IMO Shortlist 2003

– Geometry

1 Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

2 Three distinct points A, B, and C are fixed on a line in this order. Let Γ be a circle passing through A and C whose center does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

3 Let ABC be a triangle and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote by I_A, I_B, I_C the excenters of the triangle ABC. Prove that P is the circumcenter of the triangle $I_AI_BI_C$.

Proposed by C.R. Pranesachar, India

4 Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

5 Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Proposed by Hojoo Lee, Korea

Contributors: iandrei, sebadollahi, darij grinberg, grobber, vinoth_90_2004, orl, Valiowk, Fedor Petrov, Anonymous, iura, Myth, pluricomplex, flip2004, rope0811, Dapet, jmerry, heartwork, hxtung

6 Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

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7 Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all of these three semicircles has radius t. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Alternative formulation. In a triangle ABC, construct circles with diameters BC, CA, and AB, respectively. Construct a circle w externally tangent to these three circles. Let the radius of this circle w be t.
Prove: $\frac{s}{2} < t \le \frac{s}{2} + \frac{1}{2}\left(2 - \sqrt{3}\right) r$, where r is the inradius and s is the semiperimeter of triangle ABC.

Proposed by Dirk Laurie, South Africa

– Number Theory

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1 Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, ... is defined as follows:

$$x_i = \begin{cases} 2^i & \text{if } 0 \le i \le m-1; \\ \sum_{j=1}^{m} x_{i-j} & \text{if } i \ge m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m .

Proposed by Marcin Kuczma, Poland

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2 Each positive integer a undergoes the following procedure in order to obtain the number $d = d\left(a\right)$:

(i) move the last digit of a to the first position to obtain the numb er b;
(ii) square b to obtain the number c;
(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.)

Find all numbers a for which $d\left(a\right) = a^2$.

Proposed by Zoran Sunic, USA

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3 Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

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4 Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$: [i]there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.[/i]

Proposed by Laurentiu Panaitopol, Romania

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5 An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

Proposed by Hojoo Lee, Korea

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6 Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

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7 The sequence a_0, a_1, a_2, ... is defined as follows:

$$a_0 = 2, \qquad a_{k+1} = 2a_k^2 - 1 \quad \text{for } k \geq 0.$$

Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

Hi guys ,

Here is a nice problem:

Let be given a sequence a_n such that $a_0 = 2$ and $a_{n+1} = 2a_n^2 - 1$. Show that if p is an odd prime such that $p|a_n$ then we have $p^2 \equiv 1 \pmod{2^{n+3}}$

Here are some futher question proposed by me :Prove or disprove that :
1) $gcd(n, a_n) = 1$
2) for every odd prime number p we have $a_m \equiv \pm 1 \pmod{p}$ where $m = \frac{p^2-1}{2^k}$ where $k = 1$ or 2

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Thanks kiu si u

Edited by Orl.

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8 Let p be a prime number and let A be a set of positive integers that satisfies the following conditions:

(i) the set of prime divisors of the elements in A consists of $p-1$ elements;

(ii) for any nonempty subset of A, the product of its elements is not a perfect p-th power.

What is the largest possible number of elements in A ?

– Algebra

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1 Let a_{ij} $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$.

Prove the existence of positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \qquad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \qquad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are either all negative, all positive, or all zero.

Proposed by Kiran Kedlaya, USA

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2 Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that
(i) $f(0) = 0, f(1) = 1$;
(ii) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

Proposed by A. Di Pisquale & D. Matthews, Australia

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3 Consider pairs of the sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots, \qquad b_1 \geq b_2 \geq b_3 \geq \cdots$$

and the sums

$$A_n = a_1 + \cdots + a_n, \quad B_n = b_1 + \cdots + b_n; \qquad n = 1, 2, \ldots.$$

For any pair define $c_n = \min\{a_i, b_i\}$ and $C_n = c_1 + \cdots + c_n$, $n = 1, 2, \ldots$.

(1) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?

(2) Does the answer to question (1) change by assuming additionally that $b_i = 1/i$, $i = 1, 2, \ldots$?

Justify your answer.

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4 Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

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5 Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy}) f(\sqrt{yz}) f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

- $f(x) < f(y)$ for all $1 \le x < y$.

Proposed by Hojoo Lee, Korea

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6 Let n be a positive integer and let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \ldots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \ge x_i y_j$ for all $1 \le i, j \le n$.

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \ge \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Edited by Orl.

Proposed by Reid Barton, USA

– Combinatorics

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1 Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

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2 Let D_1, D_2, ..., D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i.

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3 Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with non-selfintersecting boundary) having k internal right angles.

Proposed by Juozas Juvencijus Macys, Lithuania

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4 Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \geq 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

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5 Every point with integer coordinates in the plane is the center of a disk with radius $1/1000$.

(1) Prove that there exists an equilateral triangle whose vertices lie in different discs.

(2) Prove that every equilateral triangle with vertices in different discs has side-length greater than 96.

Radu Gologan, Romania

The "¿ 96" in **(b)** can be strengthened to "¿ 124". By the way, part **(a)** of this problem is the place where I used the well-known "Dedekind" theorem (`http://mathlinks.ro/viewtopic.php?t=5537`).

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6 Let $f(k)$ be the number of integers n satisfying the following conditions:

(i) $0 \leq n < 10^k$ so n has exactly k digits (in decimal notation), with leading zeroes allowed;

(ii) the digits of n can be permuted in such a way that they yield an integer divisible by 11.

Prove that $f(2m) = 10f(2m-1)$ for every positive integer m.

Proposed by Dirk Laurie, South Africa

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