Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. Let the diagonals of the square $ABCD$ intersect at S and let P be the midpoint of AB. Let M be the intersection of AC and PD and N the intersection of BD and PC. A circle is incribed in the quadrilateral $PMSN$. Prove that the radius of the circle is $MP - MS$.

2. Find the sum of all the 5-digit integers which are not multiples of 11 and whose digits are 1, 3, 4, 7, 9.

3. Let $a_1, a_2, \ldots, a_n$ be positive integers, not necessarily distinct but with at least five distinct values. Suppose that for any $1 \leq i < j \leq n$, there exist k, ℓ, both different from i and j such that $a_i + a_j = a_k + a_\ell$. What is the smallest possible value of n?

4. A student divides an integer m by a positive integer n, where $n \leq 100$, and claims that

$$\frac{m}{n} = 0 \cdot 167a_1a_2 \cdots .$$

Show the student must be wrong.

5. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

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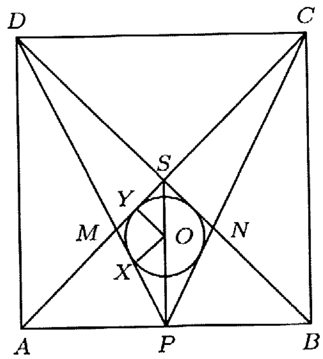
1. Let O be the centre and r the radius of the circle. Let X, Y be its points of contact with the sides PM, MS, respectively.

Since $OY \perp MS$ and $\angle YSO = \angle ASP = 45°$, $SY = YO = r$. Also $\angle OPX = \angle PDA$ (since $OP \parallel DA$) and $\angle OXP = \angle PAD = 90°$. Therefore $\triangle OXP \simeq \triangle PAD$. Hence $OX/XP = PA/AD = 1/2$. Hence $PX = 2r$. Therefore $PM - MS = 2r + MX - MY - r = r$.



2. First note that an integer is divisible by 11 if and only if the alternating sum of the digits is divisible by 11. In our case, these are the integers where 1,4 and 7 are at the odd positions. Let S be the sum of all the 5-digit integers formed by 1, 3, 4, 7, 9 and let T be the sum of those which are multiples of 11. Then

$$\begin{aligned} S &= 4!(1+3+4+7+9)(1+10+100+1000+10000) \\ &= 6399936 \\ T &= 2!2!(1+4+7)(1+100+10000) + 3!(3+9)(10+1000) = 557568. \end{aligned}$$

Thus the sum is $6399936 - 557568 = 5842368$.

3. $a_1 \le a_2 \le \cdots \le a_n$. Suppose $x < y$ are the two smallest values. Then $a_1 = x$ and let s be the smallest index such that $a_s = y$. Now there are two other terms whose sum is $x + y$. Thus we have $a_2 = x$ and $a_{s+1} = y$. Since $a_1 + a_2 = 2x$, we must have $a_3 = a_4 = x$. Similarly, by considering the largest two values $w < z$, we have $a_n = a_{n-1} = a_{n-2} = a_{n-3} = z$ and another two terms equal to w. Since there is one other value, there are at least $4 + 2 + 4 + 2 + 1 = 13$ terms. The following 13 numbers satisfy the required property: $1, 1, 1, 1, 2, 2, 3, 4, 4, 5, 5, 5, 5$. Thus the smallest possible value of n is 13.

4. We have

$$0 \cdot 167 \leq \frac{m}{n} < 0 \cdot 168 \quad \Rightarrow \quad 167n \leq 1000m < 168n.$$

Multiply by 6, we get

$$1002n \leq 6000m < 1008n \quad \Rightarrow \quad 6000m - 1000n < 8n \leq 800.$$

But $6000m - 1000n \geq 2n > 0$. Thus $6000m - 1000n \geq 1000$ since it is a multiple of 1000. We thus get a contradiction.

5. We shall prove by induction that if the original numbers are $a_1, \ldots, a_n$, $n \geq 2$, then the last number is $(1 + a_1) \cdots (1 + a_n) - 1$.

The assertion is certainly true for $n = 2$, the base case. Now suppose it is true for $n = k \geq 2$. Consider $k + 1$ numbers $a_1, \ldots, a_{k+1}$ written on the board. After one operation, we are left with k numbers. Without loss of generality, we can assume that the student erases a_k and a_{k+1} and writes $b_k = a_k + a_{k+1} + a_k a_{k+1} = (1+a_k)(1+a_{k+1}) - 1$. After a further k operations, we are left with the number

$$(1 + a_1) \cdots (1 + a_{k-1})(1 + b_k) - 1 = (1 + a_1) \cdots (1 + a_{k-1})(1 + a_k)(1 + a_{k+1}) - 1.$$

This completes the proof of the inductive step. Thus the last number is

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{2010}\right) - 1 = 2010$$

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Saturday, 25 June 2011 **0930-1230**

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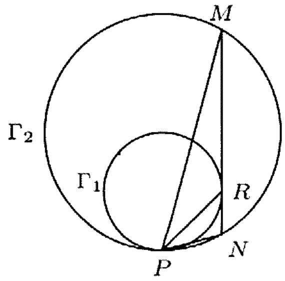
1. Suppose $a, b, c, d > 0$ and $x = \sqrt{a^2 + b^2}$, $y = \sqrt{c^2 + d^2}$. Prove that

$$xy \geq ac + bd.$$

2. Two circles Γ_1, Γ_2 with radii r_1, r_2, respectively, touch internally at the point P. A tangent parallel to the diameter through P touches Γ_1 at R and intersects Γ_2 at M and N. Prove that PR bisects $\angle MPN$.



3. Let $S_1, S_2, \ldots, S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is an integer that belongs to every S_i, $i = 1, \ldots, 2011$. (For example, $\{2, 3, 4, 5\}$ is a set of consecutive integers while $\{2, 3, 5\}$ is not.)

4. Any positive integer n can be written in the form $n = 2^a q$, where $a \geq 0$ and q is odd. We call q the *odd part* of n. Define the sequence $a_0, a_1, \ldots,$ as follows: $a_0 = 2^{2011} - 1$ and for $m \geq 0$, a_{m+1} is the odd part of $3a_m + 1$. Find a_{2011}.

5. Initially, the number 10 is written on the board. In each subsequent moves, you can either (i) erase the number 1 and replace it with a 10, or (ii) erase the number 10 and replace it with a 1 and a 25 or (iii) erase a 25 and replace it with two 10. After sometime, you notice that there are exactly one hundred copies of 1 on the board. What is the least possible sum of all the numbers on the board at that moment?

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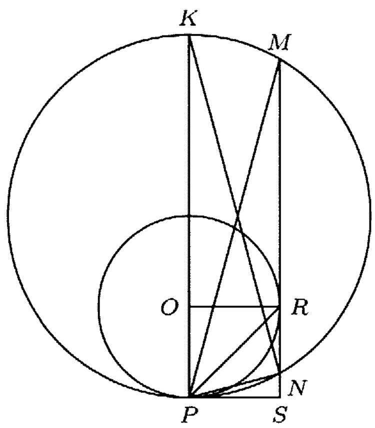
(Junior Section, Round 2 solutions)

1. It's equivalent to prove $x^2y^2 \geq (ac+bd)^2$ as all the numbers are nonnegative. This is true since

$$\begin{aligned} x^2y^2 &= (a^2+b^2)(c^2+d^2) \\ &= (ac)^2+(bd)^2+a^2d^2+b^2c^2 \\ &\geq (ac)^2+(bd)^2+2(ac)(bd) \qquad \text{AM-GM} \\ &= (ac+bd)^2. \end{aligned}$$

2. Let the tangent at P meet the tangent at R at the point S. Let O be the centre of Γ_1. Then $ORST$ is a square. Hence $\angle KPR = \angle RPS = 45°$. Also $\angle NPS = \angle NKP = \angle PMS = \angle MPK$. Thus $\angle MPR = \angle RPN$.



3. Let $a_i = \max S_i$, $b_i = \min S_i$ and suppose that $t_1 = \min\{t_i\}$. For each j, if $S_1 \cap S_j \neq \emptyset$, then $a_1 \geq b_j$. Therefore $a_1 \in S_j$.

Note: Problem 4 in the Senior Section is the general version.

4. Replace 2011 by any positive odd integer n. We first show by induction that $a_m = 3^m 2^{n-m} - 1$ for $m = 0, 1, \ldots, n-1$. This is certainly true for $m = 0$. Suppose it's true for some $m < n-1$. Then $3a_m + 1 = 3^{m+1}2^{n-m} - 2$. Since $n - m > 1$, the odd part is $3^{m+1}2^{n-m-1} - 1$ which is a_{m+1}. Now $a_{n-1} = 3^{n-1}2^1 - 1$. Thus

$3a_{n-1}+1=3^n2-2=2(3^n-1)$. When n is odd, $3^n \equiv -1 \pmod 4$. Thus $4 \nmid 3^n-1$. Hence the odd part of $2(3^n-1)$ is $\frac{3^n-1}{2}$ and this is the value of a_n.

5. Suppose the number of times that operations (i), (ii) and (iii) have been performed are x, y and z, respectively. Then the number of 1, 10 and 25 are $y-x$, $1+x-y+2z$ and $y-z$, respectively, with $-x+y=100$. Thus the sum is

$$S = y-x+10(1+x-y+2z)+25(y-z) = -890+5(5y-z).$$

Since we want the minimum values of S, y has to be as small as possible and z as large as possible. Since

$$y-x=100,\ 1+x-y+2z \geq 0, \quad y-z \geq 0$$

we get, from the first equation, $y \geq 100$, from the second inequality, $2z \geq 99$ or $z \geq 50$ and $y \geq z$ from the third. Thus the minimum is achieved when $y=100$, $x=0$ and $z=100$. Thus minimum $S=1100$.

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Singapore Mathematical Olympiad (SMO) 2012

(Junior Section, Round 2)

Saturday, 23 June 2012 **0930-1230**

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1. Let O be the centre of a parallelogram $ABCD$ and P be any point in the plane. Let M, N be the midpoints of AP, BP, respectively and Q be the intersection of MC and ND. Prove that O, P and Q are collinear.

2. Does there exist an integer A such that each of the ten digits $0, 1, \ldots, 9$ appears exactly once as a digit in exactly one of the numbers A, A^2, A^3.

3. In $\triangle ABC$, the external bisectors of $\angle A$ and $\angle B$ meet at a point D. Prove that the circumcentre of $\triangle ABD$ and the points C, D lie on the same straight line.

4. Determine the values of the positive integer n for which the following system of equations has a solution in positive integers $x_1, x_2, \ldots, x_n$. Find all solutions for each such n.

$$x_1 + x_2 + \cdots + x_n = 16 \tag{1}$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 \tag{2}$$

5. Suppose $S = \quad a_1, a_2, \ldots, a_{15}$ is a set of 15 distinct positive integers chosen from $2, 3, \ldots, 2012$ such that every two of them are coprime. Prove that S contains a prime number. (Note" Two positive integers m, n are coprime if their only common factor is 1.)

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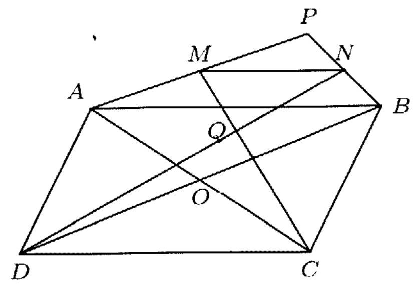
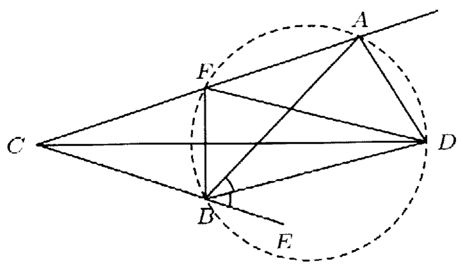
(Junior Section, Round 2 solutions)

1. Since $MN \parallel AB \parallel CD$, we have $\triangle MQN \sim \triangle CDQ$. Hence $MN = AB/2 = CD/2$. Thus $QM = CQ/2$. In $\triangle ACP$, CM is a median and Q divides CM in the ratio 1:2. Thus Q is the centroid. Hence the median PO passes through Q.



2. Since the total number of digits in A, A^2 and A^3 is 10, the total number digits in $32, 32^2, 32^3$ is 11 and the total number of digits in $20, 20^2, 20^3$ is 9, any solution A must satisfy $21 \le A \le 31$. Since the unit digits of A, A^2, A^3 are distinct, the unit digit of A can only be $2, 3, 7, 8$. Thus the only possible values of A are $22, 23, 27, 28$. None of them has the desired property. Thus no such number exists.

3. Note that CD bisects $\angle C$. If $CA = CB$, then CD is the perpendicular bisector of AB. Thus the circumcentre of $\triangle ABD$ is on CD.



If $CA \neq CB$, we may assume that $CA > CB$. Let E be a point on CB extended and F be the point on CA so that $CF = CB$. Then, since CD is the perpendicular bisector of

BF, we have $\angle AFD = \angle DBE = \angle DBA$. Thus $AFBD$ is a cyclic quadrilateral, i.e., F is on the circumcircle of $\triangle ABD$. The circumcentre lies on the perpendicular bisector of BF which is CD.

4. Without loss of generality, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. If $x_1 = 1$, then from (2), $n = 1$ and (1) cannot be satisfied. Thus $x_1 \geq 2$. If $x_2 = 2$, then $n = 2$ and again (1) cannot be satisfied. Thus $x_2 \geq 3$. Similarly, $x_3 \geq 4$. Thus $x_4 + \cdots + x_n \leq 7$ with $x_4 \geq 4$. Thus $n \leq 4$.

(i) $n = 1$: No solution.

(ii) $n = 2$: The only solution of $\frac{1}{x_1} + \frac{1}{x_2} = 1$ is $x_1 = x_2 = 2$ which doesn't satisfy (1). Thus there is no solution.

(iii) $n = 3$: The only solutions of $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1$ are $(x_1, x_2, x_3) = (2,3,6), (2,4,4)$ and $(3,3,3)$. They all do not satisfy (1).

(iv) $n = 4$: According to the discussion in the first paragraph, the solutions of $x_1 + \cdots + x_4 = 16$ are

$$\begin{aligned}(x_1, x_2, x_3, x_4) = &(2,3,4,7), (2,3,5,6), (2,4,4,6), (2,4,5,5),\\ &(3,3,4,6), (3,3,5,5), (3,4,4,5), (4,4,4,4).\end{aligned}$$

Only the last one satisfy (2).

Thus the system of equations has a solution only when $n = 4$ and for this n, the only solution is $x_1 = x_2 = x_3 = x_4 = 4$.

5. Suppose, on the contrary, that S contains no primes. For each i, let p_i be the smallest prime divisor of a_i. Then $p_1, p_2, \ldots, p_{15}$ are distinct since the numbers in S are pairwise coprime. The first 15 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. If p_j is the largest among $p_1, p_2, \ldots, p_{15}$, then $p_j \geq 47$ and $a_j \geq 47^2 = 47^2 = 2309 > 2012$, a contradiction. Thus S must contain a prime number.