Harvard-MIT Mathematics Tournament
March 15, 2003

Guts Round

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

1. **[5]** Simplify $\sqrt[2003]{2\sqrt{11} - 3\sqrt{5}} \cdot \sqrt[4006]{89 + 12\sqrt{55}}$.

2. **[5]** The graph of $x^4 = x^2 y^2$ is a union of n different lines. What is the value of n?

3. **[5]** If a and b are positive integers that can each be written as a sum of two squares, then ab is also a sum of two squares. Find the smallest positive integer c such that $c = ab$, where $a = x^3 + y^3$ and $b = x^3 + y^3$ each have solutions in integers (x, y), but $c = x^3 + y^3$ does not.

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4. **[6]** Let $z = 1 - 2i$. Find $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$.

5. **[6]** Compute the surface area of a cube inscribed in a sphere of surface area π.

6. **[6]** Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For how many n, $0 \leq n \leq 100$, is F_n a multiple of 13?

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7. **[6]** a and b are integers such that $a + \sqrt{b} = \sqrt{15 + \sqrt{216}}$. Compute a/b.

8. **[6]** How many solutions in nonnegative integers (a, b, c) are there to the equation

$$2^a + 2^b = c! \quad ?$$

9. **[6]** For x a real number, let $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. How many solutions are there to the equation

$$f(f(f(f(x)))) = x?$$

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10. [**7**] Suppose that A, B, C, D are four points in the plane, and let Q, R, S, T, U, V be the respective midpoints of AB, AC, AD, BC, BD, CD. If $QR = 2001, SU = 2002, TV = 2003$, find the distance between the midpoints of QU and RV.

11. [**7**] Find the smallest positive integer n such that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$ is divisible by 100.

12. [**7**] As shown in the figure, a circle of radius 1 has two equal circles whose diameters cover a chosen diameter of the larger circle. In each of these smaller circles we similarly draw three equal circles, then four in each of those, and so on. Compute the area of the region enclosed by a positive even number of circles.

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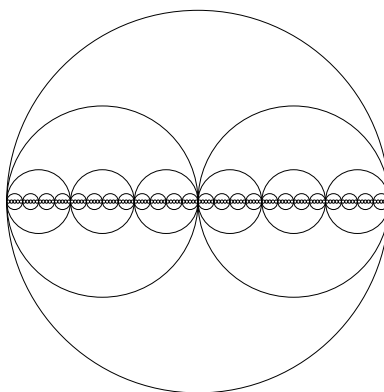
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13. [**7**] If $xy = 5$ and $x^2 + y^2 = 21$, compute $x^4 + y^4$.

14. [**7**] A positive integer will be called "sparkly" if its smallest (positive) divisor, other than 1, equals the total number of divisors (including 1). How many of the numbers $2, 3, \ldots, 2003$ are sparkly?

15. [**7**] The product of the digits of a 5-digit number is 180. How many such numbers exist?

16. [**8**] What fraction of the area of a regular hexagon of side length 1 is within distance $\frac{1}{2}$ of at least one of the vertices?

17. [**8**] There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is called a "hub" if it is directly connected to every other city. What is the largest possible number of hubs?

18. [**8**] Find the sum of the reciprocals of all the (positive) divisors of 144.

19. [**8**] Let r, s, t be the solutions to the equation $x^3 + ax^2 + bx + c = 0$. What is the value of $(rs)^2 + (st)^2 + (rt)^2$ in terms of a, b, and c?

20. [**8**] What is the smallest number of regular hexagons of side length 1 needed to completely cover a disc of radius 1?

21. [**8**] r and s are integers such that

$$3r \geq 2s - 3 \text{ and } 4s \geq r + 12.$$

What is the smallest possible value of r/s?

22. [**9**] There are 100 houses in a row on a street. A painter comes and paints every house red. Then, another painter comes and paints every third house (starting with house number 3) blue. Another painter comes and paints every fifth house red (even if it is already red), then another painter paints every seventh house blue, and so forth, alternating between red and blue, until 50 painters have been by. After this is finished, how many houses will be red?

23. [**9**] How many lattice points are enclosed by the triangle with vertices $(0, 99)$, $(5, 100)$, and $(2003, 500)$? Don't count boundary points.

24. [**9**] Compute the radius of the inscribed circle of a triangle with sides 15, 16, and 17.

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25. [**9**] Let ABC be an isosceles triangle with apex A. Let I be the incenter. If $AI = 3$ and the distance from I to BC is 2, then what is the length of BC?

26. [**9**] Find all integers x such that $x^2 + 6x + 28$ is a perfect square.

27. [**9**] The rational numbers x and y, when written in lowest terms, have denominators 60 and 70, respectively. What is the smallest possible denominator of $x + y$?

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28. [**10**] A point in three-space has distances $2, 6, 7, 8, 9$ from five of the vertices of a regular octahedron. What is its distance from the sixth vertex?

29. [**10**] A *palindrome* is a positive integer that reads the same backwards as forwards, such as 82328. What is the smallest 5-digit palindrome that is a multiple of 99?

30. [**10**] The sequence $a_1, a_2, a_3, \ldots$ of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5.

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31. [**10**] A cylinder of base radius 1 is cut into two equal parts along a plane passing through the center of the cylinder and tangent to the two base circles. Suppose that each piece's surface area is m times its volume. Find the greatest lower bound for all possible values of m as the height of the cylinder varies.

32. [**10**] If x, y, and z are real numbers such that $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, find the maximum possible value of $x - y + z$.

33. [**10**] We are given triangle ABC, with $AB = 9$, $AC = 10$, and $BC = 12$, and a point D on BC. B and C are reflected in AD to B' and C', respectively. Suppose that lines BC' and $B'C$ never meet (i.e., are parallel and distinct). Find BD.

34. [**12**] $OKRA$ is a trapezoid with OK parallel to RA. If $OK = 12$ and RA is a positive integer, how many integer values can be taken on by the length of the segment in the trapezoid, parallel to OK, through the intersection of the diagonals?

35. [**12**] A certain lottery has tickets labeled with the numbers $1, 2, 3, \ldots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is randomly drawn (again without replacement), and so forth, until an odd number is drawn. Then, every person whose ticket number was drawn (at any point in the process) wins a prize.

    You have ticket number 1000. What is the probability that you get a prize?

36. [**12**] A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples? (A whole uncut apple counts as one piece.)

37. [**15**] A *quagga* is an extinct chess piece whose move is like a knight's, but much longer: it can move 6 squares in any direction (up, down, left, or right) and then 5 squares in a perpendicular direction. Find the number of ways to place 51 quaggas on an 8×8 chessboard in such a way that no quagga attacks another. (Since quaggas are naturally belligerent creatures, a quagga is considered to attack quaggas on any squares it can move to, as well as any other quaggas on the same square.)

38. [**15**] Given are real numbers x, y. For any pair of real numbers a_0, a_1, define a sequence by $a_{n+2} = xa_{n+1} + ya_n$ for $n \geq 0$. Suppose that there exists a fixed nonnegative integer m such that, for every choice of a_0 and a_1, the numbers a_m, a_{m+1}, a_{m+3}, in this order, form an arithmetic progression. Find all possible values of y.

39. [**15**] In the figure, if $AE = 3$, $CE = 1$, $BD = CD = 2$, and $AB = 5$, find AG.

40. [**18**] All the sequences consisting of five letters from the set $\{T, U, R, N, I, P\}$ (with repetitions allowed) are arranged in alphabetical order in a dictionary. Two sequences are called "anagrams" of each other if one can be obtained by rearranging the letters of the other. How many pairs of anagrams are there that have exactly 100 other sequences between them in the dictionary?

41. [**18**] A hotel consists of a 2×8 square grid of rooms, each occupied by one guest. All the guests are uncomfortable, so each guest would like to move to one of the adjoining rooms (horizontally or vertically). Of course, they should do this simultaneously, in such a way that each room will again have one guest. In how many different ways can they collectively move?

42. [**18**] A tightrope walker stands in the center of a rope of length 32 meters. Every minute she walks forward one meter with probability 3/4 and backward one meter with probability 1/4. What is the probability that she reaches the end in front of her before the end behind her?

43. Write down an integer N between 0 and 10, inclusive. You will receive N points — unless some other team writes down the same N, in which case you receive nothing.

44. A *partition* of a number n is a sequence of positive integers, arranged in nonincreasing order, whose sum is n. For example, $n = 4$ has 5 partitions: $1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4$. Given two different partitions of the same number, $n = a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_l$, where $k \leq l$, the first partition is said to *dominate* the second if all of the following inequalities hold:

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\begin{aligned}
a_1 &\geq b_1; \\
a_1 + a_2 &\geq b_1 + b_2; \\
a_1 + a_2 + a_3 &\geq b_1 + b_2 + b_3; \\
&\vdots \\
a_1 + a_2 + \cdots + a_k &\geq b_1 + b_2 + \cdots + b_k.
\end{aligned}
$$

Find as many partitions of the number $n = 20$ as possible such that none of the partitions dominates any other. Your score will be the number of partitions you find. If you make a mistake and one of your partitions does dominate another, your score is the largest m such that the first m partitions you list constitute a valid answer.

45. Find a set S of positive integers such that no two distinct subsets of S have the same sum. Your score will be $\lfloor 20(2^n/r - 2) \rfloor$, where n is the number of elements in the set S, and r is the largest element of S (assuming, of course, that this number is nonnegative).

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