyclic. Since the quadrilateral $ABFG$ is cyclic, we have $CD.CG = \dfrac{CA.CG}{2} = \dfrac{CB.CF}{2} = CE.CF$.

The radical axes theorem, applied to the quadrilaterals $IGDH$, $IFEH$ and $GDEF$, yields the desired statement.



Problem 4.8.39. Let ABC be a triangle. The points D and E are the midpoints of the sides AC and BC, respectively. The point F is the foot of the altitude through C in $\triangle ABC$. The circumcircles of $\triangle DFB$ and $\triangle AFE$ intersect each other at the points F and G. Denote $GF \cap DE = H$. Prove that H is the midpoint of DE.



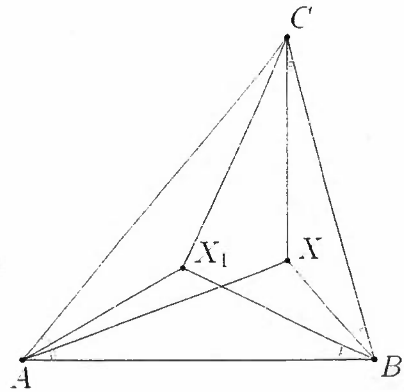
Solution. Let the reflection of D with respect to E be J and let the reflection of E with respect to D be I. Clearly, J coincides with the reflection of D with respect to the perpendicular bisector of BF. Therefore, $DJ = AB$ and the quadrilateral $JBFD$ is an isosceles trapeoid. Thus, $JBFD$ is cyclic. Analogously, I lies on the circumcircle of $\triangle AFE$ and $IE = AB$.

Now we have

$$DH.HJ = GH.HF = HE.HI,$$

and from $IE = AB = DJ$ we can easily conclude that $DH = HE$.

Problem 4.9.1. Let ABC be a triangle. Let X be an arbitrary point, not lying on any of the sides of the triangle. Prove that the isogonal conjugates of the lines AX, BX and CX with respect to $\triangle ABC$ are concurrent.



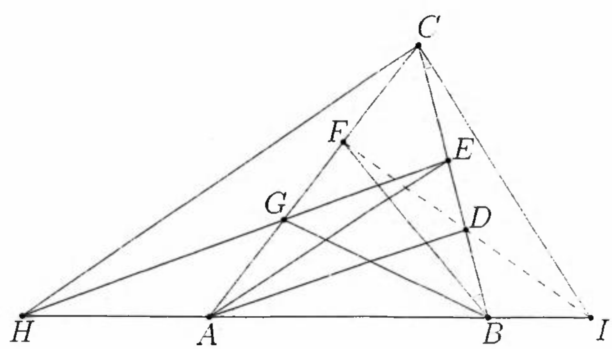
Solution. Let X_1 be the intersection point of the isogonal conjugates of the lines AX and BX with respect to $\triangle ABC$. Applying the trigonometric form of Ceva's Theorem to $\triangle ABC$ and the points X and X_1 gives

$$\frac{\sin\angle CAX}{\sin\angle XAB}\cdot\frac{\sin\angle XBA}{\sin\angle XBC}\cdot\frac{\sin\angle XCB}{\sin\angle XCA}=1,$$
$$\frac{\sin\angle CAX_1}{\sin\angle X_1AB}\cdot\frac{\sin\angle X_1BA}{\sin\angle X_1BC}\cdot\frac{\sin\angle X_1CB}{\sin\angle X_1CA}=1$$
$$\Rightarrow \frac{\sin\angle XCB}{\sin\angle XCA}\cdot\frac{\sin\angle X_1CB}{\sin\angle X_1CA}=1.$$

Therefore, $\dfrac{\sin\angle XCB}{\sin\angle XCA}=\dfrac{\sin\angle X_1CA}{\sin\angle X_1CB}$.

Hence, $\angle XCB=\angle X_1CA$ and so the isogonal conjugate of the line CX with respect to $\triangle ABC$ passes through X_1.

Problem 4.9.2. Let ABC be a triangle. The points D and E lie on the side BC and satisfy $\angle BAD=\angle EAC$. The points F and G lie on the side AC and satisfy $\angle ABG=\angle FBC$. The points H and I are chosen on the line AB, so that A is between H and B, and B is between A and I. Let $\angle HCA=\angle BCI$. If the points E, G and H are collinear, prove that the points F, D and I are collinear.



Solution. The law of sines gives

$$\frac{BD}{DC} = \frac{BD}{DA} \cdot \frac{AD}{DC} = \frac{\sin\angle BAD}{\sin\beta} \cdot \frac{\sin\gamma}{\sin\angle CAD}$$
$$= \frac{\sin\angle CAE}{\sin\gamma} \cdot \frac{\sin\beta}{\sin\angle BAE} \cdot \frac{\sin^2\gamma}{\sin^2\beta} = \frac{CE}{EB} \cdot \frac{\sin^2\gamma}{\sin^2\beta}.$$

Analogously, $\dfrac{CF}{FA} = \dfrac{AG}{GC} \cdot \dfrac{\sin^2\alpha}{\sin^2\gamma}$ and $\dfrac{AI}{IB} = \dfrac{BH}{HA} \cdot \dfrac{\sin^2\beta}{\sin^2\alpha}$.
By Menelaus' Theorem it suffices to prove that

$$\frac{BD}{DC} \cdot \frac{CF}{FA} \cdot \frac{AI}{IB} = 1.$$

But by the same theorem, applied to $\triangle ABC$ and the line EGH, it follows that

$$\frac{BD}{DC} \cdot \frac{CF}{FA} \cdot \frac{AI}{IB} = \frac{CE}{EB} \cdot \frac{AG}{GC} \cdot \frac{BH}{HA} \cdot \frac{\sin^2\gamma}{\sin^2\beta} \cdot \frac{\sin^2\alpha}{\sin^2\gamma} \cdot \frac{\sin^2\beta}{\sin^2\alpha} = 1.$$

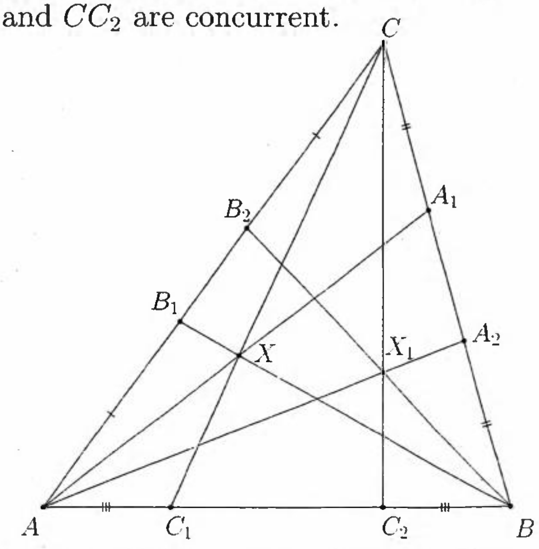
Problem 4.9.3. Let ABC be a triangle and let X be a point inside of it. The line AX intersects BC at the point A_1. The points B_1 and C_1 are defined analogously. Denote the reflection of A_1 with respect to the midpoint of BC by A_2. The points B_2 and C_2 are defined analogously. Prove that the lines AA_2, BB_2 and CC_2 are concurrent.

Solution. Ceva's Theorem, applied to $\triangle ABC$ and the points A_1, B_1 and C_1, yields

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

We have that $AC_1 = BC_2$, $BA_2 = A_1C$ and $CB_2 = AB_1$, which implies that $AC_2 = BC_1$, $BA_1 = A_2C$ and $CB_1 = AB_2$. Then

$$\frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} = 1$$

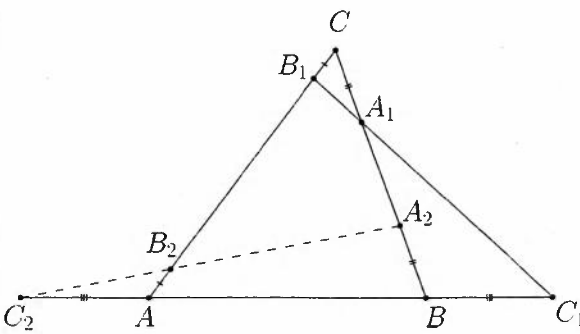


and again, Ceva's Theorem yields that the lines AA_2, BB_2 and CC_2 are concurrent. (The point of concurrency X_1 is known as the isotomic conjugate of the point X with respect to $\triangle ABC$.)

Problem 4.9.4. Let ABC be a triangle. The points A_1, B_1 and C_1 are collinear and lie on the lines BC, CA and AB, respectively. Let A_2 be the reflection of A_1 with respect to the midpoint of BC. The points B_2 and C_2 are defined analogously. Prove that the points A_2, B_2 and C_2 are collinear.

Solution. Menelaus' Theorem, applied to $\triangle ABC$ and the points A_1, B_1 and C_1, gives

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A}=1.$$



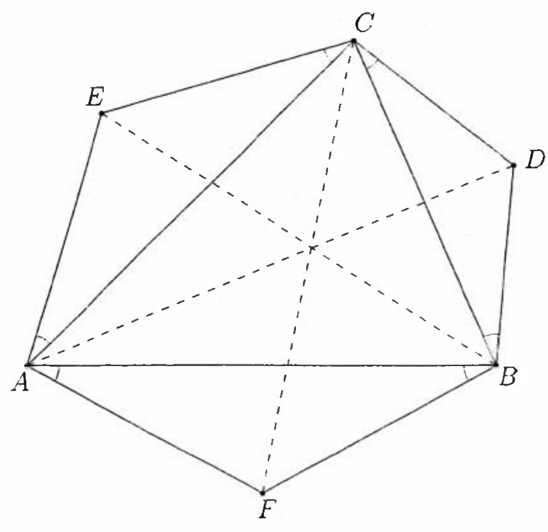
We have that $AC_1 = BC_2$, $BA_2 = A_1C$ and $CB_2 = AB_1$. Then

$$\frac{AC_2}{C_2B}\cdot\frac{BA_2}{A_2C}\cdot\frac{CB_2}{B_2A}=1$$

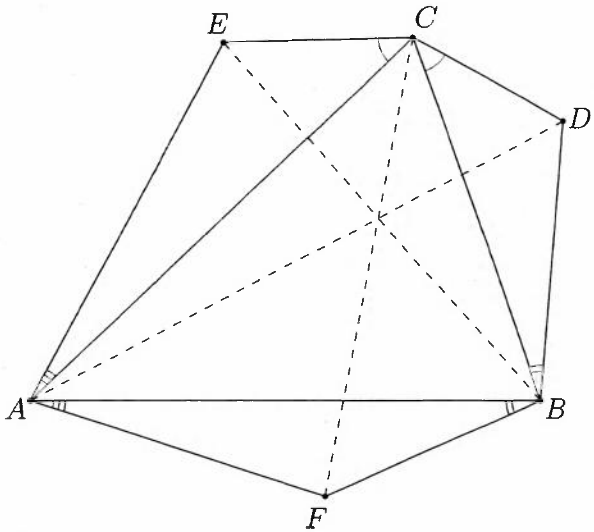
and again, Menelaus' Theorem yields that the points A_2, B_2 and C_2 are collinear.

Problem 4.9.5. Let ABC be a triangle. The triangles $\triangle BCD$, $\triangle CAE$ and $\triangle ABF$ are constructed externally on the sides BC, CA and AB, respectively, such that $\angle BCD = \angle DBC = \angle CAE = \angle ACE = \angle ABF = \angle BAF = \varphi$. Prove that the lines AD, BE and CF are concurrent.

Solution. This is a special case of Problem 4.9.6, where $\varphi = \theta = \psi$.



Problem 4.9.6. Let ABC be a triangle. The triangles $\triangle BCD$, $\triangle CAE$ and $\triangle ABF$ are constructed externally on the sides BC, CA and AB, respectively, such that $\angle BCD = \angle DBC = \varphi$, $\angle CAE = \angle ACE = \theta$ and $\angle ABF = \angle BAF = \psi$. Prove that the lines AD, BE and CF are concurrent.



Solution. The trigonometric form of Ceva's Theorem gives that it suffices to show that

$$\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\angle ACF}{\sin\angle FCB}\cdot\frac{\sin\angle CBE}{\sin\angle EBA}=1.$$

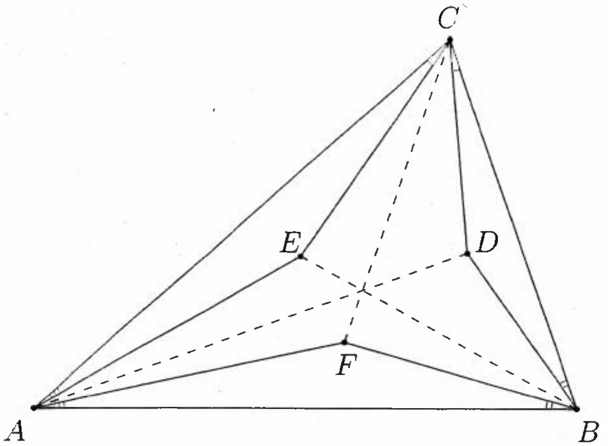
We have

$$\begin{aligned}\frac{\sin\angle BAD}{\sin\angle DAC}&=\frac{\sin\angle BAD}{\sin(\beta+\varphi)}\cdot\frac{\sin(\beta+\varphi)}{\sin(\gamma+\theta)}\cdot\frac{\sin(\gamma+\theta)}{\sin\angle DAC}\\&=\frac{BD}{AD}\cdot\frac{\sin(\beta+\varphi)}{\sin(\gamma+\theta)}\cdot\frac{AD}{CD}=\frac{\sin(\beta+\varphi)}{\sin(\gamma+\theta)}.\end{aligned}$$

Analogously, we derive similar expressions for $\dfrac{\sin\angle ACF}{\sin\angle FCB}$ and $\dfrac{\sin\angle CBE}{\sin\angle EBA}$ to get

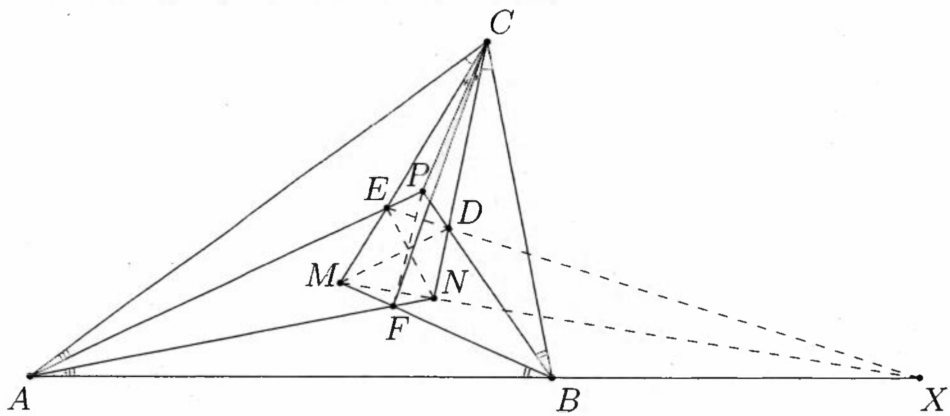
$$\begin{aligned}&\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\angle ACF}{\sin\angle FCB}\cdot\frac{\sin\angle CBE}{\sin\angle EBA}\\=&\frac{\sin(\beta+\varphi)}{\sin(\gamma+\theta)}\cdot\frac{\sin(\alpha+\psi)}{\sin(\beta+\phi)}\cdot\frac{\sin(\gamma+\theta)}{\sin(\alpha+\psi)}=1.\end{aligned}$$

Problem 4.9.7. Let ABC be a triangle. The points D, E and F lie inside of $\triangle ABC$ and satisfy $\angle BCD = \angle ECA = \varphi$, $\angle DBC = \angle ABF = \theta$ and $\angle CAE = \angle BAF = \psi$. Prove that the lines AD, BE and CF are concurrent.



Solution. The solution is analogous to that of Problem 4.9.6.

Problem 4.9.8. Let ABC be a triangle. The points D, E and F are such that $\angle BCD = \angle ECA$, $\angle DBC = \angle ABF$ and $\angle CAE = \angle BAF$. Denote $M = BF \cap CE$, $N = AF \cap CD$ and $P = AE \cap BD$. Prove that the lines MD, NE and PF are concurrent.



Solution. Desargues' Theorem (Problem 7.1), applied to $\triangle MNF$ and $\triangle EDP$, gives that it suffices to prove that the points $B = MF \cap PD$, $A = EP \cap CN$ and $X = MN \cap ED$ are collinear.

Since $\angle BAF = \angle CAP$ and $\angle FBA = \angle DBC$, the points F and P are isogonal conjugates in $\triangle ABC$ and hence $\angle PCM = \angle NCF$.

Ceva's and Menelaus' Theorems give that it suffices to show that

$$\frac{AN}{NF}\cdot\frac{FM}{MB}=\frac{AE}{EP}\cdot\frac{PD}{DB}.$$

Indeed, MN and ED would then intersect AB at the same point. We have

$$\frac{AN}{NF}=\frac{AN}{NC}\cdot\frac{NC}{NF}=\frac{\sin\angle ACN}{\sin\angle NAC}\cdot\frac{\sin\angle NFC}{\sin\angle FCN}=\frac{AC}{CF}\cdot\frac{\sin\angle ACN}{\sin\angle NCF},$$
$$\frac{FM}{MB}=\frac{FM}{MC}\cdot\frac{MC}{MB}=\frac{\sin\angle MCF}{\sin\angle MFC}\cdot\frac{\sin\angle MBC}{\sin\angle MCB}=\frac{CF}{BC}\cdot\frac{\sin\angle MCF}{\sin\angle MCB},$$
$$\frac{AE}{EP}=\frac{AE}{EC}\cdot\frac{EC}{EP}=\frac{\sin\angle ACE}{\sin\angle EAC}\cdot\frac{\sin\angle EPC}{\sin\angle PCE}=\frac{AC}{CP}\cdot\frac{\sin\angle ACE}{\sin\angle PCE},$$
$$\frac{PD}{DB}=\frac{PD}{DC}\cdot\frac{DC}{DB}=\frac{\sin\angle PCD}{\sin\angle DPC}\cdot\frac{\sin\angle DBC}{\sin\angle DCB}=\frac{CP}{BC}\cdot\frac{\sin\angle PCD}{\sin\angle DCB}.$$

Multiplying the first two equalities, we get $\dfrac{AN}{NF}\cdot\dfrac{FM}{MB}=\dfrac{AC}{BC}\cdot\dfrac{\sin\angle MCF}{\sin\angle NCF}$.
Multiplying the last two equalities, we get $\dfrac{AE}{EP}\cdot\dfrac{PD}{DB}=\dfrac{AC}{BC}\cdot\dfrac{\sin\angle PCD}{\sin\angle PCE}$.
Therefore, $\dfrac{AN}{NF}\cdot\dfrac{FM}{MB}=\dfrac{AE}{EP}\cdot\dfrac{PD}{DB}$.

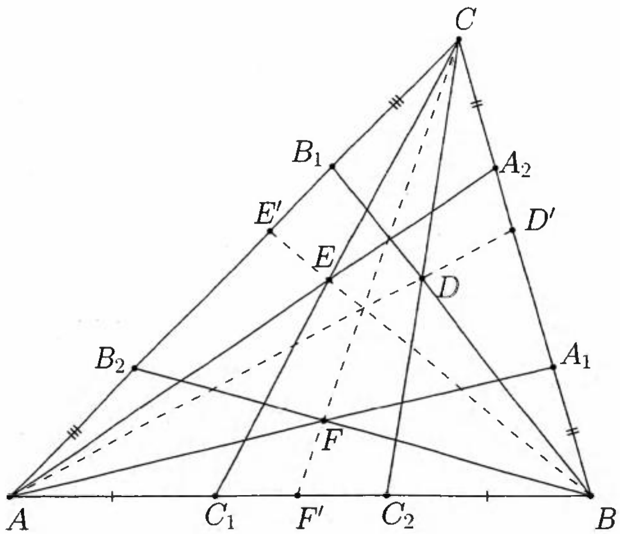
Problem 4.9.9. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $CC_2\cap BB_1=D$, $AA_2\cap CC_1=E$ and $BB_2\cap AA_1=F$. Prove that the lines AD, BE and CF are concurrent.

Solution. Let $AD\cap BC=D'$, $BE\cap AC=E'$ and $CF\cap AB=F'$. Ceva's Theorem, applied to $\triangle ABC$ and the point F, yields

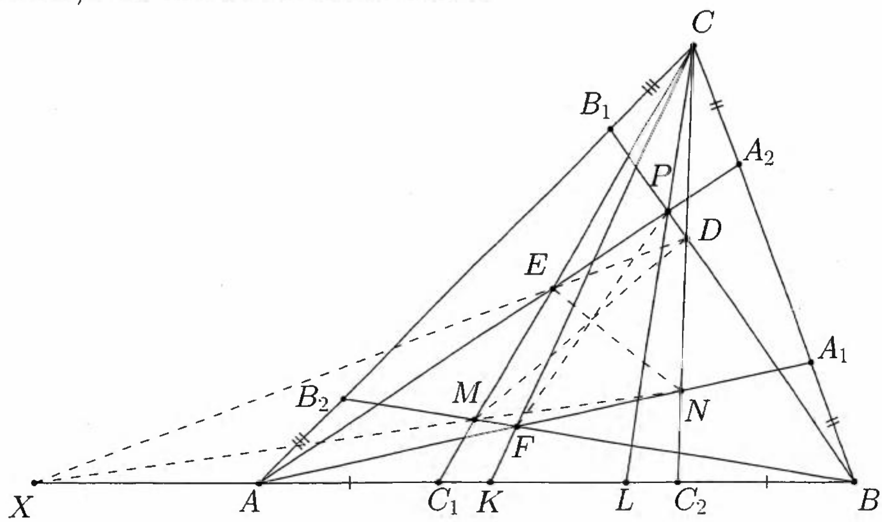
$$\frac{AF'}{F'B}=\frac{AB_2}{B_2C}\cdot\frac{CA_1}{A_1B}.$$

Analogously, $\dfrac{BD'}{D'C}=\dfrac{BC_2}{C_2A}\cdot\dfrac{AB_1}{B_1C}$ and $\dfrac{CE'}{E'A}=\dfrac{CA_2}{A_2B}\cdot\dfrac{BC_1}{C_1A}$.

Multiplying the above equations and using the given equalities between the segments gives the desired result.



Problem 4.9.10. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $CC_2 \cap BB_1 = D$, $AA_2 \cap CC_1 = E$, $BB_2 \cap AA_1 = F$, $CC_1 \cap BB_2 = M$, $CC_2 \cap BB_1 = P$ and $CC_2 \cap AA_1 = N$. Prove that the lines MD, NE and PF are concurrent.



Solution. Desargues' Theorem (Problem 7.1), applied to $\triangle EDP$ and $\triangle MNF$, gives that it suffices to prove that the points $A = EP \cap FN$, $B =$

$MF \cap PD$ and $X = ED \cap MN$ are collinear. Equivalently, we need ED and MN to intersect AB at the same point, i.e. $\frac{AN}{NF} \cdot \frac{FM}{MB} = \frac{AE}{EP} \cdot \frac{PD}{DB}$.

Let $K = CF \cap AB$ and $L = CP \cap AB$. Menelaus' Theorem, applied to $\triangle AFK$ and the line CNC_2, yields $\frac{AN}{NF} = \frac{AC_2}{C_2K} \cdot \frac{KC}{CF}$. Analogously, Menelaus' Theorem, applied to $\triangle KBF$ and the line CMC_1, yields $\frac{FM}{MB} = \frac{FC}{CK} \cdot \frac{KC_1}{C_1B}$.

The same theorem, applied to $\triangle ALP$ and the line CEC_1, gives $\frac{AE}{EP} = \frac{AC_1}{C_1L} \cdot \frac{LC}{CP}$. Again, Menelaus' Theorem, applied to $\triangle BPL$ and the line CDC_2, gives $\frac{PD}{DB} = \frac{PC}{CL} \cdot \frac{LC_2}{C_2B}$.

It remains to note that AA_1 and AA_2, as well as BB_1 and BB_2 are isotomic conjugates with respect to $\triangle ABC$, and so $F = BB_2 \cap AA_1$ and $P = BB_1 \cap AA_2$ are isotomic conjugates. Therefore, $KA = LB$ and multiplying the above equalities gives the result.

Problem 4.9.11. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $B_1C_1 \cap B_2C_2 = D$, $A_1C_1 \cap A_2C_2 = E$ and $A_1B_1 \cap A_2B_2 = F$. Prove that the lines AD, BE and CF are concurrent.

Solution. The trigonometric form of Ceva's Theorem gives that it suffices to show that

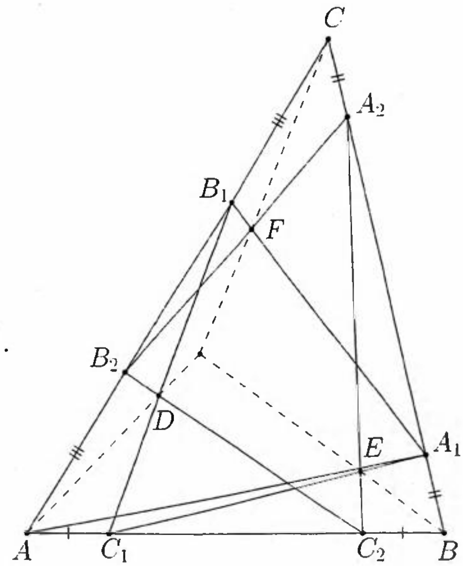
$$\frac{\sin\angle BAD}{\sin\angle DAC} \cdot \frac{\sin\angle ACD}{\sin\angle DCB} \cdot \frac{\sin\angle CBD}{\sin\angle DBA} = 1.$$

We have

$$\frac{\sin\angle C_2AD}{\sin\angle DAB_2} = \frac{\sin\angle C_2AD}{\sin\angle ADC_2} \cdot \frac{\sin\angle ADB_2}{\sin\angle DAB_2} = \frac{C_2D}{DB_2} \cdot \frac{AB_2}{AC_2}.$$

Menelaus' Theorem, applied to $\triangle AC_2B_2$ and the line B_1DC_1, yields

$$\frac{C_2D}{DB_2} = \frac{C_2C_1}{C_1A} \cdot \frac{AB_1}{B_1B_2}$$



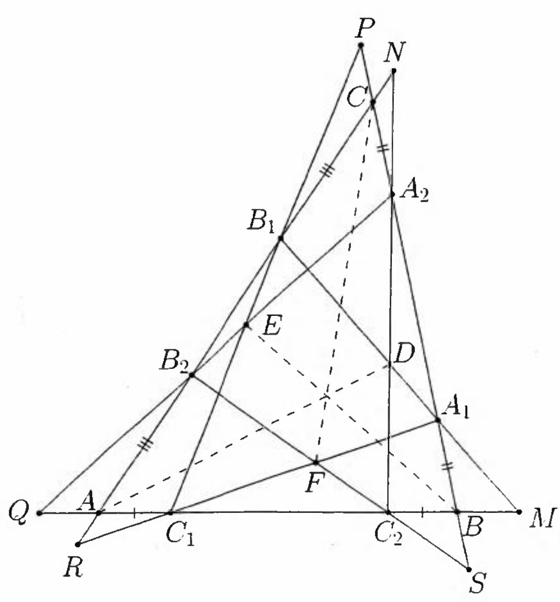
and hence

$$\frac{\sin \angle C_2AD}{\sin \angle DAB_2} = \frac{AB_2.AB_1.C_1C_2}{AC_2.AC_1.B_1B_2}.$$

Considering the analogous equalities of the sine ratios at the angles $\angle B$ and $\angle C$, we then get that

$$\prod_{cyc} \frac{\sin \angle BAD}{\sin \angle DAC} = \prod_{cyc} \frac{AB_1.AB_2.C_1C_2}{AC_2.AC_1.B_1B_2} = 1.$$

Problem 4.9.12. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $B_1C_1 \cap B_2A_2 = E$, $A_1C_1 \cap B_2C_2 = F$ and $A_1B_1 \cap A_2C_2 = D$. Prove that the lines AD, BE and CF are concurrent.



Solution. Let $A_2C_2 \cap AC = N$, $A_1C_1 \cap AC = R$, $B_1A_1 \cap AB = M$, $A_2B_2 \cap AB = Q$, $B_2C_2 \cap BC = S$ and $B_1C_1 \cap BC = P$. Problem 4.9.2 yields that $CN = AR$, $AQ = BM$ and $BS = CP$. Now

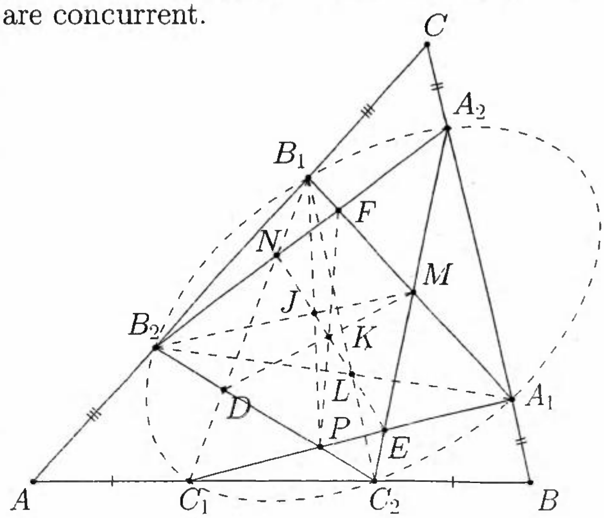
$$\frac{\sin\angle C_2AD}{\sin\angle NAD} = \frac{\sin\angle C_2AD}{\sin\angle ADC_2}\cdot\frac{\sin\angle NDA}{\sin\angle NAD} = \frac{C_2D}{DN}\cdot\frac{AN}{AC_2}.$$

Also, Menelaus' Theorem, applied to $\triangle ANC_2$ and the line B_1DM, gives $\dfrac{C_2D}{DN} = \dfrac{C_2M}{MA}\cdot\dfrac{AB_1}{B_1N}$ and hence $\dfrac{\sin\angle C_2AD}{\sin\angle NAD} = \dfrac{AN}{AM}\cdot\dfrac{AB_1}{AC_2}\cdot\dfrac{C_2M}{B_1N}$.

Analogously, we get

$$\prod_{cyc}\frac{\sin\angle C_2AD}{\sin\angle NAD} = \prod_{cyc}\frac{AN}{AM}\cdot\frac{AB_1}{AC_2}\cdot\frac{C_2M}{B_1N} = \prod_{cyc}\frac{AN}{AM}\cdot\prod_{cyc}\frac{AB_1}{AC_2}\cdot\prod_{cyc}\frac{C_2M}{B_1N} = 1.$$

Problem 4.9.13. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $B_1C_1 \cap B_2A_2 = N$, $A_1C_1 \cap B_2C_2 = P$, $A_1B_1 \cap A_2C_2 = M$, $A_1B_1 \cap A_2B_2 = F$, $C_1B_1 \cap B_2C_2 = D$ and $A_1C_1 \cap A_2C_2 = E$. Prove that the lines DM, EN and PF are concurrent.



Solution. We will firstly show that the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on an ellipse. Let the ellipse through A_1, A_2, B_1, B_2 and C_1 intersect BC again at the point X (it is easy to see that this point is well-defined). It is well known that there exists an affine transformation that maps this ellipse into a circle. Then the points A and C, as well as the points A and B have equal powers with respect to this circle.

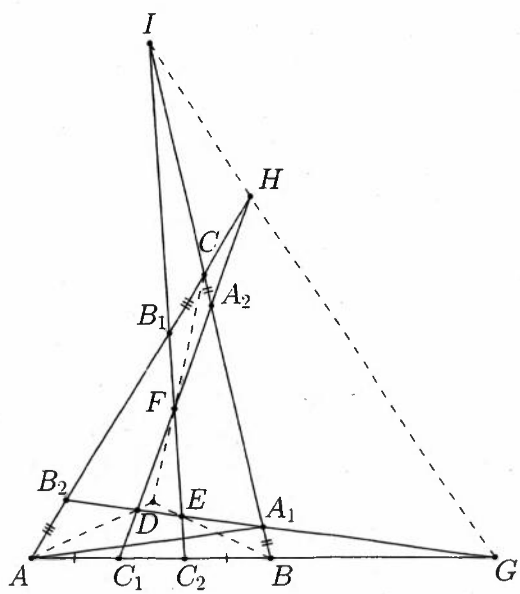
Hence, the points B and C also have equal powers with respect to the same circle, and then $CX = BA_1$. Since the order of points on a line is preserved under an affine transformation, we deduce that $X \equiv A_2$.

Pappus' Theorem yields that the points $N = B_1D \cap B_2F$, $J = B_1P \cap B_2M$ and $K = FP \cap DM$ are collinear. The same theorem gives that the points $J = B_1P \cap B_2M$, $E = MC_2 \cap PA_1$ and $L = B_2A_1 \cap B_1C_2$ are collinear.

Also, Pascal's Theorem, applied to $B_2A_2C_2B_1C_1A_1$ gives that the points N, E and L are collinear. Then the line EL intersects the line KJN at two points, i.e. these two lines coincide. In conclusion, the points N, K and E are collinear.

Problem 4.9.14. Let ABC be a triangle. The points A_1 and A_2 lie on the side BC and are equidistant from its midpoint (A_1 lies between B and A_2). The points B_1 and B_2 lie on the side AC and are equidistant from its midpoint (B_1 lies between C and B_2). The points C_1 and C_2 lie on the side AB and are equidistant from its midpoint (C_1 lies between A and C_2). Denote $A_1B_2 \cap B_1C_2 = E$, $B_1C_2 \cap A_2C_1 = F$ and $A_2C_1 \cap A_1B_2 = D$. Prove that the lines AD, BE and CF are concurrent.

Solution. Desargues' Theorem (Problem 7.1), applied to $\triangle ABC$ and $\triangle DEF$, yields that it suffices to prove that the points $G = AB \cap DE$, $H = AC \cap DF$ and $I = EF \cap BC$ are collinear.



Menelaus' Theorem, applied to $\triangle ABC$ and the line B_2A_1G, gives

$$\frac{AG}{GB} = \frac{AB_2}{B_2C} \cdot \frac{CA_1}{A_1B}.$$

Analogously, $\dfrac{BI}{IC} = \dfrac{BC_2}{C_2A} \cdot \dfrac{AB_1}{B_1C}$ and $\dfrac{CH}{HA} = \dfrac{CA_2}{A_2B} \cdot \dfrac{BC_1}{C_1A}$.

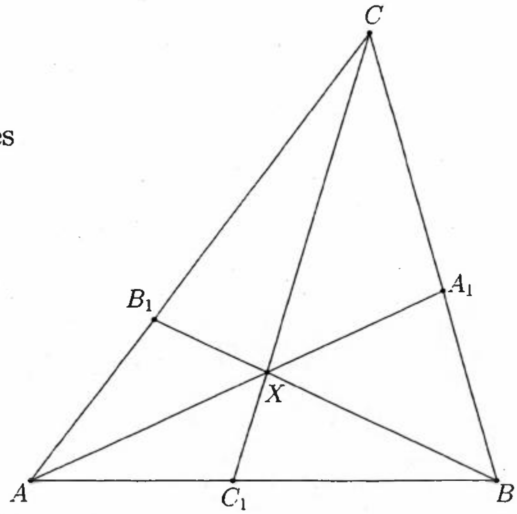
Multiplying the above equalities gives

$$\frac{AG}{GB} \cdot \frac{BI}{IC} \cdot \frac{CH}{HA} = 1$$

and the conclusion follows by Menelaus' Theorem.

Problem 4.9.15. *(Ceva's Theorem)* Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$ be points on its sides. Then the lines AA_1, BB_1 and CC_1 are concurrent if and only if $\dfrac{AC_1}{C_1B} \cdot \dfrac{BA_1}{A_1C} \cdot \dfrac{CB_1}{B_1A} = 1$.

Solution. Necessity. Let the lines AA_1, BB_1 and CC_1 intersect each other at the point X. Now we have $\dfrac{AC_1}{C_1B} = \dfrac{S_{\triangle AXC}}{S_{\triangle BXC}}$, $\dfrac{BA_1}{A_1C} = \dfrac{S_{\triangle ABX}}{S_{\triangle AXC}}$ and $\dfrac{CB_1}{B_1A} = \dfrac{S_{\triangle BXC}}{S_{\triangle ABX}}$.



Multiplying these equalities gives

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

Sufficiency. We have

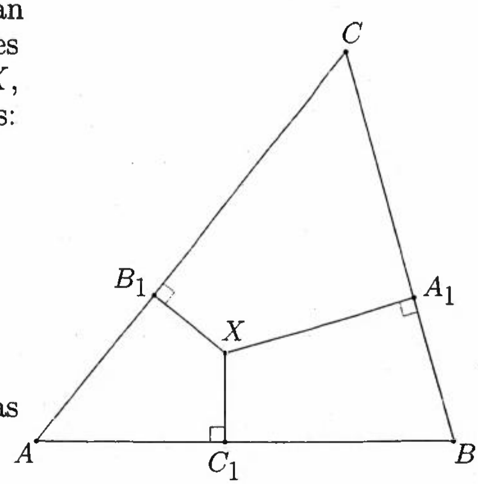
$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

Let AA_1 and BB_1 intersect at X, and let $CX \cap AB = C_2$. Then

$$\frac{AC_2}{C_2B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

Hence, $\dfrac{AC_1}{C_1B} = \dfrac{AC_2}{C_2B}$ and thus $C_1B = C_2B$. Therefore, $C_1 \equiv C_2$ and hence the lines AA_1, BB_1 and CC_1 are concurrent.

Problem 4.9.16. *(Carnot's Theorem)* Let ABC be a triangle and let $A_1 \in BC$, $B_1 \in AC$ and $C_1 \in AB$ be points on its sides. Then the perpendiculars through the points A_1, B_1 and C_1 to the sides they lie on are concurrent if and only if $AC_1^2 + BA_1^2 + CB_1^2 = C_1B^2 + A_1C^2 + B_1A^2$.



Solution. Necessity. The Pythagorean Theorem, applied to the triangles $\triangle AC_1X$, $\triangle A_1XB$, $\triangle B_1XC$, $\triangle BC_1X$, $\triangle A_1XC$, $\triangle B_1XC$, and $\triangle B_1XA$, gives:

1) $AC_1^2 = AX^2 - XC_1^2$,
2) $BA_1^2 = BX^2 - XA_1^2$,
3) $CB_1^2 = CX^2 - XB_1^2$,
4) $C_1B^2 = BX^2 - XC_1^2$,
5) $A_1C^2 = CX^2 - XA_1^2$,
6) $B_1A^2 = AX^2 - XB_1^2$,

respectively. Summing up 1, 2 and 3, as well as 4, 5 and 6, gives

$$BC_1^2 + CA_1^2 + AB_1^2 = AC_1^2 + BA_1^2 + CB_1^2 = C_1B^2 + A_1C^2 + B_1A^2.$$

Sufficiency. Suppose that $AC_1^2 + BA_1^2 + CB_1^2 = C_1B^2 + A_1C^2 + B_1A^2$. Let the perpendiculars through A and B to the sides they lie on intersect each other at the point X. Also, let C_2 be the projection of X onto AB. Then

$$AC_2^2 + BA_1^2 + CB_1^2 = C_2B^2 + A_1C^2 + B_1A^2.$$

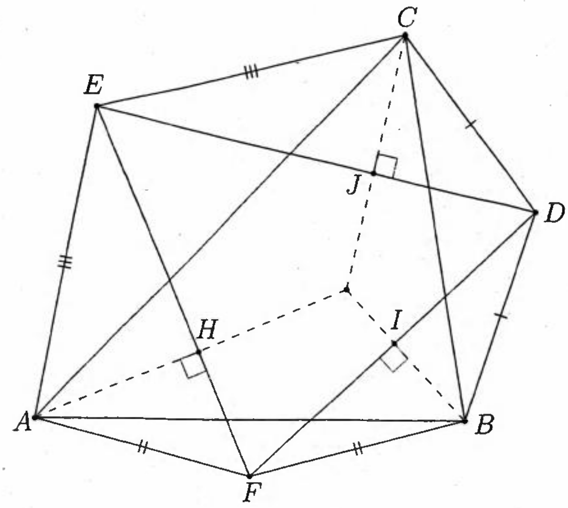
On the other hand, we have that

$$AC_1^2 + BA_1^2 + CB_1^2 = C_1B^2 + A_1C^2 + B_1A^2.$$

Subtracting the last two equalities, we get $AC_2^2 - AC_1^2 = C_2B^2 - C_1B^2$, which means that $C_2 \equiv C_1$.

Problem 4.9.17. Let ABC be a triangle. The points D, E and F are such that $BD = CD$, $CE = EA$ and $AF = FB$. Prove that the perpendiculars from A to EF, from B to DF and from C to DE are concurrent.

Solution. Denote the projections of A, B and C onto EF, FD and DE by H, I and J, respectively.



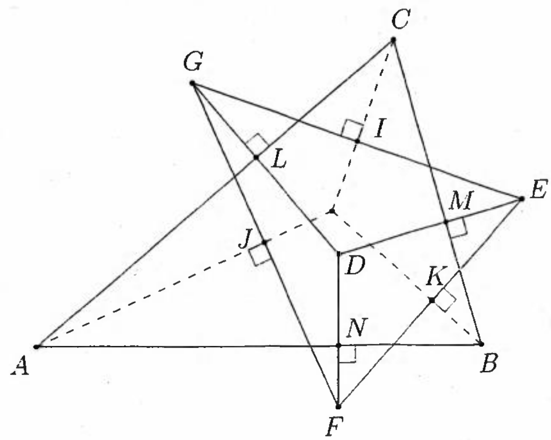
Then

$$\begin{aligned}&(DJ^2 - JE^2) + (EH^2 - HF^2) + (FI^2 - ID^2)\\ =\ &(DC^2 - CE^2) + (EA^2 - AF^2) + (FB^2 - BD^2)\\ =\ &(BF^2 - AF^2) + (AE^2 - EC^2) + (CD^2 - DB^2) = 0.\end{aligned}$$

Now Carnot's Theorem (Problem 4.9.16), applied to $\triangle EFD$, gives the desired result.

Problem 4.9.18. Let ABC be a triangle and let D be a point inside of it. The points F, E and G are chosen on the perpendiculars through D to the sides AB, BC and CA, respectively. Prove that the perpendiculars from A to GF, from B to FE and from C to GE are concurrent.

Solution. Denote the feet of the perpendiculars from A, B and C to GF, FE and GE by J, K and I, respectively. Denote the feet of the perpendiculars from F, E and G to AB, BC and CA by N, M and L, respectively.



Carnot's Theorem (Problem 4.9.16) and the Pythagorean

Theorem, applied to $\triangle AFN$, $\triangle BFN$, $\triangle BEM$, $\triangle CEM$, $\triangle CGL$ and $\triangle AGL$ give

$$\begin{aligned}&(BF^2 - AF^2) + (CE^2 - BE^2) + (AG^2 - GC^2)\\ &= AN^2 - BN^2 + BM^2 - MC^2 + CL^2 - LA^2 = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}0 &= (BF^2 - AF^2) + (CE^2 - BE^2) + (AG^2 - GC^2)\\ &= (FB^2 - BE^2) + (EC^2 - CG^2) + (GA^2 - AF^2)\\ &= (FK^2 - KE^2) + (EI^2 - IG^2) + (GJ^2 - JF^2).\end{aligned}$$

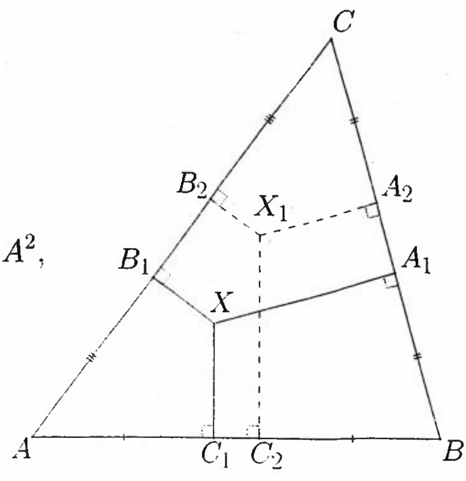
Now Carnot's Theorem, applied to $\triangle EFG$, gives the desired result.

Problem 4.9.19. Let ABC be a triangle and let X be a point inside of it. The points A_1, B_1 and C_1 are the projections of X onto the sides BC, CA and AB, respectively. Denote the reflection of A_1 with respect to the midpoint of BC by A_2. The points B_2 and C_2 are defined analogously. Prove that the perpendiculars through A_2, B_2 and C_2 to the sides these points lie on are concurrent.

Solution. Carnot's Theorem (Problem 4.9.16), applied to $\triangle ABC$ and the point X, gives $AC_1^2 + BA_1^2 + CB_1^2 = C_1B^2 + A_1C^2 + B_1A^2$.

Hence, using $AC_1 = BC_2$, $BA_2 = A_1C$ and $CB_2 = AB_1$, we deduce that

$$AC_2^2+BA_2^2+CB_2^2 = C_2B^2+A_2C^2+B_2A^2,$$



Carnot's Theorem yields that the perpendiculars are concurrent, as desired.

Menelaus' Theorem, applied to $\triangle ADC$, gives

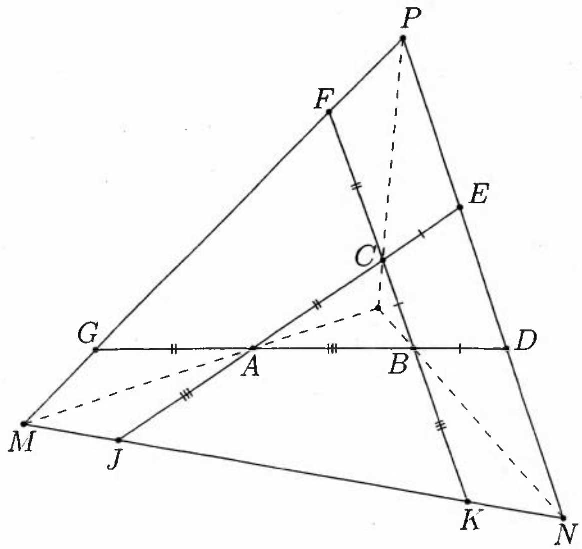
$$\frac{DM}{MC}=\frac{DB}{BA}\cdot\frac{AE}{EC}=\frac{AE}{AB}$$
$$\Rightarrow \frac{\sin\angle BAM}{\sin\angle DAC}=\frac{AC}{AB}\cdot\frac{AE}{AD}=\frac{b}{c}\cdot\frac{b+a}{c+a}$$

This and the analogous expressions for $\dfrac{\sin\angle CBN}{\sin\angle NBA}$ and $\dfrac{\sin\angle ACP}{\sin\angle PCB}$ give

$$\frac{\sin\angle BAM}{\sin\angle MAC}\cdot\frac{\sin\angle ACP}{\sin\angle PCB}\cdot\frac{\sin\angle CBN}{\sin\angle NBA}=\frac{b}{c}\cdot\frac{a+b}{a+c}\cdot\frac{a}{b}\cdot\frac{a+c}{c+b}\cdot\frac{c}{a}\cdot\frac{c+b}{a+b}=1.$$

Now the result follows by the trigonometric form of Ceva's Theorem.

Problem 4.9.24. Let ABC be a triangle. The points F and K; E and J; G and D are chosen on the lines BC, CA and AB, respectively, such that the orders on each line are $F-C-B-K$, $J-A-C-E$ and $D-B-A-G$. Also, $DB=BC=CE$, $FC=CA=AG$ and $JA=AB=BK$. Denote $ED\cap JK=N$, $JK\cap GF=M$ and $GF\cap ED=P$. Prove that the lines AM, BN and CP are concurrent.



Solution. Denote $X=BC\cap NP$, $Y=AC\cap MP$ and $Z=AB\cap MN$ (these points are not shown in the figure). Desargues' Theorem (Problem 7.1) gives that it suffices to show that the points X, Y and Z are collinear.

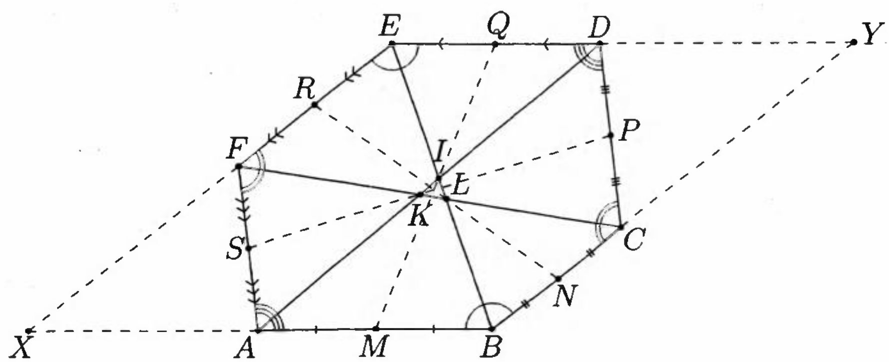
Menelaus' Theorem, applied to $\triangle ABC$ and the line EDX, gives

$$\frac{CX}{XB} = \frac{CE}{EA} \cdot \frac{AD}{DB} = \frac{c+a}{b+a}.$$

Analogously, $\dfrac{BZ}{ZA} = \dfrac{b+c}{a+c}$ and $\dfrac{AY}{YC} = \dfrac{a+b}{c+b}$. Multiplying these equalities and applying Menelaus' Theorem to $\triangle ABC$ and the points X, Y and Z gives the desired result.

Remark. If among the sides of $\triangle ABC$ there are two that are equal, then similar arguments apply in the projective plane, where the point of infinity is included.

Problem 4.9.25. Let $ABCDEF$ be a hexagon with $\angle B = \angle E$, $\angle C = \angle F$ and $\angle A = \angle D$. Prove that if the points M, N, P, Q, R and S are the midpoints of AB, BC, CD, DE, EF and FA, respectively, then the lines MQ, NR and PS are concurrent.



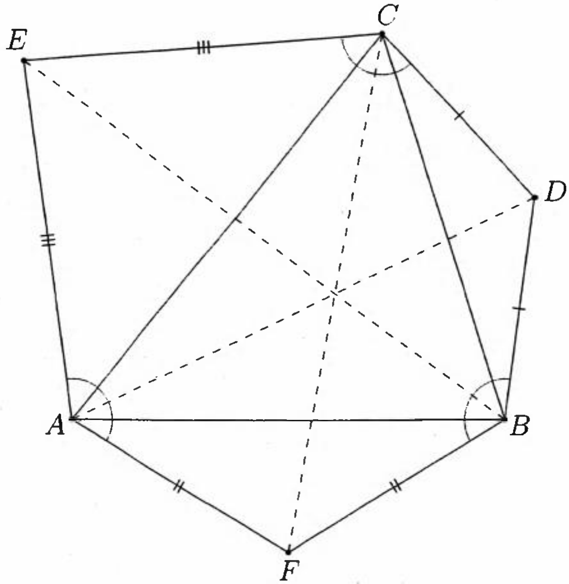
Solution. Denote $AB \cap FE = X$ and $ED \cap BC = Y$.

Using $\angle AXF = \angle CYD$, it is easy to see that the quadrilateral $XBYE$ is a parallelogram and thus $AB \parallel ED$. Analogously, $BC \parallel EF$ and $CD \parallel FA$.

Steiner's Theorem (Problem 5.2.1), applied to the trapezoid $ABDE$, gives that $EB \cap AD \cap MQ = I$ and analogously, $SP \cap AD \cap FC = K$ and $EB \cap FC \cap NR = L$. Now we will use the trigonometric form of Ceva's Theorem and the intercept theorem to prove that the lines through I, K and L are concurrent. We have

$$\frac{\sin\angle KIM}{\sin\angle MIL} \cdot \frac{\sin\angle ILR}{\sin\angle RLK} \cdot \frac{\sin\angle LKP}{\sin\angle PKI} = \frac{IB}{IA} \cdot \frac{FL}{LE} \cdot \frac{KD}{KC} = \frac{EB}{AD} \cdot \frac{FC}{BE} \cdot \frac{AD}{FC} = 1.$$

Problem 4.9.26. Let $AFBDCE$ be a hexagon, such that $\angle FBD = \angle DCE = \angle EAF = \varphi$, $AF = FB$, $BD = DC$ and $CE = EA$. Prove that the lines AD, BE and CF are concurrent.



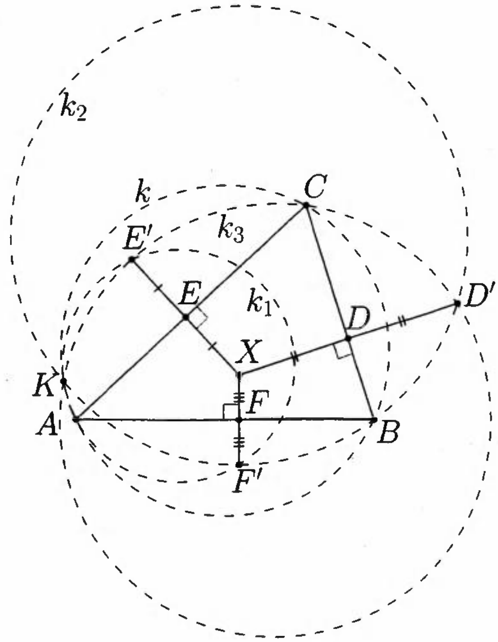
Solution. We will apply the trigonometric form of Ceva's Theorem to $\triangle ABC$. We have

$$\frac{\sin\angle BAD}{\sin\angle DAC} = \frac{\sin\angle BAD}{\sin\angle ABD}\cdot\frac{\sin\angle ABD}{\sin\angle ACD}\cdot\frac{\sin\angle ACD}{\sin\angle DAC}$$
$$= \frac{DB}{AD}\cdot\frac{\sin(\varphi - \angle ABF)}{\sin(\varphi - \angle ACE)}\cdot\frac{AD}{CD} = \frac{\sin(\varphi - \angle ABF)}{\sin(\varphi - \angle ACE)}.$$

Analogously, we get that $\dfrac{\sin\angle ACF}{\sin\angle FCB} = \dfrac{\sin(\varphi - \angle EAC)}{\sin(\varphi - \angle CBD)}$ and we also get that $\dfrac{\sin\angle CBE}{\sin\angle EBA} = \dfrac{\sin(\varphi - \angle BCD)}{\sin(\varphi - \angle FAB)}$.

Multiplying these equalities and using the given equal angles gives the desired result.

Problem 4.9.27. Let ABC be a triangle and let X be a point. The projections and reflections of X with respect to BC, CA and AB are D and D', E and E'. and F and F', respectively. Prove that the circumcircles of $\triangle AF'E'$, $\triangle BF'D'$, $\triangle CE'D$ and $\triangle ABC$ have a common point.



Solution. Denote the circumcircles of $\triangle AF'E'$, $\triangle BF'D'$, $\triangle CE'D$ and $\triangle ABC$ by k_1, k_2, k_3 and k, respectively. Set $k_1 \cap k_3 = K$.

We will show that K lies on k. We have

$$\angle AKC = \angle AKE' + \angle E'KC = 180^\circ - \angle E'F'A - \angle CD'E'.$$

On the other hand, $CE' = CX = CD'$ and $AE' = AX = AF'$. Moreover,

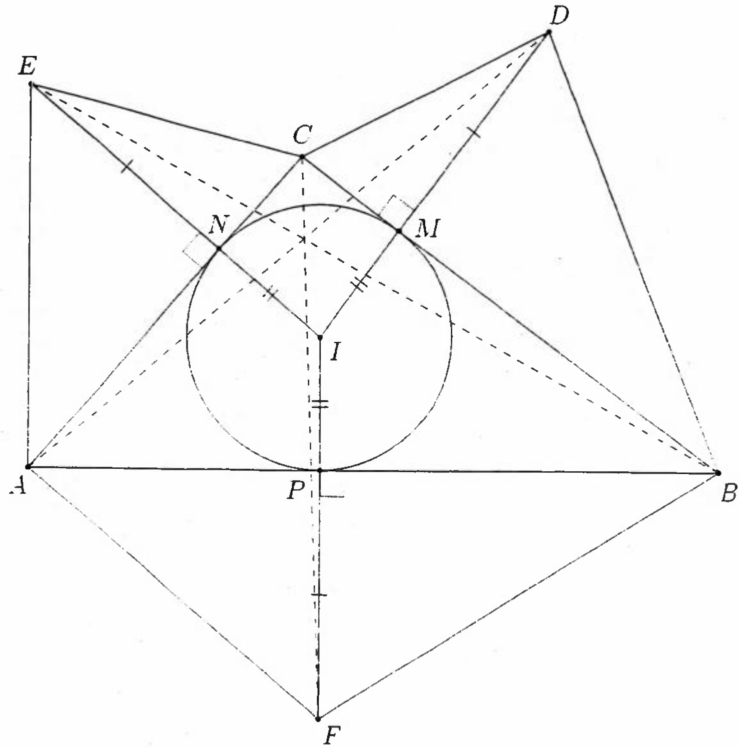
$$\angle E'AF' = \angle XAF' + \angle XAE' = 2(\angle EAX + \angle XAF) = 2\alpha$$

and analogously, $\angle E'CD' = 2\gamma$.

Hence, $\angle AKC = 180^\circ - (90^\circ - \alpha) - (90^\circ - \gamma) = \alpha + \gamma$ and thus K lies on k. Analogously, the intersection point of k_2 and k_3 and that of k_1 and k_2 lie on k. The statement follows because k_1, k_2 and k_3 cannot have two common intersection points.

Problem 4.9.28. Let ABC be a triangle with incenter I. The points F, D and E are chosen on the perpendicular lines from I to the sides AB, BC and CA, such that their distances to I are equal. Prove that the lines AD, BE and CF are concurrent.

Solution. Let the incircle of $\triangle ABC$ touch the sides BC, CA and AB at the points M, N and P, respectively. Then it is easy to see that $\triangle CMD \cong \triangle CNE$, $\triangle ANE \cong \triangle APF$ and $\triangle BPF \cong \triangle MBD$.



Hence, $CD = CE$, $EA = FA$, $FB = BD$, $\angle ECB = \angle ACD$, $\angle EAB = \angle FAC$ and $\angle FBC = \angle DBA$. The law of sines gives

$$\begin{aligned}&\frac{\sin\angle CAD}{\sin\angle DAB}\cdot\frac{\sin\angle EBA}{\sin\angle EBC}\cdot\frac{\sin\angle FCB}{\sin\angle FCA}\\ &=\frac{CD}{DB}\cdot\frac{EA}{EC}\cdot\frac{BF}{FA}\cdot\frac{\sin\angle ACD}{\sin\angle ABD}\cdot\frac{\sin\angle EAB}{\sin\angle ECB}\cdot\frac{\sin\angle FBC}{\sin\angle FAC}=1.\end{aligned}$$

The trigonometric form of Ceva's Theorem, applied to $\triangle ABC$, yields that the lines AD, BE and CF are concurrent.

Problem 4.9.29. Let ABC be a triangle. Let X be an arbitrary point, not lying on any of the sides of the triangle. Denote $D = AX \cap BC$, $E = BX \cap AC$ and $F = CX \cap AB$. The circles k_1, k_2 and k_3 are external to $\triangle ABC$. They touch the circumcircle of $\triangle ABC$ internally at the points M, N and P, and the sides BC, AC and AB at the points D, E and F, respectively. Prove that the lines AM, BN and CP are concurrent.

Solution. By Problem 6.1.1 we have that MD passes through the midpoint of the arc $\widehat{BAC}$ and hence it bisects $\angle BMC$.

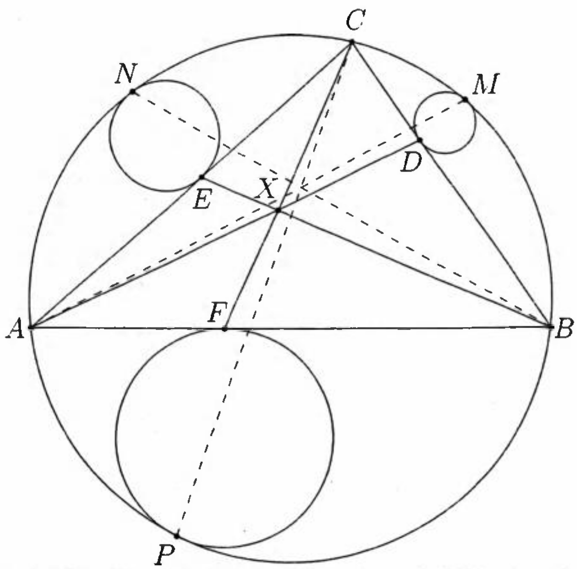
Analogously, NE bisects $\angle ANC$ and PF bisects $\angle APB$.

We have

$$\prod_{cyc} \frac{\sin \angle BAM}{\sin \angle MAC} =$$

$$= \prod_{cyc} \frac{BM}{MC} = \prod_{cyc} \frac{BD}{DC} = 1,$$

because AD, BE and CF are concurrent.

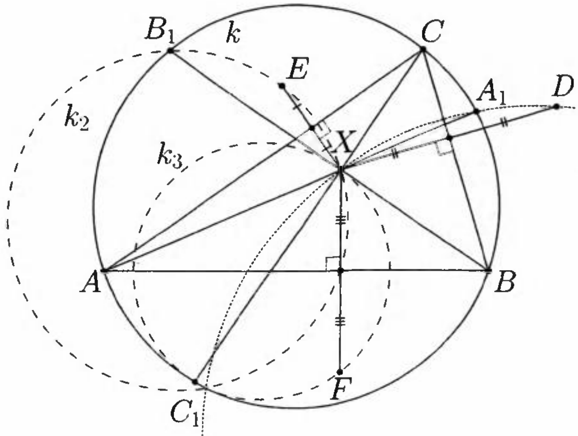


Now the trigonometric form of Ceva's Theorem gives the desired result.

Remark. Depending on the point X, the angle bisectors can be internal or external. In all cases, the proof follows the arguments above.

Problem 4.9.30. Let ABC be a triangle with circumcircle k. Let X be an arbitrary point, not lying on any of the sides of the triangle. Denote $XA \cap k = \{X, A_1\}$, $XB \cap k = \{X, B_1\}$ and $XC \cap k = \{X, C_1\}$. The points D, E and F are the reflections of X with respect to BC, CA and AB, respectively. Prove that the circumcircles of $\triangle XA_1D$, $\triangle XB_1E$ and $\triangle XC_1F$ intersect on k.

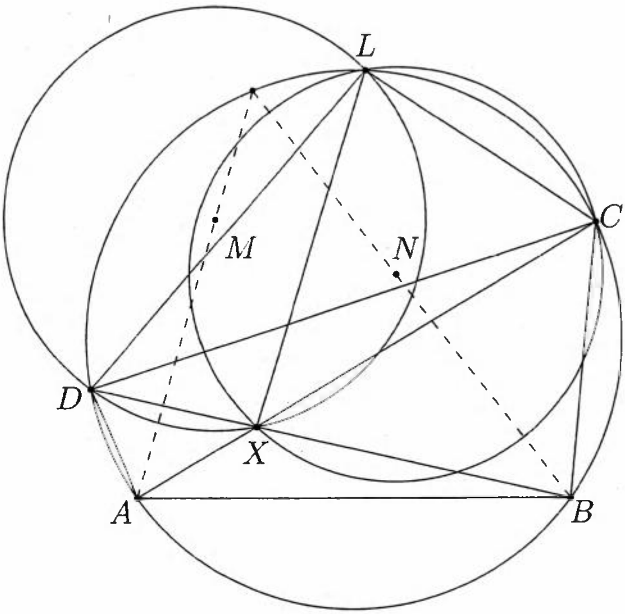
Solution. If X lies on k, the statement is clearly true. Let $X \notin k$. Denote the circumcircles of $\triangle XA_1D$, $\triangle XB_1E$ and $\triangle XC_1F$ by k_1, k_2 and k_3, respectively.



It suffices to show that two of the circles intersect on k (similarly, another two pairs of circles intersect on k, and this solves the problem because $X \notin k$).

Consider k_1 and k_2. An inversion centered at X with an arbitrary radius maps the points E and D into the circumcenters of $\triangle A'C'X$ and

$\triangle XB'C'$, where the prime symbol is used to denote the image under the inversion. Now we will prove a lemma. The problem turns into the lemma after the inversion has been applied (the notations are a bit different).



Lemma. Let $ABCD$ be a cyclic quadrilateral with diagonals intersecting at the point X. The point L is chosen arbitrarily on the arc $\widehat{CD}$. Prove that if M and N are the circumcenters of $\triangle DLX$ and $\triangle CLX$, then MA and NB intersect on k.

Proof. We have

$$\begin{aligned}\frac{\sin\angle DAM}{\sin\angle MAX} &= \frac{\sin\angle DAM}{\sin\angle MDA}\cdot\frac{\sin\angle MDA}{\sin\angle MXA}\cdot\frac{\sin\angle MXA}{\sin\angle MAX}\\ &= \frac{DM}{MA}\cdot\frac{\sin\angle MDA}{\sin\angle MXA}\cdot\frac{MA}{MX} = \frac{\sin\angle MDA}{\sin\angle MXA}\end{aligned}$$

and

$$\begin{aligned}\frac{\sin\angle XBN}{\sin\angle NBC} &= \frac{\sin\angle XBN}{\sin\angle NXB}\cdot\frac{\sin\angle NXB}{\sin\angle NCB}\cdot\frac{\sin\angle NCB}{\sin\angle CBN}\\ &= \frac{XN}{NB}\cdot\frac{\sin\angle NXB}{\sin\angle NCB}\cdot\frac{NB}{NC} = \frac{\sin\angle NXB}{\sin\angle NCB}.\end{aligned}$$

Also,

$$\begin{aligned}
&2\sin\angle MDA \sin\angle NCB \\
&= -\cos(\angle MDA + \angle NCB) + \cos(\angle MDA - \angle NCB) \\
&= -\cos(\angle MDA + \angle NCB) + \cos(\angle MDX - \angle NXC)
\end{aligned}$$

and

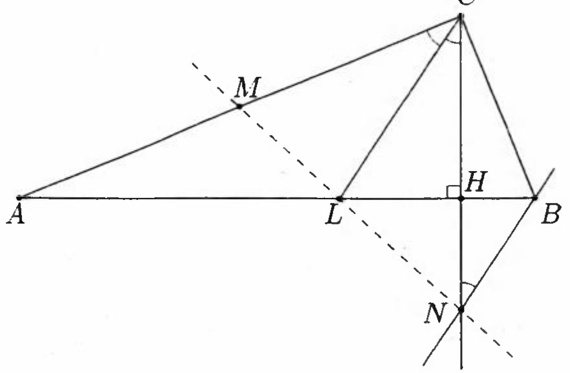
$$\begin{aligned}
&2\sin\angle MXA \sin\angle NXB \\
&= -\cos(\angle MXA + \angle NXB) + \cos(\angle MXA - \angle NXB) \\
&= -\cos(\angle MXA + \angle NXB) + \cos(\angle MDX - \angle NXC).
\end{aligned}$$

Finally, note that

$$\angle MXA + \angle NXB + \angle MDA + \angle NCB = 360^\circ.$$

Hence, the cosines of the angles $\angle MXA + \angle NXB$ and $\angle MDA + \angle NCB$ are equal.

Problem 4.10.1. Let ABC be a triangle with altitude $CH (H \in AB)$. Let CL $(L \in AB)$ be the angle bisector of $\angle ACH$. Let the line through B, that is parallel to CL, intersect CH at the point N. Let M be the midpoint of AC. Prove that the points M, L and N are collinear.



Solution. If $\angle BAC = \alpha$, then $\angle ABC = 90° - \alpha$, $\angle LCH = 45° - \dfrac{\alpha}{2}$, $\angle CLH = 45° + \dfrac{\alpha}{2}$ and $\angle LCB = 45° + \dfrac{\alpha}{2}$. Therefore, $\triangle LBC$ is isosceles.

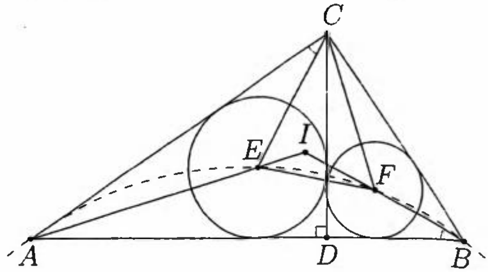
Consider the product $\dfrac{CN}{NH} \cdot \dfrac{HL}{LA} \cdot \dfrac{AM}{MC} = \dfrac{CN}{NH} \cdot \dfrac{HL}{LA}$.

The angle bisector theorem, applied to $\triangle AHC$, and the intercept theorem yield that the product above is equal to

$$\frac{LB}{BH} \cdot \frac{CH}{CA} = \frac{BC}{BH} \cdot \frac{CH}{CA} = \frac{1}{\sin\alpha} \cdot \frac{\sin\alpha}{1} = 1.$$

Menelaus' Theorem, applied to $\triangle AHC$ and the points M, L and N, yields that these points are collinear.

Problem 4.10.2. Let ABC be a triangle with $\angle ACB = 90°$, and let CD $(D \in AB)$ be an altitude. The incenters of $\triangle ADC$ and $\triangle BDC$ are E and F, respectively. Prove that the quadrilateral $ABFE$ is cyclic.



Solution. Denote $\angle ABF = \alpha$. Now, $\angle ABC = 2\alpha$ and $\angle ACD = \angle ABC = 2\alpha$.

We have that $\angle AEC = 135°$. The law of sines, applied to $\triangle AEC$ yields

$$\frac{AC}{\sin 45^\circ} = \frac{CE}{\sin(45^\circ - \alpha)}.$$

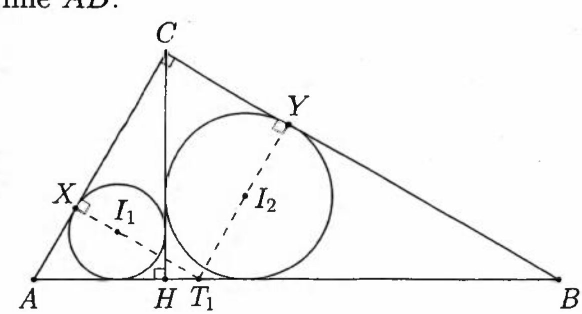
Therefore, $CE = \sqrt{2}AC.\sin(45^\circ - \alpha)$. Analogously, we get that $CF = \sqrt{2}BC.\sin\alpha$.

Note that $\angle ECD + \angle FCD = \dfrac{\angle ACB}{2} = 45^\circ$. The cotangent technique, applied to $\triangle EFC$ yields

$$\begin{aligned}\cot\angle CEF &= \frac{\dfrac{CE}{CF} - \cos\angle ECF}{\sin\angle ECF} = \frac{\dfrac{AC.\sin(45^\circ-\alpha)}{BC.\sin\alpha} - \cos 45^\circ}{\sin 45^\circ}\\ &= \frac{\sqrt{2}.\tan 2\alpha.\sin(45^\circ-\alpha)}{\sin\alpha} - 1 = \frac{2\cos\alpha(\cos\alpha - \sin\alpha)}{\cos^2\alpha - \sin^2\alpha} - 1\\ &= \frac{2\cos\alpha}{\cos\alpha + \sin\alpha} - 1 = \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \frac{\sin(45^\circ - \alpha)}{\sin(45^\circ + \alpha)}\\ &= \cot(45^\circ + \alpha).\end{aligned}$$

Hence, $\angle CEF = 45^\circ + \alpha$ and $\angle AEF = 360^\circ - 135^\circ - 45^\circ - \alpha = 180^\circ - \alpha$, which means that $\angle AEF + \angle ABF = 180^\circ - \alpha + \alpha = 180^\circ$ and thus the quadrilateral $ABFE$ is cyclic.

Problem 4.10.3. Let ABC be a triangle with $\angle ACB = 90^\circ$, and let CH ($H \in AB$) be an altitude. The incenters of $\triangle AHC$ and $\triangle BHC$ are I_1 and I_2, respectively. The points X and Y lie on the sides AC and BC, respectively, such that $I_1X \perp AC$ and $I_2Y \perp BC$. Prove that the lines XI_1 and YI_2 intersect on the line AB.



Solution. Denote $XI_1 \cap AB = T_1$. Thus, $T_1X \parallel BC$. The intercept theorem, applied to $\triangle ABC$ and the points T_1 and X, yields

$$\frac{AT_1}{AB} = \frac{AX}{AC}.$$

Let $YI_2 \cap AB = T_2$ and $T_2Y \parallel AC$. The intercept theorem, applied to $\triangle ABC$ and the points T_2 and Y, yields

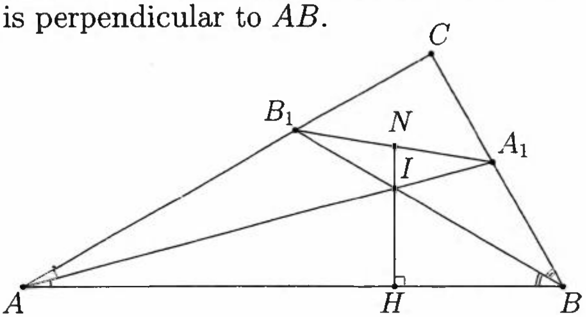
$$\frac{AT_2}{AB} = \frac{CY}{CB}.$$

The points X and Y are corresponding elements in the similar triangles $\triangle AHC$ and $\triangle CHB$. Thus,

$$\frac{AT_1}{AB} = \frac{AX}{AC} = \frac{CY}{CB} = \frac{AT_2}{AB}.$$

Hence, $T_2 \equiv T_1$ as desired.

Problem 4.10.4. Let ABC be a triangle with $\angle ACB = 90^\circ$ and angle bisectors AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$), which intersect at the point I. Prove that the line, defined by I and the midpoint of the segment A_1B_1, is perpendicular to AB.



Solution. Let H be the foot of the perpendicular from I to AB and let $IH \cap A_1B_1 = N$.

The law of sines, applied to $\triangle AIB$, yields $\dfrac{AI}{BI} = \dfrac{\sin\left(45^\circ - \dfrac{\alpha}{2}\right)}{\sin\dfrac{\alpha}{2}}$.

The same theorem, applied to $\triangle AB_1I$ and to $\triangle A_1BI$, yields $B_1I = \dfrac{AI.\sin\dfrac{\alpha}{2}}{\cos\left(45^\circ - \dfrac{\alpha}{2}\right)}$ and $A_1I = \dfrac{BI.\sin\left(45^\circ - \dfrac{\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}}$.

Thus,

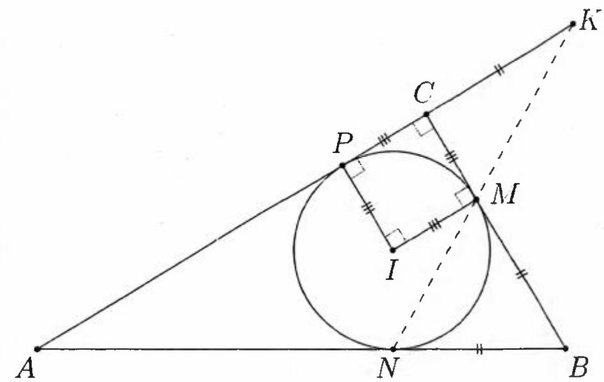
$$\frac{B_1I}{A_1I} = \frac{AI}{BI}\cdot\frac{\sin\dfrac{\alpha}{2}.\cos\dfrac{\alpha}{2}}{\sin\left(45^\circ - \dfrac{\alpha}{2}\right).\sin\left(45^\circ + \dfrac{\alpha}{2}\right)} = \frac{\cos\dfrac{\alpha}{2}}{\sin\left(45^\circ + \dfrac{\alpha}{2}\right)}.$$

Hence,

$$B_1I.IN.\sin\left(45^\circ + \frac{\alpha}{2}\right) = A_1I.IN.\sin\left(90^\circ - \frac{\alpha}{2}\right) \Rightarrow S_{\triangle A_1IN} = S_{\triangle B_1IN}.$$

It follows that $A_1N = B_1N$, and so N is the midpoint of A_1B_1, as desired.

Problem 4.10.5. Let ABC be a triangle with $\angle ACB = 90°$ and incircle ω, which touches AB, BC and CA at the points N, M and P, respectively. A point K is chosen on the line AC, such that $CK = BM$ and C lies between K and A. Prove that the points M, N and K are collinear.



Solution. Let I be the center of ω. It is easy to see that the quadrilateral $IMCP$ is a square.

Since $BM = BN = s - b$, then $CK = s-b$ and thus $AK = b + s - b = s$.

Furthermore, $CM = s - c$, $MB = s - b = BN$ and $AN = s - a$.

We will prove that $\dfrac{AK}{KC} \cdot \dfrac{CM}{MB} \cdot \dfrac{BN}{NA} = 1$. It is equivalent to proving that

$$\frac{s}{s-b} \cdot \frac{s-c}{s-b} \cdot \frac{s-b}{s-a} = 1 \Leftrightarrow \frac{s}{s-b} \cdot \frac{s-c}{s-a} = 1.$$

The latter is equivalent to proving that $a^2 + b^2 = c^2$, which is true by the Pythagorean Theorem. Menelaus' Theorem, applied to $\triangle ABC$ and the points N, M and K, yields that these points are collinear.

Problem 4.10.6. Let ABC be a triangle with $\angle BAC = 90°$. Let M and N be the midpoints of AC and BC, respectively. The point $D \in BC$ is arbitrary. Let k be the circumcircle of $\triangle ABD$. The tangent lines to k at the points A and D intersect at the point F. Prove that the points N, M and F are collinear.

Solution. Let k intersect AC again at the point E, and let $DE \cap AB = P$. We have that the polar lines of the points C, F and P pass through the intersection point of the diagonals of the quadrilateral $ABDE$. Therefore, the points C, F and P are collinear.

We have that $\angle PDC = \angle PAC = 90°$. Thus, the quadrilateral $PADC$ is cyclic. Now we have

$$\angle PCA = \angle PDA = \angle EDA = \angle CAF \Rightarrow CF = FA.$$

Since $\triangle PAC$ is right-angled we conclude that $CF = FA = FP$.

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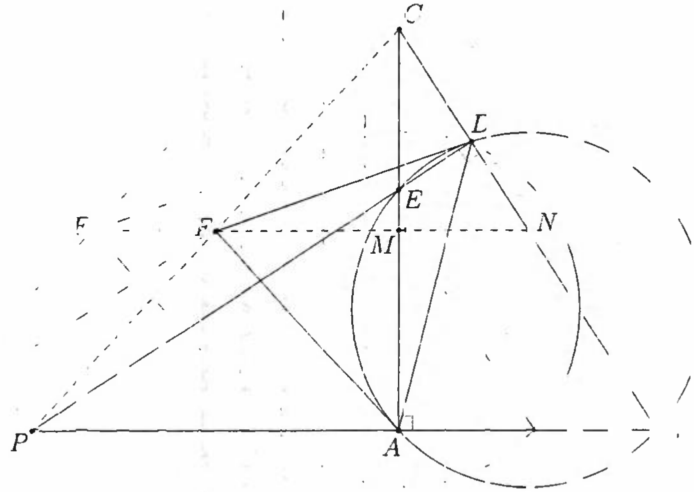
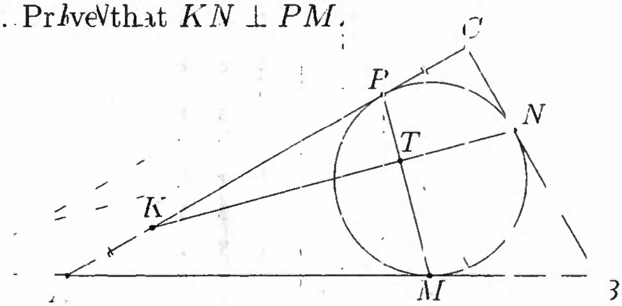


Considering the midsegments of $\triangle APC$ and $\triangle ABC$ we get that $FM \parallel PA$ and $MN \parallel AB$, so the points F, M and N are collinear.

Problem 4.107. Let ABC be a triangle with $\angle$ [illegible] 90° and an incircle ω that touches AB, BC and CA at the points M, N and P, respectively. Let K be the reflection of P with respect to the midpoint of the side AC. Prove that $KN \perp PM$.

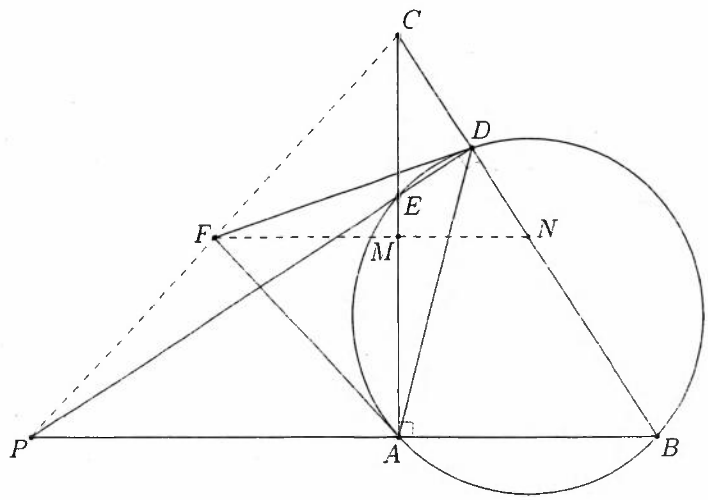


Solution. Considering $\triangle AMP$, we have that $AP = s - a$, $\angle PAM = \alpha$ and $\angle APM = 90^\circ - \dfrac{\alpha}{2}$. Let $T = PM \cap KN$. $\angle KPT = 90^\circ - \dfrac{\alpha}{2}$.

Denote $\angle NKC = x$. We have $AP = KC = s - a$. On the other hand, $CN = s - c$.

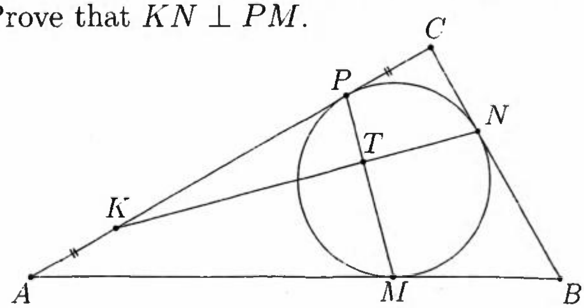
Thus, $\tan x = \dfrac{CN}{KC} = \dfrac{4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}}{4R\cos\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}} =$ [illegible].

Hence, $\angle PTK = 180^\circ - \angle TKP - \angle TPK = 180^\circ -$ [illegible] $= 90^\circ$.



Considering the midsegments of $\triangle APC$ and $\triangle ABC$, we get that $FM \parallel PA$ and $MN \parallel AB$, so the points F, M and N are collinear.

Problem 4.10.7. Let ABC be a triangle with $\angle ACB = 90°$ and an incircle ω that touches AB, BC and CA at the points M, N and P, respectively. Let K be the reflection of P with respect to the midpoint of the side AC. Prove that $KN \perp PM$.



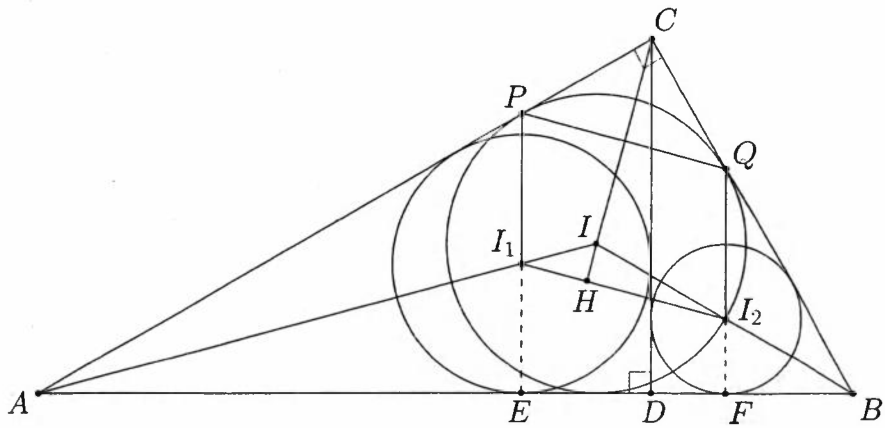
Solution. Considering $\triangle AMP$, we have that $AP = AM = s - a$. $\angle PAM = \alpha$ and $\angle APM = 90° - \dfrac{\alpha}{2}$. Let $T = PM \cap KN$. Then $\angle KPT = 90° - \dfrac{\alpha}{2}$.

Denote $\angle NKC = x$. We have $AP = KC = s - a$. On the other hand, $CN = s - c$.

Thus, $\tan x = \dfrac{CN}{KC} = \dfrac{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}{4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = \tan\dfrac{\alpha}{2}$ and so $x = \dfrac{\alpha}{2}$.

Hence, $\angle PTK = 180° - \angle TKP - \angle TPK = 180° - \dfrac{\alpha}{2} - \left(90° - \dfrac{\alpha}{2}\right) = 90°$.

Problem 4.10.8. Let ABC be a triangle with $\angle ACB = 90°$. Let CD be its altitude. Let I_1 and I_2 be the incenters of $\triangle ADC$ and $\triangle BDC$, respectively. Let P and Q be the points of tangency of the sides AC and BC to the incircle of $\triangle ABC$, respectively. Prove that the quadrilateral I_1I_2QP is a parallelogram.



Solution. Let I be the incenter of $\triangle ABC$. Let the projections of I_1 and I_2 onto AB be E and F, respectively.

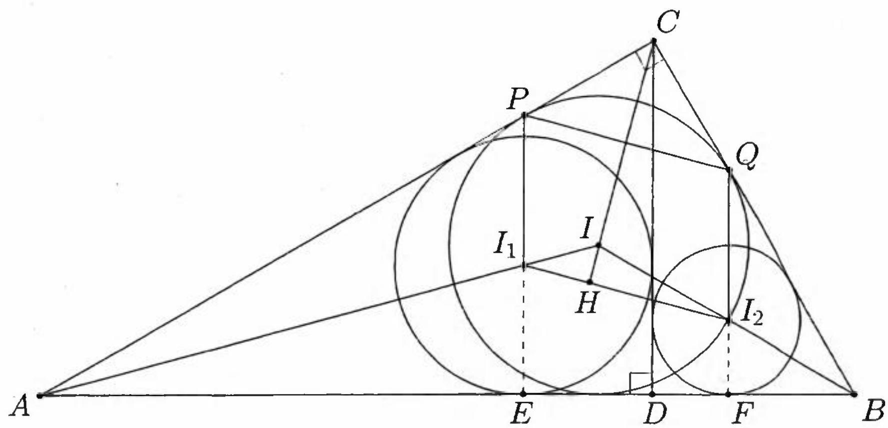
We have that $\triangle BDC \sim \triangle BCA$ with a ratio of $\cos\beta$. The points F and Q are corresponding points in the two triangles, so $\dfrac{BF}{BQ} = \cos\beta$. Thus, $\angle QFB = 90°$ and hence the points Q, I_2 and F are collinear.

Analogously, the points P, I_1 and E are collinear. Therefore, $QI_2 \parallel PI_1$. We have that $CI \perp PQ$. We will prove that $I_1I_2 \perp CI$ and the desired $PQ \parallel I_1I_2$ will follow.

Let $CI \cap I_1I_2 = H$. Problem 4.10.2 yields that ABI_2I_1 is cyclic. Then

$$\angle IHI_2 = \angle BIC - \angle II_2I_1 = 90° + \frac{\angle BAC}{2} - \angle BAI = 90°.$$

Problem 4.10.8. Let ABC be a triangle with $\angle ACB = 90^\circ$. Let CD be its altitude. Let I_1 and I_2 be the incenters of $\triangle ADC$ and $\triangle BDC$, respectively. Let P and Q be the points of tangency of the sides AC and BC to the incircle of $\triangle ABC$, respectively. Prove that the quadrilateral I_1I_2QP is a parallelogram.



Solution. Let I be the incenter of $\triangle ABC$. Let the projections of I_1 and I_2 onto AB be E and F, respectively.

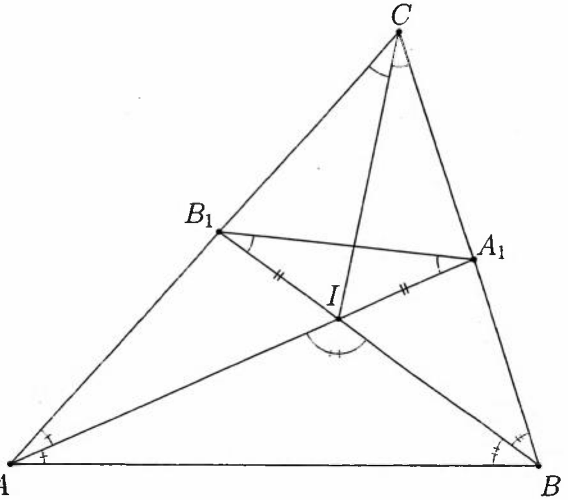
We have that $\triangle BDC \sim \triangle BCA$ with a ratio of $\cos\beta$. The points F and Q are corresponding points in the two triangles, so $\frac{BF}{BQ} = \cos\beta$. Thus, $\angle QFB = 90^\circ$ and hence the points Q, I_2 and F are collinear.

Analogously, the points P, I_1 and E are collinear. Therefore, $QI_2 \parallel PI_1$. We have that $CI \perp PQ$. We will prove that $I_1I_2 \perp CI$ and the desired $PQ \parallel I_1I_2$ will follow.

Let $CI \cap I_1I_2 = H$. Problem 4.10.2 yields that ABI_2I_1 is cyclic. Then

$$\angle IHI_2 = \angle BIC - \angle II_2I_1 = 90^\circ + \frac{\angle BAC}{2} - \angle BAI = 90^\circ.$$

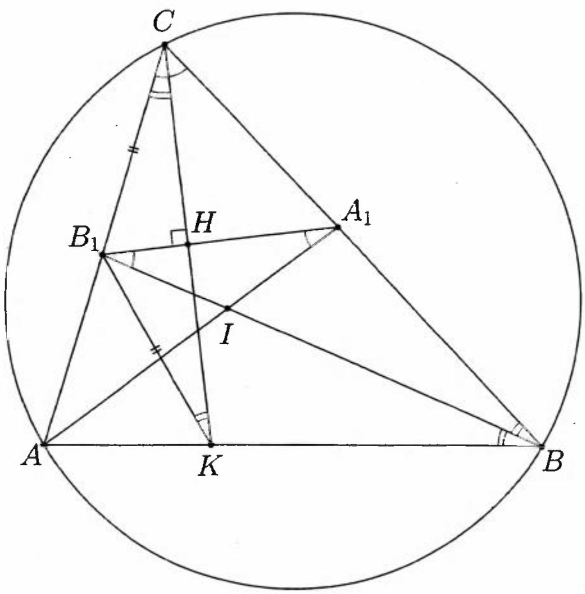
Problem 4.11.1. Let ABC be a triangle with $\angle ACB = 60°$. The angle bisectors AA_1 $(A_1 \in BC)$ and BB_1 $(B_1 \in AC)$ intersect at the point I. Prove that $A_1I = B_1I$.



Solution. It is clear that $\angle A_1IB_1 = 120°$.

Then the quadrilateral A_1CB_1I is cyclic. Since CI bisects $\angle A_1CB_1$, it follows that $\widehat{A_1I} = \widehat{B_1I}$. Therefore, $A_1I = B_1I$.

Problem 4.11.2. Let ABC be a triangle with $\angle ACB = 60°$. The angle bisectors AA_1 $(A_1 \in BC)$ and BB_1 $(B_1 \in AC)$ intersect at the point I. Prove that the reflection of C with respect to the line A_1B_1 lies on the line AB.



Solution. Let $H \in A_1B_1$ be a point such that $CH \perp A_1B_1$ and $CH \cap AB = K$. It suffices to prove that $CH = HK$.

We have $\angle B_1CH = 90^\circ - \angle A_1B_1C$. As in Problem 4.11.1, we see that the quadrilateral CB_1IA_1 is cyclic, which implies $\angle B_1CH = 90^\circ - \angle CIA_1$.

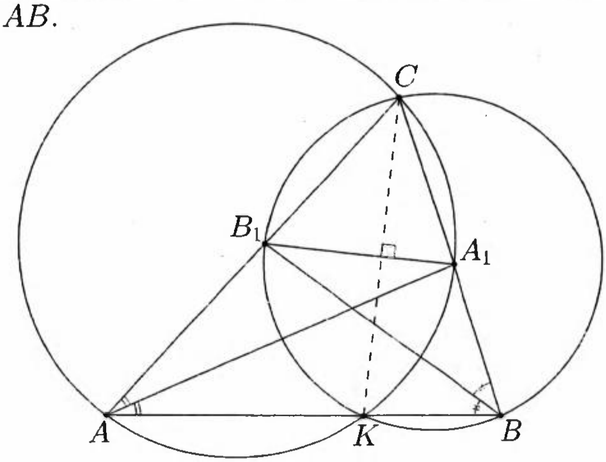
On the other hand,

$$\begin{aligned}\angle B_1BK = \angle B_1BC &= 180^\circ - \angle ICB - \angle CIA_1 - \angle A_1IB \\ &= 180^\circ - 30^\circ - \angle CIA_1 - 60^\circ \\ &= 90^\circ - \angle CIA_1 = \angle B_1CH.\end{aligned}$$

Therefore, the quadrilateral BKB_1C is cyclic and $B_1K = B_1C$. Thus, $CH = HK$.

Problem 4.11.3. Let ABC be a triangle with $\angle ACB = 60^\circ$. The angle bisectors AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) intersect at the point I. Prove that the second intersection point of the circumcircles of $\triangle AA_1C$ and $\triangle BB_1C$ lies on the line AB.

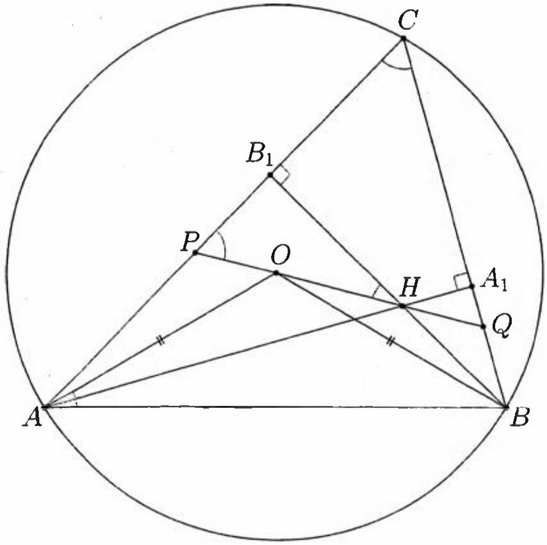
Solution. Let K be the reflection of C with respect to A_1B_1. Problem 4.11.2 implies that K lies on AB. In the proof of 4.11.2 we showed that K lies on the circumcircles of $\triangle AA_1C$ and $\triangle BB_1C$, as desired.



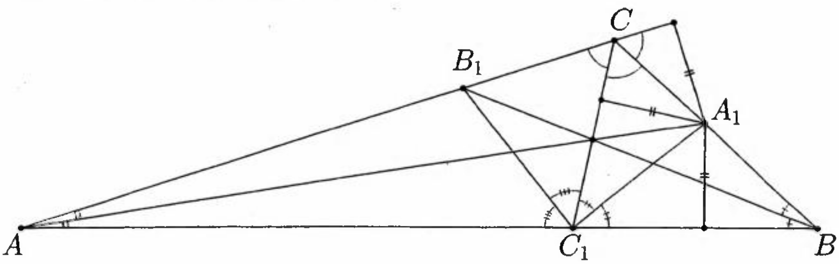
Problem 4.11.4. Let ABC be a triangle with circumcenter O and $\angle ACB = 60^\circ$. Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in the triangle, intersecting at the point H. The line OH intersects the lines AC and BC at the points P and Q. Prove that $\triangle PQC$ is equilateral.

Solution. We have that $\angle AOB = \angle AHB = 120^\circ$, so the quadrilateral $ABHO$ is cyclic. Also, $\angle BAO = 30^\circ$, thus $\angle AHO = 30^\circ$.

Note that $\angle AHB_1 = 60^\circ$. Therefore, $\angle B_1HP = 30^\circ$, which implies that $\angle QPC = 60^\circ$ and that $\triangle PQC$ is equilateral.



Problem 4.11.5. Let ABC be a triangle with $\angle ACB = 120°$. The lines AA_1 $(A_1 \in BC)$, BB_1 $(B_1 \in AC)$ and CC_1 $(C_1 \in AB)$ are its angle bisectors. Prove that $\angle A_1C_1B_1 = 90°$.



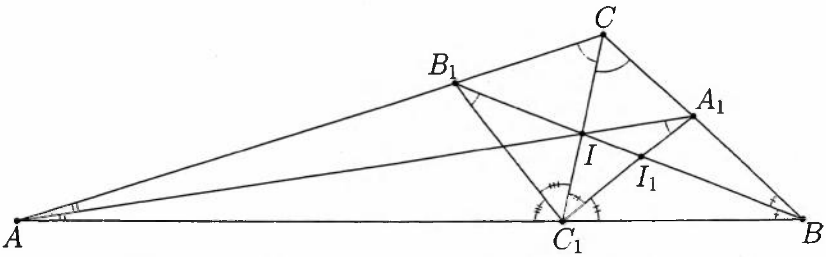
Solution. It is clear that CA_1 is an external angle bisector of $\triangle ACC_1$. Also, the point A_1 lies on the bisector of $\angle BAC$.

Therefore, A_1 is the A-excenter of $\triangle ACC_1$. Thus, C_1A_1 is the angle bisector of $\angle BC_1C$.

Analogously, C_1B_1 is the angle bisector of $\angle AC_1C$. Hence, $\angle A_1C_1B_1 = \angle B_1C_1C + \angle A_1C_1C = \frac{1}{2}\angle AC_1B = 90°$.

Problem 4.11.6. Let ABC be a triangle with $\angle ACB = 120°$. The lines AA_1 $(A_1 \in BC)$, BB_1 $(B_1 \in AC)$ and CC_1 $(C_1 \in AB)$ are its angle bisectors. Prove that $\angle AA_1C_1 = BB_1C_1 = 30°$.

Solution. Let I be the incenter of $\triangle ABC$. As in the proof of 4.11.5, we can show that C_1A_1 bisects $\angle BC_1C$. Let $C_1A_1 \cap BI = I_1$.

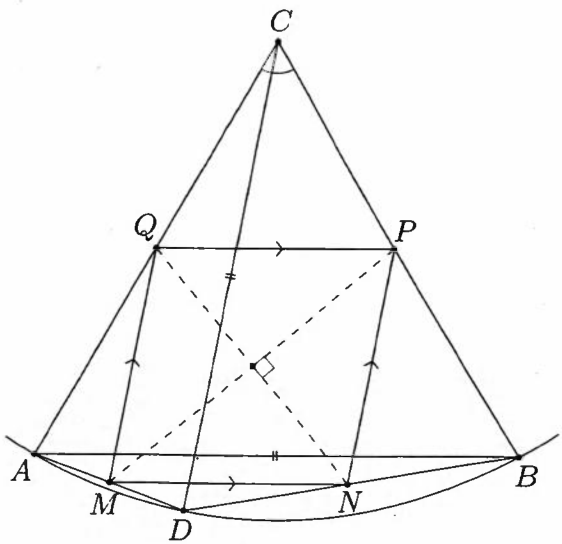


Then the point I_1 is the incenter of $\triangle BCC_1$. Since $\angle BCC_1 = 60°$, the quadrilateral CII_1A_1 is cyclic. Hence, $\angle IAI_1 = \angle ICI_1 = 30°$. Analogously, $\angle IB_1C_1 = 30°$.

Problem 4.11.7. Let k be a circle with center C. The points A and B on k are such that $\angle ACB = 60°$. For an arbitrary point $D \in \widehat{AB}$, the points M, N, P and Q are the midpoints of the segments AD, DB, BC and CA. Prove that $MP \perp QN$.

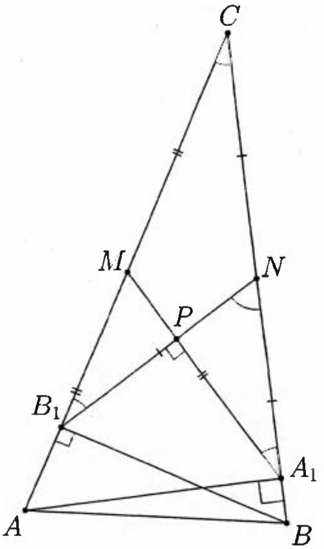
Solution. The given conditions imply that $\triangle ABC$ is equilateral. Note that PQ and MN are midsegments in $\triangle ABC$ and $\triangle ABD$, respectively.

Thus, $PQ \parallel AB \parallel MN$ and $PQ = MN = \dfrac{AB}{2}$. On the other hand, $CD = AB$. We have that MQ and PN are midsegments in $\triangle ADC$ and $\triangle BDC$, respectively.



We deduce that $MQ \parallel CD \parallel NP$ and $MQ = NP = \dfrac{CD}{2} = \dfrac{AB}{2} = MN$. Therefore, $MNPQ$ is a rhombus and $MP \perp QN$.

Problem 4.11.8. Let ABC be a triangle with $\angle ACB = 30°$. Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in the triangle, and let M and N be the midpoints of AC and BC, respectively. Prove that $A_1M \perp B_1N$.

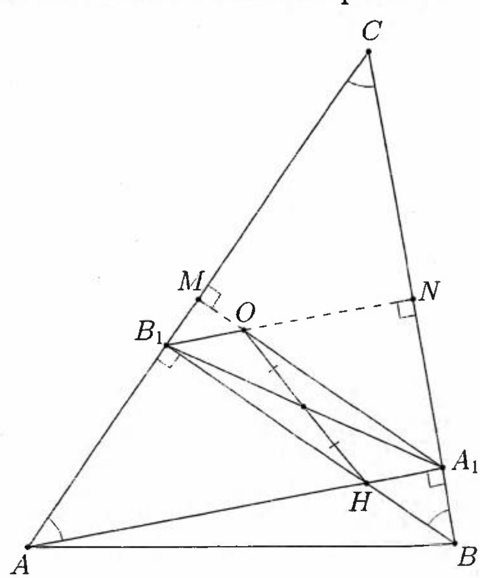


Solution. Since A_1M and B_1N are medians to the hypothenuses in $\triangle AA_1C$ and $\triangle BB_1C$, respectively, we have $\angle NB_1C = \angle ACB = \angle MA_1C = 30°$.

Hence, $\angle B_1NA_1 = 60°$. Now, if P is the intersection point of the lines A_1M and B_1N, we have $\angle PNA_1 = 60°$.

We have shown that $\angle MA_1C = 30°$ and so, $\angle PA_1N = 30°$. Therefore, $\angle A_1PN = 90°$.

Problem 4.11.9. Let ABC be a triangle with circumcenter O and $\angle ACB = 45°$. Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in the triangle, and let H be its orthocenter. Prove that the midpoint of OH lies on the line A_1B_1.



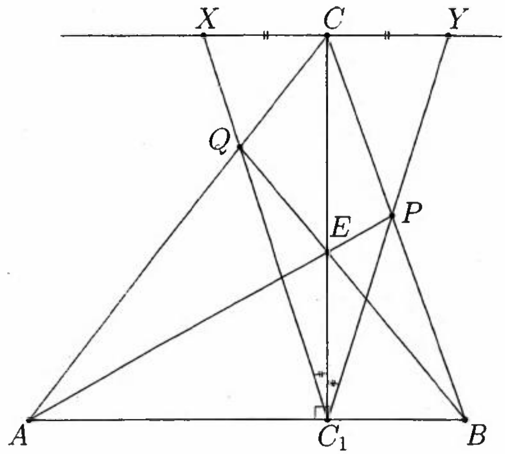
Solution. We have that $\angle A_1AC = 45°$ and $\angle B_1BC = 45°$.

Thus, $AA_1 = CA_1$ and $AO = CO$. Hence, the line A_1O is the perpendicular bisector of AC. In particular, $A_1O \parallel HB_1$.

Analogously, $B_1O \parallel HA_1$. The quadrilateral HA_1OB_1 is a parallelogram. Hence, the line A_1B_1 bisects the segment HO.

Problem 4.12.1. Let ABC be a triangle with an altitude CC_1 ($C_1 \in AB$). An arbitrary point P is chosen on the side BC. The lines AP and CC_1 intersect at the point E, and the line BE intersects AC at the point Q. Prove that C_1C is the angle bisector of $\angle PC_1Q$.

Solution. Consider the line l through C, parallel to AB. Let the lines C_1Q and C_1P intersect l at the points X and Y, respectively.



We have $\triangle AC_1Q \sim \triangle CXQ$ and $\triangle BC_1P \sim \triangle CYP$. These similarities give

$$CY = \frac{BC_1.PC}{BP}$$

and

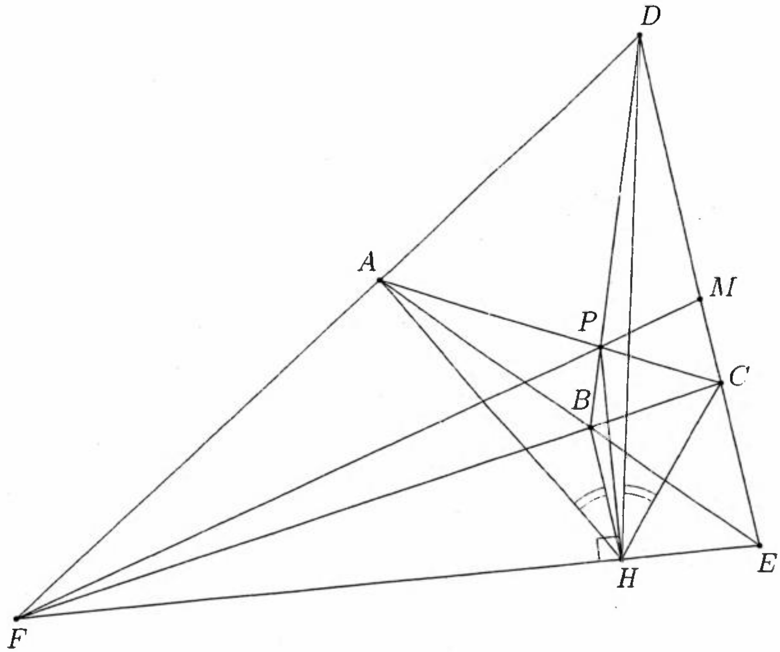
$$CX = \frac{AC_1.CQ}{AQ}.$$

Therefore, $\dfrac{CX}{CY} = \dfrac{AC_1}{C_1B} \cdot \dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} = 1$. The last equality follows from Ceva's Theorem, applied to $\triangle ABC$ and the points C_1, P and Q.

Thus, $CX = CY$. Note that $\angle XCC_1 = 90°$, due to $l \parallel AB$. Hence, in $\triangle XC_1Y$, the line C_1C is both a median and an altitude. Therefore, it also bisects $\angle PC_1Q$.

Problem 4.12.2. Let $ABCD$ be a quadrilateral. The rays AB and DC intersect at the point E and the rays DA and CB intersect at the point F. Let $AC \cap BD = P$ and let H be the foot of the perpendicular from P to EF. Prove that $\angle AHB = \angle DHC$.

Solution. Let $FP \cap DC = M$. We apply the law of sines to $\triangle CEH$, $\triangle CPH$, $\triangle HAP$, $\triangle HAF$ and $\triangle EDF$, as well as Menelaus' Theorem and Ceva's Theorem, and we get



$$\frac{\operatorname{tg}\angle EHC}{\operatorname{tg}\angle AHF} = \frac{\sin\angle EHC}{\sin\angle PHC}\cdot\frac{\sin\angle PHA}{\sin\angle AHF}$$

$$= \frac{CE}{CP}\cdot\frac{\sin\angle CEH}{\sin\angle CPH}\cdot\frac{AP}{AF}\cdot\frac{\sin\angle HPA}{\sin\angle HFA} = \frac{CE}{CP}\cdot\frac{\sin\angle CEH}{\sin\angle HFA}\cdot\frac{AP}{AF}$$

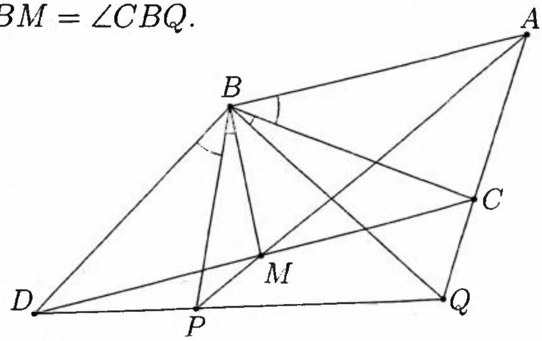
$$= \frac{CE}{CP}\cdot\frac{DF}{DE}\cdot\frac{AP}{AF} = \frac{CE}{ED}\cdot\frac{DF}{FA}\cdot\frac{AP}{PC}$$

$$= \frac{CE}{ED}\cdot\frac{DM}{MC} = \frac{FA}{AD}\cdot\frac{CB}{BF}\cdot\frac{FB}{BC}\cdot\frac{DA}{AF} = 1.$$

Therefore, $\angle EHC = \angle AHF$. Analogously, we get that $\angle EHD = \angle BHF$. Hence, $\angle AHB = \angle DHC$.

Note. The problem can also be solved using the properties of harmonic division.

Problem 4.12.3. The triangles ABC and BDP have equal angles at the vertex B and they have the same orientation. Let $DP \cap AC = Q$ and $AP \cap DC = M$. Prove that $\angle PBM = \angle CBQ$.



Solution. Let $\angle DBP = \angle CBA = \alpha$, $\angle PBM = \beta_1$ and $\angle QBC = \beta_2$. It suffices to prove that $\cot\beta_1 = \cot\beta_2$.

We have

$$\frac{\sin\angle DBM}{\sin\angle PBM} = \frac{DM}{PM}\cdot\frac{\sin\angle BDM}{\sin\angle BPM} = \frac{\sin\angle APQ}{\sin\angle CDQ}\cdot\frac{\sin\angle BDC}{\sin\angle BPA}$$

$$= \frac{\sin\angle APQ}{\sin\angle BPA}\cdot\frac{\sin\angle BDC}{\sin\angle CDQ} = \frac{AQ}{AB}\cdot\frac{\sin\angle AQD}{\sin\angle ABP}\cdot\frac{BC}{CQ}\cdot\frac{\sin\angle CBD}{\sin\angle AQD}$$

$$= \frac{AQ}{CQ}\cdot\frac{BC}{AB} = \frac{\sin\angle ABQ}{\sin\angle CBQ}\cdot\frac{AB}{BC}\cdot\frac{BC}{AB} = \frac{\sin\angle ABQ}{\sin\angle CBQ}.$$

Hence, $\dfrac{\sin(\alpha+\beta_1)}{\sin\beta_1} = \dfrac{\sin(\alpha+\beta_2)}{\sin\beta_2}$.

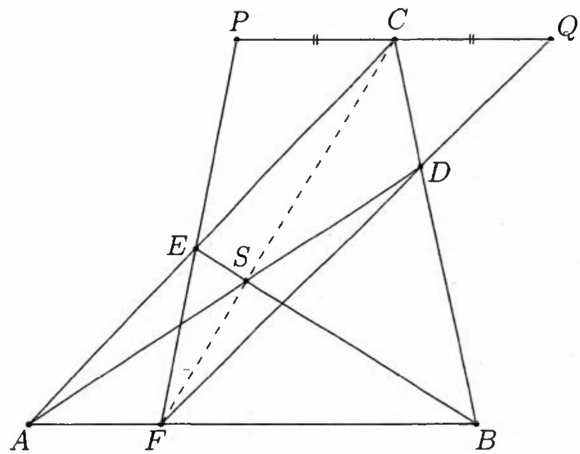
Therefore, $\sin\alpha\cot\beta_1 + \cos\alpha = \sin\alpha\cot\beta_2 + \cos\alpha$, which implies that $\cot\beta_1 = \cot\beta_2$.

Problem 4.12.4. Let D and E be arbitrary points on the sides BC and AC of the triangle ABC, respectively. The lines AD and BE intersect at the point S. Let $F \in AB$ be an arbitrary point, and let l be a line through C, parallel to AB. The lines FE and FD intersect l at the points P and Q, respectively. Prove that if $CP = CQ$, then the points C, S and F are collinear.

Solution. Note that $\triangle PEC \sim \triangle FEA$ and $\triangle CDQ \sim \triangle BDF$. Thus, $\dfrac{CP}{AF} = \dfrac{CE}{EA}$ and $\dfrac{CQ}{FB} = \dfrac{CD}{DB}$.

Hence,

$$\frac{CP}{CQ} = \frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC}.$$

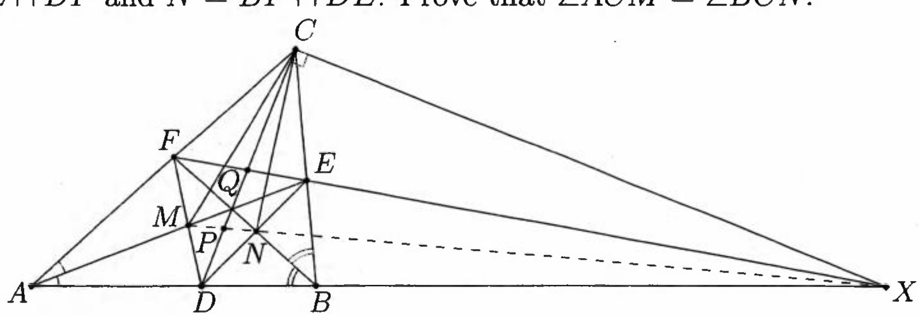


The condition $CP = CQ$ implies

$$\frac{CE}{EA} \cdot \frac{AF}{FB} \cdot \frac{BD}{DC} = 1.$$

Ceva's Theorem, applied to $\triangle ABC$ and the points F, D and E, yields that the lines AD, BE and CF are concurrent, so the points C, S and F are collinear.

Problem 4.12.5. The lines AE ($E \in BC$), BF ($F \in AC$) and CD ($D \in AB$) are the angle bisectors of the triangle ABC. Let $M = AE \cap DF$ and $N = BF \cap DE$. Prove that $\angle ACM = \angle BCN$.



Solution. We assume that the lines EF and AB are not parallel, because otherwise the desired equality easily follows.

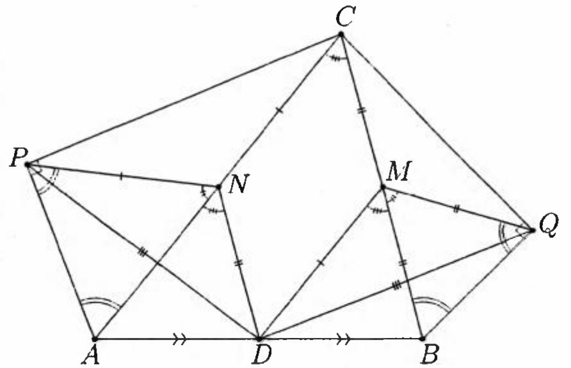
Let X be their intersection point. Problem 4.3.1 yields that CX is the external angle bisector of $\angle ACB$. Thus, $\angle DCX = 90°$.

Let $Q = FE \cap CD$. We have $(A, D, B, X) = 1 \Rightarrow (F, Q, E, X) = 1$ due to the projective property of the cross-ratio (Problem 7.10), applied to the point C. Since the lines DQ, EM and FN are concurrent and $(F, Q, E, X) = 1$, we deduce that the points M, N and X are collinear. Let $MN \cap CD = P$.

Again, the projective property of the cross-ratio with respect to the point D gives $(F, Q, E, X) = 1$. Hence, $(M, P, N, X) = 1$. Since $\angle PCX = 90°$, it follows that $\angle MCP = \angle NCP$.

Since CD is the angle bisector of $\angle ACB$, the desired equality follows.

Problem 4.12.6. The point D is the midpoint of the side AB of the triangle ABC. The triangles APC and BQC are constructed externally of the triangle, such that $\angle APC = \angle BQC = 90°$ and $\angle PAC = \angle QBC$. Prove that $PD = QD$.

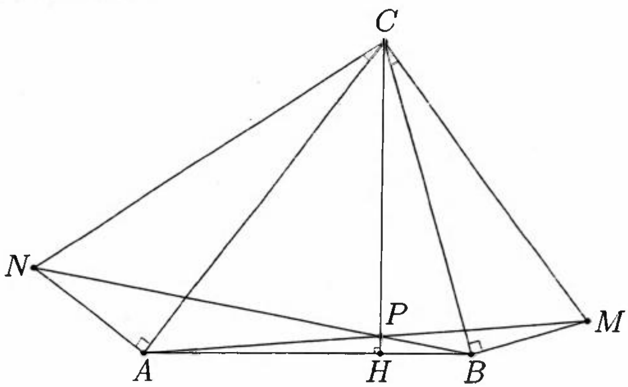


Solution. Let N and M be the midpoints of the sides AC and BC, respectively.

Since DN and DM are midsegments in $\triangle ABC$, we have $\angle AND = \angle BMD = \gamma$. Also, the triangles $\triangle PNA$ and $\triangle QMB$ are isosceles and $\angle PAC = \angle QBC$. Hence, $\angle PNA = \angle QMB$.

Therefore, $\angle PND = \angle DMQ$. Moreover, $MQ = BM = DN$ and $PN = AN = DM$. We deduce that $\triangle PND \cong \triangle DMQ$ and therefore $DP = DQ$.

Problem 4.12.7. Let ABC be a triangle. The points M and N are external for $\triangle ABC$ and such that the lines CN and CM are isogonally conjugate with respect to $\angle ACB$, and $\angle NAC = \angle MBC = 90°$. Let CH ($H \in AB$) be an altitude in $\triangle ABC$. Prove that the lines AM, BN and CH are concurrent.



Solution. Set $\angle NCA = \angle MCB = \varphi$. The trigonometric form of

Ceva's Theorem, applied twice to $\triangle ABC$, with respect to N and M, yields

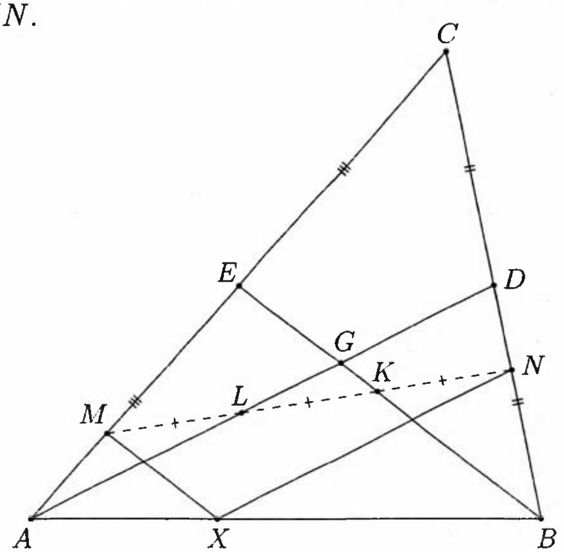
$$\frac{\sin\angle NBC}{\sin\angle NBA}=\frac{\sin(\gamma+\varphi)}{\cos\alpha\sin\varphi},$$
$$\frac{\sin\angle MAB}{\sin\angle MAC}=\frac{\cos\beta\sin\varphi}{\sin(\gamma+\varphi)}.$$

Let $P = BN \cap AM$. The same theorem, applied to $\triangle ABC$ and the point P, yields

$$\frac{\sin\angle CBP}{\sin\angle PBA}\cdot\frac{\sin\angle BAP}{\sin\angle CAP}\cdot\frac{\sin\angle ACP}{\sin\angle BCP}=1\Rightarrow\frac{\sin\angle ACP}{\sin\angle BCP}=\frac{\cos\alpha}{\cos\beta}.$$

Therefore, CP is an altitude in $\triangle ABC$.

Problem 4.12.8. Let ABC be a triangle. The points D and E are the midpoints of the sides BC and CA, respectively. The lines through an arbitrary point $X \in AB$, that are parallel to AD and BE, intersect BC and CA at the points N and M, respectively. The segment MN intersects AD and BE at the points L and K, respectively. Prove that $ML = LK = KN$.



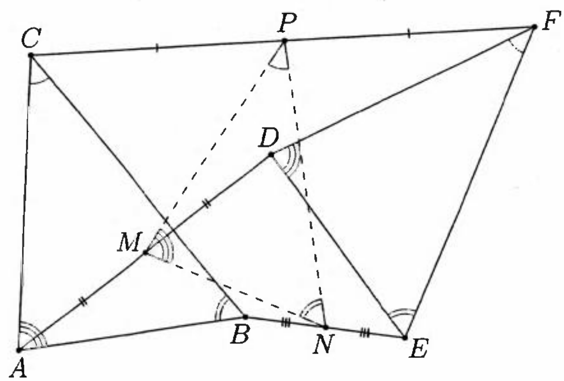
Solution. Menelaus' Theorem, applied to $\triangle MNC$ and the line EK, yields

$$1=\frac{ME}{EC}\cdot\frac{CB}{BN}\cdot\frac{NK}{KM}=\frac{ME}{EA}\cdot\frac{2BD}{BN}\cdot\frac{NK}{KM}$$
$$=\frac{XB}{AB}\cdot\frac{2AB}{BX}\cdot\frac{NK}{KM}=\frac{2NK}{KM},$$

which implies $KN = \dfrac{KM}{2}$, i.e. $KN = \dfrac{MN}{3}$.

Analogously, $ML = \dfrac{MN}{3}$, and we conclude that $ML = LK = KN$.

Problem 4.12.9. Let ABC and DEF be similar triangles. Denote the midpoints of the segments AD, BE and CF by M, N and P, respectively. Prove that $\triangle MNP \sim \triangle ABC$.



Solution. We will use rotation of vectors. When a vector is multiplied by $e^{i\varphi}$, it means that it is rotated by the angle φ counterclockwise. Let $\overrightarrow{AB}.k.e^{i\varphi} = \overrightarrow{DE}$. The given similarity implies $\overrightarrow{BC}.k.e^{i\varphi} = \overrightarrow{EF}$ and $\overrightarrow{CA}.k.e^{i\varphi} = \overrightarrow{FD}$.

Hence,

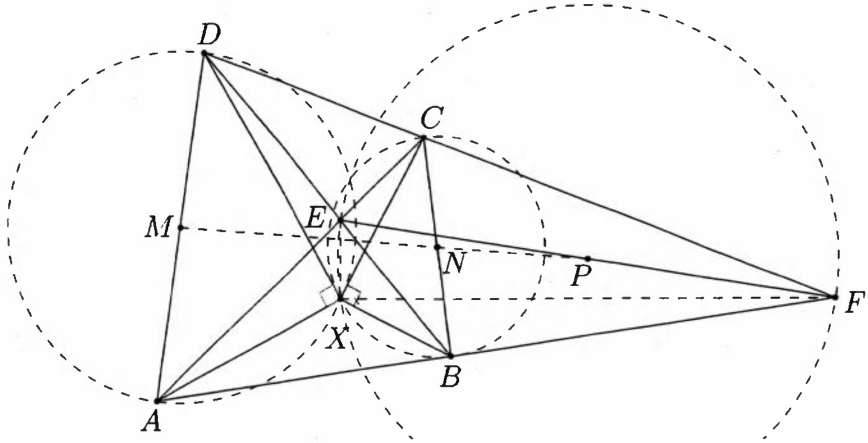
$$\overrightarrow{MN} = \frac{\overrightarrow{AB} + \overrightarrow{DE}}{2} = \overrightarrow{AB} \cdot \frac{1 + e^{i\varphi}}{2}.$$

Analogously, $\overrightarrow{NP} = \dfrac{\overrightarrow{BC} + \overrightarrow{EF}}{2} = \overrightarrow{BC} \cdot \dfrac{1 + e^{i\varphi}}{2}$ and $\overrightarrow{PM} = \dfrac{\overrightarrow{CA} + \overrightarrow{FD}}{2} = \overrightarrow{CA} \cdot \dfrac{1 + e^{i\varphi}}{2}$.

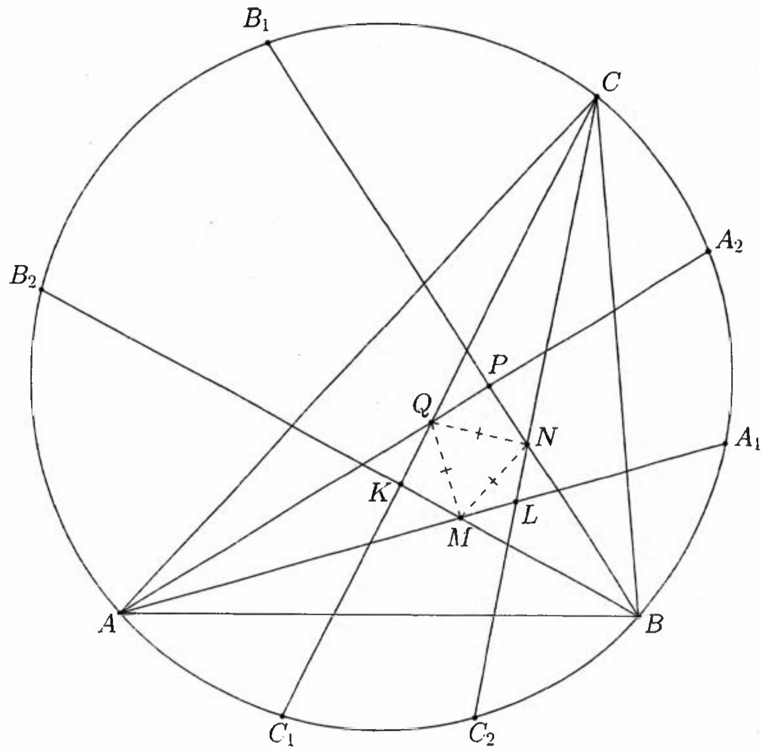
Therefore, the same composition of homothety and rotation has been applied to each of the vectors $\overrightarrow{MN}$, $\overrightarrow{NP}$ and $\overrightarrow{PM}$, with respect to the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$. We conclude that $\triangle MNP \sim \triangle ABC$.

Problem 4.12.10. Let $\angle AXD$ and $\angle BXC$ be right angles and let $AC \cap BD = E$ and $AB \cap DC = F$. Prove that $\angle FXE = 90°$.

Solution. The statement is a direct consequence of Problem 3.10.



Problem 4.12.11. *(Morley's Theorem)* Let ABC be a triangle. Its angle trisectors intersect as shown in the figure. Prove that $\triangle QMN$ is equilateral.



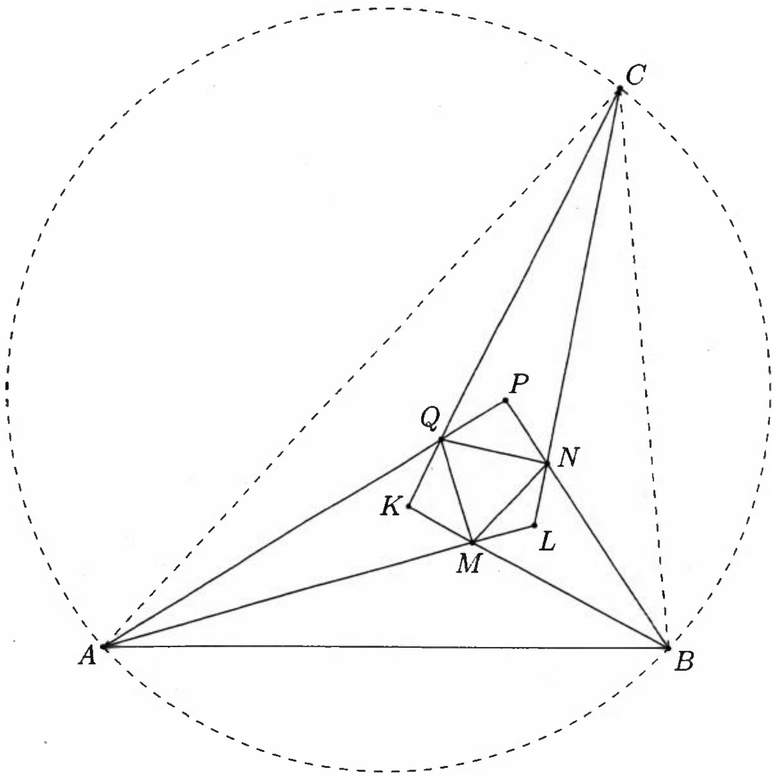
Solution. We will use the notations on the figure. Problem 4.12.12

gives $MK = KQ$, $QP = PN$ and $NL = LM$. Thus, we have

$$\begin{aligned}\angle NQM &= 180^\circ - \angle CQP - \angle PQN - \angle KQM \\ &= 180^\circ - \angle CAQ - \angle ACQ - \left(90^\circ - \frac{\angle QPN}{2}\right) - \left(90^\circ - \frac{\angle QKM}{2}\right) \\ &= 180^\circ - \frac{\alpha}{3} - \frac{\gamma}{3} - 180^\circ + \frac{\angle QPN + \angle QKM}{2} \\ &= \frac{\gamma + \frac{\alpha}{3} + \frac{\beta}{3} + \alpha + \frac{\beta}{3} + \frac{\gamma}{3}}{2} - \frac{\alpha}{3} - \frac{\gamma}{3} = \frac{\alpha}{3} + \frac{\gamma}{3} + \frac{\beta}{3} = 60^\circ.\end{aligned}$$

Analogously, we deduce that $\angle QNM = \angle QMN = 60^\circ$. Therefore, $\triangle QMN$ is equilateral.

Problem 4.12.12. Let ABC be a triangle. Its angle trisectors intersect as shown in the figure. Prove that $PQ = PN$.



Solution. We have that $\angle BAP = \frac{2}{3}\alpha$, $\angle ABP = \frac{2}{3}\beta$ and $AB = c$.

Let M be the incenter of $\triangle ABP$ and let N' be a point on BP, such that $\angle N'MB = 120^\circ - \frac{\beta}{3} - \frac{\gamma}{3}$. Let Q' be the reflection of N' with respect to the line PM.

Since PM is the angle bisector of $\angle BPA$, we have $Q' \in AP$. Let $L' \in AM$ lie on the perpendicular bisector of MN', and let $K' \in BM$ lie on the perpendicular bisector of MQ'. We have

$$\angle L'MN' = \angle L'N'M = \frac{\alpha}{3} + \frac{\gamma}{3}, \ \angle L'N'B = \frac{\beta}{3} + \frac{\gamma}{3},$$
$$\angle AMK' = \angle PN'Q' = \frac{\alpha+\beta}{3}, \ \angle MN'B = \angle AQ'M = 120^\circ - \frac{\alpha+\beta}{3},$$
$$\angle AMQ' = 60^\circ + \frac{\beta}{3}, \ \angle K'MQ' = \angle K'Q'M = \frac{\beta}{3} + \frac{\gamma}{3}.$$

Therefore,

$$\angle Q'MN' = \angle MN'Q' = \angle N'Q'M = 60^\circ.$$

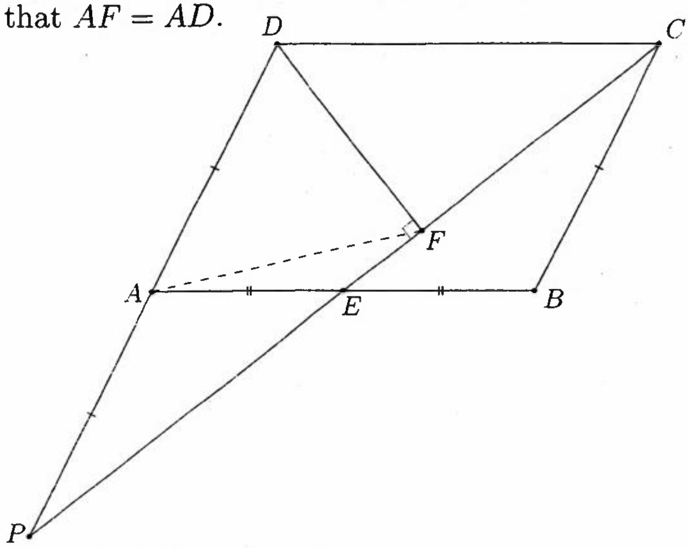
Now it is easy to see that $\triangle K'MN' \cong \triangle K'Q'N'$. Hence, $K'N'$ is the angle bisector of $\angle BK'Q'$. Let $K'Q' \cap L'N' = C'$. Then

$$\angle BN'C' = 180^\circ - \frac{\beta}{3} - \frac{\gamma}{3} = 90^\circ + \frac{\angle BK'C'}{2}.$$

This implies that N' is the incenter of $\triangle BK'C'$. Analogously, Q' is the incenter of $\triangle AL'C'$.

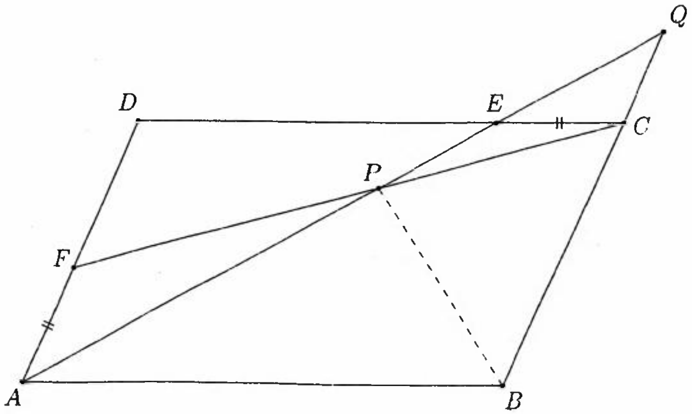
Thus, $\angle C'BA = \beta$ and $\angle C'AB = \alpha$, which implies that $C' \equiv C$. Now we deduce that $N' \equiv N$, $L' \equiv L$, $K' \equiv K$ and $Q' \equiv Q$. It follows that $PN = PQ$.

Problem 5.1.1. Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and let F be the foot of the perpendicular from D to the line EC. Prove that $AF = AD$.



Solution. Let $CE \cap DA = P$. Since $DC = 2AE$ and $DC \parallel AE$, we have that AE is a midsegment in $\triangle PCD$. Hence, $AP = AD$ and thus AF is the median to the hypotenuse in the right-angled $\triangle PFD$. We deduce that $AF = AP = AD$.

Problem 5.1.2. Let $ABCD$ be a parallelogram. The points E and F are chosen on the segments DC and AD, respectively, so that $EC = AF$. Let $AE \cap CF = P$. Prove that BP is the angle bisector of $\angle ABC$.

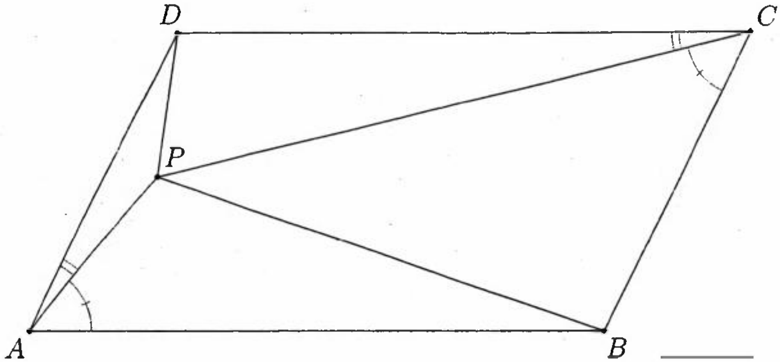


Solution. Let $AE \cap BC = Q$. Then the intercept theorem yields

$$\frac{AP}{PQ} = \frac{AF}{CQ} = \frac{EC}{CQ} = \frac{AB}{BQ}.$$

Now, the angle bisector theorem yields that BP bisects $\angle ABC$.

Problem 5.1.3. Let $ABCD$ be a parallelogram. A point P is chosen inside of $ABCD$, such that $\angle DAP = \angle DCP$. Prove that $\angle PBA = \angle PDA$.



Solution. We have

$$\angle BAP = \angle BAD - \angle PAD = \angle BCD - \angle DCP = \angle BCP.$$

Denote $\angle ABC = \angle ADC = \alpha$, $\angle PBA = \beta$ and $\angle PDA = \beta_1$. The trigonometric form of Ceva's Theorem, applied to the quadrilateral $ABCD$ and the point P, yields

$$\frac{\sin \angle BAP}{\sin \angle PAD} \cdot \frac{\sin \beta_1}{\sin(\alpha - \beta_1)} \cdot \frac{\sin \angle DCP}{\sin \angle PCB} \cdot \frac{\sin(\alpha - \beta)}{\sin \beta} = 1$$

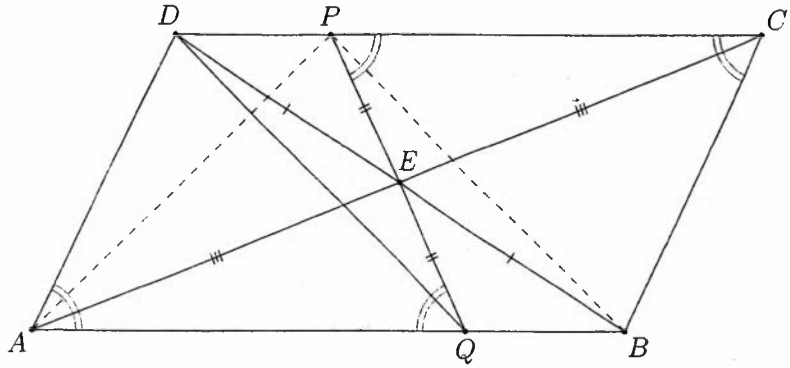
$$\Rightarrow \frac{\sin(\alpha - \beta)}{\sin \beta} = \frac{\sin(\alpha - \beta_1)}{\sin \beta_1} \Rightarrow \cot \beta = \cot \beta_1 \Rightarrow \beta = \beta_1.$$

Therefore, $\angle PBA = \angle PDA$.

Note. The trigonometric form of Ceva's Theorem of a quadrilateral is analogous to that of a triangle. However, it is valid only in one direction, i.e. even if the respective equality is true, this does not imply that the lines are concurrent. The proof consists of applying the law of sines four times.

The problem can also be solved using translation. For example, consider a translation that sends DC to AB, and where it sends $\triangle DCP$.

Problem 5.1.4. Let $ABCD$ be a parallelogram with $AC \cap BD = E$. Let P be a point on CD, such that $\angle EPC = \angle BCD$. Prove that $AP = BP$.



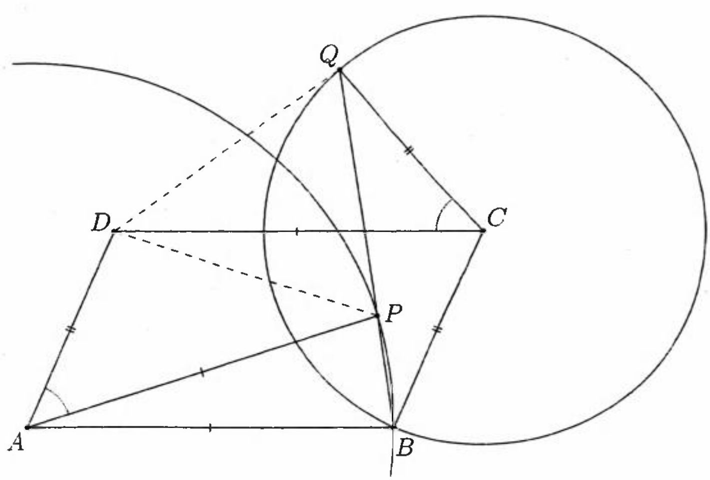
Solution. Let $PE \cap AB = Q$. The properties of the parallelogram yield $PE = EQ$. Hence, $QBPD$ is also a parallelogram.

Moreover, we have $\angle QAD = \angle BCD = \angle EPC = \angle PQA$. Hence, $AQPD$ is an isosceles trapezoid and therefore $BP = DQ = AP$.

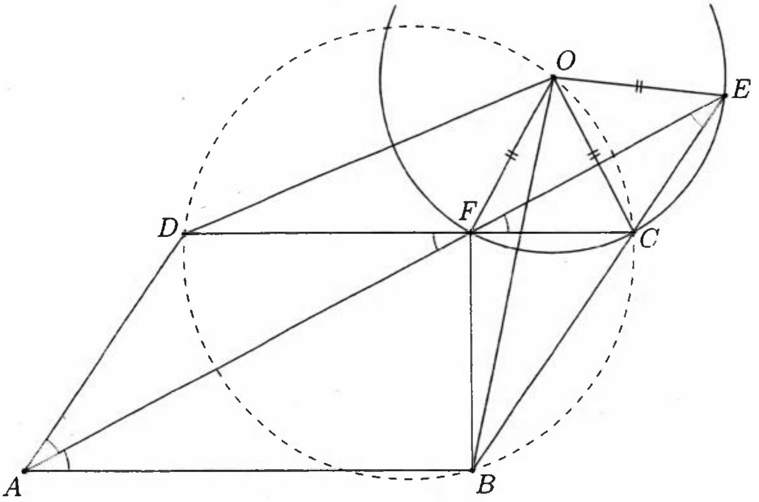
Problem 5.1.5. Let $ABCD$ be a parallelogram. The circles $k_1(A; AB)$ and $k_2(C; CB)$ are constructed. Let $P \in k_1$ be a point inside of $ABCD$, and let $BP \cap k_2 = \{B, Q\}$. Prove that $DP = DQ$.

Solution. We have $CD = AB = AP$ and $CQ = CB = AD$. Moreover,

$$\begin{aligned}
\angle DAP &= \angle DAB - \angle PAB = 180^\circ - \angle ABC - 180^\circ + 2\angle PBA \\
&= 2\angle PBA - \angle ABC = 2(\angle ABC - \angle QBC) - \angle ABC \\
&= \angle ABC - \angle QBC - \angle BQC = 180^\circ - \angle BCD - 180^\circ + \angle BCQ \\
&= \angle BCQ - \angle BCD = \angle DCQ \Rightarrow \Delta DAP \cong \Delta QCD \Rightarrow DQ = DP.
\end{aligned}$$



Problem 5.1.6. Let $ABCD$ be a parallelogram. The angle bisector of $\angle BAD$ intersects the line BC and the segment DC at the points E and F, respectively. Let O be the circumcenter of $\triangle EFC$. Prove that the quadrilateral $BDOC$ is cyclic.

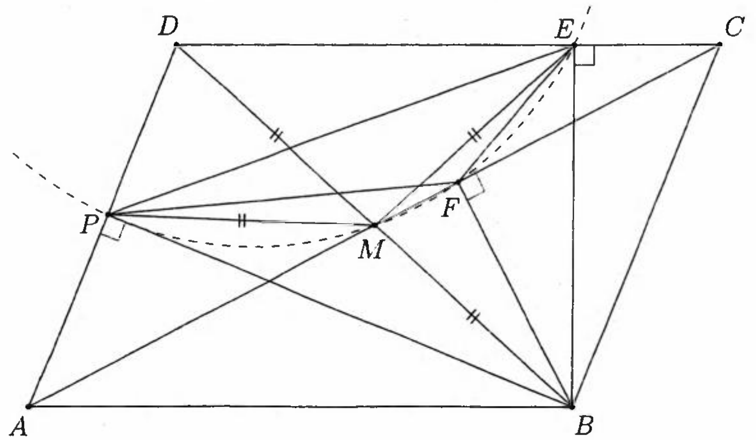


Solution. We have $\angle EFC = \angle DFA = \angle FAB = \angle FAD = \angle FEC$.

Hence, $CF = CE$ and $CD = BA = BE$. Since $\triangle EFC$ is isosceles, we have $\triangle COF \cong \triangle COE$ and $\angle OCD = \angle OEB$. Also, $OE = OC$. Thus, $\triangle DCO \cong \triangle BEO$ and $\angle ODC = \angle OBE$.

We deduce that the quadrilateral $BDOC$ is cyclic.

Problem 5.1.7. Let $ABCD$ be a parallelogram with $AC \cap BD = M$. Let the feet of the perpendiculars from B to the lines AD, AC and CD be P, F and E, respectively. Prove that the quadrilateral $PMFE$ is cyclic.



Solution. Note that $\angle BPD = \angle BED = 90°$, so $BM = MD = MP = ME$. Moreover, we have

$$\angle EFC = \angle EBC = 90° - \angle ECB = 90° - \angle BAP = \angle PBA = \angle PFA.$$

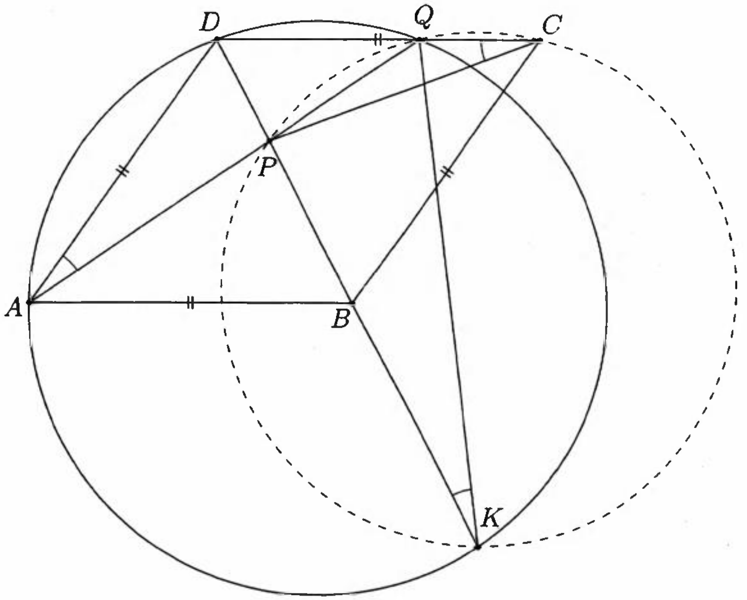
Hence, FM is an external bisector in $\triangle PFE$.

So the point M lies on the perpendicular bisector of EP and on the external bisector mentioned above. It follows that M is the midpoint of the arc $\widehat{PFE}$ of the circumcircle of $\triangle PFE$.

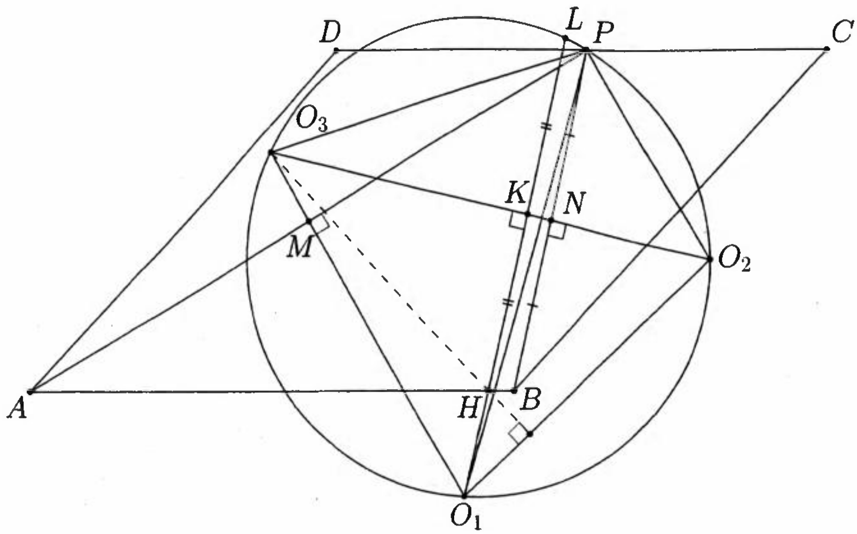
In conclusion, the quadrilateral $PMFE$ is cyclic.

Problem 5.1.8. Let $ABCD$ be a rhombus. The point P lies on BD. The line AP intersects the segment DC at the point Q. The circumcircle of $\triangle ADQ$ intersects BD for the second time at the point K. Prove that the quadrilateral $CQPK$ is cyclic.

Solution. Since $ABCD$ is a rhombus, we have $CD = DA$ and $\angle ADP = \angle CDP$. Hence, $\triangle ADP \cong \triangle CDP$ and $\angle QCP = \angle DCP = \angle DAP = \angle PKQ$. Therefore, the quadrilateral $CQPK$ is cyclic.



Problem 5.1.9. Let $ABCD$ be a parallelogram. The point P lies on the segment CD. Let O_1, O_2, O_3 be the circumcenters of $\triangle ADP$, $\triangle BPC$ and $\triangle ABP$, respectively. Prove that the orthocenter of $\triangle O_1O_2O_3$ lies on AB.



Solution. Let M and N be the midpoints of AP and BP, respectively. Then $\angle AMO_1 = 90°$ and $\angle O_2NP = 90°$. We have

$$\angle O_3O_2P = \angle NO_2P = \angle BCP = 180° - \angle ADP = \angle MO_1P = \angle O_3O_1P.$$

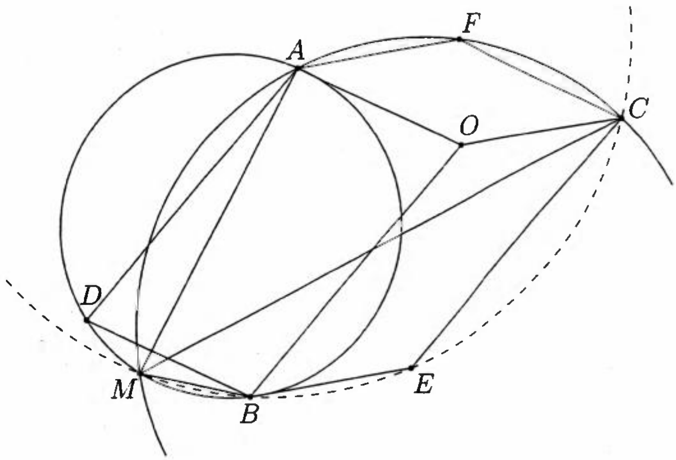
This implies that the quadrilateral $O_3O_1O_2P$ is cyclic. Let the altitude from O_1 to O_2O_3 intersect AB at the point H, O_2O_3 – at the point K, and the circumcircle of $O_3O_1O_2P$, for the second time – at the point L. Since O_3MNP is cyclic, we have

$$\begin{aligned}\angle KHB &= 180° - \angle ABP = \angle BAP + \angle APB \\ &= \angle NO_3P + \angle MO_3N = \angle O_1O_3P = \angle O_1LP.\end{aligned}$$

Thus, $HBPL$ is an isosceles trapezoid and $LK = KH$. This implies that the points L and H are symmetric with respect to O_2O_3. Consequently, $\angle HO_3O_2 = \angle LO_3O_2 = \angle LO_1O_2 = 90° - \angle O_3O_2O_1$.

This gives $O_3H \perp O_1O_2$. But by definition we have $O_1H \perp O_2O_3$. We deduce that $H \in AB$ is the orthocenter of $\triangle O_1O_2O_3$.

Problem 5.1.10. Let OA, OB and OC be line segments. The points D, E and F are such that the quadrilaterals $AOBD$, $BOCE$ and $AOCF$ are parallelograms. Prove that the circumcircles of $\triangle ABD$, $\triangle BEC$ and $\triangle ACF$ have a common point.



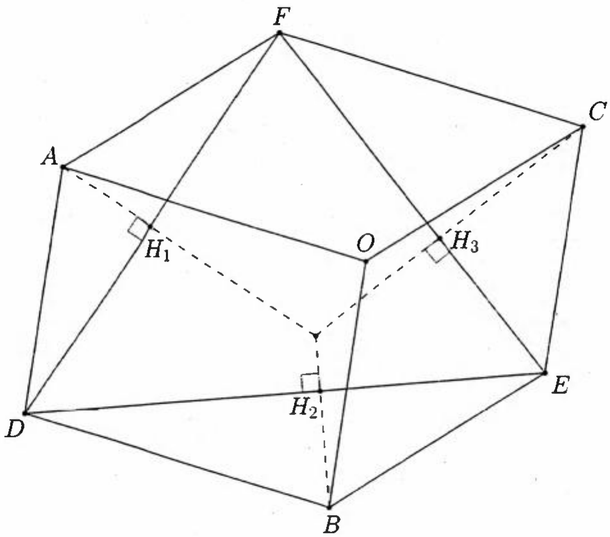
Solution. Let M be the second intersection point of the circumcircles of $\triangle ABD$ and $\triangle AFC$. It suffices to prove that the quadrilateral $BECM$ is cyclic. Note that

$$\begin{aligned}\angle BMC &= \angle BMA - \angle CMA = \angle BDA - (180^\circ - \angle CFA)\\ &= \angle BOA - (180^\circ - \angle COA) = \angle BOA + \angle COA - 180^\circ\\ &= 180^\circ - \angle BOC = 180^\circ - \angle BEC.\end{aligned}$$

Problem 5.1.11. The segments OA, OB and OC are given. The points D, E and F are chosen, such that the quadrilaterals $AOBD$, $BOCE$ and $AOCF$ are parallelograms. Let H_1 be the foot of the perpendicular from A to DF, H_2 – from B to DE, and H_3 – from C to EF. Prove that the lines AH_1, BH_2 and CH_3 are concurrent.

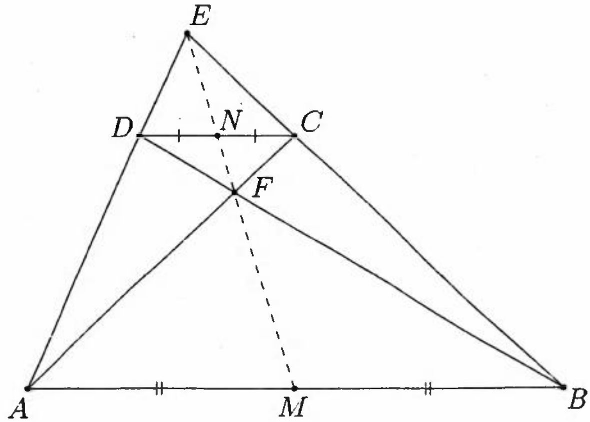
Solution. By the Pythagorean Theorem, we have

$$\begin{aligned}&H_3F^2 - H_3E^2 + H_2E^2 - H_2D^2 + H_1D^2 - H_1F^2\\ &= CF^2 - CE^2 + BE^2 - BD^2 + AD^2 - AF^2\\ &= OA_2 - OB^2 + OC^2 - OA^2 + OB^2 - OC^2 = 0.\end{aligned}$$



Carnot's Theorem (Problem 4.9.16), applied to $\triangle ABC$ and the points H_1, H_2 and H_3, yields that the lines AH_1, BH_2 and CH_3 are concurrent.

Problem 5.2.1. *(Steiner's Theorem)* Let $ABCD$ ($AB \parallel CD$) be a trapezoid and let M and N be the midpoints of AB and CD, respectively. Denote $AC \cap BD = F$ and $AD \cap BC = E$. Then the points M, F, N and E are collinear.

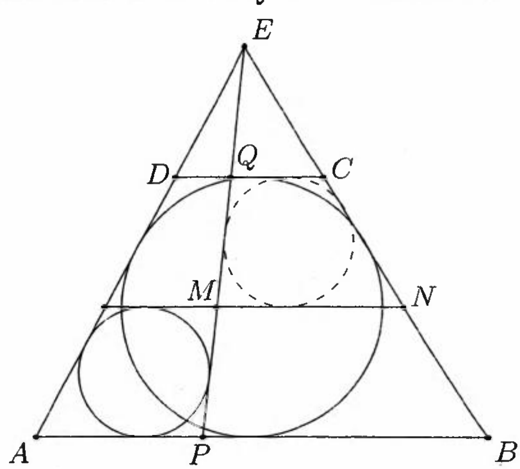


Solution: The fact that $AB \parallel CD$ proves the existence of a homothety with center F, such that $C \to A$ and $D \to B$. Therefore, $CD \to AB$. Hence, the image of the midpoint of CD is the midpoint of AB.

So $N \to M$. Therefore, the points M, F and N are collinear.

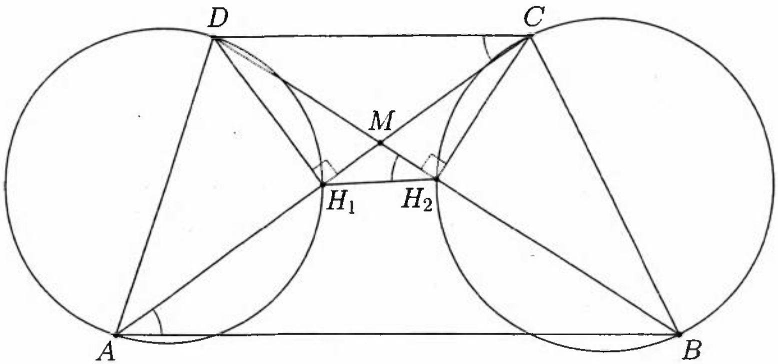
Also, the homothety with center E, such that $C \to B$ and $D \to A$, sends N to M. Therefore, the points M, F, N and E are collinear.

Problem 5.2.2. Let $ABCD$ ($AB \parallel CD$) be a circumscribed trapezoid. Let $AD \cap BC = E$. The point P lies on the segment AB and $EP \cap CD = Q$. The line, parallel to AB and tangent to the incircle of $\triangle APE$, intersects EP and EB at the points M and N, respectively. Prove that the quadrilateral $MNCQ$ is circumscribed.



Solution. The problem is analogous to Problem 5.4.1. Indeed, choose the point B in the configuration of Problem 5.4.1 to be infinite. This choice preserves the circles in the diagram.

Problem 5.2.3. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. The diagonals AC and BD intersect at M. The circles k_1 and k_2 with diameters AD and BC, respectively, do not intersect each other. Prove that M lies on the radical axis of k_1 and k_2.



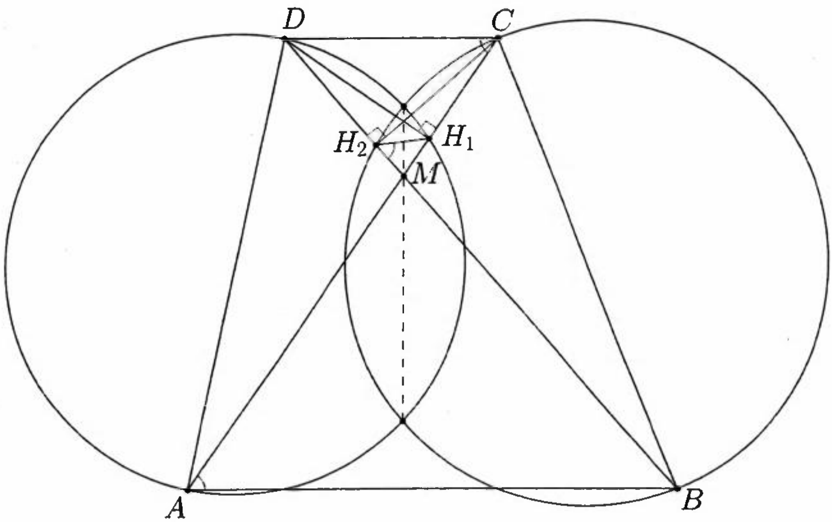
Solution. Denote the feet of the perpendiculars from C to BD and from D to AC by H_2 and H_1, respectively. Then $H_2 \in k_2$ and $H_1 \in k_1$. Moreover, the quadrilateral CDH_1H_2 is cyclic. Now we have

$$\angle H_1AB = \angle DCH_1 = \angle DH_2H_1.$$

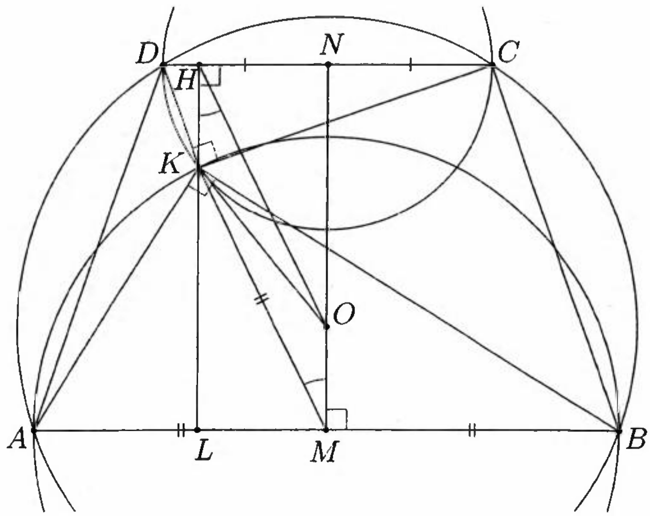
Thus, ABH_2H_1 is also cyclic. Now, M is the radical center of the circumcircle of ABH_2H_1, k_1 and k_2 (or just notice that $MH_1.MA = MH_2.MB$).

Problem 5.2.4. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. The diagonals AC and BD intersect at M. The circles k_1 and k_2 with diameters AD and BC, respectively, intersect each other. Prove that M lies on the radical axis of k_1 and k_2.

Solution. The solution is completely analogous to that of Problem 5.2.3.



Problem 5.2.5. A trapezoid $ABCD$ ($AB \parallel CD$) is inscribed in a circle with center O and radius R. Let M be the midpoint of AB. Let one of the intersection points of the circles with diameters AB and CD be K. The point H is the foot of the perpendicular from K to CD. Prove that $KH = OM$.



Solution. Since the trapezoid is inscribed, it is also isosceles. Let N be the midpoint of CD and let $KH \cap AB = L$.

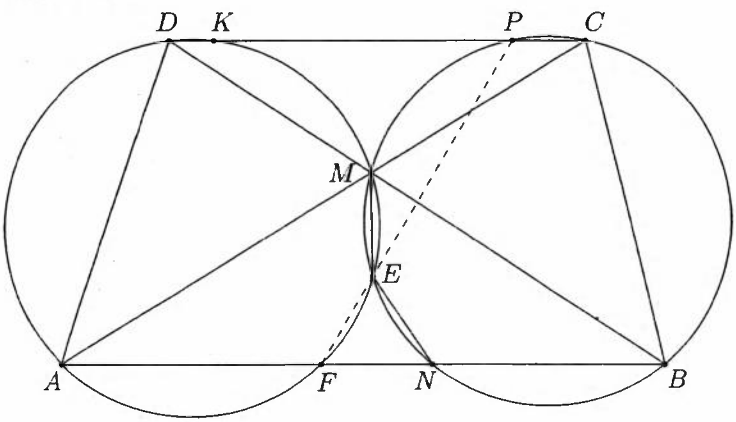
Considering the power of the point H, we get $R^2 - OH^2 = DH.CH = KH^2$, where the last equality follows from $\triangle CHK \sim \triangle KHD$. Moreover, the Pythagorean Theorem gives $R^2 - OM^2 = AM^2 = KM^2$.

These equalities imply $R^2 = KM^2 + OM^2 = KH^2 + OH^2$. Now we apply the law of cosines two times:

$$\begin{aligned} OK^2 &= OM^2 + MK^2 - 2OM.OK\cos\angle OMK, \\ OK^2 &= OH^2 + KH^2 - 2OH.KH\cos\angle OHK \\ &\Rightarrow OM.MK\cos\angle OMK = HO.KH\cos\angle OHK \\ &\Rightarrow OM.MK\sin\angle KML = KH.HO\sin\angle OHN \\ &\Rightarrow OM.KL = KH.ON \Rightarrow \frac{OM}{ON} = \frac{KH}{KL}. \end{aligned}$$

Also, $OM + ON = KH + KL$. Therefore, $OM = KH$.

Problem 5.2.6. Let $ABCD$ ($AB \parallel CD$) be a trapezoid with $AC \cap BD = M$. The circumcircle of $\triangle AMD$ intersects the segments AB and CD for the second time at the points F and K, respectively. The circumcircle of $\triangle BMC$ intersects the segments AB and CD for the second time at the points N and P, respectively. Let the intersection points of these two circles be M and E. Prove that the triples of points F, E, P and N, E, K are collinear.

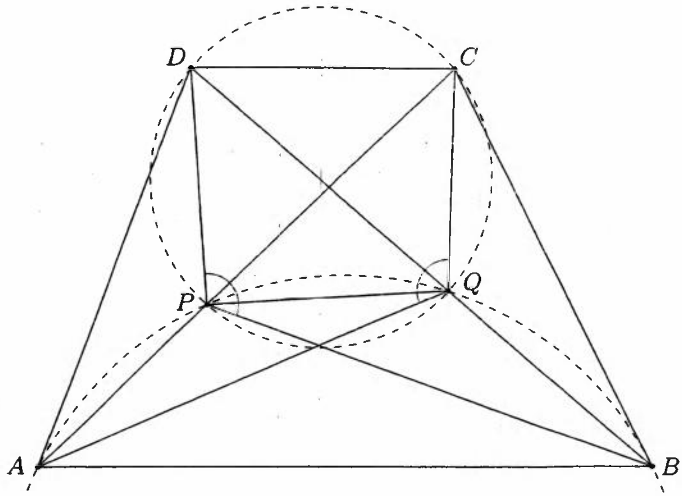


Solution. The quadrilateral $NPCB$ is cyclic, and so it is an isosceles trapezoid. Therefore, $\widehat{PN} = \widehat{BC}$. We have

$$\begin{aligned} \angle NEP &= 180^\circ - \angle BMC = \angle AMB = \angle AME + \angle BME \\ &= \angle EFN + \angle ENF = 180^\circ - \angle FEN. \end{aligned}$$

We deduce that the points F, E and P are collinear. The proof for the other triple N, E, K is analogous.

Problem 5.2.7. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. The points P and Q lie on the segments AC and BD, respectively. If $\angle BPD = \angle AQC$, prove that the quadrilateral $PQCD$ is cyclic.



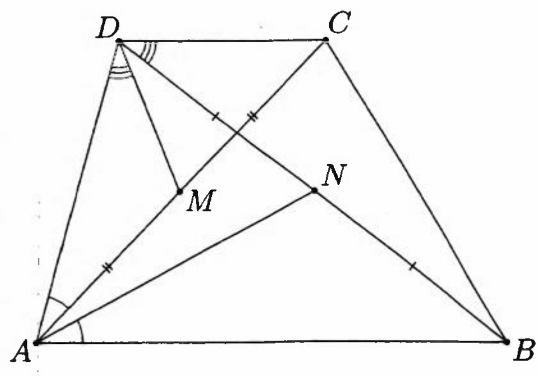
Solution. Let $Q' \in BD$ be a point such that $APQ'B$ is cyclic and $Q' \not\equiv B$. We have $\angle PQ'D = \angle PAB = \angle PCD$ and hence $PQ'CD$ is also cyclic. Therefore, $\angle BPD = \angle BPC + \angle DPC = \angle AQ'D + \angle DQ'C = \angle AQ'C$.

In conclusion, $Q' \equiv Q$ and the quadrilateral $PQCD$ is cyclic.

Problem 5.2.8. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. Let M and N be the midpoints of AC and BD, respectively. If $\angle NAB = \angle CAD$, prove that $\angle ADM = \angle BDC$.

Solution. We have

$$\frac{\sin\angle BAC}{\sin\angle CAD} = \frac{\sin\angle DAN}{\sin\angle NAB}$$
$$\Rightarrow \frac{BC}{CD}\cdot\frac{\sin\angle ABC}{\sin\angle ADC} = \frac{AB}{AD}$$
$$\Rightarrow \frac{BC}{AB}\cdot\frac{\sin\angle BCD}{\sin\angle BAD} = \frac{CD}{AD}$$
$$\Rightarrow \frac{\sin\angle CDB}{\sin\angle ADB} = \frac{\sin\angle ADM}{\sin\angle CDM}.$$



Note that $\angle CDB + \angle ADB = \angle ADC = \angle ADM + \angle CDM$. Therefore, $\cot\angle CDB = \cot\angle ADM$ and $\angle CDB = \angle ADM$.

Problem 5.2.9. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. Let $AC \cap BD = F$ and $AD \cap BC = E$. The circumcircles k_1 and k_2 of $\triangle AFD$ and $\triangle BCF$, respectively, intersect for the second time at the point P. Prove that $\angle AEF = \angle BEP$.

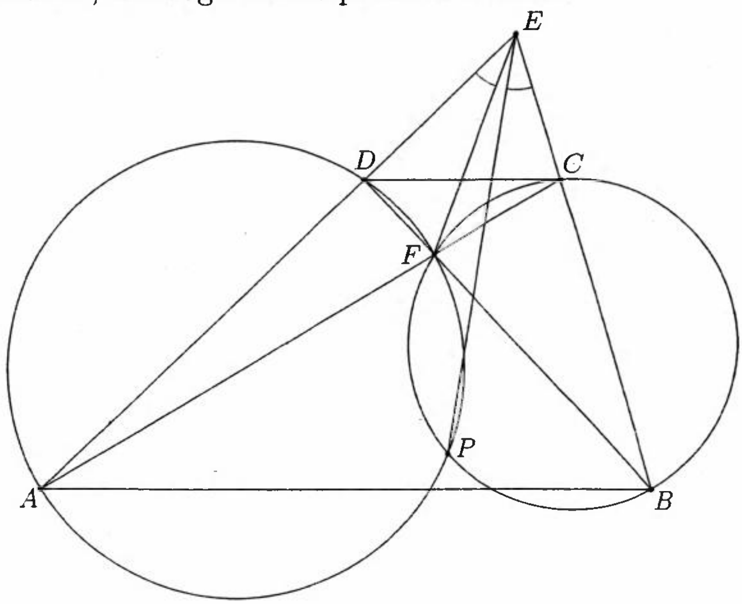
Solution. Consider the composition φ of inversion with center E and radius $\sqrt{EC.EA}$, and symmetry with respect to the angle bisector of $\angle AEB$.

Since $AB \parallel CD$, the equality $EC.EA = ED.EB$ holds. Therefore, $\varphi(A) = C$ and $\varphi(B) = D$. Hence, the image of AC is the circumcircle of $\triangle EAC$. Analogously, the image of BD is the circumcircle of $\triangle BED$.

Miquel's Theorem (Problem 2.33) yields that the point P lies on both circles. Hence, $AC \cap BD = F = \varphi(P)$.

Therefore, EF and EP are isogonal conjugates with respect to $\angle AEB$.

Note. The map used above is known as inversion plus symmetry and is widely used in geometry problems. Sometimes, the radius of the inversion does not matter, although in this problem it does.



Problem 5.2.10. Let $ABCD$ ($AB \parallel CD$) be a trapezoid. Let $AD \cap BC = E$. The circumcircles k_1 and k_2 of $\triangle BCD$ and $\triangle ABD$ have centers O_1 and O_2, respectively. Prove that $\angle AEO_2 = \angle BEO_1$.

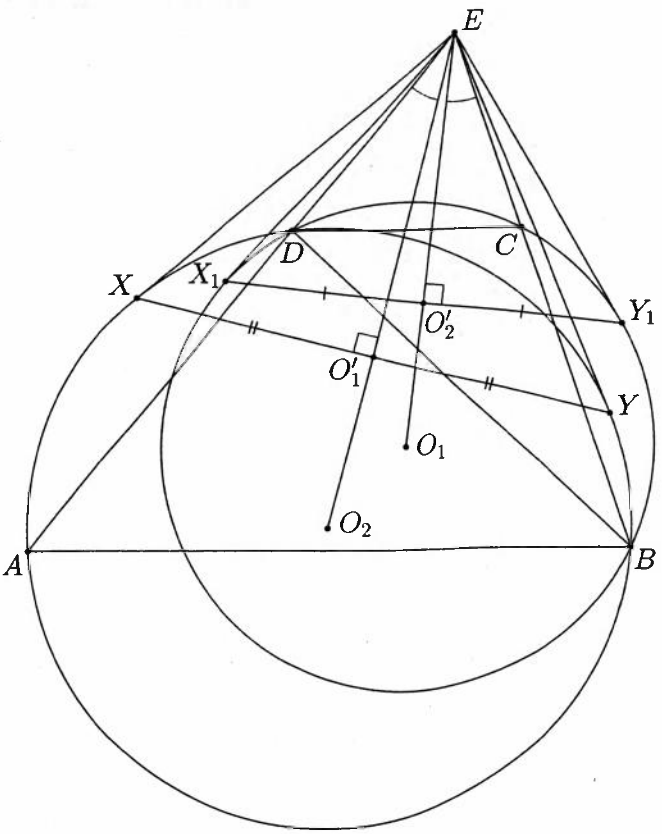
Solution. As in the previous problem, consider the composition φ of inversion with center E and radius $\sqrt{EC.EA}$, and symmetry with respect to the angle bisector of $\angle AEB$.

Since $AB \parallel CD$, the equality $EC.EA = ED.EB$ holds. Hence, $\varphi(A) = C$ and $\varphi(B) = D$. Now we have that $\varphi(k_1) = k_2$.

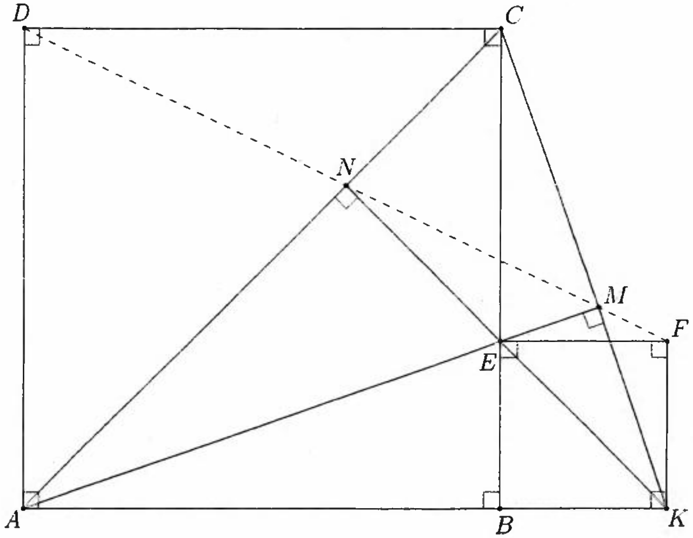
We will use the well-known fact that if we have an inversion with center O and the image of a circle c_1 is a circle c_2, then the image of the center of c_1 is not the center of c_2. It is the point that is defined as follows – the image of O after an inversion with respect to c_2.

The points $X, Y \in k_2$ are chosen, such that EX and EY touch k_2. Similarly, the points $X_1, Y_1 \in k_1$ are chosen, such that EX_1 and EY_1 touch k_1. Denote the midpoints of XY and X_1Y_1 by O_1' and O_2', respectively.

Now, the fact that $\varphi(k_1) = k_2$ yields $\varphi(O_1) = O_1'$ and $\varphi(O_2) = O_2'$. Since E, O_1' and O_2 are collinear, the lines EO_1 and EO_2 are isogonally conjugate with respect to $\angle AEB$.



Problem 5.3.1. Let $ABCD$ be a square and let E be a point on the segment BC. The square $BKFE$ is constructed, such that it is external to $ABCD$. Let $AE \cap CK = M$ and $KE \cap AC = N$. Prove that the points D, N, M and F are collinear.



Solution. We have $AB = BC$ and $BE = BK$. Hence, $\triangle ABE \cong \triangle CBK$. Thus, $\angle BCK = \angle BAM$. This implies that the quadrilateral $ABMC$ is cyclic, which yields $\angle AMK = \angle CBA = 90°$.

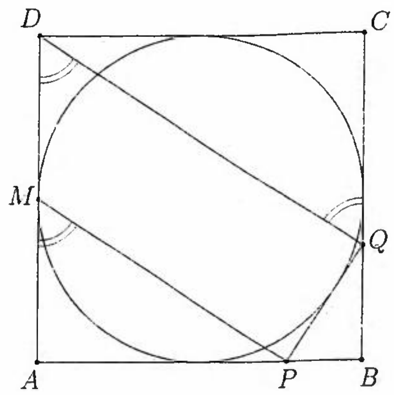
Also, $\angle EFK = \angle EMK = 90°$, so the quadrilateral $EMFK$ is cyclic. Since $CE \perp AK$ and $AE \perp CK$, the point E is the orthocenter of $\triangle AKC$, which yields $\angle ANK = 90°$. Consequently, the quadrilateral $AKMN$ is cyclic.

Now, $\angle NMA = \angle NKA = 45° = \angle EKF = 180° - \angle EMF$ implies that the points N, M and F are collinear. We have that NM is the radical axis of the circumcircles of $AKMN$ and $MENC$.

It suffices to show that D also lies on this radical axis. We have $\angle DCN = 45° = \angle BEK = \angle NEC = \angle CMN$. Thus, DC touches the circumcircle of $\triangle CNM$.

Also, $\angle DAN = 45° = \angle NKA$ and DA touches the circumcircle of $\triangle NKA$. Therefore, the point D has equal powers with respect to these two circles.

Problem 5.3.2. Let $ABCD$ be a square. Its incircle k touches the side AD at the point M. The point P lies on the segment AB and satisfies $\angle PMA = \alpha > 45°$. The point Q lies on the segment BC, such that $QD \parallel PM$. Prove that the line PQ is tangent to k.



Solution. Without loss of generality, let $AB = 2$. Then $AM = MD = 1$.

We have $\angle DQC = \angle ADQ = \alpha$ and $AP = \operatorname{tg}\alpha$. Hence, $\operatorname{tg}\alpha < 2$ and $CQ = \dfrac{2}{\operatorname{tg}\alpha} > 1$, which yields $PB = 2 - \operatorname{tg}\alpha$ and $QB = 2 - \dfrac{2}{\operatorname{tg}\alpha}$.

Now the Pythagorean Theorem gives

$$PQ = \sqrt{(2 - \operatorname{tg}\alpha)^2 + \left(2 - \frac{2}{\operatorname{tg}\alpha}\right)^2}$$

$$= \sqrt{4 - 4\operatorname{tg}\alpha + \operatorname{tg}^2\alpha + 4 - \frac{8}{\operatorname{tg}\alpha} + \frac{4}{\operatorname{tg}^2\alpha}}$$

$$= \sqrt{(2)^2 + (\operatorname{tg}\alpha)^2 + \left(\frac{2}{\operatorname{tg}\alpha}\right)^2 + 2\left(\frac{2}{\operatorname{tg}\alpha}\right)(\operatorname{tg}\alpha) - 2(2)(\operatorname{tg}\alpha) - 2(2)\left(\frac{2}{\operatorname{tg}\alpha}\right)}$$

$$= \sqrt{\left(\operatorname{tg}\alpha + \frac{2}{\operatorname{tg}\alpha} - 2\right)^2} = \left|\operatorname{tg}\alpha + \frac{2}{\operatorname{tg}\alpha} - 2\right| = \operatorname{tg}\alpha + \frac{2}{\operatorname{tg}\alpha} - 2.$$

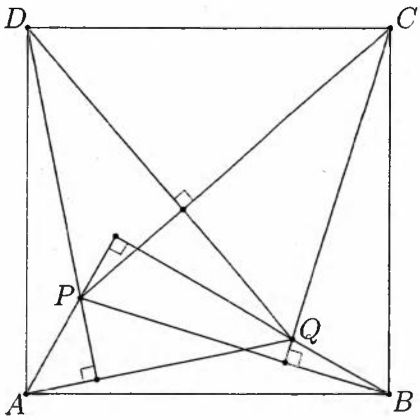
We will use the well-known formula for the area of a triangle $S = r_a(s - a)$. We apply it to $\triangle PQB$, where r_a is the radius of the B-excircle of $\triangle PQB$. We obtain

$$r_a = \frac{\left(2 - \operatorname{tg}\alpha\right)\left(2 - \dfrac{2}{\operatorname{tg}\alpha}\right)}{4 - \operatorname{tg}\alpha - \dfrac{2}{\operatorname{tg}\alpha} - \operatorname{tg}\alpha - \dfrac{2}{\operatorname{tg}\alpha} + 2} = \frac{4 + 2 - 2\operatorname{tg}\alpha - \dfrac{4}{\operatorname{tg}\alpha}}{4 + 2 - 2\operatorname{tg}\alpha - \dfrac{4}{\operatorname{tg}\alpha}} = 1.$$

This means that k coincides with the excircle mentioned above.

Problem 5.3.3. Let $ABCD$ be a square and let P be a point inside of it. Prove that the perpendiculars from A to DP, from B to AP, from C to BP, and from D to CP are concurrent.

Solution. Consider a $+90°$ rotation with center O (the center of the square).



Then $A \to B$, $B \to C$, $C \to D$ and $D \to A$. Let $P \to Q$. Then we have the following congruences:

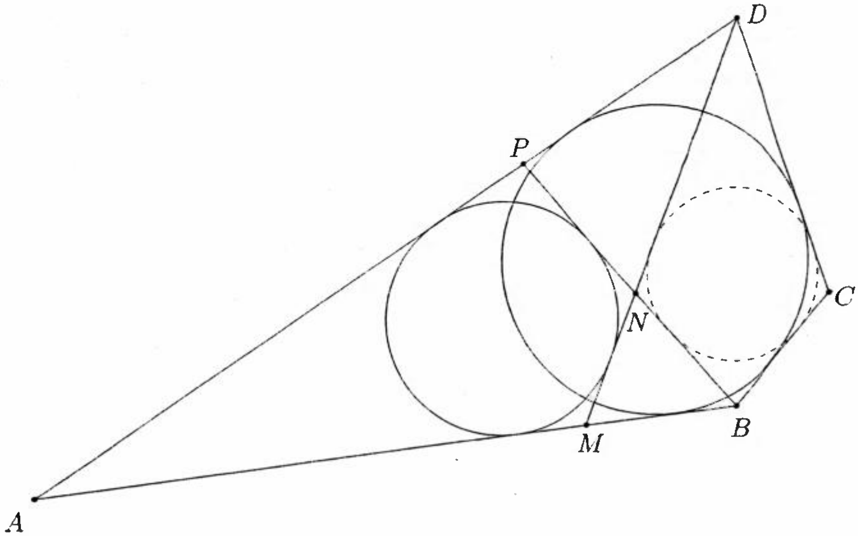
$$\triangle ABQ \cong \triangle DAP,$$
$$\triangle BCQ \cong \triangle ABP,$$
$$\triangle CDQ \cong \triangle BCP,$$
$$\triangle DAQ \cong \triangle CDP.$$

Therefore,

$$\angle ABQ + \angle BAP = \angle ABQ + 90° - \angle PAD = 90° \Rightarrow BQ \perp AP,$$
$$\angle BCQ + \angle CBP = \angle BCQ + 90° - \angle PBA = 90° \Rightarrow CQ \perp BP,$$
$$\angle CDQ + \angle DCP = \angle CDQ + 90° - \angle PCB = 90° \Rightarrow DQ \perp CP,$$
$$\angle DAQ + \angle ADP = \angle DAQ + 90° - \angle PDC = 90° \Rightarrow AQ \perp DP.$$

We deduce that the four perpendiculars under consideration intersect at the point Q.

Problem 5.4.1. Let BAD be a triangle. The circles k_1 and k_2, are inscribed in $\angle BAD$, such that they both touch the segments AB and AD and not their extensions. The tangent lines through B and D to k_1 (different from AB and AD) intersect each other at the point N. The tangent lines through B and D to k_2 (different from AB and AD) intersect each other at the point C. Prove that the quadrilateral $BCDN$ is circumscribed.



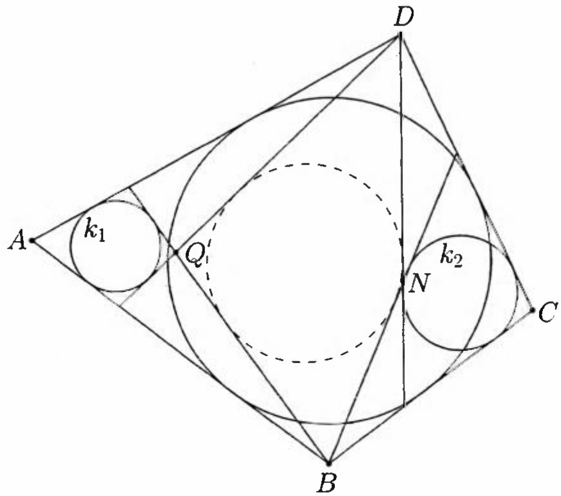
Solution. Let $BN \cap AD = P$ and let $DN \cap AB = M$. The quadrilateral $ABCD$ is circumscribed and thus $AB + CD = AD + BC$, i.e. $BC - CD = BA - AD$.

The quadrilateral $AMNP$ is also circumscribed. Thus, $BA - AD = BN - ND$, which gives $BC - CD = BA - AD = BN - ND$. Hence, $BC + ND = BN + CD$, which is equivalent to the desired statement.

Problem 5.4.2. Let $ABCD$ be a circumscribed quadrilateral. The circles k_1 and k_2 are inscribed in $\angle BAD$ and $\angle BCD$, respectively. The tangent lines through B and D to k_2 (different from DC and BC) intersect each other at the point N. The tangent lines through B and D to k_1 (different from AB and AD) intersect each other at the point Q. Prove that the quadrilateral $QBND$ is circumscribed.

Solution. Note that Problem 5.4.1, applied to the quadrilateral $ABCD$, its incircle and k_2, yields that the quadrilateral $ABND$ is circumscribed.

The same problem, applied to the quadrilateral $ABND$, its incircle and k_1, yields that the quadrilateral $QBND$ is circumscribed.

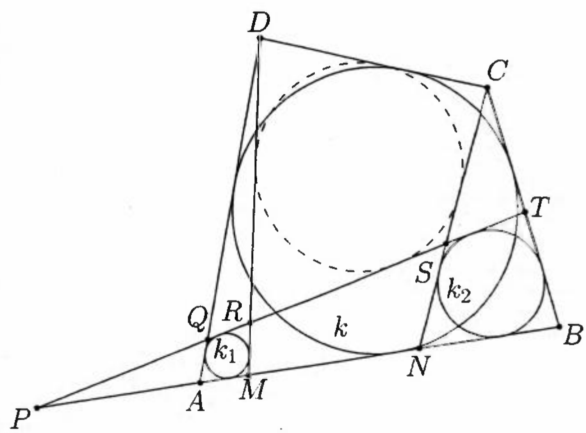


Problem 5.4.3. Let $ABCD$ be a circumscribed quadrilateral. The circles k_1 and k_2 are inscribed in $\angle BAD$ and $\angle ABC$, respectively. The tangent lines through C and D to k_2 and k_1, different from CB and AD, intersect AB at the points N and M, respectively. The common external tangent line of k_1 and k_2, different from AB, intersects the segments DM and CN in the points R and S, respectively. Prove that the quadrilateral $RSCD$ is circumscribed.

Solution. Let the line RS intersect the line AB and the segments AD and BC at the points P, Q and T, respectively.

Since the quadrilateral $AMRQ$ is circumscribed, we have

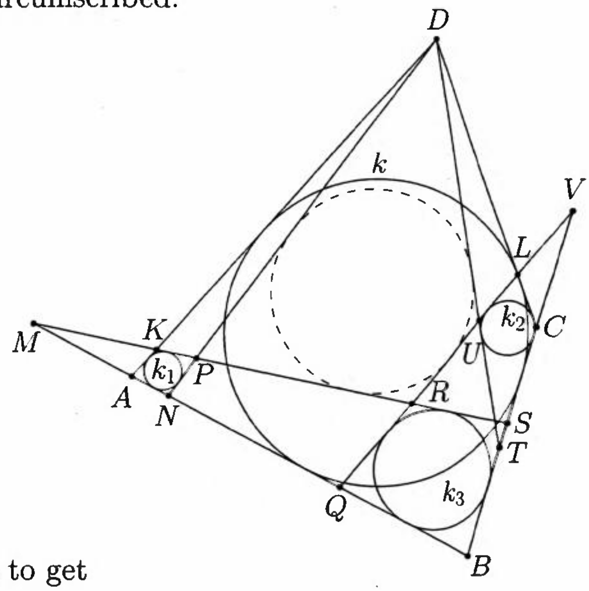
$$DR + RP = DA + AP.$$



Also, since the quadrilateral $NBTS$ is circumscribed, we have that $CS + PB = PS + CB$. Summing the last two equalities gives $DR+CS+RP+PB = DA+CB+AP+PS$, i.e. $DR + CS + PR - PS = DA + CB + AP - PB$.

Then we deduce that $DR+CS-RS = DA+CB-AB = DC$, which implies $DR + CS = RS + DC$.

Problem 5.4.4. Let $ABCD$ be a circumscribed quadrilateral. The three circles k_1, k_3 and k_2 are inscribed in $\angle BAD$, $\angle ABC$ and $\angle BCD$, respectively. The tangent lines through D to k_1 and k_2 (different from AD and DC) intersect AB and BC at the points N and T, respectively. The common external tangent lines of k_1 and k_3, and of k_2 and k_3 (different from AB and BC) intersect DN and DT at the points P and U, respectively. These tangent lines intersect each other at the point R. Prove that the quadrilateral $PRUD$ is circumscribed.



Solution. Using the notations on the figure, we have

$$DU + UV = DC + CV$$

(due to the circumscribed quadrilateral $UTCL$) and

$$RV + MB = RM + BV$$

(due to the circumscribed quadrilateral $QBSR$).

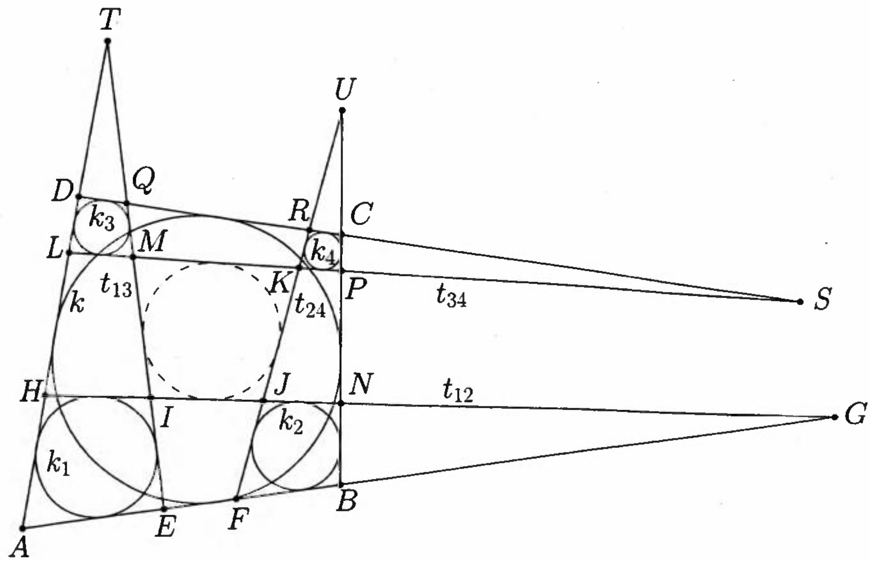
Now we subtract the second equality from the first to get

$$\begin{aligned} DU - UR &= DC + CV - RM - BV + MB \\ &= DC - CB - RM + MB. \end{aligned}$$

Analogously, we get $DP - PR = DA - AB - RV + BV$. But since the quadrilateral $ABCD$ is circumscribed, $DC - CB = DA - AB$, and since the quadrilateral $QBSR$ is circumscribed, $MB - RM = BV - RV$.

Therefore, $DU - UR = DP - PR$, which yields the desired result.

Problem 5.4.5. Let $ABCD$ be a circumscribed quadrilateral. The circles k_1, k_2, k_3, k_4 are inscribed in $\angle BAD$, $\angle ABC$, $\angle BCD$ and $\angle CDA$, respectively. Denote the common external tangent lines of k_1 and k_2, of k_2 and k_4, of k_3 and k_4, and of k_3 and k_1, different from the sides of $ABCD$, by t_{12}, t_{24}, t_{34} and t_{13}, respectively. Let $I = t_{12} \cap t_{13}$, $J = t_{12} \cap t_{24}$, $M = t_{13} \cap t_{34}$ and $K = t_{24} \cap t_{34}$. Prove that the quadrilateral $IJKM$ is circumscribed.



Solution. Using the notations on the figure, we have

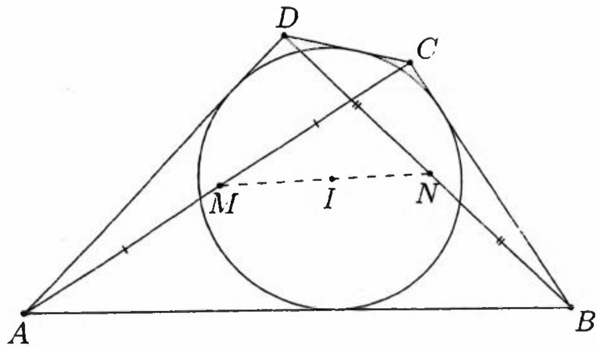
$$\begin{aligned} MK + IJ &= MS - KS + IG - JG \\ &= (SD + DT - MT) - (KU + CS - UC) + (AG + IT - AT) \\ &\quad -(GB + BU - UJ) = (SD - SC) + (DT - AT) + (IT - MT) \\ &\quad +(UJ - KU) + (UC - UB) + (AG - GB) \\ &= DC - AD + IM + JK - BC + AB = IM + JK. \end{aligned}$$

Problem 5.4.6. Let $ABCD$ be a circumscribed quadrilateral. Prove that its incenter I and the midpoints M and N of the segments AC and BD, respectively, are collinear.

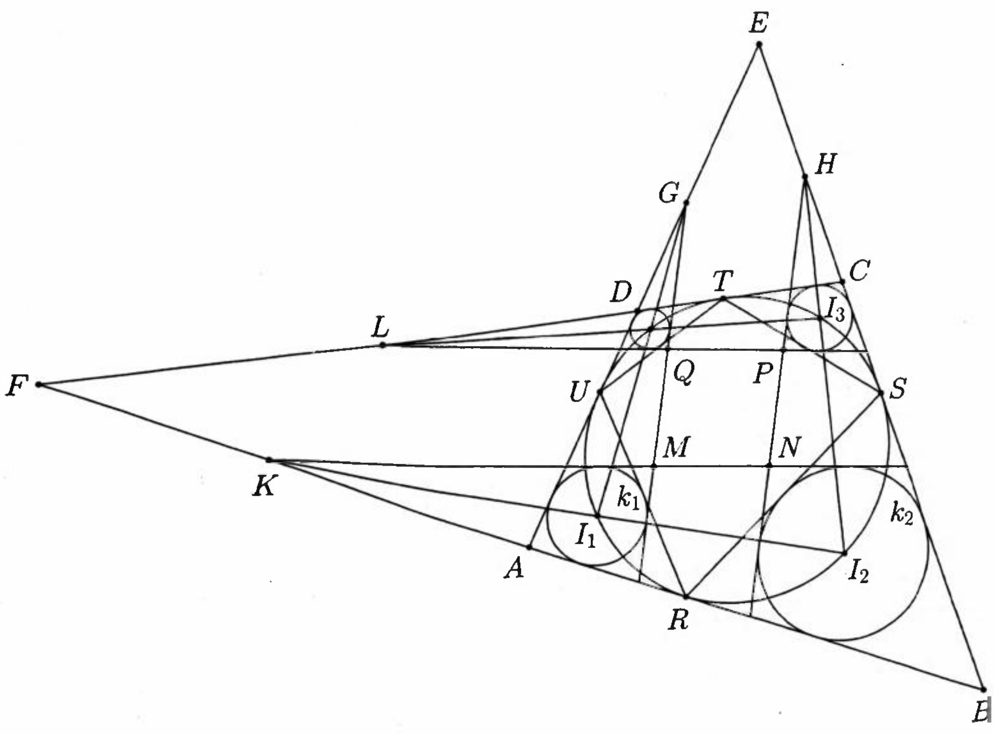
Solution. It suffices to prove that the oriented area of $\triangle MIN$ satisfies $[MIN] = 0$. We have

$$[MIN] = \frac{[MIB] + [MID]}{2} = \frac{\dfrac{[AIB] + [CIB]}{2} + \dfrac{[AID] + [CID]}{2}}{2}$$
$$= \frac{[AIB] + [CID] + [CIB] + [AID]}{4} = \frac{(BC + AD - AB - CD)r}{8}.$$

The quadrilateral $ABCD$ is circumscribed, thus $BC+AD-AB-CD = 0$, and $[MIN] = 0$ follows.



Problem 5.4.7. Let $ABCD$ be a circumscribed quadrilateral with incircle k. The circles $k_1(I_1)$, $k_2(I_2)$, $k_3(I_3)$, $k_4(I_4)$ are the incircles of $\triangle ARU$, $\triangle RBS$, $\triangle SCT$ and $\triangle UTD$, respectively, where R, S, T and U are the points of tangency of AB, BC, CD and DA to k, respectively. Denote the common external tangent lines of k_1 and k_2, of k_2 and k_3, of k_3 and k_4 and of k_4 and k_1, different from the sides of $ABCD$, by t_{12}, t_{23}, t_{34} and t_{14}, respectively. Let $M = t_{12} \cap t_{14}$, $N = t_{12} \cap t_{23}$, $P = t_{23} \cap t_{34}$ and $Q = t_{34} \cap t_{14}$. Prove that the quadrilateral $MNPQ$ is a rhombus.



Solution. Let $K = t_{12} \cap AB$, $L = t_{34} \cap CD$, $G = t_{14} \cap AD$, $H = t_{23} \cap BC$, $F = AB \cap CD$ and $E = AD \cap BC$.

Denote the angles of $ABCD$ by α, β, γ and δ. It is easy to see that the quadrilateral $I_1I_2I_3I_4$ is inscribed in k (see Problem 4.5.1).

It suffices to prove that the quadrilateral $MNPQ$ is a circumscribed parallelogram. The fact that $MN \parallel PQ$ is equivalent to $\angle QLD + \angle MKA = \angle BFC$. We have

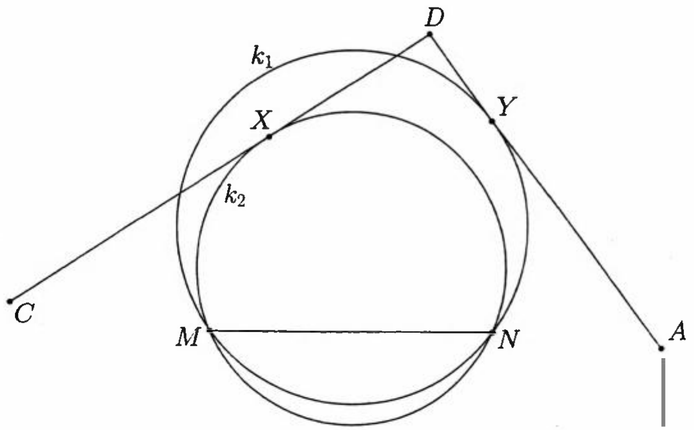
$$\begin{aligned}
\angle MKA + \angle QLD &= 2\angle I_1KR + 2\angle I_4LT \\
&= 2\angle RI_1I_2 - 2\angle I_1RA + 2\angle TI_4I_3 - 2\angle I_4TL \\
&= \angle BRS - \angle ARU + \angle CTS - \angle DTU \\
&= 90^\circ - \frac{\beta}{2} - 90^\circ + \frac{\alpha}{2} + 90^\circ - \frac{\gamma}{2} - 90^\circ + \frac{\delta}{2} \\
&= 180^\circ - \gamma - \beta = \angle BFC.
\end{aligned}$$

Hence, $MN \parallel PQ$ and analogously, $NP \parallel MQ$. Also, Problem 5.4.5 yields that the quadrilateral $MNPQ$ is circumscribed.

Problem 5.4.8. Let $ABCD$ be a circumscribed quadrilateral with incircle k. The circle k_1 is inscribed in $\angle ABC$ and intersects the segment AC at the points M and N. Prove that the circle, which passes through M and N and touches CD, also touches AD.

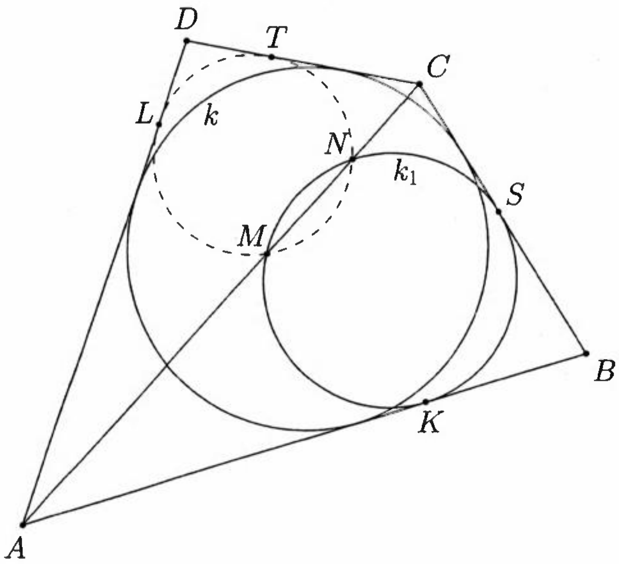
Solution. We will prove the following lemma:

Lemma. An angle $\angle CDA$ and a line MN, not containing D, are given. The circle k_1 passes through M and N and touches DA at the point Y, and the circle k_2 passes through M and N and touches CD at the point X. If $DY = DX$, then $k_1 \equiv k_2$.



Proof. Suppose that $k_1 \not\equiv k_2$. Since $DY = DX$, the powers of D with respect to these circles are equal, and so this point lies on their radical axis.

But this radical axis is the line MN, and $D \notin MN$, a contradiction.



Let the points of tangency of AB and BC to k_1 be K and S, respectively. Let L and T be points on AD and DC, respectively, such that $AK^2 = AM.AN = AL^2$ and $CS^2 = CM.CN = CT^2$.

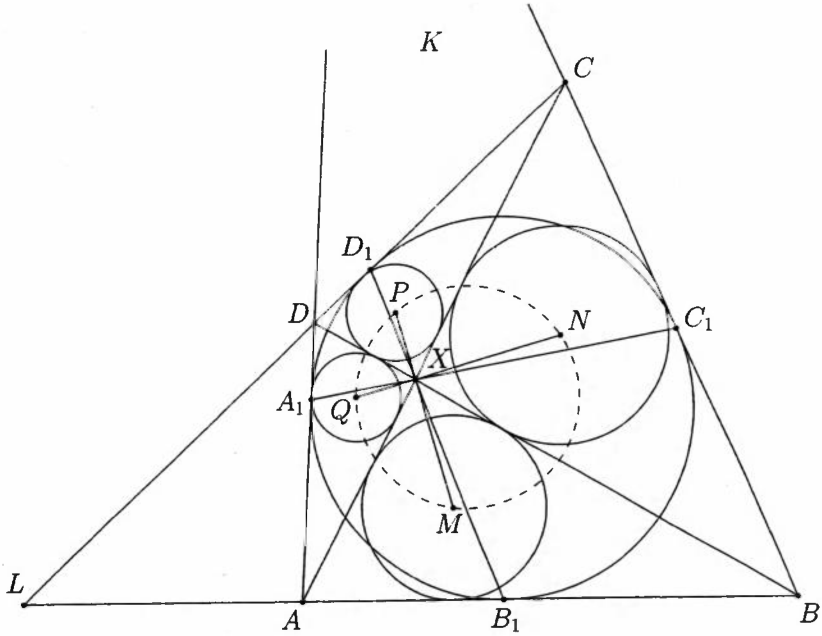
We have

$$\begin{aligned} DL - DT &= (DA - AL) - (DC - CT) = (DA - DC) + (CS - AK) \\ &= (DA - DC) + (CB - BA) = 0 \end{aligned}$$

i.e. $DL = DT$, and on the other hand, the above equalities yield that there exist circles, passing through M and N, and touching DA and DC at the points L and T.

Hence, the conditions of the lemma are satisfied, and we can use it to get the desired result.

Problem 5.4.9. Let $ABCD$ be a circumscribed quadrilateral with incircle k. Let $AC \cap BD = X$. Let $k_1(M)$, $k_2(N)$, $k_3(P)$ and $k_4(Q)$ be the incircles of $\triangle ABX$, $\triangle BCX$, $\triangle CDX$ and $\triangle DAX$, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.



Solution. Let $AB \cap CD = L$ and $AD \cap BC = K$. Denote the points of tangency of the sides AB, BC, CD and DA to k by B_1, C_1, D_1 and A_1, respectively.

Brianchon's Theorem yields $D_1B_1 \cap A_1C_1 = X$. Denote the radii of k_i and k with r_i and r, respectively, for $i = 1, 2, 3, 4$.

It is clear that $\triangle LB_1D_1$ and $\triangle A_1C_1K$ are isosceles. Therefore,

$$\begin{aligned} dist(X, AB) + dist(X, DC) &= dist(D_1, AB) \\ &= B_1D_1 \sin \angle AB_1D_1 = 2r \sin^2 \angle AB_1D_1. \end{aligned}$$

Analogously, $dist(X, AD) + dist(X, BC) = 2r \sin^2 \angle DA_1C_1$.

Hence, $\dfrac{dist(X, AB) + dist(X, DC)}{dist(X, AD) + dist(X, BC)} = \dfrac{\sin^2 \angle AB_1D_1}{\sin^2 \angle DA_1C_1}$. Considering the power of X with respect to k, we get $XB_1.XD_1 = XA_1.XC_1$. Consequently,

$$\begin{aligned}
&dist(X,AB).dist(X,CD).\sin^2\angle DA_1C_1\\
=&dist(X,AD).dist(X,BC).\sin^2\angle AB_1D_1\\
\Rightarrow&\frac{dist(X,AB)+dist(X,DC)}{dist(X,AD)+dist(X,BC)}=\frac{dist(X,AB).dist(X,CD)}{dist(X,AD).dist(X,BC)}\\
\Rightarrow&\frac{1}{dist(X,AB)}+\frac{1}{dist(X,CD)}=\frac{1}{dist(X,AD)}+\frac{1}{dist(X,BC)}\\
\Rightarrow&\frac{AB}{S_{\triangle ABX}}+\frac{CD}{S_{\triangle CDX}}=\frac{AD}{S_{\triangle ADX}}+\frac{BC}{S_{\triangle BCX}}\\
\Rightarrow&\frac{AB}{XA.XB}+\frac{CD}{XC.XD}=\frac{AD}{XA.XD}+\frac{BC}{XB.XC}\\
\Rightarrow&\frac{AB}{XA.XB}+\frac{1}{XA}+\frac{1}{XB}+\frac{CD}{XC.XD}+\frac{1}{XC}+\frac{1}{XD}\\
=&\frac{AD}{XA.XD}+\frac{1}{XA}+\frac{1}{XD}+\frac{BC}{XB.XC}+\frac{1}{XB}+\frac{1}{XC}\\
\Rightarrow&\frac{AB+XA+XB}{S_{\triangle ABX}}+\frac{CD+XC+XD}{S_{\triangle CDX}}\\
=&\frac{AD+XA+XD}{S_{\triangle ADX}}+\frac{BC+XB+XC}{S_{\triangle BCX}}\Rightarrow\frac{1}{r_1}+\frac{1}{r_3}=\frac{1}{r_2}+\frac{1}{r_4}.
\end{aligned}$$

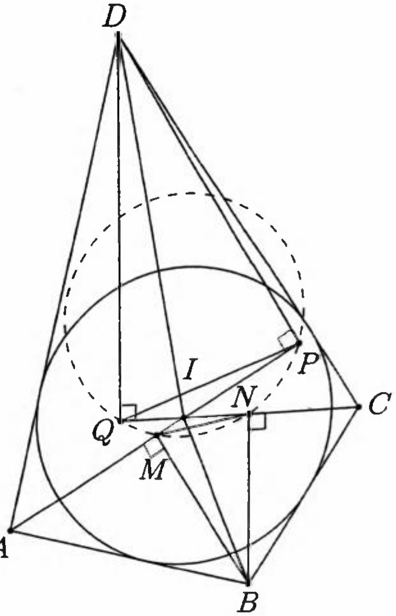
Set $\angle NXC=\varphi$. Note that

$$\begin{aligned}
&2(r_2+r_4)=\\
=&(BX+CX-BC)\operatorname{tg}\varphi+(AX+DX-AD)\operatorname{tg}\varphi\\
=&(AX+BX+CX+DX-BC-AD)\operatorname{tg}\varphi\\
=&(AX+BX+CX+DX-AB-CD)\operatorname{tg}\varphi\\
=&(AX+BX-AB)\operatorname{tg}\varphi+(CX+DX-CD)\operatorname{tg}\varphi=2(r_1+r_3)\operatorname{tg}^2\varphi\\
\Rightarrow&\frac{r_2+r_4}{r_1+r_3}=\operatorname{tg}^2\varphi.
\end{aligned}$$

But we have proved that $\dfrac{r_2+r_4}{r_1+r_3}=\dfrac{r_2r_4}{r_1r_3}$, thus $\dfrac{r_2r_4}{r_1r_3}=\operatorname{tg}^2\varphi$.

We also have $r_2=XN\sin\varphi$, $r_4=XQ\sin\varphi$, $r_1=XM\cos\varphi$ and $r_3=XP\cos\varphi$. This yields $XP.XM=XN.XQ$, as desired.

Problem 5.4.10. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. The feet of the perpendiculars from B and D to the angle bisector of $\angle BAD$ are M and P, respectively, and the feet of the perpendiculars from B and D to the angle bisector of $\angle BCD$ are N and Q, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.



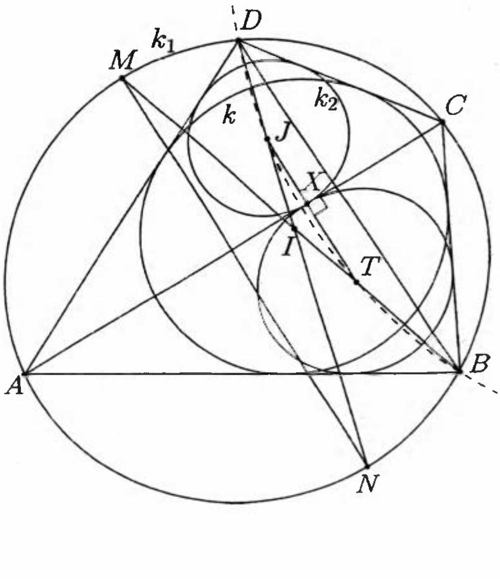
Solution. It suffices to prove that $\angle MNI = \angle MPQ$.

Note that

$$\begin{aligned}\angle MNI &= \angle MBI = 90^\circ - \angle MIB \\ &= 90^\circ - (180^\circ - \angle CID) \\ &= \angle CID - 90^\circ = \angle QDI \\ &= \angle QPI,\end{aligned}$$

and the desired statement follows.

Problem 5.4.11. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. Let $ABCD$ be also inscribed in a circle k_1. The points T and J are the incenters of $\triangle ABC$ and $\triangle ADC$, respectively. Prove that the quadrilateral $DJTB$ is cyclic.



Solution. Let the incircle k_2 of $\triangle ADC$ touch AC at the point X.

Then $CX - XA =$

$$= CD - DA = CB - BA.$$

Hence, X is the point of tangency of AC and the incircle k_3 of $\triangle ABC$. Let M and N be the midpoints of the arcs $\widehat{ADC}$ and $\widehat{ABC}$ of the circle k_1, respectively. Then $MN \parallel JT$ and thus $\angle NJT = \angle JNM = \angle DBM$, as desired.

Problem 5.4.12. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. Let $ABCD$ be also inscribed in a circle k_1. Let k touch AB, BC, CD and DA at the points M, N, P and Q, respectively. Prove that $MP \perp NQ$.

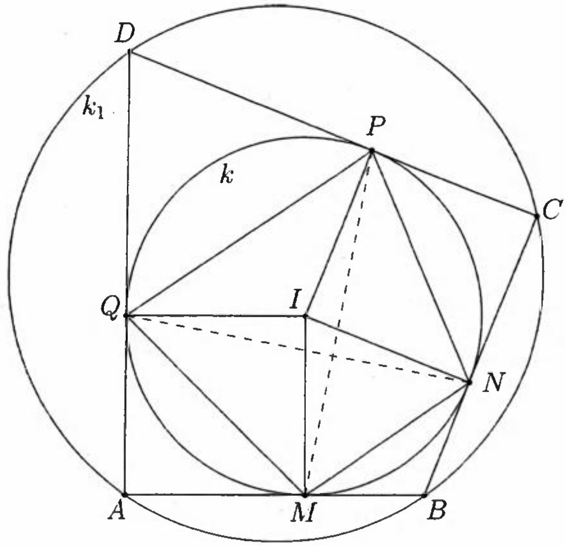
Solution. We have

$$\angle(MP, NQ) = \\ = \frac{\widehat{PN} + \widehat{QM}}{2} \\ = \frac{\angle PIN + \angle QIM}{2} \\ = \frac{360^\circ - \angle DAB - \angle DCB}{2}$$

But we know that

$$\angle DCB + \angle DAB = 180^\circ.$$

Hence, $\angle(MP, NQ) = 90^\circ$.

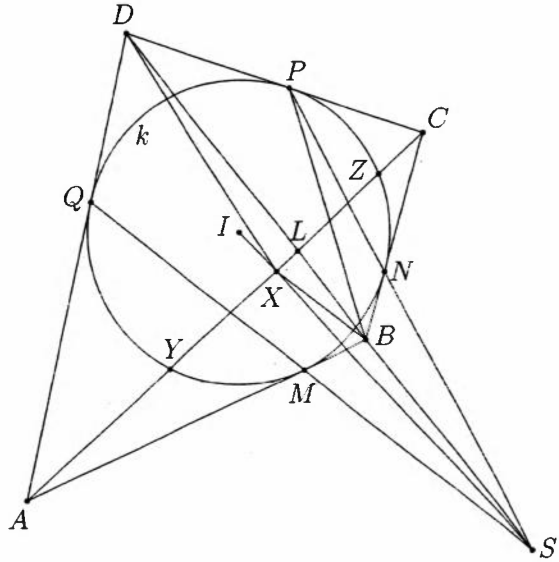


Problem 5.4.13. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. The line AC intersects k at Y and Z. The point X is the midpoint of YZ. Prove that $\angle BXC = \angle DXC$.

Solution. Let $IX \cap BD = S$. Since $\angle IXC = 90^\circ$, it suffices to prove that $(D, L, B, S) = -1$. Let $QM \cap PN = S'$.

Since the polar lines of D, B and S' with respect to k pass through the intersection point of MN and PQ, the points D, B and S' are collinear. The same idea, applied to A and C, leads to the conclusion that AC is the polar line of S'. Thus, $IS' \perp AC$ and $S' \equiv S$.

We deduce that S is the intersection point of the lines IX, BD, MQ and NP. On the other hand, Brianchon's Theorem, applied to the degenerate hexagon $ABNCPD$, yields that the lines PB, DN and AC are concurrent.

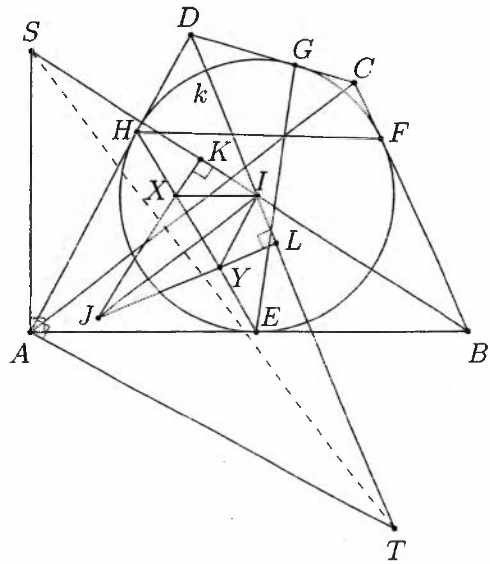


Menelaus' Theorem and Ceva's Theorem, applied to $\triangle BCD$, yield

$$\frac{BS}{SD} = \frac{BN.CP}{NC.PD} = \frac{BL}{LD},$$

whence $(D, L, B, S) = -1$.

Problem 5.4.14. Let $ABCD$ be a circumscribed quadrilateral with incircle k with center I. The line through A, perpendicular to AD, intersects the angle bisector of $\angle ADC$ at the point T, and the line through A, perpendicular to AB, intersects the angle bisector of $\angle ABC$ at the point S. Prove that $ST \perp AC$.



Solution. Let k touch AD and AB at the points H and E, respectively. It suffices to prove that the line connecting the pole of ST and I is parallel to AC.

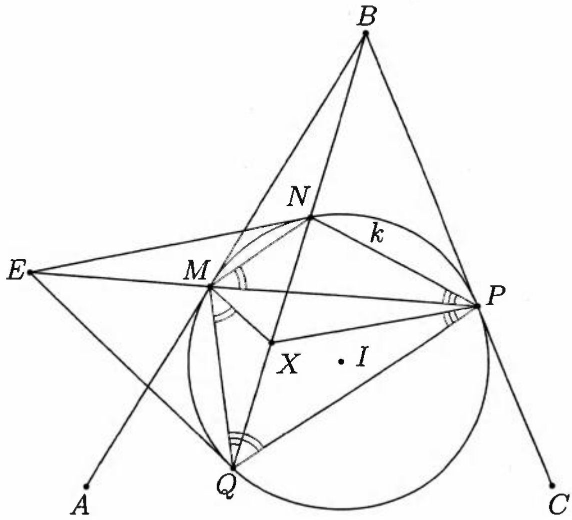
The pole of the line AT lies on the line through I, parallel to AD, and at the same time, it lies on the polar line of A with respect to k. Denote this pole by Y. Then the polar line of T is the line through Y, perpendicular to DI. Let L be the foot of this perpendicular.

Analogously, the polar line of S is the line through X, perpendicular to BI, where $X \in HE$ and $XI \parallel AB$. Let K be the foot of this perpendicular. Denote $XK \cap YL = J$. We will prove that $IJ \parallel AC$.

We have that $XI \parallel AB$ and $YI \parallel AD$. Also, $JY \perp DI$ and $HG \perp DI$. Hence, $JY \parallel HG$ and analogously, $JX \parallel EF$. This implies the existence of a 180° spiral similarity that sends the lines XI, IY, YJ and JX to the lines EA, AH, HG and FE, respectively.

Brianchon's Theorem, applied to the degenerate hexagons $AEBCGD$ and $ABFCDH$, yields that the lines EG, FH, AC, BD are concurrent. Moreover, we know that the lines HG, EF and AC are concurrent (because AC is the polar line of $EH \cap FG$), hence the spiral similarity sends the line IJ to the line AC. Therefore, $IJ \parallel AC$.

Problem 5.4.15. Let k be a circle inscribed in the angle $\angle ABC$. Let k touch AB and BC at the points M and P, respectively. A line through B intersects k at the points N and Q, such that N lies between B and Q. The point X is the midpoint of the segment NQ. Prove that $\angle MXN = \angle PXN$.



Solution. Since M and P lie on the polar line of B with respect to k, then the tangent lines at the points N and Q intersect on MP. Therefore, MP is a symmedian in $\triangle QPN$ and in $\triangle QMN$.

Problem 4.4.1 yields

$$\angle XPQ = \angle NPM = \angle MQN$$

and

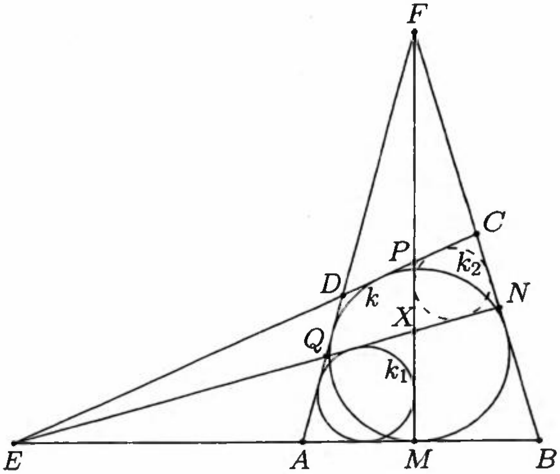
$$\angle NQP = \angle NMP = \angle QMX.$$

Summing both equalities gives the desired result.

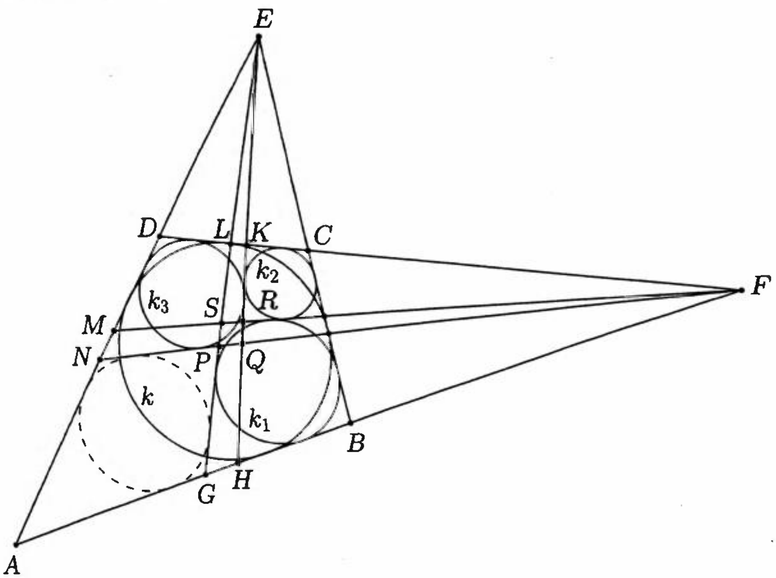
Problem 5.4.16. Let $ABCD$ be a circumscribed quadrilateral with $BC \cap AD = F$ and $BA \cap CD = E$. Let $N \in BC$ and $M \in AB$. The line FM intersects the lines EN and DC at the points X and P, respectively. Let $EX \cap AD = Q$. If the quadrilateral $AMXQ$ is circumscribed, prove that the quadrilateral $XNCP$ is circumscribed.

Solution. We have $FC + CE = FA + AE = FX + XE$. Hence, the quadrilateral $CNXP$ is circumscribed.

Note. The mutual positions of the objects in the configuration can vary, but in each case the solution is an elementary modification of the above arguments.



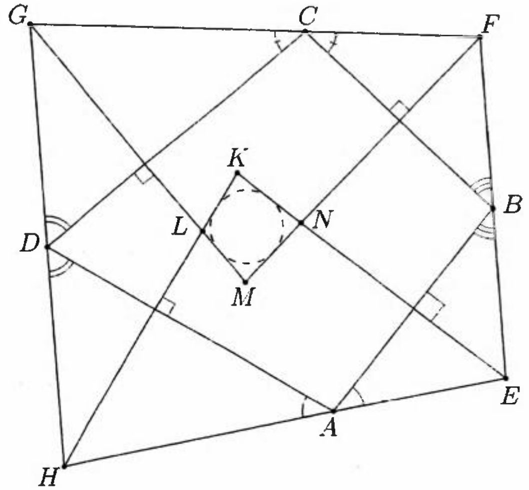
Problem 5.4.17. Let $ABCD$ be a circumscribed quadrilateral with incircle k. Let $AD \cap BC = E$ and $AB \cap CD = F$. The circles k_1, k_2 and k_3 are inscribed in $\angle ABC$, $\angle BCD$ and $\angle CDA$, respectively. One of the common internal tangent lines of k_1 and k_2 passes through F and intersects EA and the tangent lines through E to k_1 and k_3 at the points M, S and R, respectively. The tangent line through F to k_3 intersects the same lines at the points N, P and Q, respectively. One of the common internal tangent lines of k_2 and k_3 passes through E and intersects DC and AB at the points K and H. The tangent line through E to k_1 intersects DC and AB at the points L and G, respectively. Prove that the quadrilateral $AGPN$ is circumscribed.



Solution. Problem 5.4.16, applied to the quadrilateral $ABCD$ and the lines FM and EG, yields that the quadrilateral $MSLD$ is circumscribed.

The same problem, applied to the quadrilateral $NQKD$ and the lines EP and FM, yields that the quadrilateral $PQRS$ is circumscribed. Again Problem 5.4.16, applied to the quadrilateral $ABCD$ and the lines EH and FM, yields that the quadrilateral $AHRM$ is circumscribed. Finally, the same problem, applied to the quadrilateral $AHRM$ and the lines EG and FN, yields that the quadrilateral $AGPN$ is circumscribed.

Problem 5.4.18. Let $ABCD$ be a convex quadrilateral. The bisectors l_a, l_b, l_c and l_d of the external angles of $ABCD$ form the quadrilateral $EFGH$, where $l_a \cap l_b = E$, $l_b \cap l_c = F$, $l_c \cap l_d = G$ and $l_d \cap l_a = H$. The perpendicular from E to AB intersects the perpendicular from F to BC at the point N. The perpendicular from G to CD intersects the perpendicular from F to BC at the point M. The perpendicular from E to AB intersects the perpendicular from H to AD at the point K. The perpendicular from H to AD intersects the perpendicular from G to DC at the point L. Prove that the quadrilateral $LMNK$ is circumscribed.

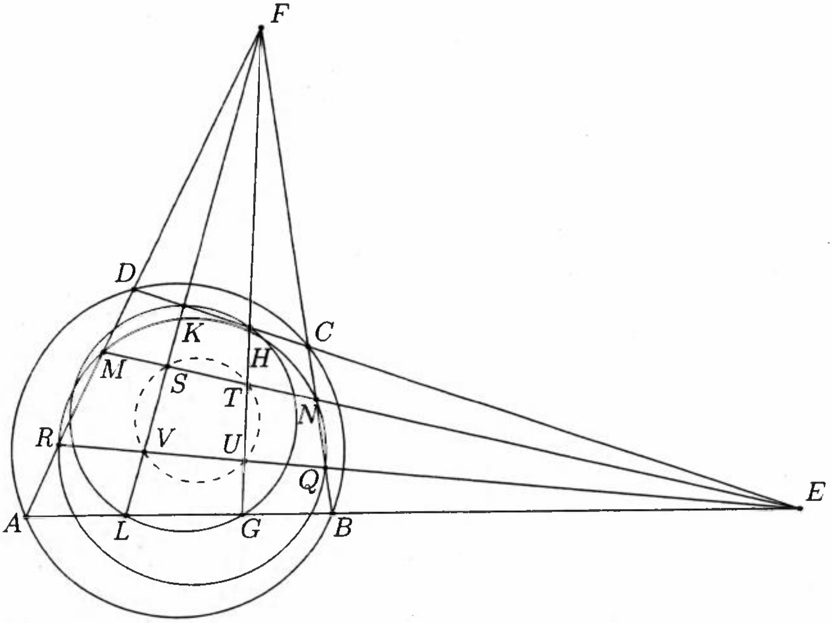


Solution. Clearly, $\angle KHA = \angle KEA$ and thus ΔKHE is isosceles, which implies that the angle bisector of $\angle LKN$ coincides with the perpendicular bisector of HE. We can make a similar argument for the other three angles.

We come to the conclusion that the four bisectors of $LMNK$ are concurrent if the quadrilateral $EFGH$ is cyclic. The latter is true because

$$\angle AEB + \angle CGD = 180^\circ - \left(90^\circ - \frac{\angle ABC}{2}\right) - \left(90^\circ - \frac{\angle BAD}{2}\right)$$
$$+ 180^\circ - \left(90^\circ - \frac{\angle BCD}{2}\right) - \left(90^\circ - \frac{\angle CDA}{2}\right) = 180^\circ.$$

Problem 5.5.1. Let $ABCD$ be a cyclic quadrilateral with $AB \cap CD = E$ and $AD \cap BC = F$. The points $K, H \in CD$ and $L, G \in AB$ are chosen, such that the quadrilateral $LGHK$ is cyclic and the triples of points F, K, L and F, H, G are collinear. The points $Q, N \in BC$ and $R, M \in AD$ are chosen, such that the quadrilateral $QNMR$ is cyclic and the triples of points E, N, M and E, Q, R are collinear. Let $RQ \cap KL = V$, $RQ \cap GH = U$, $GH \cap MN = T$ and $MN \cap KL = S$. Prove that the quadrilateral $VUTS$ is cyclic.



Solution. We have

$$\angle SVU = \angle VLG + \angle MRV - \angle LAR = \angle THC + \angle TNC - \angle BCF = \angle STH,$$

as desired.

Note. The proof does not use that any of the triples of points F, K, L; F, H, G; E, N, M and E, Q, R are collinear. The statement remains true even if none of these triples of points are collinear.

Problem 5.5.2. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The tangent lines at the points D and C intersect at the point E. The lines through E, parallel to AC and BD, intersect BC and AD at the points F and G, respectively. Prove that the quadrilateral $ABFG$ is cyclic.

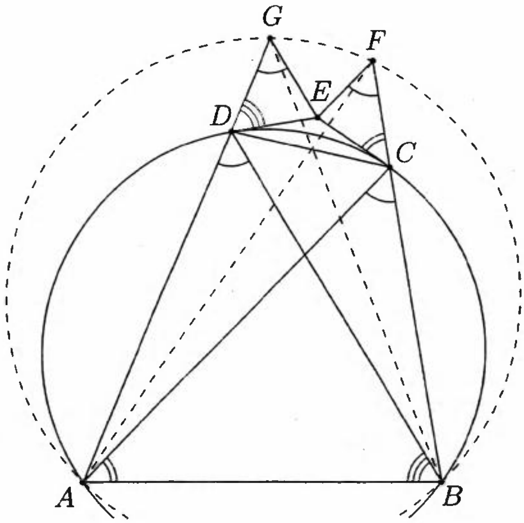
Solution. We have

$$\angle ECF = \frac{\widehat{BC}}{2} = \angle BAC$$

and $\angle EFC = \angle ACB$, which gives $\triangle ECF \sim \triangle BAC$. Analogously, we deduce that $\triangle EDG \sim \triangle ABD$. Now, $\angle ACF = \angle BDG$ and

$$\frac{DB}{DG} = \frac{AB}{DE} = \frac{AB}{EC} = \frac{AC}{CF}.$$

Therefore, $\triangle ACF \sim \triangle BDG$ and $\angle AFB = \angle AGB$.

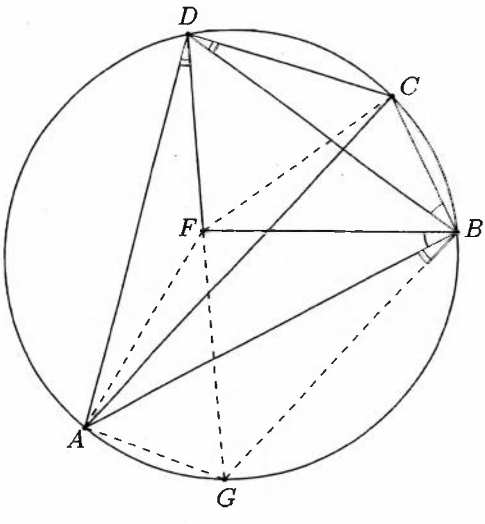


Problem 5.5.3. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The lines, symmetric to BD with respect to the angle bisectors of $\angle ADC$ and $\angle ABC$, intersect at the point F. Prove that $FA = FC$.

Solution. Let DF intersect k at the point G. Then $\widehat{AG} = \widehat{BC}$. On the other hand, $AGBC$ is an isosceles trapezoid. But

$$\angle DGB = \angle DGC + \angle CGB = \angle CBD + \angle CDB = \angle FBA + \angle ADG = \angle FBG.$$

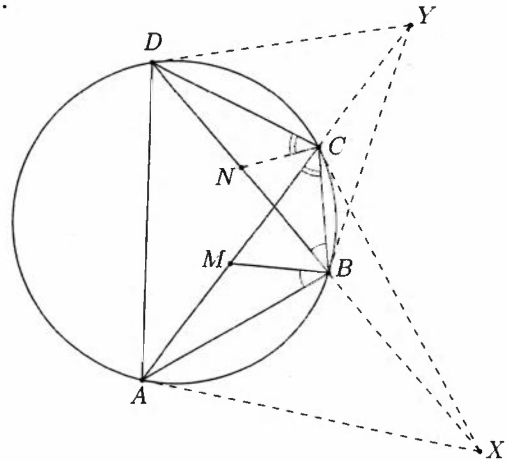
Hence, $\triangle GBF$ is isosceles and F lies on the perpendicular bisector of GB. The perpendicular bisectors of BG and AC coincide, which implies that F lies on the perpendicular bisector of AC, so $FA = FC$.



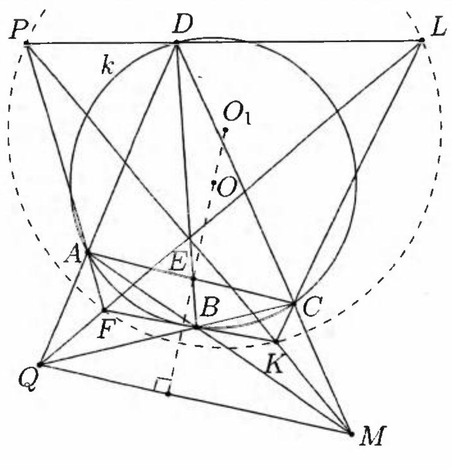
Problem 5.5.4. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Let M and N be the midpoints of AC and BD, respectively. If $\angle ABM = \angle CBD$, prove that $\angle BCA = \angle NCD$.

Solution. We have that BD is a symmedian in $\triangle ABC$ and hence it passes through the intersection point X of the tangent lines at A and C to k. Thus, the quadrilateral $ABCD$ is harmonic.

This implies that the tangent lines at B and D to k, and the line AC are also concurrent. Therefore, AC is a symmedian in $\triangle BCD$. Thus, $\angle BCA = \angle NCD$.



Problem 5.5.5. Let $ABCD$ be a cyclic quadrilateral with circumcircle k with center O. Let $AC \cap BD = E$. The external angle bisectors through A, B, C and D form the quadrilateral $FKLP$, as shown in the figure. Prove that $FKLP$ is cyclic and that its center O_1 lies on the line OE.



Solution. We have

$$\begin{aligned}\angle DLC + \angle AFB &= 360° - \angle LCD - \angle LDC - \angle FAB - \angle FBA \\ &= 360° - \angle KCB - \angle PDA - \angle DAP - \angle KBC \\ &= \angle APD + \angle BKC \\ &\Rightarrow \angle DLC + \angle AFB = \angle APD + \angle BKC = 180°.\end{aligned}$$

Hence, the quadrilateral $FKLP$ is cyclic. Let $DA \cap CB = Q$ and $AB \cap DC = M$. Then F and K are the incenters of $\triangle ABQ$ and $\triangle BMC$, respectively. Also, P is the M-excenter of $\triangle ADM$, and L is the Q-excenter of $\triangle CQD$.

Therefore, the points Q, F and L lie on the bisector of $\angle BQA$, and the points M, K and P lie on the bisector of $\angle BMC$. We have $\angle LFB = 90° - \angle FAB = 90° - \dfrac{\angle DCB}{2} = \angle BCK$.

Hence, the quadrilateral $FBCL$ is cyclic. Analogously, we obtain that the quadrilateral $KBAP$ is cyclic. Consequently, $QB.QC = QA.QD$ and $MB.MA = MK.MP$.

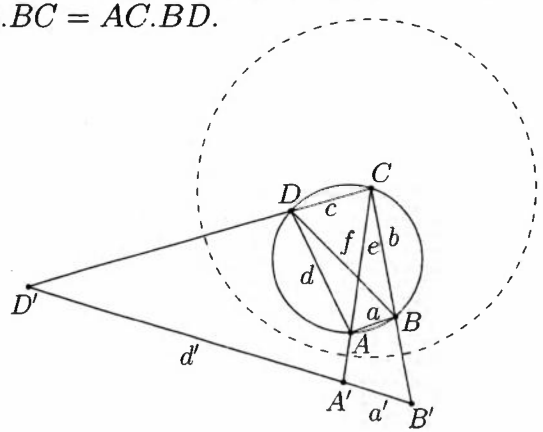
These equalities imply that M and Q lie on the radical axis of k and the circumcircle of $FKLP$. Thus, $OO_1 \perp QM$. Brocard's Theorem (Problems 4.8.24 and 4.8.25) yields that the point E is the orthocenter of $\triangle OMQ$, so $OE \perp QM$.

Therefore, the points O, O_1 and E are collinear.

Problem 5.5.6. *(Ptolemy's Theorem)* If the quadrilateral $ABCD$ is cyclic, prove that $AB.CD + AD.BC = AC.BD$.

Solution. Set $AB = a$, $DC = b$, $CB = c$, $DA = d$, $AC = e$ and $BD = f$.

Consider an inversion with center C and an arbitrary radius. Let the images of D, A and B be D', A' and B', respectively (they are collinear). Let also $D'A' = d'$ and $A'B' = a'$.



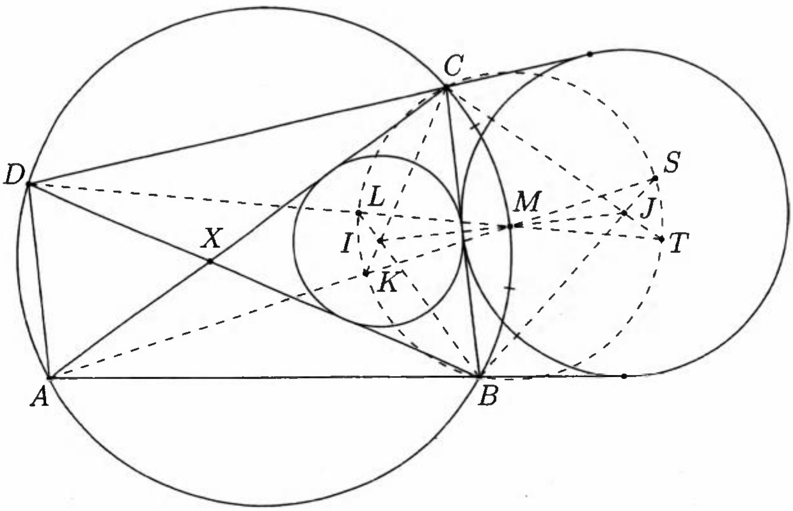
We have that the quadrilaterals $A'B'BA$ and $D'A'AD$ are cyclic. Also, $\dfrac{a}{a'} = \dfrac{b}{CA'}$, which gives $a = \dfrac{a'.b}{CA'}$.

Analogously, $d = \dfrac{d'.c}{CA'}$ and $f = \dfrac{(a'+d').b}{CD'}$.

Now the equality $a.c + b.d = e.f$ is equivalent to $\dfrac{c}{CA'} = \dfrac{e}{CD'}$, which is true.

Note. One can analogously show that every four points A, B, C and D in the plane satisfy Ptolemy's inequality $AB.CD + AD.BC \geq AC.BD$. Bretschneider's Theorem can be proven using the same idea.

Problem 5.5.8. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Its diagonals intersect at the point X. The incenter of $\triangle XBC$ is I, and the center of the circle, external to $ABCD$, that touches the lines AB and CD and the segment BC is J. Prove that if M is the midpoint of the arc $\widehat{CB}$ of k, then the points I, M and J are collinear.

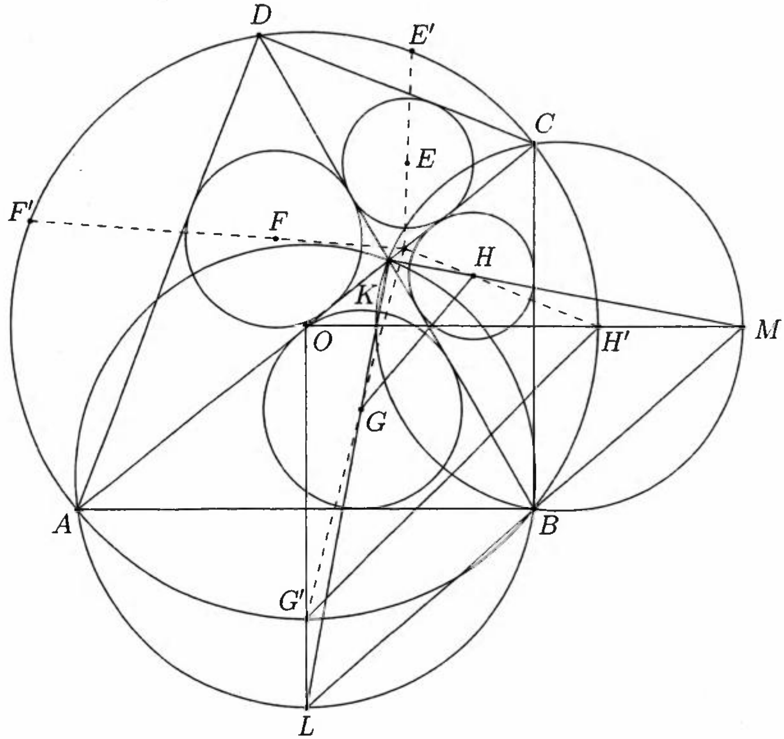


Solution. Denote the incenters of $\triangle ABC$ and $\triangle DBC$ by K and L, respectively. Let the excircles of these triangles, tangent to the segment BC, have centers S and T, respectively.

The trillium theorem (Problem 4.6.1) and its analog for excircles, applied to $\triangle ABC$, $\triangle DBC$ and the midpoint M of the arc $\widehat{CB}$ of k, yields that the points L, K, B, T, S and C lie on a circle with center M.

Finally, Pascal's Theorem, applied to the hexagon $KCTLBS$, yields that the points $LB \cap KC = I$, $LT \cap KS = M$ and $CT \cap BS = J$ are collinear.

Problem 5.5.9. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Its diagonals intersect at the point K. The incenters of $\triangle ABK$, $\triangle BCK$, $\triangle CDK$ and $\triangle DAK$ are G, H, E and F, respectively. The midpoints of the smaller arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$ are G', H', E' and F', respectively. Prove that the lines GG', HH', EE' and FF' are concurrent.



Solution. Let O be the center of k. It suffices to prove that the lines GG', HH' and OK are concurrent.

Desargues' Theorem (Problem 7.1), applied to $\triangle OG'H'$ and $\triangle KGH$, yields that it suffices to prove that the points $L = OG' \cap KG$, $M = OH' \cap KH$ and $G'H' \cap GH$ are collinear.

It is easy to see that L and M are the midpoints of the arcs $\widehat{AB}$ and $\widehat{BC}$ of the circumcircles of $\triangle ABK$ and $\triangle BCK$, respectively.

If we prove that $\dfrac{LG'}{G'O} \cdot \dfrac{OH'}{H'M} = \dfrac{LG}{GK} \cdot \dfrac{KH}{HM}$ the desired statement will follow by Menelaus' Theorem, applied to $\triangle OLM$ and $\triangle LKM$. We have

$$\frac{LG'}{G'O} \cdot \frac{OH'}{H'M} \cdot \frac{HM}{LG} = \frac{LG'}{LG} \cdot \frac{HM}{H'M} = \frac{LG'}{LB} \cdot \frac{MB}{H'M} = \frac{\sin\angle G'BL}{\sin\angle LG'B} \cdot \frac{\sin\angle MH'B}{\sin\angle H'BM}.$$

On the other hand,

$$\angle G'BL = \frac{1}{2}(\angle AG'B - \angle ALB) = \frac{1}{2}(\angle AKB - \angle ACB) = \angle KBH,$$

whence

$$\angle BG'L = 180^\circ - \angle OG'B = 90^\circ + \frac{\angle ACB}{2}.$$

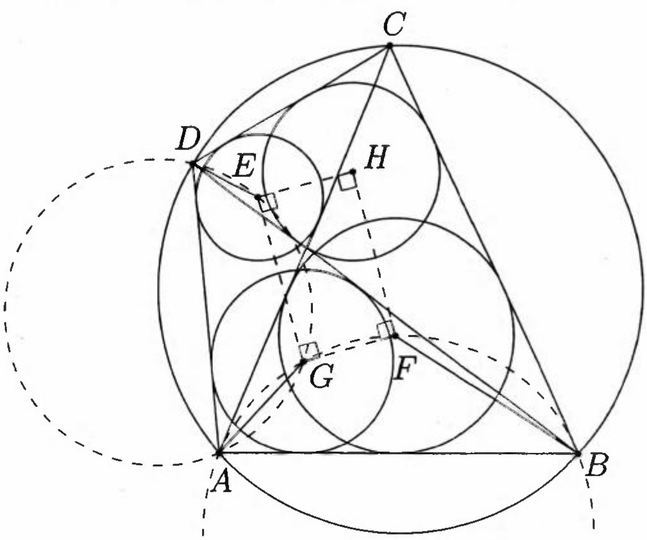
We deduce that $\dfrac{\sin\angle G'BL}{\sin\angle LG'B} = \dfrac{\sin\angle KBH}{\sin\angle KHB}$.

Analogously, $\dfrac{\sin\angle MH'B}{\sin\angle H'BM} = \dfrac{\sin\angle KBG}{\sin\angle BGK}$. Therefore,

$$\frac{\sin\angle G'BL}{\sin\angle LG'B}\cdot\frac{\sin\angle MH'B}{\sin\angle H'BM} = \frac{\sin\angle KBH}{\sin\angle KHB}\cdot\frac{\sin\angle BGK}{\sin\angle KBG} = \frac{KH}{BK}\cdot\frac{BK}{GK} = \frac{KH}{GK},$$

whence the desired equality follows.

Problem 5.5.10. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The incenters of $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$ are F, H, E and G, respectively. Prove that the quadrilateral $EGFH$ is a rectangle.



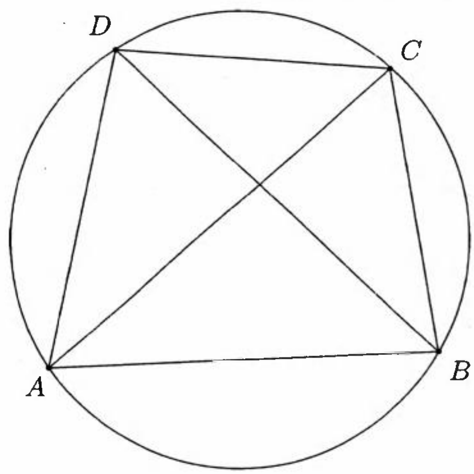
Solution. We have $\angle AGB = 90^\circ + \dfrac{\angle ADB}{2} = 90^\circ + \dfrac{\angle ACB}{2} = \angle AFB$. Hence, the quadrilateral $ABFG$ is cyclic. Analogously, the quadrilateral

$AGED$ is cyclic. Then

$$\begin{aligned}\angle FGE &= 360^\circ - \angle AGF - \angle AGE \\ &= 360^\circ - (180^\circ - \angle ADE) - (180^\circ - \angle ABF) \\ &= \frac{\angle ABC + \angle ADC}{2} = 90^\circ.\end{aligned}$$

Analogously, the other three angles of the quadrilateral $GFHE$ are right, as desired.

Second Theorem of Ptolemy. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Prove that $\dfrac{AC}{BD} = \dfrac{AD.AB + CB.CD}{DA.DC + BA.BC}$.



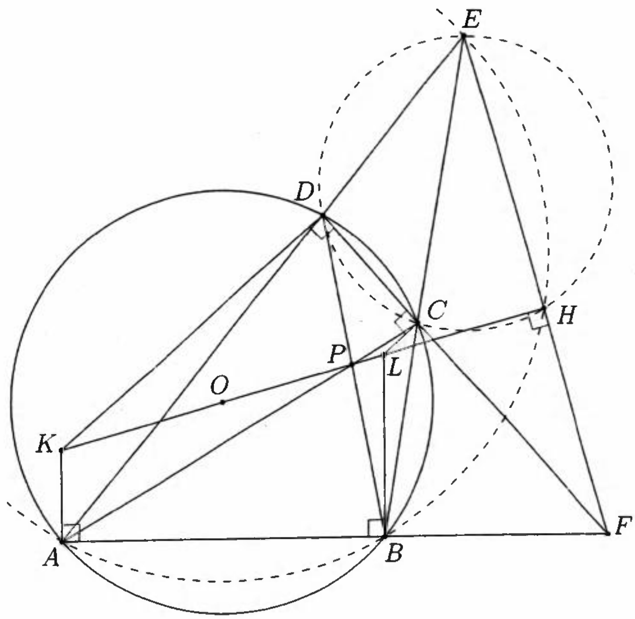
Solution. We will use the well-known formula $S = \dfrac{abc}{4R}$ for a triangle with side lengths a, b, c and circumradius R.

We have $S_{ABC} + S_{ADC} = S_{ABD} + S_{CBD}$. Hence,

$$\frac{BA.BC.AC}{4R} + \frac{DA.DC.AC}{4R} = \frac{AB.AD.BD}{4R} + \frac{CB.CD.BD}{4R},$$

and the desired statement follows.

Problem 5.6.1. Let $ABCD$ be a cyclic quadrilateral with circumcenter O and let $AD \cap BC = E$. The circumcircles of $\triangle ABE$ and $\triangle CDE$ intersect for the second time at the point H. Prove that $\angle EHO = 90°$.



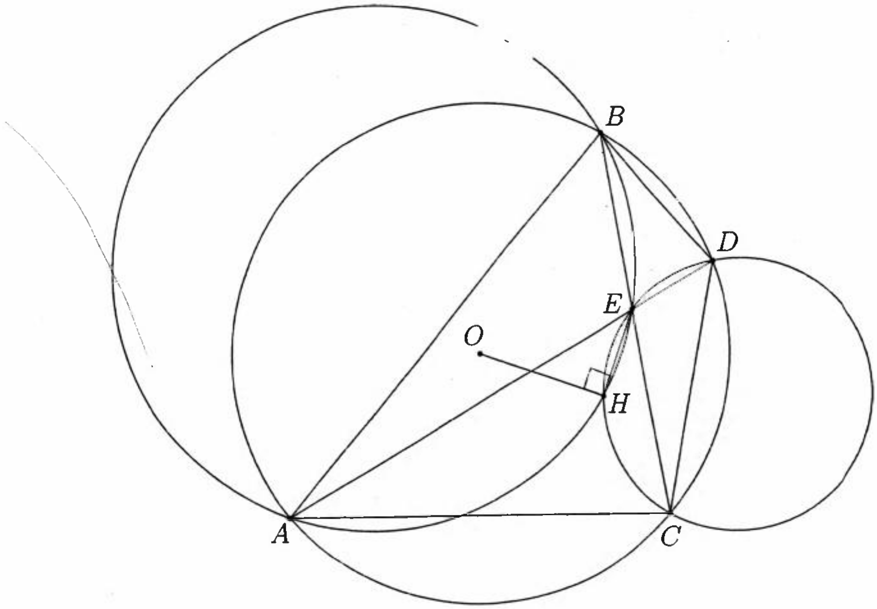
Solution. Let $AB \cap CD = F$, $AC \cap BD = P$ and $OP \cap EF = H'$.

Brocard's Theorem (Problem 4.8.25) yields that $OP \perp EF$. Thus, $OH' \perp EF$. Let K and L be points such that $AK \perp AB$, $DK \perp DC$, $LB \perp AB$ and $LC \perp CD$.

Problem 5.7.6 yields $K \in OP$ and $L \in OP$. Hence, the pentagon $FBLCH'$ is inscribed in the circle with diameter LF. Also, the pentagon $FAKDH'$ is inscribed in the circle with diameter FK.

Therefore, H' is the Miquel point (Problem 2.33) of the quadrilateral $ABCD$. Hence, the quadrilaterals $ABH'E$ and $CDEH'$ are cyclic. In conclusion, $H' \equiv H$ and $\angle EHO = 90°$.

Problem 5.6.2. Let $ABDC$ be a cyclic quadrilateral with circumcenter O and let $AD \cap BC = E$. The circumcircles of $\triangle ABE$ and $\triangle CDE$ intersect for the second time at the point H. Prove that $\angle EHO = 90°$.



Solution. This problem is essentially the same as Problem 5.6.1 but for the quadrilateral $ABDC$ instead of $ABCD$. The solution is completely analogous.

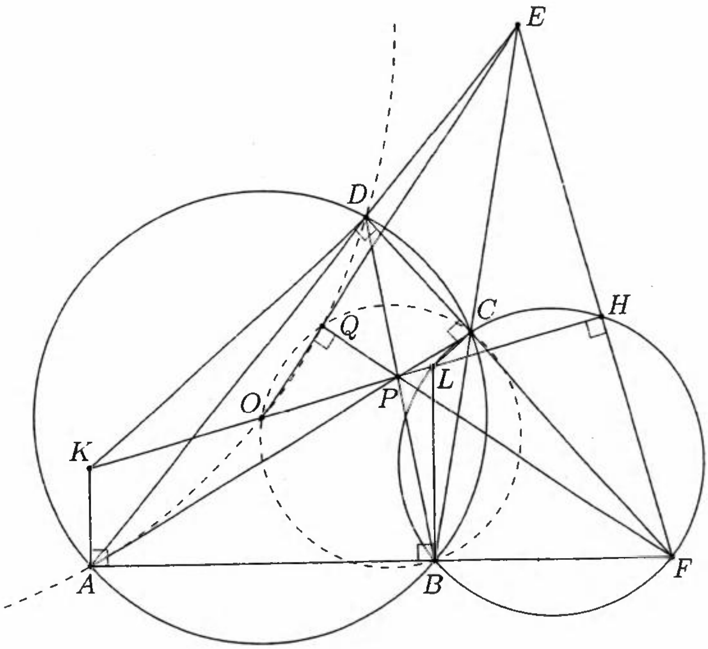
Problem 5.6.3. Let $ABCD$ be a cyclic quadrilateral with circumcenter O and let $AB \cap CD = F$. The circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect for the second time at the point Q. Prove that $\angle FQO = 90°$.

Solution. Let us use the results and notations of Problem 5.6.1. Let $FP \cap EO = Q'$.

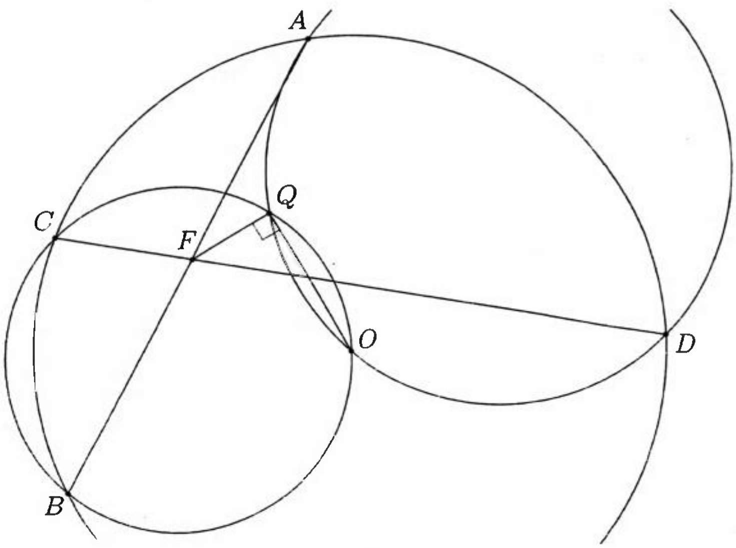
Brocard's Theorem (Problem 4.8.24) yields $FP \perp EO$. Therefore, the quadrilateral $OFHQ'$ is cyclic. On the other hand, the quadrilateral $BCHF$ is also cyclic. Hence,

$$EQ'.EO = EH.EF = EC.EB = EA.ED.$$

These equalities give that the quadrilaterals $BCQ'O$ and $ADQ'O$ are cyclic, which means that $Q' \equiv Q$ and $\angle FQO = 90°$.

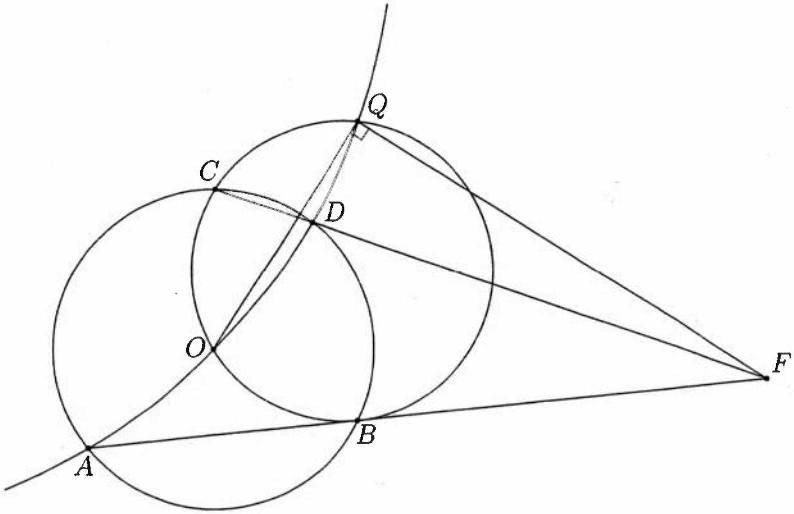


Problem 5.6.4. Let $ADBC$ be a cyclic quadrilateral with circumcenter O and let $AB \cap CD = F$. The circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect for the second time at the point Q. Prove that $\angle FQO = 90°$.



Solution. This problem is essentially the same as Problem 5.6.3 but for the quadrilateral $ADBC$ instead of $ABCD$. The solution is completely analogous.

Problem 5.6.5. Let $ABDC$ be a cyclic quadrilateral with circumcenter O and let $AB \cap CD = F$. The circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect for the second time at the point Q. Prove that $\angle FQO = 90°$.



Solution. This problem is essentially the same as Problem 5.6.3 but for the quadrilateral $ABDC$ instead of $ABCD$. The solution is completely analogous.

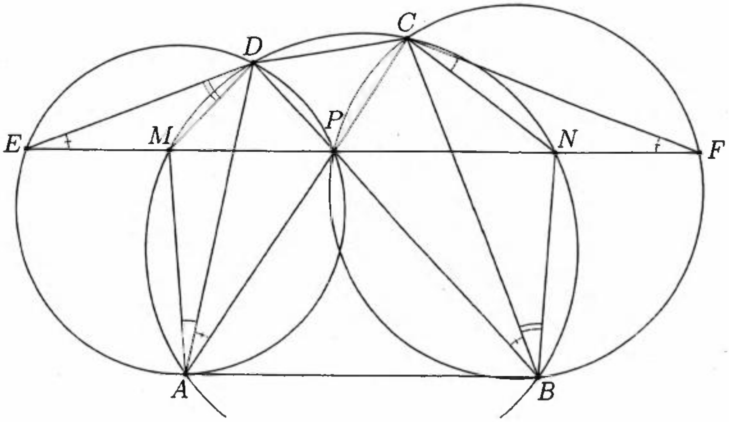
Problem 5.6.6. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Let $AC \cap BD = P$. The line l passes through P and intersects the circumcircles of $\triangle APD$ and $\triangle BCP$ for the second time at the points E and F, respectively (E and F lie outside of k). Let $l \cap k = \{M, N\}$ and let M lie between E and N. Prove that $EM = NF$.

Solution. Set $\angle DAP = \psi$, $\angle DAM = \delta$ and $\angle CBN = \varphi$.

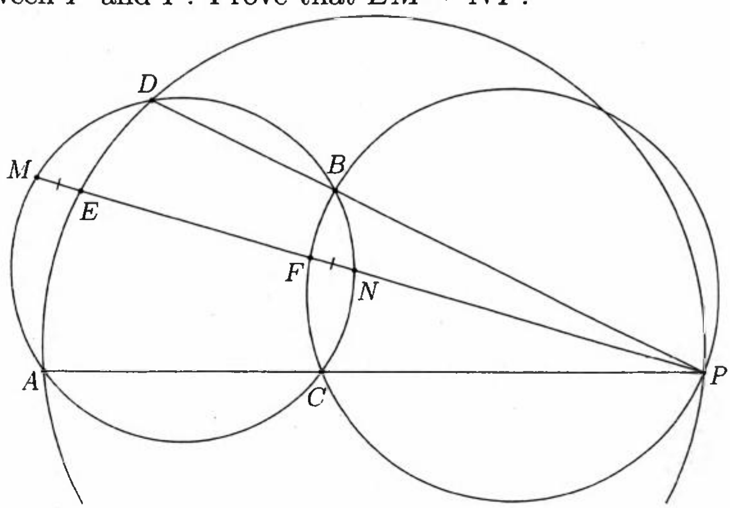
Therefore, $\angle PED = \psi = \angle DBC = \angle PFC$, $\angle FCN = \angle CNM - \psi = \angle CAM - \psi = \angle DAM = \delta$ and $\angle EDM = \angle DMN - \psi = \angle DBN - \psi = \angle CBN = \varphi$. This gives

$$\frac{NF}{EM} = \frac{CN \sin\psi \sin\delta}{MD \sin\psi \sin\varphi} = \frac{CN}{\sin\varphi} \cdot \frac{\sin\delta}{MD} = \frac{2R}{2R} = 1,$$

where R is the radius of k. Thus, $EM = NF$.

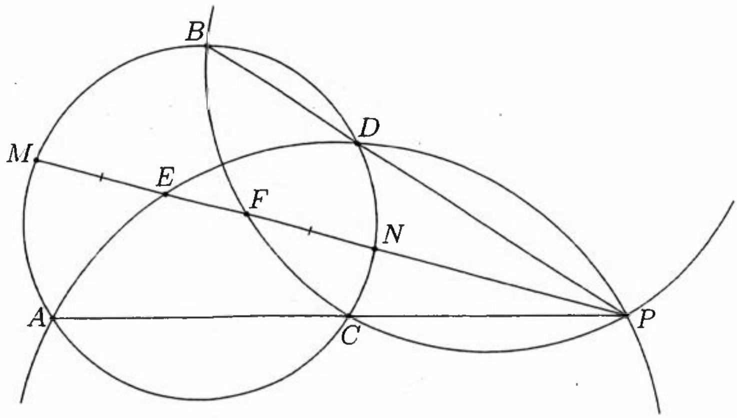


Problem 5.6.7. Let $ACBD$ be a cyclic quadrilateral with circumcircle k. Let $AC \cap BD = P$. The line l passes through P and intersects the circumcircles of $\triangle APD$ and $\triangle BCP$ for the second time at the points E and F, respectively (E and F lie inside of k). Let $l \cap k = \{M, N\}$ and let N lie between P and F. Prove that $EM = NF$.



Solution. This problem is essentially the same as Problem 5.6.6 but for a quadrilateral $ACBD$ instead of $ABCD$, and points E and F lying inside of k. The solution is completely analogous.

Problem 5.6.8. Let $ACDB$ be a cyclic quadrilateral with circumcircle k. Let $AC \cap BD = P$. The line l passes through P and intersects the circumcircles of $\triangle APD$ and $\triangle BCP$ for the second time at the points E and F, respectively (E and F lie inside of k). Let $l \cap k = \{M, N\}$ and let N lie between P and F. Prove that $EM = NF$.

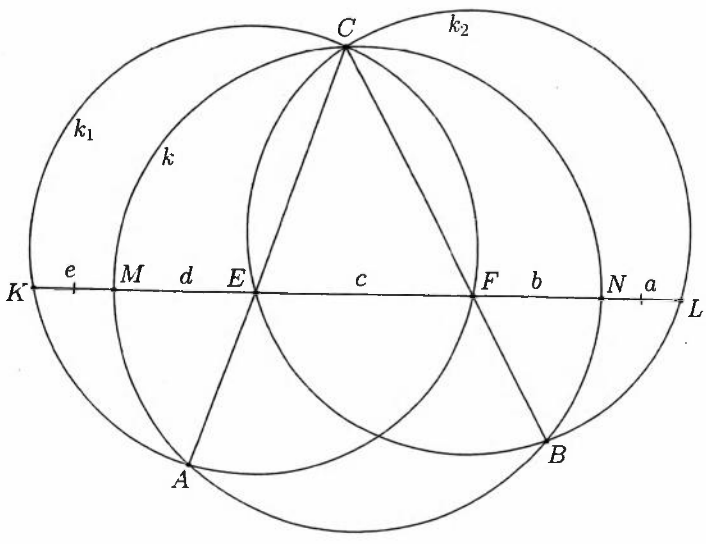


Solution. This problem is essentially the same as Problem 5.6.6 but for a quadrilateral $ACDB$ instead of $ABCD$, and points E and F lying inside of k. The solution is completely analogous.

Problem 5.6.9. Let ABC be a triangle. The points E and F lie on the segments AC and BC, respectively. Denote the circumcircles of $\triangle ABC$, $\triangle ACF$ and $\triangle BCE$ by k, k_1 and k_2, respectively. Let $EF \cap k = \{M, N\}$ and let E lie between M and F. Let also $k_1 \cap EF = \{F, K\}$ and $k_2 \cap EF = \{E, L\}$. Prove that $MK = NL$.

Solution. Set $NL = a$, $FN = b$, $EF = c$, $ME = d$ and $KM = e$. Considering the power of the point F with respect to k and k_2, we get

$$c(b+a) = CF.BF = b(c+d) \Rightarrow a = \frac{bd}{c}.$$



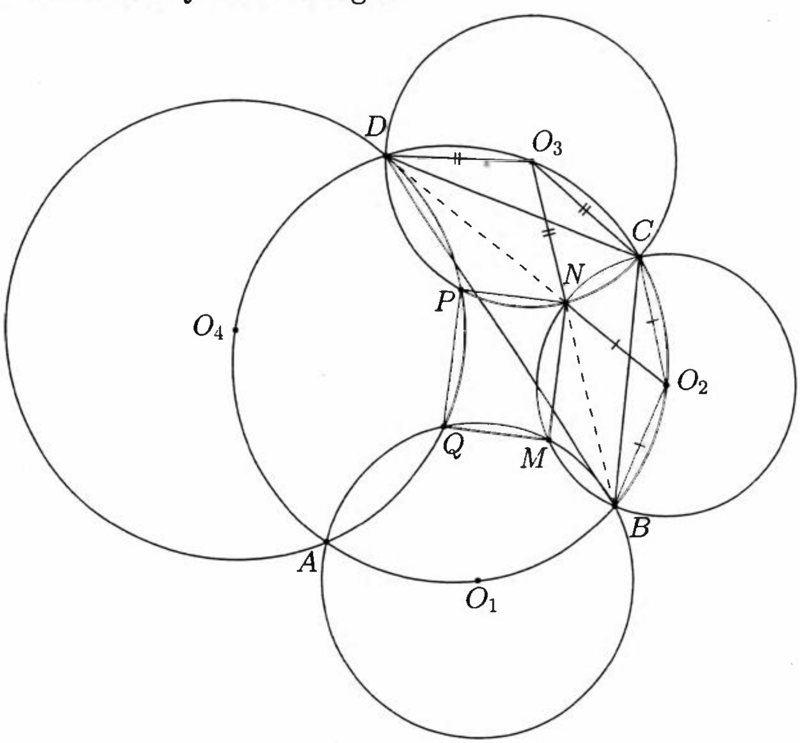
Considering the power of the point E with respect to the circles k and k_1, we get

$$c(e+d) = AE.EC = d(b+c) \Rightarrow e = \frac{bd}{c} \Rightarrow a = e \Rightarrow MK = NL.$$

Problem 5.6.10. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. The points O_1, O_2, O_3 and O_4 are the midpoints of the small arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$ and $\widehat{DA}$, respectively. For the circles $k_1(O_1;O_1A)$, $k_2(O_2;O_2B)$, $k_3(O_3;O_3C)$ and $k_4(O_4;O_4D)$, let $k_1 \cap k_2 = \{B, M\}$, $k_2 \cap k_3 = \{C, N\}$, $k_3 \cap k_4 = \{D, P\}$ and $k_4 \cap k_1 = \{A, Q\}$. Prove that the quadrilateral $MNPQ$ is a rectangle.

Solution. The trillium theorem (Problem 4.6.1), applied to $\triangle BCD$, yields that if I is its incenter, then $IO_3 = O_3C$ and $IO_2 = O_2C$. But $NO_3 = O_3C$ and $NO_2 = O_2C$. Thus, $N \equiv I$.

Analogously, the points M, Q and P are the incenters of $\triangle ABC$, $\triangle ABD$ and $\triangle ACD$, respectively. Finally, Problem 5.5.10 yields that the quadrilateral $MNPQ$ is a rectangle.

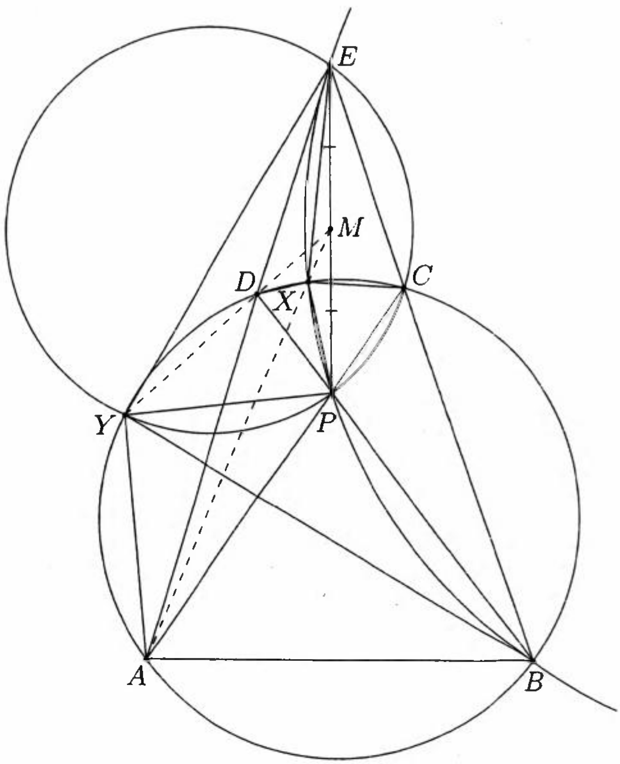


Problem 5.6.11. Let $ABCD$ be a cyclic quadrilateral with circumcircle k. Let $AD \cap BC = E$ and $AC \cap BD = P$. The point M is the midpoint of EP. The circumcircle of $\triangle BPE$ intersects k at the points B and X. The circumcircle of $\triangle CPE$ intersects k at the points C and Y. Prove that the triples of points A, X, M and D, Y, M are collinear.

Solution. We will prove that $\dfrac{\sin\angle EAM}{\sin\angle PAM} = \dfrac{\sin\angle EAX}{\sin\angle PAX}$.

This will imply $\cot\angle EAM = \cot\angle EAX$ and, consequently, $\angle EAM = \angle EAX$, which will give the collinearity of A, X and M. We have

$$\frac{\sin\angle EAM}{\sin\angle PAM} = \frac{PA}{AE} = \frac{DP.AB}{CD.AE} = \frac{DP.CD}{CD.EC} = \frac{DP}{EC}.$$



Note that $\angle XDP = \angle XDB = \angle XCE$ and $\angle XPD = \angle XEC$. Hence, $\triangle PXD \sim \triangle EXC$. Therefore,

$$\frac{\sin\angle EAM}{\sin\angle PAM} = \frac{DP}{EC} = \frac{DX}{XC} = \frac{\sin\angle EAX}{\sin\angle PAX}$$

and the points A, X and M are collinear.

For the other triple we proceed analogously, so

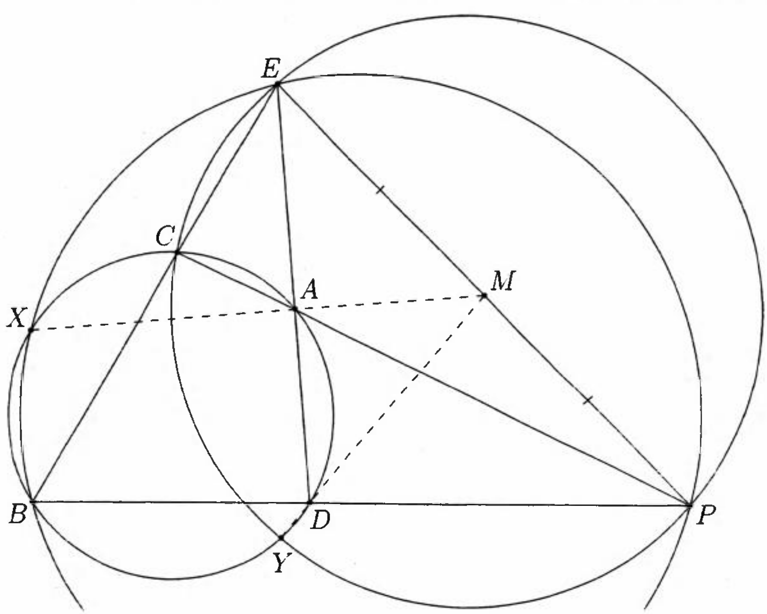
$$\frac{\sin\angle EDM}{\sin\angle PDM} = \frac{DP}{ED} = \frac{DC.AP}{AB.ED} = \frac{AP.BA}{AB.EB} = \frac{AP}{EB}.$$

But $\angle YAC = \angle YBE$. Also, $\angle YEC = \angle YPA$. Therefore, $\triangle YPA \sim \triangle YEB$ and

$$\frac{\sin\angle EDM}{\sin\angle PDM} = \frac{AP}{EB} = \frac{AY}{YB} = \frac{\sin\angle YDE}{\sin\angle YDB}.$$

This means that the points D, Y and M are collinear.

Problem 5.6.12. Let $CADB$ be a cyclic quadrilateral with circumcircle k. Let $AD \cap BC = E$ and $AC \cap BD = P$. The point M is the midpoint of EP. The circumcircle of $\triangle BPE$ intersects k at the points B and X. The circumcircle of $\triangle CPE$ intersects k at the points C and Y. Prove that the triples of points A, X, M and D, Y, M are collinear.

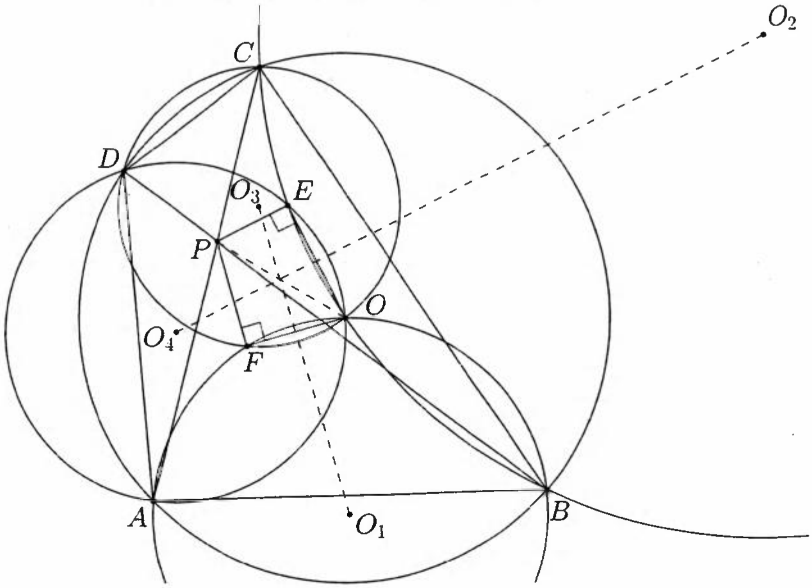


Solution. This problem is essentially the same as Problem 5.6.11 but for the quadrilateral $CADB$ instead of $ABCD$. The solution is completely analogous.

Problem 5.6.13. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. The points O_1, O_2, O_3 and O_4 are the circumcenters of $\triangle ABO$, $\triangle BCO$, $\triangle CDO$ and $\triangle DAO$, respectively. Let $AC \cap BD = P$. Prove that the lines O_1O_3, O_2O_4 and OP are concurrent.

Solution. Let the circumcircles of $\triangle ABO$ and $\triangle CDO$ intersect at the points O and F, and let the circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect at the points O and E.

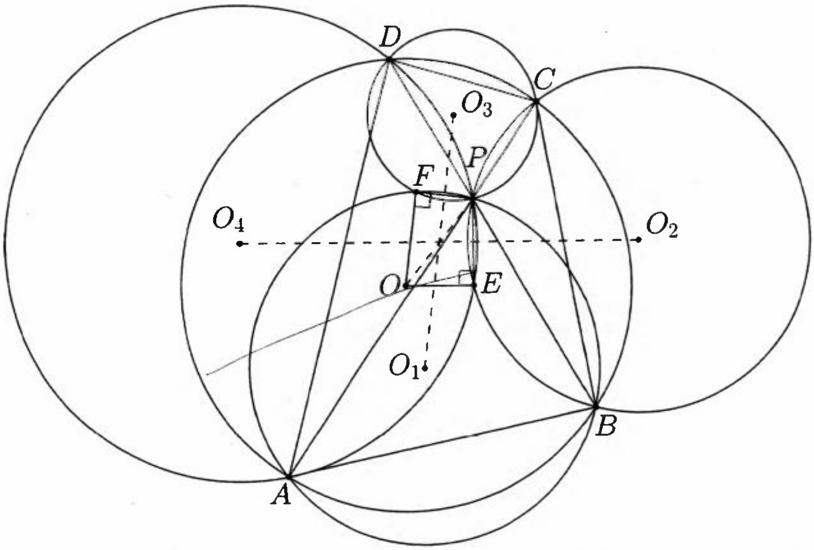
Problem 5.6.4 yields that $\angle PEO = \angle PFO = 90°$. Moreover, O_1O_3 is the perpendicular bisector of OF, and O_2O_4 is the perpendicular bisector of OE. Therefore, they intersect at the circumcenter of the quadrilateral $PEOF$, which is exactly the midpoint of OP.



Problem 5.6.14. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let $AC \cap BD = P$. The points O_1, O_2, O_3 and O_4 are the circumcenters of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$ and $\triangle DAP$, respectively. Prove that the lines O_1O_3, O_2O_4 and OP are concurrent.

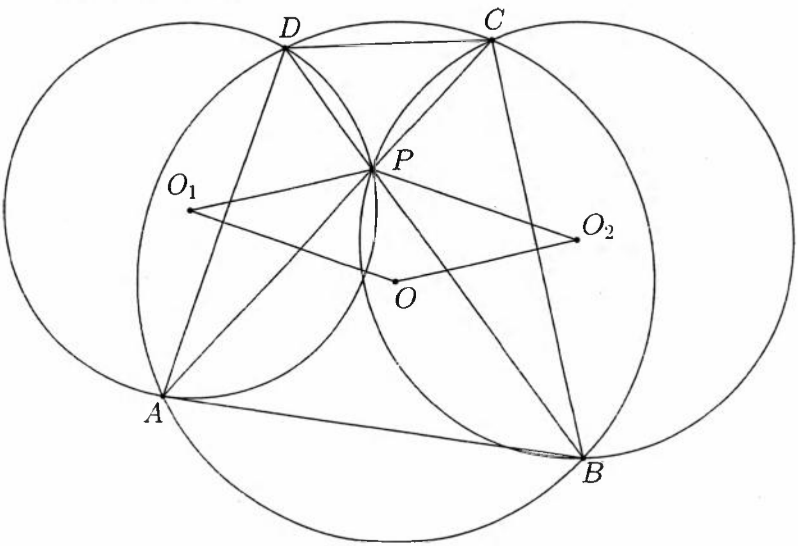
Solution. Let the circumcircles of $\triangle ABP$ and $\triangle CDP$ intersect at the points P and F, and let the circumcircles of $\triangle ADP$ and $\triangle BCP$ intersect at the points P and E.

Problem 5.6.2 yields that $\angle PEO = \angle PFO = 90°$. Moreover, O_1O_3 is the perpendicular bisector of PF, and O_2O_4 is the perpendicular bisector of PE. Hence, they intersect at the circumcenter of the quadrilateral $PEOF$, which is exactly the midpoint of OP.



Note. The problem also follows easily from Problem 5.6.15 – since O_1OO_3P and O_2OO_4P are parallelograms, their diagonals bisect and the desired statement follows.

Problem 5.6.15. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let $AC \cap BD = P$. The points O_1 and O_2 are the circumcenters of $\triangle ADP$ and $\triangle BCP$, respectively. Prove that the quadrilateral O_1OO_2P is a parallelogram.

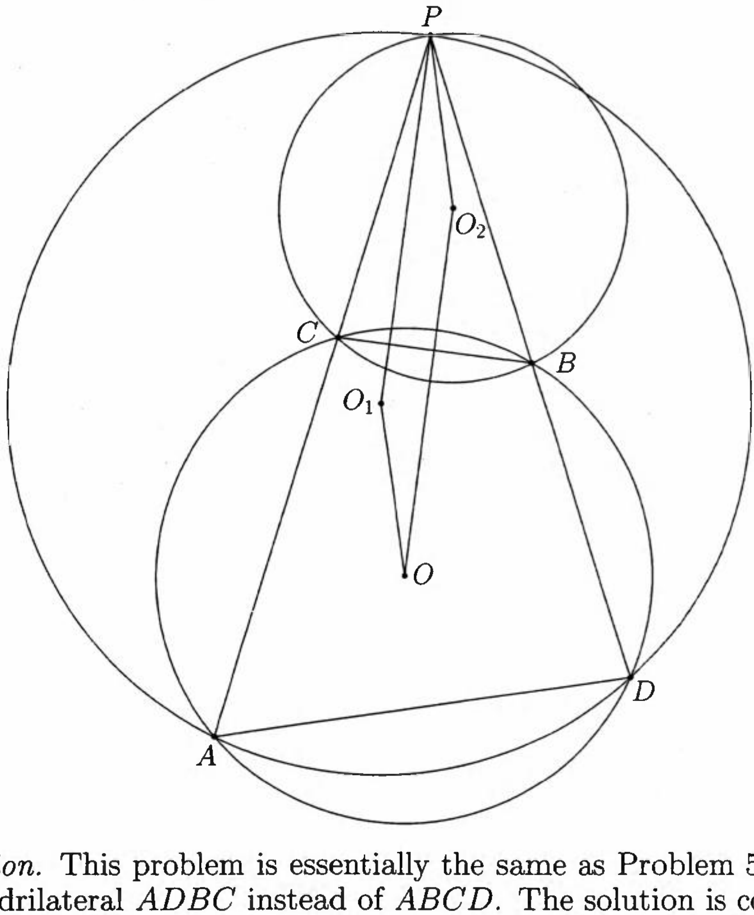


Solution. Note that $OO_1 \perp AD$, since OO_1 is the perpendicular bisector of AD. Moreover, we have

$$\angle O_2PB = 90^\circ - \angle PCB = 90^\circ - \angle ADP \Rightarrow O_2P \perp AD \Rightarrow O_2P \parallel OO_1.$$

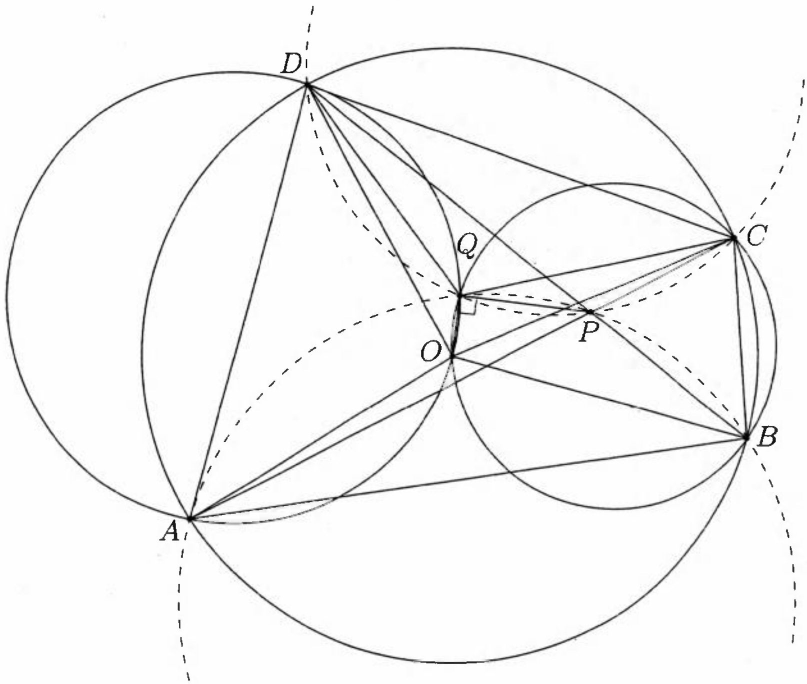
Analogously, we deduce that $OO_2 \parallel O_1P$.

Problem 5.6.16. Let $ADBC$ be a cyclic quadrilateral with circumcenter O. Let $AC \cap BD = P$. The points O_1 and O_2 are the circumcenters of $\triangle ADP$ and $\triangle BCP$, respectively. Prove that the quadrilateral O_1OO_2P is a parallelogram.



Solution. This problem is essentially the same as Problem 5.6.15 but for a quadrilateral $ADBC$ instead of $ABCD$. The solution is completely analogous.

Problem 5.6.17. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let $AC \cap BD = P$. The circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect for the second time at the point Q. Prove that the quadrilaterals $QPCD$ and $ABPQ$ are cyclic.



Solution. We have

$$\begin{aligned}\angle CQD &= 360^\circ - \angle CQO - \angle DQO = \angle OBC + \angle OAD \\ &= 90^\circ - \angle BAC + 90^\circ - \angle DBA = 180^\circ - \angle PAB - \angle PBA \\ &= \angle APB = \angle CPD.\end{aligned}$$

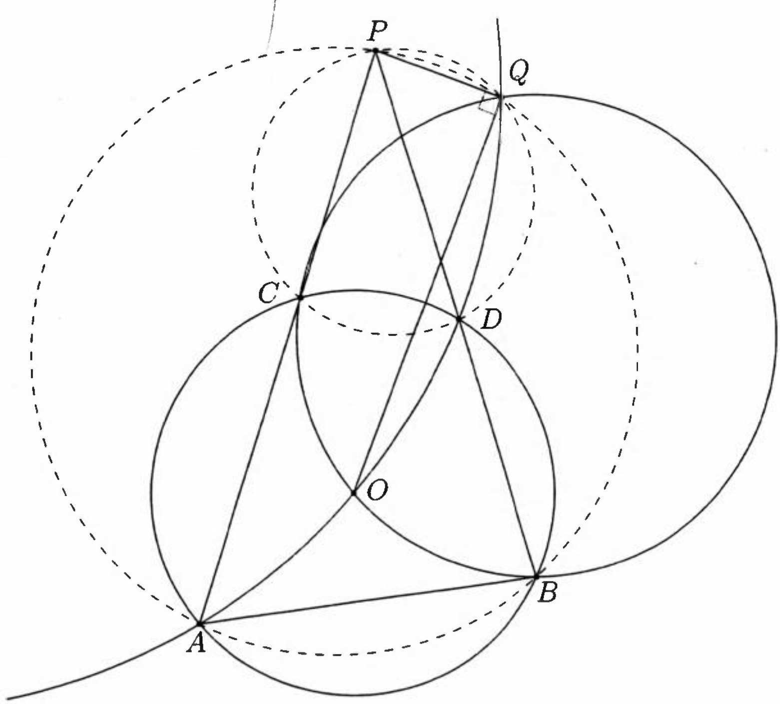
Thus, the quadrilateral $CDQP$ is cyclic. Analogously, the quadrilateral $ABPQ$ is also cyclic.

Note. This problem implies that the right angles in Problems 5.6.2 and 5.6.4 are actually the same angle. Moreover, the statements of these problems easily follow from the proof of this problem.

Indeed, we have

$$\angle OPQ = \angle OQC - \angle CQP = 180^\circ - \angle OBC - \angle CDP = 90^\circ.$$

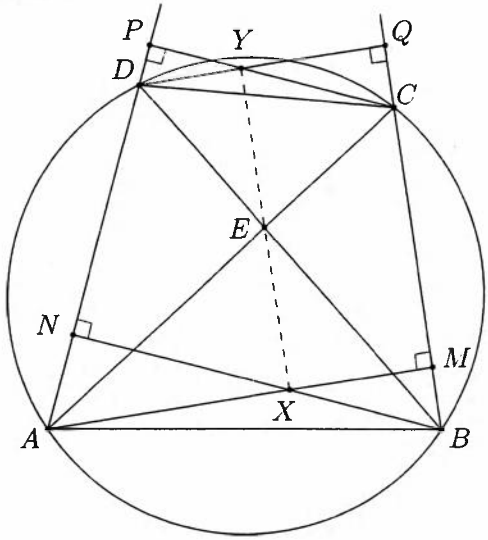
Problem 5.6.18. Let $ABDC$ be a cyclic quadrilateral with circumcenter O. Let $AC \cap BD = P$. The circumcircles of $\triangle ADO$ and $\triangle BCO$ intersect for the second time at the point Q. Prove that the quadrilaterals $QPCD$ and $ABPQ$ are cyclic.



Solution. This problem is essentially the same as Problem 5.6.17 but for a quadrilateral $ABDC$ instead of $ABCD$. The solution is completely analogous.

Note. Analogously to Problem 5.6.17, one can use this problem to prove Problems 5.6.1 and 5.6.5, where the right angles are once again the same angle.

Problem 5.7.1. Let $ABCD$ be a quadrilateral with $AC \cap BD = E$. Let M and Q be the projections of the points A and D onto the line BC, respectively. Let N and P be the projections of the points B and C onto the line AC, respectively. If $DQ \cap CP = Y$ and $AM \cap BN = X$, prove that the points X, Y and E are collinear.



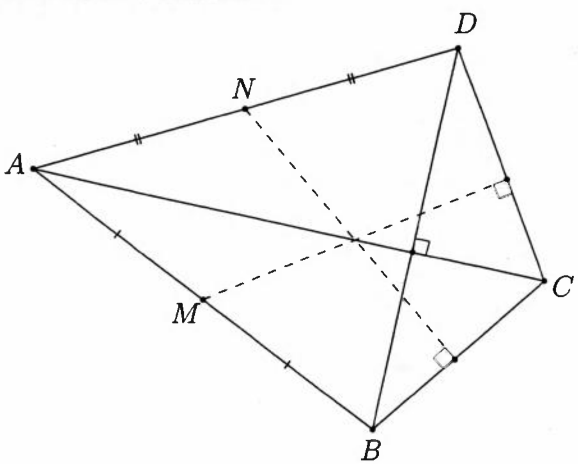
Solution. We have that the quadrilaterals $AMCP$ and $BNDQ$ are inscribed in the circles k_1 and k_2, respectively. On the other hand, $\triangle DPY \sim \triangle CQY$, which gives $DY.YQ = PY.YC$.

Also, $\triangle AXN \sim \triangle BXM$, so $AX.MX = BX.XN$.

Considering the power of the point E, we get $AE.EC = ED.EB$.

These equalities mean that the powers of X with respect to the circles k_1 and k_2 are equal, and the same follows for the points Y and E. In conclusion, the points X, Y and E lie on the radical axis of k_1 and k_2.

Problem 5.7.2. Let $ABCD$ be a quadrilateral with perpendicular diagonals. Let M and N be the midpoints of AB and AD, respectively. Prove that the perpendicular from N to BC and the perpendicular from M to CD intersect on the line AC.

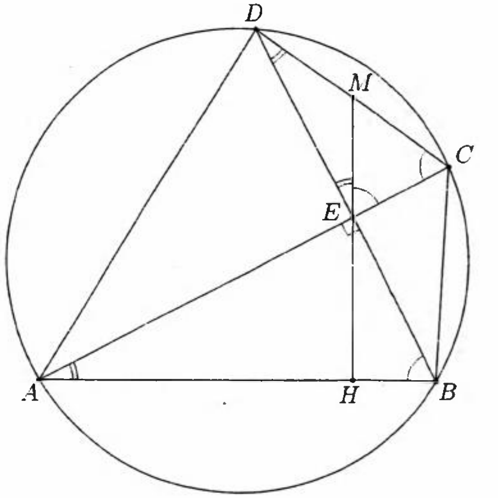


Solution. Consider a homothety with center A and coefficient 2.

The line AC is fixed, and the perpendiculars from N to BC and from M to CD are sent to the altitudes from D and B in $\triangle BCD$, respectively.

Hence, these three lines intersect at the orthocenter of $\triangle BCD$, so the pre-image of their intersection lies on the line AC.

Problem 5.7.3. Let $ABCD$ be a cyclic quadrilateral with $AC \cap BD = E$ and $AC \perp BD$. Let M be the midpoint of the side DC. Prove that $ME \perp AB$.



Solution. Let $ME \cap AB = H$.

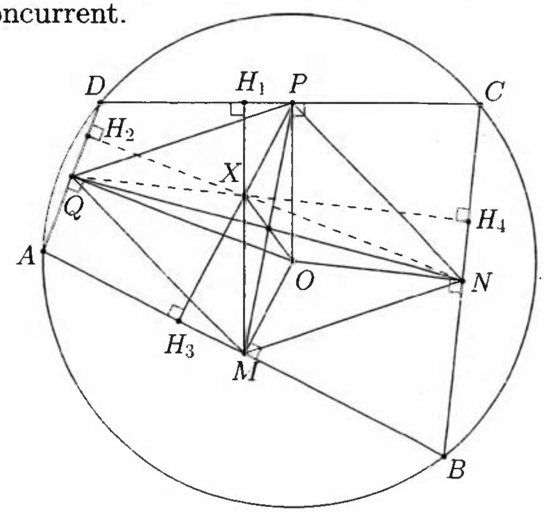
Set $\angle EDM = x$. Since EM is a median to the hypotenuse in $\triangle DEC$, we have $\angle DEM = x$ and $\angle BEH = x$.

Thus,

$$\angle ACD = 90^\circ - \angle EDC = 90^\circ - x.$$

But $\angle HBE = \angle ACD = 90^\circ - x$ and from $\angle BEH = x$, we conclude that $\angle BHE = 90^\circ$, i.e. $ME \perp AB$.

Problem 5.7.4. Let $ABCD$ be a cyclic quadrilateral. Let M, N, P and Q be the midpoints of AB, BC, CD and DA, respectively. The points H_1, H_2, H_3 and H_4 are the feet of the perpendiculars from M, N, P and Q to CD, DA, AB and BC, respectively. Prove that the lines MH_1, NH_2, PH_3 and QH_4 are concurrent.



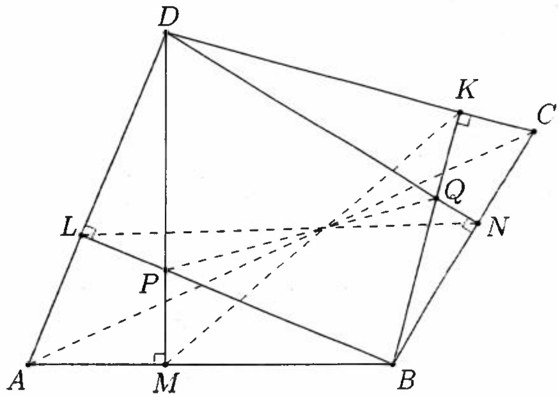
Solution. Midsegments give that PN is parallel and equal to MQ.

Therefore, the quadrilateral $PNMQ$ is a parallelogram. Denote the circumcenter of $ABCD$ by O.

Now we have that $OP \perp CD$, $ON \perp BC$, $OM \perp AB$ and $OQ \perp AD$.

If $MH_1 \cap PH_3 = X$, then the quadrilateral $POMX$ is a parallelogram. Thus, the midpoint of OX coincides with the midpoint of PM, which coincides with the midpoint of NQ. Hence, the quadrilateral $ONXQ$ is also a parallelogram. Consequently, $NX \perp AD$ and $QX \perp BC$, which gives $QH_4 \cap NH_2 = X$.

Problem 5.7.5. Let $ABCD$ be a quadrilateral. Denote the feet of the perpendiculars from D to AB and BC by M and N, respectively, and the feet of the perpendiculars from B to CD and DA by K and L, respectively. Denote the orthocenters of $\triangle ABD$ and $\triangle BCD$ by P and Q, respectively. Prove that the lines LN, KM, AC and PQ are concurrent.



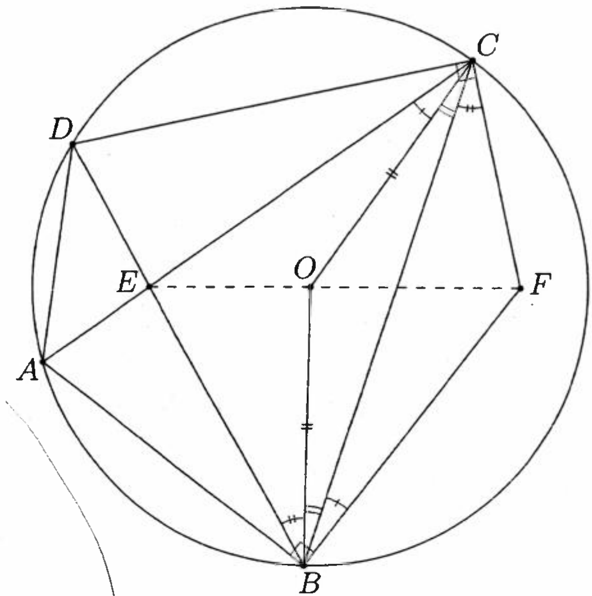
Solution. Let k be the circle with diameter BD. Then the hexagon $BMLDKN$ is inscribed in k.

Desargues' Theorem (Problem 7.1), applied to $\triangle MPL$ and $\triangle KQN$, yields that the lines PQ, KM and LN are concurrent if and only if the points $MP \cap KQ$, $LP \cap NQ$ and $ML \cap KN$ are collinear. The latter, however, follows by Pascal's Theorem, applied to the hexagon $BKNDML$. Analogously, Desargues' Theorem, applied to $\triangle AML$ and $\triangle CKN$, yields that the lines AC, KM and LN are concurrent if and only if the points $AL \cap CN$, $AM \cap CK$ and $ML \cap KN$ are collinear. This is true once again by Pascal's Theorem, applied to the hexagon $BMLDKN$.

Problem 5.7.6. Let $ABCD$ be a cyclic quadrilateral with circumcenter O, and let $AC \cap BD = E$. Let F be a point such that $CF \perp CD$ and $FB \perp AB$. Prove that the points E, O and F are collinear.

Solution. We have $OB = OC$ and $\angle OCB = \angle OBC$. Also,

$$\begin{aligned}\angle ECO &= \angle DCO - \angle DCA \\ &= 90^\circ - \angle DBC - \angle DBA = \angle CBF.\end{aligned}$$



Analogously, we deduce that $\angle BCF = \angle OBE$. This implies $\angle EBF = \angle ECF$. Now the law of sines yields

$$\begin{aligned}\frac{\sin\angle CEO}{\sin\angle OEB} &= \frac{OC}{OB}\cdot\frac{\sin\angle ECO}{\sin\angle EBO} = \frac{\sin\angle CBF}{\sin\angle BCF}\\ &= \frac{CF}{BF} = \frac{CF}{BF}\cdot\frac{\sin\angle ECF}{\sin\angle EBF} = \frac{\sin\angle CEF}{\sin\angle FEB}.\end{aligned}$$

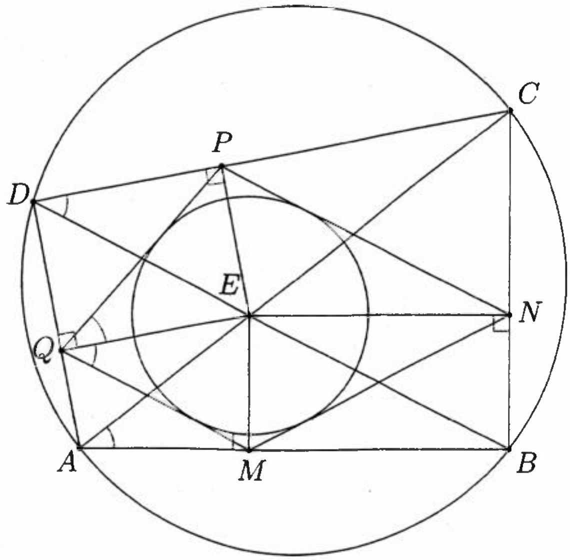
Note that $\angle CEO + \angle OEB = \angle CEF + \angle FEB$. Hence, $\angle CEO = \angle CEF$ and the points E, O and F are collinear.

Problem 5.7.7. Let $ABCD$ be a cyclic quadrilateral with $AC \cap BD = E$. Let M, N, P and Q be the projections of E onto the lines AB, BC, CD and DA, respectively. Prove that the quadrilateral $MNPQ$ is circumscribed.

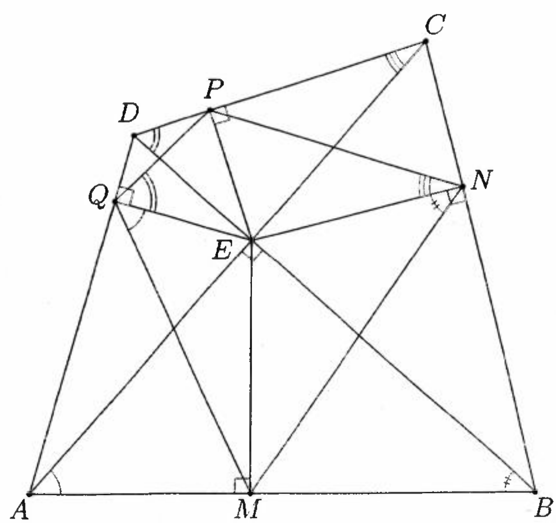
Solution. We have that the quadrilaterals $AMEQ$ and $QEPD$ are cyclic. Hence, $\angle MQE = \angle MAE$ and $\angle EDP = \angle EQP$.

But since the quadrilateral $ABCD$ is also cyclic, we have $\angle EDP = \angle EAM$ and hence $\angle MQE = \angle PQE$. This implies that QE is the angle bisector of $\angle MQP$. Analogously, we deduce that PE, NE and ME bisect $\angle QPN$, $\angle PNM$ and $\angle NMQ$, respectively.

In conclusion, the bisectors of the angles of the quadrilateral $MNPQ$ concur at E. Thus, the quadrilateral $MNPQ$ is circumscribed.



Problem 5.7.8. Let $ABCD$ be a convex quadrilateral with $AC \cap BD = E$ and $AC \perp BD$. Let M, N, P and Q be the projections of E onto the lines AB, BC, CD and DA, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.



Solution. We have that the quadrilaterals $AMEQ$, $QEPD$, $NEPC$ and $NEMB$ are cyclic. Therefore,

$$\angle MQE = \angle MAE,$$
$$\angle EDP = \angle EQP,$$
$$\angle ECP = \angle ENP,$$
$$\angle MNE = \angle MBE.$$

Further, we have that $\angle MAE + \angle MBE = 90°$ and $\angle PCE + \angle PDE = 90°$.

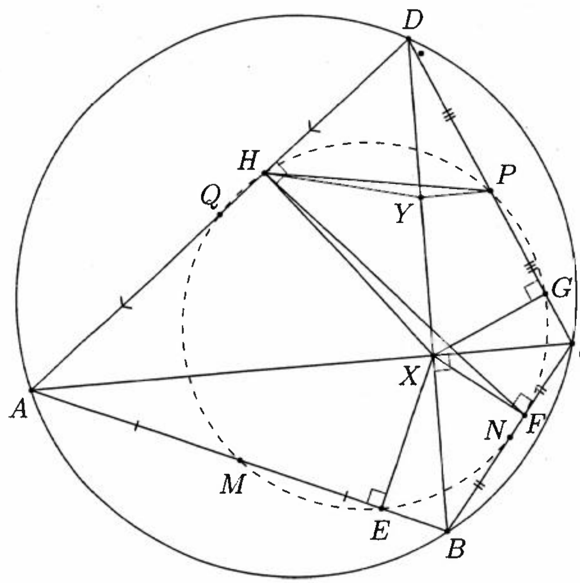
Thus,

$$\begin{aligned}\angle MQP + \angle MNP &= (\angle MQE + \angle PQE) + (\angle ENP + \angle ENM)\\ &= (\angle MAE + \angle MBE) + (\angle PCE + \angle PDE) = 180°.\end{aligned}$$

Therefore, the quadrilateral $MNPQ$ is cyclic.

Problem 5.7.9. Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals, and let $AC \cap BD = X$. Let the feet of the perpendiculars from X to AB, BC, CD and DA be E, F, G and H, respectively. If the midpoints of AB, BC, CD and DA are M, N, P, Q, respectively, prove that the points E, F, G, H, M, N, P and Q are concyclic.

Solution. We have $\angle FGH = \angle XGH + \angle XGF = \angle XDH + \angle XCF$ and $\angle FEH = \angle XEH + \angle XEF = \angle XAH + \angle XBF$. Hence, $\angle FGH + \angle FEH = 180°$ and the quadrilateral $EFGH$ is cyclic.



It suffices to prove that P lies on its circumcircle (for the other midpoints the proof is analogous). Let Y be the midpoint of DX. Then $\dfrac{PY}{FX} = \dfrac{XC}{2FX} = \dfrac{XD}{2HX} = \dfrac{YH}{HX}$. Moreover,

$$\begin{aligned}\angle PYH &= 90° + \angle DYH = 90° + 180° - 2\angle ADX \\ &= 90° + 90° - \angle ADX + 90° - \angle BCX \\ &= 90° + \angle DXH + \angle CXF = \angle FXH.\end{aligned}$$

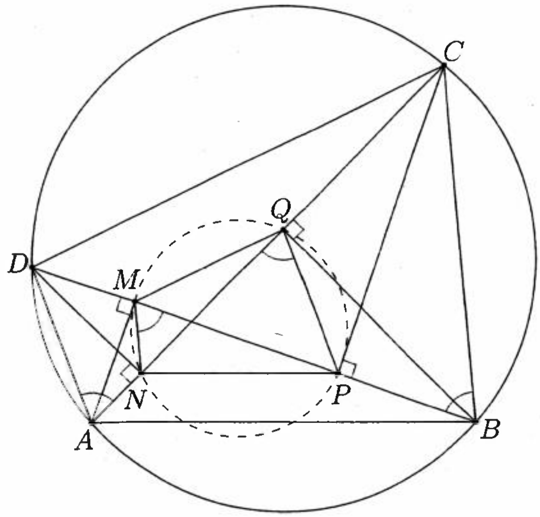
Thus, $\triangle HXF \sim \triangle HYP$. Therefore,

$$\angle PHF = \angle YHX = \angle DXH = \angle CXF = \angle CGF.$$

This implies that P lies on the circumcircle of $EFGH$.

Note. This circle is known as the Eight-point circle.

Problem 5.7.10. Let $ABCD$ be a cyclic quadrilateral. Let M and P be the projections of A and C onto BD, respectively, and let N and Q be the projections of D and B onto AC, respectively. Prove that the quadrilateral $MNPQ$ is cyclic.



Solution. We have that the quadrilaterals $ANMD$ and $BPQC$ are cyclic.

Then $\angle DAN = \angle NMP$ and $\angle PBC = \angle PQN$.

The quadrilateral $ABCD$ is cyclic, and thus we have

$$\angle DAN = \angle CBP$$

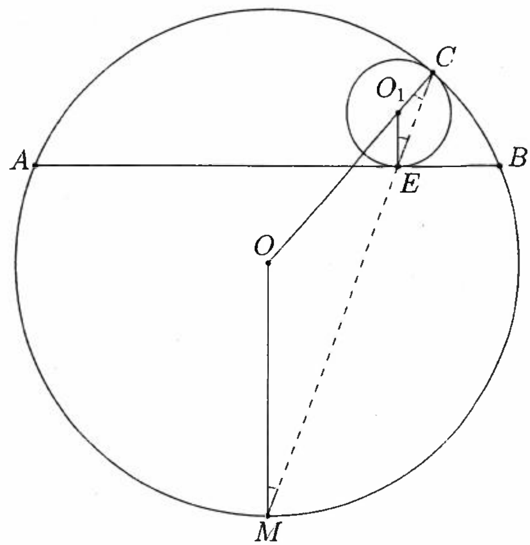
and so $\angle NMP = \angle NQP$, which means that the quadrilateral $MNPQ$ is cyclic.

Problem 6.1.1. Let k be a circle and let AB be a chord in it. The circle k_1 touches AB and k at the points E and C, respectively. Let $CE \cap k = M \neq C$. Prove that M is the midpoint of the arc $\widehat{AB}$ which does not contain C.

Solution. Denote the centers of k and k_1 by O and O_1, respectively. Then O, O_1 and C are collinear. Note that

$$\angle OMC = \angle OCM = \angle O_1EC$$

from the isosceles triangles MOC and EO_1C.



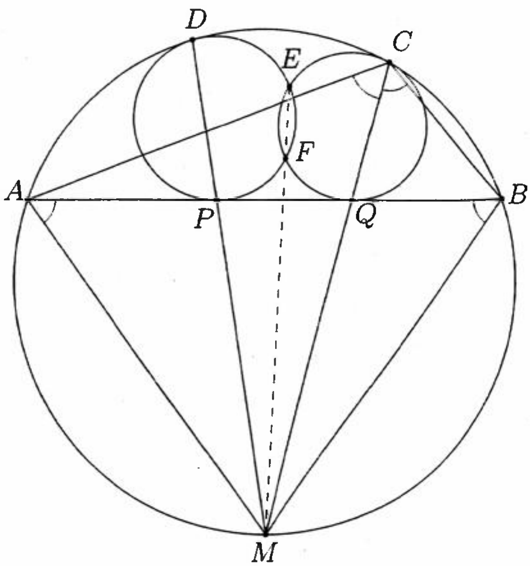
Hence, $MO \parallel EO_1$. Since $O_1E \perp AB$, we conclude that $MO \perp AB$.

Therefore, O lies on the perpendicular bisector of AB. We get that MO is the perpendicular bisector of AB. Hence, $\triangle ABM$ is an isosceles triangle.

Problem 6.1.2. Let k be a circle and let AB be a chord in it. The circle k_1 touches AB and k at P and D, respectively. The circle k_2 touches AB and k at Q and C, respectively. Assume that k_1 and k_2 lie in the same half-plane with respect to the line AB. Let M be the midpoint of the arc $\widehat{AB}$, which does not contain C. Prove that M lies on the radical axis of k_1 and k_2.

Solution. Problem 6.1.1 yields that the points D, P and M, as well as the points C, Q and M are collinear. Observe that CM is the angle bisector of $\angle ACB$. Hence, $\angle ACM = \angle BCM = \angle ABM = \angle BAM$.

We obtain that $\triangle BMC \sim \triangle OMB$. Therefore, $MQ.MC = MB^2$. Analogously, $MP.MD = MA^2 = MB^2$. Thus, the powers of M with respect to the circles k_1 and k_2 are equal, as required.



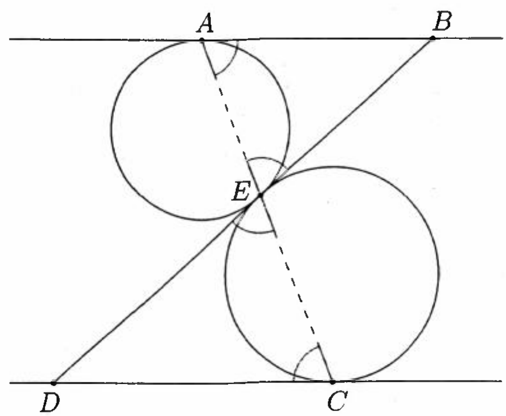
Problem 6.1.3. Let l_1 and l_2 be two parallel lines, $A \in l_1$ and $C \in l_2$. Let k_1 and k_2 be circles touching l_1 and l_2 at A and C, respectively, and let they touch each other externally at E. Prove that the points A, E and C are collinear.

Solution. Let l be the common internal tangent line of the circles k_1 and k_2. Let $l \cap l_1 = B$ and $l \cap l_2 = D$.

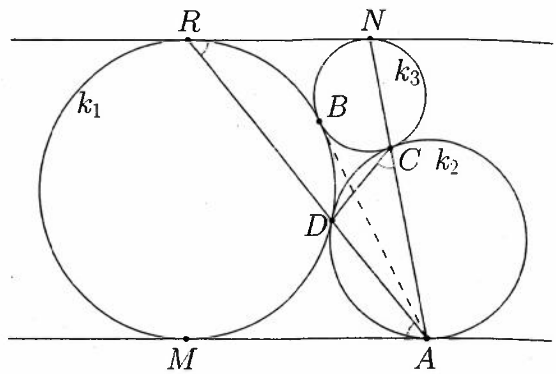
Observe that $CD = ED$, $AB = BE$ and $\angle CDE = \angle ABE$. Therefore,

$$\angle DEC = \angle BEA,$$

which implies that the points A, E and C are collinear.



Problem 6.1.4. Let l_1 and l_2 be two parallel lines. A circle k_1 touches l_1 and l_2 at the points M and R, respectively. A circle k_2 touches l_1 and k_1 externally at the points A and D, respectively. A circle k_3 touches l_2, k_1 and k_2 at the points N, B and C, respectively. Prove that AB touches both k_1 and k_3.



Solution. Problem 6.1.3 yields that the points A, C and N, as well as the points A, D and R are collinear.

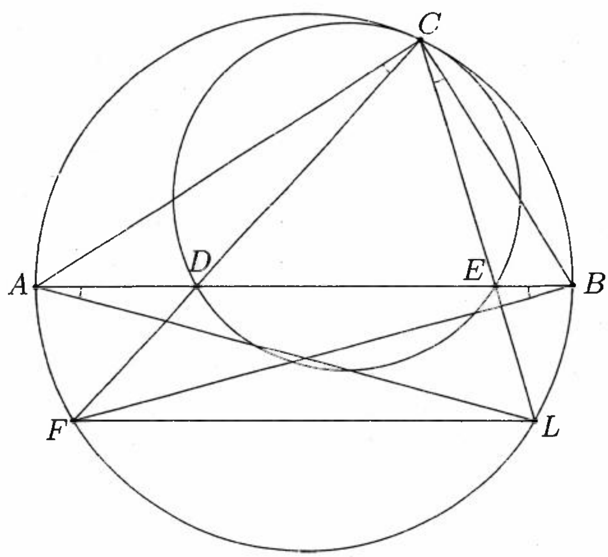
Note that $\angle ARN = \angle MAR$ and $\angle MAD = \angle ACD$.

Hence, $\angle DRN = \angle DCA$ and so $DCNR$ is a cyclic quadrilateral. Thus, we have $AD.AR = AC.AN$ and we obtain that A lies on the radical axis of k_1 and k_3, which is their common internal tangent line at B.

Problem 6.1.5. Let k and k_1 be circles that touch each other internally at C. Let k_1 be the smaller of the two circles. Let $D, E \in k_1$ and $DE \cap k = \{A, B\}$. Assume that D lies between A and E. Prove that $\angle ACD = \angle BCE$.

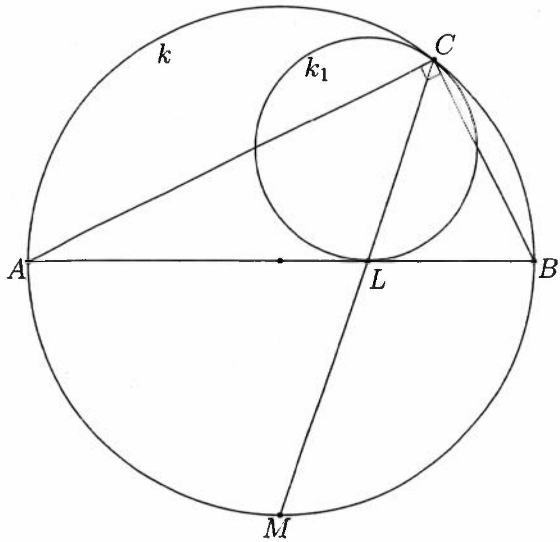
Solution. Consider the homothety h with center C, such that $h(k_1) = k$. Let $h(D) = F$ and $h(E) = L$. Then h sends the line AB to the line FL, hence $FL \parallel AB$. The quadrilateral $LFAB$ is a cyclic trapezoid and thus $AF = BL$.

We obtain $\angle ACF = \angle BCL$ and $\angle ACD = \angle BCE$.



Problem 6.1.6. Let k be a circle with diameter AB. A circle k_1 touches the line AB and the circle k internally at the points L and C, respectively. Prove that $\angle(AC, CL) = 45°$.

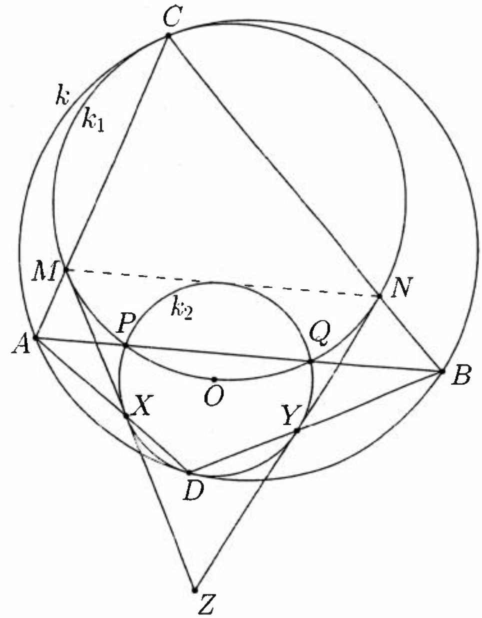
Solution. Problem 6.1.1 yields that CL is an angle bisector of $\angle ACB$. Since AB is a diameter, we have $\angle ACB = 90°$. Therefore, $\angle ACL = 45°$.



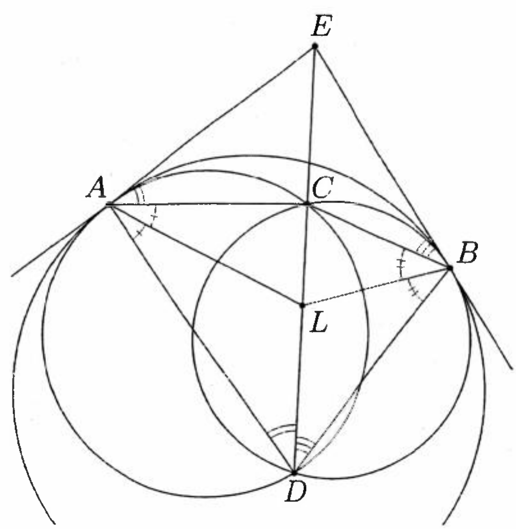
Problem 6.1.7. The circles k and k_1 touch each other internally at the point C. Let k_1 be the smaller of the two circles. A circle $k_2(O)$, where $O \in k_1$, touches k. Let $k_1 \cap k_2 = \{P, Q\}$. Let $PQ \cap k = \{A, B\}$ and let P lie between A and Q. Denote the second intersection points of AC and BC with k_1 by M and N, respectively. Prove that MN touches k_2.

Solution. Let $k_2 \cap k = D$, $DA \cap k_2 = X$ and $DB \cap k_2 = Y$. Then Problem 6.10.11 yields that MX and NY are the common external tangent lines of k_1 and k_2. Denote the intersection point of these tangents by Z.

Now Problem 4.5.1, applied to $\triangle MNZ$, yields that MN touches k_2.



Problem 6.1.8. Let k_1 and k_2 be circles that touch another circle k internally at the points A and B, respectively. Let $k_1 \cap k_2 = \{C, D\}$. Prove that the intersection point L of the angle bisectors of $\angle CAD$ and $\angle CBD$ lies on CD.



Solution. The angle bisector theorem, applied to $\triangle ACD$ and $\triangle BCD$, yields that it suffices to show that $\dfrac{AC}{AD} = \dfrac{BC}{BD}$.

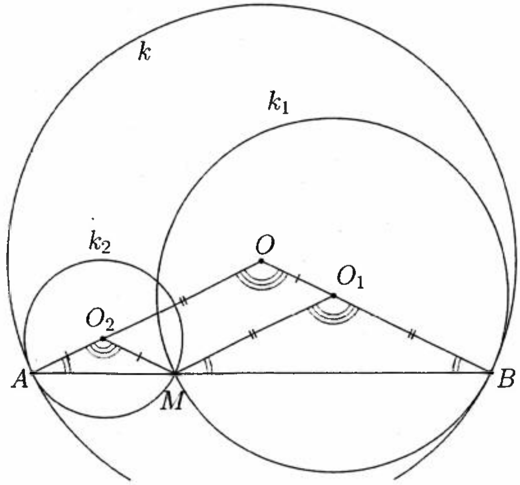
Let E be the intersection point of the tangent lines to k at A and B. Observe that E lies on the radical axis of k_1 and k_2. Hence, $E \in CD$.

Note that $\triangle EAC \sim \triangle EDA$ and $\triangle EBC \sim \triangle EDB$.

Thus,

$$\frac{AC}{AD} = \frac{AE}{ED} = \frac{BE}{ED} = \frac{BC}{BD}.$$

Problem 6.1.9. Let $k(O,r)$ be a circle and let $A, B \in k$. A circle $k_1(O_1, r_1)$ touches k at A and intersects the segment AB at M. The circle $k_2(O_2, r_2)$ touches k at B and passes through M. Prove that $r_1 + r_2 = r$.



Solution. It is clear that the points A, O_2 and O, as well as the points B, O_1 and O are collinear. Then

$$\begin{aligned}\angle MAO_2 &= \angle BAO = \angle ABO \\ &= \angle MBO_1 = \angle BMO_1.\end{aligned}$$

Hence, $AO \parallel MO_1$.

Analogously, $BO \parallel MO_2$, which means that O_2MO_1O is a parallelogram.

We deduce that $OO_2 = O_1M$. Since OO_2, AO_2 and OA are the radii of k_1, k_2 and k, respectively, we get that $r_1 + r_2 = r$.

Problem 6.1.10. *(Casey's Theorem.)* Let k be a circle and let k_1, k_2, k_3 and k_4 touch k. Let t_{ij} be the length of the common external tangent segment of k_i and k_j. Let l_{ij} be the length of the common internal tangent segment of k_i and k_j. Prove that if the points of tangency of k_1, k_2, k_3 and k_4 to k are A_1, A_2, A_3 and A_4, respectively, and if they lie in this order along the circle, then one of the following equalities holds:

$$t_{13}t_{24} = t_{12}t_{34} + t_{14}t_{23}$$

$$l_{13}t_{24} = l_{12}t_{34} + l_{14}t_{23}$$

$$l_{13}l_{24} = t_{12}t_{34} + l_{14}l_{23}$$

$$t_{13}t_{24} = l_{12}l_{34} + l_{14}l_{23}$$

The first one holds when k_1, k_2, k_3 and k_4 touch k in the same way (either only internally or only externally).

The second one holds when k touches k_1 in one way, and touches k_2, k_3 and k_4 in the other way.

The third one holds when k touches k_1 and k_2 in one way and touches k_3 and k_4 in the other way.

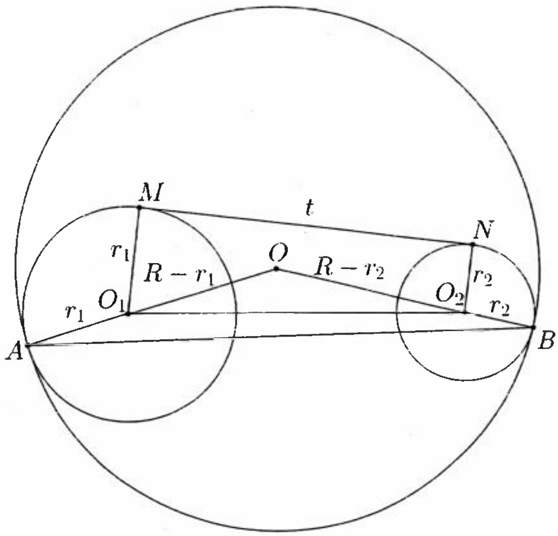
The fourth one holds when k touches k_1 and k_3 in one way and touches k_2 and k_4 in the other way.

The converse of Casey's Theorem. If we have one of the four equalities above for the lengths of the tangent segments among four circles, prove that there exists a circle k, such that it touches the four circles in the way described above.

Solution. We will prove Casey's Theorem only in the case when all circles touch k internally, and its converse only in the case when $t_{13}t_{24} = t_{12}t_{34} + t_{14}t_{23}$, because the proofs of the other cases are analogous. Approaching the problems, let us prove some lemmas:

Lemma 1. Two circles $k_1(O_1; r_1)$ and $k_2(O_2; r_2)$ are given, and they touch $k(O; R)$ internally at the points A and B, respectively. Let the length of their common tangent segment be t. Then

$$t = \frac{AB\sqrt{(R-r_1)(R-r_2)}}{R}.$$



Proof. Let one of the common external tangent lines of k_1 and k_2 touch k_1 at the point M and k_2 at the point N. Let $\angle AOB = \varphi$.

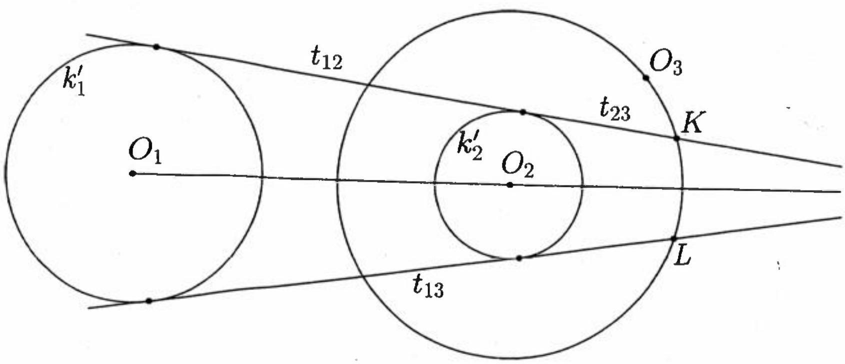
The Pythagorean Theorem, applied in the right-angled trapezoid O_1O_2MN, yields the equality

$$O_1O_2^2 = (r_1 - r_2)^2 + t^2.$$

We apply the law of cosines to $\triangle OO_1O_2$ and $\triangle OAB$ to get consecutively

$$
\begin{aligned}
AB^2 &= 2R^2(1-\cos\varphi) = 2R^2\left(1 - \frac{(R-r_1)^2+(R-r_2)^2-O_1O_2^2}{2(R-r_1)(R-r_2)}\right) \\
&= \frac{2R^2}{2(R-r_1)(R-r_2)}(2(R-r_1)(R-r_2) - 2R^2 - r_1^2 - r_2^2 \\
&\quad + 2R(r_1+r_2) + r_1^2 + r_2^2 - 2r_1r_2 + t^2) \\
&= \frac{R^2}{(R-r_1)(R-r_2)}(2R^2 - 2R(r_1+r_2) + 2r_1r_2 - 2R^2 \\
&\quad + 2R(r_1+r_2) - 2r_1r_2 + t^2) \\
&= \frac{R^2t^2}{(R-r_1)(R-r_2)} \Rightarrow t = \frac{AB\sqrt{(R-r_1)(R-r_2)}}{R}.
\end{aligned}
$$

Lemma 2. Let $k_1(O_1;r_1)$, $k_2(O_2;r_2)$ and $k_3(O_3;r_3)$ be circles. Let t_{ij} denote the length of the common external tangent segment of k_i and k_j. If $t_{12}+t_{23}=t_{13}$, then there exists a line that touches k_1, k_2 and k_3.



Proof. Without loss of generality, let k_3 be the smallest circle (if k_2 is the smallest one, the proof is analogous). We decrease the radii of the three circles by r_3, such that k_3 turns into a point – O_3, and k_1 and k_2 turn into the smaller circles k_1' and k_2'. The lengths of the common tangent segments remain the same, and the property that there exists/does not exist a line which touches the three circles is preserved.

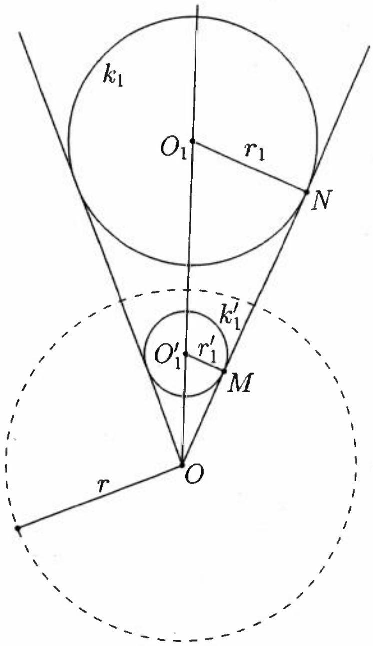
Assume that O_3 does not lie on the common external tangent line of k_1' and k_2'. We know that the locus of points, such that the length of the tangent segment from the points to a given circle is constant, is a circle concentric to it. Let us construct a circle with center O_2 and radius O_2O_3

and let it intersect the common external tangent lines of k_1' and k_2' at the points L and K, such that they do not lie on the tangent segments, as shown in the figure.

The lengths of the tangent segments from K and from O_3 to k_2' are equal. Now the equality $t_{12} + t_{23} = t_{13}$ gives that the tangent segments drawn from the points K and O_3 to k_1' have the same length. Therefore, these points lie on a circle, concentric to k_1'.

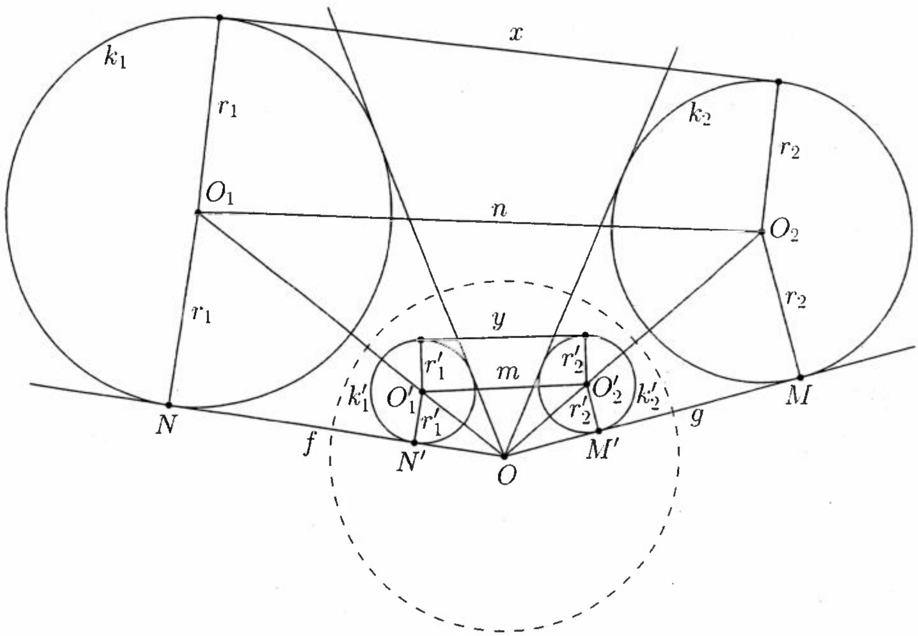
Hence, O_1O_2 is the perpendicular bisector of KO_3, which means that $O_3 \equiv L$, which is a contradiction with the assumption that O_3 does not lie on the common external tangent line of k_1' and k_2'.

Lemma 3. Let φ be an inversion with center O and radius r. It sends the circle $k_1(O_1; r_1)$ to the circle $k_1'(O_1'; r_1')$. Let t_{Ok_1} be the length of the tangent segment from the point O to k_1. Then $r_1' = \dfrac{r^2 r_1}{t_{Ok_1}^2}$.



Proof. We will use the notations on the figure. The properties of inversion yield $OM = \dfrac{r^2}{t_{Ok_1}}$, and the intercept theorem gives the equality $\dfrac{OM}{ON} = \dfrac{MO_1'}{NO_1}$. Therefore, $r_1' = \dfrac{r^2 r_1}{t_{Ok_1}^2}$.

Lemma 4. Let φ be an inversion with center O and radius r. It sends the circle $k_1(O_1; r_1)$ to the circle $k_1'(O_1'; r_1')$, and the circle $k_2(O_2; r_2)$ to the circle $k_2'(O_2'; r_2')$. Let x be the length of the common external tangent segment of k_1 and k_2. Let y be the length of the common external tangent segment of k_1' and k_2'. Let f and g be the lengths of the tangent segments from the point O to k_1 and k_2, respectively. Then $y = \dfrac{xr^2}{fg}$.



Proof. Denote $O_1O_2 = n$ and $O_1'O_2' = m$.

Then (see the figure) we can express y by using the Pythagorean Theorem. We have

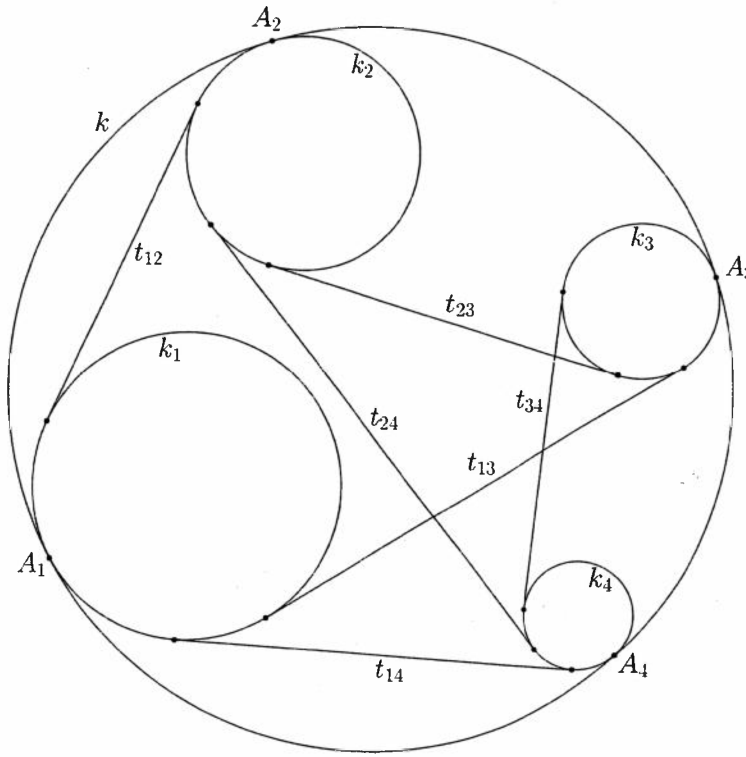
$$y^2 = m^2 - r_1'^2 - r_2'^2 + 2r_1'r_2'$$

We apply the law of cosines to $\triangle OO_1O_2$ and $\triangle OO_1'O_2'$ and we get

$$m^2 = OO_1'^2 + OO_2'^2 - 2OO_1'OO_2'\frac{OO_1^2 + OO_2^2 - n^2}{2OO_1OO_2}.$$

The Pythagorean Theorem and Lemma 3 yield $OO_1' = \sqrt{\dfrac{r^4}{f^2} + \dfrac{r^4 r_1^2}{f^4}}$ and $OO_2' = \dfrac{r^2 OO_2}{g^2}$. Now we can calculate y. We consecutively have

$$
\begin{aligned}
y^2 &= \frac{r^4 OO_1^2}{f^4} + \frac{r^4 OO_2^2}{g^4} - 2\frac{r^2 OO_1}{f^2} \cdot \frac{r^2 OO_2}{g^2} \cdot \frac{OO_1^2 + OO_2^2 - n^2}{2OO_1OO_2} \\
&\quad - r_1'^2 - r_2'^2 + 2r_1'r_2' \\
&= \frac{r^4 OO_1^2}{f^4} + \frac{r^4 OO_2^2}{g^4} - \frac{r^4}{f^2g^2}(OO_1^2 + OO_2^2 - n^2) - \frac{r^4 r_1^2}{f^4} - \frac{r^4 r_2^2}{g^4} + 2\frac{r^4 r_1 r_2}{f^2 g^2} \\
&= \frac{r^4(f^2 + r_1^2)}{f^4} + \frac{r^4(g^2 + r_2^2)}{g^4} - \frac{r^4}{f^2g^2}(f^2 + g^2 + r_1^2 + r_2^2 - n^2) - \frac{r^4 r_1^2}{f^4} \\
&\quad - \frac{r^4 r_2^2}{g^4} + 2\frac{r^4 r_1 r_2}{f^2 g^2} \\
&= \frac{r^4}{f^2g^2}(-r_1^2 - r_2^2 + x^2 + r_1^2 + r_2^2 - 2r_1r_2 + 2r_1r_2) = \frac{r^4 x^2}{f^2 g^2}.
\end{aligned}
$$



We can now prove Casey's Theorem. If r_1, r_2, r_3, r_4 and R are the radii of k_1, k_2, k_3, k_4 and k, respectively, then Lemma 1 yields

$$t_{13} = \frac{A_1A_3\sqrt{(R-r_1)(R-r_3)}}{R},$$

$$t_{24} = \frac{A_2A_4\sqrt{(R-r_2)(R-r_4)}}{R},$$

$$t_{12} = \frac{A_1A_2\sqrt{(R-r_1)(R-r_2)}}{R},$$

$$t_{34} = \frac{A_3A_4\sqrt{(R-r_3)(R-r_4)}}{R},$$

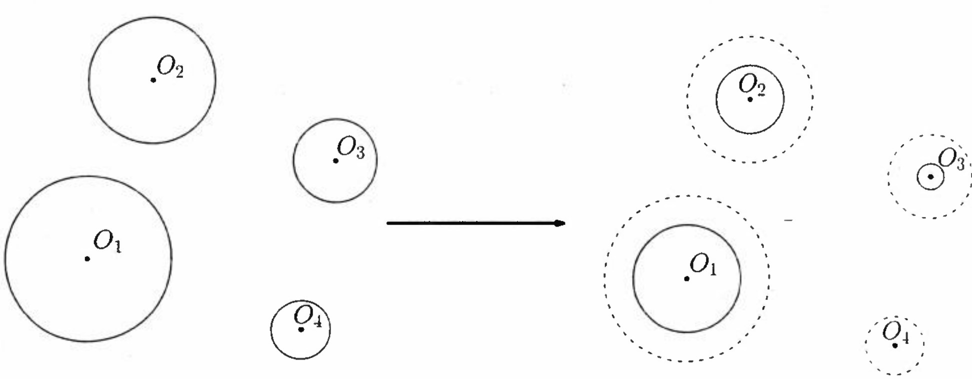
$$t_{14} = \frac{A_1A_4\sqrt{(R-r_1)(R-r_4)}}{R},$$

$$t_{23} = \frac{A_2A_3\sqrt{(R-r_2)(R-r_3)}}{R}.$$

Therefore,

$$t_{13}t_{24} - t_{12}t_{34} - t_{14}t_{23} =$$
$$= \frac{\sqrt{(R-r_1)(R-r_2)(R-r_3)(R-r_4)}}{R}(A_1A_3.A_2A_4 -$$
$$- A_1A_2.A_3A_4 - A_1A_4.A_2A_3).$$

The last expression is equal to 0 because of the Ptolemy's Theorem (Problem 5.5.6). This completes the proof of Casey's Theorem.



Now we will prove the converse of Casey's Theorem. Without loss of generality, let r_4 be the smallest of the radii of the four circles. We can decrease the radii of these circles by r_4, such that the lengths of the common external tangent segments remain the same, and the property that there exists a circle, which touches/does not touch all of them is preserved.

Let the three new circles (the fourth one is a point) be c_1, c_2 and c_3. Consider an inversion φ with center O_4 and radius R. We will use the prime symbol to denote the images of the objects after φ has been applied to them.

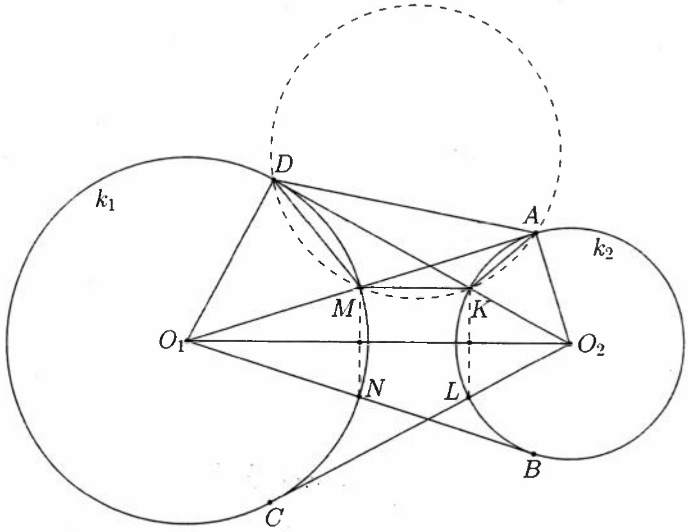
It suffices to prove that there exists a line that touches c'_1, c'_2 and c'_3 because the inversion guarantees that the image of that line (which is a circle) would touch c_1, c_2 and c_3 and would pass through O_4. Increasing the radii back by r_4 would give us the desired circle.

Let r'_1, r'_2 and r'_3 be the radii of c'_1, c'_2 and c'_3, respectively. Let t'_{ij} be the length of the common external tangent segment of c'_i and c'_j.

We have that $t_{13}t_{24} = t_{12}t_{34} + t_{14}t_{23}$. Also, Lemmas 3 and 4 yield that $t_{24} = R\sqrt{\dfrac{r_2}{r'_2}}$ and $t_{13} = \dfrac{t'_{13}t_{14}t_{34}}{R^2} = t'_{13}\sqrt{\dfrac{r_1r_3}{r'_1r'_3}}$.

We can calculate the lengths of the other tangent segments analogously. After substituting and simplifying, we get that $t'_{12}+t'_{23} = t'_{13}$. Now Lemma 2 proves the existence of the desired line.

Problem 6.2.1. Let $k_1(O_1)$ and $k_2(O_2)$ be two nonintersecting circles. The tangent segments from O_1 to k_2 – O_1A and O_1B ($A, B \in k_2$), intersect k_1 at the points M and N, respectively. The tangent segments from O_2 to k_1 – O_2C and O_2D ($C, D \in k_1$), intersect k_2 at the points L and K, respectively. Prove that $MN = KL$.

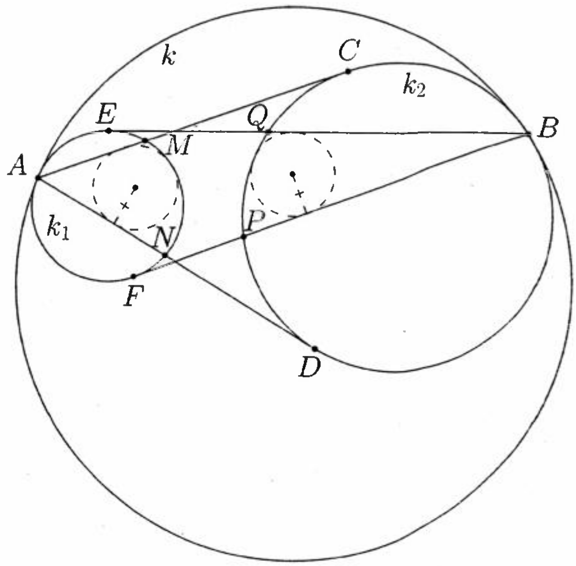


Solution. The configuration is symmetric with respect to the line O_1O_2. Hence, this line is the perpendicular bisector of both MN and KL. Therefore, it suffices to prove that $MK \parallel O_1O_2$. The quadrilateral O_1DAO_2 is cyclic because $\angle O_1DO_2 = \angle O_1AO_2 = 90°$.

Thus, $\angle KAM = \dfrac{\angle DO_2A}{2} = \dfrac{\angle DO_1A}{2} = \angle MDK$, which means that the quadrilateral $DMKA$ is cyclic, and hence $\angle MKD = \angle MAD = \angle O_1O_2D$, or $MK \parallel O_1O_2$.

Problem 6.2.2. Let k, k_1 and k_2 be circles with radii r, r_1 and r_2, respectively. The circles k_1 and k_2 do not intersect each other and they touch k internally at the points A and B, respectively. The tangent lines AC and AD from A to k_2 ($C, D \in k_2$) intersect k_1 at the points M and N, respectively. The tangent lines BE and BF from B to k_1 ($E, F \in k_1$) intersect k_2 at the points Q and P, respectively. Let r' be the radius of the circle that touches the segments AM and AN, and the arc $\widehat{MN}$ of k_1. Let r'' be the radius of the circle that touches the segments BP and BQ, and the arc $\widehat{PQ}$ of k_2. Prove that $r' = r''$.

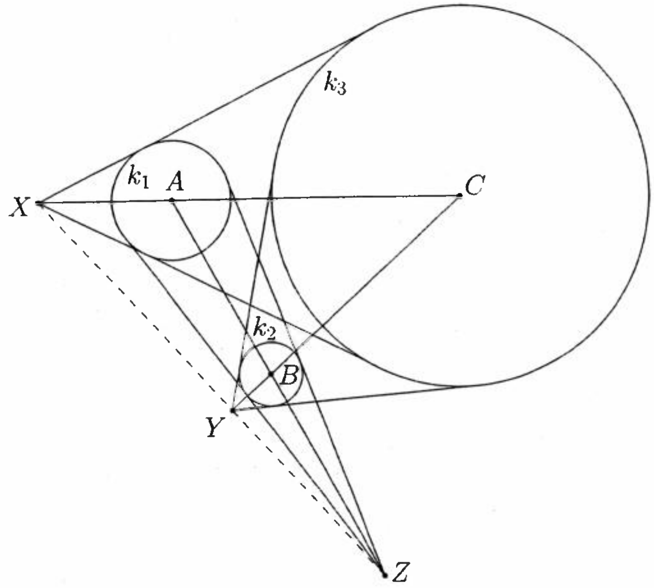
Solution. Consider the homothety centered at A that sends k_1 to k. It sends the incircle of the curvilinear triangle ANM to k_2 and hence $\frac{r'}{r_2} = \frac{r_1}{r}$, so $r' = \frac{r_1 . r_2}{r}$.



Analogously, considering the homothety centered at B that sends k_2 to k, we get

$$r'' = \frac{r_1 . r_2}{r} = r'.$$

Problems 6.2.3 and 6.2.4. *(Monge's Theorem)* Let $k_1(A)$, $k_2(B)$ and $k_3(C)$ be circles. Let X be the external (internal) homothetic center of k_1 and k_3, let Y be the external (internal) homothetic center of k_2 and k_3, and let Z be the external (internal) homothetic center of k_1 and k_2. Prove that the points X, Y and Z are collinear.

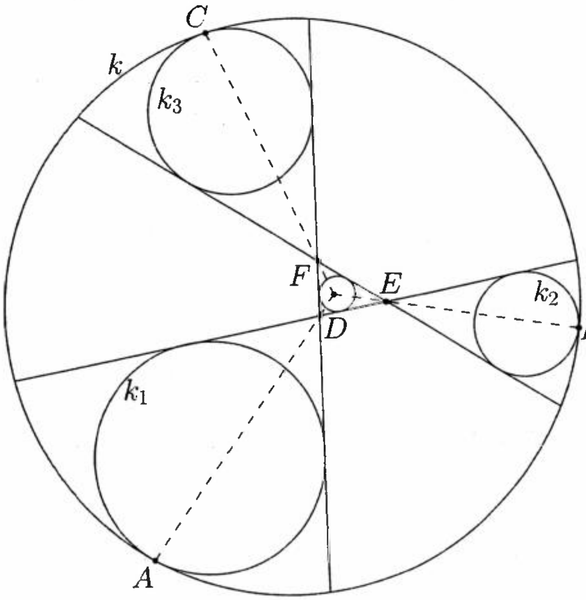


Solution. Menelaus' Theorem, applied to $\triangle ABC$ yields that it suffices to prove that $\frac{AX}{XC} \cdot \frac{CY}{YB} \cdot \frac{BZ}{ZC} = 1$.

We have that $\frac{AX}{XC} = \frac{r_1}{r_3}$, $\frac{CY}{YB} = \frac{r_3}{r_2}$ and $\frac{BZ}{ZC} = \frac{r_2}{r_1}$, so we obtain the desired relationship.

Note. The difference between Problems 6.2.3 and 6.2.4 is that in the latter one we take two internal and one external homothetic centers, instead of three external ones. This difference, however, does not change the solution's idea and its implementation.

Problem 6.2.5. Let k, k_1, k_2 and k_3 be circles. The circles k_1, k_2 and k_3 do not intersect each other and they touch k internally at the points A, B and C, respectively. The common external tangent lines of each pair from k_1, k_2 and k_3, that separate them from the third circle, form $\triangle DEF$, such that the two tangent lines, touching k_1, intersect at the point D, the two tangent lines, touching k_2, intersect at the point E, and the two tangent lines, touching k_3, intersect at the point F. Prove that the lines AD, BE and CF are concurrent.



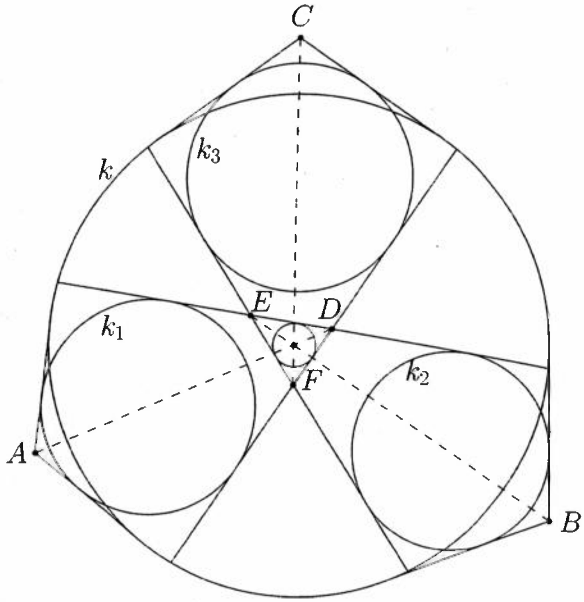
Solution. Problem 6.2.4, applied to the circles k, k_1 and the incircle of $\triangle DEF$, yields that A, D and the internal homothetic center of k and the incircle of $\triangle DEF$ are collinear.

Analogously, BE and CF also pass through the internal homothetic center of k and the incircle of $\triangle DEF$.

Problem 6.2.6. Let k, k_1, k_2 and k_3 be circles. The circles k_1, k_2 and k_3 do not intersect each other, and k does not contain each of them fully. The common external tangent lines of each pair from k_1, k_2 and k_3, that separate them from the third circle, form $\triangle DEF$, such that the two tangent lines, touching k_1, intersect at the point D, the two tangent lines, touching k_2, intersect at the point E, and the two tangent lines, touching k_3, intersect at the point F. The external homothetic centers of k and k_1, of k and k_2, and of k and k_3 are A, B and C, respectively. Prove that the lines AD, BE and CF are concurrent.

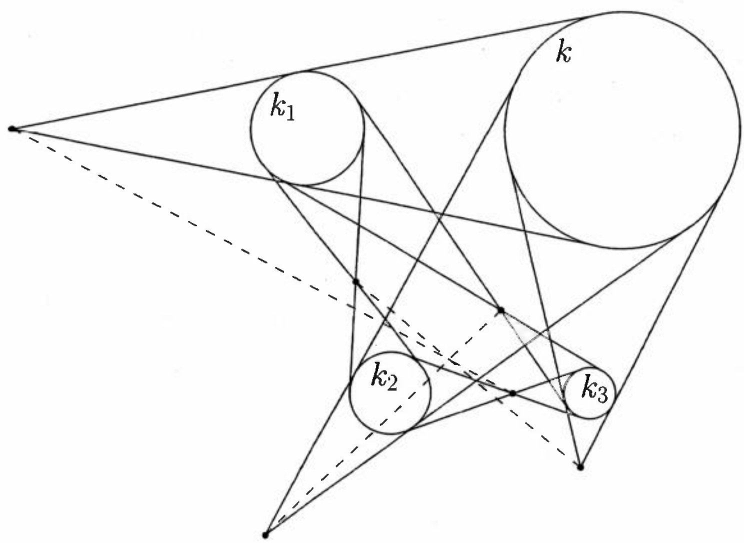
Solution. Problem 6.2.3, applied to the circles k, k_1 and the incircle of $\triangle DEF$, yields that A, D and the external homothetic center of k and the incircle of $\triangle DEF$ are collinear.

Analogously, BE and CF also pass through the external homothetic center of k and the incircle of $\triangle DEF$.



In the next few problems we will use the notations $S(k_1, k_2)$ and $L(k_1, k_2)$, corresponding respectively to the external and the internal homothetic centers of the circles k_1 and k_2.

Problem 6.2.7. Let k, k_1, k_2 and k_3 be circles. Prove that the lines $S(k, k_1)L(k_2, k_3)$, $S(k, k_2)L(k_1, k_3)$ and $S(k, k_3)L(k_1, k_2)$ are concurrent.



Solution. Desargues' Theorem (Problem 7.1) yields that it suffices to prove that the intersection points

$$S(k,k_1)S(k,k_2)\cap L(k_1,k_3)L(k_2,k_3),$$

$$S(k,k_2)S(k,k_3)\cap L(k_1,k_3)L(k_1,k_3),$$

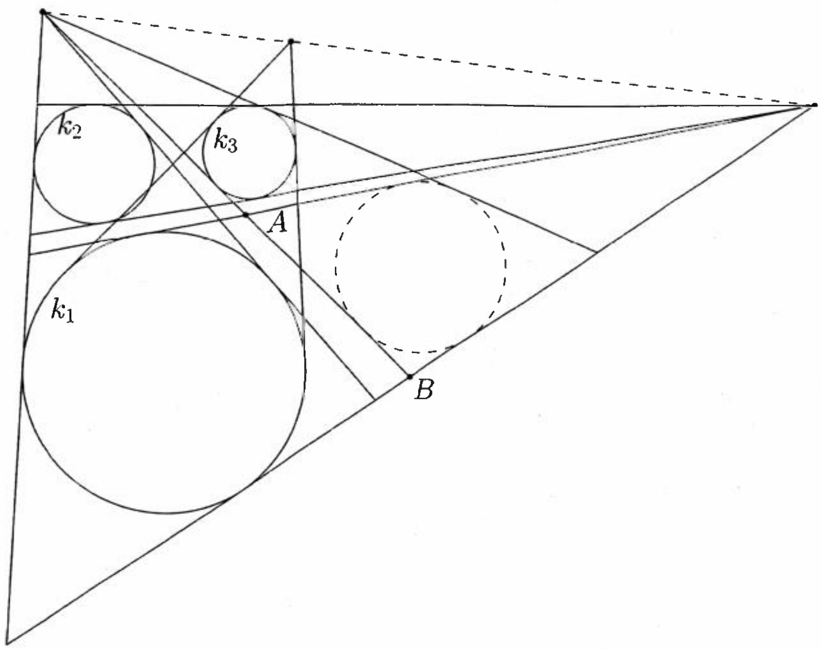
$$S(k,k_1)S(k,k_3)\cap L(k_2,k_3)L(k_1,k_2)$$

are collinear. Monge's Theorem (Problems 6.2.3 and 6.2.4) yields that $S(k,k_1)$, $S(k,k_2)$ and $S(k_1,k_2)$ are collinear, as well as that $L(k_1,k_3)$, $L(k_2,k_3)$ and $S(k_1,k_2)$ are collinear. Therefore,

$$S(k,k_1)S(k,k_2)\cap L(k_1,k_3)L(k_2,k_3)=S(k_1,k_2).$$

Analogously, $S(k,k_2)S(k,k_3)\cap L(k_1,k_3)L(k_1,k_2)=S(k_2,k_3)$ and also, $S(k,k_1)S(k,k_3)\cap L(k_2,k_3)L(k_1,k_2)=S(k_1,k_3)$. Applying again Monge's Theorem, we see that the points $S(k_1,k_2)$, $S(k_2,k_3)$ and $S(k_1,k_3)$ are collinear.

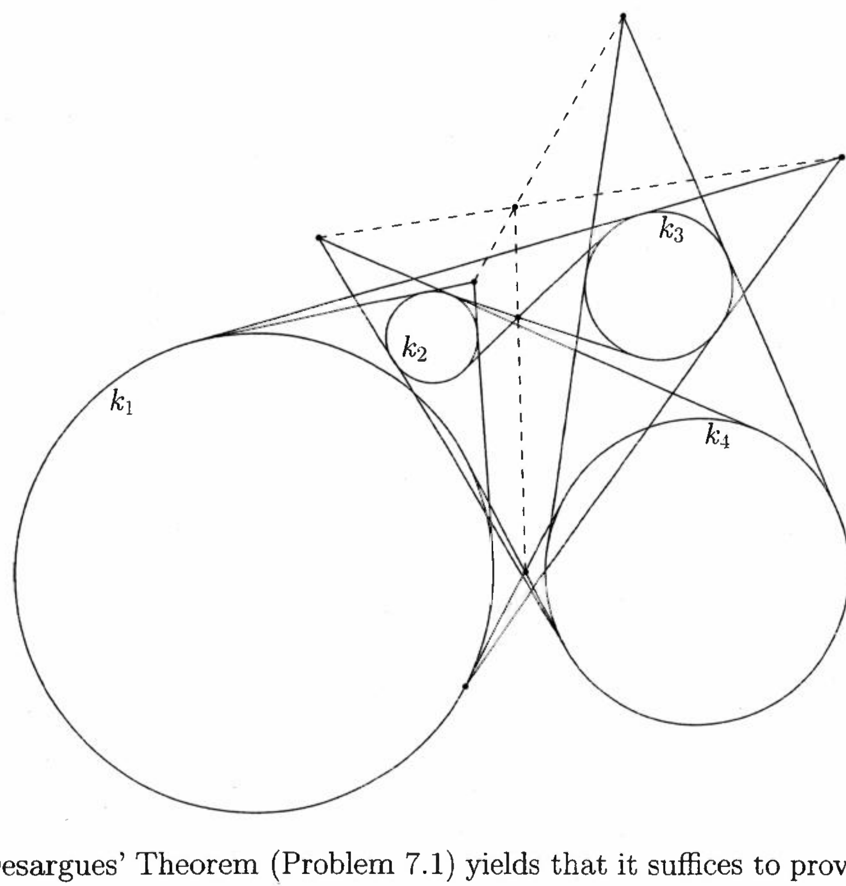
Problem 6.2.8. Let k_1, k_2 and k_3 be circles. We draw the tangent lines from $S(k_2,k_3)$ to k_1 and from $S(k_1,k_2)$ to k_3. Prove that the convex quadrilateral, formed by these two pairs of lines is circumscribed.



Solution. Let one of the tangent lines from $S(k_1,k_2)$ to k_3 intersect the tangent lines from $S(k_2,k_3)$ to k_1 at the points A and B. We will prove that the circle k, which touches AB, $S(k_2, k_3)A$ and $S(k_2, k_3)B$ also touches the desired fourth line.

Monge's Theorem yields that the points $S(k_1, k_2)$, $S(k_2, k_3)$ and $S(k_3, k_1)$ are collinear. On the other hand, the same theorem, applied to k, k_1 and k_3, yields that the points $S(k, k_1) \equiv S(k_2, k_3)$, $S(k_1, k_3)$ and $S(k, k_3)$ are collinear. It follows that $S(k, k_3) = S(k_1, k_3)S(k_2, k_3) \cap AB = S(k_1, k_2)$, as desired.

Problem 6.2.9. Let k_1, k_2, k_3 and k_4 be circles. Prove that the lines $S(k_1, k_3)S(k_2, k_4)$, $S(k_1, k_2)S(k_3, k_4)$ and $L(k_1, k_4)L(k_2, k_3)$ are concurrent.



Desargues' Theorem (Problem 7.1) yields that it suffices to prove that the points

$$S(k_1, k_3)S(k_1, k_2) \cap S(k_3, k_4)S(k_2, k_4),$$

$$S(k_1, k_2)L(k_1, k_4) \cap S(k_3, k_4)L(k_2, k_3),$$

$$S(k_2, k_4)L(k_2, k_3) \cap S(k_1, k_3)L(k_1, k_4)$$

are collinear. Monge's Theorem (Problems 6.2.3 and 6.2.4) yields that the points $S(k_1, k_3)$, $S(k_1, k_2)$ and $S(k_2, k_3)$ are collinear, as well as that the points $S(k_3, k_4)$, $S(k_2, k_3)$ and $S(k_2, k_4)$ are collinear. Thus,

$$S(k_1, k_3)S(k_1, k_2) \cap S(k_3, k_4)L(k_2, k_4) = S(k_2, k_3).$$

Analogously, we see that

$$S(k_1, k_2)L(k_1, k_4) \cap S(k_3, k_4)L(k_2, k_3) = L(k_2, k_4)$$

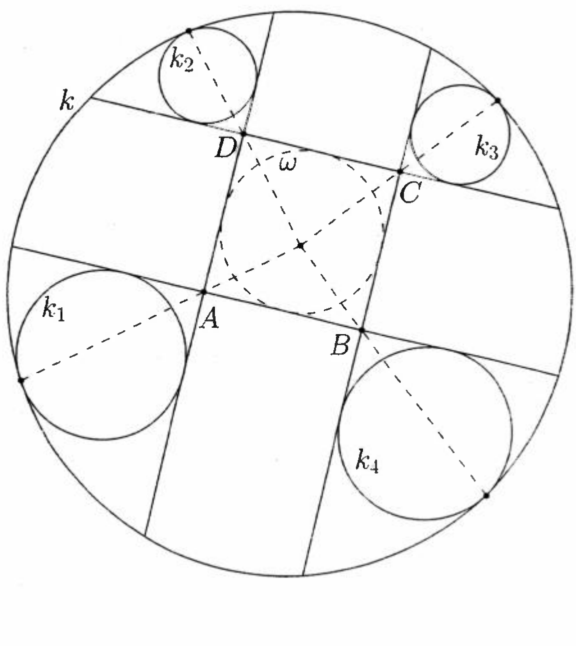
and also

$$S(k_2, k_4)L(k_2, k_3) \cap S(k_1, k_3)L(k_1, k_4) = L(k_3, k_4).$$

Now Monge's Theorem yields that the points $S(k_2, k_3)$, $L(k_2, k_4)$ and $L(k_3, k_4)$ are collinear.

Problem 6.2.10. Let $ABCD$ be a square and let k be a circle that contains it. The circles k_1, k_2, k_3 and k_4 are inscribed in the curvilinear triangles, formed by the intersections of the square's lines and the circle k. Let the circle with respect to the vertex A be k_1, let the circle with respect to the vertex B be k_2, let the circle with respect to the vertex C be k_3, and let the circle with respect to the vertex D be k_4. Prove that the lines $S(k, k_1)A$, $S(k, k_2)B$, $S(k, k_3)C$ and $S(k, k_4)D$ are concurrent.

Solution. Let ω be the incircle of the square $ABCD$.

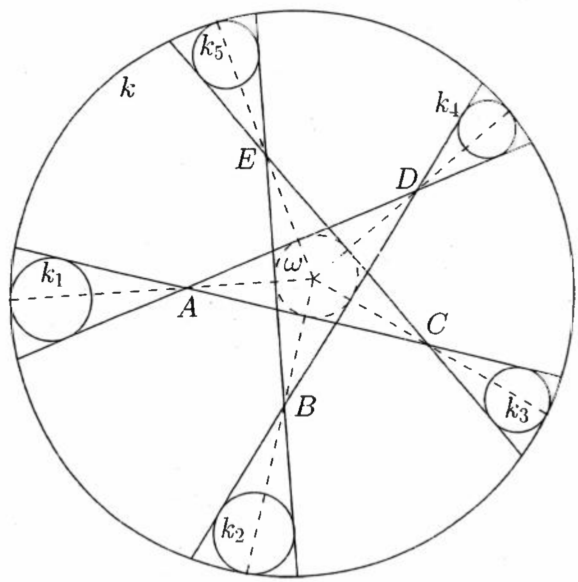


Monge's Theorem (see Problems 6.2.3 and 6.2.4), applied to the circles k, ω and k_1, yields that the points A, $S(k, k_1)$ and $L(k, \omega)$ are collinear, and so the line $S(k, k_1)A$ passes through $L(k, \omega)$.

Analogously, we obtain that the lines $S(k, k_2)B$, $S(k, k_3)C$ and $S(k, k_4)D$ also pass through $L(k, \omega)$.

Problem 6.2.11. Let $ABCDE$ be a regular pentagon, and let k be a circle that contains it. The circles k_1, k_2, k_3, k_4 and k_5 are inscribed in the curvilinear triangles, formed by the intersections of the pentagon's diagonals and the circle k. Let the circle with respect to the vertex A be k_1, let the circle with respect to the vertex B be k_2, let the circle with respect to the vertex C be k_3, let the circle with respect to the vertex D be k_4, and let the circle with respect to the vertex E be k_5. Prove that the lines $S(k,k_1)A$, $S(k,k_2)B$, $S(k,k_3)C$, $S(k,k_4)D$ and $S(k,k_5)E$ are concurrent.

Solution. Let ω be the circle that touches all diagonals of $ABCDE$.

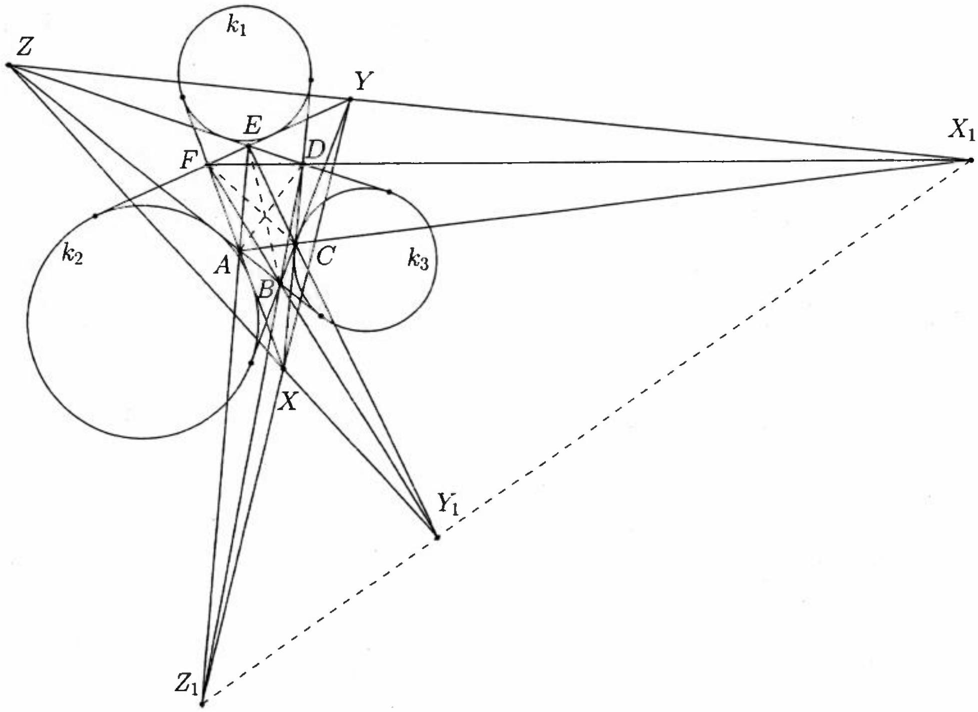


Monge's Theorem (see Problems 6.2.3 and 6.2.4), applied to the circles k, ω and k_1, yields that the points $A, S(k,k_1)$ and $L(k,\omega)$ are collinear, and so the line $S(k,k_1)A$ passes through $L(k,\omega)$.

Analogously, we obtain that the lines $S(k,k_2)B$, $S(k,k_3)C$, $S(k,k_4)D$ and $S(k,k_5)E$ also pass through $L(k,\omega)$.

Note. The statement of Problem 6.2.10 remains true if we replace the square with a circumscribed quadrilateral. One can generalize Problems 6.2.10 and 6.2.11 by considering any circumscribed n-gon.

Problem 6.3.1. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. All their common internal tangent lines form the hexagon $ABCDEF$. Prove that the lines AD, BE and CF are concurrent.



Solution. Let $AB \cap DE = Z$, $AF \cap DC = X$ and $FE \cap BC = Y$. We apply Desargues' Theorem (Problem 7.1) to $\triangle FAZ$ and $\triangle DCY$.

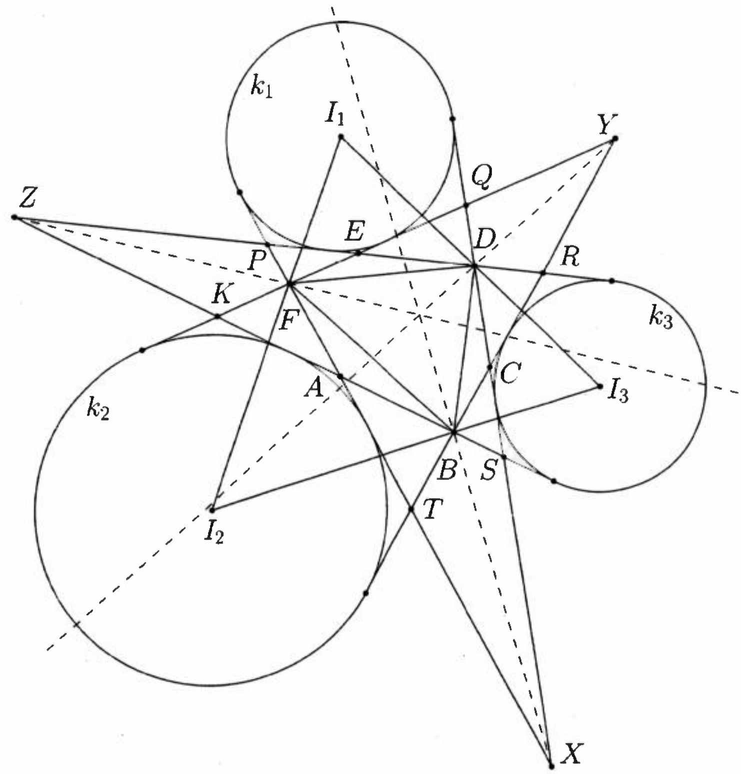
We have that $FA \cap DC = X$, $AZ \cap CY = B$. Also, the lines XB, FZ and DY are concurrent (see Problem 6.3.2). Therefore, the lines ZY, FD and AC are concurrent. Let their intersection point be X_1.

Analogously, the lines FB, EC and XZ intersect at the point Y_1, and the lines BD, AE and XY intersect at the point Z_1.

We apply Desargues' Theorem to $\triangle XYZ$ and $\triangle BDF$. Since the lines XB, FZ and DY are concurrent (see Problem 6.3.2), we get that the points X_1, Y_1 and Z_1 are collinear.

We apply Desargues' Theorem to $\triangle FDB$ and $\triangle CAE$. Since the points X_1, Y_1 and Z_1 are collinear, we conclude that the lines AD, BE and CF are concurrent.

Problem 6.3.2. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. All their common external tangent lines form the hexagon $ABCDEF$. Let $AB \cap DE = Z$, $AF \cap DC = X$ and $FE \cap BC = Y$. Prove that the lines ZF, XB and YD are concurrent.



Solution. Let $AF \cap DE = P$, $FE \cap DC = Q$, $ED \cap BC = R$, $CD \cap AB = S$, $BC \cap AF = T$ and $BA \cap EF = K$. Let the centers of k_1, k_2 and k_3 be I_1, I_2 and I_3, and let their radii be r_1, r_2 and r_3, respectively. We have

$$\frac{I_2B}{BI_3} \cdot \frac{I_3D}{DI_1} \cdot \frac{I_1F}{FI_2} = \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} \cdot \frac{r_1}{r_2} = 1,$$

$$\angle FI_1E = \frac{\angle FPE}{2} = \angle XI_1D, \angle EI_1D = \frac{\angle EQD}{2} = \angle FI_1X,$$
$$\angle DI_3C = \frac{\angle DRC}{2} = \angle BI_3Z, \angle CI_3B = \frac{\angle CSB}{2} = \angle DI_3Z,$$
$$\angle BI_2A = \frac{\angle BTA}{2} = \angle FI_2Y, \angle BI_2Y = \frac{\angle AKF}{2} = \angle AI_2F.$$

Now we apply the law of sines and we consecutively get

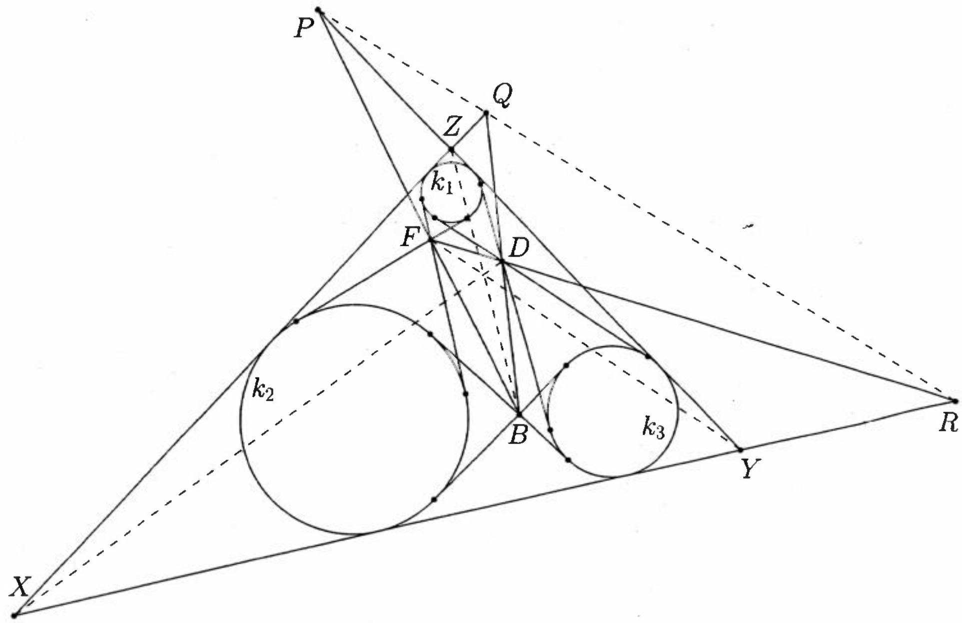
$$
\begin{aligned}
&\frac{\sin\angle DFZ}{\sin\angle ZFB}\cdot\frac{\sin\angle FBX}{\sin\angle XBD}\cdot\frac{\sin\angle BDY}{\sin\angle YDF}\\
=&\frac{DZ}{ZB}\cdot\frac{BY}{YF}\cdot\frac{FX}{XD}\cdot\frac{\sin\angle FDE}{\sin\angle FBA}\cdot\frac{\sin\angle BFA}{\sin\angle BDC}\cdot\frac{\sin\angle DBC}{\sin\angle DFE}\\
=&\frac{DZ}{ZB}\cdot\frac{BY}{YF}\cdot\frac{FX}{XD}\cdot\frac{FE}{ED}\cdot\frac{DC}{CB}\cdot\frac{BA}{AF}\\
=&\frac{\sin\angle DI_3Z}{\sin\angle BI_3Z}\cdot\frac{DI_3}{I_3B}\cdot\frac{\sin\angle BI_2Y}{\sin\angle FI_2Y}\cdot\frac{BI_2}{I_2F}\cdot\frac{\sin\angle FI_1X}{\sin\angle XI_1D}\cdot\frac{FI_1}{I_1D}\cdot\\
&\cdot\frac{\sin\angle FI_1E}{\sin\angle EI_1D}\cdot\frac{FI_1}{I_1D}\cdot\frac{\sin\angle DI_3C}{\sin\angle CI_3B}\cdot\frac{DI_3}{I_3B}\cdot\frac{\sin\angle BI_2A}{\sin\angle AI_2F}\cdot\frac{BI_2}{I_2F}\\
=&\frac{I_2B^2}{BI_3^2}\cdot\frac{I_3D^2}{DI_1^2}\cdot\frac{I_1F^2}{FI_2^2}\cdot\frac{\sin\angle DI_3Z}{\sin\angle CI_3B}\cdot\frac{\sin\angle BI_2Y}{\sin\angle AI_2F}\cdot\frac{\sin\angle FI_1X}{\sin\angle EI_1D}\cdot\\
&\cdot\frac{\sin\angle FI_1E}{\sin\angle XI_1D}\cdot\frac{\sin\angle DI_3C}{\sin\angle BI_3Z}\cdot\frac{\sin\angle BI_2A}{\sin\angle FI_2Y}=1.
\end{aligned}
$$

Hence, the trigonometric form of Ceva's Theorem, applied to $\triangle BDF$ and the lines FZ, BX and DY, gives the desired result.

Problem 6.3.3. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. The common external homothetic centers of each pair of circles are F, D and B, as shown in the figure. For each pair of circles, one of their common external tangent lines is drawn – the one that leaves all three circles completely in the same half-plane. These tangent lines form the triangle XYZ, as shown in the figure. Prove that the lines ZB, XD and YF are concurrent.

Solution. Let $XZ \cap BD = Q$, $BF \cap YZ = P$ and $FD \cap XY = R$. Desargues' Theorem (Problem 7.1), applied to $\triangle BFD$ and $\triangle ZYX$, yields that our statement is equivalent to the statement that the points P, Q and R are collinear.

We will prove that these three points are the external homothetic centers of the three circles, which, by Monge's Theorem (Problem 6.2.3), would give the desired result.



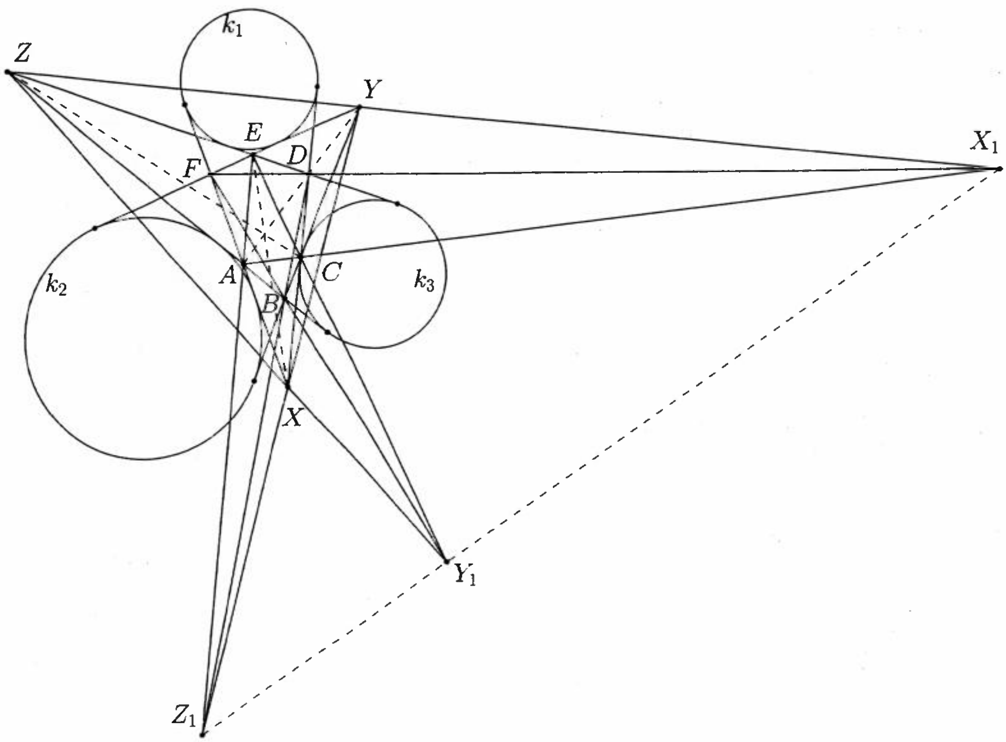
Problem 6.2.4 yields that R is the external homothetic center of k_2 and k_3. Analogously, P is the external homothetic center of k_1 and k_3, and Q is the external homothetic center of k_2 and k_1.

Problem 6.3.5. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. All their common internal tangent lines form the hexagon $ABCDEF$. Let $AB \cap DE = Z$, $AF \cap DC = X$ and $FE \cap BC = Y$. Prove that the lines AY, EX and CZ are concurrent.

Solution. We apply Desargues' Theorem (Problem 7.1) to $\triangle FAZ$ and $\triangle DCY$. We have that $FA \cap DC = X$, $AZ \cap CY = B$. In addition, the lines XB, FZ and DY are concurrent (see Problem 6.3.2). Therefore, the lines ZY, FD and AC are concurrent. Let their intersection point be X_1. Analogously, the lines FB, EC and XZ intersect at the point Y_1, and the lines BD, AE and XY intersect at the point Z_1.

We apply Desargues' Theorem to $\triangle XYZ$ and $\triangle BDF$. Since the lines XB, FZ and DY are concurrent (see Problem 6.3.2), then the points X_1, Y_1 and Z_1 are collinear.

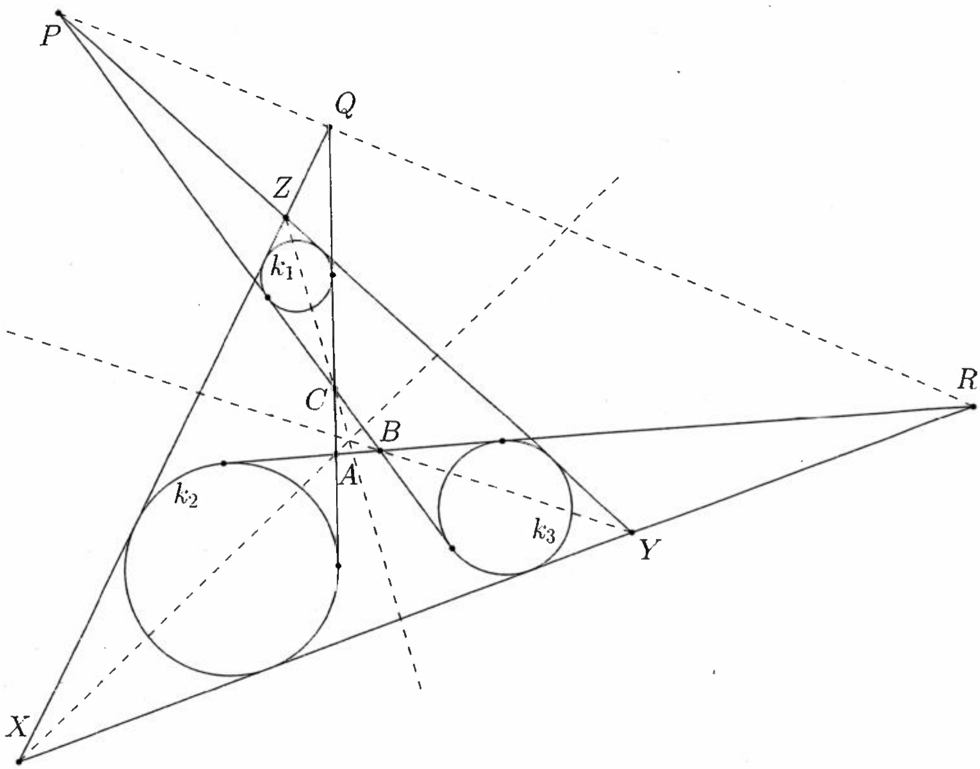
Next, we apply Desargues' Theorem to $\triangle XYZ$ and $\triangle EAC$. Since the points X_1, Y_1 and Z_1 are collinear, then the lines AY, EX and CZ are concurrent.



Problem 6.3.6. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. All their common external tangent lines are drawn. For each pair of circles, let us consider one of their common external tangent lines – the one that leaves all three circles completely in the same half-plane. These three tangent lines form the triangle XYZ, as shown in the figure. The other three tangent lines form the triangle ABC, as shown in the figure. Prove that the lines AX, BY and CZ are concurrent.

Solution. Let $XZ \cap AC = Q$, $BC \cap YZ = P$ and $AB \cap XY = R$. Monge's Theorem (Problem 6.2.3) yields that the points P, Q and R are collinear.

Now Desargues' Theorem (Problem 7.1), applied to the triangles ABC and XYZ, yields that the lines AX, BY and CZ are concurrent.



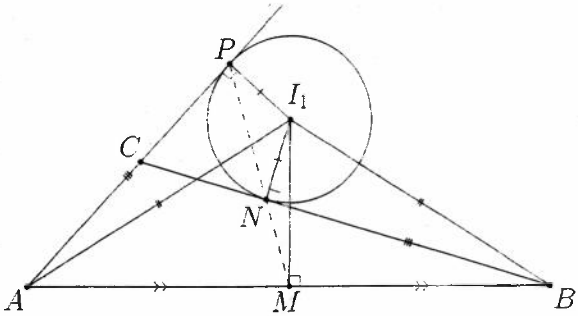
Problem 6.3.8. Let k_1, k_2 and k_3 be circles, such that their disks do not intersect each other. Let their centers be I_1, I_2 and I_3, and let their radii be r_1, r_2 and r_3, respectively. Let B be the internal homothetic center of k_2 and k_3, let D be the internal homothetic center of k_1 and k_3, and let F be the internal homothetic center of k_1 and k_2. Prove that the lines I_1B, I_2D and I_3F are concurrent.

Solution. (See the figure of Problem 6.3.2) We have that the points I_1, F and I_2, as well as the points I_2, B and I_3, and the points I_3, D and I_1 are collinear. Then

$$\frac{I_2B}{BI_3} \cdot \frac{I_3D}{DI_1} \cdot \frac{I_1F}{FI_2} = \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} \cdot \frac{r_1}{r_2} = 1.$$

Now Ceva's Theorem, applied to $\triangle I_1I_2I_3$, yields the desired result.

Problem 6.4.1 Let ABC be a triangle. A circle $k_1(O_1)$ touches BC and AC at the points N and P, respectively. The point I_1 lies on the perpendicular bisector of AB. Denote the midpoint of AB by M. Prove that the points M, N and P are collinear.



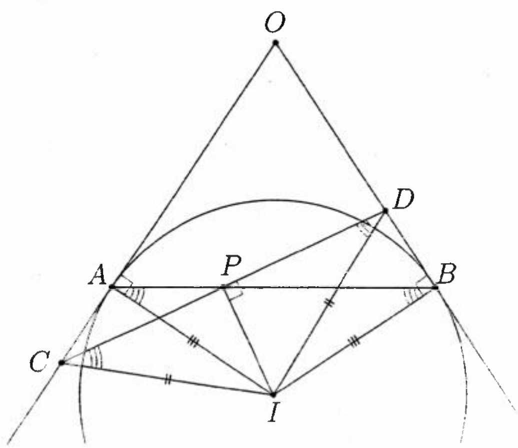
First solution. Since I_1 belongs to the perpendicular bisector of AB, we have $AI_1 = BI_1$. Note that $I_1P = I_1N$.

Hence, $\triangle AI_1P \cong \triangle BI_1N$ and the eqalities $AP = BN$ and $CN = CP$ hold.

It follows that $\dfrac{AP}{PC} \cdot \dfrac{CN}{NB} \cdot \dfrac{BM}{MA} = 1$. Menelaus' Theorem, applied to $\triangle ABC$, yields that the points M, N and P are collinear.

Second solution. Since I_1 lies on the external bisector of $\angle ACB$ and also on the perpendicular bisector of AB, then it is the midpoint of the arc $\widehat{ACB}$ of the circumcircle of $\triangle ABC$. Thus, the points M, N and P are collinear because they lie on the Simson's line (Problem 3.11) of the point I_1, which lies on the circumcircle of $\triangle ABC$.

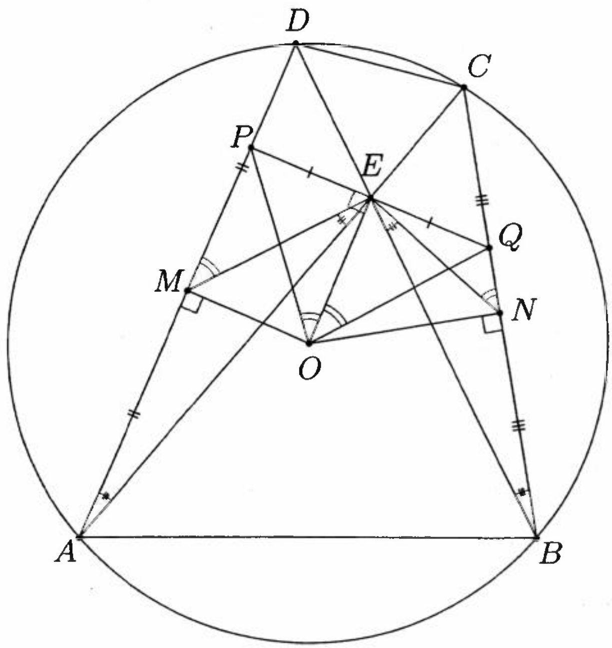
Problem 6.4.2. Let ABO be a triangle and let $AO = BO$. A circle $k_1(I)$ touches OA and OB at the points A and B, respectively. A point P is chosen on the side AB. A line l passes through P, such that $IP \perp l$. Let the line l intersect OA and OB at the points C and D, respectively. Prove that $PC = PD$.



Solution. Observe that $\angle IAB = \angle IBA$. The quadrilaterals $PIBD$ and $PICA$ are cyclic.

Hence, $\angle PDI = \angle PBI = \angle IAP = \angle ICP$, which means that $IC = ID$. Thus, $\triangle CID$ is an isosceles triangle, and IP is both an altitude and a median in it. Therefore, $CP = DP$.

Problem 6.4.3. *(The butterfly theorem)* Let $ABCD$ be a quadrilateral inscribed in a circle with center O. The diagonals of $ABCD$ intersect at the point E. A line perpendicular to EO at E intersects AD and BC at the points P and Q, respectively. Prove that $PE = QE$.

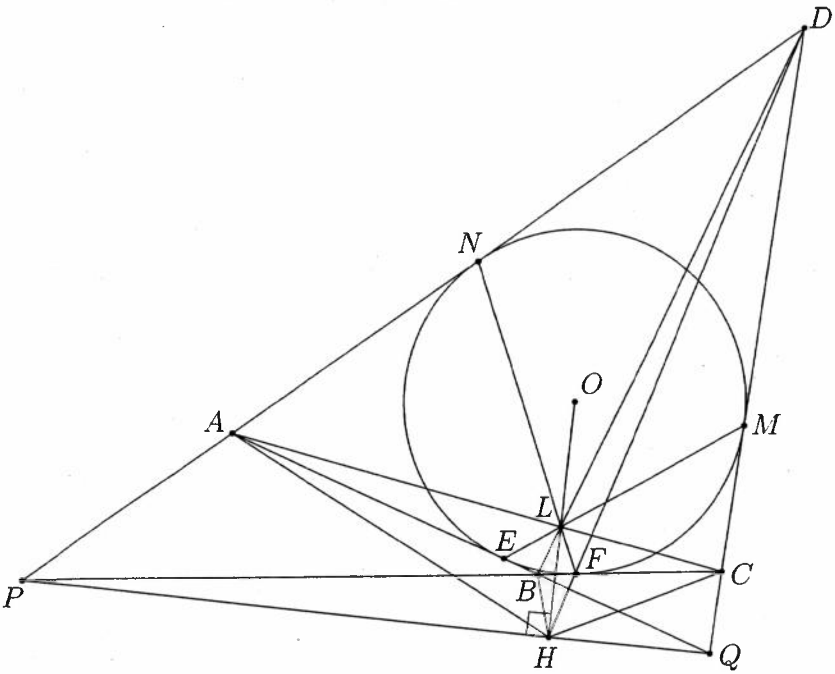


Solution. Note that $\angle DAE = \angle CBE$ and $\angle AED = \angle BEC$. Hence, $\triangle AED \sim \triangle BEC$. Denote the midpoints of AD and BC by M and N, respectively. Then EM and EN are the corresponding medians of the similar triangles $\triangle ADE$ and $\triangle BCE$. Thus, $\angle EMD = \angle ENC$.

On the other hand, $OM \perp AD$. Hence, $OMPE$ is cyclic and $\angle DME = \angle PME = \angle POE$. Analogously, $\angle EOQ = \angle CNE$. We have $\angle DME = \angle CNE$ and thus $\angle POE = \angle QOE$.

We obtained that the line OE is both an altitude and an angle bisector in $\triangle POQ$. Hence, it is a median as well.

Problem 6.4.4. Let $k(O)$ be a circle. The points P and Q are external with respect to k. Denote the orthogonal projection of O onto PQ by H. The four tangent lines from P and Q to k form the quadrilateral $ABCD$. Prove that the line HO bisects both $\angle BHD$ and $\angle AHC$.



Solution. Let the points of tangency of AB, BC, CD and DA to k be E, F, M and N, respectively. Brianchon's Theorem, applied to the circumscribed hexagon $AEBCMD$, yields that the lines AC, BD and EM are concurrent. Let their intersection point be L.

Analogously, L lies on NF. Let π_k be a pole-polar transformation with respect to k. Then $\pi_K(PQ)$ is the intersection point of $\pi_k(P) = NF$ and $\pi_k(Q) = EM$.

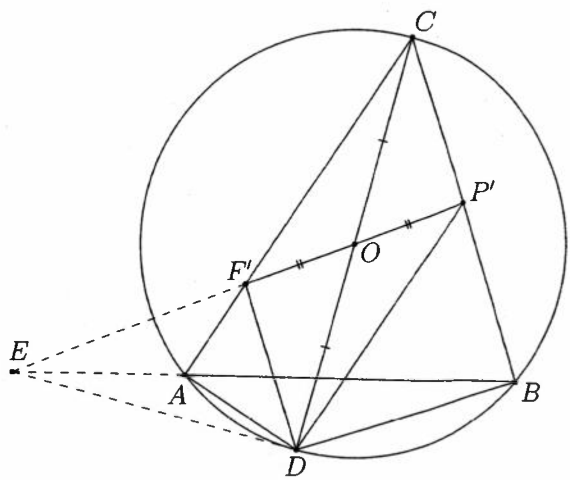
Hence, $\pi_k(AB) = L$. Then $OL \perp PQ$ and $L \in OH$. Now Problem 4.12.2, applied to the quadrilateral $ABCD$, yields the desired result.

Problem 6.4.5. Let ABC be a triangle. Let $k(O)$ be the circumcircle of the triangle and let D be the reflection of C with respect to O. The tangent line to k at the point D intersects AB at the point E. Let OE intersect AC and BC at the points F and P, respectively. Prove that $FO = OP$.

Solution. Let $F' \in AC$ be a point, such that $F'D \parallel BC$. Let $P' \in BC$ be a point, such that $DP' \parallel AC$. Then clearly the quadrilateral $DF'CP'$ is a parallelogram and the point O bisects its diagonals.

Let $F'P' \cap AB = E'$.

We will prove that the point E coincides with the point E', which will lead us to $P \equiv P'$ and $F \equiv F'$, so the desired statement will follow.



We have that $CD = 2R$, $\angle F'CD = \angle CDP' = 90° - \beta$, $\angle P'CD = \angle CDF' = 90° - \alpha$ and by the law of sines,

$$CF' = \frac{2R\cos\alpha}{\sin\gamma},\ CP' = \frac{2R\cos\beta}{\sin\gamma}.$$

Therefore,

$$AF' = b - CF' = \frac{2R}{\sin\gamma}(\sin\beta\sin\gamma - \cos\alpha) = \frac{2R\cos\beta\cos\gamma}{\sin\gamma},$$

$$BP' = a - CP' = \frac{2R}{\sin\gamma}(\sin\alpha\sin\gamma - \cos\beta) = \frac{2R\cos\alpha\cos\gamma}{\sin\gamma}.$$

Menelaus' Theorem, applied to $\triangle ABC$ and the line $P'F'$, yields

$$\frac{BE'}{E'A} = \frac{\dfrac{2R\cos\alpha\cos\gamma}{\sin\gamma}}{\dfrac{2R\cos\beta}{\sin\gamma}} \cdot \frac{\dfrac{2R\cos\alpha}{\sin\gamma}}{\dfrac{2R\cos\beta\cos\gamma}{\sin\gamma}} = \frac{\cos^2\alpha}{\cos^2\beta}.$$

On the other hand, we have that $\angle EAD = \angle EDB = 90° + \alpha$. Also, $\angle ABD = \angle ADE = 90° - \beta$. The law of sines, applied to $\triangle EAD$ and $\triangle EBD$, yields

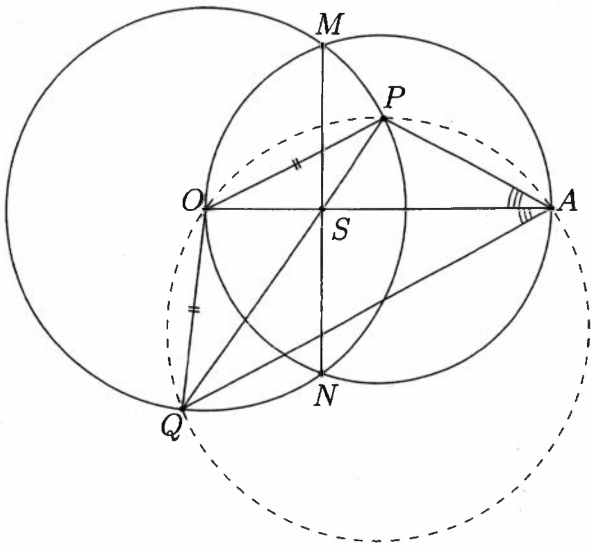
$$\frac{BE}{EA} = \frac{\sin\angle BDE}{\sin\angle ADE} \cdot \frac{\sin\angle EAD}{\sin\angle ABD} = \frac{\cos^2\alpha}{\cos^2\beta} = \frac{BE'}{E'A}.$$

Hence, $E \equiv E'$, as required.

Problem 6.4.6. Let $k(O)$ be a circle. The circle k_1 passes through O. Let $k_1 \cap k = \{M, N\}$. Let $A \in k_1$ and $P \in k$ be arbitrary points. Denote one of the intersection points of k and the reflection of the line PA with respect to OA by Q. Let $S = MN \cap AO$. Prove that the points Q, S and P are collinear.

Solution. In $\triangle APQ$, the line OA is a bisector of $\angle PAQ$ and O lies on the perpendicular bisector of PQ.

Hence, O lies on the circumcircle of $\triangle APQ$. Observe that the lines MN, PQ and AO are the radical axes of the circles k, k_1 and the circumcircle of the quadrilateral $QAPO$. By the radical axes theorem (Problem 6.5.1), they intersect at S.

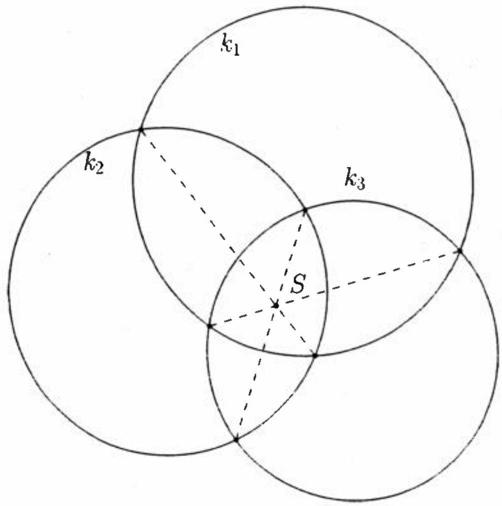


Problems 6.5.1 and 6.5.2 *(The radical axes theorem)* Let k_1, k_2 and k_3 be three non-concentric circles. Prove that the three radical axes of each pair of circles are either concurrent or parallel.

Solutions. Denote the radical axes by $\rho(k_1,k_2)$, $\rho(k_2,k_3)$ and $\rho(k_1,k_3)$.

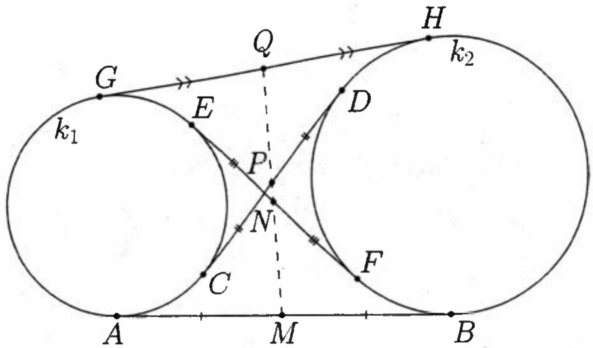
If $\rho(k_1,k_2) \parallel \rho(k_2,k_3)$, then the centers of the circles are collinear, and the third radical axis will be parallel to the first two.

If the three radical axes are not parallel, then $S = \rho(k_1,k_2) \cap \rho(k_2,k_3)$ exists and it follows that S has the same power with respect to the pairs of circles k_1 and k_2, and k_2 and k_3, which means that $S \in \rho(k_1,k_3)$.



Problem 6.5.3. Let k_1 and k_2 be circles, such that their disks do not intersect each other. The points A, C, E and G lie on k_1. The points B, D, F and H lie on k_2. The lines AB, GH, EF and CD are the common external and internal tangent lines of the circles, respectively. Prove that the midpoints of AB, CD, EF and GH are collinear.

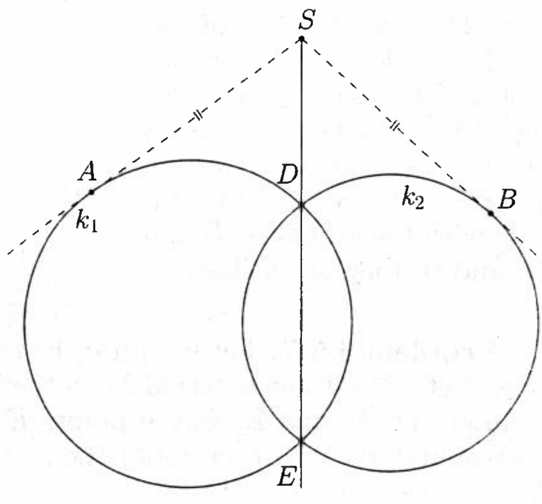
Solution. We will use the notations on the figure. Since the powers of M with respect to k_1 and k_2 are equal (due to $MA^2 = MB^2$), we conclude that M belongs to the radical axis of k_1 and k_2. Analogously, we see that the points N, P and Q lie on the same line.



Problem 6.5.4. Let k_1 and k_2 be circles. Assume that $k_1 \cap k_2 = \{D, E\}$. Let S be an arbitrary point on the line DE external for the segment DE. Let the tangent lines from S to k_1 and k_2 be SA ($A \in k_1$) and SB ($B \in k_2$). Prove that $AS = SB$.

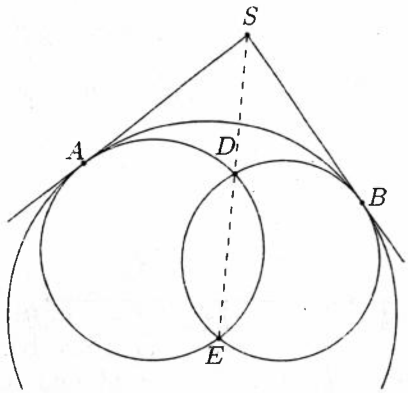
Solution. Observe that the line DE is the radical axis of k_1 and k_2, and that $S \in DE$.

Hence, the powers of S with respect to k_1 and k_2 are equal. Since these powers are respectively equal to AS^2 and SB^2, we conclude that $AS^2 = SB^2$.

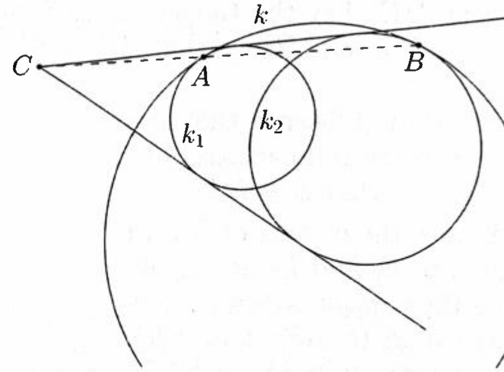


Problem 6.5.5. Let k be a circle. The circles k_1 and k_2 touch k at the points A and B, respectively. Let S be the intersection point of the tangent lines to k at A and B. Prove that S lies on the radical axis of k_1 and k_2.

Solution. Observe that $\rho(k_1, k) \equiv AS$ and $\rho(k_2, k) \equiv BS$. Since $S = \rho(k_1, k) \cap \rho(k_2, k)$, we conclude that S is the radical center of the three circles.



Problem 6.5.6. Let k be a circle. The circles k_1 and k_2 touch k internally at the points A and B, respectively. Denote the external homothetic center of k_1 and k_2 by C. Prove that the points A, B and C are collinear.



Solution. Monge's Theorem (Problem 6.2.3), applied to the circles k_1, k_2 and k, yields that the external homothetic centers of the pairs (k_1, k_2), (k_2, k) and (k_1, k) are collinear. These centers coincide with the points A, B and C, and so they are collinear.

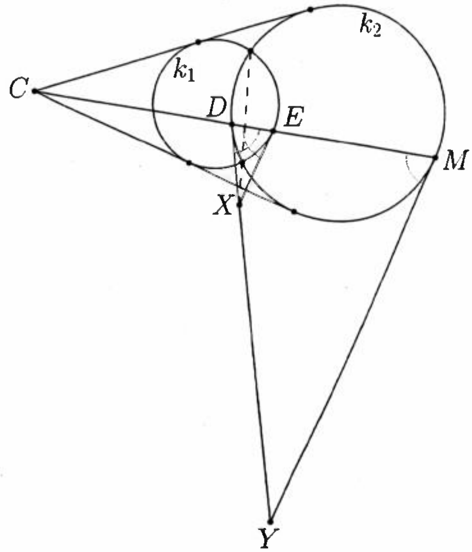
Problem 6.5.7. Let k_1 and k_2 be two circles and let k_1 be the smaller one. Let C be their external homothetic center. A line passing through C intersects k_1 and k_2 at the points E and D. These two points do not correspond to each other under the homothety h with center C, such that $h(k_1) = k_2$. Let X be the intersection point of the tangent lines to k_1 and k_2 at the points E and D, respectively. Prove that X lies on the radical axis of k_1 and k_2.

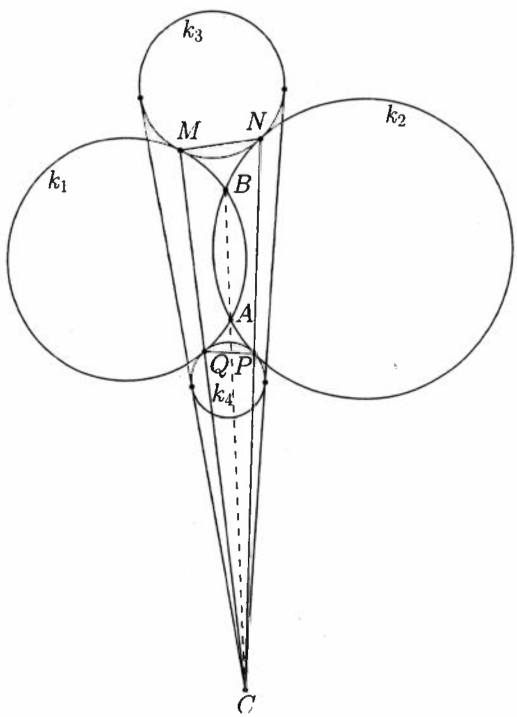
Solution. Let M be the second intersection point of DE and k_2. Let Y be the intersection of DM and the tangent line to k_2 at the point M.

Since $h(E) = M$, we have $EX \parallel MY$. Therefore,

$$\angle EDX = \angle DMY = \angle DEX.$$

Thus, $DX = XE$, and so X lies on the radical axis of k_1 and k_2.



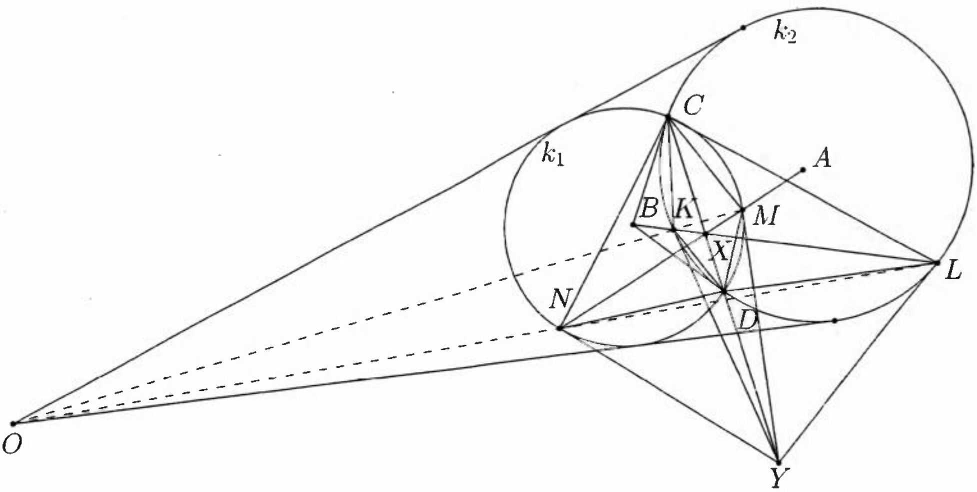


Problem 6.5.8. Let k_1 and k_2 be circles and let $k_1 \cap k_2 = \{A, B\}$. The circles k_3 and k_4 touch k_1 and k_2 externally. Let C be the external homothetic center of k_3 and k_4. Prove that the points A, B and C are collinear.

Solution. Let $k_1 \cap k_3 = M$, $k_2 \cap k_3 = N$, $k_2 \cap k_4 = P$ and $k_1 \cap k_4 = Q$. Applying Problem 6.2.4, we get that the points M, Q and C, as well as the points N, P and C are collinear.

Problem 6.8.1, applied to all four given circles, yields that $PQMN$ is cyclic. The radical axes theorem (Problem 6.5.1), applied to the circles k_1, k_2 and $(PQMN)$, yields that C lies on $\rho(k_1, k_2)$. Since $\rho(k_1, k_2) \equiv AB$, it follows that the points A, B and C are collinear. Note that it is not necessary for k_1 and k_2 to intersect each other. The statement remains true if these circles do not have common points and none of them lies inside of the other one.

Problem 6.5.9. Let k_1 and k_2 be circles and let $k_1 \cap k_2 = \{C, D\}$. The tangent lines to k_1 at the points C and D intersect each other at the point A. Point B is defined analogously for the circle k_2. Let $X \in CD$ be such that $BX \cap k_1 = \{K, L\}$ and $AX \cap k_1 = \{M, N\}$, as shown in the figure. Prove that the lines KM and NL intersect at O, where O is the external homothetic center of k_1 and k_2.



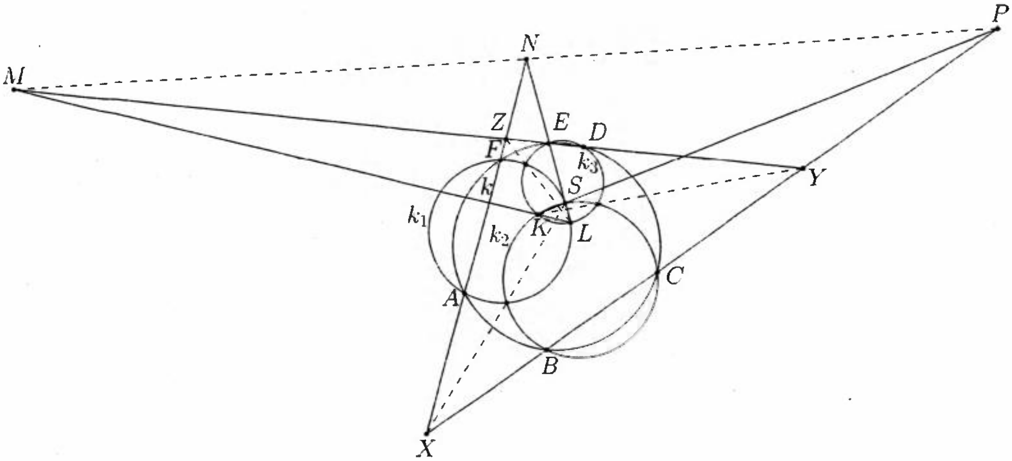
Solution. The quadrilateral $NDMC$ is harmonic because $NM \cap t(D, k_1) \cap t(C, k_1) = A$. Analogously, the quadrilateral $DLCK$ is harmonic.

Let $t(M, k_1) \cap t(N, k_1) = Y$. Then $Y \in CD$ and $(C, X, D, Y) = 1$. Let $t(K, k_2) \cap t(L, k_2) \cap CD = Y'$. Hence, $(C, X, D, Y') = 1$ and $Y \equiv Y'$.

So $Y \in CD$ and CD is the radical axis of k_1 and k_2. Hence, $YN = YL$. Then there exists a circle ω_1 that touches YN and YL at the points N and L, respectively. Monge's Theorem, applied to the circles k_1, k_2 and ω_1, yields that the points O, N and L are collinear.

Analogously, the points O, K and M are collinear.

Problem 6.5.10. The points A, B, C, D, E and F lie on a circle k in this order. The circles k_1, k_2 and k_3 pass through the pair of points (A, F), (B, C) and (D, E), respectively. Let $k_1 \cap k_2 = S$ and let S lie in the interior of k_3. Analogously, let $k_1 \cap k_3 = L$ and $k_2 \cap k_3 = K$, where L and K lie on the interior of the circles k_2 and k_1, respectively. Let $KS \cap BC = P$, $LS \cap AF = N$ and $KL \cap ED = M$. Prove that the points M, N and P are collinear.



Solution. Let $AF \cap BC = X$, $BC \cap ED = Y$ and $AF \cap ED = Z$.

Desargues' Theorem (Problem 7.1), applied to ΔXYZ and ΔSKL, yields that it suffices to show that the lines XS, YK and ZL are concurrent.

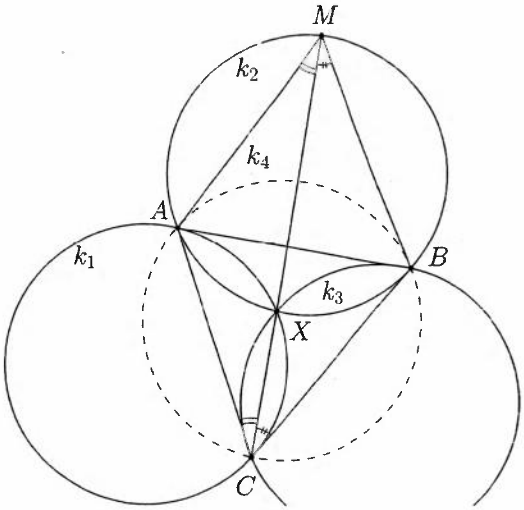
Since XY and XZ are the radical axes of the pairs of circles (k, k_2) and (k, k_1), we conclude that X lies on the radical axis of k_1 and k_2.

Hence, XS is the radical axis of k_1 and k_2. Analogously, YK is the radical axis of k_2 and k_3, and also ZL is the radical axis of k_3 and k_1.

Applying Problem 6.5.1, we get that the lines XS, YK and ZL are concurrent.

Problem 6.6.1. Let k_1, k_2 and k_3 be three congruent circles with radii R. Let $k_1 \cap k_2 = \{A, X\}$, $k_2 \cap k_3 = \{B, X\}$ and $k_3 \cap k_1 = \{C, X\}$. Prove that the circumradius of $\triangle ABC$ equals R.

Solution. Let CX intersect k_2 for the second time at the point M. Denote the circumradius of $\triangle ABC$ by R_1. Since the radii of k_1, k_2 and k_3 are all equal to R, then $\angle XCB = \angle XMB$ and $\angle XCA = \angle XMA$. Hence, $\angle AMB = \angle ACB$.



The law of sines, applied to $\triangle ABM$ and $\triangle ABC$, yields

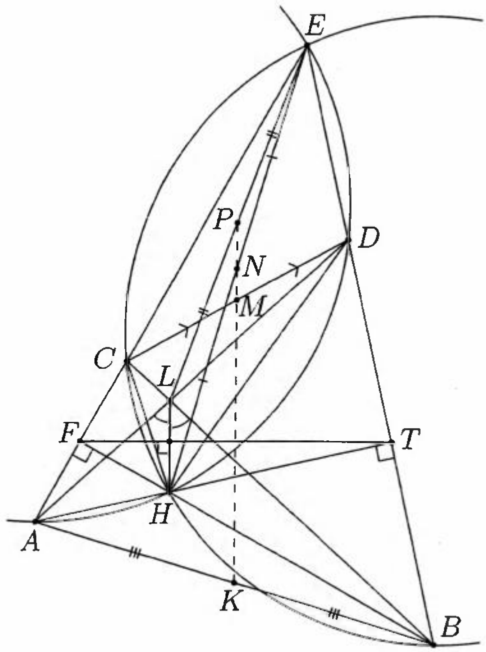
$$2R_1 = \frac{AB}{\sin \angle ACB} = \frac{AB}{\sin \angle AMB} = 2R.$$

Thus, $R_1 = R$.

Problem 6.6.2. Let k_1 and k_2 be two circles with equal radii and let $k_1 \cap k_2 = \{E, H\}$. The points $C \in k_2$ and $D \in k_1$ are chosen, such that C and D lie inside of k_1 and k_2, respectively. Let $EC \cap k_1 = \{A, E\}$ and $ED \cap k_2 = \{B, E\}$. Prove that the midpoints K, M and N of AB, CD and EH, respectively, are collinear.

Solution. Let $AD \cap BC = L$. Denote the midpoint of EL by P. Problem 3.8, applied to the quadrilateral $ECLD$, yields that the points K, M and P are collinear.

We will prove that N lies on the same line. It suffices to show that $MK \parallel LH$ because PN is a midsegment in $\triangle LHE$.



Denote the midpoints of AC and BD by F and T, respectively. Now

$$BC = 2R\sin\angle BEC = 2R\sin\angle AED = AD.$$

Note that R is the radius of the circles k_1 and k_2. Consider the midsegments

$$FM = KT = \frac{AD}{2} = \frac{BC}{2} = FK = MT.$$

Thus, $FKTM$ is a rhombus and $MK \perp FT$. We are left to prove that $FT \perp LH$.

Observe that $BH = 2R\sin\angle BEH = 2R\sin\angle DEH = DH$ and $AH = 2R\sin\angle AEH = 2R\sin\angle CEH = CH$. Hence,

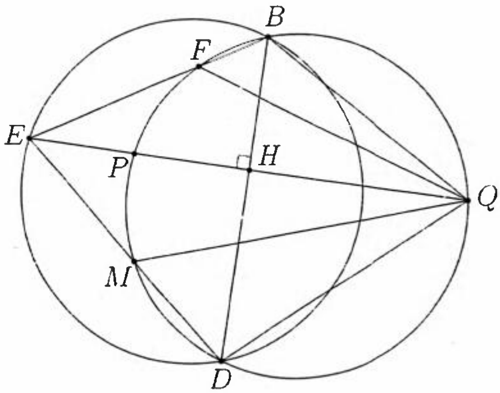
$$\sin\angle ECH = \frac{EH}{2R} = \sin\angle EDH$$
$$\Rightarrow \angle CAH = \angle ACH = \angle HDB = \angle HBD.$$

It follows that $\triangle ACH \sim \triangle BDH$ and $\dfrac{HF}{HT} = \dfrac{HC}{HB}$. Then $\triangle HFT \sim \triangle HCB$.

The angle between the lines BC and FT is equal to the one between BH and HT, i.e. to $\angle BHT$. On the other hand, $\triangle HCB \cong \triangle HAD$ and hence the quadrilaterals $AHLC$ and $BHLD$ are cyclic.

Since $AH = CH$ and $BH = DH$, we conclude that LH bisects $\angle ALB = 2\angle HBT$. Thus, the angle between the lines LH and BL equals $90^\circ - \angle BHT$. Hence, $FT \perp LH$.

Problem 6.6.3. Let k_1 and k_2 be two circles with equal radii, and let $k_1 \cap k_2 = \{B, D\}$. A point $E \in k_1$ is chosen, such that E does not lie inside of k_2. The lines EB and ED intersect k_2 for the second time at the points F and M, respectively. Denote the midpoint of the smaller arc $\widehat{FM}$ by P. Prove that $EP \perp BD$.



Solution. Let H be the orthogonal projection of E onto BD and Q be the reflection of E with respect to BD.

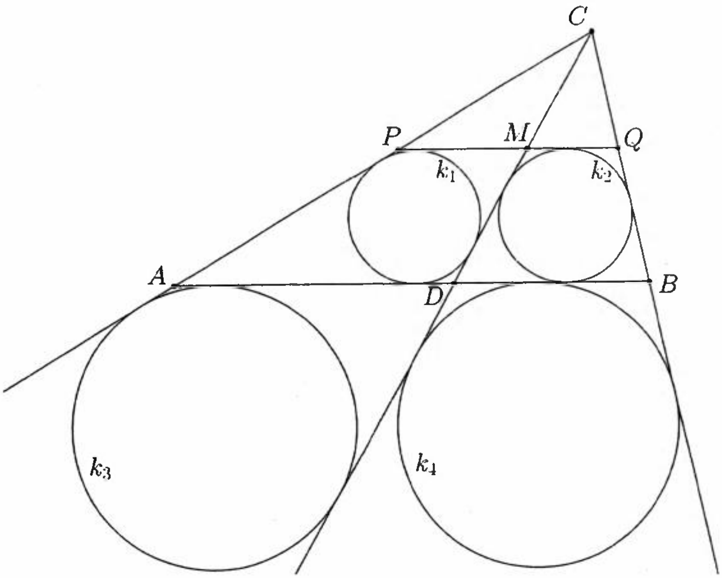
It is clear that $Q \in k_2$ because k_1 and k_2 have equal radii. We will prove that $\angle FQE = \angle EQM$. Let $EH \cap k_2 = \{P, Q\}$.

Observe that

$$\begin{aligned}\angle FQE &= \angle HQB - \angle BQF = (90^\circ - \angle DBQ) - (180^\circ - \angle BFQ - \angle FBQ) \\ &= (90^\circ - \angle FBD) - (180^\circ - \angle BDQ - \angle FBD - \angle DBQ) \\ &= 90^\circ - \angle DQB.\end{aligned}$$

Analogously, $\angle EQM = 90^\circ - \angle DQB$.

Problem 6.6.4. Let ABC be a triangle. The point D lies on the segment AB, such that the radii of the incircles of $\triangle ADC$ and $\triangle DBC$ are equal. Prove that the C-excircles of $\triangle ADC$ and $\triangle BDC$ have equal radii.



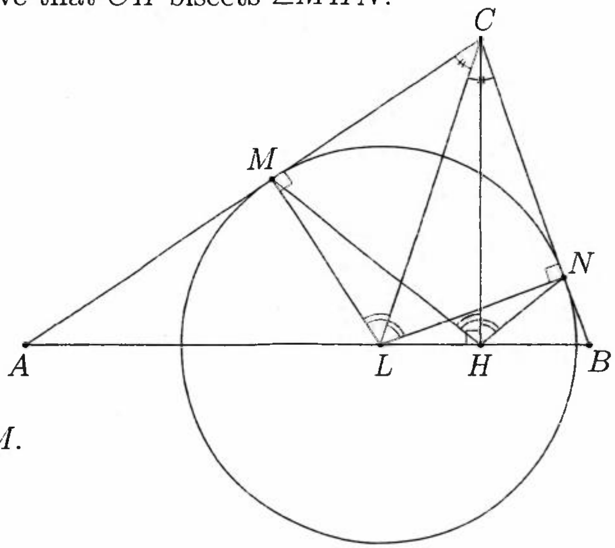
Solution. Denote the incircles of $\triangle ADC$ and $\triangle DBC$ by k_1 and k_2, respectively. Let r be the radius of these circles. The common external tangent line of k_1 and k_2 (different from AB) intersects AC and BC at the points P and Q, respectively. Let $PQ \cap CD = M$.

Denote the C-excircles of $\triangle ADC$ and $\triangle BDC$ by k_3 and k_4, respectively. Since the radii of k_1 and k_2 are equal, we have $PQ \parallel AB$.

Consider a homothety h centered at C, such that $h(PQ) = AB$. Then $h(M) = D$, $h(k_1) = k_3$ and $h(k_2) = k_4$.

Hence, the circles k_3 and k_4 have equal radii.

Problem 6.7.1. Let ABC be a triangle with an altitude CH ($H \in AB$). A circle $k_1(L)$, where $L \in AB$, touches AC and BC at the points M and N, respectively. Prove that CH bisects $\angle MHN$.



Solution. Note that $\triangle MLC \cong \triangle NLC$. Hence, $\angle MLC = \angle NLC$.

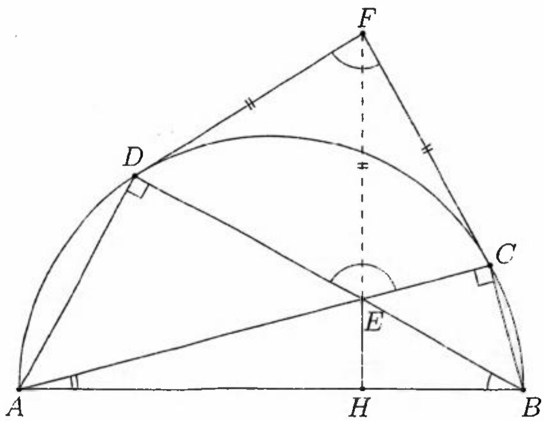
It is clear that the points M, C, N, H and L lie on the circle with diameter CL.

We get

$$\angle CHN = \angle CLN = \angle CLM = \angle CHM.$$

Hence, CH is the bisector of $\angle MHN$.

Problem 6.7.2. The points A, B, C and D lie on a circle with diameter AB. The tangent lines to the circle at the points C and D intersect each other at the point F. Let $E = AC \cap BD$. Prove that $FE \perp AB$.



Solution. Let $H = FE \cap AB$.

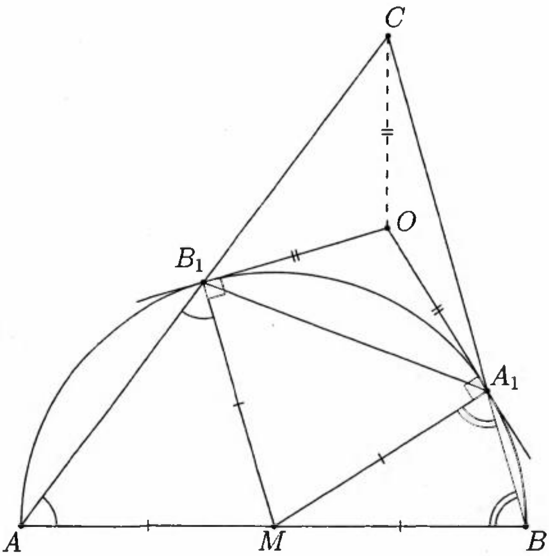
Denote $\angle ABD = \alpha$ and $\angle BAC = \beta$. Observe that $\angle ADB = \angle ACB = 90°$ and $\angle BAD = 90° - \alpha = \angle BDF$. Analogously, $\angle ACF = 90° - \beta$. We obtain $\angle CED = \angle AEB = 180° - \alpha - \beta$. It follows from the quadrilateral $DECF$ that $\angle DFC = 2\alpha + 2\beta$, which means that

$$2(180° - \angle CED) = \angle DFC.$$

Hence, E lies on the circle centered at F with radius FD. Thus, $FD = FE$. Therefore, $\angle HEB = \angle DEF = \angle EDF = 90° - \alpha$, $\angle HBD = \alpha$ and $FE \perp AB$.

Problem 6.7.3. Let ABC be a triangle with $\angle ACB < 90°$. The circle k_1 with diameter AB and center M intersects AC and BC at the points B_1 and A_1, respectively. Denote the intersection point of the tangent lines to k_1 at A_1 and B_1 by O. Prove that O is the circumcenter of $\triangle A_1B_1C$.

Solution. It is easy to see that $\angle BMA_1 = 180° - 2\beta$ and $\angle AMB_1 = 180° - 2\alpha$.



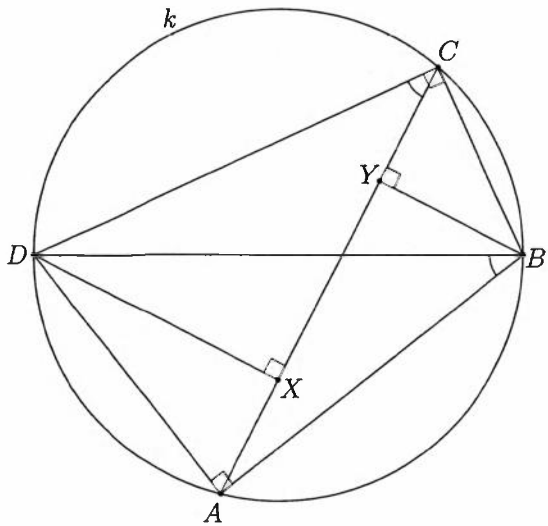
Hence, $\angle A_1MB_1 = 2\alpha + 2\beta - 180° = 180° - 2\gamma$. We obtain $\angle A_1OB_1 = 360° - 2\alpha - 2\beta = 2\gamma$.

Denote the circumcenter of $\triangle A_1B_1C$ by O_1. Then we have $\angle A_1O_1B_1 = 2\gamma = \angle A_1OB_1$.

The points O and O_1 lie on the perpendicular bisector of A_1B_1. Thus, $\triangle A_1B_1O_1 \cong \triangle A_1B_1O$ and $O \equiv O_1$, as desired.

Problem 6.7.4. Let $ABCD$ be a quadrilateral with $\angle DAB = \angle DCB = 90°$. Let X and Y be the orthogonal projections of D and B onto AC, respectively. Prove that $AX = CY$.

Solution. Denote $\angle DCA = \alpha$ and $\angle CAB = \beta$. We get $\angle BCY = 90° - \alpha$ and $\angle CBY = \alpha$.



Also, $\angle XAD = 90° - \beta$ and $\angle XDA = \beta$.

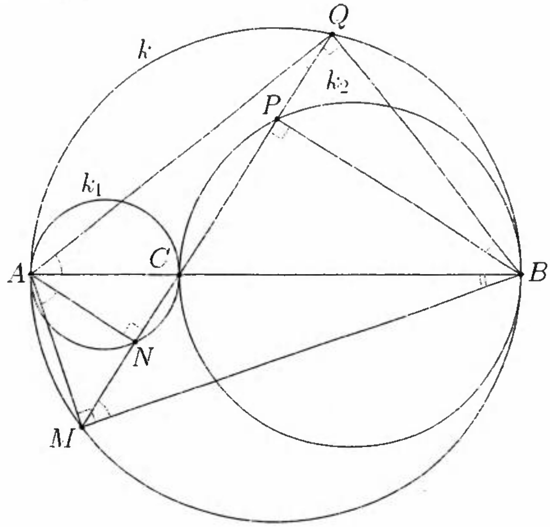
Let R be the circumradius of $ABCD$. Then

$$AX = AD \sin\beta = 2R \sin\alpha \sin\beta,$$

$$CY = BC \sin\alpha = 2R \sin\beta \sin\alpha.$$

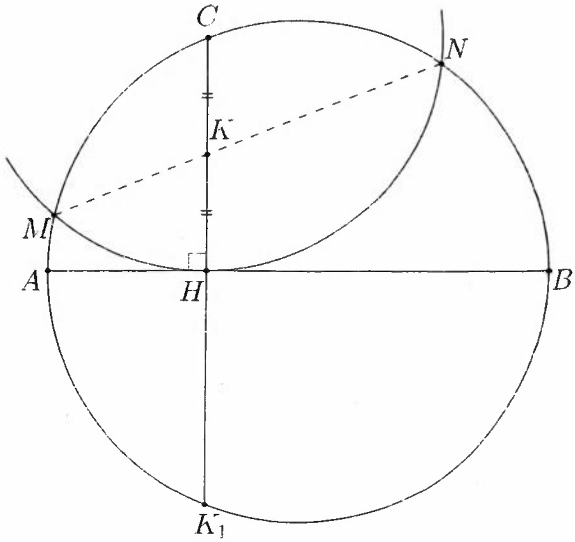
Hence, $AX = CY$.

Problem 6.7.5. Let k be a circle with diameter AB. The circles k_1 and k_2 touch each other externally at the point C. Moreover, the centers of k_1 and k_2 lie on the segment AB, and those circles touch k internally at the points A and B, respectively. Let $M \in k$ be an arbitrary point and let $Q = MC \cap k$. Let k_1 and k_2 intersect MC for the second time at the points N and P, respectively. Prove that $MN = PQ$.



Solution. Observe that $\angle ANC = \angle BPC$. Now we get the construction of Problem 6.7.4 and the result follows from there.

Problem 6.7.6. Let k be a circle with diameter AB. Let $C \in k$ be an arbitrary point. Denote the orthogonal projection of C onto AB by H. Let $k_1(C, CH)$ be a circle. If $k \cap k_1 = \{M, N\}$, prove that the points K, M and N are collinear.

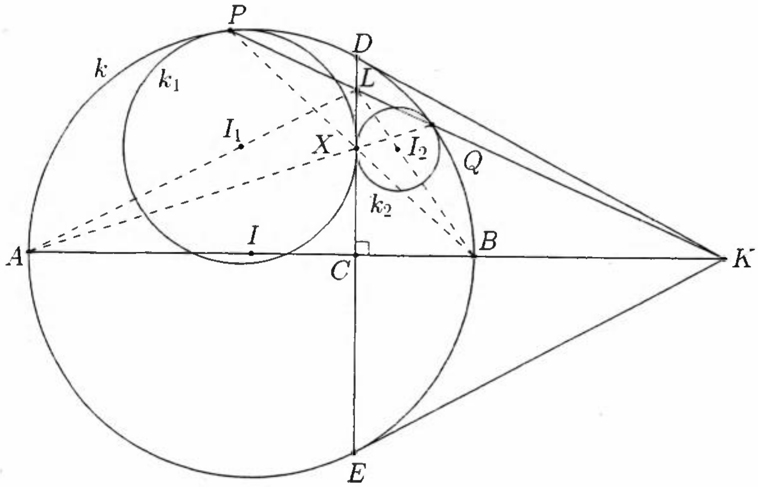


Solution. Consider the inversion $I(C, CH)$. Observe that $I(M) = M$ and $I(N) = N$. Then $I(k) = MN$.

Let $K_1 = I(K)$. Hence, $CK_1 = \dfrac{CH^2}{CK}$. Since $2CK = CH$, we have $CK_1 = 2CH$.

Note that $I(CK) = CK$ and $K_1 \in CK$. In addition, K_1 is the reflection of C with respect to H. Therefore, K_1 lies on k. So K lies on MN.

Problem 6.7.7. Let $k(I)$ be a circle with diameter AB. Let $D \in k$ be an arbitrary point. Denote the orthogonal projection of D onto AB by C. An arbitrary point X lies on DC. The circles $k_1(I_1)$ and $k_2(I_2)$ touch DC at X and k at P and Q, respectively. Prove that the lines AI_1, BI_2, CD and PQ are concurrent.



Solution. Let $DC \cap k = \{D, E\}$. Applying Problem 6.1.1, we obtain that PX and QX are the bisectors of $\angle DPE$ and $\angle DQE$, respectively. Observe that B and A lie on these lines, respectively.

Thus, $\dfrac{DP}{PE} = \dfrac{DX}{XE} = \dfrac{DQ}{QE}$ and the quadrilateral $PEQD$ is harmonic. Let K be the intersection point of the tangent lines to k at D and E. Hence, $K \in PQ$. Let $PQ \cap DE = L$.

It suffices to prove that the points A, I_1 and L are collinear. Applying Menelaus' Theorem, we get that it suffices to show that $\dfrac{KL}{LP} \cdot \dfrac{PI_1}{I_1I} \cdot \dfrac{IA}{AK} = 1$.

Consider the homothety h centered at P, such that $h(k_1) = k$. We obtain $\dfrac{PI_1}{I_1I} = \dfrac{PX}{XB}$.

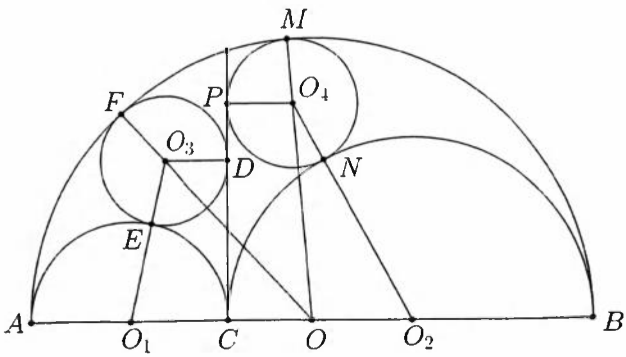
Menelaus' Theorem, applied to $\triangle BKP$ and the line LXC, yields $\dfrac{KL}{LP} \cdot \dfrac{PX}{XB} \cdot \dfrac{BC}{CK} = 1$. Therefore, $\dfrac{KL}{LP} \cdot \dfrac{PI_1}{I_1I} \cdot \dfrac{BC}{CK} = 1$.

We are left to prove that $\dfrac{IA}{AK} = \dfrac{BC}{CK}$.

Problem 4.5.1 gives us that DB bisects $\angle CDK$, and so

$$\frac{AK}{IA} = 1 + \frac{IK}{ID} = 1 + \frac{DK}{DC} = 1 + \frac{KB}{BC} = \frac{CK}{BC}.$$

Problem 6.7.8. The points A, C and B lie on a line in this order. Consider the semicircles k, k_1 and k_2 with diameters AB, AC and BC and centers O, O_1 and O_2, respectively, lying in the same half-plane with respect to AB. Denote the common internal tangent line of k_1 and k_2 by l. A circle $k_3(O_3, r_3)$ touches externally k_1, internally k and the line l. A circle $k_4(O_4, r_4)$ touches externally k_2, internally k and the line l. Prove that $r_3 = r_4$.



Solution. Denote all the points of tangency as shown in the figure.

Let the radii of k, k_1 and k_2 be R, r_1 and r_2, respectively. It is clear that $R = r_1 + r_2$. We are going to express PC^2 in two different ways – by using the right-angled trapezoids CO_2O_4P and COO_4P. We get

$$(r_2 + r_4)^2 - (r_2 - r_4)^2 = CP^2 = (R - r_4)^2 - (R - 2r_1 - r_4)^2.$$

Now a little algebra shows that

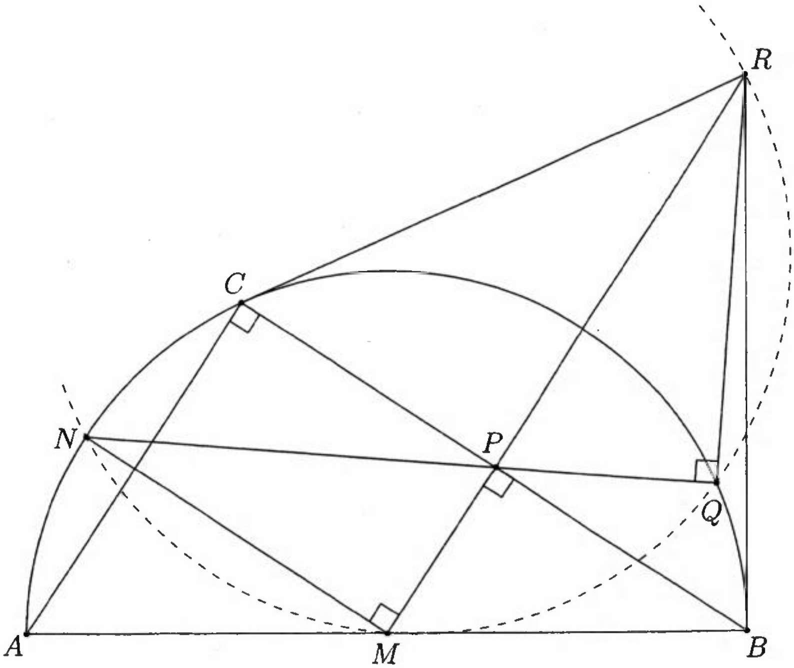
$$4r_2r_4 = 4Rr_1 - 4r_1r_4 - 4r_1^2,$$

whence

$$r_4 = \frac{r_1(R - r_1)}{r_1 + r_2} = \frac{r_1r_2}{r_1 + r_2}.$$

Analogously, $r_3 = \dfrac{r_1r_2}{r_1 + r_2} = r_4.$

Problem 6.7.9. Let ABC be a triangle with $\angle ACB = 90°$. The tangent lines to the circumcircle of $\triangle ABC$ at B and C intersect each other at the point R. Let P and N be the midpoints of BC and the arc $\widehat{AC}$, respectively. Denote the second intersection point of NP and the circumcircle of $\triangle ABC$ by Q. Prove that $\angle NQR = 90°$.



Solution. Denote the midpoint of AB by M. Hence, M is the center of the circumcircle of $\triangle ABC$, and the points M, P and R are collinear. On the other hand, MN contains the midpoint of AC.

Hence, $MN \parallel BC$ and $MP \parallel AC$, which means that $\angle NMP = 90°$.

It suffices to show that the quadrilateral $NMQR$ is cyclic.

Consider the inversion $I(M, MC)$. We will prove that the reflections of N, Q and R are collinear. Observe that $I(N) = N$, $I(Q) = Q$ and $I(R) = P$.

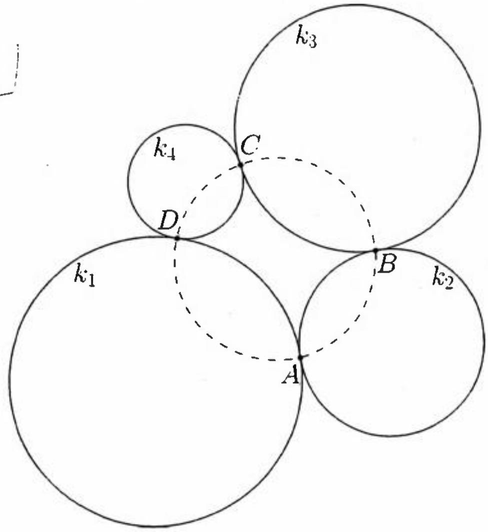
Since the points N, Q and P are collinear, we conclude that the points N, Q and R lie on a circle that contains the center of the inversion M.

Problem 6.8.1. Let k_1, k_2, k_3 and k_4 be circles. Let k_1 touch k_2 and k_4 externally at the points A and D, respectively. Let k_3 touch k_2 and k_4 externally at the points B and C, respectively. Prove that the quadrilateral $ABCD$ is cyclic.

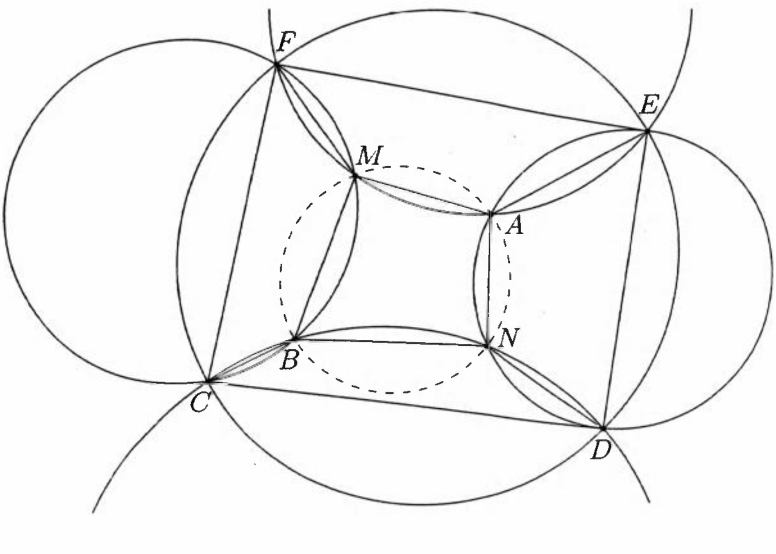
Solution. Consider an inversion centered at A with an arbitrary radius.

Now the statement follows from Problem 6.1.3.

Note. The problem has another easy solution. Draw the common tangent lines at the points A, B, C and D and do the angle chasing.



Problem 6.8.2. Let $CDEF$ be a cyclic quadrilateral. Let A and B be two internal points, such that the circumcircles of $\triangle DAE$ and $\triangle CBD$ intersect for the second time at the point N, which is internal to $CDEF$. Let the circumcircles of $\triangle EFA$ and $\triangle CBF$ intersect for the second time at the point M, which is internal to $CDEF$. Prove that the quadrilateral $AMBN$ is cyclic.

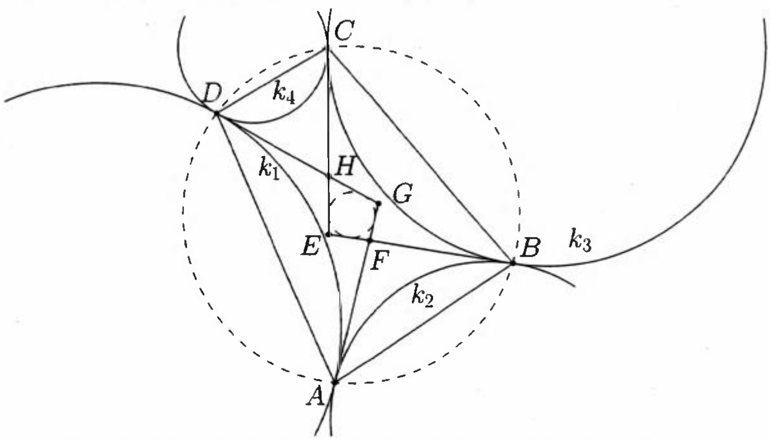


Solution. We consecutively have

$$\begin{aligned}\angle AMB &= 360^\circ - \angle AMF - \angle BMF = \angle AEF + \angle BCF\\ &= 180^\circ - \angle AED - \angle BCD = \angle AND + \angle BND - 180^\circ\\ &= 180^\circ - \angle ANB.\end{aligned}$$

Therefore, the quadrilateral $AMBN$ is cyclic.

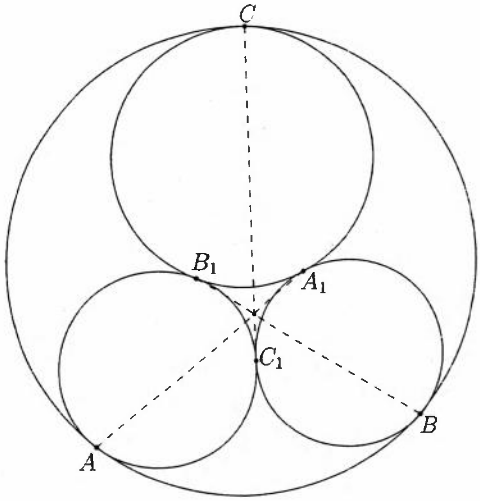
Problem 6.8.3. Let k_1, k_2, k_3 and k_4 be circles. Let k_1 touch k_2 and k_4 externally at the points A and D, respectively. Let k_3 touch k_2 and k_4 externally at the points B and C, respectively. The common internal tangent lines at the points A, B, C and D form the quadrilateral $FEHG$. Prove that this quadrilateral is circumscribed.



Solution. Problem 6.8.1 yields that the quadrilateral $ABCD$ is cyclic. Then the perpendicular bisectors of the sides AB, BC, CD and DA are concurrent. Also, $FA = FB$ because they are tangent segments to k_2. Thus, the perpendicular bisector of AB coincides with the angle bisector of $\angle AFB$, which means that the angle bisector of $\angle EFG$ coincides with the perpendicular bisector of AB.

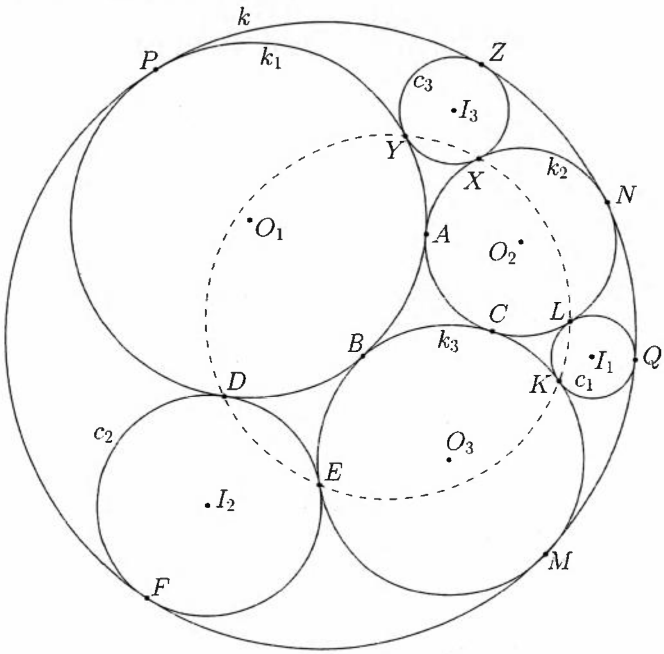
Analogously, the bisectors of the other three angles of the quadrilateral $FEHG$ coincide with the corresponding perpendicular bisectors of the sides of $ABCD$, which are concurrent. Therefore, the quadrilateral $EFGH$ is circumscribed.

Problem 6.8.4. Let k be a circle. The circles k_1, k_2 and k_3 touch k internally at the points A, B and C, respectively. Let k_1 touch k_2 externally at the point C_1, let k_2 touch k_3 externally at the point A_1 and let k_3 touch k_1 externally at the point B_1. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.



Solution. This problem is a special case of Problem 6.2.7.

Problem 6.8.5. Let k, k_1, k_2, k_3, c_1, c_2 and c_3 be circles. Each one of k_1, k_2 and k_3 touches each of the other two externally. The circle k touches these three circles internally. Each of the circles c_1, c_2 and c_3 touches two of k_1, k_2 and k_3 externally, and touches k internally, as shown in the figure. Using the notatons on the figure, prove that the points X, Y, D, E, K and L lie on a circle.



Solution. We apply an inversion centered at the point P with an arbitrary radius. We will use the prime symbol to denote the images of the objects after the inversion has been applied to them.

The circles k and k_1 map to parallel lines, and the circles c_2, k_2, k_3 and c_3 map to circles with equal radii, with each circle touching the following one, as well as the lines k' and k'_1.

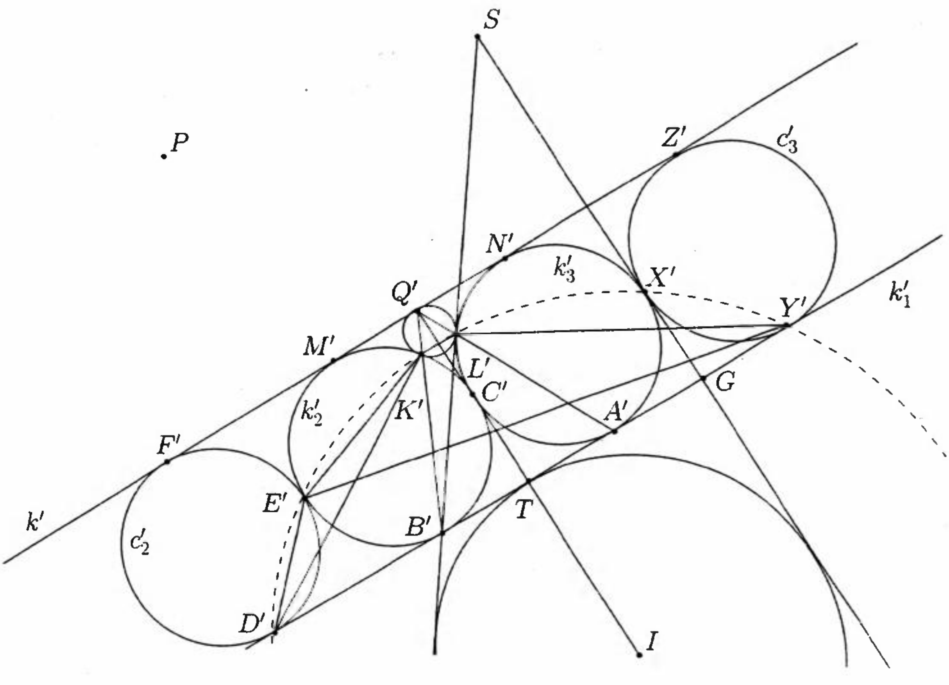
We have that c'_1 touches the line k' and the circles k'_2 and k'_3. We will prove that the points X', Y', D', E', K' and L' lie on a circle.

Problem 6.1.3 yields that the points Q', K' and B', as well as the points Q', L' and A', are collinear. Problem 6.1.4 yields that $B'L'$ is the common internal tangent line of the circles c'_1 and k'_3.

Let us use the common internal tangent lines of the circles to define the points G, S and T, as in the figure. Let us consider $\triangle B'GS$.

We have that $B'T = TA' = A'G$ and also A' is a point of tangency of its incircle. Therefore, T is a point of tangency of its S-excircle. Let this circle have a center I.

Then the perpendicular bisectors of the segments $X'Y'$, $X'L'$ and $K'L'$ pass through the point I. Thus, the quadrilateral $K'L'X'Y'$ is cyclic. We have that the quadrilaterals $X'L'K'E'$ and $X'Y'E'D'$ are isosceles trapezoids, and so they are cyclic. Therefore, the hexagon $Y'X'L'K'E'D'$ is cyclic.

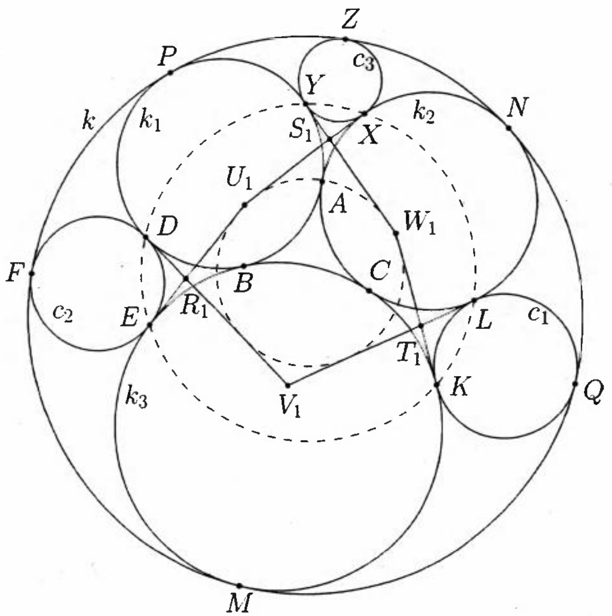


Problem 6.8.6. Let k, k_1, k_2, k_3, c_1, c_2 and c_3 be circles. Each one of k_1, k_2 and k_3 touches each of the other two externally. The circle k touches these three circles internally. Each of the circles c_1, c_2 and c_3 touches two of k_1, k_2 and k_3 externally, and touches k internally, the points of tangency being denoted as shown in the figure. The tangent lines to c_2 at D and E intersect at the point R_1, the tangent lines to c_3 at X and Y intersect at the point S_1, and the tangent lines to c_1 at L and K intersect at the point T_1. The tangent lines to k_2 at X and to k_3 at E intersect at the point U_1, the tangent lines to k_1 at D and to k_2 at L intersect at the point V_1, and the tangent lines to k_1 at Y and to k_3 at K intersect at the point W_1. Prove that the hexagon $R_1V_1T_1W_1S_1U_1$ is circumscribed.

Solution. Problem 6.8.5 yields that the hexagon $DYXLKE$ is cyclic. Then the internal angle bisectors of the hexagon $R_1V_1T_1W_1S_1U_1$ at the vertices S_1, T_1 and R_1 coincide with the perpendicular bisectors of YX, LK and ED, respectively. These three lines have a common point – the circumcenter of the hexagon $DYXLKE$ – point O.

Moreover, the angle bisector of the acute angle formed by the lines YS_1 and DR_1 coincides with the perpendicular bisector of DY, which passes through the point O. Therefore, O is equidistant from the lines S_1W_1 and R_1V_1, but it lies on the internal angle bisectors of the hexagon $R_1V_1T_1W_1S_1U_1$ at the vertices S_1 and R_1.

Hence, the point O lies on the angle bisector of $\angle S_1U_1R_1$. Analogously, O lies on the remaining internal bisectors of the angles of the hexagon $R_1V_1T_1W_1S_1U_1$, which means that this hexagon is circumscribed.



Problem 6.8.7. Let k, k_1, k_2, k_3, $c_1(I_1)$, $c_2(I_2)$ and $c_3(I_3)$ be circles. Each one of k_1, k_2 and k_3 touches each of the other two externally. The circle k touches these three circles internally. Each of the circles c_1, c_2 and c_3 touches two of k_1, k_2 and k_3 externally, and touches k internally, the points of tangency being denoted as shown in the figure. Prove that the lines MI_3, NI_2 and PI_1 are concurrent.

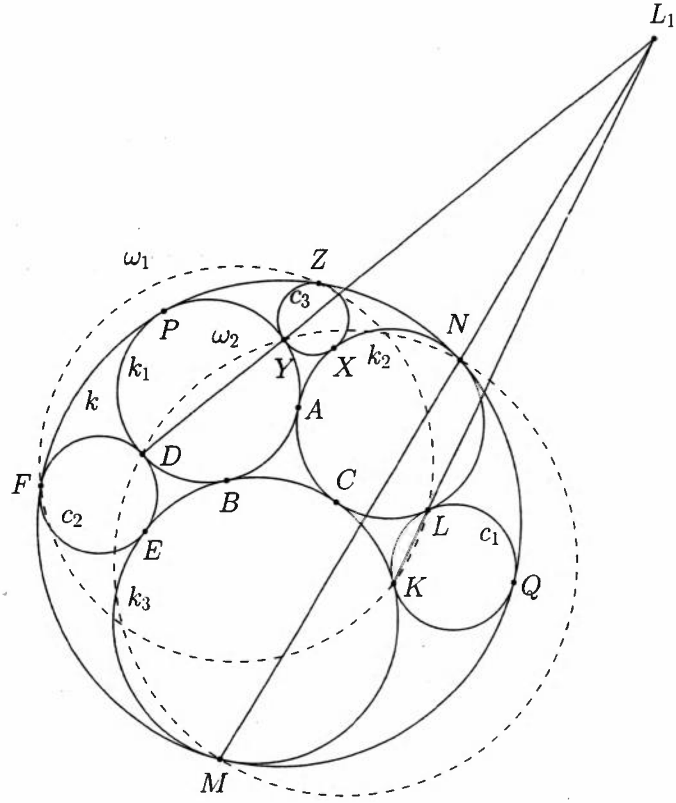
Solution. We will prove the following lemma.

Lemma. The external homothetic centers of the pairs of circles (c_2, c_3) and (k_2, k_3) coincide.

Proof. Problem 6.8.1, applied to the circles k, k_2, c_1 and k_3, yields that the quadrilateral $NLKM$ is cyclic.

Problem 6.8.1, applied to the circles k, c_3, k_1 and c_2, yields that the quadrilateral $ZYDF$ is also cyclic.

Consider an inversion centered at P with an arbitrary radius. The images of k and k_1 are two parallel lines, and the images of the remaining circles create the configuration shown in the figure. We will use the prime symbol to denote the images of the objects after the inversion is applied.



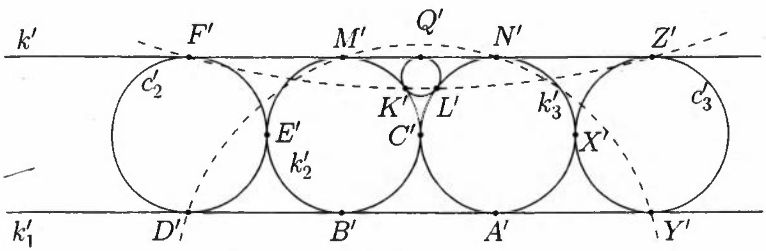
The points D', Y', N' and M' form an isosceles trapezoid, which means that they lie on a circle. Thus, the quadrilateral $DYNM$ is inscribed in a circle – ω_2. Analogously, the quadrilateral $KLZF$ is inscribed in a circle – ω_1.

The radical axes theorem (Problem 6.5.1), applied to k, ω_1 and the circumcircle of $NLKM$, yields that FZ, MN and KL are concurrent. The same theorem, applied to the circles k, ω_2 and the circumcircle of $ZYDF$, yields that the lines DY, MN and FZ are concurrent.

Let $L_1 = FZ \cap MN$. Then L_1 lies on the lines KL and DY. Monge's Theorem, applied to the triples of circles (k_1, c_2, c_3) and (k, c_2, c_3), yields that the external homothetic center of c_2 and c_3 is the intersection point of the lines FZ and DY.

Analogously, we get that the external homothetic center of k_2 and k_3 is the intersection point of the lines MN and KL.

Therefore, the external homothetic centers of the pairs of circles (c_2, c_3) and (k_2, k_3) coincide and the lemma is proved.



By the lemma and by Monge's Theorem, the points $L_1 = MN \cap I_2I_3$, $L_3 = NP \cap I_1I_2$ and $L_2 = MP \cap I_1I_3$ are collinear, because they are the external homothetic centers of the pairs of circles (c_2, c_3), (c_1, c_2) and (c_1, c_3), respectively.

Desargues' Theorem (Problem 7.1), applied to $\triangle MNP$ and $\triangle I_3I_2I_1$, yields that the lines MI_3, NI_2 and PI_1 are concurrent.

Problem 6.8.8. Let k, $k_1(O_1)$, $k_2(O_2)$, $k_3(O_3)$, c_1, c_2 and c_3 be circles. Each one of k_1, k_2 and k_3 touches each of the other two externally. The circle k touches these three circles internally. Each of the circles c_1, c_2 and c_3 touches two of k_1, k_2 and k_3 externally, and touches k internally, the points of tangency being denoted as shown in the figure. Prove that the lines QO_1, FO_2 and ZO_3 are concurrent.

Solution. We apply the results that we obtained while proving Problem 6.8.7. If L_1, L_2 and L_3 are the external homothetic centers of the pairs of circles among k_1, k_2 and k_3, then $FZ \cap O_2O_3 = L_1$, $QZ \cap O_1O_3 = L_2$ and $FQ \cap O_1O_2 = L_3$.

Monge's Theorem (Problem 6.2.3) yields that these three points are collinear. Finally, Desargues' Theorem (Problem 7.1), applied to $\triangle QFZ$ and $\triangle O_1O_2O_3$, yields that the lines QO_1, FO_2 and ZO_3 are concurrent.

Problem 6.8.9. Let k, $k_1(O_1)$, $k_2(O_2)$, $k_3(O_3)$, $c_1(I_1)$, $c_2(I_2)$ and $c_3(I_3)$ be circles. Each one of k_1, k_2 and k_3 touches each of the other two externally. The circle k touches these three circles internally. Each of the circles c_1, c_2 and c_3 touches two of k_1, k_2 and k_3 externally, and touches k internally, the points of tangency being denoted as shown in the figure. Prove that the lines I_1O_1, I_2O_2 and I_3O_3 are concurrent.

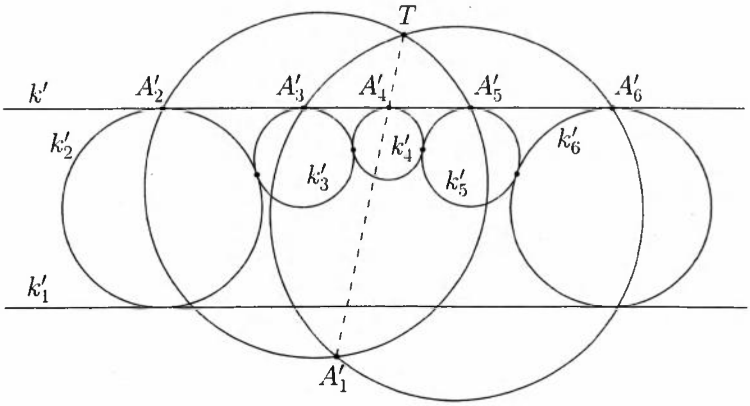
Solution. This problem follows from Desargues' Theorem (Problem 7.1) and the fact that the homothetic centers of (c_1, c_2) and (k_1, k_2), of (c_1, c_3) and (k_1, k_3), and of (c_2, c_3) and (k_2, k_3) coincide, which we proved in Problem 6.8.7.

Note. We did not include figures for Problems 6.8.8 and 6.8.9 because the figures for Problem 6.8.7 are sufficient.

Problem 6.8.10. Let k be a circle. The circles k_1, k_2, k_3, k_4, k_5 and k_6 touch k internally. Also, the circle k_i touches the circles k_{i-1} and k_{i+1} externally ($k_7 \equiv k_1$). Let k_i touch k at the point A_i. Prove that the lines A_1A_4, A_2A_5 and A_3A_6 are concurrent.

Solution. Consider an inversion centered at A_1 with an arbitrary radius. We will use the prime symbol to denote the images of the objects after the inversion has been applied to them. The problem is now reformulated as follows:

Let k' and k'_1 be two parallel lines, and let k'_2 and k'_6 be two equal circles that touch k' and k'_6. Let k'_3, k'_4 and k'_5 be circles that all touch k', while each circle touches its two neighbouring circles externally (two circles are neighbours if their indices are consecutive integers). Let each circle k'_i touch k' at the point A'_i. Prove that the radical axis of the circumcircles of $\triangle A'_1A'_2A'_5$ and $\triangle A'_1A'_3A'_6$ is the line $A'_1A'_4$, where A'_1 is an arbitrary point.

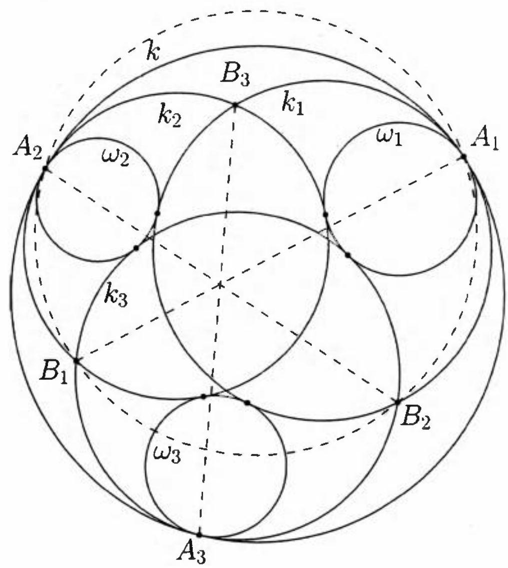


We will prove that the point A'_4 lies on the radical axis of these two circles, which is equivalent to $A'_2A'_4.A'_4A'_5 - A'_3A'_4.A'_4A'_6 = 0$.

Let each circle k'_i has a radius of r'_i. The Pythagorean Theorem gives that $A'_iA'_{i+1} = 2\sqrt{r'_ir'_{i+1}}$. Since $r'_2 = r'_6$, we have

$$\begin{aligned}
A'_2A'_4.A'_4A'_5 - A'_3A'_4.A'_4A'_6 &= 4\Big(\big(\sqrt{r'_2r'_3} + \sqrt{r'_3r'_4}\big)\sqrt{r'_4r'_5} \\
&\quad - \sqrt{r'_3r'_4}\big(\sqrt{r'_4r'_5} + \sqrt{r'_5r'_6}\big)\Big) \\
&= 4\Big(\sqrt{r'_3r'_4r'_5}\big(\sqrt{r'_4} + \sqrt{r'_2}\big) - \sqrt{r'_3r'_4r'_5}\big(\sqrt{r'_4} + \sqrt{r'_6}\big)\Big) \\
&= 0.
\end{aligned}$$

Problem 6.8.11. Let the points A_1, A_2 and A_3 lie on the circle k. The circle ω_1 touches k internally at the point A_1. The circle k_2 touches k internally at the point A_2 and touches ω_1 externally. The circle ω_3 touches k internally at the point A_3 and touches k_2 externally. The circle k_1 touches k internally at the point A_1 and touches ω_3 externally. The circle ω_2 touches k internally at the point A_2 and touches k_1 externally. The circle k_3 touches k internally at the point A_3 and touches ω_2 externally. Let the circles k_1, k_2 and k_3 intersect at the points B_1, B_2 and B_3, as shown in the figure. Prove that the lines A_1B_1, A_2B_2 and A_3B_3 are concurrent.



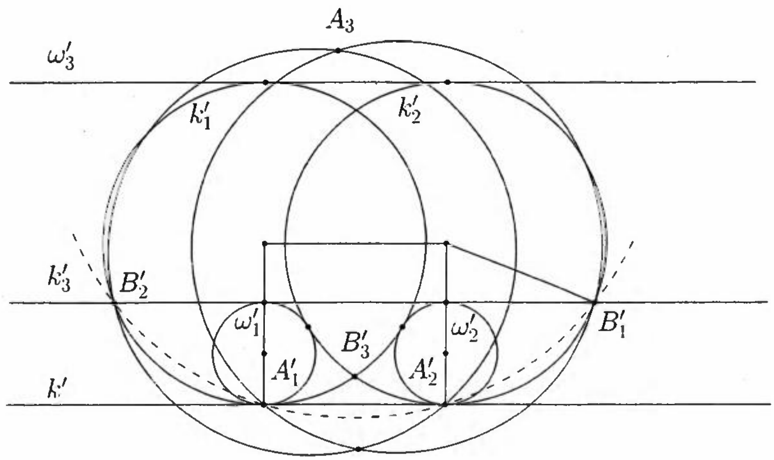
Solution. Problem 10.8 yields that the circle ω_1 touches the circle k_3 externally.

Consider an inversion centered at A_3 with an arbitrary radius. We will use the prime symbol to denote the images of the objects after the inversion has been applied to them. The problem is now reformulated as follows:

The lines ω_3', k_3' and k' are parallel. The circles ω_1' and ω_2' touch k' at the points A_1' and A_2', and they also touch k_3'. The circles k_1' and k_2' touch k' at the points A_1' and A_2', and they also touch ω_3'. Furthermore, k_1' touches ω_2', and k_2' touches ω_1'. Let A_3' be an arbitrary point. Let the intersection point of k_1' and k_2' be B_3', such that it lies between the lines k' and k_3'. The points B_1' and B_2' are the most distant intersection points of the four, formed by k_3' and k_2', and by k_3' and k_1'.

Considering the symmetry with respect to the radical axis of k_1' and k_2', we get that $A_1B_2 = A_2B_1$. Hence, the quadrilateral $A_1'B_2'B_1'A_2'$ is an isosceles trapezoid. Therefore, the quadrilateral $A_1'B_2'B_1'A_2'$ is cyclic, so the quadrilateral $A_1B_2B_1A_2$ is also cyclic.

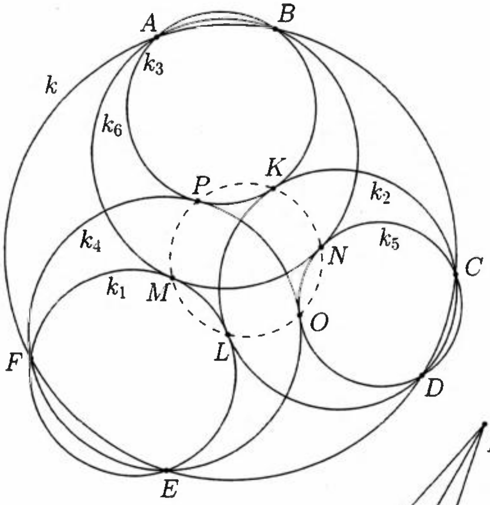
Analogously, the quadrilaterals $A_1B_3B_1A_3$ and $A_3B_2B_3C_2$ are cyclic. The radical axes theorem (Problem 6.5.1), applied to the circumcircles of the quadrilaterals $A_1B_2B_1A_2$, $A_1B_3B_1A_3$ and $A_3B_2B_3C_2$, yields that the lines A_1B_1, A_2B_2 and A_3B_3 are concurrent.

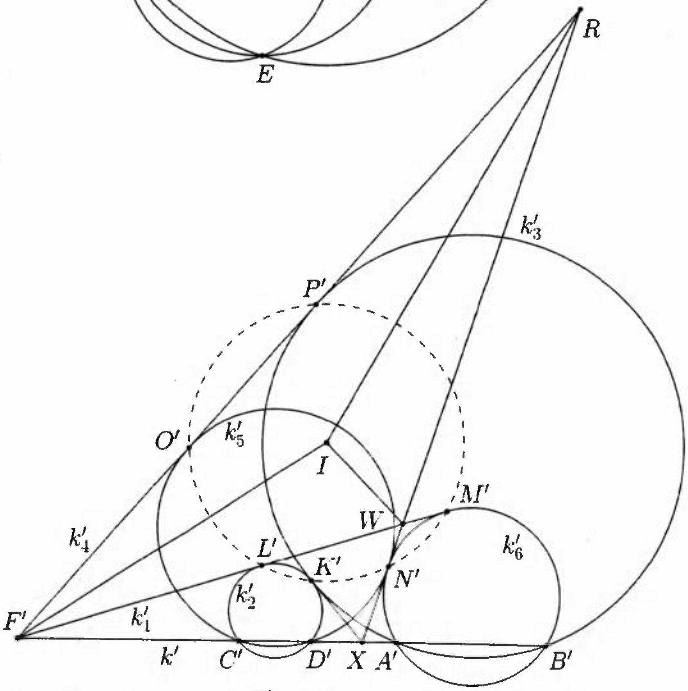


Problem 6.8.12. Let the points A, B, C, D, E and F lie on the circle k. The circle k_1 passes through the points E and F. The circle k_2 passes through the points C and D, and touches k_1 at the point L. The circle k_3 passes through the points A and B, and touches k_2 at the point K. The circle k_4 touches k_3 at the point P and passes through the points E and F. The circle k_5 touches k_4 at the point O and passes through the points C and D. Problem 10.8 gives that there exists a circle k_6, such that it touches k_1 and k_5, and such that it passes through the points A and B. Let this circle touch k_1 and k_5 at the points M and N, respectively. Prove that the hexagon $MLONKP$ is cyclic.

Solution. Consider an inversion centered at E. We will use the prime symbol to denote the images of the objects after the inversion has been applied to them. The problem is now reformulated as follows:

Let k', k_1' and k_4' be three lines passing through the point F'. The points C' and D' are chosen on k'. The circles k_2' and k_5' touch the lines k_1' and k_4' at the points L' and O', respectively, and they both pass through the points C' and D'. The circle k_6' touches the line k_1' as well as the circle k_5' at the points M' and N', respectively, and it intersects k' at the poins A' and B'. Problem 10.8 yields that there exists a circle k_3', that touches k_2' and k_4' at the points K' and P', respectively, such that it passes through the points A' and B'. Prove that the points M', L', O', N', K' and P' lie on a circle.

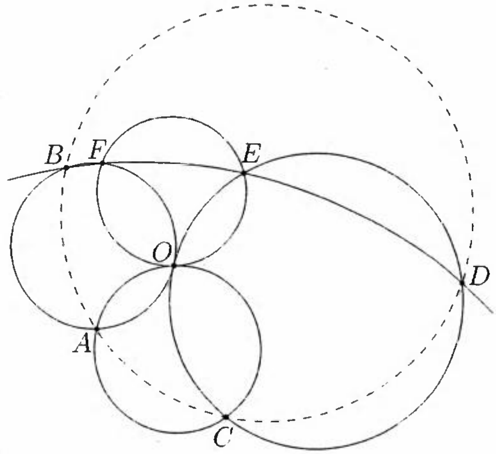




Considering the power of F' with respect to the four circles, it follows that the quadrilateral $O'L'M'P'$ is an isosceles trapezoid, and so it is cyclic. Let the tangent line at K' to k_2' intersect k' at the point X. Then considering the power of the point X, we get that the line XN' is the common internal tangent line of k_6' and k_5'. Let XN' intersect k_1' and k_4' at the points W and R, respectively. Consider the angle bisectors of $\triangle F'WR$.

They coincide with the perpendicular bisectors of $L'O'$, $O'N'$ and $M'N'$, and so they intersect at the circumcenter of $O'L'N'M'$. Thus, the quadrilateral $O'L'M'P'$ is cyclic. Then N' lies on the circumcircle of $O'L'M'P'$. Analogously, Y lies on the circumcircle of $O'L'M'P'$.

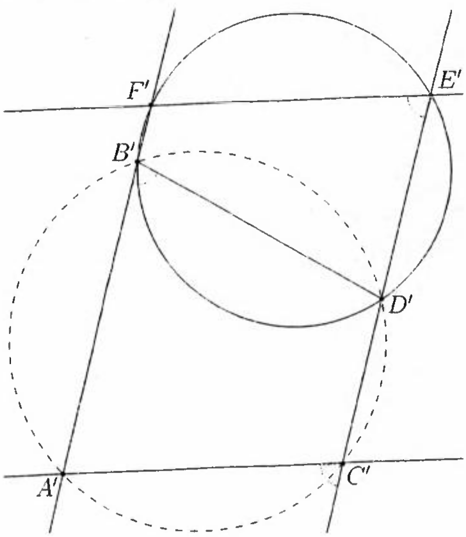
Problem 6.8.13. Let k_1 and k_2 be circles which touch each other externally at the point O. The circles k_3 and k_4 also touch each other externally at the point O. Let F be the second intersection point of k_1 and k_3, let A be the second intersection point of k_1 and k_4, let E be the second intersection point of k_2 and k_3, and let C be the second intersection point of k_2 and k_4. Let k_5 be a circle that passes through the points F and E and intersects k_1 and k_2 for the second time at the points B and D, respectively. Prove that the points B, A, C and D lie on a circle.



Solution. Consider an inversion centered at O with an arbitrary radius. The images of k_1 and k_3 are two parallel lines, and the images of k_2 and k_4 are also two parallel lines.

We will use the prime symbol to denote the images of the objects after the inversion has been applied to them. The problem is now reformulated as follows:

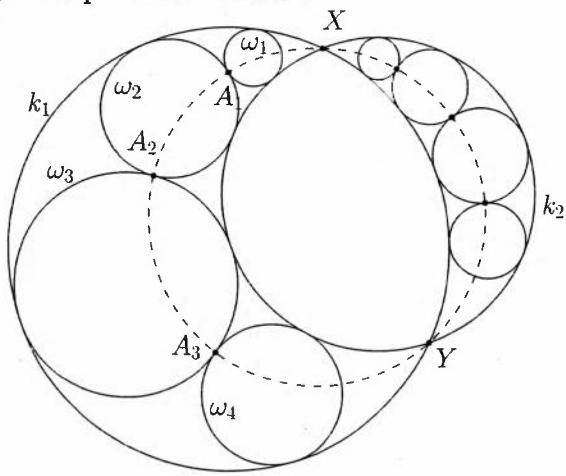
Let $A'C'E'F'$ be a parallelogram. The points $B' \in A'F'$ and $D' \in C'E'$ are chosen such that the quadrilateral $F'E'D'B'$ is cyclic. Then the quadrilateral $A'C'D'B'$ is cyclic.



We have that $\angle A'B'D' = \angle F'E'D' = 180^\circ - \angle A'C'D'$. Therefore, the quadrilateral $A'C'D'B'$ is cyclic.

Note. This solution does not make use of the fact that $A'F' \parallel C'E'$. Thus, the condition that k_1 and k_2 touch each other at the point O can be replaced with the condition that they just pass through it.

Problem 6.8.14. Let k_1 and k_2 be circles that intersect each other at the points X and Y. A chain of circles, each touching the next one $\omega_1, \omega_2, \ldots$ is constructed, such that each of these circles touches externally one of k_1 and k_2, and internally the other one. For all $i \in \mathbb{N}$, we denote the point of tangency of ω_i and ω_{i+1} by A_i. Prove that the points $A_1, A_2, \ldots$ lie on a circle that passes through the points X and Y.

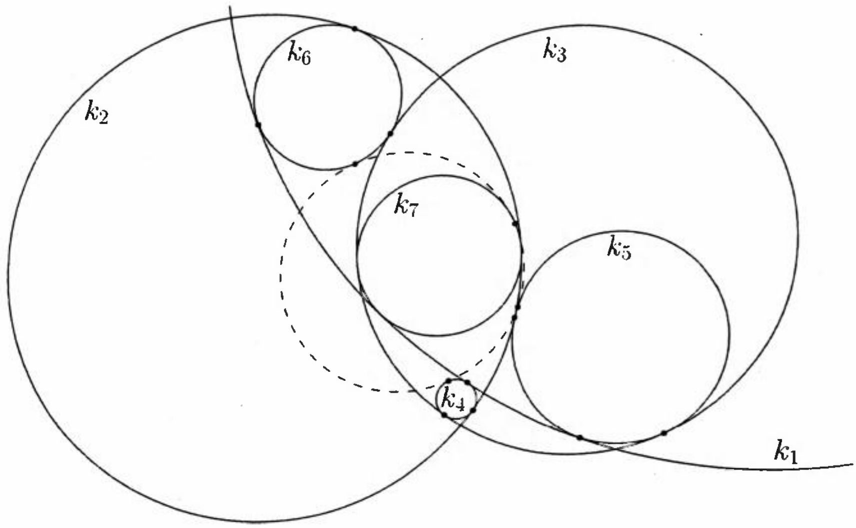


Solution. Let us consider an inversion centered at X with an arbitrary radius. The images of the circles k_1 and k_2 are two intersecting lines. The chain of circles $\omega_1, \omega_2, \ldots$ maps to a chain of circles that touch the images of k_1 and k_2.

Let us denote these images by k_1' and k_2', respectively. The images of the points of tangency of ω_i and ω_{i+1} lie on one of the angle bisectors defined by $\angle(k_1', k_2')$, i.e. the images of $A_1, A_2, \ldots$ are collinear.

Thus, before the inversion, the points A_i lie on a circle that passes through X and Y.

Problem 6.8.15. Let k_1, k_2 and k_3 be circles that intersect each other at six points. Let the circle k_4 touch k_2 and k_3 internally and k_1 externally. Let the circle k_5 touch k_1 and k_3 internally and k_2 externally. Let the circle k_6 touch k_2 and k_1 internally and k_3 externally. Let the circle k_7 touch k_1, k_2 and k_3 internally, such that it smaller than each of the circles k_1, k_2 and k_3. Prove that there exists a circle that touches k_4, k_5 and k_6 externally, and touches k_7 internally.

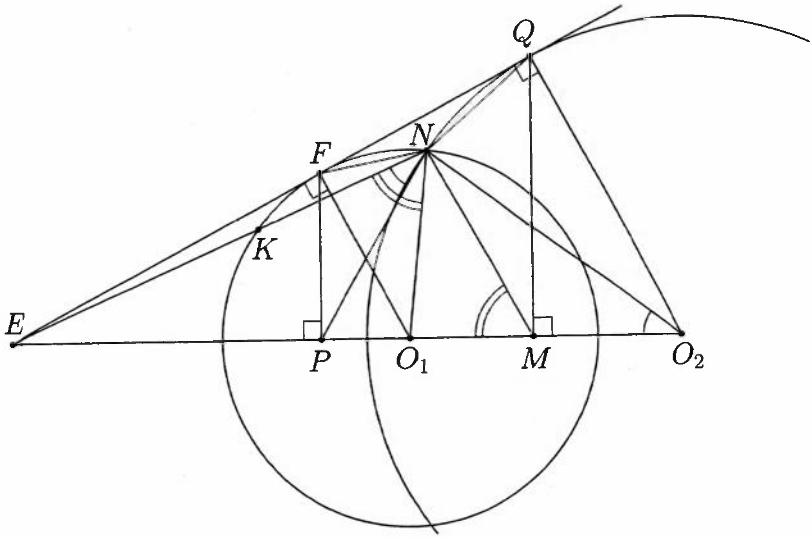


Solution. We will apply Casey's Theorem (Problem 6.1.10) three times, and its converse once. Let t_{ij} denote the length of the common external tangent segment of k_i and k_j. Let l_{ij} denote the length of the common internal tangent segment of k_i and k_j. We have

$$t_{57}l_{46} = l_{45}t_{67} + l_{47}t_{56},$$
$$l_{56}t_{47} = l_{45}t_{67} + l_{57}t_{46},$$
$$l_{46}t_{57} = l_{56}t_{47} + t_{45}l_{67}.$$

We sum the second and the third equality and then subtract the first to get $l_{47}t_{56} = t_{45}l_{67} + t_{46}l_{57}$. Now, the converse of Casey's Theorem yields that the desired circle exists.

Problem 6.9.1. Let N be one of the intersection points of the circles $k_1(O_1)$ and $k_2(O_2)$. The points $F \in k_1$ and $Q \in k_2$ are chosen, such that the points N, F and Q lie in the same half-plane with respect to O_1O_2, and such that FQ touches both circles. Let P and M be the projections of F and Q onto the line O_1O_2, respectively. Prove that $\angle PNO_1 = \angle MNO_2$.



Solution. Let $O_1O_2 \cap FQ = E$ and $EN \cap k_1 = \{N, K\}$. There exists a homothety h centered at E, such that $h(k_1) = k_2$.

Hence, $h(F) = Q$ and $h(K) = N$. Thus, $\widehat{KF} = \widehat{NQ}$ and we obtain $\angle EQN = \angle ENF$. Then $\triangle FNE \sim \triangle NQE$ and $EF.EQ = EN^2$.

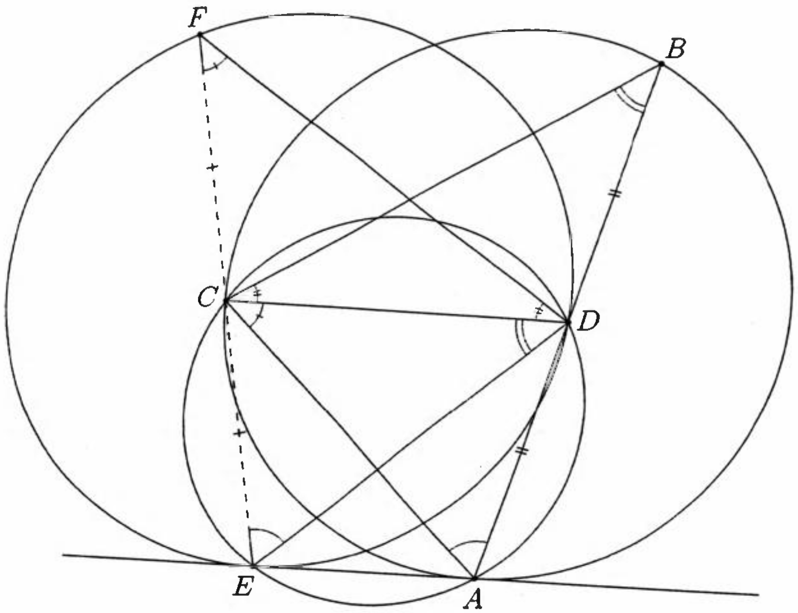
Observe that $\angle FPE = \angle QME = \angle O_1FE = \angle O_2QE = 90°$. Hence, the quadrilaterals FPO_2Q and FO_1MQ are cyclic.

Therefore, $EN^2 = EF.EQ = EP.EO_2 = EM.EO_1$.

It follows that $\triangle ENP \sim \triangle EO_2N$ and $\triangle EMN \sim \triangle ENO_1$. Hence, $\angle ENP = \angle EO_2N$ and $\angle EMN = \angle ENO_1$. Finally,

$$\begin{aligned}\angle PNO_1 &= \angle ENO_1 - \angle ENP \\ &= \angle EMN - \angle EO_2N = \angle MNO_2.\end{aligned}$$

Problem 6.9.2. Let k_1 and k_2 be two circles that intersect each other. Let EA be their external common tangent line ($E \in k_1$, $A \in k_2$). Let $F \in k_1$, $B \in k_2$, $C \in k_2$ and $D \in k_1$ be points such that C and D are the midpoints of EF and AB, respectively. Prove that the points E, A, D and C are concyclic.



Solution. Consider $\triangle ABC$ that has a median CD and a circumcircle k_2. Let the tangent line to k_2 at A intersect the circumcircle of $\triangle CDA$ at the point E'. Let k_1' be the circle that passes through the point D and touches AE' at E'.

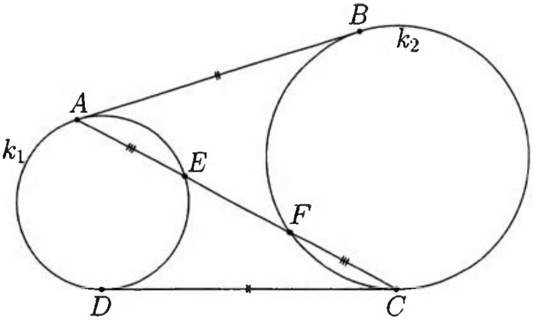
Let F' be the second intersection point of $E'C$ and k_1'. It suffices to show that $F'C = CE'$ because then $k_1' \equiv k_1$, $E' \equiv E$ and $F' \equiv F$. We have

$$\begin{aligned}
\angle DCA = \angle DE'A = \angle DF'E',\ \angle CBA = \angle CAE' = \angle CDE',\\
\angle F'E'D = \angle CAB,\ \angle BCD = 180^\circ - \angle CBA - \angle DCA - \angle BAC\\
= 180^\circ - \angle CDE' - \angle E'F'D - \angle DE'F' = \angle CDF'.
\end{aligned}$$

The law of sines, applied to $\triangle BCD$, $\triangle ACD$, $\triangle E'DC$ and $\triangle F'DC$, yields

$$\frac{F'C}{CE'} = \frac{\sin\angle F'DC}{\sin\angle CDE'} \cdot \frac{\sin\angle F'E'D}{\sin\angle E'F'D} = \frac{\sin\angle BCD}{\sin\angle CBA} \cdot \frac{\sin\angle CAB}{\sin\angle DCA} = \frac{BD}{DA} = 1.$$

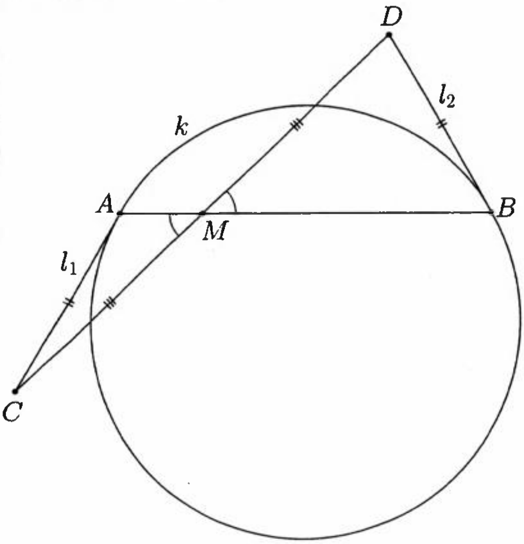
Problem 6.9.3. Let k_1 and k_2 be two circles. Let $A, D \in k_1$ and $B, C \in k_2$ be points, such that AB and DC are the common external tangent lines of k_1 and k_2. The line AC intersects k_1 and k_2 for the second time at E and F, respectively. Prove that $AE = CF$.



Solution. Due to symmetry, we have $AB = CD$.

Consider the power of A with respect to k_2. We get $AB^2 = AF.AC$. Analogously, $CD^2 = CE.CA$. Thus, $CE.CA = AF.CA$ and therefore $AE = FC$.

Problem 6.9.4. Let AB be a chord in the circle k. The points C and D lie on the tangent lines to k at A and B, respectively. Let $AC = BD$. The points D and C lie in different half-planes with respect to the line AB. Let $M = CD \cap AB$. Prove that $DM = CM$.

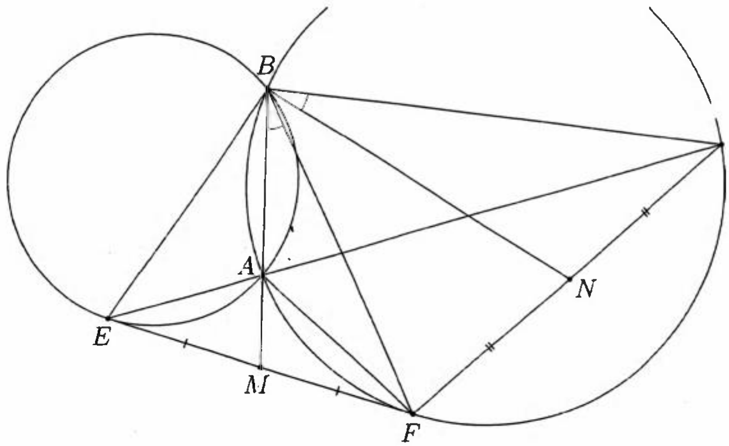


Solution. We have $\angle AMC = \angle BMD$ and $\angle CAM = 180° - \angle MBD$. Therefore, $\sin\angle AMC = \sin\angle BMD$ and $\sin\angle CAM = \sin\angle MBD$.

The law of sines, applied to $\triangle AMC$ and $\triangle BMD$, yields

$$\frac{AC}{CM} = \frac{\sin\angle AMC}{\sin\angle CAM} = \frac{\sin\angle BMD}{\sin\angle MBD} = \frac{BD}{MD}.$$

Problem 6.9.5. Let k_1 and k_2 be circles such that $k_1 \cap k_2 = \{A, B\}$. Let $E \in k_1$ and $F \in k_2$ be points such that A lies inside of $\triangle BEF$, and EF is one of the common external tangent lines of k_1 and k_2. Let $EA \cap k_2 = \{A, C\}$. Let N be the midpoint of FC. Prove that $\angle CAF = \angle NBA$.



Solution. Let $AB \cap EF = M$. Consider the power of M with respect to k_1 and k_2. We have $MF^2 = MA.MB = ME^2$, so M is the midpoint of EF. The alternate segments theorem yields $\angle BEM = 180° - \angle BAE = \angle BAC = \angle BFC$.

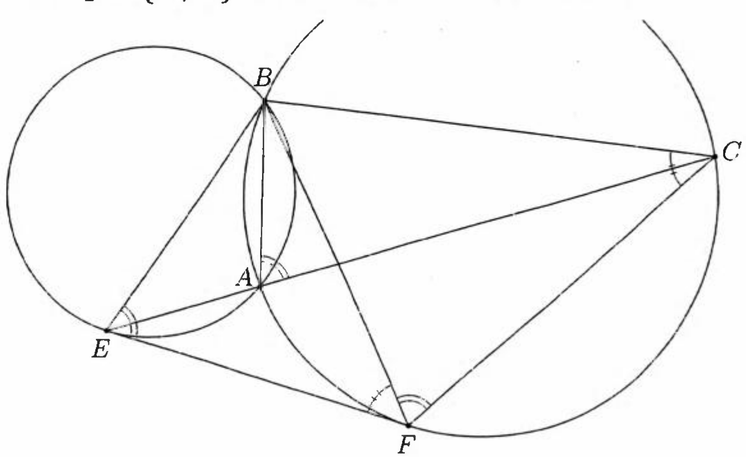
Therefore,

$$\angle BCF = \angle BFE \Rightarrow \triangle CFB \sim \triangle FEB \Rightarrow \angle CBN = \angle FBA.$$

The last equality is true because BN and BM are corresponding medians in two similar triangles.

Finally, $\angle CAF = \angle CBF = \angle NBA$.

Problem 6.9.6. Let k_1 and k_2 be circles that intersect at the points A and B. Let $E \in k_1$ and $F \in k_2$ be points, such that A lies inside of $\triangle BEF$, and EF is one of the common external tangent lines of k_1 and k_2. Let $EA \cap k_2 = \{A, C\}$. Prove that $\angle FBC = \angle FBE$.



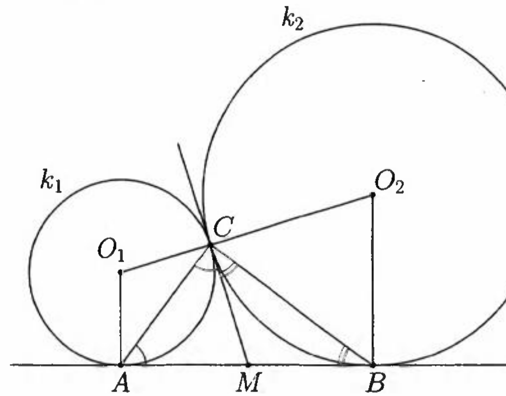
Solutions. The alternate segments theorem yields

$$\angle BEF = 180° - \angle BAE = \angle BAC = \angle BFC,$$
$$\angle BCF = \angle BFE \Rightarrow \angle FBC = \angle FBE.$$

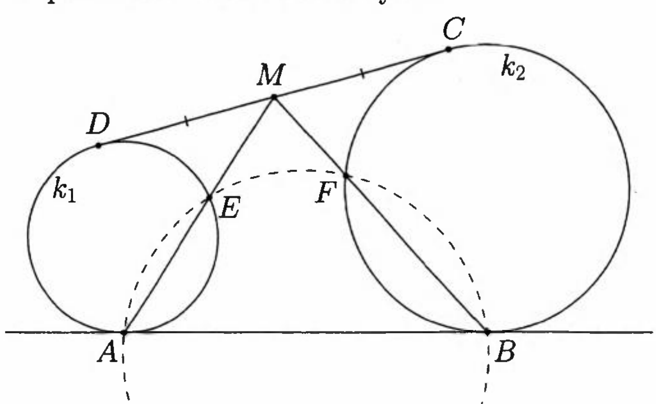
Problem 6.9.7. Let k_1 and k_2 be circles that touch each other externally at the point C. The points $A \in k_1$ and $B \in k_2$ are chosen, such that AB is one of the common external tangent lines of k_1 and k_2. Prove that $\angle ACB = 90°$.

Solution. Let M be the intersection point of AB and the common internal tangent line of the circles at C.

Since $MC = MA$ and $MC = MB$, we obtain that the points A, B and C lie on a circle centered at M. Hence, $\angle ACB = 90°$.

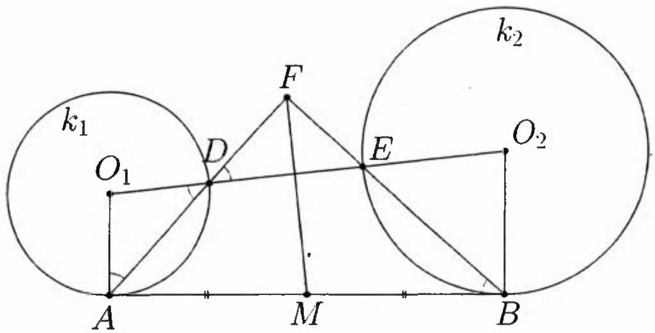


Problem 6.9.8. Let k_1 and k_2 be circles such that none of them lies inside of the other one. The points $A, D \in k_1$ and $B, D \in k_2$ are such that AB and CD are the common external tangent lines of k_1 and k_2. Let M be the midpoint of CD. Let $k_1 \cap MA = \{A, E\}$ and $k_2 \cap MB = \{B, F\}$. Prove that the quadrilateral $AEFB$ is cyclic.



Solution. Consider the power of M with respect to k_1 and k_2. We get $MF.MB = MC^2 = MD^2 = ME.MA$. Therefore, the quadrilateral $AEFB$ is cyclic.

Problem 6.9.9. Let $k_1(O_1)$ and $k_2(O_2)$ be circles such that their disks do not intersect each other. The line l is a common external tangent line of the two circles, and it touches k_1 and k_2 at the points A and B, respectively ($A \neq B$). Let M be the midpoint of AB. Let $D = O_1O_2 \cap k_1$ and $E = O_1O_2 \cap k_2$, such that D and E lie on the segment O_1O_2. Let F be the intersection point of AD and BE. Prove that $FM \perp O_1O_2$.



Solution. Observe that $\angle AO_1D + \angle EO_2B = 180°$. Then $\angle O_1AD + \angle O_1DA + \angle O_2BE + \angle O_2EB = 180°$.

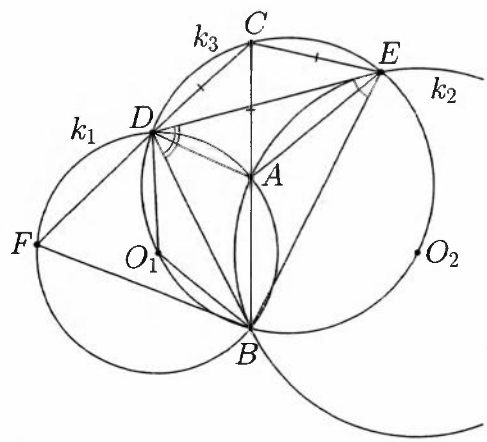
Since $O_1D = O_1A$ and $O_2B = O_2E$, we obtain $\angle EBO_2 + \angle DAO_1 = 90°$ and $\angle EBA = 90° - \angle EBO_2 = \angle DAO_1 = \angle ADO_1$.

Thus, the quadrilateral $ABED$ is cyclic and hence $FD.FA = FE.FB$. Then F lies on the radical axis of k_1 and k_2. Note that M lies on the same radical axis.

It follows that MF is the radical axis of the two circles. Thus, $MF \perp O_1O_2$.

Note. If we define both D and E to not lie on the segment O_1O_2, the problem statement still holds. We can further generalize the problem by defining exactly one of D and E to lie on the segment O_1O_2. In this case, l needs to be defined as the internal tangent line, instead of the external one.

Problem 6.10.1. Let $k_1(O_1)$ and $k_2(O_2)$ be circles that intersect each other at the points A and B. Let k_3 be the circumcircle of $\triangle O_1BO_2$, and let k_3 intersect k_1 and k_2 at the points D and E, respectively. Let C be the second intersection point of k_3 and AB. Prove that $CD = CA = CE$.

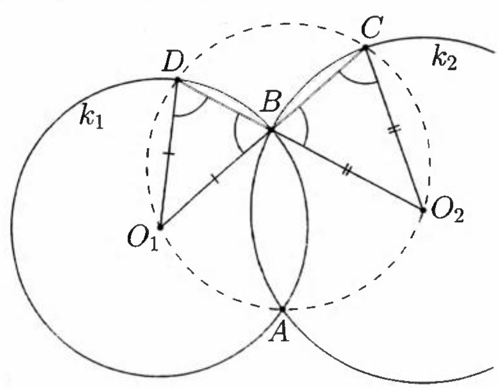


Solution. Let $\angle DEB = \alpha$. Then $\angle BO_1D = 180° - \alpha$. Hence, $\angle BAD = 90° + \dfrac{\alpha}{2}$. Denote $\angle BDE = \beta$. Then $\angle BAE = 90° + \dfrac{\beta}{2}$.

Let A_1 be the incenter of $\triangle BDE$. Then $\angle BA_1D = 90° + \dfrac{\alpha}{2}$ and $\angle BA_1E = 90° + \dfrac{\beta}{2}$. Therefore, $A \equiv A_1$.

Hence, A is the incenter of $\triangle BDE$, and C lies on its circumcircle. Now Problem 4.6.1 yields the desired equality.

Problem 6.10.2. Let $k_1(O_1)$ and $k_2(O_2)$ be circles that intersect each other at the points A and B. Let $O_2B \cap k_1 = \{B, D\}$ and $O_1B \cap k_2 = \{B, C\}$. Prove that the quadrilateral O_1O_2CD is cyclic.



Solution. Observe that $O_1D = O_1B$ and $O_2B = O_2C$. Hence,

$$\angle O_1DO_2 = \angle O_1BD$$
$$= \angle O_2BC = \angle O_2CO_1.$$

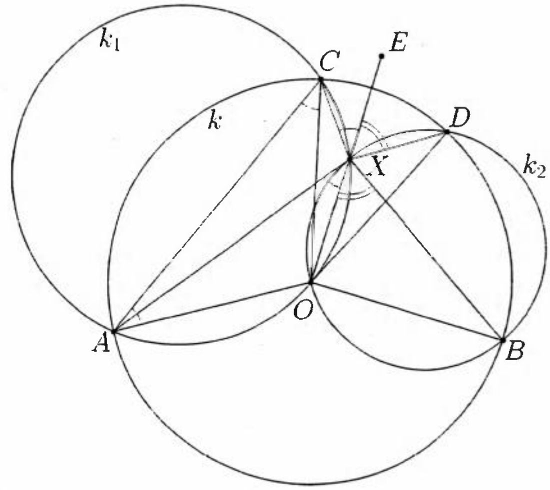
Thus, the quadrilateral O_1O_2CD is cyclic.

Problem 6.10.3. Let $k(O)$ be a circle and let X be a point. The points A and B lie on k, and the circumcircles of $\triangle AOX$ and $\triangle BOX$ intersect k at the points C and D, respectively. Prove that $\angle AXB = \angle CXD$.

Solution. Let E be a point on the ray $OX^{\rightarrow}$ after X.

Hence,

$$\angle CXE = \angle OAC = \angle OCA = \angle OXA.$$



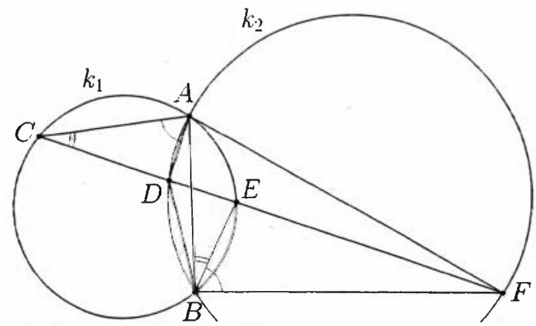
Analogously, we obtain $\angle DXE = \angle OXB$.

Summing the above equations, we get $\angle AXB = \angle CXD$.

Problem 6.10.4. Let k_1 and k_2 be circles that intersect each other at the points A and B. The points C, D, E and F are collinear and lie in this order, such that $E, C \in k_1$ and $D, F \in k_2$. Prove that $\angle CAD = \angle FBE$.

Solution. We consecutively have

$$\begin{aligned}\angle FBE &= \angle FBA - \angle EBA \\ &= \angle FDA - \angle ECA \\ &= \angle CAD.\end{aligned}$$

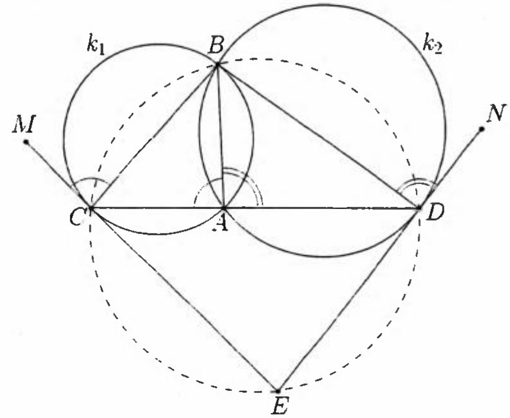


Problem 6.10.5. Let k_1 and k_2 be circles that intersect each other at the points A and B. The point C lies on k_1 and the line AC intersects k_2 for the second time at the point D. The tangent lines to k_1 and k_2 at C and D, respectively, intersect each other at the point E. Prove that the quadrilateral $BCED$ is cyclic.

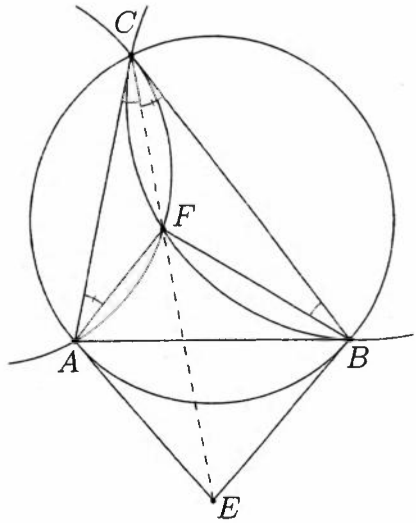
Solution. Let M and N lie on the extensions of EC and ED, as shown in the figure.

The alternate segments theorem yields

$$\begin{aligned}\angle BDN &= \angle DAB \\ &= 180^\circ - \angle CAB \\ &= 180^\circ - \angle MCB.\end{aligned}$$



Problem 6.10.6. Let $\triangle ABC$ be a triangle with circumcircle k. The tangent lines to k at A and B intersect at the point E. The circle k_1 passes through the points A and C and touches the line BC. The circle k_2 passes through the points B and C and touches the line AC. Let $k_1 \cap k_2 = \{C, F\}$. Prove that the points C, F and E are collinear.



Solution. The alternate segments theorem yields

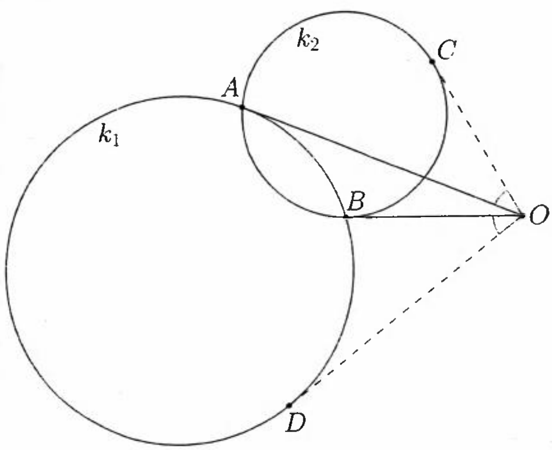
$$\angle ACF = \angle CBF$$

and

$$\angle BCF = \angle CAF.$$

The statement of the problem is now the same as that of Problem 4.4.6.

Problems 6.10.7. Let k_1 and k_2 be circles that intersect each other at the points A and B. The tangent lines to k_1 and k_2 at A and B, respectively, intersect at the point O. Let $D \in k_1$ and $C \in k_2$ are points, such that OD and OC are the other tangent lines to k_1 and k_2, respectively. Prove that $\angle COA = \angle DOB$.



Solution. Let the mapping φ be a composition of inversion $I(O, \sqrt{OA.OB})$ and symmetry with respect to the angle bisector of $\angle AOB$.

Then $\varphi(A) = B$, $\varphi(B) = A$, $\varphi(k_1) = k_2$ and $\varphi(k_2) = k_1$, because k_1 gets transformed into a circle that touches OB at B and that passes through A.

Thus, $\varphi(OC) = OD$, and so the lines OC and OD are symmetric with respect to the angle bisector of $\angle AOB$, implying that $\angle COA = \angle DOB$.

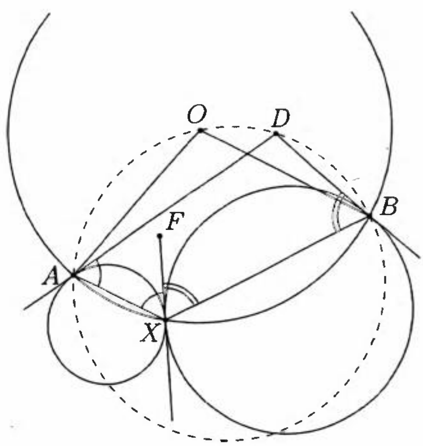
Problem 6.10.8. Let k_1 and k_2 be circles that touch each other externally at the point X. The point D lies outside of both circles. The points $A \in k_1$ and $B \in k_2$ are such that DA and DB are tangent lines to k_1 and k_2, respectively. Denote the circumcenter of $\triangle ABX$ by O. Prove that the quadrilateral $ABDO$ is cyclic.

Solution. Let F be a point such that FX is the common internal tangent line of k_1 and k_2.

Observe that $\angle FXA = \angle DAX$ and $\angle DBX = \angle BXF$. Hence, we consecutively have

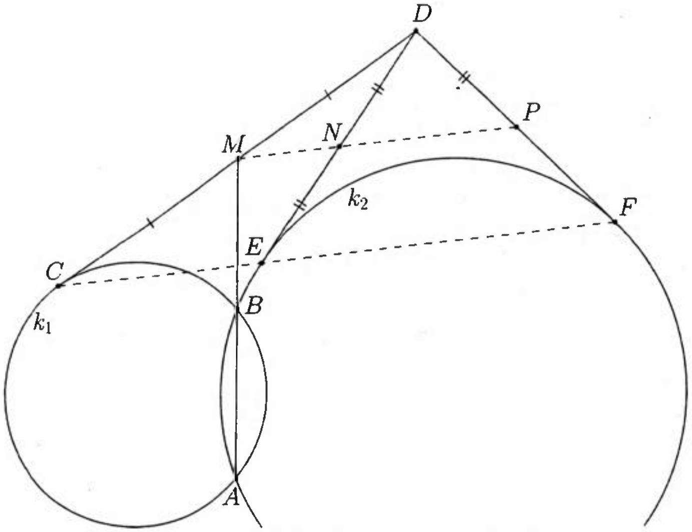
$$\begin{aligned}\angle ADB &= 360° - \angle DBX - \angle BXF - \angle AXF - \angle XAD \\ &= 2(180° - \angle AXB) = \angle AOB.\end{aligned}$$

Thus, the quadrilateral $ABDO$ is cyclic.



Problem 6.10.9. Let k_1 and k_2 be circles that intersect each other at the points A and B. Let C be an arbitrary point on k_1. The tangent line to k_1 at C intersects AB at M. Denote the reflection of C with respect to the point M by D. Assume that D lies outside of k_2. Prove that the point C lies on the polar line of D with respect to k_2.

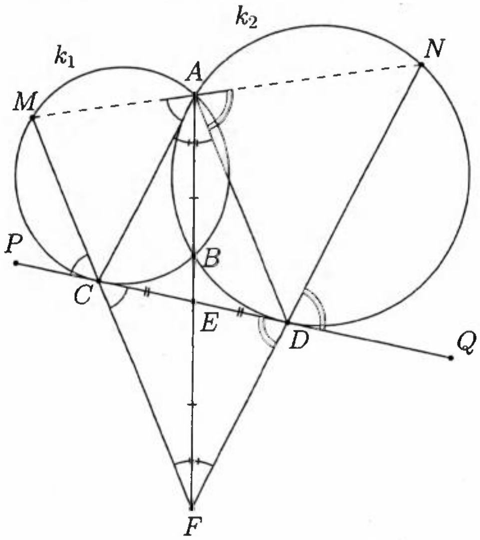
Solution. Let DE and DF be the tangent lines to k_2, where E, $F \in k_2$. Hence, EF is the polar line of D with respect to k_2. Denote the midpoints of DE and DF by N and P, respectively. Then N and P lie on the radical axis of the circle k_2 and the point D. Thus, NP is the radical axis of k_2 and D.



On the other hand, $MB.MA = MC^2 = MD^2$ and we obtain that M lies on the same radical axis. Thus, the points M, N and P are collinear.

Consider the homothety $h(D, 2)$. Then $h(M) = C$, $h(N) = E$ and $h(P) = F$. Therefore, the points C, E and F are collinear.

Problem 6.10.10. Let k_1 and k_2 be circles that intersect each other at the points A and B. The points C and D lie on k_1 and k_2, respectively, and CD is the common external tangent line of the circles which is closer to B. Let $AB \cap CD = E$. Denote the reflection of A with respect to the point E by F. Let M be the second intersection point of k_1 and FC, and let N be the second intersection point of k_2 and FD. Prove that the points M, A and N are collinear.



Solution. Since E lies on the radical axis of k_1 and k_2, we have that $EC = ED$.

Since $AE = EF$, we obtain that the quadrilateral $CADF$ is a parallelogram. Thus, $\angle CAD = \angle CFD$. On the other hand,

$$\angle MAC = \angle FCD,$$

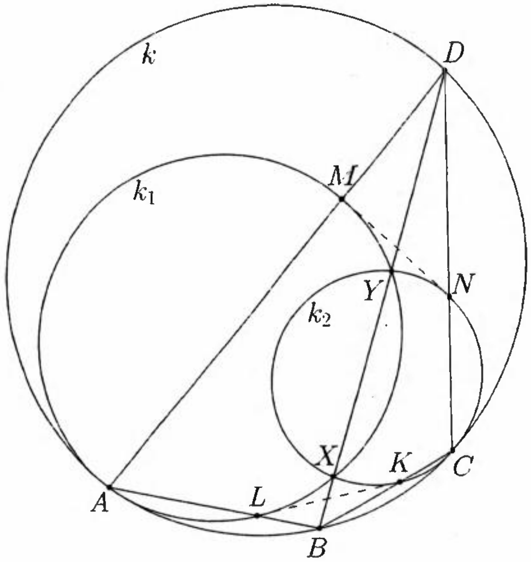
$$\angle DAN = \angle CDF.$$

Now observe that

$$\angle MAC + \angle CAD + \angle DAN = \angle FCD + \angle CFD + \angle CDF = 180^\circ,$$

which proves the desired statement.

Problem 6.10.11. Let k, k_1 and k_2 be circles, such that k_1 and k_2 touch k internally at the points A and B, respectively. Let $k_1 \cap k_2 = \{X, Y\}$. The line XY intersects k at the points B and D. Let $DA \cap k_1 = M$, $DC \cap k_2 = N$, $BA \cap k_1 = L$, and $BC \cap k_2 = K$. Prove that the lines MN and LK touch k_1 and k_2.



Solution. Consider the inversion $I(D, \sqrt{DM.DA})$. Since D lies on the radical axis of k_1 and k_2, it follows that $I(k_1) = k_1$ and $I(k_2) = k_2$. On the other hand, $I(M) = A$ and $I(N) = C$.

Thus, $I(MN) = k$ because the image of MN is the circle k, tangent to k_1 and k_2, passing through A and B. Hence, MN touches k_1 and k_2.

Analogously, after considering the inversion $I(B, \sqrt{BA.BC})$ we prove that LK touches k_1 and k_2.

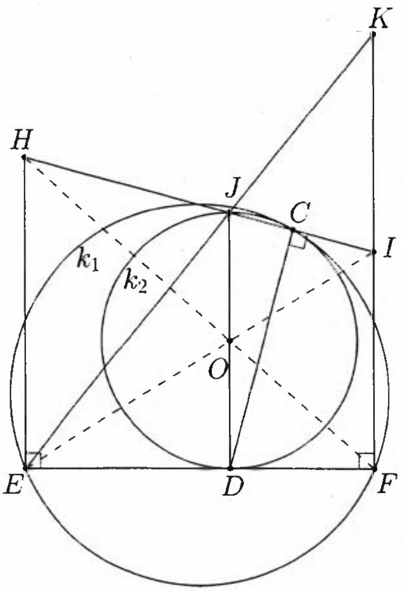
Problem 6.10.12. Let $k_2(O)$ be a circle that is internally tangent to the circle k_1 at the point C. The chord EF at k_1 is tangent to k_2 at the point D. The line passing through C, perpendicular to DC, intersects the lines through E and F, perpendicular to EF, at H and I, respectively. Prove that $EI \cap FH = O$.

Solution. Let $HI \cap k_2 = \{C, J\}$ and $EJ \cap FI = K$. The segment JD is a diameter of k_2 and $JD \perp EF$. It suffices to show that $IK = IF$ because of Steiner's Theorem (Problem 5.2.1), applied to the trapezoid $DFKJ$.

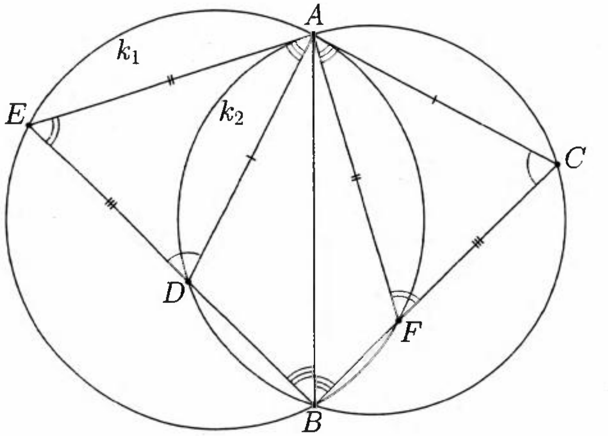
The intercept theorem yields $\dfrac{HE}{IK} = \dfrac{HJ}{JI} = \dfrac{ED}{DF}$.

Applying the result of Problem 6.1.1, we obtain that CD is a bisector of $\angle ECF$. Consider the cyclic quadrilaterals $EDCH$ and $CDFI$. It follows that $\angle EHD = \angle ECD = \angle DCF = \angle DIF$ and hence $\Delta EHD \sim \Delta FID$.

Thus, $\dfrac{ED}{DF} = \dfrac{HE}{FI}$ and $KI = IF$.



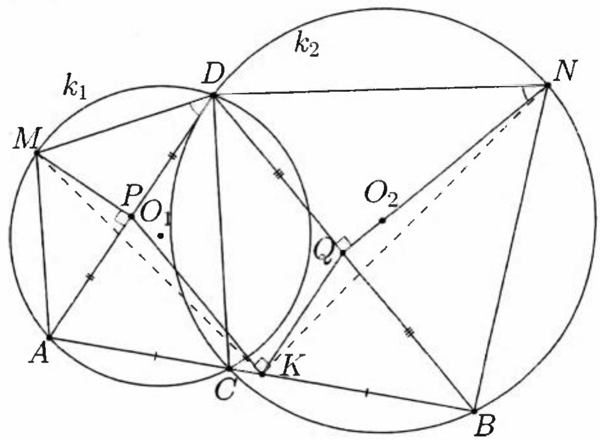
Problem 6.10.13. Let k_1 and k_2 be circles that intersect each other at the points A and B. The points E and C lie on k_1 and k_2, respectively, such that the lines EA and CA touch k_2 and k_1, respectively. Let $BE \cap k_2 = \{D, B\}$ and $BC \cap k_1 = \{F, B\}$. Prove that $ED = CF$.



Solution. Consider the cyclic quadrilateral $CBDA$. We have $\angle ADE = \angle ACB$. On the other hand, $\angle BEA = \angle CFA$ because the quadrilateral $BEAF$ is cyclic. Then $\triangle EAD \sim \triangle FAC$. Hence, $\angle EAD = \angle FAC$. Note that $\angle DBA = \angle EAD = \angle FAC = \angle FBA$, due to the alternate segments theorem. Thus, BA is the bisector of $\angle EBC$, and so $DA = CA$.

The segments DA and CA are corresponding elements in the similar $\triangle EAD$ and $\triangle FAC$. Hence, $\triangle EAD \cong \triangle FAC$ and so $ED = CF$.

Problem 6.10.14. Let k_1 and k_2 be circles that intersect each other at the points C and D. A line passing through C intersects k_1 and k_2 at the points A and B, respectively. Denote the midpoints of the arcs $\widehat{AD}$ and $\widehat{DB}$ of k_1 and k_2, which do not contain C by M and N, respectively. Prove that $\angle MKN = 90°$.



Solution. Let P and Q be the midpoints of AD and DB, respectively. Then $\angle APK = \angle ADB = \angle KQB$ and $\angle MPK = \angle KQN$. Since

$$\angle MDP = \frac{\angle ACD}{2} = \frac{\angle DNB}{2} = \angle DNQ,$$

we have $\triangle MPD \sim \triangle DQN$, and so $\dfrac{MP}{PK} = \dfrac{MP}{DQ} = \dfrac{DP}{NQ} = \dfrac{KQ}{QN}$. Hence, $\triangle KPM \sim \triangle NQK$. We obtain that

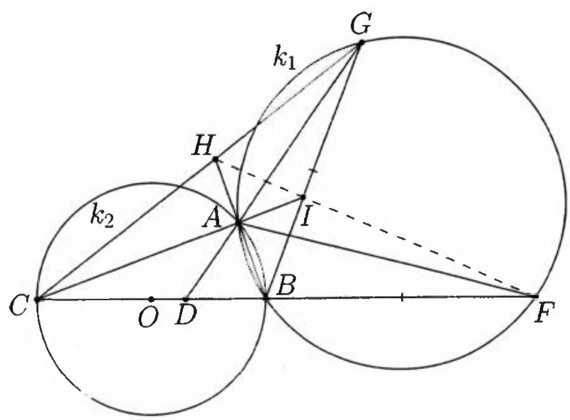
$$\begin{aligned}\angle MKN &= \angle MKP + \angle PKQ + \angle QKN \\ &= 180° - \angle KQN + \angle KQB = 90°.\end{aligned}$$

Problem 6.10.15. Let the circles k_1 and $k_2(O)$ intersect each other at the points A and B. Denote the reflection of B with respect to O by C. Let $BC \cap k_1 = \{B, F\}$. The point $G \in k_1$ is such that $BG = BF$. Let $AB \cap CG = H$ and $BG \cap CA = I$. Prove that the points H, I and F are collinear.

Solution. Let GA intersect BC at D. We will prove that $(C, D, B, F) = 1$ and the desired result will follow.

Note that B is the midpoint of the arc $\widehat{GAF}$ of k_1, and $\angle IAB = 90°$. Hence, $\angle FAI = \angle GAI = \angle CAD$.

But $\angle CAB = 90°$, and so AB is the internal biector of $\angle DAF$ and AC is the external bisector of the same angle. Hence, $(C, D, B, F) = 1$.



Problem 6.10.16. Let $k(O)$ be a circle with a chord AB. Let D be an arbitrary point on k. The circumcircle of $\triangle ADO$ intersects AB for the second time at the point E. Prove that $DE = BE$.

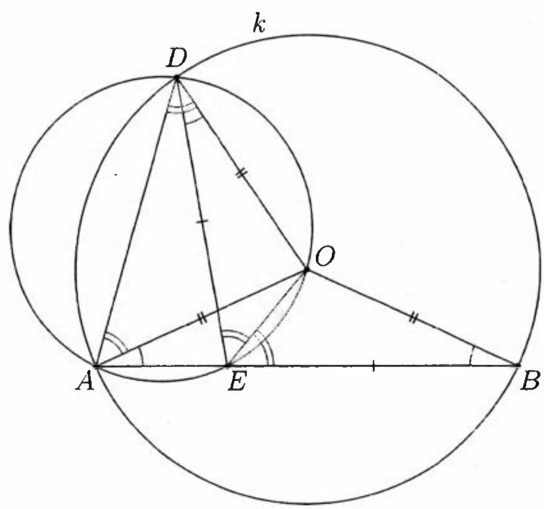
Solution. We have

$$\angle OEB = \angle ODA$$
$$= \angle OAD = \angle OED.$$

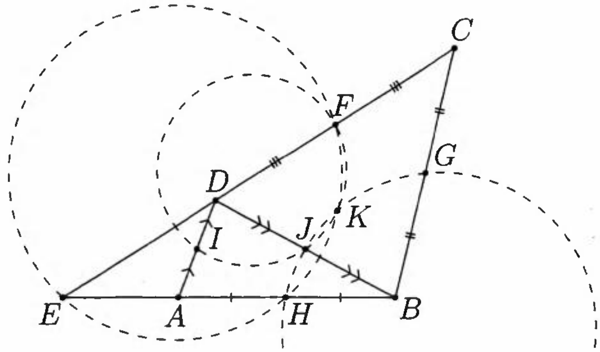
Also,

$$\angle OBE = \angle OAE = \angle ODE.$$

Hence, $\triangle OEB \cong \triangle OED$, which means that $DE = BE$.



Problem 6.10.17. Let $ABCD$ be a convex quadrilateral, such that $BA \cap CD = E$. Denote the midpoints of AB, BC, CD, DA and DB by H, G, F, I and J, respectively. Prove that the circumcircles of $\triangle HJG$, $\triangle IJF$ and $\triangle EHF$ have a common point.



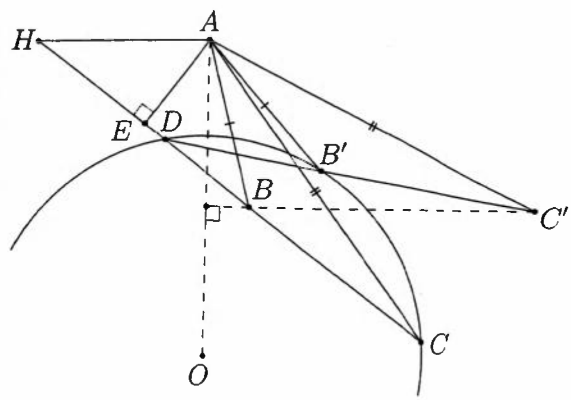
Solution. Let $k(HJG) \cap k(IJF) = \{J, K\}$. Then we consecutively have

$$\begin{aligned}\angle HKF &= \angle HKJ + \angle JKF = \angle HGJ + 180^\circ - \angle JIF \\ &= \angle JGB - \angle HGB + 180^\circ - \angle JID + \angle FID \\ &= 180^\circ + \angle DCB - \angle ACB + \angle CAD - \angle DAB \\ &= 180^\circ + \angle DCA - \angle CAB = 180^\circ - \angle FEH,\end{aligned}$$

which means that the quadrilateral $EHKF$ is cyclic.

Problem 6.10.18. Let $\triangle ABC \cong \triangle AB'C'$. Let $B'C' \cap BC = D$ and let O be the circumcenter of $\triangle DB'C$. Prove that $AO \perp BC'$.

Solution. Let E be the orthogonal projection of A onto the line BC, and let H be the reflection of B with respect to the point E. We will prove that



$$C'A^2 - BA^2 = C'O^2 - BO^2,$$

and the desired result will follow. Note that

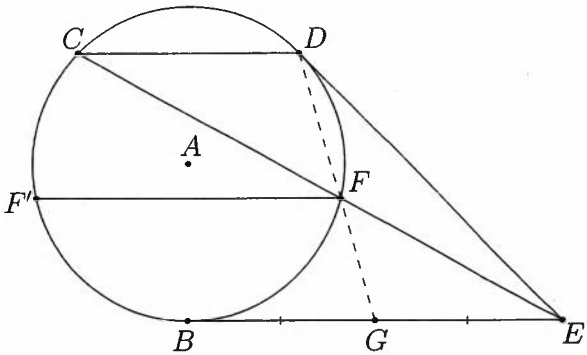
$$\begin{aligned}C'O^2 - BO^2 &= C'B'.C'D + R^2 - (R^2 - BD.BC) \\ &= CB(C'D + BD) = CB.(CD + DB').\end{aligned}$$

On the other hand, $C'A^2 - BA^2 = CA^2 - BA^2 = CE^2 - BE^2 = CB(CD + DH)$.

It remains to show that $DH = DB'$.

We have $\angle AB'D = \angle ABD = \angle AHD$. The quadrilateral $DBB'A$ is cyclic, so $\angle ADB' = \angle ABB' = \angle AB'B = \angle ADH$. Hence, $\triangle DAH \cong \triangle DAB'$, which means that $DH = DB'$.

Problem 6.10.19. Let $k(A)$ be a circle and let the line l touch it at the point B. A line parallel to l intersects k at the points C and D. The tangent lines to k at B and D intersect at E. Let $CE \cap k = \{C, F\}$. Denote the midpoint of BE by G. Prove that the points D, F and G are collinear.



Solution. Let the line through F, parallel to CD, intersect k for the second time at the point F'.

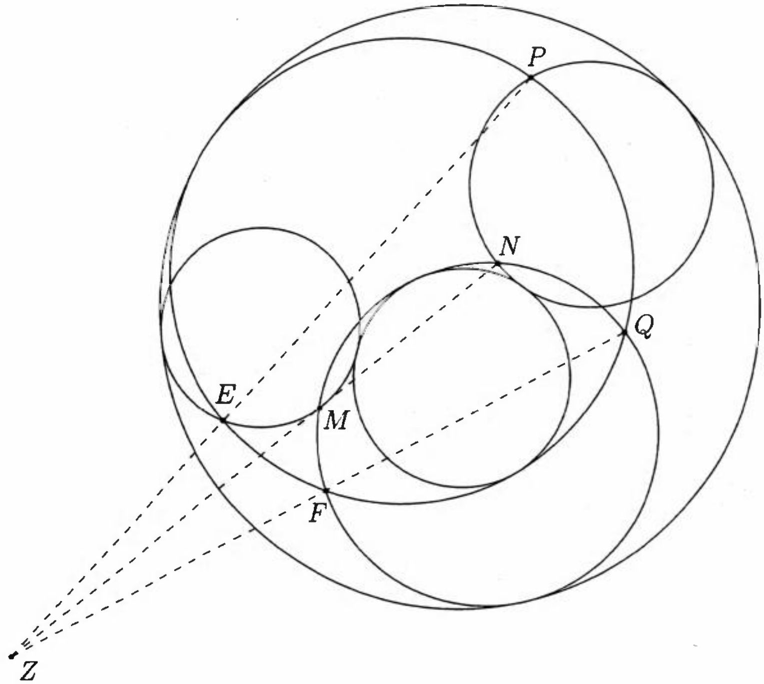
We are going to project the quadrilateral $BF'CD$ onto the line BE with respect to the point F. The point F' maps to the point of infinity, C maps to E, and B is fixed. If we show that the quadrilateral $BF'CD$ is harmonic, it will follow that D maps to the midpoint of BE.

Consider the symmetry with respect to the perpendicular bisector of CD. The quadrilateral $BF'CD$ is symmetric to $BFCD$. On the other hand, the quadrilateral $BFCD$ is harmonic because $t(D) \cap t(B) \cap CF = E$ (see Problems 4.4.1 and 4.4.4).

Hence, the quadrilateral $BF'CD$ is also harmonic.

Problem 6.10.20. Let k and k_1 be circles, such that k_1 lies inside of k. The circles k_2 and k_3 touch k internally and k_1 externally. The circles k_4 and k_5 touch k internally and k_1 externally. Denote the intersection points of the circles as shown in the figure. Prove that the lines FQ, MN and EP are concurrent.

Solution. Consider an inversion I with center F and an arbitrary radius. We will use the prime symbol to denote the images of the objects after the inversion has been applied to them. Then $I(k_2) = k_2'$ and $I(k_3) = k_3'$, where k_2' and k_3' are two lines intersecting each other at Q'.

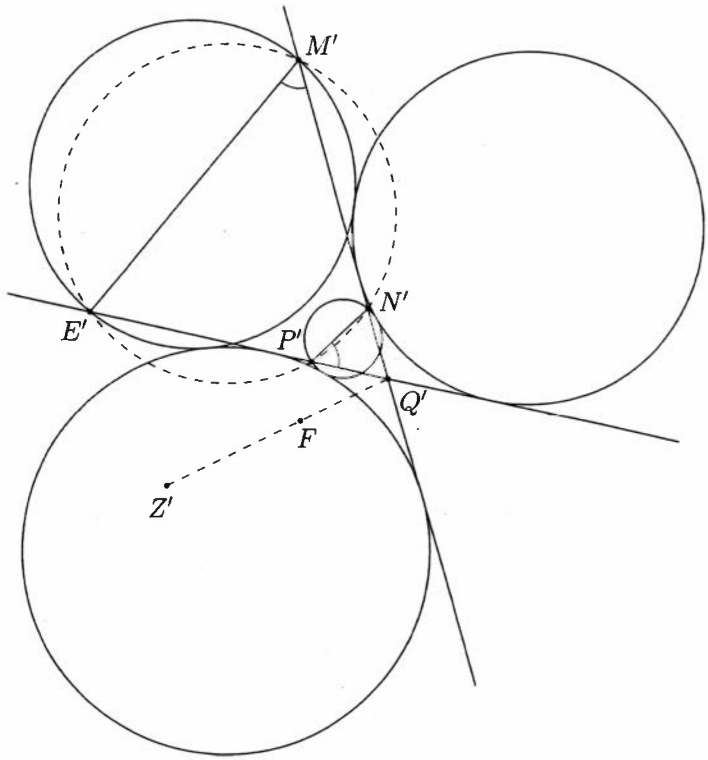


The circles k and k_1 are mapped to the circles k' and k_1', which are tangent to the lines above, as shown in the figure. The circles k_4 and k_5 are mapped to the circles k_4' and k_5', which are tangent to k' and k_1', as shown.

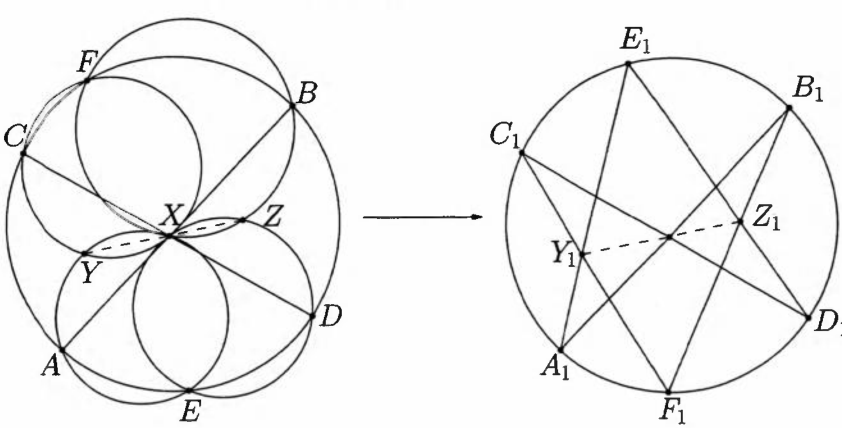
After we map the intersection points, we get the configuration of Problem 4.7.19, which gives us that $\angle N'P'Q' = \angle E'M'N'$. Hence, the quadrilateral $P'N'M'E'$ is cyclic. We are going to finish the solution in two ways.

The first one: The quadrilateral $P'N'M'E'$ is cyclic and hence the quadrilateral $PNME$ is cyclic. The radical axes theorem (Problem 6.5.1) yields that the lines FQ, MN and EP are concurrent.

The second one: Note that $Q'N'.Q'M' = Q'P'.Q'E'$. Hence, Q' lies on the radical axes of the circumcircles of $\triangle M'N'F$ and $\triangle E'P'F$. Denote the second intersection point of these circles by Z'. Then $Q' \in Z'F$ and so the lines FQ, MN and EP are concurrent.



Problem 6.10.21. Let AB and CD be two chords in the circle k that intersect each other at the point X. Points E and F are chosen on the arcs $\widehat{AD}$ and $\widehat{BC}$, respectively. The circumcircles of $\triangle FCX$ and $\triangle AEX$ intersect for the second time at the point Y, and the circumcircles of $\triangle BFX$ and $\triangle DXE$ intersect for the second time at the point Z. Prove that the points X, Y and Z are collinear.



Solution. Consider an inversion $I(X)$ with an arbitrary radius.

The circle $(ADBC)$ transforms into a circle k' passing through X. The other points that lie on k map on k'. Let $I(A) = A_1$, $I(B) = B_1$, etc. The circumcircles of $\triangle CFX$, $\triangle BFX$, $\triangle DEX$ and $\triangle AEX$ pass through the center of the inversion, so they transform into the lines C_1F_1, B_1F_1, D_1E_1 and A_1E_1, respectively.

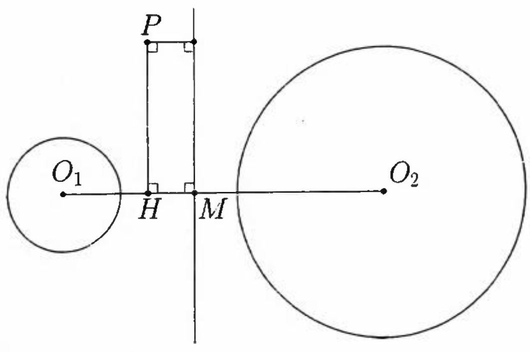
Let $I(Y) = Y_1$ and $I(Z) = Z_1$. Pascal's Theorem, applied to the inscribed hexagon $A_1C_1E_1B_1D_1F_1$, yields that the points Y_1, Z_1 and X are collinear. Hence, the points X, Y and Z are collinear.

Problem 6.10.22. Let $ABCD$ be a cyclic quadrilateral, inscribed in the circle k. The line l forms equal angles with the pair of lines (AB, CD) while intersecting them. Let us denote the circle that touches the pair (AB, CD) at their points of intersection with l by k_2. The circle k_1 touches the line BD at its point of intersection with l, and k_1 also touches the line AC. Prove that k_2, k_1 and k have a common radical axis.

Solution. Angle chasing gives that l forms equal angles with the pair (AC, BD). Therefore, the circle k_1 touches the line AC at its point of intersection with l. First, we will prove the following two lemmas:

Lemma 1. Two circles $k_1(O_1, r)$ and $k_2(O_2, r)$ are given. If P is a point and l is the radical axis of the two circles, then

$$| pow(P, k_1) - pow(P, k_2) |= 2O_1O_2.dist(P, l).$$



Proof. Without loss of generality, let $pow(P, k_1) \le pow(P, k_2)$.

Let $O_1O_2 \cap l = M$ and let H be the foot of the perpendicular from P to O_1O_2. Let $O_1H = x$, $HM = y$ and $MO_2 = z$.

We have to prove that

$$PO_2^2 - R^2 - PO_1^2 + r^2 = 2(x+y+z)y.$$

We have that $PO_2^2 - PO_1^2 = HO_2^2 - HO_1^2 = (y+z)^2 - x^2$.

Furthermore, since the point M lies on the radical axis of k_1 and k_2, we have that $MO_1^2 - r^2 = MO_2^2 - R^2$, or $r^2 - R^2 = (x+y)^2 - z^2$. Thus,

$$\begin{aligned} PO_2^2 - R^2 - PO_1^2 + r^2 &= (y+z)^2 - x^2 + (x+y)^2 - z^2 \\ &= y^2 + 2yz + 2xy + y^2 = 2y(x+y+z), \end{aligned}$$

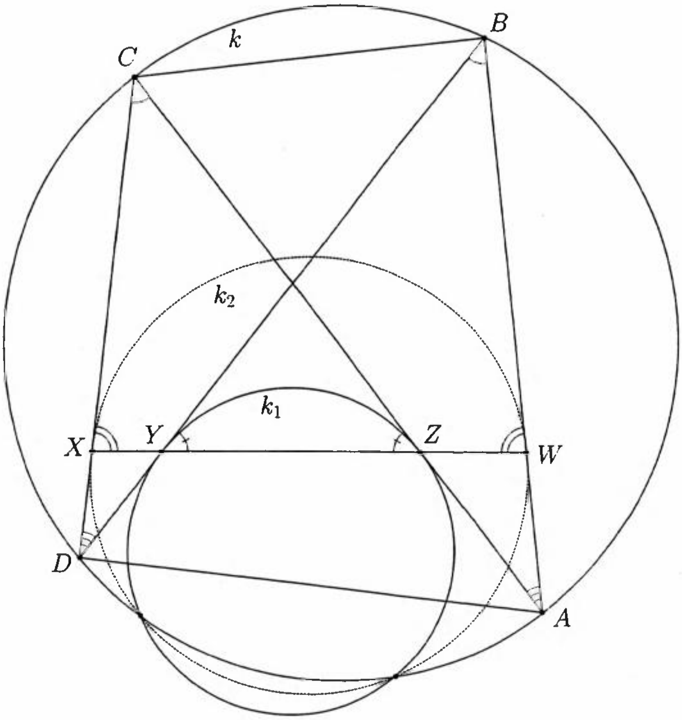
and the lemma is proved.

Lemma 2. Let $k_1(O_1)$ and $k_2(O_2)$ be circles. The locus of points P, such that $\dfrac{pow(P,k_1)}{pow(P,k_2)} = const$, is a circle and the three circles have a common radical axis.

Proof. Let $k(O)$ is a circle, such that k, k_1 and k_2 have a common radical axis l. Let $P \in k$. We have

$$\frac{pow(P,k_1)}{pow(P,k_2)} = \frac{|\, pow(P,k_1) - pow(P,k)\,|}{|\, pow(P,k_2) - pow(P,k)\,|} = \frac{2OO_1.dist(P,l)}{2OO_2.dist(P,l)} = \frac{OO_1}{OO_2} = const.$$

The constant does not depend on the position of the point P. Moreover, you can observe (for example by considering the points on the line O_1O_2) that this constant is different for different circles k. Lemma 2 is proven.

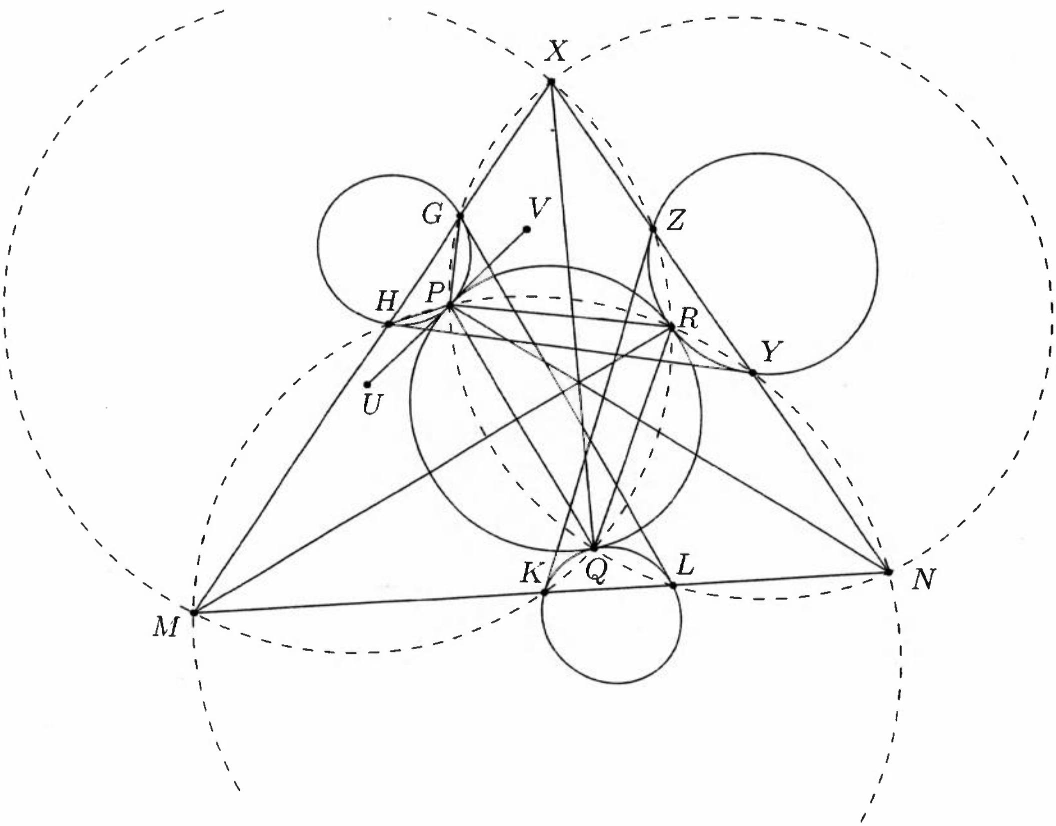


We will use the notations from the figure. Applying the law of sines, we get

$$\frac{DY}{DX} = \frac{\sin\angle DXY}{\sin\angle DYX} = \frac{\sin\angle AWZ}{\sin\angle AZW} = \frac{AZ}{AW}$$
$$= \frac{\sin\angle BWY}{\sin\angle WYB} = \frac{BY}{WB} = \frac{\sin\angle CXZ}{\sin\angle CZX} = \frac{CZ}{CX}.$$

Therefore, since the points A, B, C and D lie on the circle k, it follows by Lemma 2 that the circles k, k_1 and k_2 have a common radical axis.

Problem 6.10.23. Let k_1, k_2 and k_3 be three circles such that their disks do not intersect each other. For each pair of circles, the external tangent line that is closer to the third circle is drawn. The six points of tangency are shown in the figure. Let k be the circle that touches the three circles externally at the points P, Q and R. Prove that the hexagons $MHPRYN$, $NLQPGX$ and $XZRQKM$ are cyclic.



Solution. First, we will prove that the quadrilateral $MHYN$ is cyclic. Analogously, it will follow that the quadrilaterals $NLGX$ and $MKZX$ are also cyclic. We consecutively have

$$\begin{aligned}\angle MNY &= 180^\circ - \angle NKZ - \angle NZK \\ &= 180^\circ - \angle MLG - \angle HYX \\ &= 180^\circ - (180^\circ - \angle LMG - \angle LGM) - (180^\circ - \angle YHX - \angle HXY) \\ &= 2\angle GHY + (180^\circ - \angle MNY) - 180^\circ \\ &= 2\angle GHY - \angle MNY.\end{aligned}$$

Therefore, $\angle MNY = \angle GHY$. Now let us intersect the circumcircle of $MHYN$ and the circumcircle of $\triangle HGP$ at the point P', different from H. We will prove that the pentagon $XGP'LN$ is cyclic and this will imply that the circumcircles of $MHYN$ and $XGLN$ intersect at the point P' that lies on the circumcircle of $\triangle HGP$.

The analogous statements for the points Q' and R' will also follow. After that, we will prove that the circumcircle of $\triangle P'Q'R'$ touches the other three circles externally, and the desired result will follow. We consecutively have

$$\begin{aligned}\angle XNP' &= \angle YNM - \angle P'NM = \angle YHG - \angle P'HG \\ &= 180^\circ - \angle HP'G - \angle P'HG = \angle P'GH.\end{aligned}$$

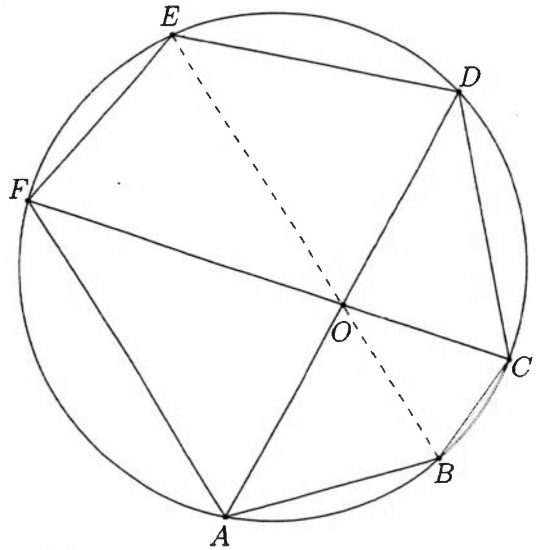
Thus, the circumcircles of $MHYN$ and $XGLN$ intersect at the point P' that lies on the circumcircle of $\triangle HGP$. Now we will prove that $P' \equiv P$.

Let us draw the tangent line to the circumcircle of $\triangle HGP'$ at the point P' and let us choose the points U and V on it, such that the point P' lies between them, as shown in the figure. We will prove that the line UV touches the circumcircle of $\triangle P'Q'R'$.

It will be enough to prove that $\angle VP'R' = \angle P'Q'R'$. We consecutively have

$$\begin{aligned}\angle VP'R' &= \angle GP'Q' - \angle GP'V - \angle R'P'Q' \\ &= 180^\circ - \angle GXQ' - \angle GHP' - \angle R'P'Q' \\ &= 180^\circ - \angle MXQ' - \angle MNP' - \angle R'P'Q' \\ &= 180^\circ - \angle MR'Q' - \angle MR'P' - \angle R'P'Q' \\ &= 180^\circ - \angle Q'R'P' - \angle R'P'Q' = \angle P'Q'R'.\end{aligned}$$

Problem 6.10.24. Let $ABCDEF$ be a cyclic hexagon. Prove that the lines AD, BE and CF are concurrent if and only if the equality $DE.FA.BC = EF.AB.CD$ holds.



Solution. Let AD, BE an CF be concurrent.

The trigonometric form of Ceva's Theorem, applied to $\triangle ACE$ and the lines AD, CF and EB, and a few applications of the law of sines are enough to prove the desired equality.

Conversely, let the equality $DE.FA.BC = EF.AB.CD$ hold. We will show that the lines AD, BE and CF are concurrent.

Let the diagonals AD and CF intersect at the point O. Let us denote the second intersection point of BO and the circumcircle of $ABCDEF$ by E'. The hexagon $ABCDE'F$ is cyclic and the lines AD, BE' and CF are concurrent. Therefore, $DE'.FA.BC = E'F.AB.CD$.

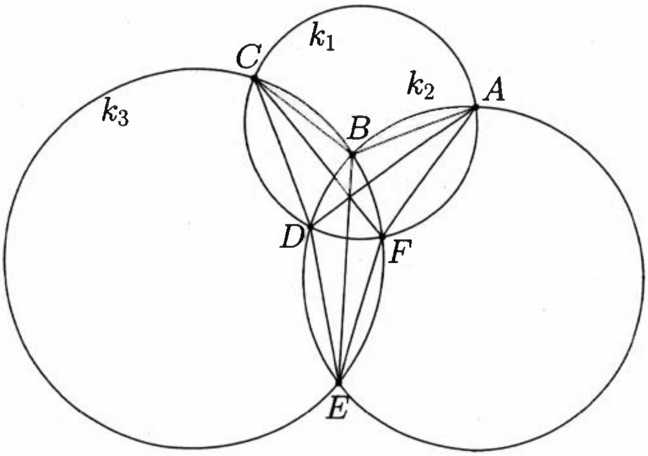
On the other hand, $DE.FA.BC = EF.AB.CD$. Thus, $\dfrac{DE'}{E'F} = \dfrac{DE}{EF}$ and we also have that $\angle FED = \angle FE'D$, which means that $\triangle FED \sim \triangle FE'D$, and so $\angle EFD = \angle E'FD$. Hence, $E \equiv E'$ and so the lines AD, BE and CF are concurrent.

Problem 6.10.25. Let k_1, k_2 and k_3 be three circles. Let $k_1 \cap k_2 = \{A, D\}$, $k_2 \cap k_3 = \{C, F\}$ and $k_3 \cap k_1 = \{B, E\}$ (see the figure). Prove that $AB.CD.EF = BC.DE.FA$.

Solution. The radical axes theorem (Problem 6.5.1) yields that the lines AD, BE and CF are concurrent.

Applying the trigonometric form of Ceva's Theorem to $\triangle BDF$ and the lines DA, BE and FC, we get

$$\frac{\sin\angle FDA}{\sin\angle ADB}\cdot\frac{\sin\angle DBE}{\sin\angle EBF}\cdot\frac{\sin\angle BFC}{\sin\angle CFD} = 1.$$



By the law of sines, we have

$$\frac{CD}{AF} = \frac{\sin \angle CFD}{\sin \angle FDA},$$

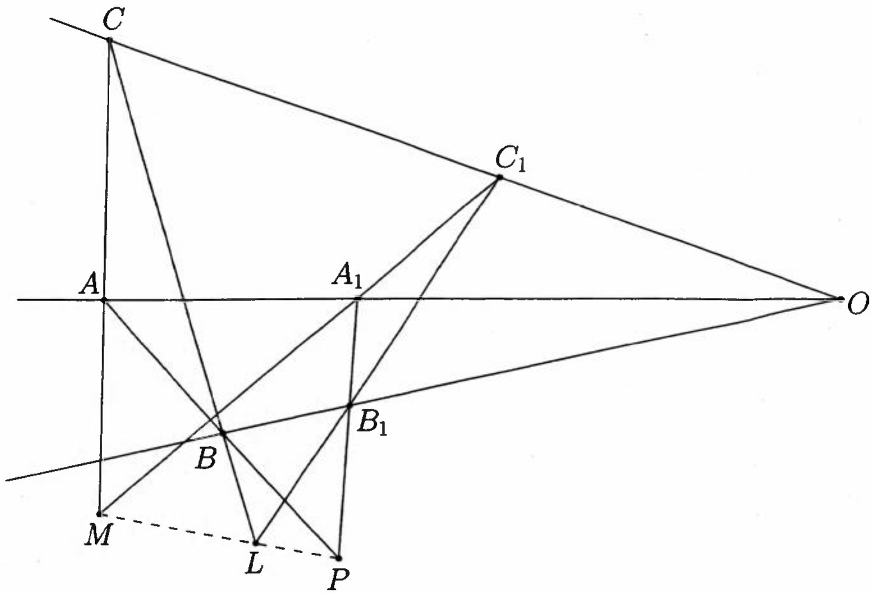
$$\frac{FE}{CB} = \frac{\sin \angle EBF}{\sin \angle BFC},$$

$$\frac{BA}{DE} = \frac{\sin \angle ADB}{\sin \angle DBE}.$$

Multiplying these equalities, we get

$$AB.CD.EF = BC.DE.FA.$$

Problems 7.1 and 7.2. *(Desargues' Theorem)* Let ABC and $A_1B_1C_1$ be triangles, such that the lines AA_1, BB_1 and CC_1 are concurrent at the point O. Let $M = AC \cap A_1C_1$, $L = BC \cap B_1C_1$ and $P = AB \cap A_1B_1$. Prove that the points L, M and P are collinear.



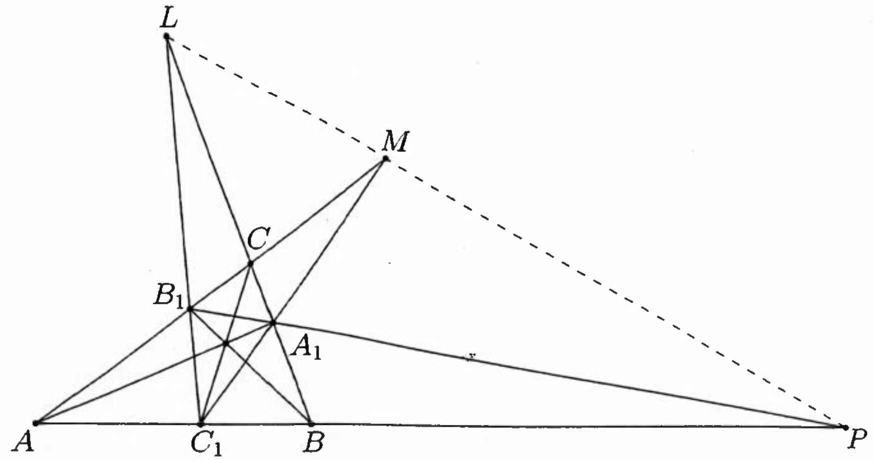
Solution. Menelaus' Theorem, applied to $\triangle OA_1C_1$ and the points M, A and C, yields $\dfrac{OC}{CC_1} \cdot \dfrac{C_1M}{MA_1} \cdot \dfrac{A_1A}{AO} = 1.$

The same theorem, applied to $\triangle OB_1C_1$ and the points L, B and C, gives $\dfrac{OC}{CC_1} \cdot \dfrac{C_1L}{LB_1} \cdot \dfrac{B_1B}{BO} = 1.$

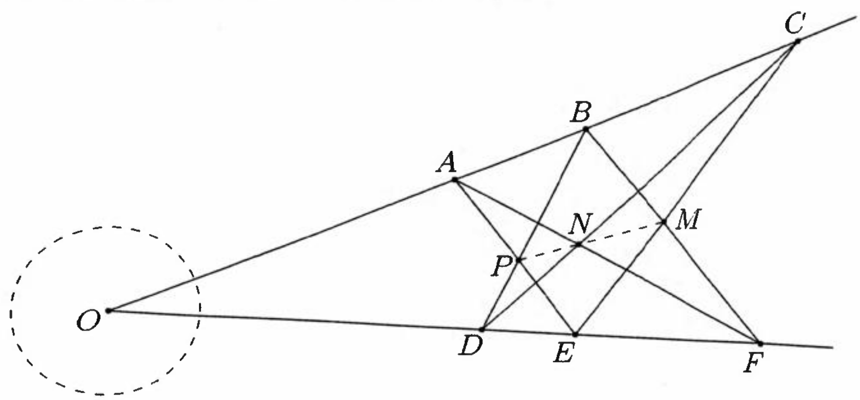
Applying it to $\triangle OA_1B_1$ and the points P, B and A, we get that $\dfrac{OA}{AA_1} \cdot \dfrac{A_1P}{PB_1} \cdot \dfrac{B_1B}{BO} = 1.$

Now we multiply the first and the third equality and the we divide the result by the second one. Hence, $\dfrac{C_1M}{MA_1} \cdot \dfrac{A_1P}{PB_1} \cdot \dfrac{B_1L}{LC_1} = 1$. Then Menelaus' Theorem, applied to $\triangle A_1B_1C_1$, yields that the points L, P and M are collinear.

Problem 7.2 is essentially the same, but in the case when O lies inside of $\triangle ABC$. The proof is conducted analogously.



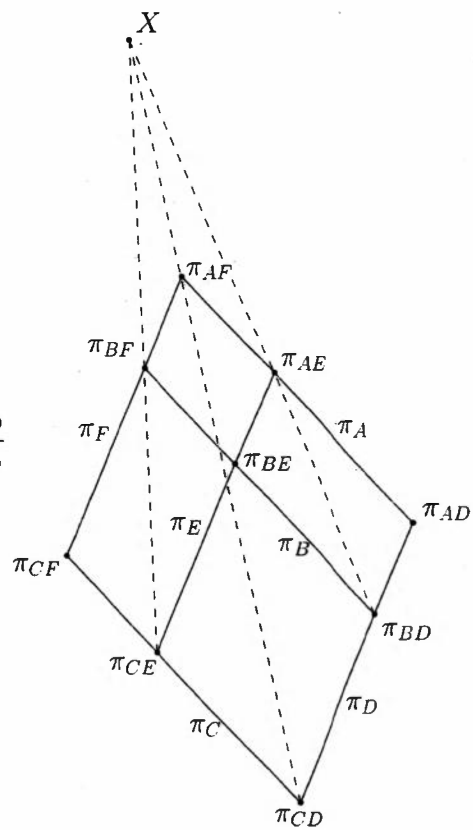
Problem 7.3. *(Pappus' Theorem)* The points A, B and C lie on a line (in this order). The points D, E and F lie on another line (in this order). Let $AE \cap BD = P$, $AF \cap CD = N$ and $BF \cap CE = M$. Prove that the points P, N and M are collinear.



Solution. We will consider the statement in the case when the two lines intersect. A good exercise for the reader is to solve the problem in the case when they are parallel.

Let the lines intersect at the point O and let k be a circle with center O. Consider a pole-polar transformation π with respect to k. The polar lines of A, B and C are π_A, π_B and π_C, which are parallel because O, A, B and C are collinear. Analogously, the lines π_D, π_E and π_F are also parallel. The line π_P passes through the points π_{AE} and π_{BD}, due to $AE \cap BD = P$. We define the lines π_N and π_M analogously. Hence, it suffices to prove that the lines $\pi_{AE}\pi_{BD}$, $\pi_{AF}\pi_{CD}$ and $\pi_{CE}\pi_{BF}$ are concurrent.

Let $\pi_{AE}\pi_{BD} \cap \pi_{AF}\pi_{CD} = X$ and $\pi_{CE}\pi_{BF} \cap \pi_{AF}\pi_{CD} = X'$. We will prove that $X \equiv X'$. We apply Menelaus' Theorem to $\triangle\pi_{AF}\pi_{CD}\pi_{AD}$ and to $\triangle\pi_{AF}\pi_{CD}\pi_{CF}$.



Then

$$\frac{\pi_{CD}X}{X\pi_{AF}} \cdot \frac{\pi_{AF}\pi_{AE}}{\pi_{AE}\pi_{AD}} \cdot \frac{\pi_{AD}\pi_{BD}}{\pi_{BD}\pi_{CD}} = 1$$

and

$$\frac{\pi_{CD}X'}{X'\pi_{AF}} \cdot \frac{\pi_{AF}\pi_{BF}}{\pi_{BF}\pi_{CF}} \cdot \frac{\pi_{CF}\pi_{CE}}{\pi_{CE}\pi_{CD}} = 1.$$

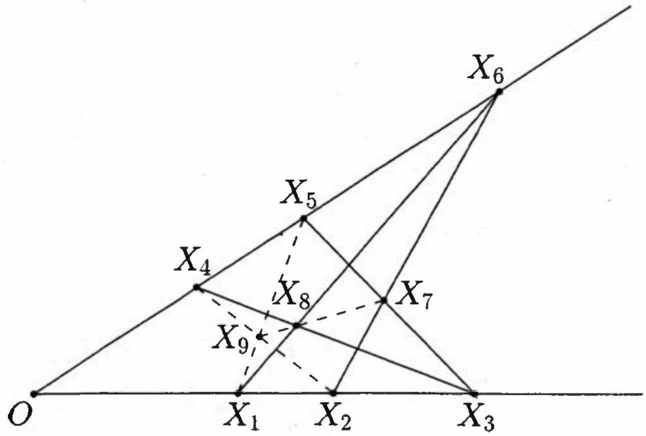
Dividing these equalities gives

$$\frac{\pi_{AF}\pi_{AE}}{\pi_{AE}\pi_{AD}} \cdot \frac{\pi_{AD}\pi_{BD}}{\pi_{BD}\pi_{CD}} \cdot \frac{\pi_{BF}\pi_{CF}}{\pi_{AF}\pi_{BF}} \cdot \frac{\pi_{CE}\pi_{CD}}{\pi_{CF}\pi_{CE}} = \frac{\pi_{CD}X'}{X'\pi_{AF}} \cdot \frac{\pi_{AF}X}{X\pi_{CD}}.$$

The respective parallelograms give $\pi_{AF}\pi_{AE} = \pi_{CF}\pi_{CE}$, $\pi_{AE}\pi_{AD} = \pi_{CE}\pi_{CD}$, $\pi_{AF}\pi_{BF} = \pi_{AD}\pi_{BD}$ and $\pi_{BF}\pi_{CF} = \pi_{BD}\pi_{CD}$.

Therefore, $\dfrac{\pi_{CD}X}{X\pi_{AF}} = \dfrac{\pi_{CD}X'}{X'\pi_{AF}}$ and we conclude that $X \equiv X'$.

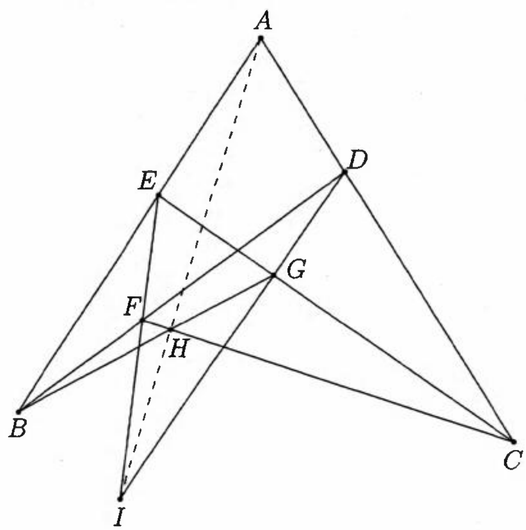
Problem 7.4. Let O be a point. Two distinct rays with origin O are given. Let X_1, X_2 and X_3 be points lying (in this order) on one of the rays, and let X_4, X_5 and X_6 be points lying (in this order) on the other one. Let $X_2X_6 \cap X_3X_5 = X_7$ and $X_1X_6 \cap X_3X_4 = X_8$. Prove that the lines X_1X_5, X_2X_4 and X_7X_8 are concurrent.



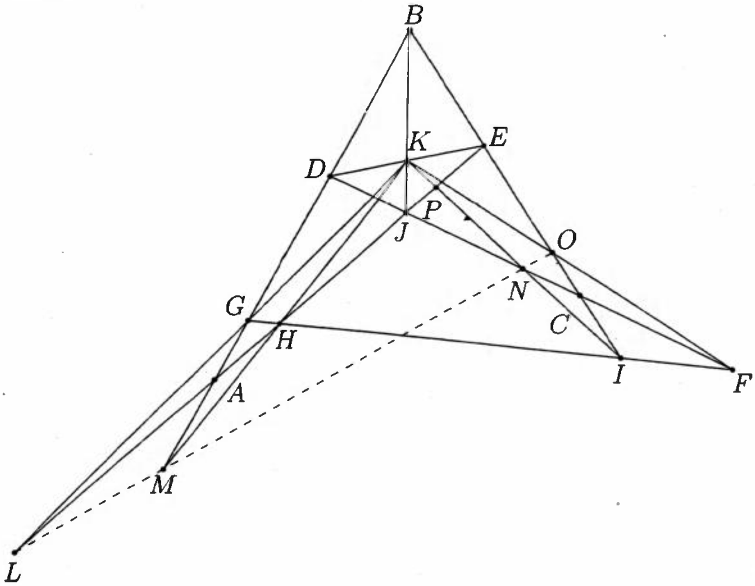
Solution. Let $X_9 = X_1X_5 \cap X_2X_4$. Pappus' Theorem (Problem 7.3), applied to the collinear triples of points X_1, X_2, X_3 and X_4, X_5, X_6, gives that the points X_7, X_8 and X_9 are collinear. So X_9 is the intersection point of the lines X_1X_5, X_2X_4 and X_7X_8, as desired.

Problem 7.5. Let A be a point. Two distinct rays with origin A are given. The points E and B lie on the first one (E lies between A and B) and the points D and C lie on the second one (D lies between A and C). Let $F \in BD$, $G \in EC$ and $CF \cap BG = H$. Prove that the lines EF, DG and AH are concurrent.

Solution. Let $EF \cap DG = I$. Applying Pappus' Theorem to $BG \cap CF$, $BE \cap CD$ and $EF \cap DG$, yields that the points A, H and I are collinear and the desired statement follows.



Problem 7.6. Let $DJEB$ be a quadrilateral. Let $BD \cap EJ = A$, $BE \cap DJ = C$ and $BJ \cap DE = K$. The line l intersects the lines AE, AB, CB and CD at the points H, G, I and F, respectively. The lines KH, KG, KI and KF intersect the lines AB, AE, DC and CB at the points M, L, N and O, respectively. Prove that the points M, L, N and O are collinear.



Solution. We will show that the points L, M and N are collinear (the proof is analogous for the other triples). Menelaus' Theorem, applied to $\triangle AJD$, yields that it suffices to prove that $\frac{AL}{LJ} \cdot \frac{JN}{ND} \cdot \frac{DM}{MA} = 1$.

The same theorem, applied to $\triangle ADE$, gives $\frac{DM}{MA} = \frac{DK}{KE} \cdot \frac{EH}{HA}$.

Analogously, considering $\triangle AJB$ we deduce that $\frac{AL}{LJ} = \frac{AG}{GB} \cdot \frac{BK}{KJ}$. If $JE \cap KI = P$, the same theorem, applied to $\triangle DEJ$, yields that $\frac{JN}{ND} = \frac{JP}{PE} \cdot \frac{EK}{KD}$. So the desired equality is equivalent to

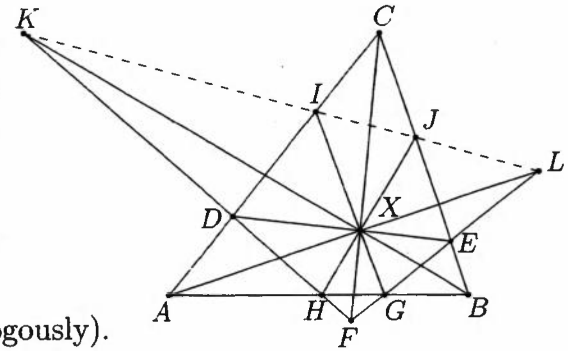
$$\frac{EH}{HA} \cdot \frac{AG}{GB} \cdot \frac{BK}{KJ} \cdot \frac{JP}{PE} = 1.$$

Menelaus' Theorem, applied to $\triangle ABE$ yields

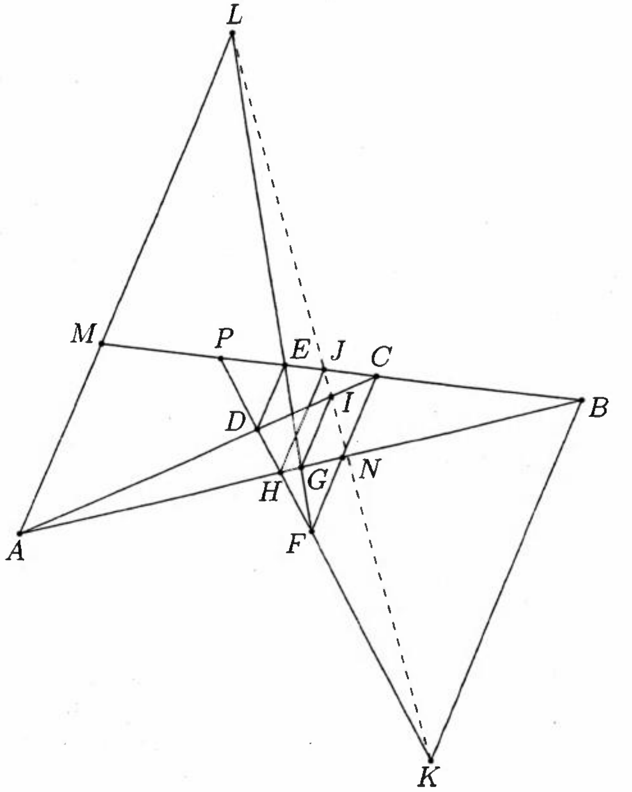
$$\frac{EH}{HA} \cdot \frac{AG}{GB} = \frac{EI}{IB}.$$

So it suffices to prove the equality $\dfrac{EI}{IB} \cdot \dfrac{BK}{KJ} \cdot \dfrac{JP}{PE} = 1$, which follows by the same theorem, applied to $\triangle JEB$.

Problem 7.7. Let ABC be a triangle. The points D and E lie on the sides CA and CB, respectively. Let F be a point and let $DE \cap CF = X$, $FD \cap AB = H$, $FE \cap AB = G$, $BX \cap FD = K$, $AX \cap EG = L$, $XH \cap BC = J$ and $XG \cap AC = I$. Prove that the points K, L, J and I are collinear.



Solution. Let us consider the problem in the projective plane. We send X to the point of infinity. then the lines AL, DE, HJ, GI, CF and BK become parallel. We will prove that the points L, J and K are collinear (the collinearity of the points L, I and K follows analogously).

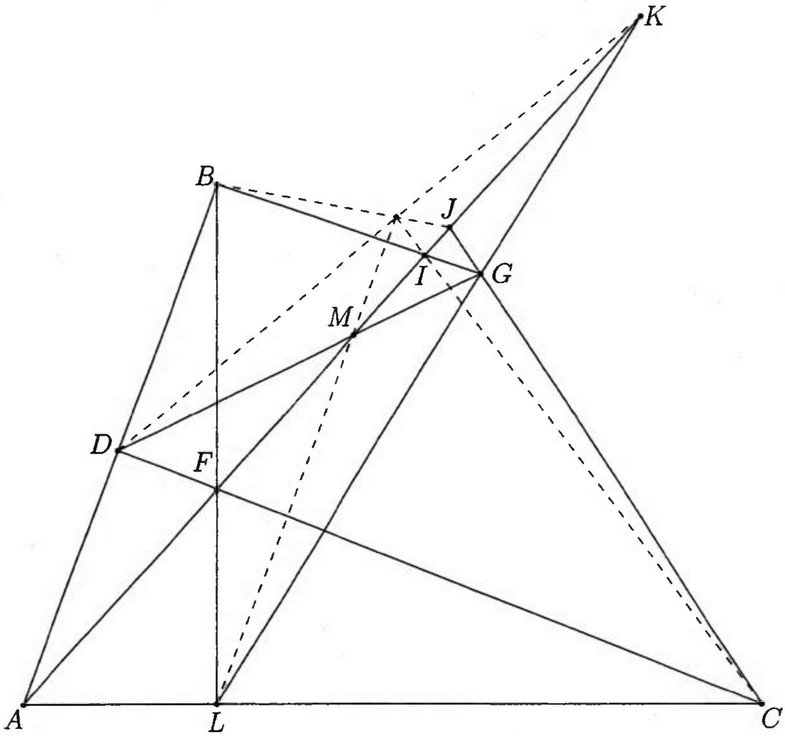


It suffices to show that $\dfrac{BK}{ML} = \dfrac{BJ}{JM}$.

We have $\dfrac{ML}{CF} = \dfrac{ME}{EC} = \dfrac{\frac{CE.AD}{CD}}{CE} = \dfrac{AD}{CD}$. Let $KF \cap BC = P$. Then

$$\frac{BJ}{JM} = \frac{BH}{HA} = \frac{BP}{PC} \cdot \frac{CD}{DA} = \frac{BK}{CF} \cdot \frac{CD}{AD} = \frac{BK}{ML}.$$

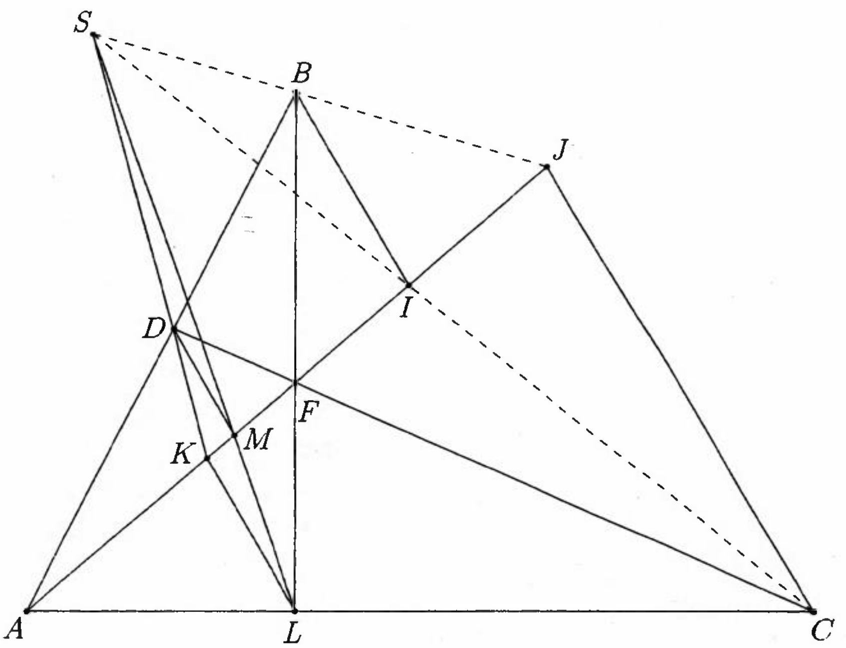
Problem 7.8. Let $ALFD$ be a quadrilateral will non-parallel sides. Let G be a point that does not lie on the line AF. Let $AD \cap FL = B$ and $AL \cap FD = C$. The lines GB, GD, GL and GC intersect AF at the points I, M, K and J, respectively. Prove that the lines BJ, DK, LM and CI are concurrent.



Solution. Let us consider the problem in the projective plane, sending the point G to the point of infinity (in a direction different from that of the line AF). Then the lines KL, DM, BI and CJ become parallel. Let $KD \cap LM = S$. Menelaus' Theorem, applied to $\triangle AML$, yields that the points S, I and C are collinear if and only if

$$\frac{AI}{IM}\cdot\frac{MS}{SL}\cdot\frac{LC}{CA}=1 \Leftrightarrow \frac{BI}{BI-DM}\cdot\frac{DM}{KL}\cdot\frac{LF}{FB}\cdot\frac{BD}{DA}=1$$
$$\Leftrightarrow \frac{BI}{BI-DM}\cdot\frac{DM}{KL}\cdot\frac{LK}{IB}\cdot\frac{BI-DM}{DM}=1$$

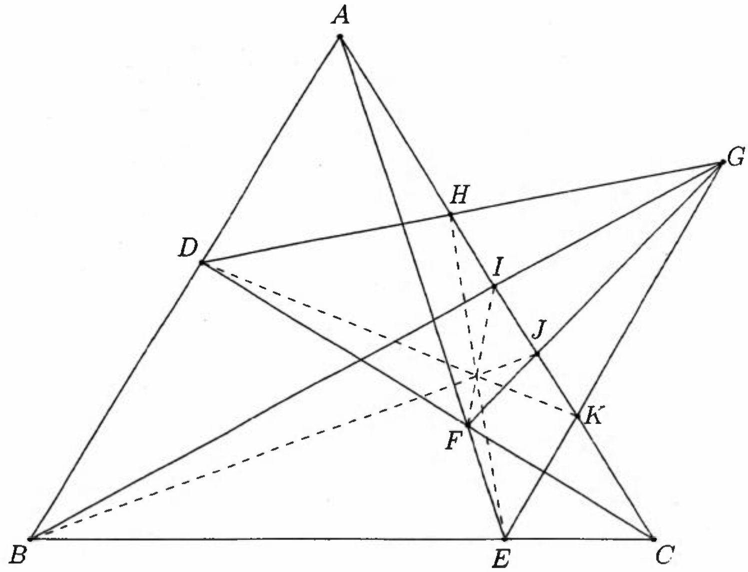
(we used Menelaus' Theorem for $\triangle ALB$ and the intercept theorem).



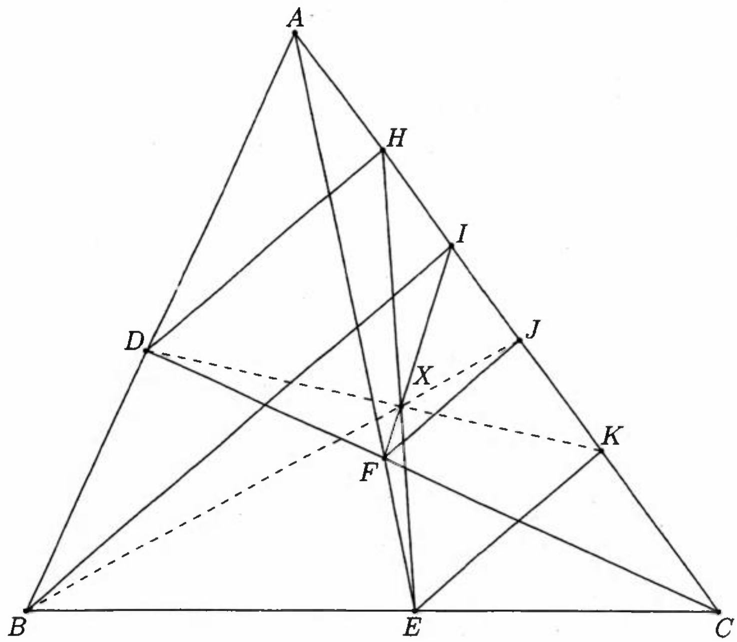
Analogously, we prove that the points J, B and S are collinear.

Problem 7.9. Let $BEFD$ be a convex quadrilateral. Let $BD\cap EF = A$ and $BE\cap DF = C$. The point G is arbitrary. The lines GD, GB, GF and GE intersect AC at the points H, I, J and K, respectively. Prove that the lines KD, JB, IF and HE are concurrent.

Solution. Let us consider the problem in the projective plane, sending the point G to the point of infinity. Then the lines DH, BI, FJ and EK become parallel. Let $HE\cap FI = X$. We will prove that the points D, X and K are collinear (the collinearity of the points B, X and J follows analogously).



Menelaus' Theorem, applied to $\triangle FCI$, yields that the points D, X and K are collinear if $\dfrac{FX}{XI} \cdot \dfrac{IK}{KC} \cdot \dfrac{CD}{DF} = 1$.



The same theorem, applied to $\triangle FIA$, gives

$$\frac{FX}{XI} = \frac{FE}{EA} \cdot \frac{AH}{HI} = \frac{FE}{EA} \cdot \frac{AD}{DB}.$$

It remains to prove that

$$\frac{FE}{EA}\cdot\frac{AD}{DB}\cdot\frac{IK}{KC}\cdot\frac{CD}{DF}=\frac{FE}{EA}\cdot\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CD}{DF}=1.$$

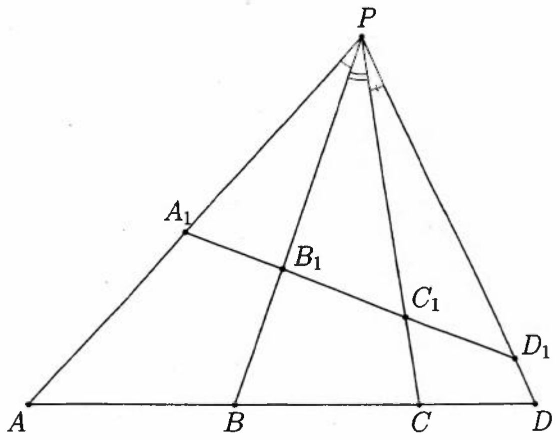
Again, Menelaus' Theorem, applied to $\triangle FAD$ and $\triangle BCD$, gives

$$\frac{FE}{EA}\cdot\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CD}{DF}=\frac{FC}{CD}\cdot\frac{DB}{BA}\cdot\frac{AD}{DB}\cdot\frac{BE}{EC}\cdot\frac{CD}{DF}=\frac{CF}{FD}\cdot\frac{DA}{AB}\cdot\frac{BE}{EC}=1.$$

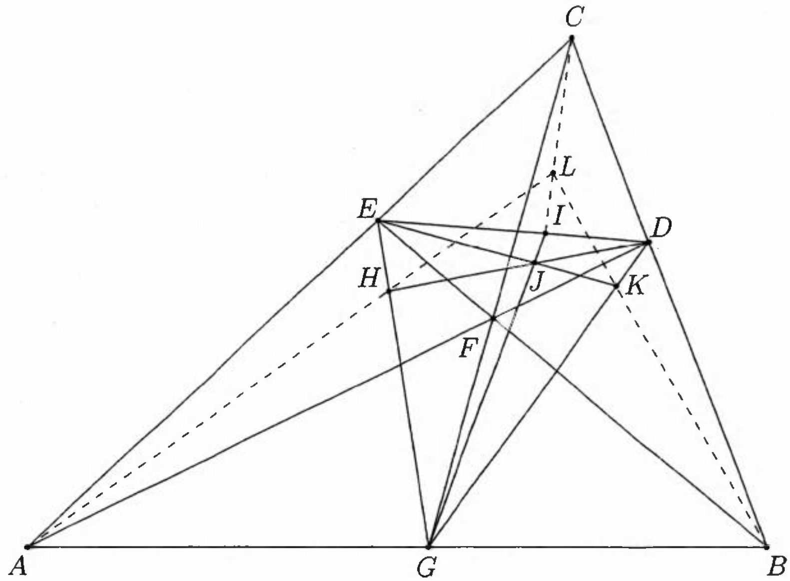
Problem 7.10. Let l be a line and $P \notin l$ be a point. The points A, B, C and D lie (in this order) on l. The collinear points A_1, B_1, C_1 and D_1 lie on the lines PA, PB, PC and PD, respectively. Prove that $(A,B,C,D)=(A_1,B_1,C_1,D_1)$.

Solution. Let $\angle APB=\alpha$, $\angle BPC=\beta$ and $\angle CPD=\gamma$. Then

$$\begin{aligned}(A,B,C,D)&=\frac{AB}{BC}\cdot\frac{CD}{DA}=\frac{S_{\triangle ABP}}{S_{\triangle BCP}}\cdot\frac{S_{\triangle CDP}}{S_{\triangle ADP}}\\&=\frac{AP.BP.\sin\alpha}{BP.CP.\sin\beta}\cdot\frac{CP.DP.\sin\gamma}{AP.DP.\sin(\alpha+\beta+\gamma)}\\&=\frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\gamma}{\sin(\alpha+\beta+\gamma)}\\&=\frac{A_1P.B_1P.\sin\alpha}{B_1P.C_1P.\sin\beta}\cdot\frac{C_1P.D_1P.\sin\gamma}{A_1P.D_1P.\sin(\alpha+\beta+\gamma)}\\&=\frac{S_{\triangle A_1B_1P}}{S_{\triangle B_1C_1P}}\cdot\frac{S_{\triangle C_1D_1P}}{S_{\triangle A_1D_1P}}=\frac{A_1B_1}{B_1C_1}\cdot\frac{C_1D_1}{D_1A_1}=(A_1,B_1,C_1,D_1).\end{aligned}$$



Problem 7.11. Let ABC be a triangle. The points D, E and G lie on the sides BC, CA and AB, respectively, such that $AD \cap BE \cap CG = F$. The point J lies inside of $\triangle DEG$. Let $GJ \cap DE = I$, $EJ \cap DG = K$ and $DJ \cap EG = H$. Prove that the lines AH, CI and BK are concurrent.



Solution. The trigonometric form of Ceva's Theorem, applied to $\triangle ABC$, yields that it suffices to prove that

$$\frac{\sin \angle EAH}{\sin \angle HAG} \cdot \frac{\sin \angle GBK}{\sin \angle KBD} \cdot \frac{\sin \angle DCI}{\sin \angle ICE} = 1.$$

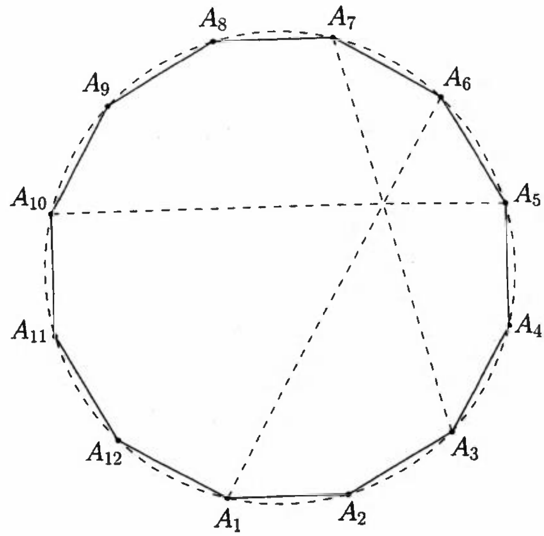
But $\dfrac{\sin \angle EAH}{\sin \angle HAG} = \dfrac{\sin \angle EAH}{\sin \angle AHE} \cdot \dfrac{\sin \angle AHG}{\sin \angle HAG} = \dfrac{EH}{AE} \cdot \dfrac{AG}{GH} = \dfrac{EH}{HG} \cdot \dfrac{AG}{AE}$. Using analogous equalities, we have

$$\begin{aligned}\frac{\sin \angle EAH}{\sin \angle HAG} \cdot \frac{\sin \angle GBK}{\sin \angle KBD} \cdot \frac{\sin \angle DCI}{\sin \angle ICE} &= \frac{EH}{HG} \cdot \frac{GA}{AE} \cdot \frac{GK}{KD} \cdot \frac{DB}{BG} \cdot \frac{DI}{IE} \cdot \frac{CE}{CD} \\ &= \frac{EH}{HG} \cdot \frac{GK}{KD} \cdot \frac{DI}{IE} \cdot \frac{GA}{AE} \cdot \frac{DB}{BG} \cdot \frac{CE}{CD} = 1,\end{aligned}$$

which is true because Ceva's Theorem, applied to $\triangle ABC$ and $\triangle DEF$, gives that $\dfrac{GA}{AE} \cdot \dfrac{DB}{BG} \cdot \dfrac{CE}{CD} = 1$ and $\dfrac{EH}{HG} \cdot \dfrac{GK}{KD} \cdot \dfrac{DI}{IE} = 1$, respectively.

Throughout the following ten problems, we will denote a regular polygon with n vertices by $A_1A_2\dots A_n$. In the proofs, we will implicitly use Problem 6.10.24 and the fact that any regular polygon is cyclic.

Problem 8.1. In a regular 12-gon, prove that the lines A_1A_6, A_3A_7 and A_5A_{10} are concurrent.

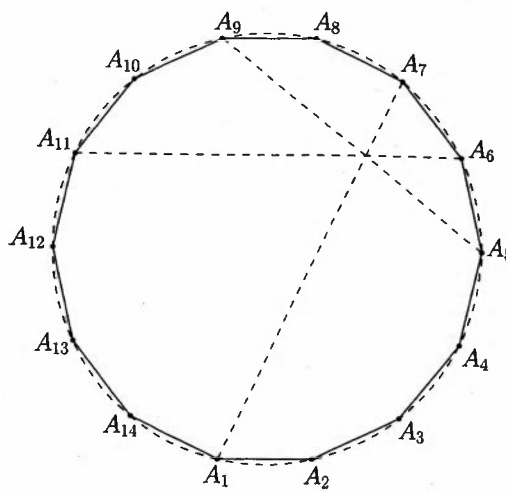


Solution. It suffices to prove that

$$\frac{A_1A_3}{A_3A_5}\cdot\frac{A_5A_6}{A_6A_7}\cdot\frac{A_7A_{10}}{A_{10}A_1}=1.$$

But it is easy to see that $A_1A_3 = A_3A_5$. Also, $A_5A_6 = A_6A_7$ and $A_7A_{10} = A_{10}A_1$, hence the desired equality follows.

Problem 8.2. In a regular 14-gon, prove that the lines A_1A_7, A_5A_9 and A_6A_{11} are concurrent.

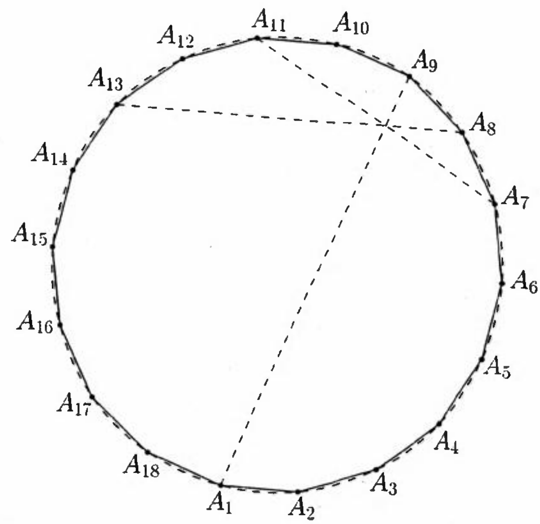


Solution. It suffices to prove that

$$\frac{A_1A_5}{A_5A_6}\cdot\frac{A_6A_7}{A_7A_9}\cdot\frac{A_9A_{11}}{A_{11}A_1}=1.$$

But $A_1A_5 = A_1A_{11}$, $A_5A_6 = A_6A_7$ and $A_7A_9 = A_{11}A_9$ and the desired equality follows.

Problem 8.3. In a regular 18-gon, prove that the lines A_1A_9, A_7A_{11} and A_8A_{13} are concurrent.

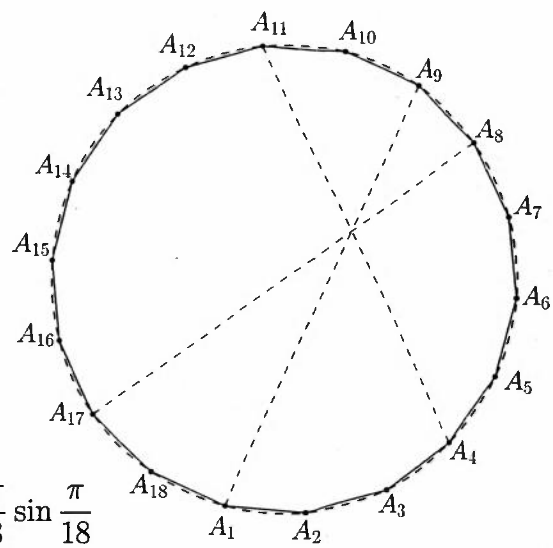


Solution. It suffices to prove that

$$\frac{A_1A_7}{A_7A_8} \cdot \frac{A_8A_9}{A_9A_{11}} \cdot \frac{A_{11}A_{13}}{A_{13}A_1} = 1.$$

But note that $A_1A_7 = A_1A_{13}$, $A_7A_8 = A_8A_9$ and $A_9A_{11} = A_{11}A_{13}$. The desired equality follows.

Problem 8.4. In a regular 18-gon, prove that the lines A_1A_9, A_4A_{11} and A_8A_{17} are concurrent.



Solution. It suffices to prove that

$$\frac{A_1A_4}{A_4A_8} \cdot \frac{A_8A_9}{A_9A_{11}} \cdot \frac{A_{11}A_{17}}{A_{17}A_1} = 1.$$

This is equivalent to

$$\frac{\sin\frac{3\pi}{18}}{\sin\frac{4\pi}{18}} \cdot \frac{\sin\frac{\pi}{18}}{\sin\frac{2\pi}{18}} \cdot \frac{\sin\frac{6\pi}{18}}{\sin\frac{2\pi}{18}} = 1$$

$$\Leftrightarrow \sin^2\frac{2\pi}{18}\sin\frac{4\pi}{18} = \sin\frac{3\pi}{18}\sin\frac{6\pi}{18}\sin\frac{\pi}{18}$$

$$\Leftrightarrow 2\cos\frac{\pi}{18}\sin\frac{2\pi}{18}\sin\frac{4\pi}{18} = \sin\frac{3\pi}{18}\sin\frac{6\pi}{18}$$

$$\Leftrightarrow \left(\sin\frac{3\pi}{18} + \sin\frac{5\pi}{18}\right)\sin\frac{2\pi}{18} = \sin\frac{3\pi}{18}\sin\frac{6\pi}{18}$$

$$\Leftrightarrow \sin\frac{2\pi}{18}\sin\frac{5\pi}{18} = \sin\frac{3\pi}{18}\left(\sin\frac{6\pi}{18} - \sin\frac{2\pi}{18}\right)$$

$$\Leftrightarrow \sin\frac{2\pi}{18}\sin\frac{5\pi}{18} = \sin\frac{2\pi}{18}\cos\frac{4\pi}{18}.$$

The last equality is obvious and the proof is completed.

Problem 8.5. In a regular 18-gon, prove that the lines A_1A_8, A_3A_9 and A_7A_{16} are concurrent.

Solution. It suffices to prove that

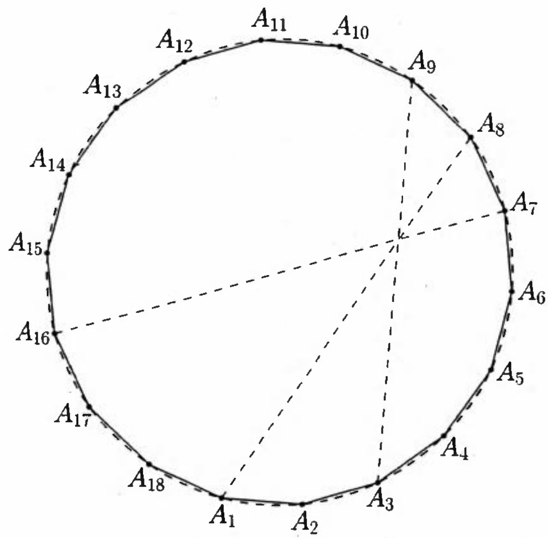
$$\frac{A_1A_3}{A_3A_7} \cdot \frac{A_7A_8}{A_8A_9} \cdot \frac{A_9A_{16}}{A_{16}A_1} = 1.$$

But $A_7A_8 = A_8A_9$, so this is equivalent to

$$\frac{\sin\frac{2\pi}{18}}{\sin\frac{4\pi}{18}} \cdot \frac{\sin\frac{7\pi}{18}}{\sin\frac{3\pi}{18}} = 1$$

$$\Leftrightarrow 2\sin\frac{2\pi}{18}\cos\frac{2\pi}{18} = \sin\frac{4\pi}{18},$$

which completes the proof.



Problem 8.6. In a regular 18-gon, prove that the lines A_1A_7, A_3A_8 and A_6A_{15} are concurrent.

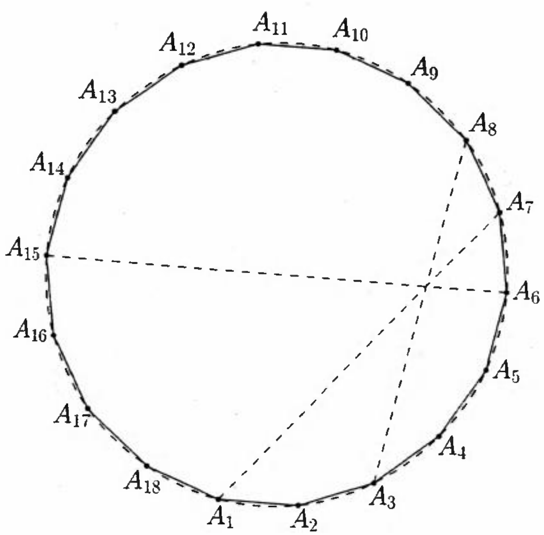
Solution. It suffices to prove that

$$\frac{A_1A_3}{A_3A_6} \cdot \frac{A_6A_7}{A_7A_8} \cdot \frac{A_8A_{15}}{A_{15}A_1} = 1.$$

But $A_6A_7 = A_7A_8$, so this is equivalent to

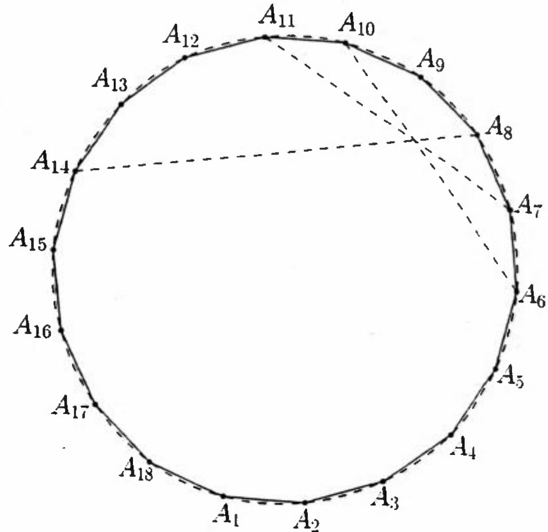
$$\frac{\sin\frac{4\pi}{18}}{\sin\frac{2\pi}{18}} \cdot \frac{\sin\frac{3\pi}{18}}{\sin\frac{7\pi}{18}} = 1$$

$$\Leftrightarrow \sin\frac{2\pi}{18}\cos\frac{2\pi}{18} = \sin\frac{7\pi}{18}\sin\frac{2\pi}{18}.$$



The last equality is obvious and the proof is completed.

Problem 8.7. In a regular 18-gon, prove that the lines A_6A_{10}, A_7A_{11} and A_8A_{14} are concurrent.



Solution. It suffices to prove that

$$\frac{A_6A_7}{A_7A_8}\cdot\frac{A_8A_{10}}{A_{10}A_{11}}\cdot\frac{A_{11}A_{14}}{A_{14}A_6}=1.$$

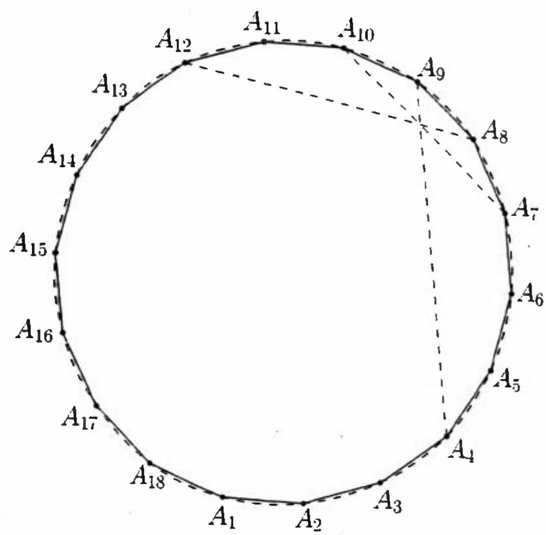
But $A_6A_7 = A_7A_8$ and the statement is equivalent to

$$\frac{\sin\frac{2\pi}{18}}{\sin\frac{\pi}{18}}\cdot\frac{\sin\frac{3\pi}{18}}{\sin\frac{10\pi}{18}}=1$$

$$\Leftrightarrow 2\sin\frac{\pi}{18}\cos\frac{\pi}{18}=\sin\frac{2\pi}{18},$$

which completes the proof.

Problem 8.8. In a regular 18-gon, prove that the lines A_4A_9, A_7A_{10} and A_8A_{12} are concurrent.



Solution. It suffices to prove that

$$\frac{A_4A_7}{A_7A_8}\cdot\frac{A_8A_9}{A_9A_{10}}\cdot\frac{A_{10}A_{12}}{A_{12}A_4}=1.$$

But $A_8A_9 = A_9A_{10}$, so this is equivalent to

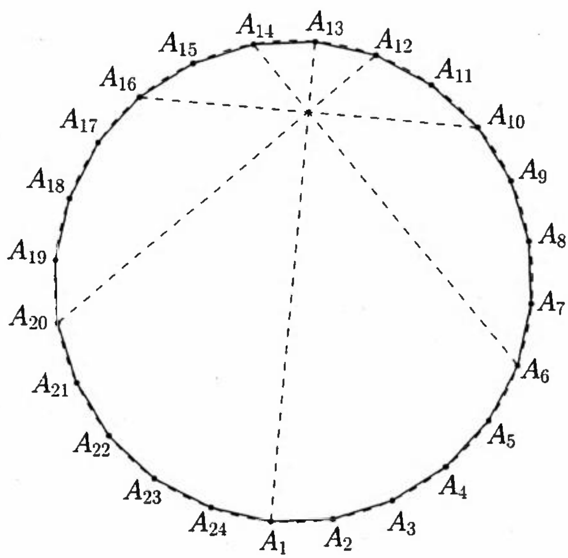
$$\frac{\sin\frac{3\pi}{18}}{\sin\frac{\pi}{18}}\cdot\frac{\sin\frac{2\pi}{18}}{\sin\frac{10\pi}{18}}=1$$

$$2\sin\frac{\pi}{18}\cdot\cos\frac{\pi}{18}=\sin\frac{2\pi}{18},$$

which completes the proof.

Problem 8.9. In a regular 24-gon, prove that the lines A_1A_{13}, A_6A_{14}, $A_{10}A_{16}$ and $A_{12}A_{20}$ are concurrent.

Solution. Note that A_1A_{13} is the common perpendicular bisector of $A_{12}A_{14}$ and A_6A_{20}, which gives that the lines A_1A_{13}, A_6A_{14} and $A_{12}A_{20}$ are concurrent.



We will prove that the lines A_6A_{14}, $A_{10}A_{16}$ and $A_{12}A_{20}$ are concurrent.

It suffices to prove that

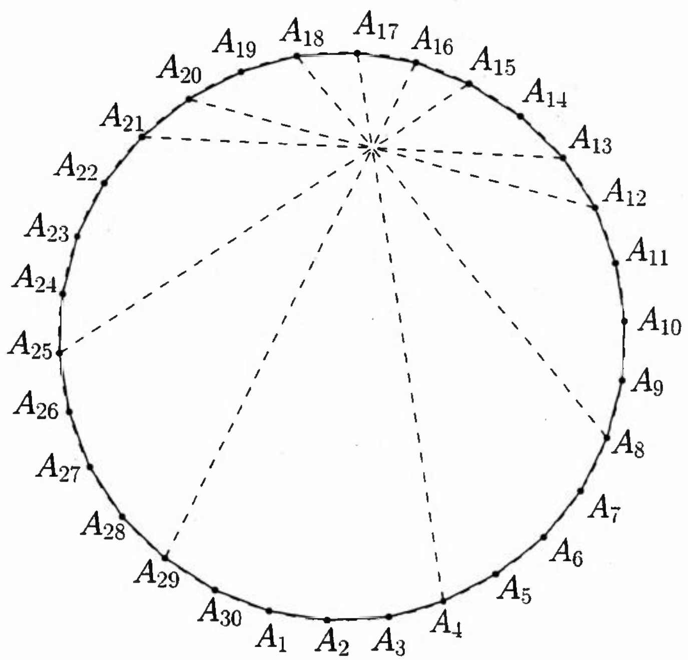
$$\frac{A_6A_{10}}{A_{10}A_{12}} \cdot \frac{A_{12}A_{14}}{A_{14}A_{16}} \cdot \frac{A_{16}A_{20}}{A_{20}A_6} = 1.$$

This is equivalent to

$$\frac{\sin^2 \dfrac{4\pi}{24}}{\sin^2 \dfrac{2\pi}{24}} \cdot \frac{\sin \dfrac{2\pi}{24}}{\sin \dfrac{10\pi}{24}} = 1 \Leftrightarrow 4\cos^2 \frac{2\pi}{24} . \sin \frac{\pi}{24} = \cos \frac{2\pi}{24} \Leftrightarrow \sin \frac{4\pi}{24} = \sin \frac{\pi}{6} = \frac{1}{2},$$

and the proof is completed.

Problem 8.10. In a regular 30-gon, prove that the lines A_4A_{17}, A_8A_{18}, $A_{12}A_{20}$, $A_{13}A_{21}$, $A_{15}A_{25}$ and $A_{16}A_{29}$ are concurrent.



Solution. We divide the solution into several steps.

1.) We will prove that the lines A_4A_{17}, A_8A_{18} and $A_{16}A_{29}$ are concurrent.

It suffices to prove that $\dfrac{A_4A_8}{A_8A_{16}} \cdot \dfrac{A_{16}A_{17}}{A_{17}A_{18}} \cdot \dfrac{A_{18}A_{29}}{A_{29}A_4} = 1$.

This is equivalent to

$$\frac{\sin\dfrac{4\pi}{30}}{\sin\dfrac{8\pi}{30}} \cdot \frac{\sin\dfrac{11\pi}{30}}{\sin\dfrac{5\pi}{30}} = 1 \Leftrightarrow \sin\frac{11\pi}{30} = \cos\frac{4\pi}{30}.$$

The last equality is true and the statement in 1.) follows.

2.) We will prove that the lines A_8A_{18}, $A_{12}A_{20}$ and $A_{16}A_{29}$ are concurrent.

This is equivalent to

$$\frac{\sin\dfrac{9\pi}{30}}{\sin\dfrac{4\pi}{30}} \cdot \frac{\sin\dfrac{4\pi}{30}}{\sin\dfrac{2\pi}{30}} \cdot \frac{\sin\dfrac{2\pi}{30}}{\sin\dfrac{9\pi}{30}} = 1,$$

which is obviously true.

3.) We will prove that the lines A_4A_{17}, A_8A_{18} and $A_{13}A_{21}$ are concurrent.

It suffices to prove that

$$\frac{\sin\frac{4\pi}{30}}{\sin\frac{5\pi}{30}}\cdot\frac{\sin\frac{4\pi}{30}}{\sin\frac{\pi}{30}}\cdot\frac{\sin\frac{3\pi}{30}}{\sin\frac{13\pi}{30}}=1\Leftrightarrow 2\sin^2\frac{4\pi}{30}\sin\frac{3\pi}{30}=\sin\frac{\pi}{30}\cos\frac{2\pi}{30}$$

$$\Leftrightarrow 8\cos\frac{\pi}{30}\sin^2\frac{4\pi}{30}\sin\frac{3\pi}{30}=\sin\frac{4\pi}{30}\Leftrightarrow 8\cos\frac{\pi}{30}\sin\frac{4\pi}{30}\sin\frac{3\pi}{30}=1$$

$$\Leftrightarrow 4\left(\sin\frac{5\pi}{30}+\sin\frac{3\pi}{30}\right)\sin\frac{3\pi}{30}=1\Leftrightarrow 2\sin\frac{3\pi}{30}+4\sin^2\frac{3\pi}{30}=1$$

$$\Leftrightarrow \sin\frac{3\pi}{30}=\frac{\sqrt{5}-1}{4}.$$

The last equality is true and the statement in 3.) follows.

4.) Finally, we will prove that the lines $A_{15}A_{25}$, A_4A_{17} and $A_{13}A_{21}$ are concurrent.

It suffices to prove that $\dfrac{A_4A_{13}}{A_{13}A_{15}}\cdot\dfrac{A_{15}A_{17}}{A_{17}A_{21}}\cdot\dfrac{A_{21}A_{25}}{A_{25}A_4}=1.$

But $A_{13}A_{15}=A_{15}A_{17}$, $A_{17}A_{21}=A_{21}A_{25}$ and $A_{25}A_4=A_4A_{13}$, so the statement in 4.) follows.

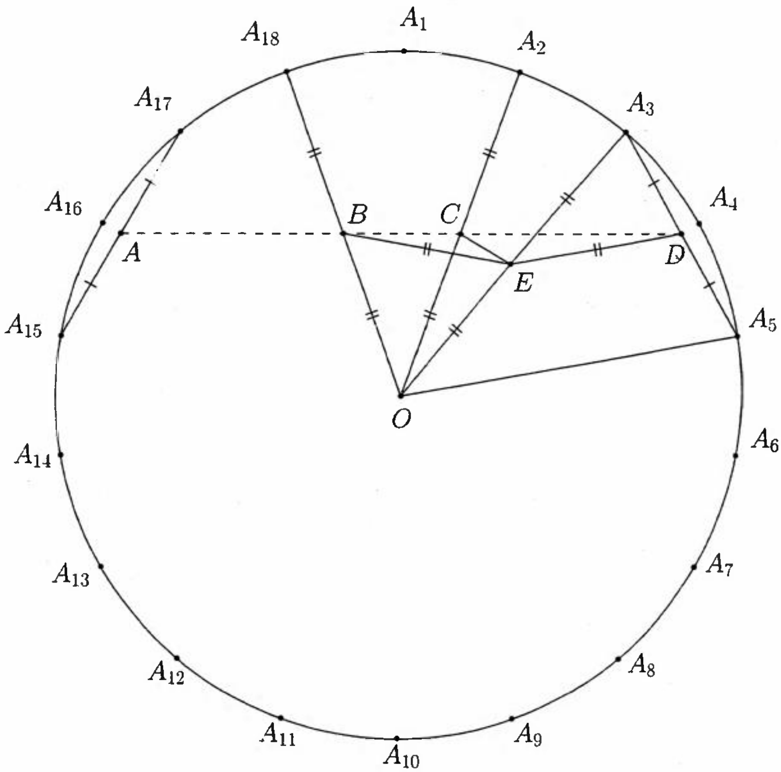
Combining the four steps, we see that the lines A_4A_{17}, A_8A_{18}, $A_{16}A_{29}$, $A_{12}A_{20}$, $A_{13}A_{21}$ and $A_{15}A_{25}$ are concurrent.

Problem 8.11. Let $A_1A_2A_3...A_{18}$ be a regular 18-gon with center O. Let A, B, C and D be the midpoints of $A_{15}A_{17}$, OA_{18}, OA_2 and A_3A_5, respectively. Prove that the points A, B, C and D are collinear.

Solution. We will prove that the points B, C and D are collinear (the fact that A lies on this line follows analogously).

Let E be the midpoint of OA_3. Then BC is a midsegment in $\triangle OA_2A_{18}$, CE – in $\triangle OA_2A_3$ and DE – in $\triangle OA_3A_5$.

Now we have that $\angle BCO=70°$, $\angle OCE=\angle OEC=80°$, $\angle CED=\angle CEA_3+\angle A_3ED=100°+\angle A_3OA_5=140°$ and $ED=\dfrac{OA_5}{2}=\dfrac{OA_3}{2}=OE$.



Moreover, the law of sines yields

$$\frac{CE}{ED} = \frac{CE}{EO} = \frac{\sin 20^\circ}{\sin 80^\circ} = \frac{2\sin 10^\circ \cos 10^\circ}{\cos 10^\circ} = 2\sin 10^\circ.$$

The cotangent technique, applied to $\triangle CED$, gives

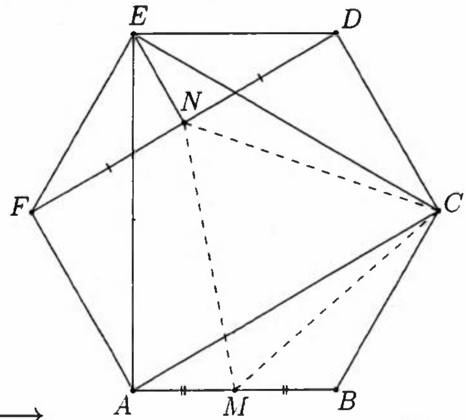
$$\begin{aligned}
\cot \angle ECD = \frac{\dfrac{CE}{ED} - \cos \angle CED}{\sin \angle CED} &= \frac{2\sin 10^\circ - \cos 140^\circ}{\sin 140^\circ} \\
= \frac{2\sin 10^\circ + \sin 50^\circ}{\sin 40^\circ} &= \frac{\sin 10^\circ + (\sin 10^\circ + \sin 50^\circ)}{\sin 40^\circ} \\
= \frac{\sin 10^\circ + 2\sin 30^\circ \cos 20^\circ}{\sin 40^\circ} &= \frac{\sin 10^\circ + \sin 70^\circ}{\sin 40^\circ} \\
= \frac{2\sin 40^\circ \cos 30^\circ}{\sin 40^\circ} &= \cot 30^\circ \Rightarrow \angle ECD = 30^\circ.
\end{aligned}$$

It follows that

$$\angle BCD = \angle BCO + \angle OCE + \angle ECD = 70^\circ + 80^\circ + 30^\circ = 180^\circ.$$

Therefore, the points B, C and D are collinear.

Problem 8.12. Let $ABCDEF$ be a regular hexagon. The points M and N are the midpoints of AB and DF, respectively. Prove that $\triangle MCN$ is equilateral.



Solution. It is clear that $\triangle ACE$ is equilateral. We have

$$2\overrightarrow{MC} = \overrightarrow{AC} + \overrightarrow{BC}.$$

A $+60°$ rotation of all vectors in the above equality gives

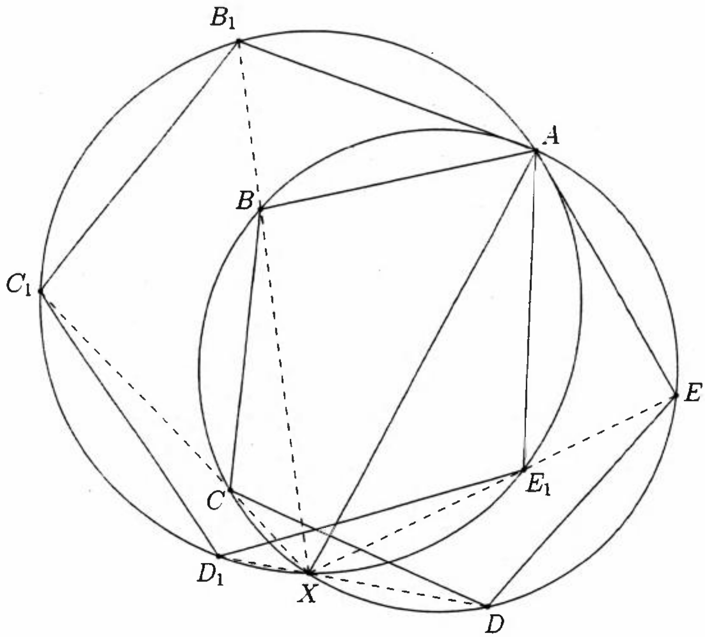
$$2\overrightarrow{a} = \overrightarrow{AE} + \overrightarrow{AF},$$

where $\overrightarrow{a}$ is the image of the vector $\overrightarrow{MC}$ under a $+60°$ rotation. Since $\overrightarrow{AE} = \overrightarrow{BD}$, we obtain $2\overrightarrow{a} = \overrightarrow{BD} + \overrightarrow{AF} = 2\overrightarrow{MN}$, whence $\overrightarrow{a} = \overrightarrow{MN}$.

Therefore, $\triangle CMN$ is equilateral.

Another possible approach uses the fact that $\triangle CAM \cong \triangle CEN$.

Problem 8.13. Let $ABCDE$ and $AB_1C_1D_1E_1$ be regular pentagons with the same orientation and a common vertex A. Prove that the lines BB_1, CC_1, DD_1 and EE_1 are concurrent.



Solution. Let X be the second intersection point of the circumcircles of the two pentagons. We will prove that X is the intersection point of the lines BB_1, CC_1, DD_1 and EE_1. We have

$$\angle DXA = \angle DCA = 72^\circ.$$

Also,

$$\angle D_1XA = \angle D_1E_1A = 108^\circ$$

Hence, $X \in DD_1$. Note that

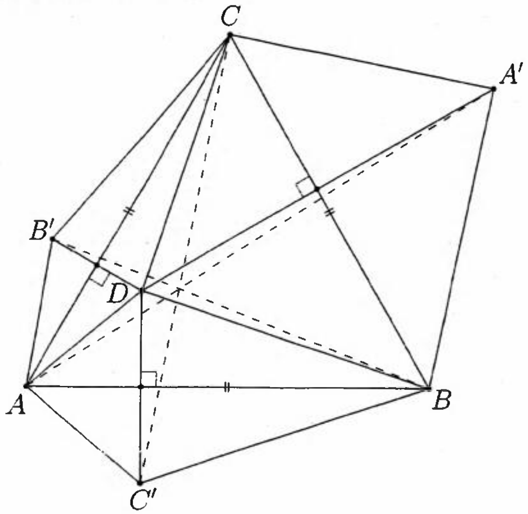
$$\angle EXD = \angle EAD = 36^\circ$$

and

$$\angle E_1XD = \angle E_1AD_1 = 36^\circ$$

Therefore, $\angle EXD = \angle E_1XD$, which implies that the points E, E_1 and X are collinear. The collinearity of the triples B, B_1, X and C, C_1, X follows analogously.

Problem 8.1.1. Let ABC be an equilateral triangle. The point D lies inside of the triangle. The symmetric points of D with respect to the sides AB, BC and AC are C', A' and B', respectively. Prove that the lines AA', BB' and CC' are concurrent.



Solution. Note that

$$\begin{aligned}\angle ABA' + \angle CBC' &= 2\angle ABC + \angle CBA' + \angle ABC' \\ &= 2\angle ABC + \angle CBD + \angle ABD = 3\angle ABC = 180^\circ.\end{aligned}$$

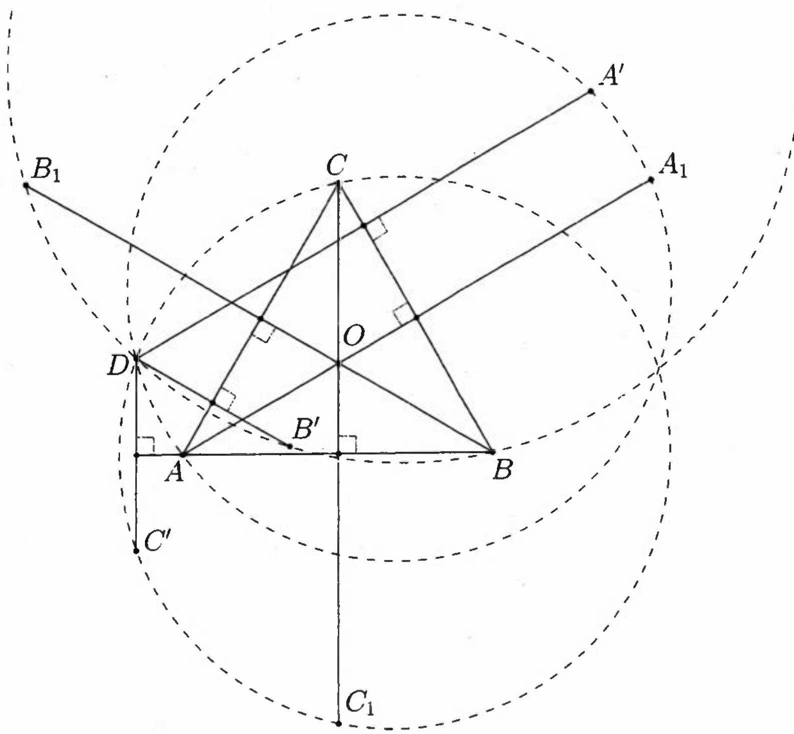
Therefore, $\sin\angle ABA' = \sin\angle CBC'$ and analogously, $\sin\angle BCB' = \sin\angle ACA'$ and $\sin\angle BAB' = \sin\angle CAC'$. Thus, we have

$$\begin{aligned}&\frac{\sin\angle ABB'}{\sin\angle B'BC}\cdot\frac{\sin\angle BCC'}{\sin\angle C'CA}\cdot\frac{\sin\angle CAA'}{\sin\angle A'AB} \\ &= \frac{AB'}{B'C}\cdot\frac{\sin\angle BAB'}{\sin\angle BCB'}\cdot\frac{BC'}{C'A}\cdot\frac{\sin\angle CBC'}{\sin\angle CAC'}\cdot\frac{CA'}{A'B}\cdot\frac{\sin\angle ACA'}{\sin\angle ABA'} \\ &= \frac{AD}{CD}\cdot\frac{\sin\angle BAB'}{\sin\angle ACA'}\cdot\frac{BD}{AD}\cdot\frac{\sin\angle CBC'}{\sin\angle BAB'}\cdot\frac{CD}{BD}\cdot\frac{\sin\angle ACA'}{\sin\angle CBC'} = 1.\end{aligned}$$

The trigonometric form of Ceva's Theorem gives that the lines AA', BB' and CC' are concurrent.

Note. The statement of the problem remains true if the point D does not lie inside of $\triangle ABC$.

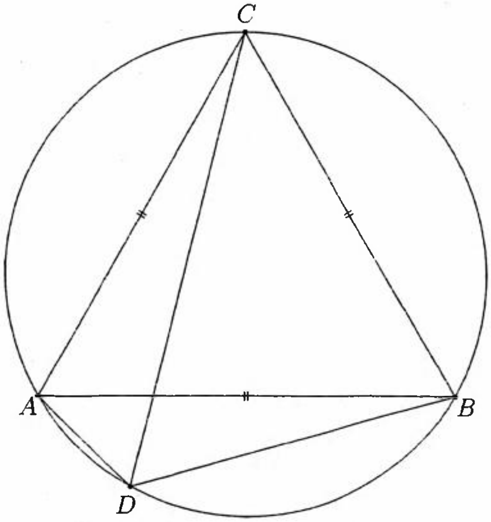
Problem 8.1.2. Let ABC be an equilateral triangle. The point D lies outside of the triangle. The symmetric points of D with respect to the lines AB, BC and AC are C', A' and B', respectively. Prove that the circumcircles of $\triangle DAA'$, $\triangle DBB'$ and $\triangle DCC'$ have a second common point different from D.



Solution. It suffices to show that the three circles are coaxial (have a common radical axis). Clearly, D has equal powers with respect to each of them. If we prove that the circumcenter O of $\triangle ABC$ has the same property, then the circles will either have a common radical axis (which will complete the proof) or will have two radical centers (a contradiction, see Problem 6.5.1).

Let A_1 be the reflection of A with respect to BC, B_1 be the reflection of B with respect to AC, and C_1 be the reflection of C with respect to AB. Then the quadrilaterals $ADA'A_1$, $BDB'B_1$ and $CDC'C_1$ are isosceles trapezoids. In particular, they are cyclic. Hence, the power of O with respect to the three circles is $AO.OA_1 = BO.OB_1 = CO.OC_1$, as desired.

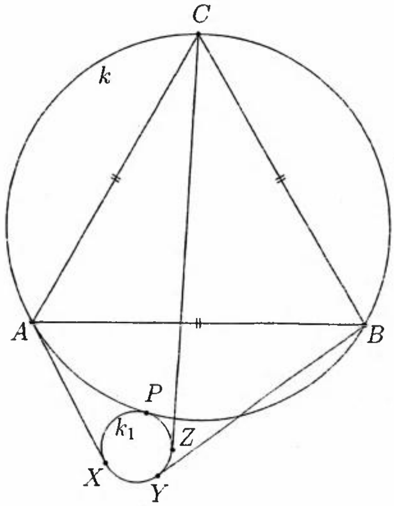
Problem 8.1.3. Let ABC be an equilateral triangle. The point D lies on the smaller arc $\widehat{AB}$ of its circumcircle. Prove that $AD + BD = CD$.



Solution. Let the sidelength of $\triangle ABC$ be a. The quadrilateral $ACBD$ is cyclic. By Ptolemy's Theorem (Problem 5.5.6) we get that

$$AD.BC + BD.AC = CD.AB$$
$$\Leftrightarrow a.AD + a.BD = a.CD$$
$$\Leftrightarrow AD + BD = CD.$$

Problem 8.1.4. Let ABC be an equilateral triangle with circumcircle k. The point P lies on the smaller arc $\widehat{AB}$ of k. The circle k_1 touches k externally at the point P. The lines AX, BY and CZ touch k_1 at the points X, Y and Z, respectively. Prove that $AX + BY = CZ$.



Solution. Let the sidelength of $\triangle ABC$ be a. Casey's Theorem (Problem 6.1.10), applied to A, k_1, B and C (points are circles with zero radius), yields

$$AX.BC + BY.CA = CZ.AB$$
$$\Leftrightarrow a.AX + a.BY = a.CZ$$
$$\Leftrightarrow AX + BY = CZ.$$

Problem 8.1.5. Let ABC and $A_1B_1C_1$ be two equilateral triangles with a common center O and different orientations. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.

Solution. A 120° rotation centered at O gives

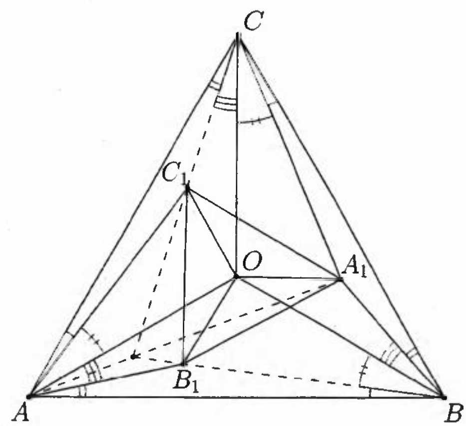
$$\angle AOB_1 = \angle BOA_1 = \angle COC_1.$$

Since $AO = BO = CO$ and $A_1O = B_1O = C_1O$, we get that

$$\triangle AB_1O \cong \triangle BA_1O \cong \triangle CC_1O.$$

Analogously, we deduce that

$$\triangle AC_1O \cong \triangle BB_1O \cong \triangle CA_1O.$$



Now we get $\angle A_1BC = \angle C_1CA$ and $\angle A_1CB = \angle B_1BA$.

The trigonometric form of Ceva's Theorem, applied to $\triangle ABC$ and the point A_1 gives

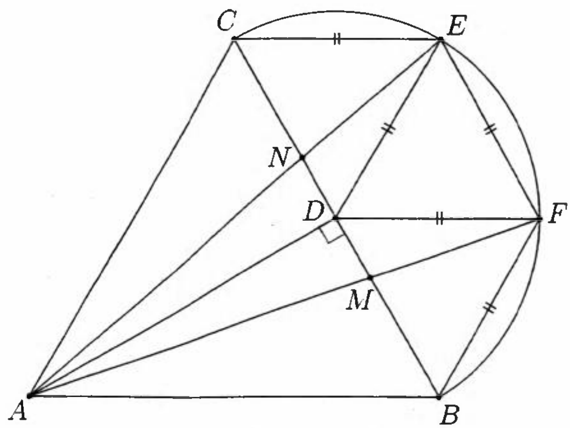
$$\frac{\sin\angle A_1AB}{\sin\angle A_1AC}\cdot\frac{\sin\angle ACA_1}{\sin\angle A_1CB}\cdot\frac{\sin\angle CBA_1}{\sin\angle A_1BA} = 1$$
$$\Rightarrow \frac{\sin\angle A_1AB}{\sin\angle A_1AC}\cdot\frac{\sin\angle CBB_1}{\sin\angle B_1BA}\cdot\frac{\sin\angle ACC_1}{\sin\angle C_1CB} = 1$$
$$\Rightarrow \frac{\sin\angle A_1AB}{\sin\angle A_1AC}\cdot\frac{\sin\angle ACC_1}{\sin\angle C_1CB}\cdot\frac{\sin\angle CBB_1}{\sin\angle B_1BA} = 1.$$

The last equality and the same theorem give that the lines AA_1, BB_1 and CC_1 are concurrent.

Problem 8.1.6. Let ABC be an equilateral triangle. The points N and M trisect the side BC (N is between C and M). The points E and F trisect the outside arc $\widehat{BC}$ of the circle with diameter BC (E lies between C and F on the arc). Prove that the triples of points A, M, F and A, N, E are collinear.

Solution. We will only prove that the points A, M and F are collinear (the proof for the other triple is analogous).

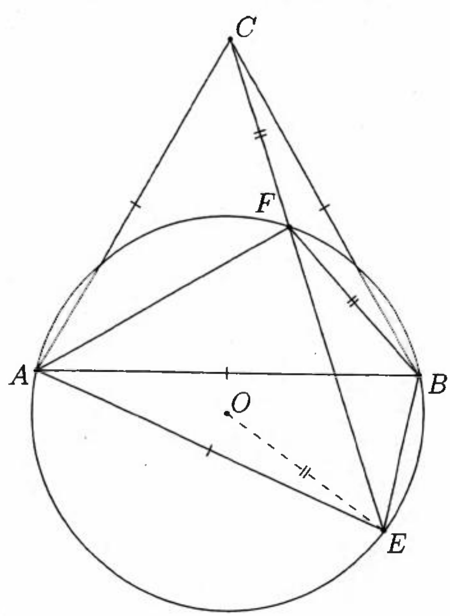
Let $AF \cap BC = M'$. We aim to prove that $M \equiv M'$. If D is the midpoint of BC (i.e. the center of the circle), then $\widehat{CE} = \widehat{EF} = \widehat{FB}$ gives that $\triangle BFD$, $\triangle FDE$ and $\triangle EDC$ are equilateral.



Therefore, $DF = DB = \dfrac{BC}{2} = \dfrac{AB}{2}$ and $DF \parallel AB$. Finally, the intercept theorem gives

$$\frac{BM'}{M'D} = \frac{AB}{DF} = 2 \Rightarrow BM' = \frac{2}{3}BD = \frac{1}{3}BC \Rightarrow M \equiv M'.$$

Problem 8.1.7. Let ABC be an equilateral triangle. Let E be a point such that A is the circumcenter of $\triangle BEC$. Let the circumradius of $\triangle BEC$ be R. The circumcircle of $\triangle ABE$ intersects the segment CE at the point F. Prove that $CF = R$.



Solution. Since A is the circumcenter of $\triangle BEC$, we get

$$\angle BCF = \frac{\angle BAE}{2} = \frac{\angle BFE}{2}.$$

Therefore,

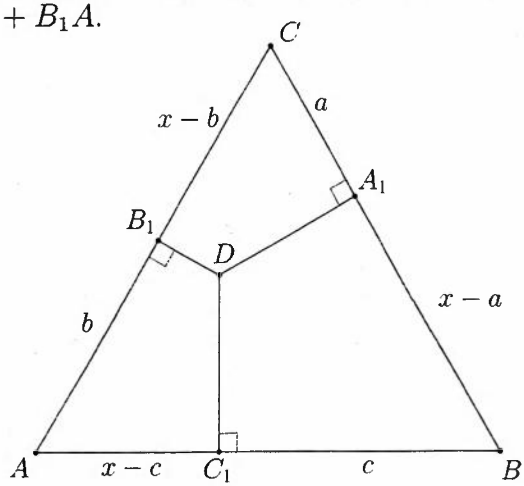
$$\angle BCF = \angle CBF \Rightarrow CF = FB.$$

We also have that

$$\angle BEF = \angle BEC = \frac{\angle BAC}{2} = 30^\circ.$$

Finally, the law of sines, applied to $\triangle BEF$, gives $CF = FB = 2R \sin 30^\circ = R$.

Problem 8.1.8. Let ABC be an equilateral triangle. The point D lies inside of the triangle. The feet of the perpendiculars from D to the sides BC, AC and AB are A_1, B_1 and C_1, respectively. Prove that $AC_1 + BA_1 + CB_1 = C_1B + A_1C + B_1A$.



Solution. Denote

$$AB = BC = CA = x,$$
$$BC_1 = c,$$
$$CA_1 = a,$$
$$AB_1 = b.$$

Carnot's Theorem (Problem 4.9.16) yields

$$a^2 + b^2 + c^2 = (x-a)^2 + (x-b)^2 + (x-c)^2$$
$$\Rightarrow a + b + c = \frac{3}{2}x \Rightarrow (x-a) + (x-b) + (x-c) = \frac{3}{2}x = a + b + c.$$

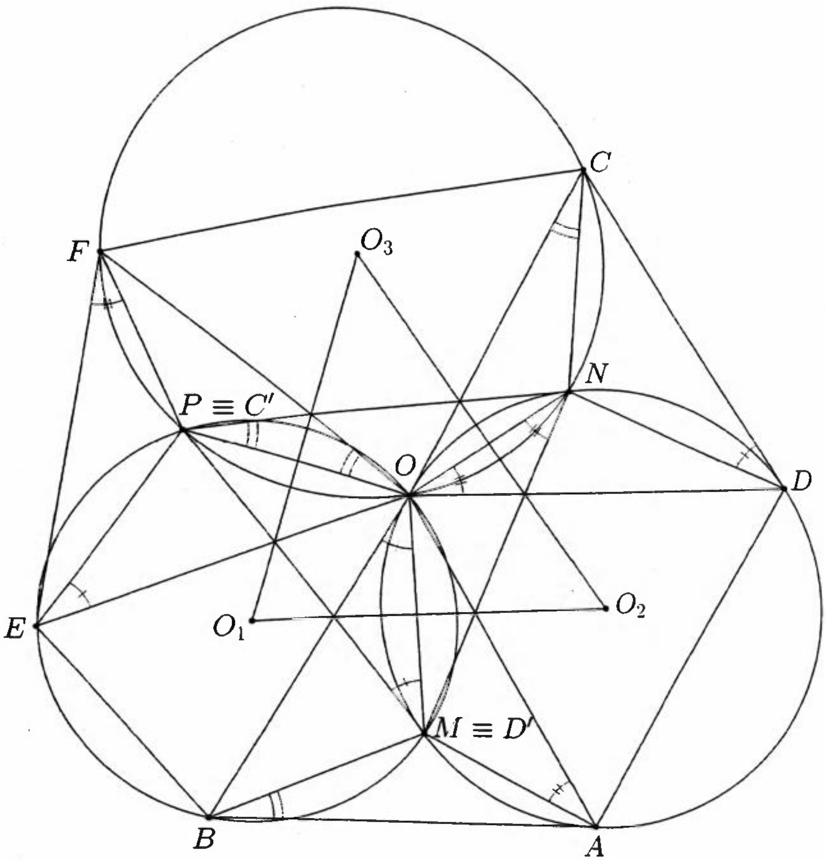
Therefore, $AC_1 + BA_1 + CB_1 = C_1B + A_1C + B_1A$.

Problem 8.1.10. Let ABO, CDO and EFO be congruent equilateral triangles, as shown in the figure. The circumcircles of $\triangle FCO$ and $\triangle EOB$ intersect at the points O and P. The circumcircles of $\triangle FCO$ and $\triangle AOD$ intersect at the points O and N. The circumcircles of $\triangle AOD$ and $\triangle EOB$ intersect at the points O and M. Prove that $\triangle MNP$ is equilateral and congruent to the three given triangles.

Solution. Let the circumcenters of $\triangle BOE$, $\triangle AOD$ and $\triangle COF$ be O_1, O_2 and O_3, respectively. Consider the symmetry with respect to the line O_2O_3. It is clear that $N \to O$. Let $D \to D'$ and $C \to C'$.

It is easy to see that D' lies on the circumcircle of $\triangle AOD$. Moreover, C' lies on the circumcircle of $\triangle COF$.

Thus, $\triangle D'ON \cong \triangle DNO$ and $\triangle C'ON \cong \triangle CNO$. Hence, $C'N = OC = OD = ND'$ and we also have $\angle C'ND' = \angle C'NO + \angle D'NO = \angle CON + \angle DON = 60°$. Therefore, $\triangle NC'D' \cong \triangle COD$.



We will prove that $C' \equiv P$ and $D' \equiv M$. It suffices to show that the points C' and D' lie on the circumcircle of $\triangle EOB$.

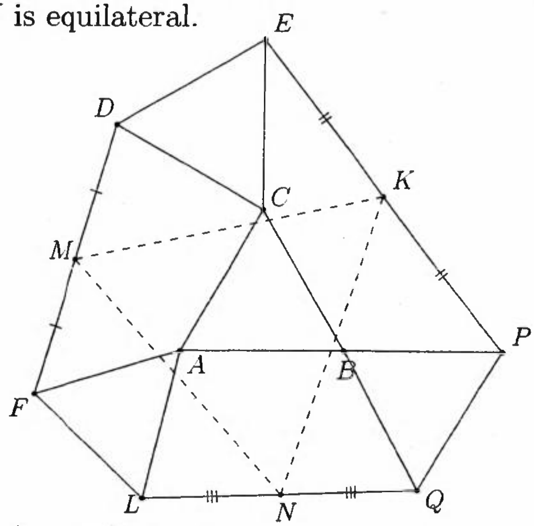
Note that $\angle NOD = \angle OND' = \angle OAD'$, $\angle OD'A = 180^\circ - \angle ODA = 180^\circ - \angle OAD = \angle OND$ and $OA = OD$. Hence, $\triangle OND \cong \triangle AD'O$.

Furthermore, we have that $\triangle OCD \cong \triangle ABO$, so $\triangle D'BO \cong \triangle NCD$. Therefore, $\angle CND = \angle OD'B$. Denote $\angle ODA = \angle OAD = \alpha$ and $\angle OCF = \angle OFC = \beta$. Then $\angle ONC = 180^\circ - \beta$ and $\angle OND = 180^\circ - \alpha$. Hence, $\angle OD'B = \angle CND = \alpha + \beta$. On the other hand,

$$\begin{aligned}\angle OEB &= 90^\circ - \frac{\angle EOB}{2} = 90^\circ - \frac{180^\circ - \angle AOD - \angle COF}{2} \\ &= \frac{\angle AOD + \angle COF}{2} = \frac{180^\circ - 2\alpha + 180^\circ - 2\beta}{2} \\ &= 180^\circ - \alpha - \beta = 180^\circ - \angle OD'B.\end{aligned}$$

Therefore, D' lies on the circumcircle of $\triangle OEB$. Analogously, C' lies on the same circle.

Problem 8.1.11. Let ABC be an equilateral triangle. The equilateral triangles $\triangle CED$, $\triangle AFL$ and $\triangle BQP$ have the same orientation as $\triangle ABC$. Let M, K and N be the midpoints of DF, EP and LQ, respectively. Prove that $\triangle MKN$ is equilateral.



Solution. We will use vector rotation. We have

$$\overrightarrow{MK} = \frac{\overrightarrow{DE} + \overrightarrow{FP}}{2} = \frac{\overrightarrow{DE} + \overrightarrow{FA} + \overrightarrow{AB} + \overrightarrow{BP}}{2}.$$

We rotate each vector in this equality by $+60°$, so it transforms into

$$\overrightarrow{N'K} = \frac{\overrightarrow{CE} + \overrightarrow{LA} + \overrightarrow{AC} + \overrightarrow{QP}}{2} = \frac{\overrightarrow{LE} + \overrightarrow{QP}}{2} = \overrightarrow{NK} \Rightarrow N' \equiv N.$$

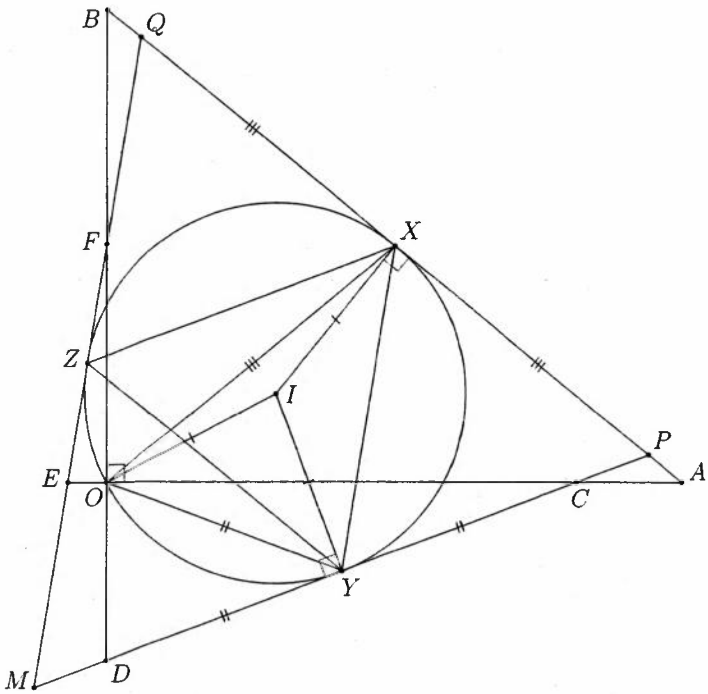
(Here we have defined N' to be the point such that $\triangle N'KM$ is equilateral and has the same orientation as $\triangle ABC$.)

Problem 8.1.12. Two perpendicular lines x and y intersect each other at the point O. Let k be a circle with center I that passes through O. The points $A \in x$ and $B \in y$ are chosen, such that the points I and A lie in the same half-plane with respect to the line y, the points I and B lie in the same half-plane with respect to the line x, AB touches k at the point X and $XA = XB$. The points $C \in x$, $D \in y$, $E \in x$ and $F \in y$ are defined analogously. Prove that the intersections of the lines AB, CD and EF are the vertices of an equilateral triangle.

Solution. We will use the notations from the figure. Denote $\angle AOI = \alpha$, $\angle OCD = \varphi$ and $\angle BAO = \psi$. The triangles $\triangle COD$ and $\triangle AOB$ are right-angled and hence $CY = YD = YO$ and $AX = XB = XO$. Therefore,

$$\angle COY = \varphi \Rightarrow \angle OYD = 2\varphi \Rightarrow \angle OYI = 90° - 2\varphi,$$

$$\Rightarrow \angle YOI = 90° - 2\varphi \Rightarrow \varphi + \alpha = 90° - 2\varphi \Rightarrow \varphi = 30° - \frac{\alpha}{3}.$$



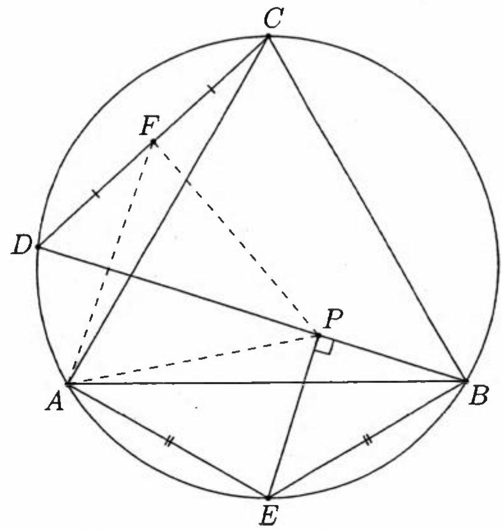
Also, we have that $\angle AOX = \psi$, and thus $\angle OXB = 2\psi$. Hence, $\angle OXI = 90° - 2\psi$, and therefore $\angle XOI = 90° - 2\psi$. It follows that $\psi - \alpha = 90° - 2\psi$, and thus $\psi = 30° + \dfrac{\alpha}{3}$. Hence,

$$\angle MPQ = \angle OCD + \angle BAO = \varphi + \psi = 30° - \frac{\alpha}{3} + 30° + \frac{\alpha}{3} = 60°.$$

Analogously, we deduce that $\angle MQP = 60°$. Therefore, $\triangle MPQ$ is equilateral, which means that $\triangle XYZ$ is equilateral too.

Problem 8.1.13. Let ABC be an equilateral triangle. The point D lies on the smaller arc $\overset{\frown}{AC}$ of the circumcircle of $\triangle ABC$, and the point E bisects the smaller arc $\overset{\frown}{AB}$. Let P be the foot of the perpendicular from E to BD, and let F be the midpoint of the segment CD. Prove that $\triangle APF$ is equilateral.

Solution. If $\angle DBC = \alpha$, then $\angle PEB = \alpha$. Let R be the circumradius of $\triangle ABC$. The law of sines yields that



$$CF = \frac{CD}{2} = \frac{2R\sin\alpha}{2}$$
$$= R\sin\alpha = EB\sin\alpha = PB.$$

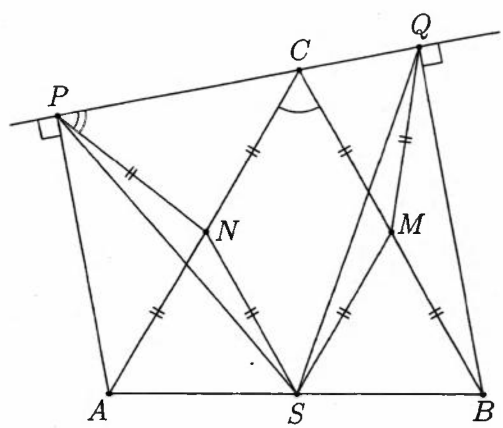
Moreover, note that

$$\angle ABP = \angle ACF.$$

Since $AB = AC$, we get that $\triangle ABP \cong \triangle ACF$. Hence, $AP = AF$ and $\angle FAP = \angle CAB = 60°$. Therefore, $\triangle APF$ is equilateral.

Problem 8.1.14. Let ABC be an equilateral triangle and let l be an arbitrary line through the vertex C. Let P and Q be the feet of the perpendiculars from A and B to l, respectively. If S is the midpoint of AB, prove that $\triangle PQS$ is equilateral.

Solution. Let M and N be the midpoints of BC and AC, respectively. Considering the midsegments and the fact that $\triangle BQC$ and $\triangle APC$ are right-angled, we get that $MQ = MB = MC = MS = SN = AN = NC = NP$.

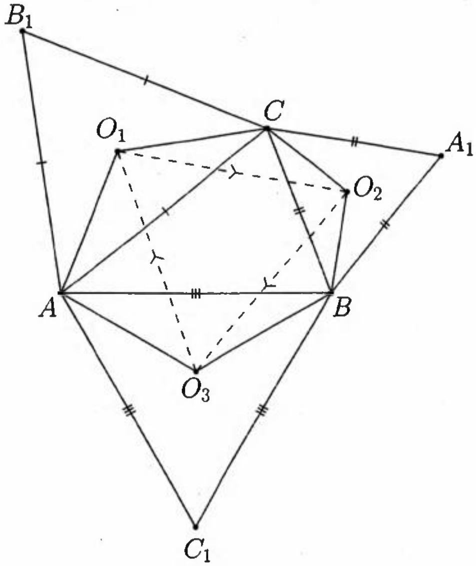


Moreover,

$$\begin{aligned}\angle CMQ &= 180° - 2\angle MCQ\\ &= 180° - 2(120° - \angle PCN)\\ &= 2\angle PCN - 60°\\ &= 120° - \angle PNC.\end{aligned}$$

Hence, $\angle SMQ = 120° + \angle CMQ = 240° - \angle PNC = \angle PNS$. We deduce that $\triangle PNS \cong \triangle QMS$ and $PS = SQ$. But $\angle PSQ = \angle NSM = \angle NCM = 60°$ and therefore $\triangle PQS$ is equilateral.

Problem 8.1.15. Let ABC be a triangle. The equilateral triangles ABC_1, A_1BC and AB_1C are constructed externally to its sides. These three triangles have circumcenters O_3, O_2 and O_1, respectively. Prove that $\triangle O_1O_2O_3$ is equilateral.



Solution. Note that $\angle O_3BO_2 = \angle ABA_1$. Also, $\triangle ABO_3 \sim \triangle BA_1O_2$. Hence, $\dfrac{BO_3}{BO_2} = \dfrac{BA}{BA_1}$. It follows that $\triangle ABA_1 \sim \triangle O_3BO_2$.

Therefore, $O_3O_2 = \dfrac{AA_1.O_3B}{AB}$.

Analogously, $O_1O_2 = \dfrac{BB_1.O_1C}{AC}$.

We have that $CA_1 = CB$, $CA = CB_1$ and $\angle ACA_1 = \angle B_1CB$. Therefore, $\triangle B_1CB \cong \triangle ACA_1$, which gives us that $AA_1 = BB_1$.

Also, $\triangle ABO_3 \sim \triangle ACO_1$ and hence $\dfrac{O_3O_2}{O_1O_2} = \dfrac{O_3B}{O_1C} \cdot \dfrac{AC}{AB} = 1$.

Analogously, $O_1O_2 = O_1O_3$.

Problem 8.1.16. Let ABC be a triangle. The equilateral triangles ABF, ACD and BCE are constructed externally. Let M and N be the midpoints of DF and EF, respectively. Prove that $\triangle MNC$ is equilateral.

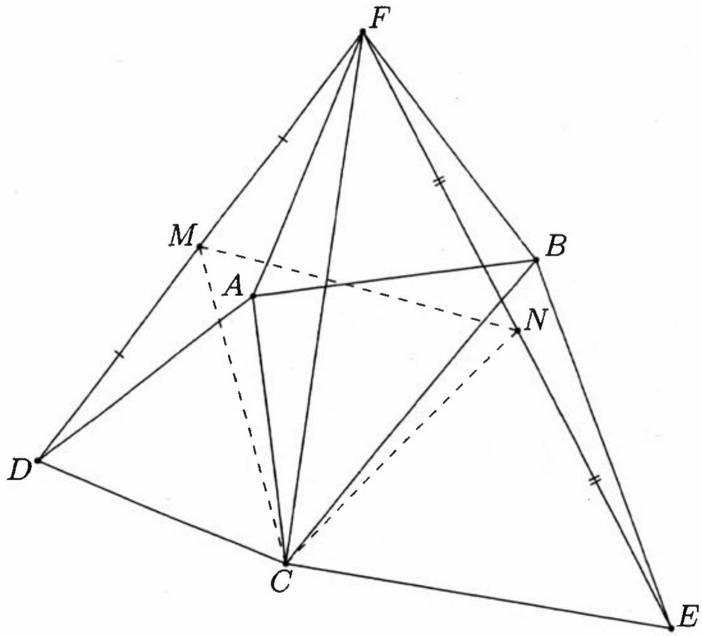
Solution. Let M' be the point such that $\triangle NCM'$ is equilateral and has the same orientation as $\triangle NCM$. Then we have

$$\overrightarrow{CN} = \frac{1}{2}(\overrightarrow{CE} + \overrightarrow{CF}) = \frac{\overrightarrow{CE} + \overrightarrow{CA} + \overrightarrow{AF}}{2}.$$

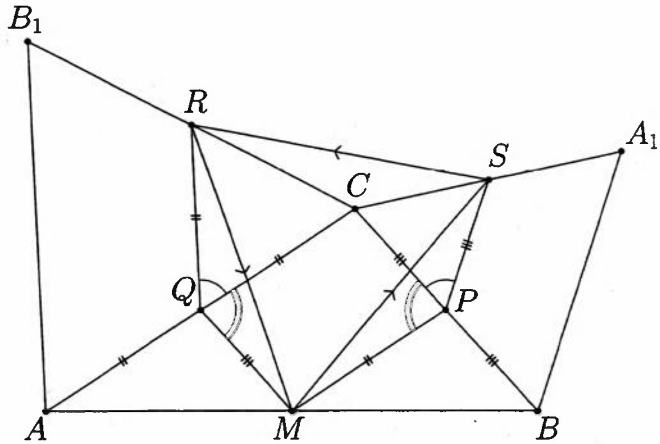
A $+60°$ rotation of all vectors in the above equality gives

$$\overrightarrow{CM'} = \frac{\overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{BF}}{2} = \frac{1}{2}(\overrightarrow{CD} + \overrightarrow{CF}) = \overrightarrow{CM}.$$

Therefore, $M' \equiv M$ and $\triangle MNC$ is equilateral.



Problem 8.1.17. Let ABC be a triangle. The equilateral triangles BCA_1 and AB_1C are constructed externally to its sides. Let R, S and M be the midpoints of B_1C, A_1C and AB, respectively. Prove that $\triangle MSR$ is equilateral.



Solution. Let P and Q be the midpoints of BC and AC, respectively. Considering the midsegments, we get $MQ \parallel BC$ and $MP \parallel AC$, which implies that $\angle MQC = \angle MPC$. On the other hand, RQ and SP are midsegments in $\triangle ACB_1$ and $\triangle A_1BC$, respectively. Then $\angle RQC = \angle SPC = 60°$. Now we have that $RQ = \dfrac{AC}{2} = MP$ and $SP = \dfrac{BC}{2} = QM$.

We deduce that $\angle RQM = \angle MPS$ and hence $\triangle MQR \cong \triangle SPM$. Therefore, $MR = MS$. Moreover,

$$\begin{aligned}\angle SMR &= \angle PMQ - \angle PMS - \angle RMQ \\ &= \angle ACB - \angle PMS - \angle MSP \\ &= \angle ACB - 180^\circ + \angle MPC + \angle CPS \\ &= 180^\circ - 180^\circ + 60^\circ = 60^\circ.\end{aligned}$$

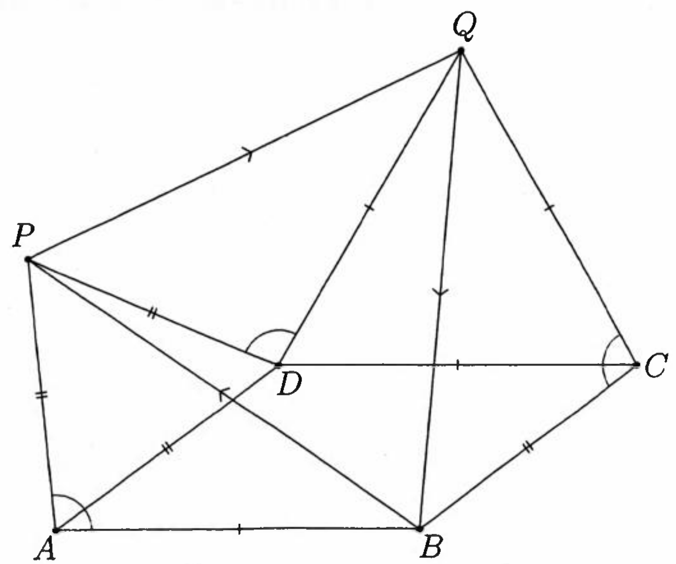
We conclude that $\triangle MSR$ is equilateral.

Problems 8.1.18 and 8.1.19. Let $ABCD$ be a parallelogram. The equilateral triangles CDQ and DAP are constructed externally. Prove that $\triangle BPQ$ is equilateral.

Solution. It is easy to see that $AB = DC = CQ = DQ$ and $BC = AD = AP = PD$. Also,

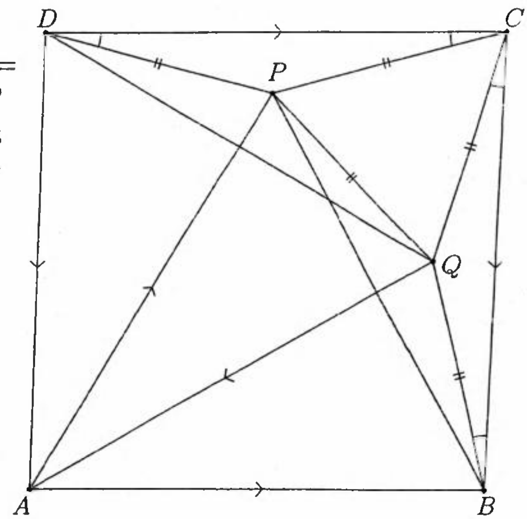
$$\begin{aligned}\angle BCQ &= \angle BCD + 60^\circ = \angle BAD + 60^\circ = \angle BAP \\ &= 180^\circ - \angle ADC + 60^\circ = 360^\circ - \angle ADC - 120^\circ = \angle PDQ,\end{aligned}$$

and hence $\triangle BAP \cong \triangle QCB \cong \triangle QDP$. We deduce that $BP = BQ = PQ$. Therefore, $\triangle BPQ$ is equilateral.



Note. In Problem 8.1.19 the parallelogram $ABCD$ is a square.

Problem 8.1.20. Let $ABCD$ be a square. The equilateral triangles ABP and ADQ are constructed internally. Prove that $\triangle CPQ$ is equilateral.



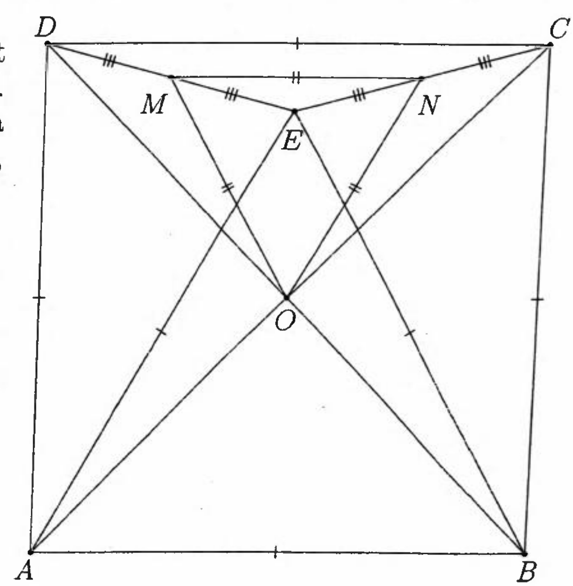
Solution. We have that $\angle PAD = 30°$, $AP = AD$, $\angle PBC = 30°$ and $BP = BC$. It follows that $\triangle PAD \cong \triangle PBC$ and hence $PC = PD$.

Now it is easy to see that $\triangle CPD \cong \triangle CQB$. Then

$$PC = PD = CQ = QB.$$

Note that $\angle QCP = 60°$ and $CP = CQ$, which means that $\triangle CPQ$ is equilateral.

Problem 8.1.21. Let $ABCD$ be a square with center O. The equilateral triangle ABE is constructed internally. Let M and N be the midpoints of ED and EC, respectively. Prove that $\triangle OMN$ is equilateral.



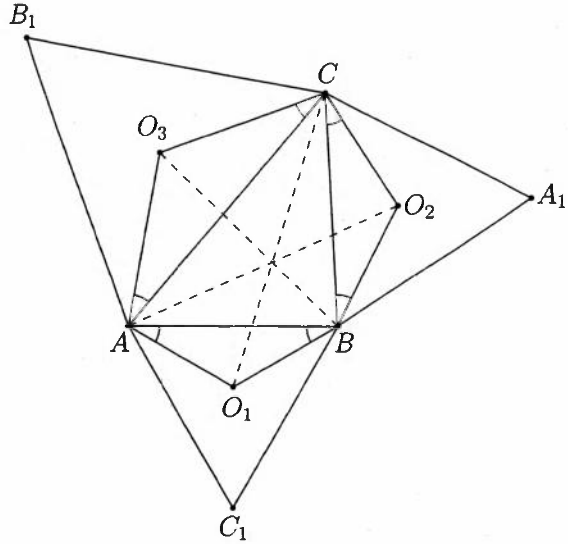
Solution. We have that $AE = BE = AB = CD$. On the other hand, OM is a midsegment in $\triangle BED$. Thus, $OM = \dfrac{BE}{2}$.

Also, ON is a midsegment in $\triangle AEC$. Hence, $ON = \dfrac{AE}{2}$.

Therefore, $ON = OM$. The midsegment MN gives $MN = \dfrac{CD}{2}$.

In conclusion, $OM = ON = MN$ and $\triangle OMN$ is equilateral.

Problem 9.1. Let ABC be a triangle. The equilateral triangles ABC_1, A_1BC and AB_1C are constructed externally. These three triangles have circumcenters O_1, O_2 and O_3, respectively. Prove that the lines AO_2, BO_3 and CO_1 are concurrent.



Solution. It is easy to see that

$$\angle O_1AB = \angle O_1BA = 30^\circ,$$
$$\angle O_2BC = \angle O_2CB = 30^\circ,$$
$$\angle O_3CA = \angle O_3AC = 30^\circ.$$

Now Problem 4.9.5 yields that the lines AO_2, BO_3 and CO_1 are concurrent.

Problem 9.2. Let $ABCD$ be a convex quadrilateral with diagonals equal in length. The equilateral triangles ABM, BCN, CDK and DAL are constructed externally. These four triangles have centers O_1, O_2, O_3 and O_4, respectively. Prove that $O_1O_3 \perp O_2O_4$.

Solution. Note that $\dfrac{AO_4}{AL} = \dfrac{1}{\sqrt{3}} = \dfrac{AO_1}{AB}$. Hence, if we consider two spiral similarities with center A, ratio $\sqrt{3}$ and angles $\pm 30^\circ$, the image of ΔO_1AO_4 is going to be ΔBAL under the first and ΔMAD under the second. Therefore, $BL = MD = \sqrt{3}O_1O_4$.

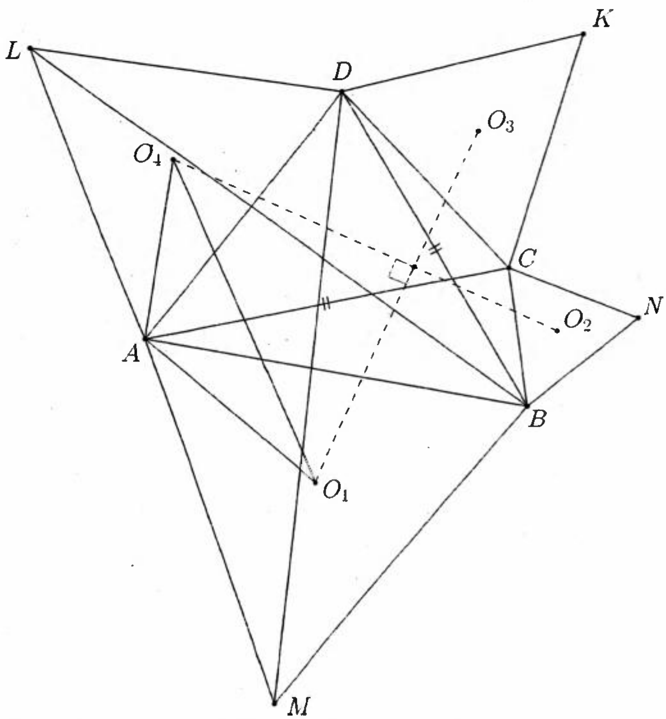
Analogously, $DN = BK = \sqrt{3}O_2O_3$, $MC = AN = \sqrt{3}O_1O_2$ and $AK = LC = \sqrt{3}O_3O_4$.

To prove the desired statement we will use the well-known lemma that $O_1O_3 \perp O_2O_4$ if and only if

$$O_1O_2^2 + O_3O_4^2 = O_1O_4^2 + O_2O_3^2.$$

Its proof consists of applying the law of cosines four times and manipulating the equalities. The required equality is equivalent to

$$BL^2 + MD^2 + DN^2 + BK^2 = MC^2 + AN^2 + AK^2 + CL^2.$$



Considering the difference of both sides, applying the law of cosines eight times, and using area formulas and dot products, we get

$$\begin{aligned}
&BL^2 + MD^2 + DN^2 + BK^2 - MC^2 - AN^2 - AK^2 - CL^2 \\
&= BD^2 + DL^2 - 2BD.DL\cos(60^\circ + \angle ADB) \\
&\quad + BD^2 + MB^2 - 2BD.MB\cos(60^\circ + \angle ABD) \\
&\quad + BD^2 + BN^2 - 2BD.BN\cos(60^\circ + \angle DBC) \\
&\quad + BD^2 + DK^2 - 2BD.DK\cos(60^\circ + \angle BDC) \\
&\quad - AC^2 - AM^2 + 2AM.AC\cos(60^\circ + \angle CAB) \\
&\quad - AC^2 - CN^2 + 2AC.CN\cos(60^\circ + \angle BCA) \\
&\quad - AC^2 - CK^2 + 2CK.AC\cos(60^\circ + \angle ACD) \\
&\quad - AC^2 - AL^2 + 2AL.AC\cos(60^\circ + \angle DAC) \\
&= 2(-AD.BD\cos(60^\circ + \angle ADB) - BD.AB\cos(60^\circ + \angle ABD) \\
&\quad - BD.BC\cos(60^\circ + \angle DBC) - BD.DC\cos(60^\circ + \angle BDC) \\
&\quad + AB.AC\cos(60^\circ + \angle BAC) + BC.AC\cos(60^\circ + \angle BCA) \\
&\quad + CD.AC\cos(60^\circ + \angle ACD) + AD.AC\cos(60^\circ + \angle DAC)) =
\end{aligned}$$

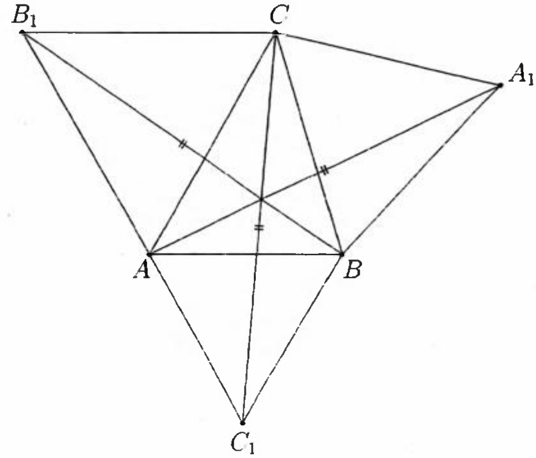
$$\begin{aligned}
&= 2\cos 60^\circ(-AD.BD\cos\angle ADB - BD.AB\cos\angle ABD \\
&\quad - BD.BC\cos\angle DBC - BD.DC\cos\angle BDC \\
&\quad + AB.AC\cos\angle BAC + BC.AC\cos\angle BCA \\
&\quad + CD.AC\cos\angle ACD + AD.AC\cos\angle DAC) \\
&\quad + 2\sin 60^\circ(2S_{ABD} + 2S_{ABD} + 2S_{BCD} + 2S_{BCD} \\
&\quad - 2S_{ABC} - 2S_{ABC} - 2S_{ACD} - 2S_{ACD}) \\
&= 2\cos 60^\circ(-\overrightarrow{DA}.\overrightarrow{DB} - \overrightarrow{BA}.\overrightarrow{BD} - \overrightarrow{BC}.\overrightarrow{BD} - \overrightarrow{DC}.\overrightarrow{DB} \\
&\quad + \overrightarrow{AB}.\overrightarrow{AC} + \overrightarrow{CB}.\overrightarrow{CA} + \overrightarrow{CD}.\overrightarrow{CA} + \overrightarrow{AD}.\overrightarrow{AC}) \\
&= 2\cos 60^\circ(\overrightarrow{BD}(\overrightarrow{DA} - \overrightarrow{BA} - \overrightarrow{BC} + \overrightarrow{DC}) + \overrightarrow{AC}(\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{CD} + \overrightarrow{AD})) \\
&= 2\cos 60^\circ(\overrightarrow{BD}(\overrightarrow{DB} + \overrightarrow{DB}) + \overrightarrow{AC}(\overrightarrow{AC} + \overrightarrow{AC})) \\
&= 4\cos 60^\circ(-(\overrightarrow{BD})^2 + (\overrightarrow{AC})^2) = 4\cos 60^\circ(-BD^2 + AC^2) = 0.
\end{aligned}$$

Problem 9.3. Let ABC be a triangle. The equilateral triangles ABC_1, A_1BC and AB_1C are constructed externally. Prove that $AA_1 = BB_1 = CC_1$.

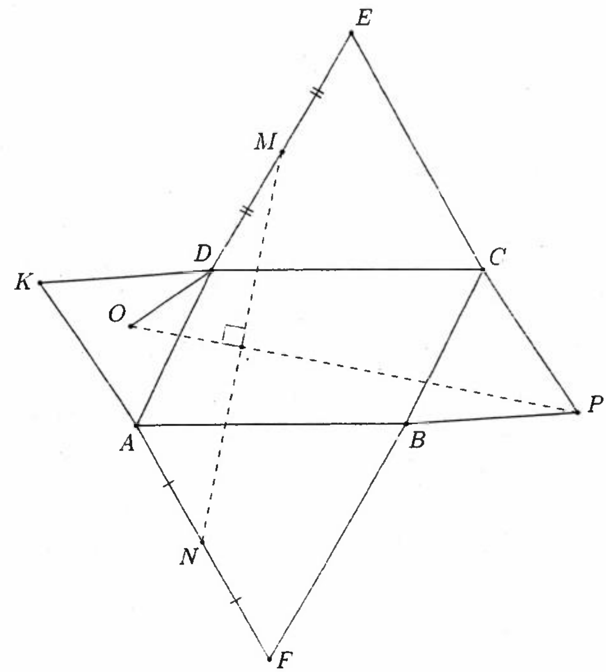
Solution. Note that $AB = BC_1$, $BC = BA_1$ and $\angle ABA_1 = \angle C_1BC$. These equalities imply that

$$\triangle ABA_1 \cong \triangle C_1BC.$$

Hence, $CC_1 = AA_1$. Analogously, we get that $AA_1 = BB_1$.



Problem 9.4. Let $ABCD$ be a parallelogram. The equilateral triangles ABF, BCP, CDE and DAK are constructed externally. Let O be the center of $\triangle DAK$, and let M and N be the midpoints of DE and AF, respectively. Prove that $OP \perp MN$.

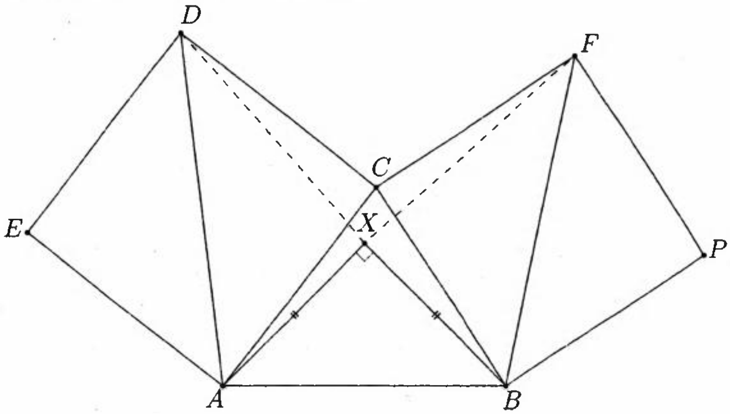


Solution. It suffices to prove that $\overrightarrow{NM}.\overrightarrow{PO} = 0$. Denote $\angle BAD = \alpha$ and $AB = a$, $AD = b$. We consecutively have

$$\begin{aligned}
\overrightarrow{NM}.\overrightarrow{PO} &= (\overrightarrow{NA} + \overrightarrow{AD} + \overrightarrow{DM})(\overrightarrow{PC} + \overrightarrow{CD} + \overrightarrow{DO}) \\
&= \left(\frac{\overrightarrow{FA}}{2} + \overrightarrow{AD} + \frac{\overrightarrow{DE}}{2}\right)\left(\overrightarrow{PC} + \overrightarrow{CD} + \frac{\overrightarrow{DK}}{3} + \frac{\overrightarrow{DA}}{3}\right) \\
&= \frac{\overrightarrow{FA}.\overrightarrow{PC}}{2} + \frac{\overrightarrow{FA}.\overrightarrow{CD}}{2} + \frac{\overrightarrow{FA}.\overrightarrow{DK}}{6} + \frac{\overrightarrow{FA}.\overrightarrow{DA}}{6} \\
&\quad + \overrightarrow{AD}.\overrightarrow{PC} + \overrightarrow{AD}.\overrightarrow{CD} + \frac{\overrightarrow{AD}.\overrightarrow{DK}}{3} + \frac{\overrightarrow{AD}.\overrightarrow{DA}}{3} \\
&\quad + \frac{\overrightarrow{DE}.\overrightarrow{PC}}{2} + \frac{\overrightarrow{DE}.\overrightarrow{CD}}{2} + \frac{\overrightarrow{DE}.\overrightarrow{DK}}{6} + \frac{\overrightarrow{DE}.\overrightarrow{DA}}{6} \\
&= \frac{ab\cos(\alpha - 60^\circ)}{2} + \frac{a^2\cos 60^\circ}{2} + \frac{ab\cos\alpha}{6} + \frac{ab\cos(60^\circ + \alpha)}{6} \\
&\quad + b^2\cos 60^\circ + ab\cos(180^\circ - \alpha) + \frac{b^2\cos 120^\circ}{3} - \frac{b^2}{3} \\
&\quad + \frac{ab\cos\alpha}{2} + \frac{a^2\cos 120^\circ}{2} + \frac{ab\cos(60^\circ + \alpha)}{6} + \frac{ab\cos(240^\circ - \alpha)}{6} =
\end{aligned}$$

$$\begin{aligned}
&= \frac{ab\cos(60^\circ-\alpha)}{2}+\frac{ab\cos\alpha}{6}+\frac{ab\cos(60^\circ+\alpha)}{6}-ab\cos\alpha\\
&\quad+\frac{ab\cos\alpha}{2}+\frac{ab\cos(60^\circ+\alpha)}{6}-\frac{ab\cos(60^\circ-\alpha)}{6}\\
&= \frac{ab}{6}(-2\cos\alpha+2\cos(60^\circ-\alpha)+2\cos(60^\circ+\alpha))\\
&= \frac{ab}{6}(-2\cos\alpha+2(\cos 60^\circ\cos\alpha+\sin 60^\circ\sin\alpha)\\
&\quad+2(\cos 60^\circ\cos\alpha-\sin 60^\circ\sin\alpha))\\
&= \frac{ab}{2}(-2\cos\alpha+\cos\alpha+\cos\alpha)=0.
\end{aligned}$$

Problem 9.5. Let ABC be a triangle. The squares $ACDE$ and $BCFP$ are constructed externally. Prove that the squares with diagonals AB and DF have a common vertex.



Solution. Let X be a point such that $\angle AXB = 90^\circ$ and $AX = BX$. Let C and X lie in the same half-plane with respect to AB. It suffices to prove that $DX = FX$ and $\angle DXF = 90^\circ$.

Note that $\angle XBF = \angle XBC + 45^\circ = \beta$. Analogously, $\angle DAX = \alpha$. Moreover, $XB = XA = \dfrac{c}{\sqrt{2}}$, $AD = b\sqrt{2}$ and $BF = a\sqrt{2}$.

The law of cosines gives

$$\begin{aligned}
XF^2 - XD^2 &= \frac{c^2}{2}+2a^2-2ac\cos\beta-\frac{c^2}{2}-2b^2+2bc\cos\alpha\\
&= 2a^2-2b^2-(a^2+c^2-b^2)+(b^2+c^2-a^2)=0,
\end{aligned}$$

whence $XF = XD$.

The cotangent technique, applied to $\triangle AXD$ and $\triangle BXF$, gives

$$\cot \angle BXF + \cot \angle AXD = \frac{\dfrac{BX}{BF} - \cos\beta}{\sin\beta} + \frac{\dfrac{AX}{AD} - \cos\alpha}{\sin\alpha}$$

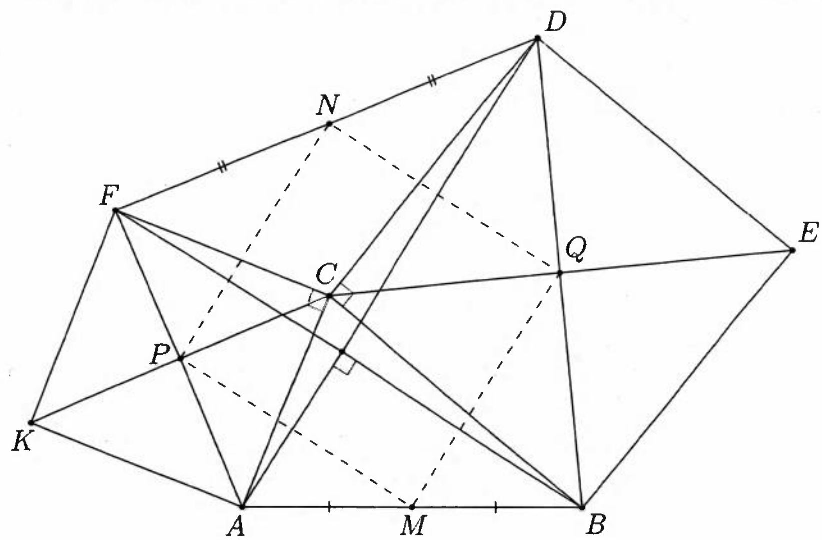
$$= \frac{\dfrac{c}{2a} - \cos\beta}{\sin\beta} + \frac{\dfrac{c}{2b} - \cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\gamma - 2\sin\alpha\cos\beta}{2\sin\alpha\sin\beta} + \frac{\sin\gamma - 2\sin\beta\cos\alpha}{2\sin\beta\sin\alpha}$$

$$= \frac{\sin\gamma - (\sin\alpha\cos\beta + \sin\beta\cos\alpha)}{\sin\alpha\sin\beta} = 0 \Rightarrow \cot\angle BXF + \cot\angle AXD = 0.$$

Therefore, $\angle BXF + \angle AXD = 180°$, and so $\angle DXF = 90°$.

Problem 9.6. Let ABC be a triangle. The squares $ACFK$ and $BCDE$ are constructed externally. Let P and Q be the centers of $ACFK$ and $BCDE$, respectively. The points M and N are the midpoints of AB and FD, respectively. Prove that the quadrilateral $MQNP$ is a square.



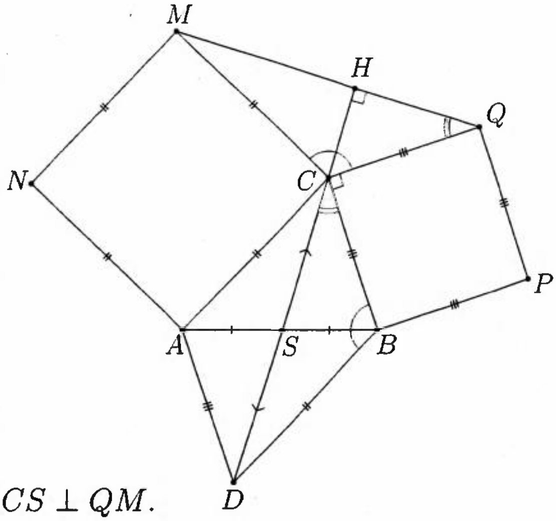
Solution. We have that $CF = CA$, $CB = CD$ and $\angle ACD = 90° + \angle ACB = \angle BCF$, which implies that $\triangle BCF \cong \triangle DCA$.

Therefore, the angle between the lines AD and BF is equal to the angle between the lines AC and CF. Hence, $AD \perp BF$.

Since the points M, Q, N and P are the midpoints of the sides of $ABDF$, the respective midsegments give $NQ = \frac{BF}{2} = \frac{AD}{2} = MQ = MP = PN$.

Moreover, $NQ \parallel BF$ and $MQ \parallel AD$. Thus, the angle between the lines NQ and QM equals the angle between the lines BF and AD (which is right). In conclusion, $MQNP$ is a rhombus with a right angle. Hence, it is a square.

Problem 9.7. Let ABC be a triangle. The squares $ACMN$ and $BCQP$ are constructed externally. Let S be the midpoint of AB. Prove that $CS \perp QM$.

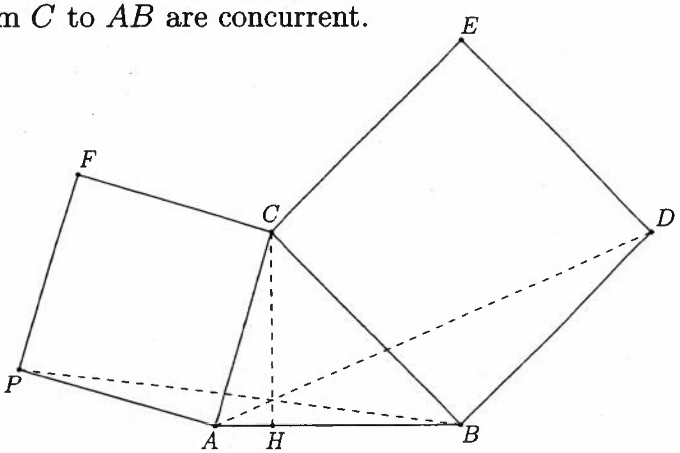


Solution. We have that $\angle MCQ = 180° - \gamma$.

Let D be the symmetric point of C with respect to the point S. Then $ACBD$ is a parallelogram. Thus, $BD = AC = MC$ and $\angle CBD = 180° - \gamma = \angle MCQ$.

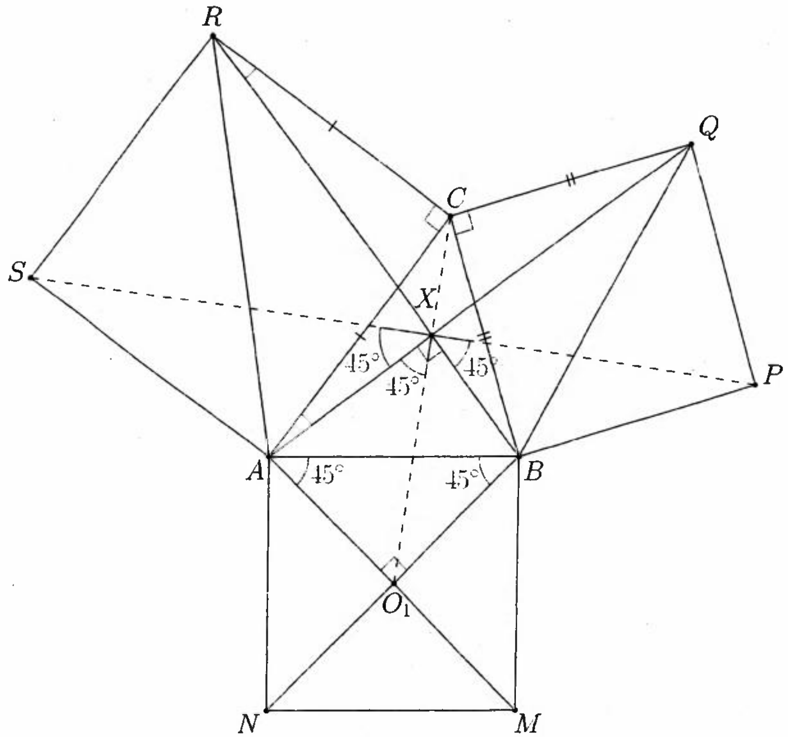
We deduce that $\triangle MCQ \cong \triangle DBC$. Let $DC \cap MQ = H$. Then $\angle CHQ = 180° - \angle MQC - \angle HCQ = 180° - \angle DCB - \angle HCQ = 90°$, giving $CS \perp QM$.

Problem 9.8. Let ABC be a triangle. The squares $BCED$ and $ACFP$ are constructed externally. Prove that the lines AD, BP and the altitude from C to AB are concurrent.



Solution. This problem is a special case of Problem 4.12.7 ($\varphi = 45°$).

Problem 9.9. Let ABC be a triangle. The squares $ABMN$, $BPQC$ and $ACRS$ are constructed externally. Let O_1 be the center of $ABMN$. Prove that the lines CO_1, BR, AQ and PS are concurrent.

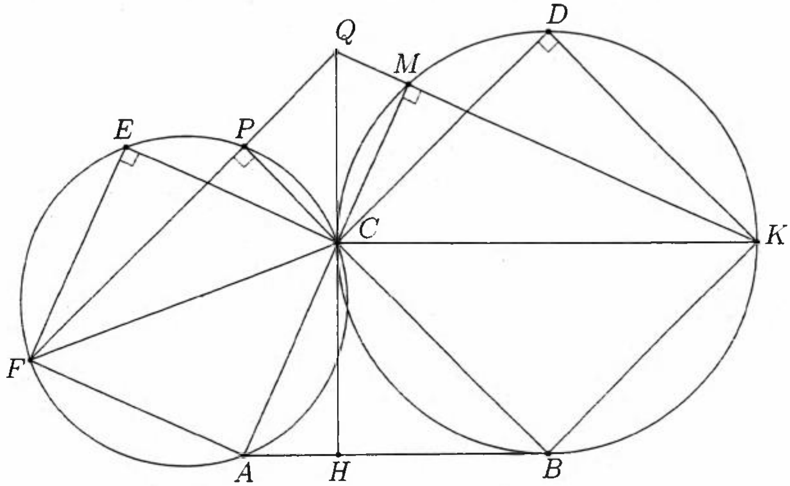


Solution. Let $AQ \cap BR = X$. Then $\triangle BCR \cong \triangle QCA$ due to the equalities $\angle BCR = \angle QCA$, $BC = QC$ and $AC = CR$. We deduce that $\angle CAQ = \angle CRB$, which implies that the quadrilateral $CRAX$ is cyclic. Thus, the pentagon $AXCRS$ is cyclic.

Analogously, the pentagon $CXBPQ$ is cyclic too. Now we have that $\angle AXC = 135° = \angle BXC$ because $\angle ARC = \angle BQC = 45°$. Hence, $\angle AXB = 90°$ and $\angle AO_1B = 90°$, which means that the quadrilateral AO_1BX is cyclic. Therefore, $\angle AXO_1 = \angle BXO_1 = 45°$ and $\angle AXC + \angle AXO_1 = 180°$. Thus, the points C, X and O_1 are collinear.

Note that $\angle SXP = \angle SXA + \angle AXB + \angle BXP = \angle SRA + 90° + \angle PQB = 45° + 90° + 45° = 180°$, which implies that the points S, X and P are also collinear.

Problem 9.10. Let ABC be a triangle. The squares $BCDK$ and $ACEF$ are constructed externally. Prove that the perpendiculars from K to AC, from F to BC and from C to AB are concurrent.



Solution. Denote the foot of the perpendicular from K to the line AC by M, and the feet of the perpendicular from F to the line BC by P.

Let $KM \cap FP = Q$ and $QC \cap AB = H$. We will prove that CH is an altitude in $\triangle ABC$. It suffices to prove the equality $\angle CQM = \alpha$. We have

$$\begin{aligned}\angle CFQ &= 90^\circ - \angle PCF \\ &= 90^\circ - (180^\circ - 45^\circ - \gamma) \\ &= 90^\circ - \angle KCM = \angle QKC.\end{aligned}$$

Therefore,

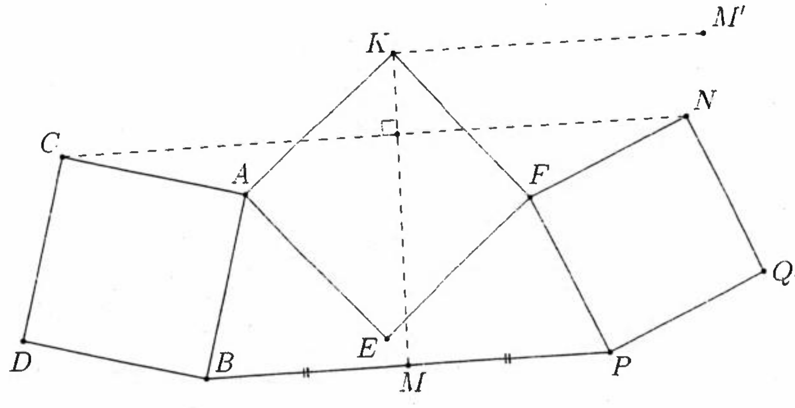
$$\frac{\sin \angle CQK}{\sin \angle CQF} = \frac{CK}{CF} \cdot \frac{\sin \angle QKC}{\sin \angle QFC} = \frac{CB}{CA} = \frac{\sin \alpha}{\sin \beta}.$$

Moreover,

$$\angle PQM = 180^\circ - \angle PCM = 180^\circ - \gamma = \alpha + \beta.$$

This implies that $\cot \angle CQM = \cot \alpha$, and so $\angle CQM = \alpha$.

Problem 9.11. Let $ABDC$, $AKFE$ and $FNQP$ be squares with the same orientation. The midpoint of BP is M. Prove that $CN \perp KM$.



Solution. We will use vector rotation. We have

$$\overrightarrow{KM} = \frac{1}{2}(\overrightarrow{KB} + \overrightarrow{KP}) = \frac{1}{2}(\overrightarrow{KA} + \overrightarrow{AB} + \overrightarrow{KF} + \overrightarrow{FP}).$$

A $+90°$ rotation of each vector in the equality gives

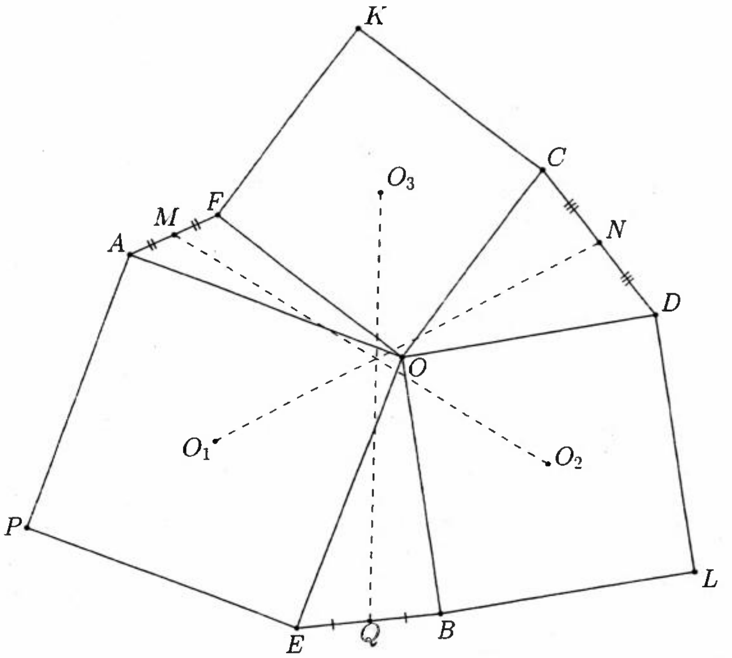
$$\overrightarrow{KM'} = \frac{1}{2}(\overrightarrow{AE} + \overrightarrow{CA} + \overrightarrow{EF} + \overrightarrow{FN}) = \frac{1}{2}\overrightarrow{CN},$$

where M' is a point such that $M'K \perp KM$ because the image of $\overrightarrow{KM}$ under this rotation is $\overrightarrow{KM'}$. We conclude that $KM' \parallel CN$ and $CN \perp KM$.

Note. We also proved the equality $2KM = CN$.

Problem 9.13. Let $OAPE$, $OBLD$ and $OCKF$ be squares with a common vertex O and the same orientation. Let their centers be O_1, O_2 and O_3, respectively. Denote the midpoints of EB, CD and AF by Q, N and M, respectively. Prove that the lines O_1N, O_2M and O_3Q are concurrent or parallel.

Solution. We will use complex numbers. Denote the center of the complex plane by O. As usual, the complex number of each point will be denoted as the small letter of its name. Let a, b and c be arbitrary. We have that $d = bi$, $f = ci$ and $e = ai$. Moreover, O_1 is the midpoint of AE, O_2 is the midpoint of BD and O_3 is the midpoint of CF. Therefore, $o_1 = \dfrac{a+ai}{2}$, $o_2 = \dfrac{b+bi}{2}$ and $o_3 = \dfrac{c+ci}{2}$. Note that $q = \dfrac{b+ai}{2}$, $n = \dfrac{c+bi}{2}$ and $m = \dfrac{a+ci}{2}$. Also, observe that $\overline{o_1} = \dfrac{\overline{a}+\overline{ai}}{2} = \dfrac{\overline{a}-\overline{a}i}{2}$. Analogously, we obtain the conjugates of the other complex numbers.



We have to verify the existence of a complex number z, such that

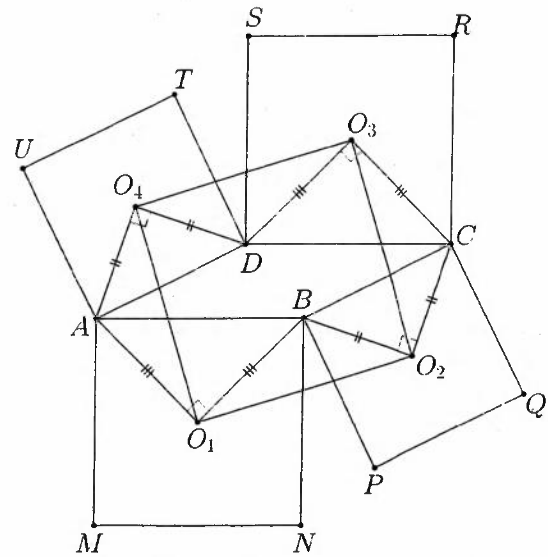
$$\left|\begin{array}{l} z(\frac{\overline{a}-\overline{a}i-\overline{c}+\overline{b}i}{2})+\overline{z}(\frac{-a-ai+c+bi}{2})+(\frac{a+ai}{2})(\frac{\overline{c}-\overline{b}i}{2})-(\frac{c+bi}{2})(\frac{\overline{a}-\overline{a}i}{2})=0 \\ z(\frac{\overline{b}-\overline{b}i-\overline{a}+\overline{c}i}{2})+\overline{z}(\frac{-b-bi+a+ci}{2})+(\frac{b+bi}{2})(\frac{\overline{a}-\overline{c}i}{2})-(\frac{a+ci}{2})(\frac{\overline{b}-\overline{b}i}{2})=0 \\ z(\frac{\overline{c}-\overline{c}i-\overline{b}+\overline{a}i}{2})+\overline{z}(\frac{-c-ci+b+ai}{2})+(\frac{c+ci}{2})(\frac{\overline{b}-\overline{a}i}{2})-(\frac{b+ai}{2})(\frac{\overline{c}-\overline{c}i}{2})=0 \end{array}\right.$$

The solutions to the first equation describe the points z which lie on the line O_1N, and the analogous statement is true for the other two equations. If no two of these equations have a common solution, then the lines O_1N, O_2M and O_3Q are parallel. Now, without loss of generality, let z_0 be a mutual solution of the first two equations, i.e. $O_1N \cap O_2M = Z_0$. We multiply each of them by (-1) and we add them. This gives

$$z_0(\frac{\overline{c}-\overline{c}i-\overline{b}+\overline{a}i}{2})+\overline{z_0}(\frac{-c-ci+b+ai}{2})+(\frac{c+ci}{2})(\frac{\overline{b}-\overline{a}i}{2})-(\frac{b+ai}{2})(\frac{\overline{c}-\overline{c}i}{2})=0,$$

which means that z_0 also satisfies the third equation, i.e. $Z_0 \in O_3Q$.

Problem 9.14. Let $ABCD$ be a parallelogram. The squares $ABNM$, $BPQC$, $DCRS$ and $ADTU$ are constructed externally. Their centers are O_1, O_2, O_3 and O_4, respectively. Prove that the quadrilateral $O_1O_2O_3O_4$ is a square.



Solution. The quadrilateral $ABCD$ is a parallelogram, and so the squares $ABNM$ and $CDSR$ are congruent.

Angle chasing gives that $\angle O_1BO_2 = 270° - \angle ABC = 90° + \angle BCD = \angle O_2CO_3$.

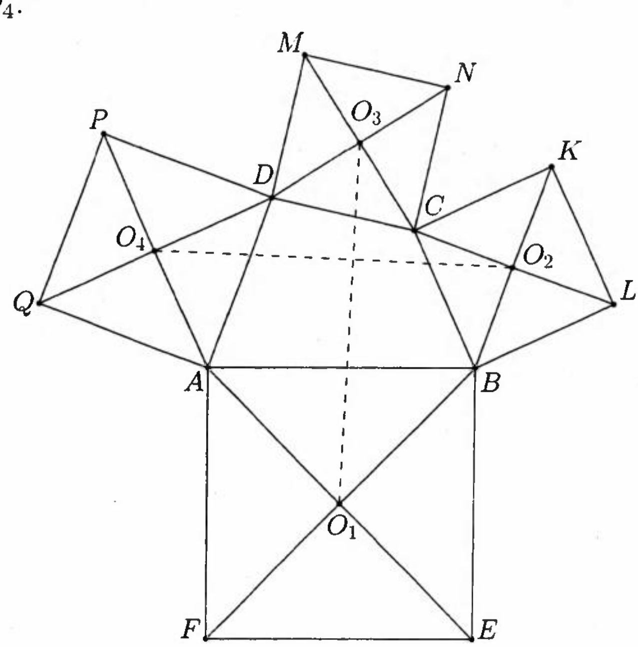
Moreover, $BO_1 = CO_3$ and $BO_2 = CO_2$.

Then $\Delta O_1BO_2 \cong \Delta O_3CO_2$ and therefore $O_1O_2 = O_2O_3$.

Analogously, we deduce that all the sides of $O_1O_2O_3O_4$ are equal.

Moreover, $\angle O_1O_2O_3 = \angle O_1O_2B + \angle BO_2O_3 = \angle CO_2O_3 + \angle BO_2O_3 = 90°$, which implies that the quadrilateral $O_1O_2O_3O_4$ is a rhombus with a right angle, i.e. a square.

Problem 9.15. Let $ABCD$ be a quadrilateral. The squares $ABEF$, $BCKL$, $CDMN$ and $ADPQ$ are constructed externally. Their centers are O_1, O_2, O_3 and O_4, respectively. Prove that $O_1O_3 \perp O_2O_4$ and that $O_1O_3 = O_2O_4$.



Solution. Firstly, we will prove that $\overrightarrow{KC}+\overrightarrow{NC}=\overrightarrow{AQ}+\overrightarrow{AF}$. Consider the equality $\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DA}+\overrightarrow{AB}=\overrightarrow{0}$.

A $-90°$ rotation of each vector in it gives

$$\overrightarrow{CK}+\overrightarrow{CN}+\overrightarrow{AQ}+\overrightarrow{AF}=\overrightarrow{0} \Rightarrow \overrightarrow{KC}+\overrightarrow{NC}=\overrightarrow{AQ}+\overrightarrow{AF}.$$

Now consider the equality

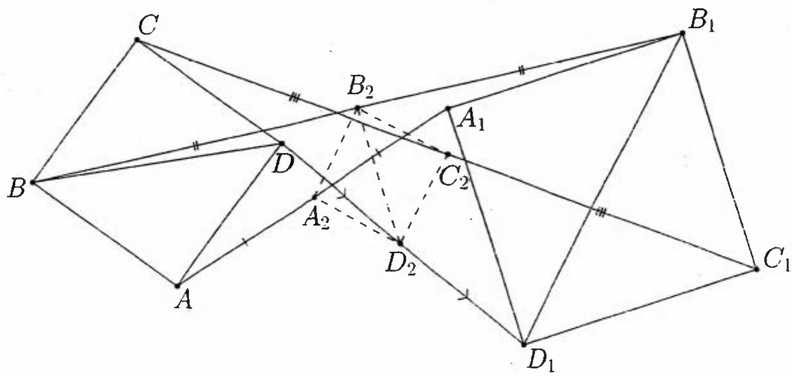
$$2\overrightarrow{O_3O_1}=2\overrightarrow{O_3D}+2\overrightarrow{DA}+2\overrightarrow{AO_1} \Rightarrow 2\overrightarrow{O_3O_1}=\overrightarrow{MD}+\overrightarrow{CD}+2\overrightarrow{DA}+\overrightarrow{AB}+\overrightarrow{AF}.$$

Again, a $-90°$ rotation of each vector in the last equality gives

$$\begin{aligned}
2\overrightarrow{a} &= \overrightarrow{CD}+\overrightarrow{CN}+2\overrightarrow{AQ}+\overrightarrow{AF}+\overrightarrow{BA}\\
&= \overrightarrow{CD}+\overrightarrow{CN}+\overrightarrow{BA}+\overrightarrow{AQ}+(\overrightarrow{AF}+\overrightarrow{AQ})\\
&= \overrightarrow{CD}+\overrightarrow{CN}+\overrightarrow{BA}+\overrightarrow{AQ}+(\overrightarrow{NC}+\overrightarrow{KC}) = \overrightarrow{CD}+\overrightarrow{BA}+\overrightarrow{AQ}+\overrightarrow{LB}\\
&= \overrightarrow{CD}+\overrightarrow{LQ}=2\overrightarrow{O_2O_4},
\end{aligned}$$

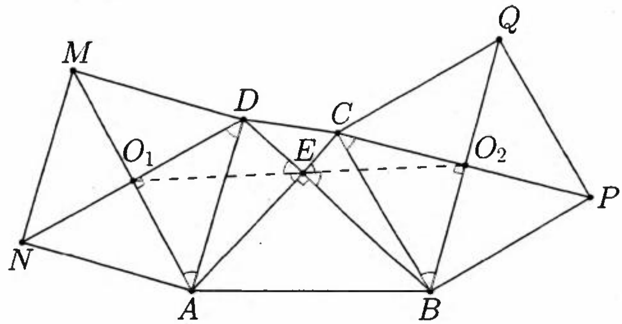
where $\overrightarrow{a}$ denotes the image of $\overrightarrow{O_3O_1}$ under a $-90°$ rotation. Therefore, $\overrightarrow{a}=\overrightarrow{O_2O_4}$, which implies that $O_1O_3 \perp O_2O_4$ and $O_1O_3=O_2O_4$.

Problem 9.16. Let $ABCD$ and $A_1B_1C_1D_1$ be squares with the same orientation. Let the midpoints of AA_1, BB_1, CC_1 and DD_1 be A_2, B_2, C_2 and D_2, respectively. Prove that the quadrilateral $A_2B_2C_2D_2$ is a square.



Solution. We will apply Problem 4.12.9 twice. Applying it to $\triangle ABD$ and $\triangle A_1B_1D_1$ gives that $\triangle A_2B_2D_2$ is isosceles and right-angled with $\angle B_2A_2D_2=90°$. Also, applying it to $\triangle CBD$ and $\triangle C_1B_1D_1$ implies that $\triangle C_2B_2D_2$ is isosceles and right-angled with $\angle B_2C_2D_2=90°$. These two facts imply that $A_2B_2C_2D_2$ is a square.

Problem 9.17. Let $ABCD$ be a convex quadrilateral, such that $AC \cap BD = E$ and $AC \perp BD$. The squares $ADMN$ and $BCQP$ are constructed externally. Their centers are O_1 and O_2, respectively. Prove that the points O_1, O_2 and E are collinear.



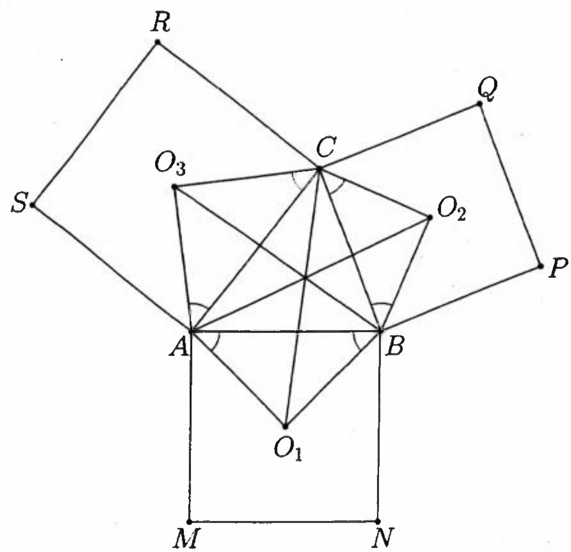
Solution. Note that $\angle AED + \angle AO_1D = 90° + 90° = 180°$, which means that the quadrilateral $AEDO_1$ is cyclic. Analogously, the quadrilateral BO_2CE is also cyclic.

Now $\angle O_1EO_2 = \angle O_1EA + \angle AEB + \angle BEO_2 = \angle O_1DA + 90° + \angle BCO_2 = 45° + 90° + 45° = 180°$, and so the points O_1, O_2 are E are collinear.

Problem 9.18. Let ABC be a triangle. The squares $ABNM$, $BPQC$ and $ACRS$ are constructed externally. Their centers are O_1, O_2 and O_3, respectively. Prove that the lines AO_2, BO_3 and CO_1 are concurrent.

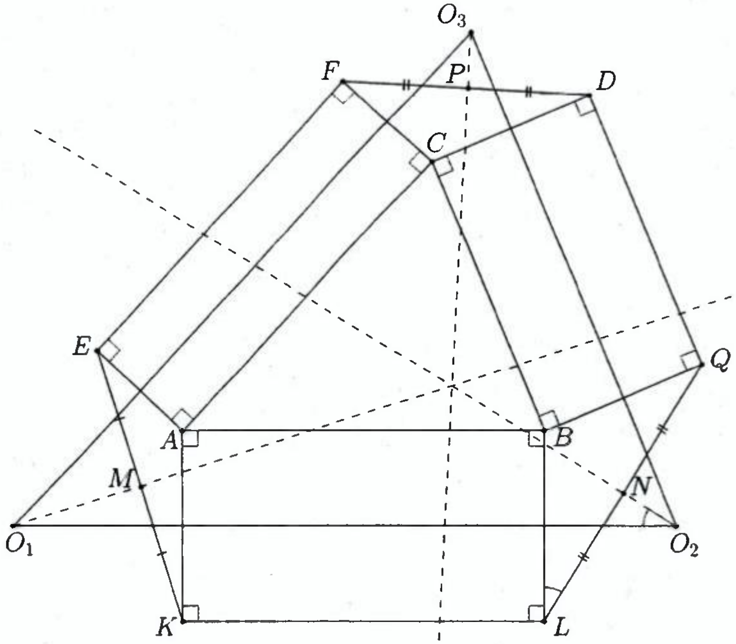
Solution. Since O_1, O_2 and O_3 are the centers of $ABNM$, $BPQC$ and $ACRS$, respectively, the measure of each of the marked angles in the figure equals 45°.

Now Problem 4.9.5 implies that the lines AO_2, BO_3 and CO_1 are concurrent.



Problem 9.19. Let ABC be a triangle. The rectangles $ABLK$, $ACFE$ and $BCDQ$ are constructed externally to its sides. Prove that the perpendicular bisectors of the segments EK, LQ and FD are concurrent.

Solution. Denote the midpoints of EK, LQ and FD by M, N and P, respectively. Also, let O_1, O_2 and O_3 be the circumcenters of $\triangle EKA$, $\triangle LQB$ and $\triangle FDC$, respectively. We have that $O_2N \perp LQ$, $O_1M \perp EK$ and $O_3P \perp FD$. Also, note that $O_1O_2 \perp AK$, $O_1O_2 \perp BL$, $O_1O_3 \perp AE$, $O_1O_3 \perp CF$, $O_2O_3 \perp BQ$ and $O_2O_3 \perp CD$.



We get that $\angle NO_2O_1 = \angle QLB$, $\angle NO_2O_3 = \angle LQB$, $\angle PO_3O_2 = \angle FDC$, $\angle PO_3O_1 = \angle DFC$, $\angle MO_1O_3 = \angle AEK$ and $\angle MO_1O_2 = \angle EKA$.

Therefore,

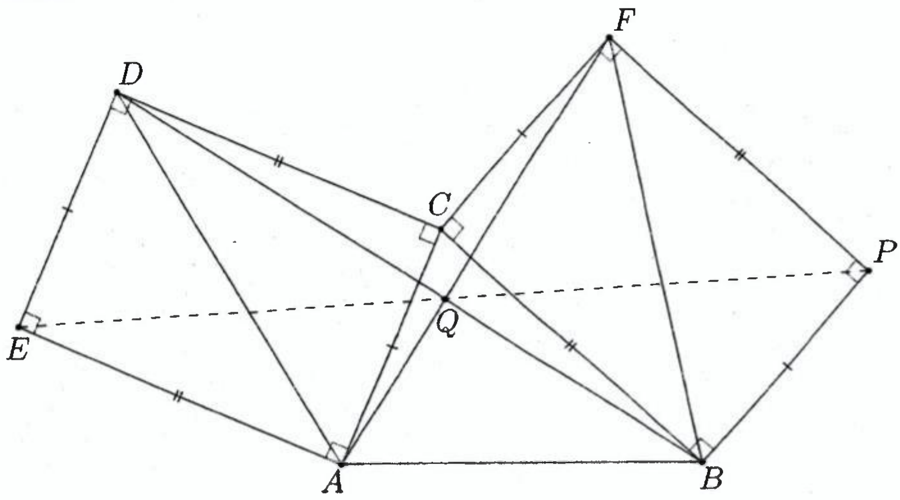
$$\frac{\sin\angle NO_2O_1}{\sin\angle NO_2O_3}\cdot\frac{\sin\angle PO_3O_2}{\sin\angle PO_3O_1}\cdot\frac{\sin\angle MO_1O_3}{\sin\angle MO_1O_2}$$

$$=\frac{\sin\angle QLB}{\sin\angle LQB}\cdot\frac{\sin\angle FDC}{\sin\angle DFC}\cdot\frac{\sin\angle AEK}{\sin\angle EKA}$$

$$=\frac{QB}{LB}\cdot\frac{FC}{DC}\cdot\frac{AK}{EA}=\frac{QB}{AK}\cdot\frac{FC}{QB}\cdot\frac{AK}{FC}=1.$$

The trigonometric form of Ceva's Theorem now yields that the lines O_1M, O_2N and O_3P are concurrent.

Problem 9.20. Let ABC be a triangle such that $\angle ACB < 90°$. The rectangles $ACDE$ and $BCFP$ are constructed externally, such that $CD = CB$ and $CF = CA$. Prove that the lines AF, BD and EP are concurrent.



Solution. Let $AF \cap BD = Q$. It suffices to prove that the points E, Q and P are collinear. Denote $\angle ACB = \gamma$.

Then $\angle BCD = \angle ACF = 90° + \gamma$, $AC = CF$ and $BC = CD$. Therefore,

$$\angle CBQ = \angle CDQ = \angle CFQ = \angle CAQ = 45° - \frac{\gamma}{2}.$$

Thus, Q is the second intersection point of the circumcircles of the two rectangles. Using that $\Delta BPF \cong \Delta DEA$, we have

$$\angle BQP = \angle BFP = \angle EAD = \angle EQD.$$

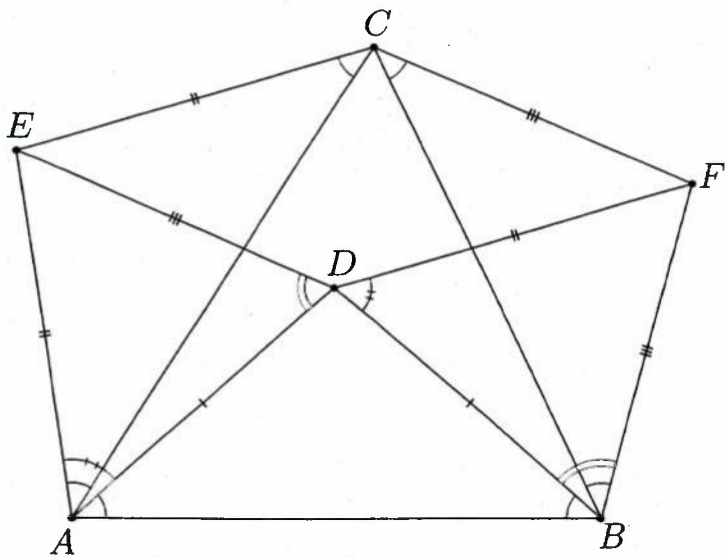
Hence, the point Q lies on the line EP.

Note. The problem statement remains true if $\angle ACB \geq 90°$. The solution in the case $\angle ACB > 90°$ is analogous to that above, and the solution in the case $\angle ACB = 90°$ is trivial.

Problem 9.21. Let ABC be a triangle. The isosceles triangles BCF ($BF = CF$) and AEC ($AE = CE$) are constructed externally, such that $\angle ACE = \angle BCF$. The point D lies inside of $\triangle ABC$, such that $AD = BD$ and $\angle BAD = \angle CAE$. Prove that $DF = EA$ and $DE = BF$.

Solution. The problem conditions imply that $\triangle ABD \sim \triangle BCF \sim \triangle ACE$. Then $\dfrac{AE}{AD} = \dfrac{AC}{AB}$.

On the other hand, we have $\angle DAE = \angle BAC$, which implies that $\triangle ABC \sim \triangle ADE$.



Analogously, $\triangle DBF \sim \triangle ABC$. Hence, $\triangle DBF \sim \triangle ADE$. This and the equality $AD = BD$ imply that actually $\triangle DBF \cong \triangle ADE$. Therefore, $DF = EA$ and $DE = BF$.

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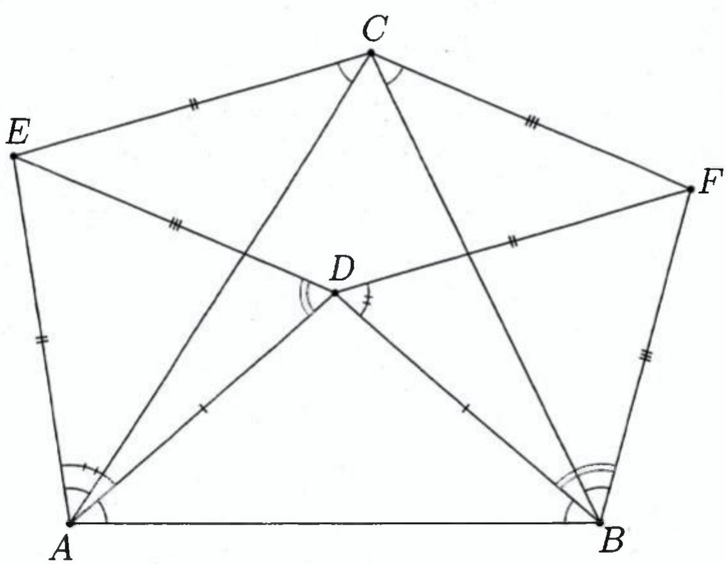

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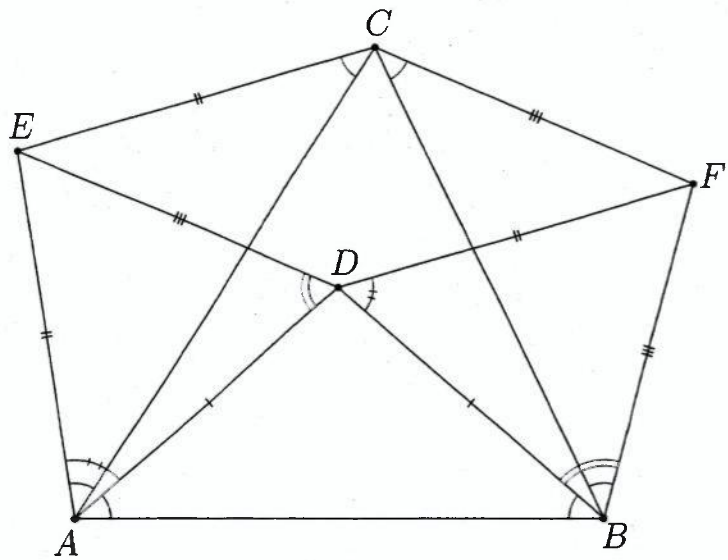


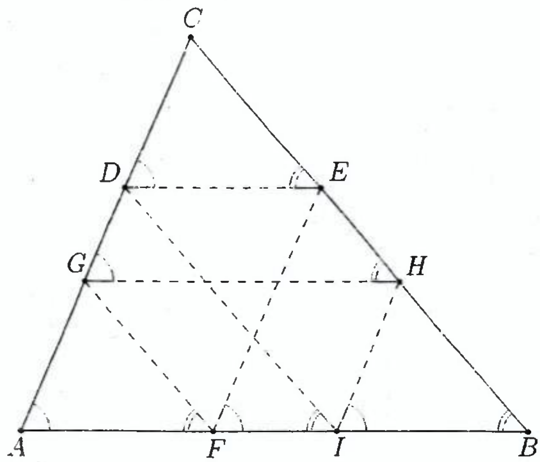
Analogously, $\triangle DBF \sim \triangle ABC$. Hence, $\triangle DBF \sim \triangle ADE$. This and the equality $AD = BD$ imply that actually $\triangle DBF \cong \triangle ADE$. Therefore, $DF = EA$ and $DE = BF$.

Problem 10.1. Let ABC be a triangle. The points $D, G \in AC$, $E, H \in BC$ and $F, I \in AB$ are such that $DI \parallel BC$, $IH \parallel AC$, $HG \parallel AB$, $GF \parallel BC$ and $FE \parallel AC$. Prove that $DE \parallel AB$.

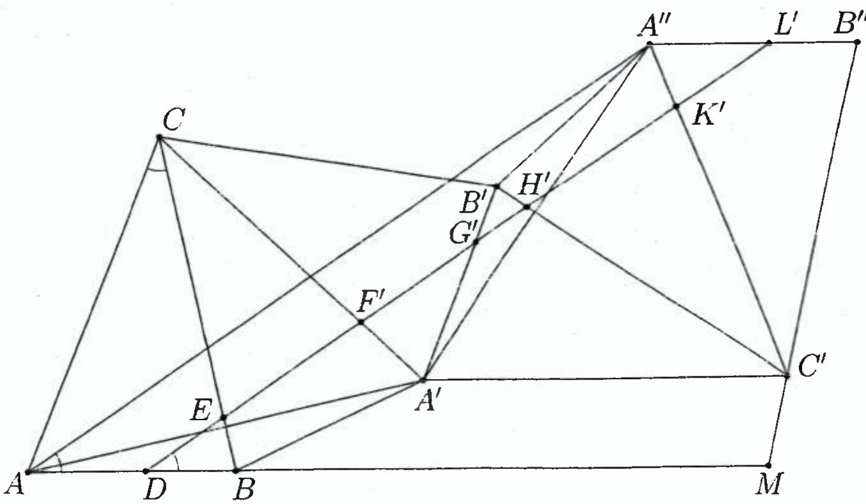
Solution. Applying the intercept theorem, we get

$$\frac{CD}{DA} = \frac{BI}{IA} = \frac{BH}{HC} = \frac{AG}{GC} = \frac{AF}{FB} = \frac{CE}{EB}.$$

By the same theorem, we conclude that $DE \parallel AB$.



Problem 10.2. Let ABC be a triangle. The points $D, G \in AB$, $E, H \in BC$ and $F, K \in AC$ are such that the quadrilaterals $ACED$, $BEFA$, $BCFG$, $ACHG$ and $ABHK$ are cyclic. Prove that the quadrilateral $BCKD$ is cyclic.



Solution. Let the reflection of A with respect to BC be A'. Let the reflection of B with respect to $A'C$ be B'. Let the reflection of C with respect to $A'B'$ be C'. Let the reflection of A' with respect to $B'C'$ be A''. Let the reflection of B' with respect to $A''C'$ be B''.

Let the line DE intersect the segments CA', $A'B'$, $B'C'$, $C'A''$ and $A''B''$ at the points F', G', H', K' and L'.

We can reformulate the statement as follows: A light ray DE, antiparallel to AC (i.e. $ADEC$ is cyclic), reflects itself with respect to the sides of $\triangle ABC$. Prove that the chain closes after exactly six steps.

By reflecting the vertices above, we have described the path of the ray, and we have constructed six congruent triangles. Hence, $A'F' = AF$, $B'G' = BG$, $C'H' = CH$ and $A''K' = AK$. To prove that the chain closes, it suffices to show that $A''L' = AD$.

Let $B''C' \cap AB = M$. We have that $\angle BA'C' = 360^\circ - 3\alpha$, $\angle MBA' = 180^\circ - 2\beta$ and $\angle A'C'M = 180^\circ - 3\gamma$. Hence, $\angle BMC' = 180^\circ - \beta$. But $\angle C'B''A'' = \beta$ and thus $AB \parallel A''B''$.

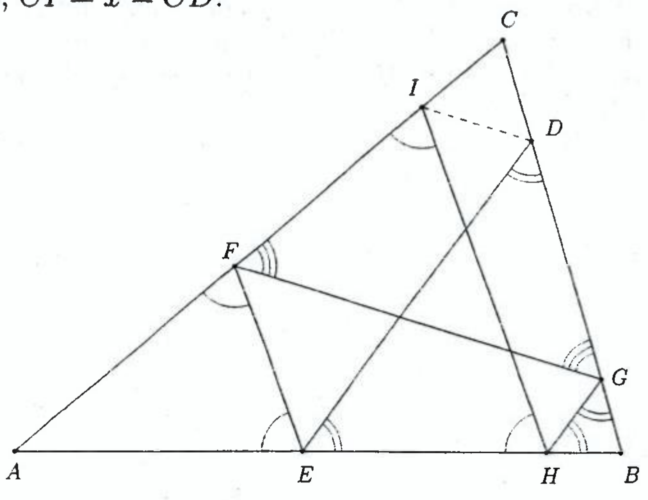
We will prove that $AA'' \parallel DL'$, which would imply that the quadrilateral $ADL'A''$ is a parallelogram, and consequently that $A''L' = AD$.

It is clear that $\triangle ABA' \cong \triangle A'B'A''$, so $AA' = A'A''$ and $\angle AA'A'' = \angle BA'B' = 2\alpha$. This gives $\angle A'AA'' = 90^\circ - \alpha$. On the other hand, $\angle BAA' = 90^\circ - \beta$ due to $AA' \perp BC$. Therefore, $\angle A''AB = \gamma = \angle EDB$ and the desired statement follows.

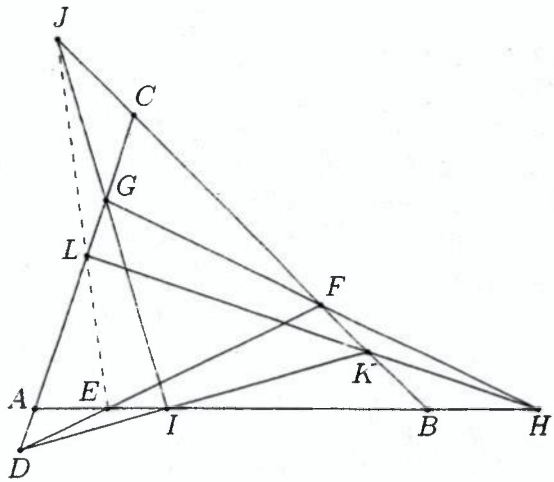
Problem 10.3. Let ABC be a triangle. The points $D, G \in BC$, $E, H \in AB$ and $F, I \in AC$ are chosen, such that $AI = AH$, $BH = BG$, $CG = CF$, $AF = AE$ and $BE = BD$. Prove that $CI = CD$.

Solution. Set $CI = x$. Then $AH = AI = b-x$, $BG = BH = c-AH = c-b+x$, $CF = CG = a-BG = a-c+b-x$, $AE = AF = b-CF = c-a+x$, $BD = BE = c - AE = a - x$ and finally $CD = x$.

Therefore, $CI = x = CD$.



Problem 10.4. Let ABC be a triangle. The point D lies on the ray $CA^{\rightarrow}$ beyond A. Two arbitrary lines through D intersect AB at the points E and I, and BC at the points F and K, respectively. The point H lies on the ray $AB^{\rightarrow}$ beyond B. The lines HK and HF intersect AC at the points L and G, respectively. If $IG \cap BC = J$, prove that the points E, L and J are collinear.



Solution. We will apply Menelaus' Theorem to $\triangle ABC$ a few times. We have

$$\frac{CD}{DA} = \frac{EB.FC}{AE.BF} = \frac{IB.KC}{AI.BK}.$$

Also,

$$\frac{BK}{KC} = \frac{BH}{HA} \cdot \frac{AL}{LC}$$

and

$$\frac{BF}{FC} = \frac{BH}{HA} \cdot \frac{AG}{GC}.$$

Then $\dfrac{BE}{EA} \cdot \dfrac{AL}{LC} = \dfrac{BI}{IA} \cdot \dfrac{AG}{GC}$, which implies that

$$\frac{BI}{IA} \cdot \frac{AG}{GC} = \frac{BJ}{JC} = \frac{BE}{EA} \cdot \frac{AL}{LC}.$$

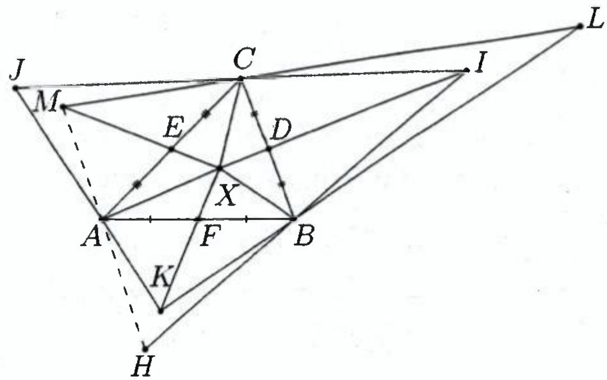
Menelaus' Theorem yields that the points E, L and J are collinear.

Problem 10.5. Let ABC be a triangle. The points X lies inside of $\triangle ABC$. The points D, E and F are the midpoints of BC, AC and AB, respectively. The point H lies on the ray $XF^{\rightarrow}$ beyond F. Let $HB \cap XD = I$, $IC \cap XE = J$, $JA \cap XF = K$, $KB \cap XD = L$ and $LC \cap XE = M$. Prove that the points H, A and M are collinear.

Solution. Menelaus' Theorem, applied to $\triangle KXJ$ and the line MAH, implies that it suffices to prove that

$$\frac{XM}{MJ} \cdot \frac{JA}{AK} \cdot \frac{KH}{HX} = 1.$$

The same theorem, applied to $\triangle JIX$ and the line MCL, yields $\dfrac{XM}{MJ} = \dfrac{XL}{LI} \cdot \dfrac{IC}{CJ}$. Analogously, we derive $\dfrac{XL}{LI} = \dfrac{XK}{KH} \cdot \dfrac{HB}{BI}$.

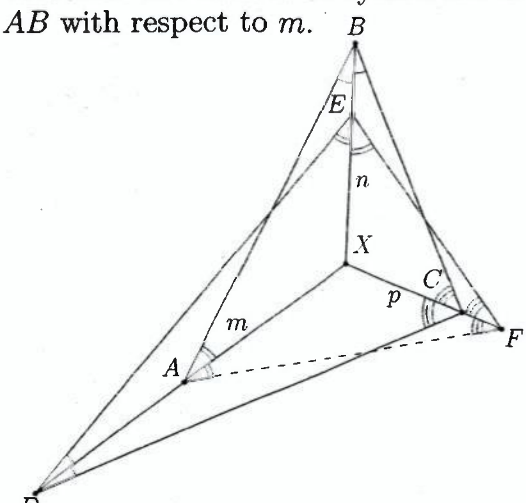


Therefore, we have to show that

$$\frac{XK}{KH}\cdot\frac{HB}{BI}\cdot\frac{IC}{CJ}\cdot\frac{JA}{AK}\cdot\frac{KH}{HX}=\frac{HB}{BI}\cdot\frac{IC}{CJ}\cdot\frac{JA}{AK}\cdot\frac{XK}{HX}=1.$$

But we have that $\dfrac{HB}{BI}=\dfrac{S_{\triangle HBX}}{S_{\triangle XBI}}=\dfrac{S_{AHBX}}{S_{XBIC}}$, $\dfrac{IC}{CJ}=\dfrac{S_{\triangle ICX}}{S_{\triangle JCX}}=\dfrac{S_{XBIC}}{S_{JAXC}}$, $\dfrac{JA}{AK}=\dfrac{S_{\triangle JAX}}{S_{\triangle AXK}}=\dfrac{S_{JAXC}}{S_{AKBC}}$ and $\dfrac{XK}{XH}=\dfrac{S_{AKBX}}{S_{AHBX}}$. The statement follows by multiplying these four equalities.

Problem 10.6. Let $m^{\rightarrow}$, $n^{\rightarrow}$ and $p^{\rightarrow}$ be three rays with a common origin X. Let $A\in m$ and $B\in n$. The line symmetric to AB with respect to n intersects p at the point C. The line symmetric to BC with respect to p intersects m at the point D. The line symmetric to CD with respect to m intersects n at the point E. Finally, the line symmetric to DE with respect to n intersects p at the point F. Prove that the line AF is symmetric to both EF with respect to p and to AB with respect to m.



Solution. Let the line symmetric to AB with respect to m intersect p at the point F'. It suffices to show that $F'\equiv F$.

Recall that a point on an angle bisector is equidistant from the arms of the angle. We have

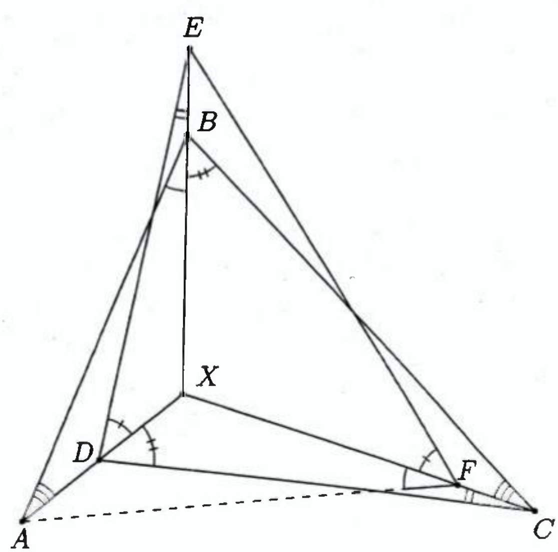
$$\begin{aligned}&dist(X,AF')=dist(X,AB)\\&=dist(X,BC)=dist(X,CD)\\&=dist(X,DE)=dist(X,EF)\end{aligned}$$

Let the projections of X onto the lines EF and AF' be H_1 and H_2, respectively. Then $XH_2 = XH_1$. We have

$$\begin{aligned}\angle H_1FX &= \angle XCB + \angle XBC - \angle XEF \\ &= \angle XCD + \angle XBA - \angle XED \\ &= \angle XCD + \angle XDE - \angle XAB \\ &= \angle XCD + \angle XDC - \angle XAF' \\ &= \angle H_2F'X.\end{aligned}$$

Therefore, $\Delta XH_1F \cong \Delta XH_2F'$, which means that $XF' = XF$, or $F \equiv F'$.

Problem 10.7. Let $m^{\rightarrow}$, $n^{\rightarrow}$ and $p^{\rightarrow}$ be three rays with a common origin X. Let $A \in m$ and $B \in n$. The point $C \in p$ satisfies $\angle BAX = \angle BCX$, the point $D \in m$ satisfies $\angle CBX = \angle CDX$, the point $E \in n$ satisfies $\angle DEX = \angle DCX$, and the point $F \in p$ satisfies $\angle EFX = \angle EDX$. Prove that $\angle XAF = \angle XEF$ and $\angle AFX = \angle ABX$.



Solution. Let $F' \in p$ be such that $\angle XBA = \angle XF'A$.

It suffices to show that the points F and F' coincide. We have

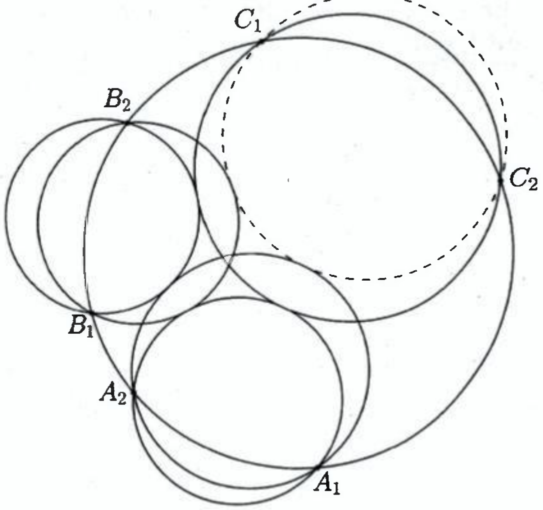
$$\begin{aligned}180^\circ &= 540^\circ - \angle AXB - \angle BXC - \angle CXA \\ &= \angle XF'A + \angle XAF' + \angle XBC + \angle XCB + \angle XDE + \angle XED \\ &= \angle XFE + \angle XEF + \angle XBA + \angle XAB + \angle XDC + \angle XCD.\end{aligned}$$

Hence, $\angle XAF' = \angle XEF$. The law of sines yields

$$\begin{aligned}\frac{XF'}{XF} &= \frac{XF'}{XA} \cdot \frac{XA}{XB} \cdot \frac{XB}{XC} \cdot \frac{XC}{XD} \cdot \frac{XD}{XE} \cdot \frac{XE}{XF} \\ &= \frac{\sin \angle XAF'}{\sin \angle XF'A} \cdot \frac{\sin \angle XBA}{\sin \angle XAB} \cdot \frac{\sin \angle XCB}{\sin \angle XBC} \cdot \\ &\quad \cdot \frac{\sin \angle XDC}{\sin \angle XCD} \cdot \frac{\sin \angle XED}{\sin \angle XDE} \cdot \frac{\sin \angle XFE}{\sin \angle XEF} = 1.\end{aligned}$$

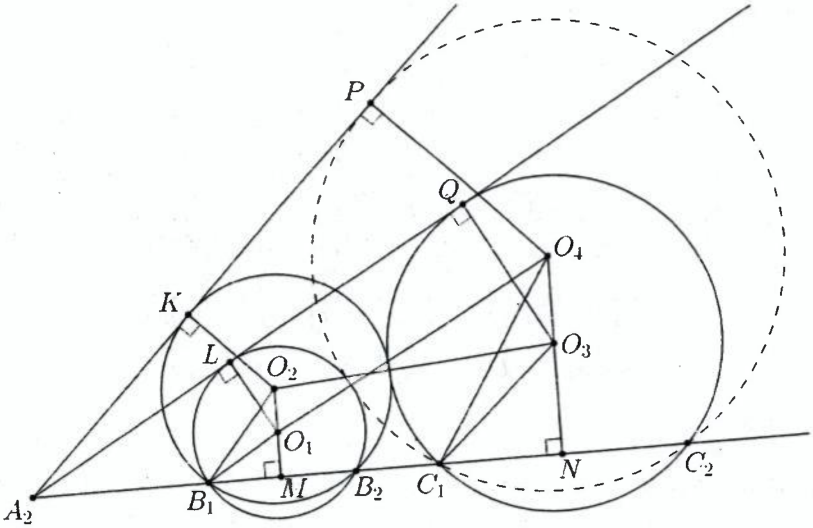
Therefore, $F \equiv F'$.

Problem 10.8. Let the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a circle k in this order. Let k_1 be a circle through A_1 and A_2. The circles k_2, k_3, k_4, k_5 and k_6 are constructed as follows: k_2 is externally tangent to k_1 and passes through B_1 and B_2, k_3 is externally tangent to k_2 and passes through C_1 and C_2, k_4 is externally tangent to k_3 and passes through A_1 and A_2, k_5 is externally tangent to k_4 and passes through B_1 and B_2, k_6 is externally tangent to k_5 and passes through C_1 and C_2. Prove that the circles k_1 and k_6 are externally tangent as well, i.e. that this chain closes after exactly six steps.



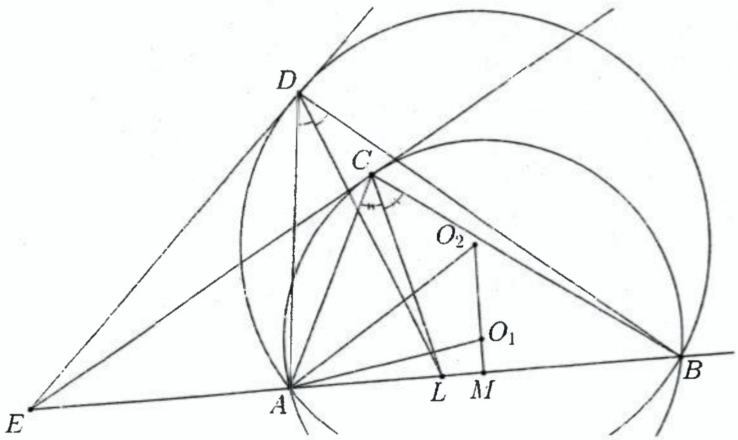
Solution. Consider an inversion centered at A_1 with an arbitrary radius. Now the problem can be reformulated as follows:

Let l, l_1 and l_2 be three rays with a common origin A_2. The points B_1, B_2, C_1 and C_2 lie on l. The points L and Q lie on l_1, and the points K and P lie on l_2. The line l_1 is a common external tangent line of the circumcircles of $\triangle B_1B_2L$ (ω_1, say) and $\triangle C_1C_2Q$ (ω_3, say). The line l_2 is a common external tangent line of the circumcircles of $\triangle B_1B_2K$ (ω_2, say) and $\triangle C_1C_2P$ (ω_4, say). Also, the circles ω_2 and ω_3 are externally tangent. Prove that the circles ω_1 and ω_4 are externally tangent.



First, we will prove the following lemma.

Lemma. Let l, l_1 and l_2 be three rays with a common origin E. The points A and B lie on l. The point C lies on l_1, and the point D lies on l_2. The line l_1 touches the circumcircle of $\triangle ABC$ (k_1, say), and the line l_2 touches the circumcircle of $\triangle ABD$ (k_2, say). Let O_1 and O_2 be the centers of k_1 and k_2, respectively. Then the value of the expression $\dfrac{AO_1 + AO_2}{O_1O_2}$ is independent of the choice of the points A and B.



Proof. Set $\angle ACB = \gamma$, $\angle ADB = \gamma_1$, $\angle BEC = \alpha$, $\angle BED = \beta$, $EA = x^2$ and $EB = y^2$.

Then $EC = ED = xy$. We have that $\triangle EBC \sim \triangle ECA$ and $\triangle EBD \sim \triangle EDA$. Hence, $\dfrac{AC}{CB} = \dfrac{EC}{EB} = \dfrac{xy}{y^2} = \dfrac{x}{y}$.

Analogously, $\dfrac{AD}{DB} = \dfrac{ED}{EB} = \dfrac{xy}{y^2} = \dfrac{x}{y}$. We deduce that $\dfrac{AC}{CB} = \dfrac{AD}{DB}$.

Therefore, the internal angle bisectors of $\angle ACB$ and $\angle ADB$ intersect on AB (at the point L). Problem 4.3.4 yields that $\triangle ECL$ and $\triangle EDL$ are isosceles. Thus, $\angle ELC = 90° - \dfrac{\alpha}{2}$ and $\angle ELD = 90° - \dfrac{\beta}{2}$.

We also have that $CL = 2xy \sin\dfrac{\alpha}{2}$ and $DL = 2xy \cos\dfrac{\beta}{2}$.

Since $\dfrac{AL}{BL} = \dfrac{AC}{CB} = \dfrac{x}{y}$ and $AL + BL = EB - EA = y^2 - x^2$, it follows that $AL = xy - x^2$ and $BL = y^2 - xy$. Now we apply the cotangent technique to $\triangle ALC$ and we get that

$$\cot\frac{\gamma}{2} = \frac{\dfrac{CL}{AL} - \sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}} = \frac{\dfrac{2xy\sin\dfrac{\alpha}{2}}{xy - x^2} - \sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}} = \frac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}\left(\frac{2y}{y-x} - 1\right) = \operatorname{tg}\frac{\alpha}{2}\cdot\frac{y+x}{y-x}.$$

Analogously, we get that $\cot\dfrac{\gamma_1}{2} = \operatorname{tg}\dfrac{\beta}{2}\cdot\dfrac{y+x}{y-x}$.

Note that $\angle AO_2M = \gamma_1$ and $\angle AO_1M = \gamma$. Hence, the expression $\dfrac{AO_1 + AO_2}{O_1O_2}$ is equal to $\dfrac{\sin\gamma + \sin\gamma_1}{\sin(\gamma - \gamma_1)}$. But

$$\frac{\sin\gamma + \sin\gamma_1}{\sin(\gamma - \gamma_1)} = \frac{\sin\left(\dfrac{\gamma_1 + \gamma}{2}\right)}{\sin\left(\dfrac{\gamma - \gamma_1}{2}\right)} = \frac{\cot\dfrac{\gamma}{2} + \cot\dfrac{\gamma_1}{2}}{\cot\dfrac{\gamma_1}{2} - \cot\dfrac{\gamma}{2}} = \frac{\operatorname{tg}\dfrac{\alpha}{2} + \operatorname{tg}\dfrac{\beta}{2}}{\operatorname{tg}\dfrac{\beta}{2} - \operatorname{tg}\dfrac{\alpha}{2}},$$

and so the value of $\dfrac{AO_1 + AO_2}{O_1O_2}$ is independent of x and y. The lemma is proven.

Denote the radius of the circle ω_i by r_i. We can assume that $r_1 \neq r_2$ because if this is not true, then $k_1 \equiv k_4$ and the problem becomes trivial.

Analogously, we assume that $r_3 \neq r_4$. Let M and N be such that $B_1M = MB_2 = x$ and $C_1N = NC_2 = y$.

The lemma gives $\dfrac{r_1+r_2}{\sqrt{r_2^2-x^2}-\sqrt{r_1^2-x^2}} = \dfrac{r_3+r_4}{\sqrt{r_4^2-y^2}-\sqrt{r_3^2-y^2}}$, or

$$(r_1+r_2)(r_4-r_3) = \left(\sqrt{r_2^2-x^2}-\sqrt{r_1^2-x^2}\right)\left(\sqrt{r_4^2-y^2}+\sqrt{r_3^2-y^2}\right)$$

Note that $A_2B_1.A_2B_2 = A^2L^2 = A_2K^2$ and $A_2C_1.A_2C_2 = A^2Q^2 = A_2P^2$. Hence, $LQ = KP$. The Pythagorean Theorem yields

$$\begin{aligned} MN^2 + \left(\sqrt{r_3^2-y^2}-\sqrt{r_1^2-x^2}\right)^2 &= O_1O_3^2 = QL^2 + (r_3-r_1)^2 \\ &= KP^2 + (r_3-r_1)^2 \\ &= O_2O_4^2 - (r_4-r_2)^2 + (r_3-r_1)^2 \\ &= MN^2 + \left(\sqrt{r_4^2-y^2}-\sqrt{r_2^2-x^2}\right)^2 \\ &\quad - (r_4-r_2)^2 + (r_3-r_1)^2. \end{aligned}$$

Set $\sqrt{r_1^2-x^2} = t_1$, $\sqrt{r_2^2-x^2} = t_2$, $\sqrt{r_3^2-y^2} = t_3$ and $\sqrt{r_4^2-y^2} = t_4$. Then we have the following two equations

$$\begin{aligned} r_2r_4 - r_1r_3 + r_1r_4 - r_2r_3 &= t_2t_4 - t_1t_3 + t_2t_3 - t_1t_4 \\ t_2t_4 - t_1t_3 &= r_2r_4 - r_1r_3. \end{aligned}$$

They give $r_1r_4 - r_2r_3 = t_2t_3 - t_1t_4$.

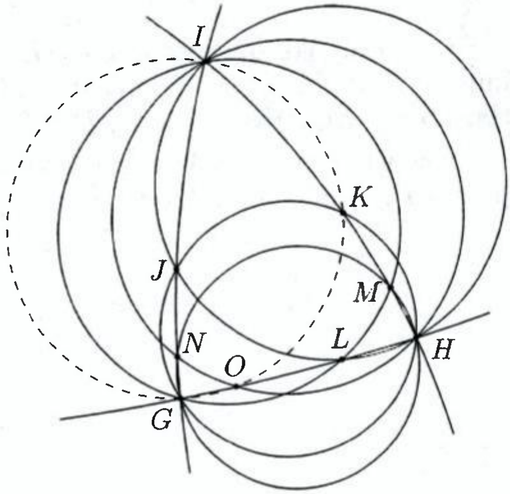
We have that $O_2O_3 = r_2+r_3$ and it suffices to prove that $O_1O_4 = r_1+r_4$. Again, the Pythagorean Theorem gives

$$\begin{aligned} O_1O_4^2 = MN^2 + (t_4-t_1)^2 &= O_2O_3^2 - (t_2-t_3)^2 + (t_4-t_1)^2 \\ &= (r_2+r_3)^2 - (t_2-t_3)^2 + (t_4-t_1)^2 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow O_1O_4^2 &= r_2^2 + r_3^2 + 2r_2r_3 - (r_2^2-x^2) - (r_3^2-y^2) + 2t_2t_3 + \\ &\quad + (r_4^2-y^2) + (r_1^2-x^2) - 2t_1t_4 \\ \Leftrightarrow O_1O_4^2 &= 2r_1r_4 + r_1^2 + r_4^2 \Leftrightarrow O_1O_4 = r_1 + r_4 \end{aligned}$$

Therefore, the circles ω_1 and ω_4 are externally tangent.

Problem 10.9. Let k_1, k_2 and k_3 be three circles intersecting at the points G, H and I, as shown in the figure. The point O lies on the smaller arc $\widehat{GH}$ of k_1. The point $N \in k_2$ is such that the quadrilateral $HONI$ is cyclic, as shown in the figure. The points M, L, J and K are defined analogously to create a chain. Prove that the quadrilateral $GOKI$ is cyclic, i.e. that the chain closes after exactly six steps.



Solution. Consider an inversion centered at H with an arbitrary radius. The problem is now equivalent to the following one:

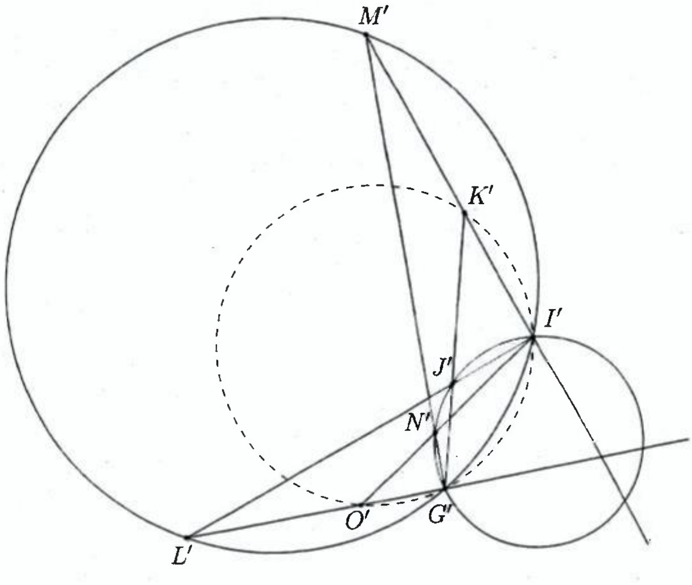
The points M', K' and I' lie on a line in this order, and the points L', O' and G' lie on another line in this order, as shown in the second figure.

Let $I'O' \cap M'G' = N'$ and $L'I' \cap K'G' = J'$. If the quadrilaterals $L'G'I'M'$ and $G'N'J'I'$ are cyclic, prove that the quadrilateral $O'G'I'K'$ is cyclic as well.

We have that

$$\angle O'G'K' = \angle L'G'M' + \angle N'G'J' = \angle L'I'M' + \angle N'I'J' = \angle O'I'K'$$

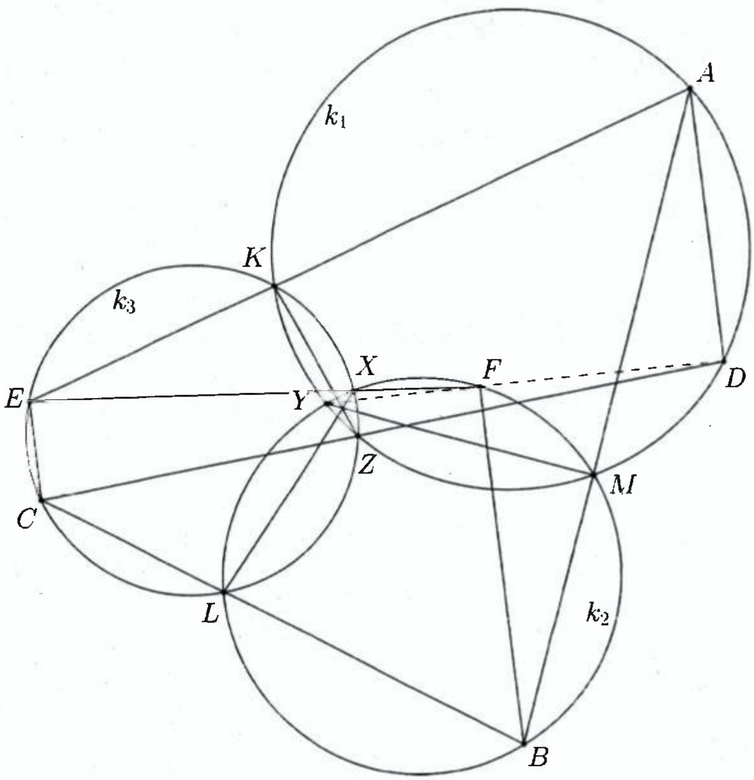
and the statement follows.



Problem 10.10. Let k_1, k_2 and k_3 be circles that intersect each other at the points X, Y, Z, M, L and K, as shown in the figure. Let $A \in k_1$. The line AM intersects k_2 for the second time at the point B. The line BL intersects k_3 for the second time at the point C. The line CZ intersects k_1 for the second time at the point D. The line AK intersects k_3 for the second time at the point E. The line EX intersects k_2 for the second time at the point F. Prove that the points Y, F and D are collinear.

Solution. We have that $\angle DAK = \angle KZC = 180° - \angle CEK$, which implies that $EC \parallel AD$. Moreover, $\angle FBL = \angle EXL = 180° - \angle ECL$ and thus $BF \parallel EC$. Hence, $AD \parallel BF$.

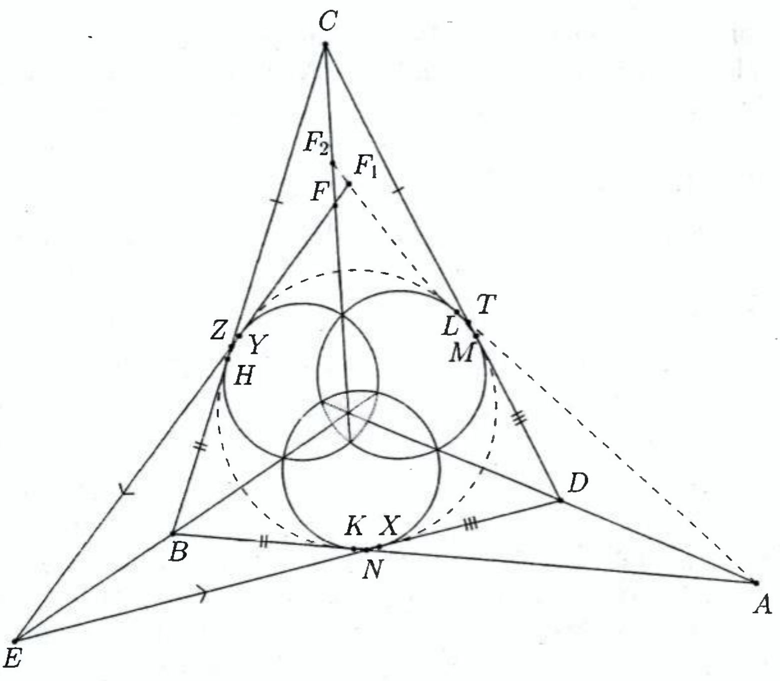
Now we have $\angle FYM = \angle FBM = \angle DAM = \angle DYM$. Therefore, the points Y, F and D are collinear.



Problem 10.11. Let k_1, k_2 and k_3 be three circles. The three radical axes are drawn and the point A is chosen on one of them (say on l – the radical axis of k_1 and k_2). One of the tangent lines to k_1 that passes through A is drawn – the one that does not intersect k_2. This tangent line intersects the radical axis of k_3 and k_1 at the point B. The points C, D, E, F and A_1 are constructed analogously, creating a chain. Prove that the chain closes after exactly six steps, i.e. that $A \equiv A_1$.

Solution. We will use the notations on the figure. We have

$$CB+ND=CH+HB+NX+DX=CM+BK+NK+DM=CD+BN.$$



Therefore, the quadrilateral $CBND$ is circumscribed around a circle k, say. Moreover, we have

$$\begin{aligned} EN+NB &= EX-XN+NK+KB = EX+KB \\ &= EY+BH = EY-ZY+ZH+BH = EZ+BZ. \end{aligned}$$

Therefore, the quadrilateral $ENBZ$ has an excircle, whose center lies on the angle bisectors of $\angle BND$ and $\angle NBC$. Thus, this circle coincides with k. Now assume that FA is not tangent to its respective circle.

Then let AL be the tangent line through A, and let it intersect the radical axis containing F at the point F_2 and the line EF at the point F_1.

Then we have

$$\begin{aligned} AT + TD &= AL - LT + TM + MD \\ &= AL + MD \\ &= AK + DX \\ &= AK - KN + NX + XD \\ &= AN + ND. \end{aligned}$$

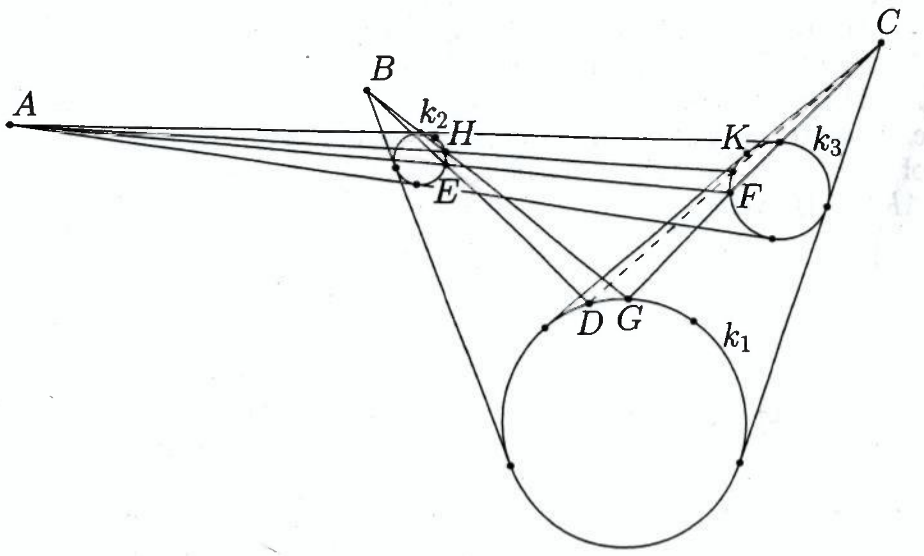
Therefore, the quadrilateral $ATDN$ has an excircle, whose center lies on the angle bisectors of $\angle CDN$ and $\angle BND$. Thus, this circle coincides with k. Hence, the quadrilateral $ABZF_1$ is circumscribed. Then

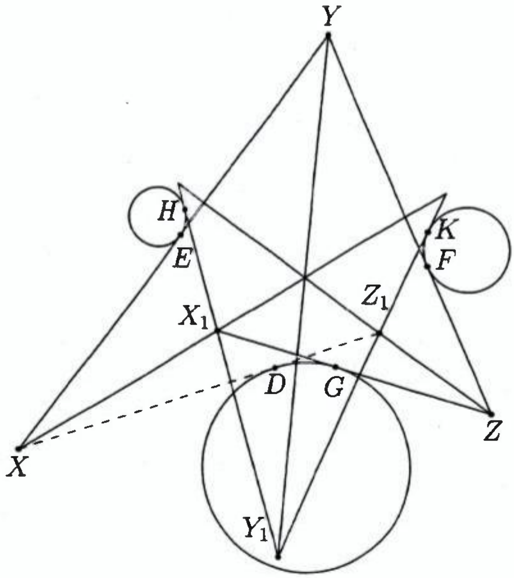
$$\begin{aligned} 0 &= AF_1 + BZ - AB - ZF_1 \\ &= AL + LF_1 + BH + HZ - AK - KB - ZY - YF_1 \\ &= LF_1 - YF_1. \end{aligned}$$

We deduce that $LF_1 = YF_1$. This means that F_1 lies on the radical axis containing F. Thus, $F_1 \equiv F_2 \equiv F$, a contradiction.

Problem 10.12. Let k_1, k_2 and k_3 be three circles, such that their disks do not intersect each other. Let the external homothetic centers of the three pairs of circles be A, B and C, as shown in the figure. A line passing through B intersects k_1 and k_2 at the points E and D. These two points do not correspond to each other under the homothety h_1 with center B, such that $h_1(k_1) = k_2$. The line AE intersects k_3 at the point F. The points E and F do not correspond to each other under the homothety h_2 with center A, such that $h_2(k_2) = k_3$. The points G, H and K are defined analogously. Prove that the points C, K and D are collinear.

Solution. We will solve the problem by applying Problem 6.5.7 six times and the analog of Problem 10.11 once (considering the other external common tangent lines of the circles). The proof of this analog is identical to that of Problem 10.11, so we will not describe it here.



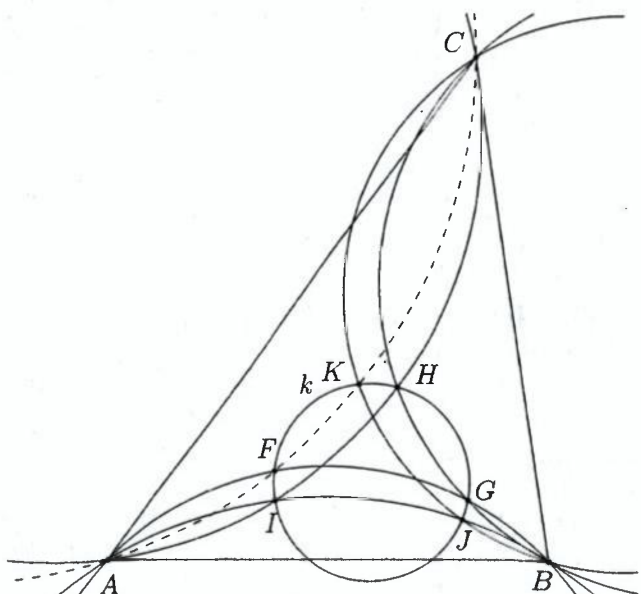


Firstly, by Problem 6.5.7, we have that the tangent lines through D and E intersect the respective radical axis at a point X. Define the points Y, Z, X_1, Y_1 and Z_1 analogously.

Note that if we prove that the tangent lines through D and K intersect at the respective radical axis, then the points C, K and D would be collinear, as desired (because for a fixed point D there exists a unique point Z_1 on the radical axis, such that DZ_1 touches k_1).

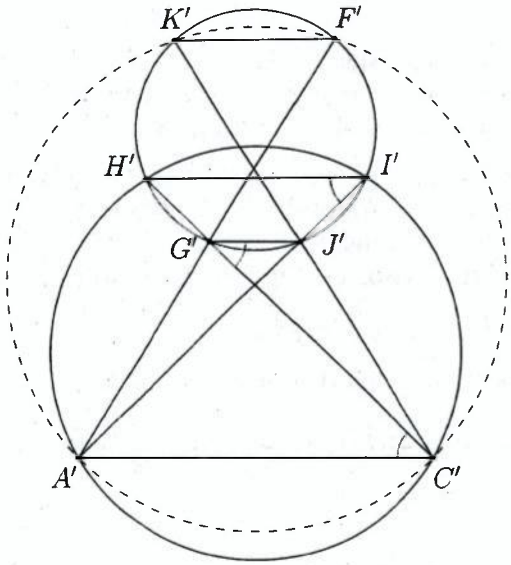
On the other hand, the point Z_1 uniquely determines the point K, which, by Problem 6.5.7, lies on the line CD. It remains to show that Z_1X touches k_1 at D. This follows from the analog of Problem 10.11 because the chain of points X, Y, Z, X_1, Y_1 and Z_1 will close itself.

Problems 10.13 and 10.14. Let ABC be a triangle and let k be an arbitrary circle. A circle through A and B intersects k at the points F and G. The circumcircle of $\triangle BGC$ intersects k at the points G and H. The circumcircle of $\triangle CHA$ intersects k at the points I and H. The circumcircle of $\triangle ABI$ intersects k at the points I and J. The circumcircle of $\triangle BCJ$ intersects k at the points J and K. Prove that the quadrilateral $AFKC$ is cyclic.

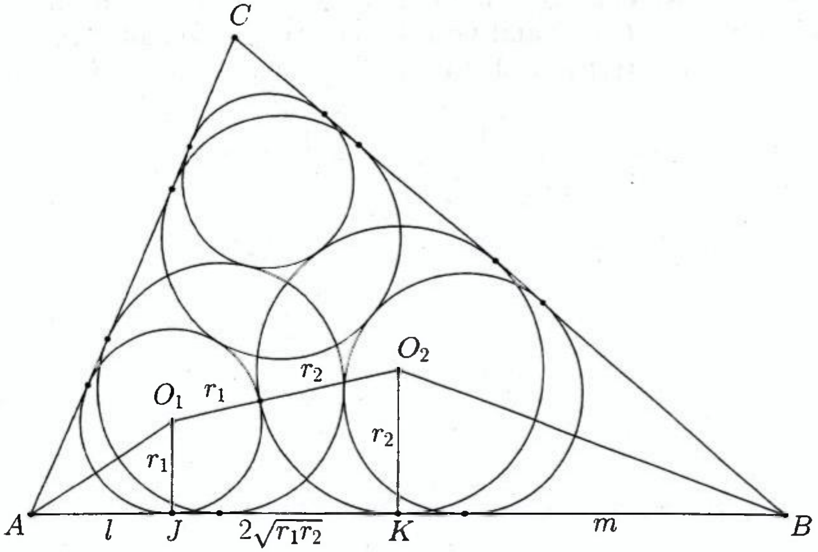


Solution. Consider an inversion with center B and an arbitrary radius. Then it follows that $\angle H'C'A' = \angle H'I'A' = \angle J'G'C'$ because $A'C'I'H'$ is a cyclic quadrilateral. Hence, $A'C' \parallel G'J'$. But then $\angle F'K'C' = \angle F'G'J' = \angle F'A'C'$, which implies that the quadrilateral $A'C'F'K'$ is cyclic. Hence, the quadrilateral $AFKC$ is cyclic.

Note. Problem 10.13 is a particular case of Problem 10.14 – when k is the incircle of $\triangle ABC$.



Problem 10.15. Let ABC be a triangle. The circles k_1, k_2, k_3, k_4, k_5 and k_6 are internal for $\triangle ABC$ and are defined as follows. The circle k_1 is inscribed in $\angle BAC$. The circle k_2 is inscribed in $\angle ABC$ and is externally tangent to k_1. The circle k_3 is inscribed in $\angle ACB$ and is externally tangent to k_2. The circle k_4 is inscribed in $\angle BAC$ and is externally tangent to k_3. The circle k_5 is inscribed in $\angle ABC$ and is externally tangent to k_4. The circle k_6 is inscribed in $\angle ACB$ and is externally tangent to k_5. Prove that the circles k_1 and k_6 are externally tangent as well, i.e. that the chain closes after exactly six steps.



Solution. Let k_7 be the circle inside of $\triangle ABC$, which is inscribed in $\angle BAC$ and is externally tangent to k_6. It suffices to show that $k_1 \equiv k_7$. Let l, m, n, l', m', n' and l'' be the lengths of the tangent segments to k_1, k_2, k_3, k_4, k_5, k_6 and k_7 from A, B, C, A, B, C and A, respectively.

Our aim is to prove that $l = l''$. Let O_1 and O_2 be the centers of k_1 and k_2, and let their radii be r_1 and r_2, respectively.

Let their points of tangency to AB be J and K, respectively. Then $AJ = l$ and $BK = m$. Also, O_1 lies on the internal bisector of $\angle BAC$. Hence, $r_1 = JO_1 = l \operatorname{tg} \frac{\alpha}{2}$. Analogously, we get $r_2 = KO_2 = m \operatorname{tg} \frac{\beta}{2}$.

Now the Pythagorean Theorem and the fact that k_1 and k_2 are externally tangent give $JK = 2\sqrt{r_1 r_2} = 2\sqrt{lm \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2}}$.

Now note that

$$c = AB = AJ + JK + KB = l + m + 2\sqrt{lm \operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2}}.$$

We make use of the following equalities $s - c = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$ and $s = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$ to get

$$\operatorname{tg} \frac{\alpha}{2} \operatorname{tg} \frac{\beta}{2} = \frac{s-c}{s} \Rightarrow c = l + m + 2\sqrt{lm \cdot \frac{s-c}{s}}.$$

Analogously, we can find equalities for l, m, n, l', m', n' and l''. Set p, q, r, p', q', r', p'', f, g, h and t to be the square roots of l, m, n, l', m', n', l'', a, b, c and s, respectively. Define the obtuse angles θ, φ and ψ as follows:

$$\cos\theta = -\sqrt{1 - \frac{a}{s}} = -\sqrt{1 - \frac{f^2}{t^2}}, \quad \sin\theta = \frac{f}{t},$$

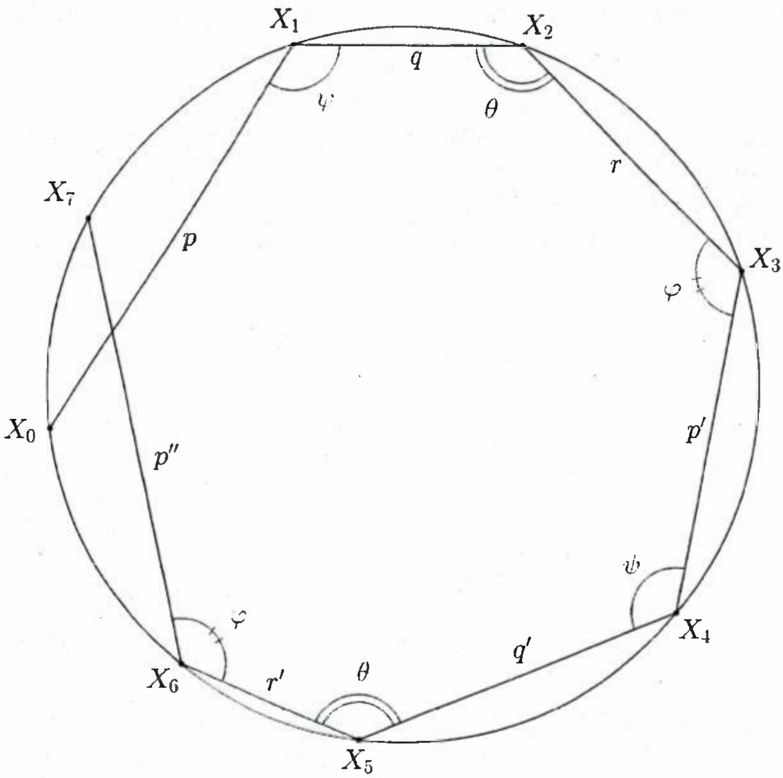
$$\cos\varphi = -\sqrt{1 - \frac{b}{s}} = -\sqrt{1 - \frac{g^2}{t^2}}, \quad \sin\varphi = \frac{g}{t},$$

$$\cos\psi = -\sqrt{1 - \frac{c}{s}} = -\sqrt{1 - \frac{h^2}{t^2}}, \quad \sin\psi = \frac{h}{t}.$$

Using these notations, the equality $c = l + m + 2\sqrt{lm \cdot \frac{s-c}{s}}$ is equivalent to $h^2 = p^2 + q^2 - 2pq\cos\psi$. Analogously, we find five similar equalities. Each of these resembles the law of cosines.

Consider the triangle with sides p and q and with angle ψ between them. The length of the third side of this triangle is exactly h, and the diameter of its circumcircle is $\dfrac{h}{\sin\psi} = t$.

The diameters of the analogous circumcircles are also equal to t. Therefore, we can glue the triangles to their common sides, so that their circumcircles coincide.



This results in an open cyclic polygon $X_0X_1...X_7$, having angles ψ at the vertices X_1 and X_4, θ at the vertices X_2 and X_5, and φ at the vertices X_3 and X_6. Now we have that $\triangle X_0X_2X_4 \cong \triangle X_3X_5X_7$ (their sides are equal). Hence, $\angle X_0X_2X_4 = \angle X_3X_5X_7$. Subtracting these angles from $\angle X_1X_2X_3$ and $\angle X_4X_5X_6$ gives

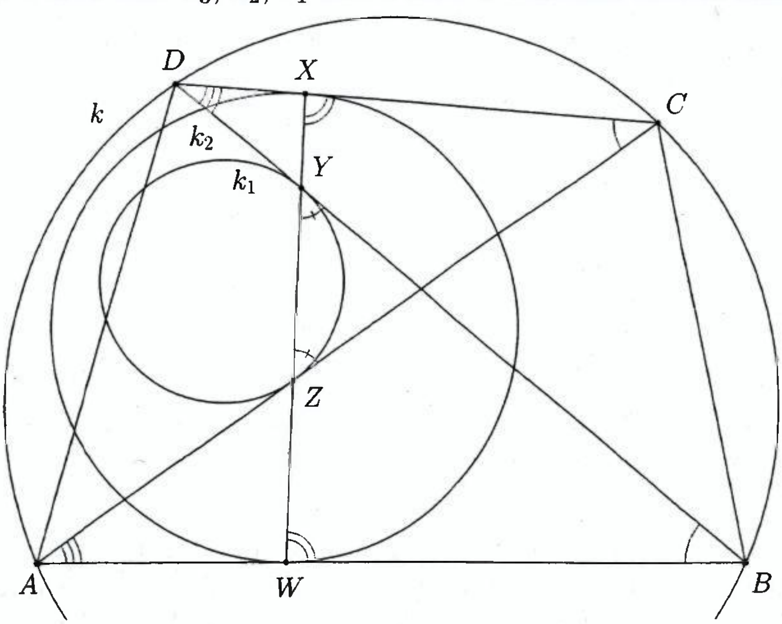
$$\angle X_0X_2X_1 + \angle X_3X_2X_4 = \angle X_3X_5X_4 + \angle X_6X_5X_7.$$

But $\angle X_3X_2X_4 = \angle X_3X_5X_4$, so $\angle X_0X_2X_1 = \angle X_6X_5X_7$. Therefore, the arcs $\widehat{X_0X_1}$ and $\widehat{X_6X_7}$ are equal, which means that $p = p''$ and $l = l''$.

Problems 10.17, 10.18, 10.19 and 10.20. Let k and ω be two circles such that ω lies inside of k. Let the points $A_i \in k$, $i = 1, 2, 3, \ldots, n$ be such that A_iA_{i+1} is tangent to ω for all positive integers $i \leq n-1$, and such that A_nA_1 is also tangent to ω. Prove that if we vary A_1 along the circle, the chain A_1, A_2, A_3, ..., A_n will always close after n steps, i.e. A_1A_n will always be tangent to ω.

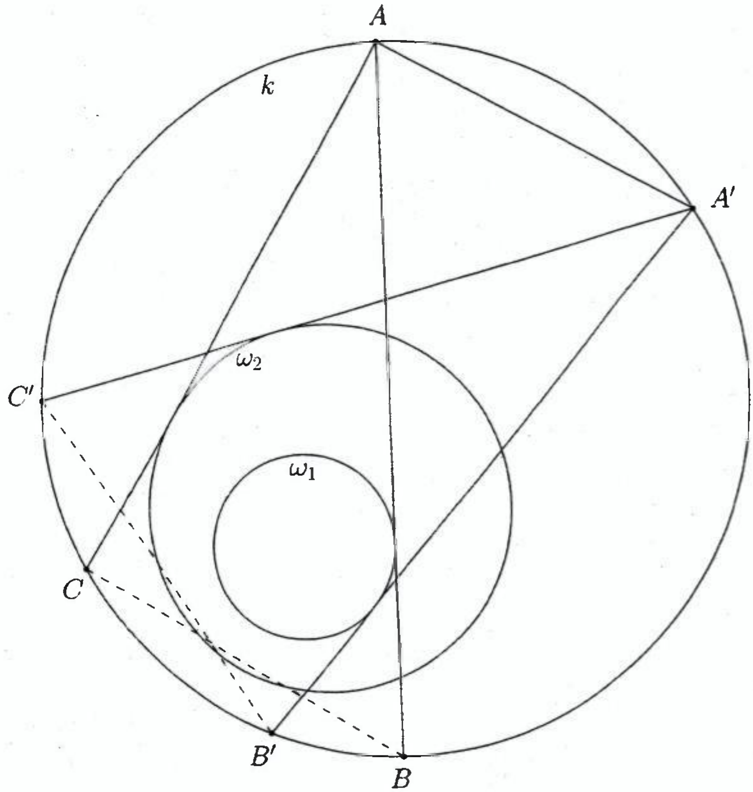
Solution. First, we will prove four lemmas. Lemma 1 and Lemma 2 are the respective ones from Problem 6.10.22.

Lemma 3. In the setting of Problem 6.10.22, define the circle k_3 as the one that touches the pair of lines (AD, BC) at their points of intersection with l (you can prove that l forms equal angles with this pair by angle chasing). Prove that k_3, k_2, k_1 and k have a common radical axis.



Proof. Analogous to the one of 6.10.22.

Lemma 4. The point A lies on the circle k. Let AB and AC be lines that touch ω_1 and ω_2 – circles inside of k, which are coaxial (have a common radical axis) with k ($B \in k$ and $C \in k$). Then if we vary A along k, the line BC is always tangent to a fixed circle ω_3, which is coaxial with k, ω_1 and ω_2.

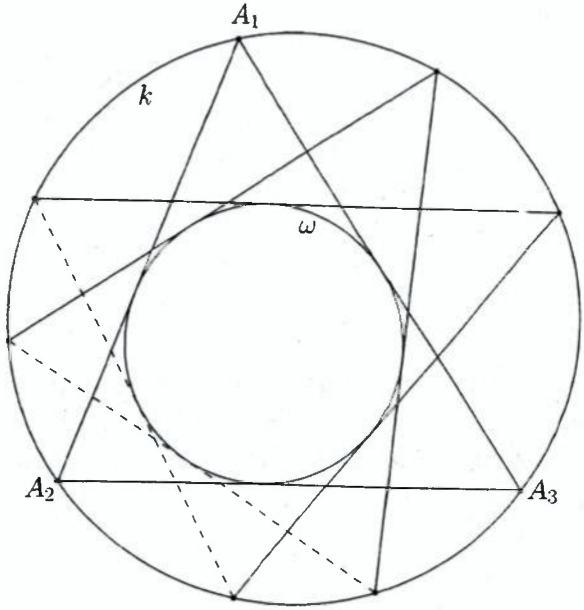


Proof. Define the points A', B' and C' as the points A, B and C for a different placement of A on k.

Applying Lemma 3 to the quadrilateral $AA'BB'$ gives that AA' and BB' are tangent to a circle from the same pencil (a group of coaxial circles forms a pencil). Analogously, the same fact holds for AA' and CC'. However, there exists only one circle from this pencil that is tangent to AA' and that lies in the same half-plane with respect to AA' as BB' and CC'.

Therefore, the lines AA', BB' and CC' are tangent to the same circle. Now we apply Lemma 3 to the quadrilateral $CC'BB'$. We get that BC and $B'C'$ are tangent to a circle from the same pencil.

Now fixing A, B and C and varying A', B' and C', the lines BC and $B'C'$ are tangent to a fixed circle (the one tangent to BC and being in the pencil). The lemma is proven.



As A_1 varies on k, we can apply Lemma 4 and get that A_1A_3 is tangent to a fixed circle from the pencil of k and ω, because A_1A_2 and A_2A_3 are tangent to a fixed circle from the same pencil. But A_1A_3 and A_3A_4 are also tangent to a fixed circle from the pencil. Now Lemma 4 gives that A_1A_4 is tangent to a fixed circle from the pencil.

In the same manner, we conclude that A_1A_n is tangent to a fixed circle from the pencil. In the starting configuration, this circle is exactly ω. Therefore, A_1A_n is always tangent to ω.

Final notes and comments

The sources of some of the problems are listed below, just the way they are listed in Akopyan's "Geometry in Figures".

2.8) A. G. Myakishev, Fourth Geometrical Olympiad in Honour of I. F. Sharygin, 2008, Correspondence round, Problem 10.

2.32) Lemoine point is the center of the dashed circle.

3.9) Here it is shown that the Aubert line is perpendicular to the Gauss line.

4.1.2) A. A. Polyansky, All-Russian Mathematical Olympiad, 2007–2008, Final round, Grade 10, Problem 6.

4.1.3) L. A. Emelyanov, All-Russian Mathematical Olympiad, 2009–2010, Regional round, Grade 9, Problem 6.

4.1.9) V. V. Astakhov, All-Russian Mathematical Olympiad, 2006–2007, Final round, Grade 10, Problem 3.

4.1.17) See also 5.7.9.

4.1.19) A. V. Smirnov, Saint Petersburg Mathematical Olympiad, 2005, Round II, Grade 10, Problem 6.

4.1.20) D. V. Prokopenko, Fifth Geometrical Olympiad in Honour of I. F. Sharygin, 2009, Correspondence round, Problem 20.

4.1.21) A. A. Polyansky, All-Russian Mathematical Olympiad, 2010–2011, Final round, Grade 10, Problem 6.

4.1.23) M. Chirija, Romanian Masters 2006, District round, Grade 7, Problem 4.

4.2.6) United Kingdom, IMO Shortlist 1996.

4.3.6) R. Kozarev, Bulgarian National Olympiad, 1997, Fourth round, Problem 5.

4.3.8) Moscow Mathematical Olympiad, 1994, Grade 11, Problem 5.

4.3.10) D. Şerbănescu and V. Vornicu, International Mathematical Olympiad, 2004, Problem 1.

4.3.14) L. A. Emelyanov, All-Russian Mathematical Olympiad, 2009–2010, Final round, Grade 10, Problem 6.

4.3.17) I. I. Bogdanov, Sixth Geometrical Olympiad in Honour of I. F. Sharygin, 2010, Final round, Grade 8, Problem 4.

4.3.19) D. V. Prokopenko, All-Russian Mathematical Olympiad, 2009–2010, Regional round, Grade 10, Problem 3.

4.3.21) M. G. Sonkin, All-Russian Mathematical Olympiad, 1999–2000, District round, Grade 8, Problem 4.

4.4.6) A. A. Zaslavsky and F. K. Nilov, Fourth Geometrical Olympiad in Honour of I. F. Sharygin, 2008, Final round, Grade 8, Problem 4.

4.4.7) France, IMO Shortlist 1970.

4.5.5) USA, IMO Shortlist 1979.

4.5.12) Twenty First Tournament of Towns, 1999–2000, Fall round, Senior A-Level, Problem 4.

4.5.14) M. A. Kungozhin, All-Russian Mathematical Olympiad, 2010–2011, Final round, Grade 11, Problem 8.

4.5.15) Personal communication from L. A. Emelyanov.

4.5.16) Bulgaria, IMO Shortlist 1996.

4.5.20) F. L. Bakharev, Saint Petersburg Mathematical Olympiad, 2005, Round II, Grade 10, Problem 6.

4.5.22) Brazil, IMO Shortlist 2006. The bold line is parallel to the base of the triangle.

4.5.23) A. A. Polansky, All-Russian Mathematical Olympiad, 2006–2007, Final round, Grade 11, Problem 2. The bold line is parallel to the base of the triangle.

4.5.29) Special case of 6.3.3.

4.5.31) This construction using circles is not rare. See 10.15.

4.5.35) D. V. Shvetsov, Sixth Geometrical Olympiad in Honour of I. F. Sharygin, 2010, Correspondence round, Problem 8.

4.5.36) M. G. Sonkin, From the materials of the Summer Conference Tournament of Towns "Circles inscribed in circular segments and tangents", 1999.

4.5.37) M. G. Sonkin, All-Russian Mathematical Olympiad, 1998–1999, Final round, Grade 9, Problem 3.

4.5.38, 4.5.39) Based on Bulgarian problem from IMO Shortlist 2009.

4.5.40) D. Djukić and A. V. Smirnov, Saint Petersburg Mathematical Olympiad, 2005, Round II, Grade 9, Problem 6. N. I. Beluhov's generalization.

4.5.41) L. A. Emelyanov, Twenty Third Tournament of Towns, 2001–2002, Spring round, Senior A-Level, Problem 5.

4.5.43) V. A. Shmarov, All-Russian Mathematical Olympiad, 2007–2008, Final round, Grade 11, Problem 7.

4.6.2) A. I. Badzyan. All-Russian Mathematical Olympiad, 2004–2005, District round, Grade 9, Problem 4.

4.6.4) V. P. Filimonov, Moscow Mathematical Olympiad, 2008, Grade 11, Problem 4.

4.6.6) V. Yu. Protasov, Third Geometrical Olympiad in Honour of I. F. Sharygin, 2006, Correspondence round, Problem 15.

4.7.6) Generalization of 4.7.1.

4.7.8) Iranian National Mathematical Olympiad, 1999.

4.7.9) Iranian National Mathematical Olympiad, 1997, Fourth round, Problem 4.

4.7.18) Nguyen Van Linh, From forum `www.artofproblemsolving.com`, Theme: "A concyclic problem" at 27 May 2010.

4.7.16) Personal communication from K. V. Ivanov.

4.8.5) Personal communication from L. A. Emelyanov and T. L. Emelyanova.

4.8.7) Personal communication from F. F. Ivlev.

4.8.8) China, Team Selection Test, 2011.

4.8.9) L. A. Emelyanov, Journal "Matematicheskoe Prosveschenie", Tret'ya Seriya, N 7, 2003, Problem section, Problem 8.

4.8.13) A. V. Smirnov, Saint Petersburg Mathematical Olympiad, 2009, Round II, Grade 10, Problem 7.

4.8.15) G. B. Feldman, Seventh Geometrical Olympiad in Honour of I. F. Sharygin, 2011, Correspondence round, Problem 22.

4.8.16) L. A. Emelyanov and T. L. Emelyanova, All-Russian Mathematical Olympiad, 2010–2011, Final round, Grade 9, Problem 2.

4.8.20) A. V. Gribalko, All-Russian Mathematical Olympiad, 2007–2008, District round, Grade 10, Problem 2.

4.8.21) Special case of 4.8.23.

4.8.27) V. P. Filimonov, All-Russian Mathematical Olympiad, 2007–2008, District round, Grade 9, Problem 7.

4.8.29) China, Team Selection Test, 2010.

4.8.30) T. L. Emelyanova, All-Russian Mathematical Olympiad, 2010–2011, Regional round, Grade 10, Problem 2.

4.8.32) A. V. Akopyan, All-Russian Mathematical Olympiad, 2007–2008, Grade 10, Problem 3.

4.8.33) D. Skrobot, All-Russian Mathematical Olympiad, 2007–2008, District round, Grade 10, Problem 8.

4.8.38) F. K. Nilov, Special case of problem from Geometrical Olympiad in Honour of I. F. Sharygin, 2008, Final round, Grade 10, Problem 7.

4.8.39) V. P. Filimonov, All-Russian Mathematical Olympiad, 2006–2007, Final round, Grade 9, Problem 6.

4.9.1) The obtained point is called the isogonal conjugate with respect to the triangle.

4.9.3) The obtained point is called the isotomic conjugate with respect to the triangle.

4.9.20) This point will be the isogonal conjugate with respect to the triangle. See 4.9.1.

4.9.26) A. A. Zaslavsky, Third Geometrical Olympiad in Honour of I. F. Sharygin, 2007, Final round, Grade 9, Problem 3.

4.10.4) D. V. Shvetsov, Sixth Geometrical Olympiad in Honour of I. F. Sharygin, 2010, Correspondence round, Problem 2.

4.10.6) A. V. Smirnov, Saint Petersburg Mathematical Olympiad, 2005, Round II, Grade 10, Problem 2.

4.11.2) D. V. Prokopenko, All-Russian Mathematical Olympiad, 2009–2010, Regional round, Grade 9, Problem 4.

4.11.9) S. L. Berlov, Saint Petersburg Mathematical Olympiad, 2007, Round II, Grade 9, Problem 2.

4.12.2) Generalization of Blanchet's Theorem (see 4.12.1).

4.12.3) A. V. Smirnov, Saint Petersburg Mathematical Olympiad, 2004, Round II, Grade 9, Problem 6.

4.12.4) The bold line is parallel to the base of the triangle.

4.12.7) USSR, IMO Shortlist 1982.

5.1.1) M. A. Volchkevich, Eighteenth Tournament of Towns, 1996—1997, Spring round, Junior A-Level, Problem 5.

5.1.2) L. A. Emelyanov, All-Russian Mathematical Olympiad, 2000–2001, District round, Grade 9, Problem 3.

5.1.4) M. V. Smurov, Nineteenth Tournament of Towns, 1997–1998, Spring round, Junior A-Level, Problem 2.

5.1.5) V. Yu. Protasov, Second Geometrical Olympiad in Honour of I. F. Sharygin, 2006, Final round, Grade 8, Problem 3.

5.1.9) L. A. Emelyanov and T. L. Emelyanova, All-Russian Mathematical Olympiad, 2010–2011, Final round, Grade 11, Problem 2.

5.2.3) S. V. Markelov, Sixteenth Tournament of Towns, 1994–1995, Spring round, Senior A-Level, Problem 3.

5.2.5) A. A. Zaslavsky, First Geometrical Olympiad in Honour of I. F.

Sharygin, 2005, Final round, Grade 10, Problem 3.

5.2.8) A. A. Zaslavsky, Third Geometrical Olympiad in Honour of I. F. Sharygin, 2007, Correspondence round, Problem 14.

5.2.10) A. V. Akopyan, Moscow Mathematical Olympiad, 2011, Problem 9.5.

5.2.2) The more general construction is illustrated in 5.4.16.

5.3.2) United Kingdom, IMO Shortlist 1979.

5.4.1–5.4.4) Special case of 5.4.5.

5.4.7) M. G. Sonkin, All-Russian Mathematical Olympiad, 1998–1999, Final round, Grade 11, Problem 3.

5.4.9) I. Wanshteyn.

5.4.10) A. A. Zaslavsky, Fourth Geometrical Olympiad in Honour of I. F. Sharygin, 2008, Correspondence round, Problem 10.

5.4.13) This construction is dual to the butterfly theorem. See 6.4.3 and 6.4.4.

5.5.2) F. V. Petrov, Saint Petersburg Mathematical Olympiad, 2006, Round II, Grade 11, Problem 3.

5.5.3) W. Pompe, International Mathematical Olympiad, 2004, Problem 5.

5.5.8) M. I. Isaev, All-Russian Mathematical Olympiad, 2006–2007, District round, Grade 10, Problem 4.

5.5.9) P. A. Kozhevnikov, All-Russian Mathematical Olympiad, 2009–2010, Final round, Grade 11, Problem 3.

5.6.1, 5.6.2) I. F. Sharygin, International Mathematical Olympiad, 1985, Problem 5.

5.6.10) Personal communication from L. A. Emelyanov.

5.6.17) A. A. Zaslavsky, Twentieth Tournament of Towns, 1998–1999, Spring round, Senior A-Level, Problem 2.

5.7.4) Poland, IMO Shortlist 1996.

6.1.7) P. A. Kozhevnikov, International Mathematical Olympiad, 1999, Problem 5.

6.2.10) R. Gologan, Rumania, Team Selection Test, 2004.

6.5.9) V. B. Mokin. XIV The A. N. Kolmogorov Cup, 2010, Personal competition, Senior level, Problem 5.

6.6.3) A. A. Zaslavsky, Second Geometrical Olympiad in Honour of I. F. Sharygin, 2006, Final round, Grade 8, Problem 3.

6.7.9) Twenty Fourth Tournament of Towns, 2002–2003, Spring round, Senior A-Level, Problem 4.

6.8.11) Constructions satisfying the condition of the figure are not rare (see 10.8).

6.9.1) I. F. Sharygin, International Mathematical Olympiad, 1983, Problem 2.

6.9.6) P. A. Kozhevnikov, Ninteenth Tournament of Towns, 1997–1998, Fall round, Junior A-Level, Problem 4.

6.9.9) M. A. Volchkevich, Seventeenth Tournament of Towns, 1995–1996, Spring round, Junior A-Level, Problem 2.

6.10.1) M. G. Sonkin, All-Russian Mathematical Olympiad, Regional round, 1994–1995, Grade 9, Problem 6.

6.10.3) A. A. Zaslavsky, P. A. Kozhevnikov, Moscow Mathematical Olympiad, 1999, Grade 10, Problem 2.

6.10.4) P. A. Kozhevnikov, All-Russian Mathematical Olympiad, 1997–1998, District round, Grade 9, Problem 2.

6.10.7) Dinu Şerbănescu, Romanian, Team Selection Test for Balkanian Mathematical Olympiad.

6.10.8) France, IMO Shortlist, 2002.

6.10.10) Twenty Fifth Tournament of Towns, 2003–2004, Spring round, Junior A-Level, Problem 4.

6.10.12) China, Team Selection Test, 2009.

6.10.13) I. Nagel, Fifteen Tournament of Towns, 1993–1994, Spring round, Junior A-Level, Problem 2. See also 4.3.15.

6.10.20) V. Yu. Protasov, Third Geometrical Olympiad in Honour of I. F. Sharygin, 2007, Final round, Grade 10, Problem 6.

6.10.21) USA, IMO Longlist 1984.

6.10.23) Personal communication from E. A. Avksent'ev. This construction is a very simple way to construct the Apollonian circle.

7.2) The obtained line is called the trilinear polar with respect to the triangle.

8.1.1) Here the points are reflections of the given point with respect to the sides of the triangle.

8.1.4) Bulgaria, IMO Longlist 1966. See also 6.1.10.

8.1.11) Hungary, IMO Longlist 1971.

8.1.12) E. Przhevalsky, Sixteenth Tournament of Towns, 1994–1995, Fall round, Junior A-Level, Problem 3.

8.1.13) I. Nagel, Twelfth Tournament of Towns, 1990–1991, Fall round, Senior A-Level, Problem 2.

9.1) If we construct our triangles in the direction of the interior, then we similarly obtain a point called the second Napoleon point.

9.2) Columbia, IMO Shortlist 1983.

9.3) See also 2.9 and 2.10.

9.9) Hungary–Israel Binational Olympiad, 1997, Second Day, Problem 2.

9.16) Belgium, IMO Longlist 1970. See alsos 4.12.9.

9.20) Twenty Seventh Tournament of Towns, 2005–2006, Spring round, Junior A-Level, Problem 3.

9.21) Belgium, IMO Shortlist 1983.

10.4) This construction is equivalent to that of the Pappus Theorem.

10.7) Personal communication from F. V. Petrov.

10.13) Personal communication from V. A. Shmarov.

10.14) Generalization of previous result.

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