Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2016

Open Section (Round 1)

Wednesday, 1 June 2016 — 0930-1200 hrs

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

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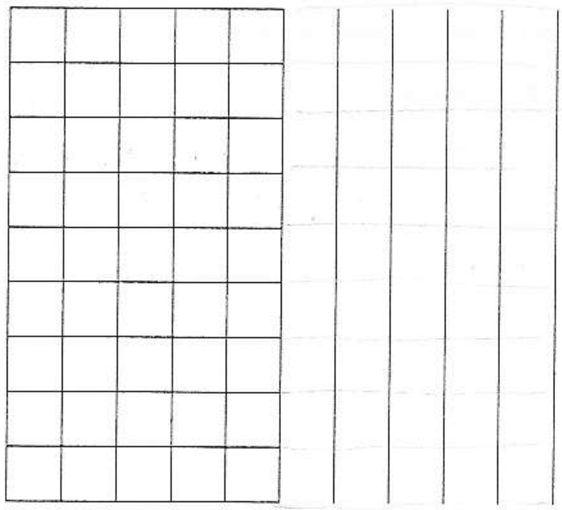
In this paper, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x. For examples, $\lfloor 5 \rfloor = 5$, $\lfloor 2.8 \rfloor = 2$, and $\lfloor -2.3 \rfloor = -3$.

1. The perimeter of a triangle ABC is 48. The point D is the midpoint of AB such that $DC = DA = 10$. Find the area of triangle CBD.

2. In an infinite geometric progression with a nonzero first term and a common ratio $\dfrac{1}{\sqrt[6]{2}}$, the sum of the first n terms equals the sum of all the remaining terms. Find n.

3. Find the difference between the largest and smallest value of x which satisfies the equation $|x-2| - |x-2016| = 2|x-1009|$.

4. The figure below shows a 10×9 rectangular board. All the small squares shown in the figure are squares of the same size.



   Find the total number of rectangles in the above figure which are not squares.

5. Find the minimum value of the function f given by
$$\mathrm{f}(x) = \sqrt{x+25} + \sqrt{36-x} + \sqrt{x}.$$

6. Let m be the number of those triangles whose longest side is 2016 and the other two sides are also of integral length. Determine $\left\lfloor \dfrac{m}{1000} \right\rfloor$.

   (Note: Two congruent triangles are considered to be the same triangle.)

7. Find $\left\lfloor \dfrac{S}{2} \right\rfloor$, where
$$S = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}}.$$

(Hint: Note that $\frac{2}{\sqrt{k}+\sqrt{k+1}} < \frac{1}{\sqrt{k}} < \frac{2}{\sqrt{k}+\sqrt{k-1}}$.)

8. Find the number of integer solutions to the equation

$$\frac{\sqrt{x}}{\sqrt{2018-x}+\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}-\sqrt{2018-x}} - \frac{1009}{x-1009} = 1.$$

9. Let $S_N = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\cdots+N}$. Find the value of $2017 \times S_{2016}$.

10. Let $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ be three unit vectors along three mutually perpendicular axes, namely, the x, y and z axes respectively, and the origin O is the intersection of the three axes. At any time $t \geq 0$ after the start of an experiment, the position of a toy plane A is located along the path $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$ and the position of another plane B is located along the path $\mathbf{r} = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$. If d is the shortest possible distance between the two toy planes, find d^2.

11. Let $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ be three unit vectors along three mutually perpendicular axes, namely, the x, y and z axes respectively, and the origin O is the intersection of the three axes. A plane mirror in the space has equation $\mathbf{r} \cdot (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 5$. A ray of light is shone along the path with equation $\mathbf{r} = 5\mathbf{i}+2\mathbf{j}+\mathbf{k}+\lambda(5\mathbf{i}+\mathbf{j})$, where λ is real. The ray of light hits the plane and is reflected along the path $\mathbf{r} = \mathbf{j}+\mathbf{k}+\mu(4\mathbf{i}+b\mathbf{j}+c\mathbf{k})$, where μ is real. Assuming that the incident ray, reflected ray and the normal at the point of incidence lie on the same plane, and that the incident ray and the reflected ray make the same angle with the mirror, find the value of $|b|+|c|$.

12. Given that $z = \lim_{\theta \to 0} \frac{\sin \theta^\circ}{\theta}$, find $\left\lfloor \frac{360z}{\pi} \right\rfloor$.

13. Find the number of ways to select four distinct integers a, b, c and d from $\{1, 2, 3, 4, \cdots, 24\}$ such that $a > b > d > c$ and that $a + c = b + d$.

14. Assume that

$$f = \sqrt{3a_1+1} + \sqrt{3a_2+1} + \sqrt{3a_3+1} \cdots + \sqrt{3a_{2016}+1}$$

where $a_1, a_2, \cdots, a_{2016}$ are real numbers such that $a_1 + a_2 + \cdots + a_{2016} = 1$. Find the maximum value of $\lfloor f \rfloor$.

15. Let p and q be integers such that the roots of the polynomial $f(x) = x^3 + px^2 + qx - 343$ are real. Determine the minimum possible value of $|p^2 - 2q|$.

16. Let $f(x) = x^{2016} + a_{2015}x^{2015} + a_{2014}x^{2014} + \cdots + a_1x^1 + a_0$ be a polynomial such that $f(i) = 2i - 1$ for all $i = 1, 2, \cdots, 2015$. Find the value of $f(0) + f(2016) - 2016!$.

17. Find the largest prime number p such that $p = a^b + 7b^a$ holds for prime numbers a and b.

18. Find the remainder when $1 \times 3 \times 5 \times \cdots \times 2017$ is divided by 1000.

19. In how many ways can the number $\frac{2017}{2016}$ be written as the product of two fractions of the form $\frac{a+2}{a}$, where a is a positive integer? [Here $\frac{a+2}{a}\frac{b+2}{b}$ and $\frac{b+2}{b}\frac{a+2}{a}$ are considered as the same product.]

20. In the triangle ABC, $\angle B = 90^\circ$ and $\angle C = 60^\circ$. Points D and E are outside the triangle ABC such that BAD and ACE are equilateral triangles. The segment DE intersects the segment AC at F. Suppose $BC = 10$. Find the length of AF.

21. A circle ω_1 of radius 8 is internally tangent to a circle ω_2 of radius 25 at a point T. A line through the centre O of ω_2 is tangent to ω_1 at S. A chord AB of ω_2 perpendicular to OS is tangent to ω_1 at Q. Find the length of AB.

22. In a rectangle $ABCD$, E and F are the midpoints of BC and CD respectively, DE intersects AF at P, DE intersects BF at Q, and AE intersects BF at R. Given the area of the triangle PQR equals 100, find the area of the triangle DQB.

23. Let ABC be an acute-angled triangle with circumcenter O. Suppose $AC = 92$, $\tan \angle OAB = \frac{1}{7}$ and $\angle CAO = 3\angle OAB$. Find the length of AB.

24. In the triangle ABC, $\angle C = 90^\circ$, $2AC > AB$, points E and F on AC and AB respectively are such that $CE = EF$. Suppose EF is the segment of minimal length that divides the area of triangle ABC into two equal halves. Given $AC = 6 + 3\sqrt{3}$, find the length of AB.

25. Determine the greatest positive constant C such that $y^5 + y^{-5} - 2 \geq C(y + y^{-1} - 2)$ for all $y > 0$.