Math Olympiad Problem Solving
Stanford University EPGY Summer Institutes 2008
Combinatorics Olympiad Problems

1. Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

2. A walk consists of a sequence of steps of length 1 taken in directions north, south, east, or west. A walk is called *self-avoiding* if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Show that
$$2^n < f(n) \le 4 \cdot 3^{n-1}.$$

3. Let n be a fixed positive integer. Find the sum of all positive integers with the following property: In base 2, it has exactly $2n$ digits consisting of n 1's and n 0's. The first digit cannot be 0.

4. Let n be an even integer not less than 4. A cube with edge n in length (an n-cube) is constructed from n^3 unit cubes. There are $\dfrac{n^3}{4}$ different colors given and exactly 4 unit cubes are colored in each of these given colors. Prove that one can choose n unit cubes of different colors, no two of which are in the same level (a level is a set of n^2 unit cubes whose centers lie in a plane parallel to one of the faces of the n-cube).

5. Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row
$$GBBGGGBGBGGGBGBGGBGG$$
we have $S = 12$. If all possible orders of these 20 people are considered, what is the average value of S? Generalize this result to a group of m boys and n girls.

6. Let n be an odd integer greater than 1. Find the number of permutations p of the set $\{1, 2, \ldots, n\}$ such that
$$|p(1) - 1| + |p(2) - 2| + \cdots + |p(n) - n| = \frac{n^2 - 1}{2}.$$

7. In a sequence of coin tosses, one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc. We denote these by TH, HH, etc. For example, in the sequence

$$HHTTHHHHTHHTTTT$$

of 15 coin tosses, we observe that there are five H,H, three HT, two TH, and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

8. Determine the smallest integer n, $n \geq 4$, for which one can choose four different numbers a, b, c, d from any n distinct integers such that $a + b - c - d$ is divisible by 20.

9. Let m and n be positive integers. Suppose that a given rectangle can be tiled by a combination of horizontal $1 \times m$ strips and vertical $n \times 1$ strips. Prove that it can be tiled using only one of the two types.

10. A subset M of $\{1, 2, 3, \ldots, 15\}$ does not contain three elements whose product is a perfect square. Determine the maximum number of elements of M.