Solving Series Using Partial Fractions

When working with sequences and series, sometimes partial fractions are needed to solve the problem. The first step is to recognize what types of sequence or series problems require the use of partial fractions. Solving the partial fraction will help set up a new series that enables to solve using limits.

Example: Find if the series converges or diverges. If convergent then find its sum.

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)}$$

Step 1: Recognize that the series requires the use of partial fractions. Usually, it involves a denominator that can be or already is factored.

Set up to solve using partial fractions $\rightarrow \rightarrow \frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3}$

Step 2: To solve partial fractions multiply both sides of our new equations by the common denominator $[(k+2)(k+3)]$. Next, solve for the coefficients, *A* and *B*.

$$[(k+2)(k+3)] \quad \left(\frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3}\right) \quad [(k+2)(k+3)]$$

$$1 = A(k+3) + B(k+2)$$

Let *k* = -2

$$1 = A[(-2)+3] + B[(-2)+2]$$
$$1 = A(1) + B(0)$$
$$1 = A(1)$$
$$A = 1$$

Solve your remaining Equations to get coefficients

Let *k* = -3

$$1 = A[(-3)+3] + B[(-3)+2]$$
$$1 = A(0) + B(-1)$$
$$1 = B(-1)$$
$$B = -1$$

Step 3 Using the coefficients found in step 2, put those values into the original equations to set up the new series.

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{A}{k+2} + \frac{B}{k+3} = \frac{1}{k+2} - \frac{1}{k+3}$$

Step 4 Write out the first few terms of the series and notice what is happening with the values. The second value is always canceling with the exception of the last term, along with the first value after the initial term. This is a Telescoping Series.

$$\sum_{k=1}^{\infty} \frac{1}{k+2} - \frac{1}{k+3} \qquad k=1 \qquad k=2 \qquad k=3 \qquad k=4 \qquad k=n$$

$$\sum_{k=1}^{\infty} = \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right] + \left[\frac{1}{5} - \frac{1}{6}\right] + \left[\frac{1}{6} - \frac{1}{7}\right] + \cdots \left[\frac{1}{k+2} - \frac{1}{k+3}\right]$$

$$\sum_{k=1}^{\infty} = \left[\frac{1}{3} - \frac{1}{k+3}\right]$$

Step 5 Evaluate the limit of the new series. The limit will tell if it converges or diverges.

$$\sum_{k=1}^{\infty} = \left[\frac{1}{3} - \frac{1}{k+3}\right] = \lim_{k\to\infty} \left[\frac{1}{3} - \frac{1}{k+3}\right]$$

$$= \frac{1}{3} - \frac{1}{\infty} = \frac{1}{3} - 0 = \frac{1}{3}$$

Since it is possible to solve for the limit, it is known that the series converges, and the sum of the series is $\frac{1}{3}$.

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{3}$$