Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Junior Section, Round 2)

1. In $\triangle ABC$, $\angle A = 2\angle B$. Let a, b, c be the lengths of its sides BC, CA, AB, respectively. Prove that

$$a^2 = b(b+c).$$

2. The set of 2000-digit integers are divided into two sets: the set M consisting all integers each of which can be represented as the product of two 1000-digit integers, and the set N which contains the other integers. Which of the sets M and N contains more elements?

3. Suppose $\overline{a_1a_2\ldots a_{2009}}$ is a 2009-digit integer such that for each $i = 1, 2, \ldots, 2007$, the 2-digit integer $\overline{a_ia_{i+1}}$ contains 3 distinct prime factors. Find a_{2008}. (Note: $\overline{xyz\ldots}$ denotes an integer whose digits are $x, y, z, \ldots$.)

4. Let S be the set of integers that can be written in the form $50m + 3n$ where m and n are non-negative integers. For example 3, 50, 53 are all in S. Find the sum of all positive integers not in S.

5. Let a, b be positive real numbers satisfying $a + b = 1$. Show that if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1x_2\ldots x_5 = 1$, then

$$(ax_1 + b)(ax_2 + b)\cdots(ax_5 + b) \geq 1.$$

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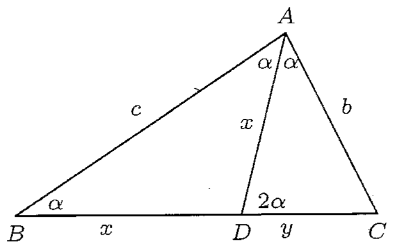
1. Let AD be the angle bisector of $\angle A$. Then $\triangle ABC \simeq \triangle DAC$. Thus $AB/DA = AC/DC = BC/AC$. Let $BD = x$ and $DC = y$. Then $c/x = b/y = a/b$. Thus $cb = ax, b^2 = ay$. Thus $b^2 + cb = ax + ay$ and hence $b(b+c) = a^2$.



2. We solve the general case of $2n$-digit integers where $n \geq 2$. There are $10^{2n} - 10^{2n-1}$ $2n$-digit integers. There are $10^n - 10^{n-1}$ n-digit integers. Consider all the products of pairs of n-digit integers. The total number P of such products satisfies

$$\begin{aligned} P &\leq 10^n - 10^{n-1} + \frac{(10^n - 10^{n-1})(10^n - 10^{n-1} - 1)}{2} \\ &= \frac{10^{2n} - 10^{2n-1} - (10^{2n-1} - 10^{2n-2} - 10^n + 10^{n-1})}{2} \\ &< \frac{10^{2n} - 10^{2n-1}}{2}. \end{aligned}$$

These products include all the numbers in M. Thus $|M| < |N|$.

3. Two-digit numbers which contain three distinct prime factors are:

$30 = 2 \cdot 3 \cdot 5, 42 = 2 \cdot 3 \cdot 7, 60 = 4 \cdot 3 \cdot 5, 66 = 2 \cdot 3 \cdot 11, 70 = 2 \cdot 5 \cdot 7, 78 = 2 \cdot 3 \cdot 13, 84 = 4 \cdot 3 \cdot 7$

From here, we conclude that $a_i = 6$ for $i = 1, 2, \ldots, 2007$ and a_{2008} is either 6 or 0.

4. If x is the smallest integer in S such that $x \equiv i \pmod 3$, then $x + 3k \in S$ and $x - 3(k+1) \notin S$ for all $k \geq 0$. We have 3 is the smallest multiple of 3 that is in S; 50 is smallest number in S that is $\equiv 2 \pmod 3$ and 100 is the smallest number in S that is $\equiv 1 \pmod 3$. Thus the positive numbers not in S are $1, 4, \ldots, 97$ and $2, 5, \ldots, 47$. Their sum is

$$\frac{33(97+1)}{2} + \frac{16(2+47)}{2} = 2009.$$

5. The left hand side is

$$\begin{aligned} & a^5 x_1 x_2 \ldots x_5 + a^4 b(x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5) \\ & \quad + a^3b^2(x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5) \\ & \quad + a^2b^3(x_1x_2 + x_1x_3 + \cdots + x_4x_5) + ab^4(x_1 + x_2 + \cdots + x_5) + b^5 \\ & \quad \geq a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = (a+b)^5 = 1. \end{aligned}$$

The last is true since by AM-GM inequality,

$$\begin{aligned} & x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_2x_3x_4x_5 \geq 5(x_1x_2x_3x_4x_5)^{4/5} = 5 \\ & x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{6/10} = 10 \\ & x_1x_2 + x_1x_3 + \cdots + x_4x_5 \geq 10(x_1x_2x_3x_4x_5)^{4/10} = 10 \\ & x_1 + x_2 + \cdots + x_5 \geq 5(x_1x_2x_3x_4x_5)^{1/5} = 5 \end{aligned}$$

(Note. For a proof of the general case, see Senior Q4)