

## CMO 2007 Solutions

1. What is the maximum number of non-overlapping  $2 \times 1$  dominoes that can be placed on an  $8 \times 9$  checkerboard if six of them are placed as shown? Each domino must be placed horizontally or vertically so as to cover two adjacent squares of the board. Diagram provided in problem.

*Solution.* Identify five subsets  $A, B, C, D, E$  of the board, where  $C$  consists of the squares occupied by the six dominoes already placed,  $B$  is the upper right corner,  $D$  is the lower left corner,  $A$  consists of the squares above and to the left of those in  $B \cup C \cup D$  and  $E$  consists of the squares below and to the right of those in  $B \cup C \cup D$ . The board can be coloured checkerboard fashion so that  $A$  has 13 black and 16 white squares,  $B$  a single white square,  $E$  16 black and 13 white squares and  $D$  a single black square. Each domino beyond the original six must lie either entirely in  $A \cup B \cup D$  or  $C \cup B \cup D$ , either of which contains at most 14 dominoes. Thus, altogether, we cannot have more than  $2 \times 14 + 6 = 34$  dominoes. This is achievable, by placing 14 dominoes in  $A \cup D$  and 14 in  $C \cup B$ .

2. You are given a pair of triangles for which

- (a) two sides of one triangle are equal in length to two sides of the second triangle, and
- (b) the triangles are similar, but not necessarily congruent.

Prove that the ratio of the sides that correspond under the similarity is a number between  $\frac{1}{2}(\sqrt{5} - 1)$  and  $\frac{1}{2}(\sqrt{5} + 1)$ .

*Solution.* If the triangles are isosceles, then they must be congruent and the desired ratio is 1. For, if they share equal side lengths, at least one of these side lengths on one triangle corresponds to the same length on the other. And if they share unequal side lengths, then either equal sides correspond or unequal sides correspond in both directions and the ratio is 1. This falls within the bounds.

Let the triangles be scalene. It is not possible for the same length to be an extreme length (largest or smallest) of both triangles. Therefore, we must have a situation in which the corresponding side lengths of the two triangles are  $(x, y, z)$  and  $(y, z, u)$  with  $x < y < z$  and  $y < z < u$ . We are given that  $y/x = z/y = u/z = r > 1$ . Thus,  $y = rx$  and  $z = ry = r^2x$ . From the triangle inequality  $z < x + y$ , we have that  $r^2 < 1 + r$ . Since  $r^2 - r - 1 < 0$  and  $r > 1$ ,  $1 < r < \frac{1}{2}(\sqrt{5} + 1)$ . The ratio of the dimensions from the smaller to the larger triangle is  $1/r$  which satisfies  $\frac{1}{2}(\sqrt{5} - 1) < 1/r < 1$ . The result follows.

3. Suppose that  $f$  is a real-valued function for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real numbers  $x$  and  $y$ .

- (a) Give a nonconstant polynomial that satisfies the condition.
- (b) Prove that  $f(x) \geq 0$  for all real  $x$ .

*Solution.* (a) Let  $f(x) = x^2 + 4$ . Then

$$\begin{aligned} f(xy) + f(y - x) - f(y + x) &= (x^2y^2 + 4) + (y - x)^2 + 4 - (y + x)^2 - 4 \\ &= (xy)^2 - 4xy + 4 = (xy - 2)^2 \geq 0. \end{aligned}$$

Thus,  $f(x) = x^2 + 4$  satisfies the condition.

(b) Consider  $(x, y)$  for which  $xy = x + y$ . Rewriting this as  $(x - 1)(y - 1) = 1$ , we find that this has the general solution  $(x, y) = (1 + t^{-1})(1 + t)$ , for  $t \neq 0$ . Plugging this into the inequality, we get that  $f(t - t^{-1}) \geq 0$  for all  $t \neq 0$ . For arbitrary real  $u$ , the equation  $t - t^{-1} = u$  leads to the quadratic  $t^2 - ut - 1 = 0$  which has a positive discriminant and so a real solution. Hence  $f(u) \geq 0$  for each real  $u$ .

*Comment.* The substitution  $v = y - x$ ,  $u = y + x$  whose inverse is  $x = \frac{1}{2}(u - v)$ ,  $y = \frac{1}{2}(u + v)$  renders the condition as  $f(\frac{1}{4}(u^2 - v^2)) + f(v) \geq f(u)$ . The same strategy as in the foregoing solution leads to the choice  $u = 2 + \sqrt{v^2 + 4}$  and  $f(v) \geq 0$  for all  $v$ .

4. For two real numbers  $a, b$ , with  $ab \neq 1$ , define the  $*$  operation by

$$a * b = \frac{a + b - 2ab}{1 - ab}.$$

Start with a list of  $n \geq 2$  real numbers whose entries  $x$  all satisfy  $0 < x < 1$ . Select any two numbers  $a$  and  $b$  in the list; remove them and put the number  $a * b$  at the end of the list, thereby reducing its length by one. Repeat this procedure until a single number remains.

(a) Prove that this single number is the same regardless of the choice of pair at each stage.

(b) Suppose that the condition on the numbers  $x$  in  $S$  is weakened to  $0 < x \leq 1$ . What happens if  $S$  contains exactly one 1?

*Solution to (b).* It is straightforward to verify that  $a * 1 = 1$  for  $a \neq 1$ , so that once 1 is included in the list, it can never be removed and so the list terminates with the single value 1.

*Solution to (a).* There are several ways of approaching (a). It is important to verify that the set  $\{x : 0 < x < 1\}$  is closed under the operation  $*$  so that it is always defined.

If  $0 < a, b < 1$ , then

$$0 < \frac{a + b - 2ab}{1 - ab} < 1.$$

The left inequality follows from

$$a + b - 2ab = a(1 - b) + b(1 - a) > 0$$

and the right from

$$1 - \frac{a + b - 2ab}{1 - ab} = \frac{(1 - a)(1 - b)}{1 - ab} > 0.$$

Hence, it will never happen that a set of numbers will contain a pair of reciprocals, and the operation can always be performed.

*Solution 1.* It can be shown by induction that any two numbers in any of the sets arise from disjoint subsets of  $S$ .

Use an induction argument on the number of entries that one starts with. At each stage the number of entries is reduced by one. If we start with  $n$  numbers, the final result is

$$\frac{\sigma_1 - 2\sigma_2 + 3\sigma_3 - \cdots + (-1)^{n-1}n\sigma_n}{1 - \sigma_2 + 2\sigma_3 - 3\sigma_4 + \cdots + (-1)^{n-1}(n-1)\sigma_n},$$

where  $\sigma_i$  is the symmetric sum of all  $\binom{n}{i}$   $i$ -fold products of the  $n$  elements  $x_i$  in the list.

*Solution 2.* Define

$$a * b = \frac{a + b - 2ab}{1 - ab}.$$

This operation is commutative and also associative:

$$a * (b * c) = (a * b) * c = \frac{a + b + c - 2(ab + bc + ca) + 3abc}{1 - (ab + bc + ca) + 2abc}.$$

Since the final result amounts to a  $*$ -product of elements of  $S$  with some arrangement of brackets, the result follows.

*Solution 3.* Let  $\phi(x) = x/(1-x)$  for  $0 < x < 1$ . This is a one-one function from the open interval  $(0, 1)$  to the half line  $(0, \infty)$ . For any numbers  $a, b \in S$ , we have that

$$\begin{aligned}\phi\left(\frac{a+b-2ab}{1-ab}\right) &= \frac{a+b-2ab}{(1-ab)-(a+b-2ab)} = \frac{a+b-2ab}{1-a-b+ab} \\ &= \frac{a}{1-a} + \frac{b}{1-b} = \phi(a) + \phi(b) .\end{aligned}$$

Let  $T = \{\phi(s) : s \in S\}$ . Then replacing  $a, b$  in  $S$  as indicated corresponds to replacing  $\phi(a)$  and  $\phi(b)$  in  $T$  by  $\phi(a) + \phi(b)$  to get a new pair of sets related by  $\phi$ . The final result is the inverse under  $\phi$  of  $\sum\{\phi(s) : s \in S\}$ .

*Solution 4.* Let  $f(x) = (1-x)^{-1}$  be defined for positive  $x$  unequal to 1. Then  $f(x) > 1$  if and only if  $0 < x < 1$ . Observe that

$$f(x * y) = \frac{1-xy}{1-x-y+xy} = \frac{1}{1-x} + \frac{1}{1-y} - 1 .$$

If  $f(x) > 1$  and  $f(y) > 1$ , then also  $f(x * y) > 1$ . It follows that if  $x$  and  $y$  lie in the open interval  $(0, 1)$ , so does  $x * y$ . We also note that  $f(x)$  is a one-one function.

To each list  $L$ , we associate the function  $g(L)$  defined by

$$g(L) = \sum\{f(x) : x \in L\} .$$

Let  $L_n$  be the given list, and let the subsequent lists be  $L_{n-1}, L_{n-2}, \dots, L_1$ , where  $L_i$  has  $i$  elements. Since  $f(x * y) = f(x) + f(y) - 1$ ,  $g(L_i) = g(L_n) - (n-i)$  regardless of the choice that creates each list from its predecessors. Hence  $g(L_1) = g(L_n) - (n-1)$  is fixed. However,  $g(L_1) = f(a)$  for some number  $a$  with  $0 < a < 1$ . Hence  $a = f^{-1}(g(L_n) - (n-1))$  is fixed.

5. Let the incircle of triangle  $ABC$  touch sides  $BC, CA$  and  $AB$  at  $D, E$  and  $F$ , respectively. Let  $\Gamma, \Gamma_1, \Gamma_2$  and  $\Gamma_3$  denote the circumcircles of triangle  $ABC, AEF, BDF$  and  $CDE$  respectively.

Let  $\Gamma$  and  $\Gamma_1$  intersect at  $A$  and  $P$ ,  $\Gamma$  and  $\Gamma_2$  intersect at  $B$  and  $Q$ , and  $\Gamma$  and  $\Gamma_3$  intersect at  $C$  and  $R$ .

(a) Prove that the circles  $\Gamma_1, \Gamma_2$  and  $\Gamma_3$  intersect in a common point.

(b) Show that  $PD, QE$  and  $RF$  are concurrent.

*Solution.* (a) Let  $I$  be the incentre of triangle  $ABC$ . Since the quadrilateral  $AEIF$  has right angles at  $E$  and  $F$ , it is concyclic, so that  $\Gamma_1$  passes through  $I$ . Similarly,  $\Gamma_2$  and  $\Gamma_3$  pass through  $I$ , and (a) follows.

(b) Let  $\omega$  and  $I$  denote the incircle and incentre of triangle  $ABC$ , respectively. Observe that, since  $AI$  bisects the angle  $FAE$  and  $AF = AE$ , then  $AI$  right bisects the segment  $FE$ . Similarly,  $BI$  right bisects  $DF$  and  $CI$  right bisects  $DE$ .

We invert the diagram through  $\omega$ . Under this inversion, let the image of  $A$  be  $A'$ , etc. Note that the centre  $I$  of inversion is collinear with any point and its image under the inversion. Under this inversion, the image of  $\Gamma_1$  is  $EF$ , which makes  $A'$  the midpoint of  $EF$ . Similarly,  $B'$  is the midpoint of  $DF$  and  $C'$  is the midpoint of  $DE$ . Hence,  $\Gamma'$ , the image of  $\Gamma$  under this inversion, is the circumcircle of triangle  $A'B'C'$ , which implies that  $\Gamma'$  is the nine-point circle of triangle  $DEF$ .

Since  $P$  is the intersection of  $\Gamma$  and  $\Gamma_1$  other than  $A$ ,  $P'$  is the intersection of  $\Gamma'$  and  $EF$  other than  $A'$ , which means that  $P'$  is the foot of the altitude from  $D$  to  $EF$ . Similarly,  $Q'$  is the foot of the altitude from  $E$  to  $DF$  and  $R'$  is the foot of the altitude from  $F$  to  $DE$ .

Now, let  $X, Y$  and  $Z$  be the midpoints of arcs  $BC, AC$  and  $AB$  on  $\Gamma$  respectively. We claim that  $X$  lies on  $PD$ .

Let  $X'$  be the image of  $X$  under the inversion, so  $I$ ,  $X$  and  $X'$  are collinear. But  $X$  is the midpoint of arc  $BC$ , so  $A$ ,  $A'$ ,  $I$ ,  $X'$  and  $X$  are collinear. The image of line  $PD$  is the circumcircle of triangle  $P'ID$ , so to prove that  $X$  lies on  $PD$ , it suffices to prove that points  $P'$ ,  $I$ ,  $X'$  and  $D$  are concyclic.

We know that  $B'$  is the midpoint of  $DF$ ,  $C'$  is the midpoint of  $DE$  and  $P'$  is the foot of the altitude from  $D$  to  $EF$ . Hence,  $D$  is the reflection of  $P'$  in  $B'C'$ .

Since  $IA' \perp EF$ ,  $IB' \perp DF$  and  $IC' \perp DE$ ,  $I$  is the orthocentre of triangle  $A'B'C'$ . So,  $X'$  is the intersection of the altitude from  $A'$  to  $B'C'$  with the circumcircle of triangle  $A'B'C'$ . From a wellknown fact,  $X'$  is the reflection of  $I$  in  $B'C'$ . This means that  $B'C'$  is the perpendicular bisector of both  $P'D$  and  $IX'$ , so that the points  $P'$ ,  $I$ ,  $X'$  and  $D$  are concyclic.

Hence,  $X$  lies on  $PD$ . Similarly,  $Y$  lies on  $QE$  and  $Z$  lies on  $RF$ . Thus, to prove that  $PD$ ,  $QE$  and  $RF$  are concurrent, it suffices to prove that  $DX$ ,  $EY$  and  $FZ$  are concurrent.

To show this, consider tangents to  $\Gamma$  at  $X$ ,  $Y$  and  $Z$ . These are parallel to  $BC$ ,  $AC$  and  $AB$ , respectively. Hence, the triangle  $\Delta$  that these tangents define is homothetic to the triangle  $ABC$ . Let  $S$  be the centre of homothety. Then the homothety taking triangle  $ABC$  to  $\Delta$  takes  $\omega$  to  $\Gamma$ , and so takes  $D$  to  $X$ ,  $E$  to  $Y$  and  $F$  to  $Z$ . Hence  $DX$ ,  $EY$  and  $FZ$  concur at  $S$ .

*Comment.* The solution uses the following result: Suppose  $ABC$  is a triangle with orthocentre  $H$  and that  $AH$  intersects  $BC$  at  $P$  and the circumcircle of  $ABC$  at  $D$ . Then  $HP = PD$ . The proof is straightforward: Let  $BH$  meet  $AC$  at  $Q$ . Note that  $AD \perp BC$  and  $BQ \perp AC$ . Since  $\angle ACB = \angle ADB$ ,

$$\angle HBC = \angle QBC = 90^\circ - \angle QCB = 90^\circ - \angle ACB = 90^\circ - \angle ADB = \angle DBP ,$$

from which follows the congruence of triangle  $HBP$  and  $DBP$  and equality of  $HP$  and  $PD$ .

*Solution 2.* (a) Let  $\Gamma_2$  and  $\Gamma_3$  intersect at  $J$ . Then  $BDJF$  and  $CDJE$  are concyclic. We have that

$$\begin{aligned} \angle FJE &= 360^\circ - (\angle DJF + \angle DJE) \\ &= 360^\circ - (180^\circ - \angle ABC + 180^\circ - \angle ACB) \\ &= \angle ABC + \angle ACB = 180^\circ - \angle FAE . \end{aligned}$$

Hence  $AFJE$  is concyclic and so the circumcircles of  $AEF$ ,  $BDF$  and  $CED$  pass through  $J$ .

(b) [Y. Li] Join  $RE$ ,  $RD$ ,  $RA$  and  $RB$ . In  $\Gamma_3$ ,  $\angle ERD = \angle ECD = \angle ACB$  and  $\angle REC = \angle RDC$ . In  $\Gamma$ ,  $\angle ARB = \angle ACB$ . Hence,  $\angle ERD = \angle ARB \implies \angle ARE = \angle BRD$ . Also,

$$\angle AER = 180^\circ - \angle REC = 180^\circ - \angle RDC = \angle BDR .$$

Therefore, triangle  $ARE$  and  $BRD$  are similar, and  $AR : BR = AE : BD = AF : BF$ . It follows that  $RF$  bisects angle  $ARB$ , so that  $RF$  passes through the midpoint of minor arc  $AB$  on  $\Gamma$ . Similarly,  $PD$  and  $QE$  are respective bisectors of angles  $BPC$  and  $CQA$  and pass through the midpoints of the minor arc  $BC$  and  $CA$  on  $\gamma$ .

Let  $O$  be the centre of circle  $\Gamma$ , and  $U, V, W$  be the respective midpoints of the minor arc  $BC, CA, AB$  on this circle, so that  $PU$  contains  $D$ ,  $QV$  contains  $E$  and  $RW$  contains  $F$ . It is required to prove that  $DU, EV$  and  $FW$  are concurrent.

Since  $ID$  and  $OU$  are perpendicular to  $BC$ ,  $ID \parallel OU$ . Similarly,  $IE \parallel OV$  and  $IF \parallel OW$ . Since  $|ID| = |IE| = |IF| = r$  (the inradius) and  $|OU| = |OV| = |OW| = R$  (the circumradius), a translation  $\vec{IO}$  followed by a dilatation of factor  $R/r$  takes triangle  $DEF$  to triangle  $UVW$ , so that these triangles are similar with corresponding sides parallel.

Suppose that  $EV$  and  $FW$  intersect at  $K$  and that  $DU$  and  $FW$  intersect at  $L$ . Taking account of the similarity of the triangles  $KEF$  and  $KVW$ ,  $LDF$  and  $LUW$ ,  $DEF$  and  $UVW$ , we have that

$$KF : FW = EF : VW = DF : UW = LF : LW ,$$

so that  $K = L$  and the lines  $DU, EV$  and  $FW$  intersect in a common point  $K$ , as desired.