The Online Math Open Spring Contest
March 22 – April 2, 2019

Online Math Open

Acknowledgments

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Contest Information

These rules supersede any rules found elsewhere about the OMO. Please send any further questions directly to the OMO Team at `OnlineMathOpenTeam@gmail.com`.

Team Registration and Eligibility

Students may compete in teams of up to four people, but no student may belong to more than one team. Participants must not have graduated from high school (or the equivalent secondary school institution in other countries). Teams need not remain the same between the Fall and Spring contests, and students are permitted to participate in whichever contests they like.
Only one member on each team needs to register an account on the website. Please check the website, `http://internetolympiad.org/pages/14-omo_info`, for registration instructions.
Note: when we say "up to four", we really do mean "up to"! Because the time limit is so long, partial teams are not significantly disadvantaged, and we welcome their participation.

Contest Format and Rules

The 2019 Spring Contest will consist of 30 problems; the answer to each problem will be an integer between 0 and $2^{31} - 1 = 2147483647$ inclusive. The contest window will be

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from 7PM ET on the start day to **7PM ET on the end day**. There is no time limit other than the contest window.

1. Four-function calculators (calculators which can perform only the four basic arithmetic operations) are permitted on the Online Math Open. **Any other computational aids, including scientific calculators, graphing calculators, or computer programs, are prohibited.** All problems on the Online Math Open are solvable without a calculator. Four-function calculators are permitted only to help participants reduce computation errors.

2. Drawing aids such as graph paper, ruler, and compass are permitted. However, electronic drawing aids, such as Geogebra and graphing calculators, are not allowed. Print and electronic publications are also not allowed.

3. Members of different teams may not communicate with each other about the contest while the contest is running.

4. Your score is the number of questions answered correctly; that is, every problem is worth one point. Ties will be broken based on the "hardest" problem that a team answered correctly. Remaining ties will be broken by the second hardest problem solved, and so on. (Problem m is harder than problem n if fewer teams solve problem m OR if the number of solves is equal and $m > n$.)

5. *Participation in the Online Math Open is free.*

Clarifications and Results

Clarifications will be posted as they are answered. For the most recent contests, they will be posted at `http://internetolympiad.org/pages/n/omo_problems`. If you have a question about problem wording, please email `OnlineMathOpenTeam@gmail.com` with "Clarification" in the subject. We have the right to deny clarification requests that we feel we cannot answer.
After the contest is over, we will release the answers to the problems within the next day. Please do not discuss the test until answers are released. If you have a protest about an answer, you may send an email to `OnlineMathOpenTeam@gmail.com` (include "Protest" in the subject). Results will be released in the following weeks. (Results will be counted only for teams that submit answers at least once. Teams that only register an account will not be listed in the final rankings.)

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1. Daniel chooses some distinct subsets of $\{1, \ldots, 2019\}$ such that any two distinct subsets chosen are disjoint. Compute the maximum possible number of subsets he can choose.

2. Let $A = (0, 0), B = (1, 0), C = (-1, 0)$, and $D = (-1, 1)$. Let $\mathcal{C}$ be the closed curve given by the segment AB, the minor arc of the circle $x^2 + (y-1)^2 = 2$ connecting B to C, the segment CD, and the minor arc of the circle $x^2 + (y-1)^2 = 1$ connecting D to A. Let $\mathcal{D}$ be a piece of paper whose boundary is $\mathcal{C}$. Compute the sum of all integers $2 \le n \le 2019$ such that it is possible to cut $\mathcal{D}$ into n congruent pieces of paper.

3. Compute the smallest positive integer that can be expressed as the product of four distinct integers.

4. Compute $\left\lceil \sum_{k=2018}^{\infty} \frac{2019! - 2018!}{k!} \right\rceil$. (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

5. Consider the set S of lattice points (x, y) with $0 \le x, y \le 8$. Call a function $f : S \to \{1, 2, \ldots, 9\}$ a *Sudoku function* if:

   - $\{f(x, 0), f(x, 1), \ldots, f(x, 8)\} = \{1, 2, \ldots, 9\}$ for each $0 \le x \le 8$ and $\{f(0, y), f(1, y), \ldots, f(8, y)\} = \{1, 2, \ldots, 9\}$ for each $0 \le y \le 8$.
   - For any integers $0 \le m, n \le 2$ and any $0 \le i_1, j_1, i_2, j_2 \le 2$, $f(3m+i_1, 3n+j_1) \ne f(3m+i_2, 3n+j_2)$ unless $i_1 = i_2$ and $j_1 = j_2$.

   Over all Sudoku functions f, compute the maximum possible value of $\sum_{0 \le i \le 8} f(i, i) + \sum_{0 \le i \le 7} f(i, i+1)$.

6. Let $A, B, C, ..., Z$ be 26 nonzero real numbers. Suppose that $T = TNYWR$. Compute the smallest possible value of
$$\left\lceil A^2 + B^2 + \cdots + Z^2 \right\rceil .$$
   (The notation $\lceil x \rceil$ denotes the least integer n such that $n \ge x$.)

7. Let $ABCD$ be a square with side length 4. Consider points P and Q on segments AB and BC, respectively, with $BP = 3$ and $BQ = 1$. Let R be the intersection of AQ and DP. If BR^2 can be expressed in the form $\frac{m}{n}$ for coprime positive integers m, n, compute $m + n$.

8. In triangle ABC, side AB has length 10, and the A- and B-medians have length 9 and 12, respectively. Compute the area of the triangle.

9. Susan is presented with six boxes $B_1, \ldots, B_6$, each of which is initially empty, and two identical coins of denomination 2^k for each $k = 0, \ldots, 5$. Compute the number of ways for Susan to place the coins in the boxes such that each box B_k contains coins of total value 2^k.

10. When two distinct digits are randomly chosen in $N = 123456789$ and their places are swapped, one gets a new number N' (for example, if 2 and 4 are swapped, then $N' = 143256789$). The expected value of N' is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the remainder when $m + n$ is divided by 10^6.

11. Jay is given 99 stacks of blocks, such that the ith stack has i^2 blocks. Jay must choose a positive integer N such that from each stack, he may take either 0 blocks or exactly N blocks. Compute the value Jay should choose for N in order to maximize the number of blocks he may take from the 99 stacks.

12. A set D of positive integers is called *indifferent* if there are at least two integers in the set, and for any two distinct elements $x, y \in D$, their positive difference $|x - y|$ is also in D. Let $M(x)$ be the smallest size of an indifferent set whose largest element is x. Compute the sum $M(2) + M(3) + \cdots + M(100)$.

13. Let $S = \{10^n + 1000 : n = 0, 1, \ldots\}$. Compute the largest positive integer not expressible as the sum of (not necessarily distinct) elements of S.

14. The sum
$$\sum_{i=0}^{1000} \frac{\binom{1000}{i}}{\binom{2019}{i}}$$
can be expressed in the form $\dfrac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$.

15. Evan has 66000 omons, particles that can cluster into groups of a perfect square number of omons. An omon in a cluster of n^2 omons has a potential energy of $\frac{1}{n}$. Evan accurately computes the sum of the potential energies of all the omons. Compute the smallest possible value of his result.

16. In triangle ABC, $BC = 3, CA = 4$, and $AB = 5$. For any point P in the same plane as ABC, define $f(P)$ as the sum of the distances from P to lines AB, BC, and CA. The area of the locus of P where $f(P) \leq 12$ is $\frac{m}{n}$ for relatively prime positive integers m and n. Compute $100m + n$.

17. Let $ABCD$ be an isosceles trapezoid with $\overline{AD} \parallel \overline{BC}$. The incircle of $\triangle ABC$ has center I and is tangent to $\overline{BC}$ at P. The incircle of $\triangle ABD$ has center J and is tangent to $\overline{AD}$ at Q. If $PI = 8$, $IJ = 25$, and $JQ = 15$, compute the greatest integer less than or equal to the area of $ABCD$.

18. Define a function f as follows. For any positive integer i, let $f(i)$ be the smallest positive integer j such that there exist pairwise distinct positive integers a, b, c, and d such that $\gcd(a, b)$, $\gcd(a, c)$, $\gcd(a, d)$, $\gcd(b, c)$, $\gcd(b, d)$, and $\gcd(c, d)$ are pairwise distinct and equal to $i, i+1, i+2, i+3, i+4$, and j in some order, if any such j exists; let $f(i) = 0$ if no such j exists. Compute $f(1) + f(2) + \cdots + f(2019)$.

19. Arianna and Brianna play a game in which they alternate turns writing numbers on a paper. Before the game begins, a referee randomly selects an integer N with $1 \leq N \leq 2019$, such that i has probability $\frac{i}{1+2+\cdots+2019}$ of being chosen. First, Arianna writes 1 on the paper. On any move thereafter, the player whose turn it is writes $a+1$ or $2a$, where a is any number on the paper, under the conditions that no number is ever written twice and any number written does not exceed N. No number is ever erased. The winner is the person who first writes the number N. Assuming both Arianna and Brianna play optimally, the probability that Brianna wins can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m + n$.

20. Let ABC be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. Suppose X and Y are points such that

    - BC and XY are parallel
    - BX and CY intersect at a point P on the circumcircle of $\triangle ABC$
    - the circumcircles of $\triangle BCX$ and $\triangle BCY$ are tangent to AB and AC, respectively.

    Then AP^2 can be written in the form $\frac{p}{q}$ for relatively prime positive integers p and q. Compute $100p + q$.

21. Define a sequence by $a_0 = 2019$ and $a_n = a_{n-1}^{2019}$ for all positive integers n. Compute the remainder when
$$a_0 + a_1 + a_2 + \cdots + a_{51}$$
is divided by 856.

22. For any set S of integers, let $f(S)$ denote the number of integers k with $0 \leq k < 2019$ such that there exist $s_1, s_2 \in S$ satisfying $s_1 - s_2 = k$. For any positive integer m, let x_m be the minimum possible value of $f(S_1) + \cdots + f(S_m)$ where $S_1, \ldots, S_m$ are nonempty sets partitioning the positive integers. Let M be the minimum of $x_1, x_2, \ldots$, and let N be the number of positive integers m such that $x_m = M$. Compute $100M + N$.

23. Let a_1, a_2, a_3, a_4, and a_5 be real numbers satisfying
$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_1 = 20,$$
$$a_1a_3 + a_2a_4 + a_3a_5 + a_4a_1 + a_5a_2 = 22.$$
Then the smallest possible value of $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2$ can be expressed as $m + \sqrt{n}$, where m and n are positive integers. Compute $100m + n$.

24. We define the binary operation $\times$ on elements of $\mathbb{Z}^2$ as
$$(a, b) \times (c, d) = (ac + bd, ad + bc)$$
for all integers a, b, c, and d. Compute the number of ordered six-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ of integers such that
$$[[[[(1, a_1) \times (2, a_2)] \times (3, a_3)] \times (4, a_4)] \times (5, a_5)] \times (6, a_6) = (350, 280).$$

25. Let S be the set of positive integers not divisible by p^4 for all primes p. Anastasia and Bananastasia play a game.

    At the beginning, Anastasia writes down the positive integer N on the board. Then the players take moves in turn; Bananastasia moves first. On any move of his, Bananastasia replaces the number n on the blackboard with a number of the form $n - a$, where $a \in S$ is a positive integer. On any move of hers, Anastasia replaces the number n on the blackboard with a number of the form n^k, where k is a positive integer. Bananastasia wins if the number on the board becomes zero.

    Compute the second-smallest possible value of N for which Anastasia can prevent Bananastasia from winning.

26. There exists a unique prime $p > 5$ for which the decimal expansion of $\frac{1}{p}$ repeats with a period of exactly 294. Given that $p > 10^{50}$, compute the remainder when p is divided by 10^9.

27. Let G be a graph on n vertices $V_1, V_2, \ldots, V_n$ and let $P_1, P_2, \ldots, P_n$ be points in the plane. Suppose that, whenever V_i and V_j are connected by an edge, P_iP_j has length 1; in this situation, we say that the P_i form an *embedding* of G in the plane. Consider a set $S \subseteq \{1, 2, \ldots, n\}$ and a configuration of points Q_i for each $i \in S$. If the number of embeddings of G such that $P_i = Q_i$ for each $i \in S$ is finite and nonzero, we say that S is a *tasty* set. Out of all tasty sets S, we define a function $f(G)$ to be the smallest size of a tasty set. Let T be the set of all connected graphs on n vertices with $n - 1$ edges. Choosing G uniformly and at random from T, let a_n be the expected value of $\frac{f(G)^2}{n^2}$. Compute $\left\lfloor 2019 \lim_{n\to\infty} a_n \right\rfloor$.

28. Let ABC be a triangle. There exists a positive real number x such that $AB = 6x^2 + 1$ and $AC = 2x^2 + 2x$, and there exist points W and X on segment AB along with points Y and Z on segment AC such that $AW = x$, $WX = x + 4$, $AY = x + 1$, and $YZ = x$. For any line ℓ not intersecting segment BC, let $f(\ell)$ be the unique point P on line ℓ and on the same side of BC as A such that ℓ is tangent to the circumcircle of triangle PBC. Suppose lines $f(WY)f(XY)$ and $f(WZ)f(XZ)$ meet at B, and that lines $f(WZ)f(WY)$ and $f(XY)f(XZ)$ meet at C. Then the product of all possible values for the length of BC can be expressed in the form $a + \dfrac{b\sqrt{c}}{d}$ for positive integers a, b, c, d with c squarefree and $\gcd(b, d) = 1$. Compute $100a + b + c + d$.

29. Let n be a positive integer and let $P(x)$ be a monic polynomial of degree n with real coefficients. Also let $Q(x) = (x + 1)^2(x + 2)^2 \ldots (x + n + 1)^2$. Consider the minimum possible value m_n of $\displaystyle\sum_{i=1}^{n+1} \frac{i^2 P(i^2)^2}{Q(i)}$. Then there exist positive constants a, b, c such that, as n approaches infinity, the ratio between m_n and $a^{2n}n^{2n+b}c$ approaches 1. Compute $\lfloor 2019abc^2 \rfloor$.

30. Let ABC be a triangle with symmedian point K, and let $\theta = \angle AKB - 90^\circ$. Suppose that θ is both positive and less than $\angle C$. Consider a point K' inside $\triangle ABC$ such that A, K', K, and B are concyclic and $\angle K'CB = \theta$. Consider another point P inside $\triangle ABC$ such that $K'P \perp BC$ and $\angle PCA = \theta$. If $\sin \angle APB = \sin^2(C - \theta)$ and the product of the lengths of the A- and B-medians of $\triangle ABC$ is $\sqrt{\sqrt{5}+1}$, then the maximum possible value of $5AB^2 - CA^2 - CB^2$ can be expressed in the form $m\sqrt{n}$ for positive integers m, n with n squarefree. Compute $100m + n$.