ANDREI NEGUȚ

Problems For the Mathematical Olympiads

FROM THE FIRST TEAM SELECTION TEST TO THE IMO

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PROBLEMS FOR THE MATHEMATICAL OLYMPIADS

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To my parents Livia and Nicu, and my brother Radu

Contents

FOREWORD

This book consists of a number of math problems, all of which are meant primarily as preparation for competitions such as the International Mathematical Olympiad. They are therefore of IMO level, and require only elementary notions of math; however, since the International Mathematical Olympiad is perhaps the most difficult exam in elementary mathematics, any participant should have with him a good knowledge and grasp of what he is dealing with. This book is not meant to teach elementary math at an IMO level, but to help a prospective participant train and enhance his understanding of these concepts.

There are many such collections of problems that are directed at IMO participants. Yet in my view there are two things which individualize this book: the first of these is in the selected problems. These are some of the most beautiful problems I encountered in my four-year preparation for such high-level math competitions, and they have been presented to me by many great professors and instructors over the years. They are neither boring nor tedious, but require a certain amount of insight and ingenuity, which I find to be the necessary quality of any 'beautiful' math problem. Moreover, I have tried not to add here very well-known problems (such as problems from past IMO's or from other important contests), as these very likely become known to every student in the first year of his Olympiad career. Instead, there is a smaller chance that the reader might already know the problems presented here, and the basis of any good preparation is to work as many new problems as possible. Any one of these problems could be given at an IMO, and I hope that this book might help them come to light.

I have branded each problem with one of 3 degrees of difficulty: E for easy, M for medium and D for difficult (the reader can find the category of a problem at the beginning of its solution). But these are just relative, as an E problem is of the level of a problem 1 at an IMO, an M problem is similar to what one should expect from problem 2 and D could be a problem 3. Therefore, a novice in the Olympiad world should not feel frustrated if he has trouble with an E problem, because on an absolute scale it can be quite difficult. Such problems are far beyond the level of regular school work.

The second thing that is important about this book is the solutions. Any good participant at an IMO needs to know not so much theory as tricks to be employed in elementary problems. It is far less useful as far as IMO's are concerned (and far more difficult) for a student to learn multivariable calculus and Lagrange multipliers than to know how to apply geometrical inversion. That is why I emphasize on all these methods, lemmas and propositions in my solutions, and I have often sacrificed succinctness of a proof to the educational value of presenting one of these methods. I have also presented a few of the concepts I have employed throughout the book in an appendix. Thus I can state that in my opinion, a potential IMO participant needs two things: ingenuity on one hand and a firm grasp on all these 'toys' and tricks on the other. Which of them is more important, I do not know yet, and I can only guess.

I would like to thank the people who invented these wonderful problems. While most of the solutions are my own work, the problems are not mine and I am in great debt to their creators. Since these problems were mainly taken from my notes and papers, their exact origin is unknown to me, and I have replaced the name of their authors by the symbols ***. It is not a fitting homage, and I apologize for this.

But because I have encountered these problems during my own preparation as an IMO participant, they have also become a part of me. Each one of them is associated to the person who showed it to me, the friends who told me their beautiful solutions, or the contests in which I have or have not solved them. I would like to thank all of the professors who helped me and who made me into what I am, and though I can't name them all, it is people like Radu Gologan, Severius Moldoveanu, Dorela Fainisi, Dan Schwartz, Calin Popescu, Mihai Baluna, Bogdan Enescu, Dinu Serbanescu and Mircea Becheanu who have shown me the most beautiful and subtle art there is. I will also not forget all the comrades and friends that have passed through the Olympiad experience with me, but they will forgive me for not naming them. They know who they are. I also cannot forget my family, who has stood by me and given me that priceless moral support which is indifferent of how well or how badly I behaved in the competitions.

I would like to thank Mircea Lascu and the Gil Publishing House for supporting me and this book on the long journey to publishing, and Prof. Radu Gologan for a great deal of help and useful advice. I also want to thank Gabriel Kreindler, Andrei Stefanescu, Andrei Ungureanu and Adrian Zahariuc for providing some of the solutions present in this book.

I wish you the best of luck in all your endeavors, mathematical or not.

Andrei Neguţ

Chapter 1
GEOMETRY

PROBLEMS

1. Let I and O be the incenter and circumcenter, respectively, of the triangle ABC. The excircle ω_A is tangent to AB, AC and BC in K, M, N respectively. If the midpoint P of KM is on the circumcircle of ABC, prove that O, I, N are collinear.

IMAR test, October 2003

2. In the convex quadrilateral $ABCD$ we have that $\angle ABC = \angle ADC = 135°$. Let M, N be on AB and AD respectively, such that $\angle MCD = \angle NCB = 90°$.If K is the second intersection of the circles (AMN) and (ABD), show that $AK \perp KC$.

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3. Let AC and BD be chords in a circle of center O, and let K be their common point. If M and N are the circumcenters of the circles (ABK) and (CDK), show that $MKON$ is a parallelogram.

***

4. Prove that in any acute triangle of sides a, b, c, semiperimeter p, inradius r and circumradius R, the following inequalities hold

$$\frac{2}{5} \leq \frac{Rp}{2aR + bc} < \frac{1}{2}$$

***

5. Let us consider an angle of vertex O and a circle tangent to in the points A and B. The parallel through A at OB will cut the circle again in C, and the line OC intersects the circle again in P. Show that the line AP splits the segment OB into two equal parts.

***

6. A triangle and a square are circumscribed around the unit square. Prove that the common area of these two figures is at least 3.4.

All-Soviet Union Olympiad 1986

7. Let $ABCD$ be a convex quadrilateral and M be the intersection of its diagonals. If O_1 and O_2 are the circumcenters of ABM and CDM, show that $4O_1O_2 > AB + CD$.

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8. In the cyclic pentagon $ABCDE$ we have $AC \parallel DE$ and $\angle AMB = \angle BMC$, where M is the midpoint of BD. Show that the line BE cuts the segment AC into two equal parts.

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9. Given is a right triangle and a finite set of points inside it. Prove that these points can be united by a broken line (non necessarily closed) such that the sum of the squares of the segments of the broken line is at most the square of the hypothenuse of the triangle.

IMAR test, 2004

10. If $A_1A_2...A_n$ is a convex n-gon, prove that we cannot have more than n closed segments A_iA_j such that any two of them intersect.

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11. Inside the triangle ABC, we will consider the distinct circles k_1, k_2, k_3, k_4 of equal radii, such that each of the k_1, k_2, k_3 are tangent to two sides of the triangle, and k_4 is tangent to the other three circles. Prove that the center of k_4 lies on the line OI (where O and I are the circumcenter and incenter of ABC).

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12. In a convex polygon, let d be the sum of the lengths of its diagonals and p be the semiperimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor - 2$$

***

13. Let $A_1...A_n$ be a polygon. Prove that we can have A_iA_{i+1} be a diameter for all $1 \leq i \leq n$ (where $A_{n+1} = A_1$) iff n is odd.

***

14. We are given the circles Γ, Γ_1 and Γ_2 of radii r, r_1, r_2 respectively, any two of them exterior and such that $r > r_1$ and $r > r_2$. The common exterior tangents of Γ and Γ_1 intersect in A_1 and those of Γ and Γ_2 intersect in A_2. The tangents from A_1 to Γ_2 and those from A_2 to Γ_1 determine a quadrilateral; prove that this quadrilateral has an inscribed circle and compute its radius in terms of r, r_1, r_2

All-Soviet Union Olympiad 1984

15. Prove that for any $n > 3$ there exists a convex non-regular polygon such that the sum of the distances from any of its interior points to its sides is constant.

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16. Let us consider in the plane a finite set of points X and an equilateral triangle T. We know that any $X' \subset X$ with $\#X = 9$ can be covered by two translations of T. Show that X itself can be covered by two translations of T.

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17. If $ABCDEF$ is a cyclic hexagon and $l(A, BDF), l(B, ACE), l(D, ABF), l(E, ABC)$ are concurrent (where $l(M, XYZ)$ is the Simson line of M with respect to XYZ), prove that $CDEF$ is a rectangle.

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18. Let P be a convex hexagon which can be split up into 27 disjoint parallelograms. Prove that P has a center of symmetry and that it can be broken up into 21 disjoint parallelograms.

***

19. Let A_1, B_1, C_1 be the midpoints of the sides of the acute-angled triangle ABC. The six lines which pass through each of these points and are perpendicular to the other two sides determine a hexagon. Prove that the area of the hexagon is half the area of ABC.

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20. Let $P_1 ... P_n$ be a convex polygon in the plane, such that for any $i \neq j$ there exists some k such that $\angle P_i P_k P_j = 60°$. Prove that our polygon is just a equilateral triangle.

***

21. Let H_a and H_b be the feet of the altitudes from A and B in the acute-angled triangle ABC, and let W_a, W_b be the feet of the bisectors from A and B. Prove that the incenter of ABC lies on $H_a H_b$ iff the circumcenter of ABC lies on $W_a W_b$.

German Olympiad 2002

22. Consider the lines d_1 and d_2 concurrent in P. For any O on $d_1 - \{P\}$, consider the circle C_1 of center O and which is tangent to d_2, and the circle C_2 which is tangent to d_1, d_2 and C_1. Find the locus of the intersection of these two circles when O travels on d_1.

***

23. Let $A_0A_1A_2$ be a triangle, and ω_1 a circle that passes through A_1 and A_2. For any $k \geq 1$, let us construct ω_k tangent to ω_{k-1} and which passes through the points A_k and A_{k+1}, where the indices are considered modulo 3 (meaning that $A_6 = A_3 = A_0$, $A_7 = A_4 = A_1$ and $A_8 = A_5 = A_2$). Prove that the circles ω_7 and ω_1 are identical.

***

24. D, E, F are the points of tangency of the incircle of triangle ABC with he sides BC, AC and AB respectively. Let X be in the interior of ABC such that the incircle of XBC touches XB, XC and BC in Z, Y and D respectively. Prove that $EFZY$ is cyclic.

IMO 1995 Shortlist

25. The point P lies in the interior of the square $ABCD$ such that $PA = 1$, $PB = 2$ and $PC = 3$. What are the possible values of the side of the square?

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26. The circle Γ_2 is interior to the circle Γ_1 and tangent to it in the point N. The points C, S, T lie on Γ_1 such that CS and CT are tangent to Γ_2 in the points M and K respectively. Let U and V be the midpoints of the arcs CS and CT, and W the second point of intersection of the circles (UMC) and (VCK). Prove that $UCVW$ is a parallelogram.

***

27. Given is a quadrilateral with an inscribed circle, and the line that splits it up into two parts of equal areas and perimeters. Prove that the line passes through the center of the inscribed circle.

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28. Let X be a point inside the triangle ABC and consider the lines through X parallel to the sides of the triangle. The parallel to AB intersects CA in M, the parallel to BC intersects AB in N and the parallel to CA intersects BC in P. The lines AP, BM and CN split the triangle into 4 triangles and 3 quadrilaterals. Prove that the sum of the areas of 3 of the triangles equals the area of the fourth.

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29. Prove that a finite number of squares with the sum of their areas $\frac{1}{2}$ fit without overlap in the unit square.

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30. Let A be a point outside the circle ω and let AB and C be the tangents from it to the circle (with B and C on ω). A line l is tangent to ω and it intersects AB and AC in P and Q respectively. If R is the intersection of BC with the parallel to AC through P, then prove that as l varies, QR passes through a fixed point.

MOSP 2004

31. If the angles between the three pairs of opposite sides of a tetrahedron are equal, prove that these angles are right.

***

32. Let P be the intersection of the diagonals in the convex quadrilateral $ABCD$ with $AB = AC = BD$. Let O and I be the circumcenter and the incenter of the triangle ABP. If these two points are different, then prove that $OI \perp CD$.

***

33. Given are 500 points inside a unit square. Show that there exist 12 of them $A_1, A_2, ..., A_{12}$ such that $A_1A_2 + A_2A_3 + ... + A_{11}A_{12} < 1$.

***

34. Prove that for any points A, B, P, Q, R in the same plane we have

$$AB + PQ + QR + RP \leq AP + AQ + AR + BP + BQ + BR$$

***

35. We have a set of 5 points in the plane, such that the area of any triangle formed by these points is at least 2. Prove that there exists such a triangle of area at least 3.

Romanian TST 2002

36. Let $ABCD$ be a parallelogram and points M, N such that $C \in (AM)$ and $D \in (BN)$. The lines NA and NC intersect the lines MB and MD in the points E, F, G, H. Prove that $EFGH$ is cyclic iff $ABCD$ is a rhombus.

***

37. Let us consider a convex polygon with $n \geq 5$ sides. Prove that there exist no more than $\dfrac{n(2n-5)}{3}$ triangles of area 1 with the vertices among the vertices of the polygon.

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38. Given is a square of side-length 20, and a set formed by its four vertices and 1999 inside points. Prove that there exist 3 of those points which determine a triangle of area at most $\dfrac{1}{10}$.

JBMO 1999

39. In the tetrahedron $ABCD$ we have that $AB, CD < 1$, $AC = BD = 1$ and $AD, BC > 1$. Prove that the radius of the sphere inscribed in the tetrahedron is less than $\dfrac{\sqrt{3}}{8}$.

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40. The points A, B, C, D lie on a circle centered at O. The lines AB and CD intersect at M, and the circles (ACM) and (BDM) intersect ar M and N. Prove that $MN \perp NO$.

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Chapter 2
NUMBER THEORY

PROBLEMS

1. Let the sequence a_n defined by $a_1 = a_2 = 97$ and the recurrence $a_{n+1} = a_n a_{n-1} + \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)}$. Prove that the number $2 + \sqrt{2 + 2a_n}$ is a perfect square.

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2. Let $x, y \in \mathbb{N}$ such that $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect square.

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3. Let $n \in \mathbb{N}$, and consider all the polynomials with coefficients 0,1,2 or 3. How many of these polynomials satisfy $P(2) = n$?

All-Soviet Union Olympiad 1986

4. Prove that for every $n \geq 2$ there exists a positive number which can be written as a sum of i squares, for all $2 \leq i \leq n$.

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5. Let $p \geq 5$ be a prime, and prove that if $1 + \frac{1}{2} + ... + \frac{1}{p} = \frac{a}{b}$, then $p^4 | ap - b$.

***

6. Consider four natural numbers a, b, c, d and the set S formed by the points $(x, y) \in (0, 1) \times (0, 1)$ which have the property that $ax + by$ and $cx + dy$ are integers. If S has 2004 elements and $(a, c) = 6$, find (b, d).

Bulgarian Olympiad 2004

7. Find all integer numbers m, n such that $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$.

All-Soviet Union Olympiad 1984

8. For every natural number n prove that the least common multiple of the numbers $1, 2, ..., n$ equals the least common multiple of $\binom{n}{1}, \binom{n}{2}, ..., \binom{n}{n}$ iff $n + 1$ is prime.

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9. Solve in $\mathbb{N}$ the equation $x^2 = y^z - 3$, for $z - 1$ non-divisible by 4.

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10. We will call a lattice point invisible iff the segment which joins it to the origin contains at least another lattice point. Prove that there are squares of any size whose inside contains only invisible points.

Romanian TST 2003

11. Let $a > 1$ and the sequence defined by $x_1 = 1$, $x_2 = a$ şi $x_{n+2} = a \cdot x_{n+1} - x_n$. Prove that this sequence has a subsequence formed by numbers which are pairwise co-prime.

***

12. Consider a set D of natural numbers with the greatest common divisor 1. Prove that there exists a bijection ϕ between $\mathbb{Z}$ and $\mathbb{Z}$ such that $|\phi(k) - \phi(k-1)| \in D$ for any integer k.

American Mathematical Monthly

13. For all n, denote by $p(k)$ the smallest prime number which divides k and by $q(k)$ the product of all primes smaller than $p(k)$. Consider the sequence defined by $x_0 = 1$ and $x_{n+1} = \dfrac{x_n p(x_n)}{q(x_n)}$. Calculate x_{2005}.

***

14. Let $a_1, ..., a_{2005}$ be non-negative integers such that for every n the number $a_1^n + ... + a_{2005}^n$ is a perfect square. What is the minimum number of zeroes among the a_i's?

Team contest

15. Let p be a prime number and f be an integer polynomial of degree d such that $f(0) = 0$, $f(1) = 1$ and $f(n)$ congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$.

***

16. Find all the natural numbers n which have a multiple with the sum of decimal digits n.

***

17. Consider the lattice in the plane, from which we may cut rectangles, but only by making cuts along the lines of the lattice. Prove that for every $m > 12 \in \mathbb{N}$ we can cut a rectangle of area greater than m out of which we cannot cut further a rectangle of area m.

All-Soviet Union Olympiad 1985

18. Prove that for an infinity of natural numbers n the numerator of the fraction $1 + \frac{1}{2} + ... + \frac{1}{n}$ has at least two prime factors.

Russian Olympiad 2002

19. Prove that there exist an infinity of n cu such that $S(3^n) \geq S(3^{n+1})$, where $S(a)$ is the sum of the digits of a in decimal notation. Moreover, there exist an infinity of m such that $S(2^m) \geq S(2^{m+1})$.

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20. Let n be a natural number greater than 1 and p be a prime such that $n|p-1$ and $p|n^3 - 1$. Prove that $4p - 3$ is a perfect square.

MOSP 2004

21. Find all natural numbers n which satisfy $n^2|3^n + 1$.

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22. Find the natural numbers n for which

$$n|1^n + 2^n + ... + (n-1)^n - 1$$

IMAR test, 2004

23. What are the natural values which $\dfrac{a^2+ab+b^2}{ab-1}$ can take on when $a, b \in \mathbb{N}$, not both 1?

Romanian TST 2004

24. Let $a \neq b$ be rational numbers such that for an infinity of natural numbers n we have $a^n - b^n \in \mathbb{Z}$. Prove that a and b are integers.

Mathlinks Contest

25. Prove that all the terms of the sequence given by $a_1 = a_2 = a_3 = a_4 = a_5 = 1$ and $a_{n+6}a_{n+1} = a_{n+5}a_{n+2} + a_{n+4}a_{n+3}$ are natural numbers.

Russian Olympiad 2002

26. Prove that the equation $x^3 + y^3 + z^3 = 2$ has an infinity of solutions in $\mathbb{Z}$.

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27. Find all natural numbers n for which the equation $a + b + c + d = n\sqrt{abcd}$ has a solution in natural numbers.

Vietnam Olympiad 2002

Chapter 3
COMBINATORICS

PROBLEMS

1. Let $A_1, ..., A_k$ be subsets of $\{1, 2, ..., n\}$, each of them with at least $\frac{n}{2}$ elements, and such that for every $i \neq j$ we have $\#\{A_i \bigcap A_j\} \leq \frac{n}{4}$. Prove that

$$\# \bigcup_{i=1}^{k} A_i \geq \frac{k}{k+1} n$$

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2. Prove that the smallest number of colors required to color the finite regions of a planar graph such that no two neighboring regions are colored in the same way is 2 if and only if all the vertices of the graph have even degree.

***

3. There is an ant which walks on the infinite lattice, but it can only walk upwards or to the right. There is an infinite sequence of points (which the ant knows) $A_{n\,n>0}$ such that immediately after the ant takes its i-th step, the point A_i becomes poisoned. Prove that the ant can always find an infinite path which always avoids stepping on a previously poisoned point.

***

4. A 10×10 checkerboard has every field colored in one of 15 given colors, such that each color is used for at least 6 squares. Prove that there are three identically colored, equally-distanced fields on the same row or column.

IMAR test, 2004

5. What is the greatest number of knights which can be placed on a 5×5 chessboard such that each of them attacks exactly 2 of the others?

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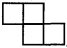
6. A $(2n + 1) \times (2n + 1)$ chessboard is covered by pieces of the shapes ,  and , which we can always rotate by $90°$ and reflect across the horizontal and vertical axis. Prove that at least $4n + 2$ trominoes (3-square pieces) are used.

***

7. Let $P_1, ..., P_n$ be distinct 2-element subsets of $\{1, 2, ..., n\}$ such that if $P_i \bigcap P_j \neq 0$; then $\{i, j\} = P_k$ for some k. Prove that every one of the a_i's appears in exactly 2 of the P_j's.

***

8. The numbers $1, 2, ..., 2005$ are divided into 6 disjoint subsets. Prove that there are numbers a, b, c (not necessarily distinct) belonging to one of the subsets such that $a + b = c$.

Romanian-Hungarian Training Camp

9. Given a graph with n vertices, not bipartite and without triangles, prove that there exists a vertex with degree at most $\frac{2n}{5}$.

Bulgarian Olympiad 2004

10. The natural numbers are colored with two colors. Prove that there exists a sequence of natural numbers $k_1 < k_2 < ... < k_n < ...$ with the property that the sequence $2k_1 < k_1 + k_2 < 2k_2 < k_2 + k_3 < 2k_3 < ...$ is of the same color.

***

11. We have $n + 1$ natural weights, the sum of which is $2n$. We are given a scale and at every step we place one of the weights (in order from the heaviest to the lightest weight) in that arm of the scale which has the less weight on it. If at one step, both arms have the same weight on them, we put the next weight on any of them. Prove that at the end of the process, the scale will be in equilibrium.

All-Soviet Union Olympiad 1984

12. In the square with vertices $A = (0,0)$, $B = (0,n)$, $C = (n,n)$, $D = (n,0)$ consider a broken line which goes from A to C by passing only once though all the lattice points inside or on the frontier of our square. We will color in black all the interior regions from which we can reach the sides AB or BC without crossing the broken line. Show that the black area is $\frac{n^2}{2}$.

***

13. The numbers from 1 to n are placed in an arbitrary order. We will perform the following operation: if the first number in the sequence is k, then we switch the order if the first k numbers in our sequence (so the first number becomes the k-th, the second becomes the $k-1$-th etc.) Prove that after a finite number of operations the first number will be 1.

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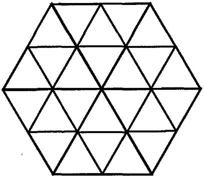
14. Consider a regular hexagon which we will divide into 6 equilateral triangles via its diagonals. Then, each of these triangles will be further divided into 4 congruent equilateral triangles, thus obtaining a network of 24 triangles. In the vertices of this network, let us write the numbers $1, 2, ..., 19$. We will call a small triangle with the numbers in its vertices $a < b < c$ if we can go from a to c via

b, by using the triangle's sides in a counterclockwise direction. Prove that there are at least 7 positive triangles.



Romanian-Hungarian Training Camp

15. For every $k \geq 3$ prove that from any graph we can erase at most $\frac{1}{k-1}$ of its edges such that the remaining graph contains no k-cliques.

Romanian-Hungarian Training Camp

16. Find the smallest and the largest number of oriented triangles (triangles in which the edges form a cycle in the order of the arrows) in a complete oriented graph with n vertices.

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17. There are n natives on an island, and any two of them are either friends, or enemies. Each of them has a necklace of colored beads (where a necklace contains any non-negative number of beads), such that evert two friends must have a bead of the same color in their necklaces, and any two enemies only have beads of different colors. What is the smallest number of colors for beads required to make the necklaces, indifferent of the relations between the natives?

IMAR test, 2003

18. For $n \geq 2$, the distinct natural numbers $a_1, ..., a_n, b_1, ..., b_n$ have the property that the $\binom{n}{2}$ sums $a_i + a_j$ are the same as the $\binom{n}{2}$ sums $b_i + b_j$ (in some order). Prove that n is a power of 2.

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19. The numbers $1, 2, ..., 100$ are written in a 10×10 table such that at the intersection of row i with column j we have the number $10(i-1)+j$. At every step we are allowed to make the following move: pick a number and two of its opposing neighbors (meaning the two neighbors directly above and below it or the two neighbors directly to the left and to the right with it) and either decrease the number by 2 and increase those neighbors by 1 each, or increase the number by 2 and decrease both the neighbors by 1. After a finite number of such steps, we obtain the initial numbers in our table. Prove that they must be in the original order.

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20. The numbers $1, 2, ..., 2n$ are divided into two sequences: $a_1 < a_2 < ... < a_n$ and $b_1 > b_2 > ... > b_n$. Prove that $|a_1 - b_1| + ... + |a_n - b_n| = n^2$.

All-Soviet Union Olympiad 1985

21. Given are 4 congruent right triangles, and a move consists of selecting a triangle and splitting it into two others by cutting along the altitude (the one that corresponds to the right angle). Prove that after a finite number of paths, we will always have two congruent triangles, regardless of how the cuts were done.

***

22. Let m, n be non-zero natural numbers such that $m > 1$ and $n > 2m$. Find the greatest natural number k for which there exist disjoint subsets $A_1, ..., A_k$ of the set $\{1, 2, ..., n\}$ such that $\#A_i \in \{m, m-1\}$ for all i.

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23. Prove that the set of numbers $\{1, 2, ..., 2005\}$ can be colored with two colors such that any of its arithmetic progressions with 18 terms contains both colors.

***

24. Let us consider the lattice in the plane,and in every point of y-coordinate non-positive we place a checker. At every step, we can make a "jump" by taking a checker, removing one of its neighbors and placing the checker two units in the direction of the removed neighbor. Show that none of our checkers will ever make it to the line $y = 5$.

***

25. Can the natural numbers from 0 to $2^n - 1$ be written on a circle such that the last $n-1$ binary digits 2 of one of them are the first $n-1$ binary digits of his neighbor in the clockwise direction?

Romanian-Hungarian Training Camp

26. Out of the 77-element subsets of $\{1, 2, ..., 2005\}$, are there more of them those with even element-sum or those with odd element-sum?

***

27. We have 3 chess clubs with n players each, such that every player has played at least $n+1$ others from the other two clubs. Prove that there are three players from different clubs which have all played each other.

***

28. A good n-path is one that starts from $(0,0)$ and is composed of n climbs of the form $(x,y) \to (x+1, y+1)$ and n descents of the form $(x,y) \to (x+1, y-1)$ such that the path never goes below the x-axis. A return is a sequence of descents which has a climb right before it and which terminates on the x-axis. Prove that the number of good $n-1$-paths equals the number of good n-paths with no return of even length.

Putnam 2003

29. In a $n \times n$ matrix with non-zero natural entries we are allowed to perform the following operations: multiplying a row by 2 or subtracting 1 from all the entries of a column. Show that we can obtain the zero matrix after a sequence of such steps.

***

30. Let a, b be distinct natural numbers. A set $\{x, y, z\}$ is called (a, b)-adapted if $\{z - y, y - x\} = \{a, b\}$. Prove that the set of natural numbers can be written as a disjoint union of $\{a, b\}$-adapted sets.

77 de ture

31. We have a 100×100 board, all of whose fields are either infected or healthy. Every day, all healthy cells which have at least two infected neighbors become infected. Find the smallest n such that there is an initial distribution of n infected fields which will eventually infect the entire board.

***

32. In a graph G, any subgraph G' of order 4 contains a vertex connected by exactly one other vertex of G'. Find the greatest possible number of vertices that G can have.

Team contest, 2004

33. A rectangle may be split up into disjoint rectangles, all of which have an integer dimension. Prove that the big rectangle has an integer dimension.

***

34. Prove that in a $m \times n$ matrix with elements 0 and 1, the maximal number of 1's which are all on different lines and columns equals the minimum number of lines and columns which will cover all the 1's in the matrix.

***

35. All the digits are used in the decimal expression of x, and there is a $n > 1$ such that there are at most $n + 8$ different segments of n consecutive digits in the decimal expression of x. Prove that x is rational.

All-Soviet Union Olympiad 1983

36. Two players play a game in which they alternatively write down numbers between 1 and 1000 such that they never write a divisor of a number previously written. Which one of them has a winning strategy no matter what the other does?

All-Soviet Union Olympiad 1987

37. Prove that a graph with $2n$ sides and $n^2 + 1$ edges contains at least n triangles.

***

Chapter 4
ALGEBRA

PROBLEMS

1. All the roots of the polynomial $X^n + a_{n-1}X^{n-1} + ... + a_0$ are in the interval $(0, 1)$, where $n \geq 3$. Prove that $\sum_{k=0}^{n-2} ka_k > 0$.

***

2. Let λ be a real number such that the inequality
$$0 < \sqrt{2002} - \frac{a}{b} < \frac{\lambda}{ab}$$
holds for an infinity of pairs (a, b) of natural numbers. Prove that $\lambda \geq 5$.

77 de ture

3. For any natural number $n \geq 4$ and non-negative numbers $a_1, ..., a_n$ with sum 1, prove that
$$a_1a_2 + a_2a_3 + ... + a_na_1 \leq \frac{1}{4}$$

***

4. If a, b, c are the side-lengths of a triangle, then prove that
$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \frac{a}{c} + \frac{c}{b} + \frac{b}{a} + 3$$

***

5. For any $0 < a_1 \leq a_2 \leq ... \leq a_n$ and $0 < b_1 \leq b_2 \leq ... \leq b_n$ prove that
$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) \leq (a_1b_1 + ... + a_nb_n)^2 \cdot \frac{1}{4}\left(\sqrt{\frac{a_nb_n}{a_1b_1}} + \sqrt{\frac{a_1b_1}{a_nb_n}}\right)^2$$

***

6. Prove that for any $a, b > 0$ we have that
$$\frac{(a+b)^2}{2} + \frac{a+b}{4} \geq a\sqrt{b} + b\sqrt{a}$$

All-Soviet Union Olympiad 1984

7. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy, for any $x, y \in \mathbb{R}$,
$$f(x - f(y)) = f(x) + xf(y) + f(f(y)) - 1$$

IMO 1999

8. The positive numbers a, b, c, A, B, C satisfy $a + A = b + B = c + C = k$. Prove that $aB + bC + cA \leq k^2$.

All-Soviet Union Olympiad 1987

9. Consider the natural numbers $a_1 < a_2 < ... < a_n$ such that $\sum \frac{1}{a_i} \leq 1$. Prove that for any $x > 0$ we have that

$$\left(\sum \frac{1}{a_i^2 + x}\right)^2 \leq \frac{1}{2} \frac{1}{a_1(a_1 - 1) + x}$$

***

10. Consider the positive numbers $x_1, x_2, ..., x_n$ which satisfy $\sum x_i = \sum \frac{1}{x_i}$. Prove that

$$\sum \frac{1}{n - 1 + x_k} \leq 1$$

***

11. Given positive numbers x, y, z with $xy+yz+zx = 3$, prove that for any $a, b, c > 0$ the following relation holds

$$\sum \frac{a}{b+c}(y + z) \geq 3$$

***

12. Find all integer polynomials P for which $(x^2 + 6x + 10)P^2(x) - 1$ is the square of an integer polynomial.

Vietnam Olympiad 2002

13. Find the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that for every $x, y \in \mathbb{R}$ we have

$$f(x^2 + f(y)) = y + f^2(x)$$

***

14. Find the largest k such that $a, b, c > 0$ and $kabc > a^3 + b^3 + c^3$ implies that a, b, c are the side-lengths of a triangle.

***

15. Prove that if $(a_n)_{n \in \mathbb{N}}$ is a strictly increasing and unbounded sequence of positive numbers, then the sequence

$$\frac{a_2 - a_1}{a_2} + \frac{a_3 - a_2}{a_3} + ... + \frac{a_n - a_{n-1}}{a_n}$$

is unbounded.

All-Soviet Union Olympiad 1985

16. Find all the functions $f : \mathbb{R} - \{0\} \to \mathbb{R}$ with the property that for all x, y we have

$$xf\left(x+\frac{1}{y}\right)+yf(y)+\frac{y}{x}=yf\left(y+\frac{1}{x}\right)+xf(x)+\frac{x}{y}$$

Iran Olympiad 2002

17. Find the functions $f : \mathbb{Q} \to \mathbb{Q}$ which verify

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in \mathbb{Q}$.

***

18. Let x, y, z be positive numbers such that $\sum \frac{1}{x} = 1$. Prove that

$$\sum \sqrt{x+yz} \geq \sqrt{xyz}+\sum \sqrt{x}$$

77 de ture

19. Does there exist a finite set M of non-zero real numbers such that for every natural n, there exists a polynomial of degree greater than n with all its coefficients and roots in M?

77 de ture

20. Let $P(z) \in \mathbb{C}[X]$ be a monic polynomial with all its roots strictly inside the unit disk. Prove that there exists a z with $|z| = 1$ such that $|P(z)| \geq 1$.

***

21. Let $x, y, z \geq 0$ be real numbers such that $x + y + z + xyz = 4$. Prove that

$$x+y+z \geq xy+yz+zx$$

Team contest, 2004

22. The sequence a_n is defined by $a_1 = 2$, $a_2 = 3$ and for every $n \geq 2$ either $a_{n+1} = 2a_{n-1}$, or $a_{n+1} = 3a_n - 2a_{n-1}$. Prove that no number between 1600 and 2000 can be a term of the sequence.

***

23. Let $x_1, ..., x_n$ be natural numbers, such that none of them is the beginning part of any other, as written in base 10. Prove that

$$\frac{1}{x_1}+...+\frac{1}{x_n}<3$$

Baltic Way 2000

24. The sequence $(a_n)_n$ is defined by

$$a_1=56,\ a_{n+1}=a_n-\frac{1}{a_n}$$

for $n \geq 1$. Prove that there exists k, $1 \leq k \leq 2005$ such that $a_k < 0$.

77 de ture

25. Consider the real polynomials f and g of degree 2. Prove that if $f(x) \in \mathbb{Z}$ implies $g(x) \in \mathbb{Z}$, then there are $a, b \in \mathbb{Z}$ such that $g = af + b$.

***

26. Given a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(xy+x+y) = f(xy)+f(x)+f(y)$ for all $x, y \in \mathbb{R}$, prove that $f(x+y) = f(x) + f(y)$ also holds for all x, y.

***

27. Are there functions $f : \mathbb{Q}_+ \to \mathbb{Q}_+$ such that $f(xf(y)) = \dfrac{f(x)}{y}$ holds for all $x, y \in \mathbb{Q}_+$?

***

28. Prove that for any positive numbers $a_1, ..., a_n$ we have:

$$\frac{1}{a_1} + \frac{2}{a_1 + a_2} + ... + \frac{n}{a_1 + ... + a_n} < 4\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$$

All-Soviet Union Olympiad 1986

29. Find all functions $f : \mathbb{N} \to \mathbb{N}$ with the property that $mf(n) + nf(m) = (m + n)f(m^2 + n^2)$ for all $m, n \in \mathbb{N}$.

***

30. Let $a_1, ..., a_n$ be real numbers, and S be a subset of $\{1, 2, ..., n\}$. Prove that

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + ... + a_j)^2$$

Romanian TST 2004

31. If the polynomial f of degree n has real roots, non-negative coefficients and the first and last coefficients 1, then $f(2) \geq 3^n$.

***

32. Consider the sequence $a_0 = 0$, $a_1 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$. Prove that $\dfrac{a_n}{n}$, written as an irreducible fraction, has both numerator and denominator odd.

***

33. Prove that if the real numbers a, b, c have sum 1, then they verify

$$10|a^3 + b^3 + c^3 - 1| \leq 9|a^5 + b^5 + c^5 - 1|$$

***

34. Are there bounded functions $f : \mathbb{R}_+ \to \mathbb{R}$, with $f(1) = 1$ and which satisfy, for all $x \neq 0$, the relation

$$f\left(x + \frac{1}{x^2}\right) = f(x) + f\left(\frac{1}{x}\right)^2?$$

IMO 1995 Shortlist

35. Find the functions $f : \mathbb{N} \to \mathbb{N}$ (non-zero natural numbers) that satisfy

$$f(m + f(n)) = n + f(m + 2005)$$

for all $m, n \in \mathbb{N}^*$.

***

36. Prove that for any $x, y, z > 0$ with sum 3, we have that

$$\frac{(1-x)^2}{1-x^4} + \frac{(1-y)^2}{1-y^4} + \frac{(1-z)^2}{1-z^4} \geq 0$$

***

37. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $f(x^2 - y^2) = (x - y)(f(x) + f(y))$ holds for all $x, y \in \mathbb{R}$.

***

38. Prove that if the polynomial $X^n + X^{n-1} + \ldots + X + 1$ can be written as the product of two monic polynomials with real non-negative coefficients, then those coefficients are all 0 or 1.

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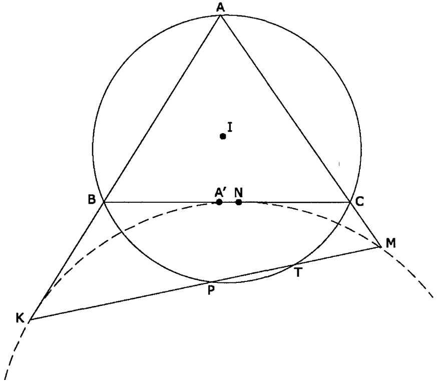
Chapter 1
GEOMETRY
SOLUTIONS

1. M We can suppose WLOG that $b \geq c$. Let T be the second meeting point of line PM with circle (ABC). Let us denote $KP = MP = \alpha$ and $PT = \beta$; we know that $KB = p - c$ and $MC = p - b$. Because of the power of the points K and M with respect to the circle (ABC), we have that

$$\alpha(\alpha + \beta) = p(p - c)$$

$$\alpha(\alpha - \beta) = p(p - b)$$



Adding the two relations will give us

$$\alpha^2 = \frac{p(2p - b - c)}{2} = \frac{pa}{2}$$

This means that $KM^2 = 4\alpha^2 = 2ap$, but by expressing KM with the cosine law in triangle AKM ($AK = AM = p$ and $\angle MAK = \angle A$) we get:

$$2pa = KM^2 = 2p^2 - 2p^2 \cos A \Rightarrow a = p\frac{2bc - (b^2 + c^2 - a^2)}{2bc} \Rightarrow abc = 2p(p-b)(p-c) \quad (1)$$

Let D be the projection of I on BC and let us look at the intersection N' of IO with BC. We want to prove that $N' = N$ and for this it will be enough to prove that $N'C = p - b$. But $BD = p - b$, so it is enough to prove that $A'D = A'N'$, where A' is the midpoint of BC. So we need to show that:

$$ID = 2OA' \Leftrightarrow r = 2R\cos A \Leftrightarrow \frac{r}{2R} = \frac{b^2 + c^2 - a^2}{2bc} \Leftrightarrow \frac{2S^2}{pabc} = \frac{b^2 + c^2 - a^2}{2bc} \Leftrightarrow$$

$$\Leftrightarrow 4p(p-a)(p-b)(p-c) = pa(b^2 + c^2 - a^2)$$

But by making the substitution $abc = 2p(p-b)(p-c)$ from (1), we get:

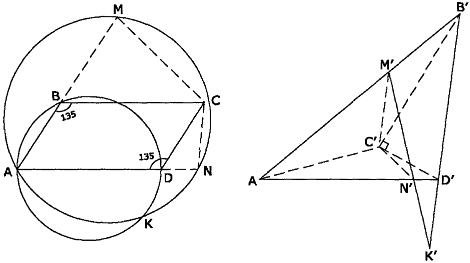
$$2(p-a)bc = p(b^2 + c^2 - a^2) \Leftrightarrow 2(p-a)bc - 2pbc = p((b-c)^2 - a^2) \Leftrightarrow$$

$$\Leftrightarrow -2abc = p(b-c-a)(b-c+a) \Leftrightarrow 2p(p-b)(p-c) = abc$$

which is true again, by (1). What follows is that $N = N'$, so the points O, N and I are collinear.

2. D We will use inversion to solve the problem. First of all, note that $\angle A \leq 360° - 2 \cdot 135° = 90°$. From the quadrilaterals $ANCB$ and $AMCD$ we have that $\angle ANC = \angle AMC = 135° - \angle A < 135°$ and therefore B and D will be situated on the segments AM and AN. Apply inversion around A and denote the inverses of our points by $'$.

   The points B' and M' will be on the line AB, with M' between A and B'; similarly, D' and N' will be on AD, with N' between A and D'. The fact that the angle between AB and BC equals $135°$ tells us that the angle between AB' and the circle $(AB'C')$ is equal to the same $135°$. This angle will be $\angle B'C'A$, and therefore $\angle B'C'A = 135°$; similarly, $\angle D'C'A = 135°$ and therefore $\angle B'C'D' = 90°$.



But the angle $\angle MCD = 90°$ tells us that the angle between the circles $(AC'M')$ and $(AC'D')$ is $90°$, and that is exactly the angle $\angle AM'C' + \angle AD'C'$. So we will have $\angle M'C'D' = \angle AM'C' + \angle AD'C' + A = 90° + A$, and similarly $\angle N'C'B' = 90° + A$. Because $\angle B'C'D' = 90°$, this gives us $\angle M'C'B' = \angle N'C'D' = A$ and therefore $\angle N'C'A = \angle M'C'A = 135° - \angle A$. And finally, K (the intersection of the circles (AMN) and (ABD)) turns into K', the intersection of the lines $M'N'$ and $B'D'$. Let $T' = K'B' \bigcap AC'$.

The angle between AK and KC equals the angle between AK' and the circle $(AK'C')$, meaning the angle $\angle K'C'A$. To prove that $AK \perp KC$ it is therefore enough to show that $AC' \perp C'K'$.

Menelaos' Theorem for the triangle $AB'D'$ with the transversal $K'M'N'$ gives us

$$\frac{K'B'}{K'D'}\frac{N'D'}{N'A}\frac{M'A}{M'B'} = 1 \tag{1}$$

But let us evaluate these ratios. The Law of Sines applied in the triangles $C'N'D'$, $C'N'A$ and $C'D'A$ tells us that

$$\frac{N'D'}{C'N'} = \frac{\sin \angle A}{\sin \angle C'D'A}$$
$$\frac{N'A}{C'N'} = \frac{\sin(135 - \angle A)}{\sin \angle C'AD'}$$
$$\frac{\sin \angle C'D'A}{\sin \angle C'AD'} = \frac{AC'}{C'D'}$$

By dividing the first two relations and replacing $\frac{\sin \angle C'D'A}{\sin \angle C'AD'}$ with the value we obtained for it from the third equality, we get that

$$\frac{N'D'}{N'A} = \frac{\sin \angle A}{\sin(135 - \angle A)}\frac{C'D'}{AC'}$$

and similarly, we have that

$$\frac{M'B'}{M'A} = \frac{\sin\angle A}{\sin(135 - \angle A)}\frac{C'B'}{AC'}$$

Introducing these values in (1) yields

$$\frac{K'B'}{K'D'} = \frac{C'B'}{C'D'} \tag{2}$$

But in triangle $C'B'D'$, $C'A = C'T'$ is a bisector, and so we have

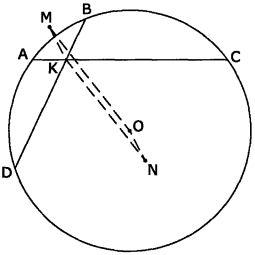
$$\frac{C'B'}{C'D'} = \frac{T'B'}{T'D'} \tag{3}$$

From (2) and (3) we have that the points C', K', T' are all on the same Apollonius circle of the segment $B'D'$. But as $K', T' \in B'D'$, they are on the same diameter in that circle, and therefore $C'K' \perp C'T' = C'A$, exactly what we needed to prove.

3. E Note that $ON \perp CD$, as both O and N are on the perpendicular bisector CD. But let $KM \bigcap CD = P$ and we will have

$$\angle DPK = \angle KCD + \angle CKP = \angle KCD + \angle AKM = \angle KCD + \frac{\pi}{2} - \angle ABK = \frac{\pi}{2}$$

as $\angle ABK = \angle KCD$. Therefore $KM \perp CD$, so KM and ON will be parallel; similarly we can prove that KN and OM will be parallel, so $MKNO$ will be a parallelogram.



4. M Let D be the foot of the altitude from A, and D will be on the side BC since the triangle has only acute angles. Now, by summing up $AD + BD > AB$ and $CD + AD > AC$ we get

$$2h_a + a > b + c \Rightarrow h_a > p - a \Rightarrow \frac{2S}{a}\cdot > p - a \Rightarrow 2S > a(p-a) \Rightarrow \frac{abc}{2R} > a(p-a) \Rightarrow$$

$$\Rightarrow bc + 2aR > 2pR \Rightarrow \frac{pR}{bc + 2aR} < \frac{1}{2}$$

For the other part, we have the following equivalences:

$$\frac{2}{5} \leq \frac{Rp}{2aR+bc} \Leftrightarrow 2aR+bc \leq \frac{5pR}{2} \Leftrightarrow 4aR+\frac{8SR}{a} \leq 5pR \Leftrightarrow$$

$$\Leftrightarrow 4a^2+4ah_a \leq 5ap \Leftrightarrow (a+h_a) \leq \frac{5p}{4} \qquad (1)$$

But $b=\sqrt{h_a^2+BD^2}$ and $c=\sqrt{h_a^2+CD^2}$ and by Minkowski's inequality we have $b+c \geq \sqrt{4h_a^2+a^2}$. Then, we will have

$$\frac{5p}{4}=\frac{5(a+b+c)}{8} \geq \frac{5(a+\sqrt{4h_a^2+a^2})}{8} \geq a+h_a \Leftrightarrow 5\sqrt{4h_a^2+a^2} \geq 3a+8h_a \Leftrightarrow$$

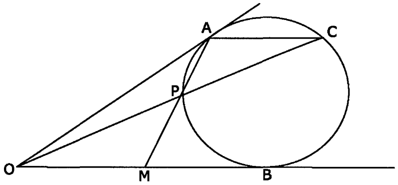
$$\Leftrightarrow 100h_a^2+25a^2 \geq 64h_a^2+9a^2+48ah_a \Leftrightarrow 36h_a^2+16a^2 \geq 48ah_a$$

which holds for all positive a and h_a because of the $AM-GM$ inequality. This tells us that (1) is true, and thus so is our conclusion.

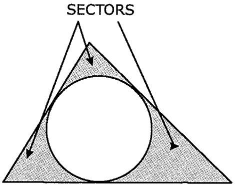
5. E Let M be the intersection of AP and OB and note that $\angle OPM=\angle APC=\pi-\angle OAC=\angle AOB$. This means that the triangles OMP and AMO are similar, and therefore

$$\frac{OM}{AM}=\frac{MP}{MO} \Rightarrow OM^2=AM \cdot MP$$

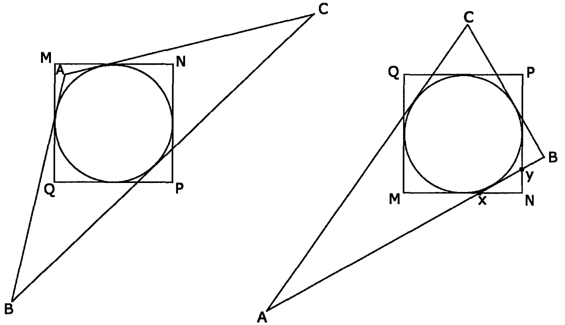
But from the power of M with respect to the circle, $MA \cdot MP = MB^2$, so we have the desired $OM=BM$.



6. D First of all, if we remove the circle from the inside of the triangle, we will be left with 3 sectors, each of which we be called "the sector of a particular vertex". The same observation goes for the square, except that we get 4 sectors.



Let the triangle be ABC and the square $MNPQ$. We will study two cases: whether or not there are vertices of the triangle inside the square. If there are such vertices, then the angle corresponding in the triangle to that vertex must be non-acute (since it is tangent to the circle and inside the right angle of the square). So there can only be one such vertex; let it be A, and let M be that vertex such that A lies in the sector of M.



We will now show that N and Q are situated inside our triangle. If N weren't inside the triangle, then the point C would have to be on the same side of NP as the circle. But in this case, the second tangents from B and C to the circle (i.e. those tangents not containing A) could only meet on different sides of MN as the circle. This is a contradiction since the circle is inscribed inside the triangle, so N and Q must lie inside the triangle.

But N and Q cannot lie in the sector of A, so one must lay in each of the sectors of B and C (obviously, they cannot both lie in the same sector or the square wouldn't be circumscribed around the circle). So the sectors of N and Q are completely inside the triangle. Thus, the common part of the two polygons is at least π (the circle) plus $2(1-\frac{\pi}{4})$ (the areas of the sectors of N and Q). This means that the common part has area at least

$$2+\frac{\pi}{2} > 3.4$$

The second case that must be studied is when none of the A, B, C lie inside the square. Then, the lines AB, BC, CA break the plane up into 7 parts. Call the parts that contain only one triangle vertex on their boundary of "type 1" and those that contain exactly two vertices of "type 2". In the three of them of type 1 there can be no vertices of the square (as then we would have vertices of the triangle inside the square). Moreover, there can be no two vertices of the square in the same region of type 2 (because then the square wouldn't be circumscribed around the circle anymore). Since there are 4 vertices of the square, one of them must lie inside the 7th part, which is the triangle itself. Let M lie inside the triangle, and if there were any other vertex of the square inside the triangle we would be done (because of the argument in part 1). So we can safely assume that M is the only vertex of the square inside the triangle.

Now, the three sides of the triangle will each cut one the sectors determined by the square except for M's sector. Let us estimate the area of such a cut. If we had a sector (say N's) and a triangle-side passing through it and cutting the tangents from N in X and Y, what would be the area between XY and the circle? Well, that area is $1 - \frac{\pi}{4} - \frac{NX \cdot NY}{2}$. But since XY is tangent to the circle, we have

$$XY + NX + NY = 2 \Rightarrow 2 = NX + NY + \sqrt{NX^2 + NY^2} \geq$$

$$\geq 2\sqrt{NX \cdot NY} + \sqrt{2}\sqrt{NX \cdot NY} \Rightarrow NX \cdot NY \leq \frac{2}{(\sqrt{2}+1)^2} = 2(\sqrt{2}-1)^2$$

and so our area would be $\geq 1 - \frac{\pi}{4} - (\sqrt{2}-1)^2$. Thus, our total area of intersection would be at least three such areas plus the area of the circle plus the area of M's sector, which adds up to

$$3 - \frac{3\pi}{4} - 3(\sqrt{2}-1)^2 + \pi + 1 - \frac{\pi}{4} = 4 - 3 \cdot (0.41)^2 = 3.49... > 3.4$$

Our problem is thus solved.

7. M Let us draw the two lines d_1 and d_2 through M which are perpendicular to AC and BD, and let the angle between them (which will equal $\angle AMB$) be x. Then because O_1 and O_2 are the circumcenters of the triangles ABM and CDM, it is easy to see that they cannot lie in the shaded region between d_1 and d_2 (as then we would have an impossible inequality of the form $\angle O_1MB \geq 90^\circ$). So $\angle O_1MO_2 > x$ and then what we need to prove becomes

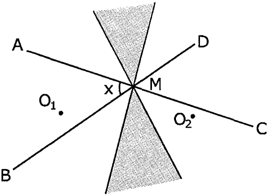
$$4\sqrt{r^2 + R^2 - 2rR\cos\angle O_1MO_2} > 2(r+R)\sin x$$

and by squaring the above relation, it is sufficient to show that

$$\Leftrightarrow 4(r^2 + R^2 - 2rR\cos x) > (1 - \cos^2 x)(r^2 + R^2 + 2Rr)$$

$$\Leftrightarrow (r^2 + R^2)(3 + \cos^2 x) > 2Rr(1 - \cos^2 x + 4\cos x)$$

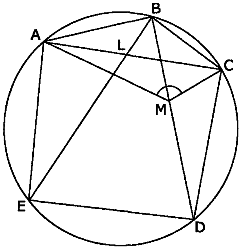
which always holds, because $r^2 + R^2 \geq 2Rr$ and $3 + \cos^2 x \geq 1 - \cos^2 x + 4\cos x \Leftrightarrow 1 + \cos^2 x \geq 2\cos x$.



8. M First of all, we should see what it means for BE to cut AC in half: let their point of intersection be L and applying the Law of Sines in triangles ALB and BLC will give us the ratio $\frac{AL}{LC} = \frac{AB}{BC} \cdot \frac{\sin ABE}{\sin EBC}$. But $AE = 2R \sin ABE$ and $EC = 2R \sin EBC$, where R is the radius of the circle $(ABCDE)$. Thus, we have

$$\frac{AL}{LC} = \frac{AB}{BC} \cdot \frac{AE}{EC}$$

and for L to be the midpoint of AC we would then need to prove that $AB \cdot AE = CB \cdot CE$. But since $AC \parallel ED$ we have $EC = AD$ and $AE = CD$, so it would suffice to prove that $AB \cdot CD = AD \cdot BC$.



Let us now turn to just the points A, B, C, D. Applying the Law of Sines in the triangles ABM and BCM give us

$$\frac{AB}{\sin AMB} = \frac{AM}{\sin ABD} = AM \cdot \frac{2R}{AD}$$
$$\frac{BC}{\sin BMC} = \frac{CM}{\sin CBD} = CM \cdot \frac{2R}{CD}$$

Dividing the two yields

$$\frac{AB}{BC} = \frac{AM}{MC} \cdot \frac{CD}{AD} \Rightarrow \frac{AM}{MC} = \frac{AB}{BC} \cdot \frac{AD}{CD} \tag{1}$$

Now, we have that

$$\cos AMB = \frac{AM^2 + BM^2 - AB^2}{2AM \cdot BM} =$$

$$= \frac{\frac{2AB^2 + 2AD^2 - BD^2}{4} + \frac{BD^2}{4} - AB^2}{AM \cdot BD} = \frac{AD^2 - AB^2}{2AM \cdot BD}$$

where we used the formula for the length of the median AM in triangle ABD. Similarly, we have $\cos BMC = \frac{CD^2 - BC^2}{2MC \cdot BD}$ and by dividing these two and employing (1) we will have

$$1 = \frac{AD^2 - AB^2}{CD^2 - BC^2} \cdot \frac{MC}{AM} = \frac{AD^2 - AB^2}{CD^2 - BC^2} \cdot \frac{CD \cdot BC}{AB \cdot AD} \Rightarrow$$

$$\Rightarrow \frac{AD^2 - AB^2}{AD \cdot AB} = \frac{CD^2 - BC^2}{CD \cdot CB} = \frac{AD}{AB} - \frac{AB}{AD} = \frac{CD}{CB} - \frac{CB}{CD}$$

Because the function $x - \frac{1}{x}$ is one-to-one on the set of positive numbers (it is increasing), we must necessarily have

$$\frac{AD}{AB} = \frac{CD}{CB} \Rightarrow AD \cdot CB = AB \cdot CD$$

which is exactly what we needed to prove.

9. D We will prove by induction after the number n of points inside the stronger statement that there is such a broken line uniting these points and whose endpoints are the two endpoints of the hypothenuse. It is true for $n = 1$, as an easy computation shows. Now, assume it true for any number of points less than n and let us prove it for n points. Let the triangle be ABC with A being the right angle.

Consider the altitude AD which breaks the original triangle into two smaller ones similar to it and let $X_1, ..., X_n$ be our points. Suppose that there are points on both sides of the altitude; then apply the induction hypothesis for both of the smaller triangles, and you will have two broken lines $CX_1X_2...X_kA$ and $AX_{k+1}...X_nB$ whose total sum of the squares is at most the square of the hypothenuse. We can replace the segments X_kA and AX_{k+1} by the segment X_kX_{k+1} and obtain our broken line; the sum of the squares now becomes even less, since $\angle X_kAX_{k+1} \leq 90^\circ \Rightarrow X_kX_{k+1}^2 < AX_k^2 + AX_{k+1}^2$.

The other case is when one of the two triangles (say ACD) contains none of the points. Then we will apply the above algorithm for the triangle ABD and get a broken line $BX_1...X_nA$ whose sum of the squares is at most AB^2. But $X_nA^2 < X_nC^2 + AC^2$ so the line $BX_1...X_nC$ will be valid, as its sum of the squares is at most $AB^2 + AC^2 = BC^2$. If we can't apply the algorithm in the previous paragraph to ABD, it must be because its own altitude leaves all the points only one side. In that case, break it into two triangles with its altitude and repeat what is said in this paragraph. Eventually, we will get one of the smaller and smaller triangles to contain more than 0 and less that n points. And then the problem will be solved.

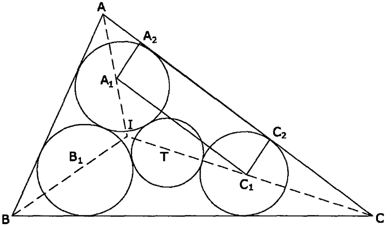
10. E Let our polygon be $A_1 ... A_n$ and we will prove the statement by induction over n. For $n = 3$ it is obvious and now we will assume it true for $n - 1$ and prove it for n. First of all, if there existed one of the points $A_1, ..., A_n$ (say A_i) who were the endpoint of at most one segment, then consider the other $n - 1$ points. Because of the induction hypothesis, among them there are at most $n - 1$ segments, and plus our at most one segment from A_i would yield at most n segments in total.

    Now, assume that each point is the endpoint of at least 2 segments and assume for the purpose of contradiction the we have at least $n + 1$ segments. These $n + 1$ segments have together at least $2n + 2$ endpoints, and by the pigeonhole principle there must be a vertex which is an endpoint to at least three segments. Let that vertex be A_1 and assume that $[A_1A_i]$, $[A_1A_j]$ and $[A_1A_k]$ are segments, with $i < j < k$. But it's easy to see that if we had another segment out of A_j except $[A_1A_j]$ that segment would not intersect either $[A_1A_i]$ or $[A_1A_k]$. Contradiction, and thus our assumption is wrong and the problem is thus solved.

    Observe that this tells us that any set of n points has at most n diameters. This is true because any two diameters of a set must intersect; indeed, if diameters AB and CD did not intersect, then either $ABCD$ would be a convex quadrilateral (and then $AC + BD > AB + CD$ contradicts the fact that AB and CD are diameters) or one of A, B, C, D lies inside the triangle formed by the other three. In this last case, suppose that D lies inside ABC and then note that at least one of CA or CB is greater than the diameter CD (which is again impossible).

11. Solution by Andrei Ungureanu

    M Let us assume the circles k_1, k_2, k_3 of radius r_0 as being tangent to the angles A, B and C respectively; let r_1 be the radius of k_4. Let A_1, B_1, C_1 and T be the centers of the circles k_1, k_2, k_3 and k_4.



But the points A_1, B_1, C_1 are on the bisectors of the angles A, B and C respectively, and therefore $AA_1 \cap BB_1 \cap CC_1 = I$.

Let A_2 and C_2 be the tangency points of the circles k_1 and k_3 with AC. We have that $A_1A_2 \parallel C_1C_2$, because both are perpendicular with AC, and that $A_1A_2 = C_1C_2$ (both are radii in the congruent circles k_1 and k_3). So $A_1A_2C_2C_1$ is a rectangle and this implies $A_1C_1 \parallel AC$.

Similarly, $A_1B_1 \parallel AB$ and $B_1C_1 \parallel BC$, so that the triangles ABC and $A_1B_1C_1$ can be obtained one from the other by a dilation. As I is the intersection of the lines AA_1, BB_1 and CC_1, it will be the two triangles' center of dilation. But T is the circumcenter of $A_1B_1C_1$ (as $TA_1 = TA_2 = TA_3 = r_0+r_1$) and through the dilation that transforms $A_1B_1C_1$ into ABC, T will be transformed into O. Since the center of dilation is I, we will have that I, T, O are collinear.

12. M Let our polygon be $A_1A_2...A_n$ and let $S_i = A_1A_{i+1} + A_2A_{i+2} + ... + A_nA_{i+n}$ (where A_{n+x} means A_x). Therefore, $d = S_2 + ... + S_k$ if $n = 2k+1$ and $d = S_2 + ... + S_{k-1} + \frac{S_k}{2}$ if $n = 2k$ (because if $n = 2k$, each of the diagonals A_iA_{i+k} is counted twice in S_k).

Obviously, we have

$$\overrightarrow{A_iA_{e+i}} = \overrightarrow{A_iA_{i+1}} + ... + \overrightarrow{A_{e+i-1}A_{e+i}} \Rightarrow A_iA_{e+i} < A_iA_{i+1} + ... + A_{e+i-1}A_{e+i}$$

If we sum this up after all i (for a certain e up to $\left\lfloor \frac{n}{2} \right\rfloor$), every side length in the RHS will appear exactly e times, so $S_e < ep$.

Now, if $n = 2k+1$, this yields

$$d = S_2 + ... + S_k < p(2 + ... + k) \Rightarrow \frac{2d}{p} < k(k+1) - 2 = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor - 2$$

For $n = 2k$ we have

$$d < p(2 + ... + k - 1) + p\frac{k}{2} \Rightarrow \frac{2d}{p} < k(k-1) + k - 2 = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor - 2$$

The left part of the inequality is trickier to prove. For that, we need the following

LEMMA 1 If $ABCD$ is a convex quadrilateral, then $AC + BD > AB + CD$. Let M be the intersection of AC and BD and we will have $AM + BM > AB$ and $CM + DM > CD$. Summing these up yields the claim of Lemma 1.

LEMMA 2 If $1 < k < n - 1$ then we have $S_k > S_1$.

This lemma can be proven easily by noting that for every i $A_iA_{i+1}A_{i+k}A_{i+k+1}$ is convex and thus we can apply Lemma 1 to it, thus yielding $A_iA_{i+k} + A_{i+1}A_{i+k+1} > A_iA_{i+1} + A_{i+k}A_{i+k+1}$. Summing these up after all i gives us the desired $S_k > S_1$.

But $2d = S_2 + S_3 + ... + S_{n-2}$, so we have $2d > (n-3)S_1$. But $S_1 = p$, so $\frac{2d}{p} > n - 3$.

13. M If $n = 2k + 1$ is odd, then it is easy to see that by putting n points at even spaces around a circle and uniting each of them to that point which is k points away in clockwise direction, we get the desired polygon.

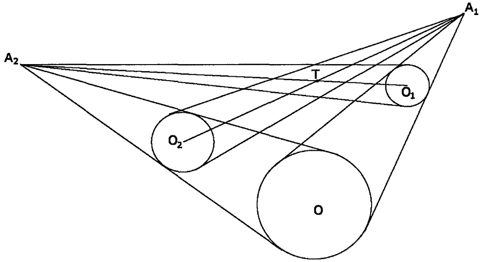
Now conversely, we must show that this is impossible in a polygon with an even number of vertices. First of all, note than a polygon with n diameters must have none of its points inside its convex hull; indeed, if it weren't so, then one of its vertices would be inside a triangle formed by three other vertices (just take the convex hull of the polygon and break it up into triangles). That vertex (say A_1) situated inside a triangle $A_iA_jA_l$ cannot be an endpoint of any diameter, because for any vertex A_x, the vector $\overrightarrow{A_xA_1}$ can be written as $k_i\overrightarrow{A_xA_i} + k_j\overrightarrow{A_xA_j} + k_l\overrightarrow{A_xA_l}$, where $0 \le k_i, k_j, k_l$ and $k_i + k_j + k_l = 1$ (barycentric coordinates). Now, that would mean that $A_xA_1 < k_iA_xA_i + k_jA_xA_j + k_lA_xA_l$, so A_xA_1 cannot be a diameter, because at least one of A_xA_i, A_xA_j and A_xA_l would be greater than it.

So a vertex like A_1 situated inside a triangle cannot be an endpoint to any diameter. Thus, the remaining $n - 1$ points form all the n diameters, which is impossible because of the remark at the end of problem 12 (that a set of m points has at most m diameters). Therefore, in order for it to have n diameters, our polygon needs to have none of its vertices inside its convex hull, and let the vertices of the convex hull be $B_1B_2...B_n$ in this order. Let us now suppose, as the hypothesis claims, that the n diameters form a cycle. It is easy to see that any two diameters need to have a common point (otherwise, we could find even longer segments than the diameters).

Let B_1B_j be a diameter, and call the two sets of vertices that are split by this diameter M and N. The other diameters that leave from B_1 (say B_1B_x), respectively B_j (say B_jB_y) must both be directed into one of the sets M or N (otherwise, these diameters would not meet). Let's assume that they both are directed into M. All the other $n-3$ diameters will now unite a point from M with a point from N (because each of them must intersect B_1B_j). So a chain of $n-3$ diameters must leave B_x and travel back and forth between M and N, ultimately reaching B_y. But $n-3$ is odd, so that chain cannot start and end in M (where B_x and B_y both are). Contradiction.

14. M SOLUTION BY GABRIEL KREINDLER

Let us denote by O, O_1 and O_2 the centers of the circles Γ, Γ_1 and Γ_2 respectively; suppose the lines A_1O_2 and A_2O_1 meet at T.



We will try to prove that T is the center of the inscribed circle in the quadrilateral determined by the intersections of the tangents from A_1 and A_2 to the circles Γ_2 and Γ_1.

We will find the distance d_1 from T to the tangents drawn from A_1 to circle Γ_2. We have that $\frac{d}{r_2} = \frac{A_1T}{A_1O_2}$ (1), but we know that

$$\frac{A_1O_1}{A_1O} = \frac{r_1}{r} \Rightarrow \frac{OO_1}{O_1A_1} = \frac{r-r_1}{r_1}$$

and that

$$\frac{O_2A_2}{A_2O} = \frac{r_2}{r}$$

Applying the theorem of Menelaos applied in triangle A_1OO_2 with the secant O_1TA_2 we find that

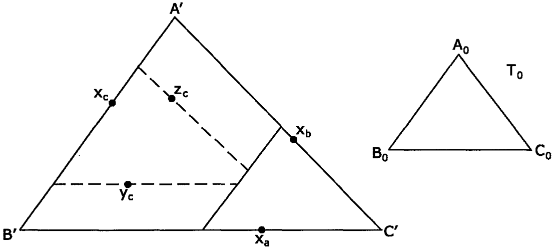
$$\frac{A_1T}{TO_2} \cdot \frac{O_2A_2}{A_2O} \cdot \frac{OO_1}{O_1A_1} = 1 \Rightarrow \frac{TO_2}{A_1T} = \frac{rr_2 - r_1r_2}{rr_1} \Rightarrow \frac{A_1T}{A_1O_2} = \frac{rr_1}{rr_1 - r_1r_2 + rr_2}$$

so by going back to relation (1), we get that $d = \dfrac{rr_1r_2}{rr_1 - r_1r_2 + rr_2}$. But this expression is symmetrical in r_1 and r_2, thus it will also be the distance from T to the tangents from A_2 and we conclude that T is the center of a circle with radius d that is inscribed in the given quadrilateral.

15. E For $n = 4$ any parallelogram verifies. For $n \geq 5$, take a regular $n-2$-gon and a line l through one of its vertices (call that vertex A) which isn't parallel to any side and which only touches the polygon in A. Then, translate this line until again in touches the polygon in only one vertex (call that vertex B). Now, move the lines just slightly closer together, and the two of them together with the original $n-2$-gon form an n-gon. We will prove that the sum of a distances from an inside point to its sides is constant.

    The sum from ant point to the two parallel line we just built is obviously constant (it equals the distance between the lines). Now, for any point in the regular $n-2$-gon, multiplying the sum of the distances to the sides by the side length will yield double the area of the polygon. So the sum of the distances from any interior point in the regular $n-2$-gon to the sides will be constant. Because this sum is constant, as is the sum of the distances to the two new parallel sides, the total sum of the distances in the n-gon we built will be constant.

16. D Let $T = A_0B_0C_0$ and let d_a, d_b, d_c be the directions of the sides of the triangle T. Through each point of our set let us draw a line parallel to every side of T, and from all the triangles determined by these 3 sets of parallels, let us pick the one which contains all our points in its interior or on its sides. So we will have a triangle $T' = ABC$ which has a point of X on each side and which contains all the points in its interior. Let X_a be that point on the side of T' which is parallel to d_a and define similarly the points X_b and X_c (if two of these points are identical, there is no problem in what follows).



Let us construct three translations of T, called T_a, T_b and T_c which are contained in T' and have one common vertex with it (T_a will correspond to that vertex which opposes direction d_a). We will now look at the regions $T' - T_a$, $T' - T_b$ and $T' - T_c$. Let Y_a, Z_a be the points of $T' - T_a$ the closest to AB and AC, respectively; define similarly Y_b, Z_b, Y_c, Z_c.

The 9 points $X_a, X_b, X_c, Y_a, Y_b, Y_c, Z_a, Z_b, Z_c$ can be covered by two translations of T. That is why two of X_a, X_b, X_c (suppose them X_a and X_b) must be covered by the same translation T_0. Note that because T_0 covers X_a and X_b we will have that $T_0 \bigcap T' \subset T_c$ (since T_0 and T_c have the same size). That is why instead of T_0 we can consider T_c, which covers at least as many points (outside of T' there are no points of our set, so this area isn't worth covering).

So T_c is one of the translations. Therefore, the other one (call it T_1), must cover the points X_c, Y_c, Z_c (which are by definition in $T' - T_c$). But any point of our set inside $T' - T_c$ is covered by T_1, because the points X_c, Y_c, Z_c are extremal in this set with respect to the directions d_a, d_b, d_c.

We have thus constructed two translation that covered the points of T_c and $T' - T_c$, so all the points are covered. The problem is therefore proven.

17. *M* We will use an important facts concerning Simson lines, which is proved in the appendix.

    LEMMA If $ABCD$ are points on a circle, the 4 Simson lines of one with respect to the other three are concurrent in the point determined by the complex number $\dfrac{a+b+c+d}{4}$ (where a, b, c, d are the complex numbers associated to our points).

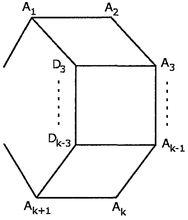
    Now the problem becomes easy. Let the common point of those 4 lines be X. The two Simson lines $l(A, BDF)$ and $l(D, ABF)$ intersect in the point given by the complex number $\dfrac{a+b+d+f}{2}$ (as the Lemma says). The other two lines intersect in $\dfrac{a+b+c+e}{2}$, and since all 4 lines are concurrent, these two points must be one and the same. But equating the two yields $c + e = d + f$, and so $CDEF$ is a parallelogram. But since it is inscribed in a circle, it must be a rectangle.

18. *D* Let us consider a certain side, which we will call S_0 and which we will assume to be at the "bottom" of the polygon (just imagine the polygon to be a rigid body sitting on the side S_0). A number of parallelograms are adjacent to S_0, and let S_1 be the set of sides of these parallelograms that are above S_0 and parallel to it. Obviously, all the segments in S_1 are parallel among themselves and the sum of their lengths will be at least that of S_0. A number of other parallelograms will be adjacent to the sides in S_1, and let S_2 be the set of their sides which are parallel to those of S_1 and lie above it. Again, the sum of the lengths of the segments of S_2 will be at least that of S_1. Repeat this algorithm, and when we reach a side of the polygon with a side of a parallelogram, consider

that side further in the next level S_j. Ultimately, we will reach an S_i which comprises only of polygon sides, and because the segments of the S_i are all parallel, they can only lie on one side. If that side is S'_0, we just proved that $S_0 \parallel S'_0$ and $|S'_0| \geq |S_0|$; but the same reasoning, starting with S'_0 will yield $|S_0| \geq |S'_0| \Rightarrow |S_0| = |S'_0|$.

So we just proved that each side of the polygon has another side, parallel and equal to it. In particular, this means that the polygon has an even number of sides $A_1A_2...A_{2k}$. Let A_iA_{i+1} be the side parallel to A_1A_2 and we will show that $i = k+1$. Because the polygon is convex, $A_2A_3, ..., A_{i-1}A_i$ must be pairwise parallel to $A_{i+1}A_{i+2}, ..., A_{2k}A_1$ respectively. Therefore, $i-2 = 2k-i \Rightarrow i = k+1$. So the side A_1A_2 will be parallel to that side which is k sides away from it, and the same remark goes for any one of the sides. Let O be the midpoint of A_1A_{k+1}. Because $A_1A_2A_{k+1}A_{k+2}$ is a parallelogram (equal and parallel opposite sides) then O will also be the midpoint of A_2A_{k+2}. In this way, we can prove that O is the midpoint of $A_3A_{k+3}, A_4A_{k+4}, ...A_{2k}A_{k-1}$ and thus O will be the center of symmetry.

We proved before that our polygon must be centrally symmetric, and therefore it must have $2k$ sides. Let us prove that the minimum number of parallelograms in which such a $2k$ sided n-gon can be broken up is $\binom{k}{2}$. First, we will show that this number is necessary. Take any two pairs of parallel sides $a \parallel b$ and $c \parallel d$ and two sequences of pairwise adjacent parallelograms, one going from a to b and the other from c to d. Eventually, these two 'strings'of parallelograms will intersect, and since the parallelograms are all distinct, they can only intersect in a whole parallelogram. So for any 2 pairs of opposite sides, there is a parallelogram with sides parallel to them. Since there are $\binom{k}{2}$ such pairs, we have at least so many parallelograms.



Now, to prove that $\binom{k}{2}$ are enough. We will prove this by induction over k (it is obvious for $k = 2$), so assume it true for $k - 1$ and let's prove it for k. Consider the parallelograms $A_1A_2A_3D_1$, $D_1A_3A_4D_2$, $D_2A_4A_5D_3$,...,

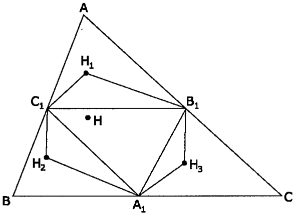
$D_{k-3}A_{k-2}A_{k-1}D_{k-2}$ and $D_{k-3}A_{k-1}A_kA_{k+1}$. There are $k-1$ of them and by removing them we are left with a centrally symmetric $2k-2$-gon. By the induction hypothesis, we can pave it using $\frac{(k-1)(k-2)}{2}$; so in total we used $\frac{(k-1)(k-2)}{2} + k - 1 = \frac{k(k-1)}{2}$. The induction is thus complete

Now, since $27 < \binom{8}{2}$, the above claims tell us that our polygon has at most $2 \cdot 7$ sides. But we just proved that to pave it, there are enough $\binom{7}{2} = 21$ parallelograms.

19. E Solution by Adrian Zahariuc

Let H_1,H_2,H_3 be the orthocenters of the triangles AB_1C_1, A_1BC_1 and A_1B_1C respectively, and let H be that of the original triangle. The hexagon the problem refers to is $A_1H_3B_1H_1C_1H_2$. We will have that

$$[A_1H_3B_1H_1C_1H_2] = [A_1B_1C_1] + [H_1B_1C_1] + [A_1H_2C_1] + [A_1B_1H_3]$$



But AB_1C_1 and ABC may be dilated into each other by a factor of 2, and through this dilation HB_1C_1 will be transformed into HBC by the same factor. So we will have $[H_1B_1C_1] = \frac{[HBC]}{4}$ and the corresponding equalities for the other vertices. Therefore,

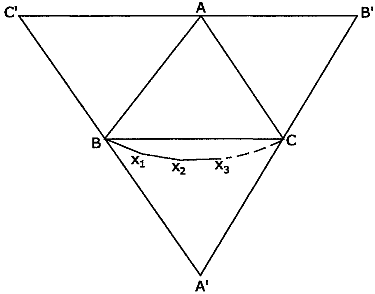
$$[A_1H_3B_1H_1C_1H_2] = [A_1B_1C_1] + \frac{[HBC] + [HAC] + [HAB]}{4} =$$

$$= \frac{[ABC]}{4} + \frac{[ABC]}{4} = \frac{[ABC]}{2}$$

20. M Assume that our polygon has at least 4 points, and note that in this case it can't have all sides and diagonals equal (only an equilateral triangle can ave that). Consider the diameter AB of our polygon, and there must be a point C such that $\angle ACB = 60°$. But AB will be the largest side in the triangle ABC, so it will oppose the largest angle. But $60°$ can be the largest angle iff all the

angles are 60°, so $AB = BC = CA$. Similarly, if DE is the segment of shortest length, there will be an F such that DEF is equilateral.

But let us go back to the triangle ABC. Draw lines through A, B and C parallel to BC, AC and AB respectively, and assume that they meet in A', B' and C'. It is easy to note that if we had vertices of our polygon outside of $A'B'C'$ the we would contradict the maximality of AB, BC and CA. Now, if one the sides of the triangle DEF (say DE) cuts one of the sides of ABC (say AB), it will have to also cut AC (as there are no points of our convex polygon inside ABC) and then $\angle DAE > \angle BAC = 60^\circ$. But DE is the shortest side in the triangle DAE (by assumption), so it will oppose the smallest angle. So $\angle DAE > 60^\circ$ is a contradiction.



Therefore the triangle DEF is completely contained in one of the triangles $A'BC$, $B'AC$ or $C'AB$, so assume that it is contained in $A'BC$. Let the vertices of our polygon inside that triangle be $B = X_0, X_1, X_2, ..., X_k = C$ in that order, and assume that the triangle DEF is $X_iX_jX_k$ with $i \leq j \leq k$. But then because ore polygon is convex $\angle BX_jC \leq \angle X_iX_jX_k = 60^\circ$. Of course, since X_j is inside $A'BC$, this can only happen when $X_j = A'$. But then the segment AX_j is greater than the diameter, which is impossible. So our polygon cannot have 4 vertices, and thus it must be an equilateral triangle.

21. *M* We wish to show the equivalence of the two conditions, but it is better to put them both into a nicer form before we do that. Let us see what it means for O to be on W_aW_b; I claim that W_aW_b is the locus of the points inside the triangle whose distance to AB equals the sum of the distances to the other two sides. Indeed, note that the W_a and W_b satisfy this property, and any other point X satisfies it iff its barycentric coordinates can be written as a convex sum of those of W_a and W_b (this is very easy to check). But this happens if and only if that point is on W_aW_b, as it is stated in the Appendix.

So we just showed that O being on W_aW_b is equivalent to its distance from AB being the sum of the other two distances, or equivalently that $\cos C = \cos A + \cos B$. But what is the condition for I to be on H_aH_b? For that, we will use the barycentric coordinates of the points H_a, H_b, I, which are $\left(0, \frac{b\cos C}{a}, \frac{c\cos B}{a}\right)$, $\left(\frac{a\cos C}{b}, 0, \frac{c\cos A}{b}\right)$ and $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$. I is on H_aH_b iff it its barycentric coordinates can be written as a convex sum of those H_a and H_b, so iff there exists an α such that

$$\frac{a}{a+b+c} = \alpha\cdot 0 + (1-\alpha)\frac{a\cos C}{b} \Leftrightarrow 1-\alpha = \frac{b}{(a+b+c)\cos C}$$

and

$$\frac{b}{a+b+c} = (1-\alpha)\cdot 0 + \alpha\frac{b\cos C}{a} \Leftrightarrow \alpha = \frac{a}{(a+b+c)\cos C}$$

We needn't write the third relation between the barycentric coordinates, as it is implied by the first two (as the sum of the barycentric coordinates is always 1). Such an α exists iff the two RHS above add up to 1, so our condition for I being on H_aH_b is that

$$a+b = (a+b+c)\cos C \Leftrightarrow \sin A + \sin B = (\sin A + \sin B + \sin C)\cos C \Leftrightarrow$$

$$2\cos\frac{C}{2}\cos\frac{A-B}{2} = \left(2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}\right)\cos C \Leftrightarrow$$

$$\Leftrightarrow \cos\frac{A-B}{2} = \left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right)\cos C \Leftrightarrow$$

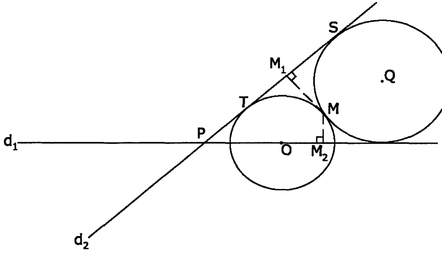
$$\Leftrightarrow \cos\frac{A-B}{2}(1-\cos C) = \sin\frac{C}{2}\cos C \Leftrightarrow$$

$$\Leftrightarrow \cos\frac{A-B}{2}\cdot 2\sin^2\frac{C}{2} = \sin\frac{C}{2}\cos C \Leftrightarrow$$

$$\Leftrightarrow 2\cos\frac{A-B}{2}\sin\frac{C}{2} = \cos C \Leftrightarrow \cos A + \cos B = \cos C$$

We thus see that the two conditions are the same, so O will be on W_aW_b iff I is on H_aH_b.

22. E We will show that the locus is formed by two straight lines that pass through P (without P itself). Indeed, take R to be the radius of the circle with center O, let T be its tangency point with d_2. The circle that is tangent to this circle will have radius r, center Q, and it will touch d_2 in S. Let M be the common point of the two circles, and let M_1 and M_2 be its projections on d_1 and d_2.



Because M lies on OQ, we will have

$$\frac{MM_1}{r} = \frac{R}{r+R}$$

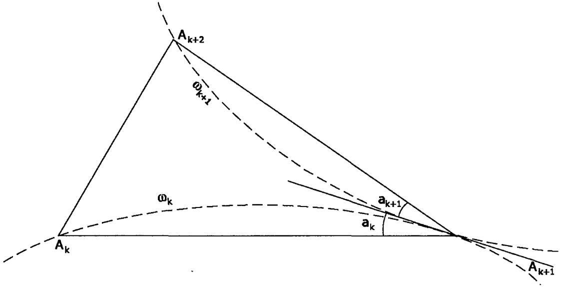
And from the right trapezoid $OQSP$ we find

$$MM_2 = \frac{2rR}{r+R}$$

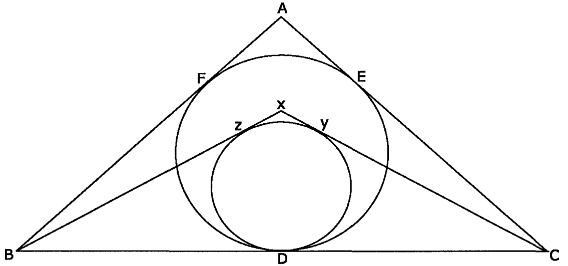
Dividing the two yields $\dfrac{MM_2}{MM_1} = 2$ and it is well-known that the locus of points with this property are two straight lines through P (one in each angle).

Conversely, we need to show that any such point M (with the ratio of its projections 2) will be a tangency point. Given such a point, we need to construct a circle that passes through it and is tangent to d_1 and d_2, and then construct a circle with the center on d_1 and tangent to the other in M. Because of the ratio in M's projections, it's easy to compute that this last circle will be tangent to d_2 (as the distance from its center to d_2 will equal its radius). Thus, we obtain for every such M a valid construction.

23. M For every k, let a_k be the angle between circle ω_k and the line A_kA_{k+1}. Because ω_{k+1} is tangent to ω_k at A_{k+1} their tangent will break the angle $\angle A_{k+1}$ into two parts, and we will have $a_{k+1} + a_k = A_{k+1}$. Multiply this by $(-1)^k$ and we will get $(-1)^k a_{k+1} + (-1)^k a_k = (-1)^k A_{k+1}$. Now, if we sum these relations for k from 1 to 6, then we get $a_7 - a_1 = A_7 - A_6 + A_5 - A_4 + A_3 - A_2$. If we keep in mind that $A_7 = A_4$, $A_6 = A_3$ and $A_5 = A_2$ (we considered these angles modulo 3), the we get $a_7 = a_1$. So the circles ω_7 and ω_1 pass through the same two points and make equal angles with the line determined by those two points. Since these circles do not touch any of the sides in their interiors (otherwise, we wouldn't be able to construct ω_8 or ω_2), their centers need to be both below line A_0A_1. That means that they must be one and the same.



24. D We will consider an inversion around D, which will map the line BC into itself, and our two circles into lines parallel to BC. But note that E, Y, D and are at the same distance from C, so they are on a circle that obviously passes through D. The inversion will turn this into a line perpendicular to BC (because the center of the circle, namely C, is on BC) and the same thing will happen to the circle determined by the points F, Z, D.



So if the images of E, F, Y, Z are E', F', Y', Z' then we just showed that $E'F' \| BC$, $Y'Z' \| BC$, $E'Y' \perp BC$ and $F'Z' \perp BC$. Therefore, $E'F'Y'Z'$ is a rectangle, so the four images are on the same circle. Inverting the figure again to obtain the original, this will turn into a circle, so our four points E, F, Z, Y are on the same circle.

25. E The easiest way to solve this problem is to consider the vertices of the square as being determined by the complex numbers $A = il$, $B = l$ and $C = -il$. Then the relations given by the hypothesis are $|p - il| = 1$, $|p - l| = 2$ and $|p + il| = 3$. These become

$$(p-il)(\overline{p}+il)=1 \Rightarrow p\overline{p}+il(p-\overline{p})=1-l^2$$

and similarly $p\overline{p}-l(p+\overline{p})=4-l^2$ and $p\overline{p}-il(p-\overline{p})=9-l^2$. If we let $-p-\overline{p}=2S$, $i(p-\overline{p})=2D$, then $p\overline{p}=S^2+D^2$ and the above relations become

$$S^2+D^2+2lD=1-l^2$$

$$S^2+D^2+2lS=4-l^2$$

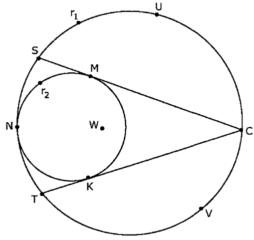
$$S^2+D^2-2lD=9-l^2$$

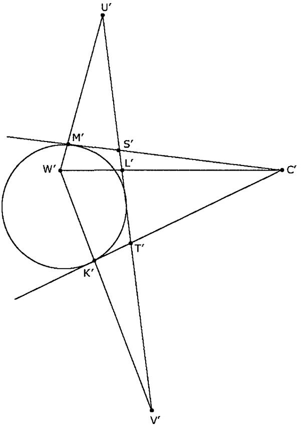
By summing up the first and the last one, we get $S^2+D^2=5-l^2$ and the first and second equations then give us $D=-\frac{2}{l}$ and $S=-\frac{1}{2l}$. Therefore

$$5-l^2=S^2+D^2=\frac{17}{4l^2} \Rightarrow 4l^4-20l^2+17=0 \Rightarrow l^2=\frac{5\pm 2\sqrt{2}}{2} \Rightarrow 2l^2=5\pm 2\sqrt{2} \quad (1)$$

But $2l^2$ is precisely the square of the side of our square. If P is inside the square, the angle APC will be obtuse, so $AC^2 \geq 3^2+1=10 \Rightarrow AB^2 \geq 5$. Of course, neither of our solutions satisfy this, so we must conclude that there are no such points P.

26. D Let us consider an inversion around C and denote the inverses of the points in our figure by $'$. The circles $(CTS),(CVK)$ and (CMU) will transform into the lines $T'S'$, $V'K'$ and $M'U'$, and W' will be the intersection of these last two lines. The circle (NKM) transforms into a circle tangent to the lines $C'T'$, $C'S'$ and $T'S'$, but it will be escribed since Γ_1 and Γ_2 were initially one inside the other.





But U' is on the line $T'S'$, on the other side of S' than T' such that

$$CU = SU \Leftrightarrow \frac{1}{CU'} = \frac{S'U'}{CU'CS'} \Rightarrow S'U' = CS'$$

Similarly, V' is on $T'S'$ such that $T'V' = CT'$. Now, to prove that $CUWV$ is a parallelogram, it suffices to show that $CV \parallel UW$ ($CU \parallel VW$ is proven in the same way). This is equivalent to proving that the angle between the lines CV and UW is 0, which in turn in equivalent to showing that the angle between line CV' and circle $(CU'W')$ is 0 (since inversion preserves angles). So we need to prove that CV' is tangent to circle $(CU'W')$; let us denote $CT' = s$, $CS' = t$, $T'S' = c$ and $CK' = V'N' = U'N' = CM' = \dfrac{s+t+u}{2} = p$.

Because $T'C = T'V'$ and $T'N' = T'K'$ it's easy to see that $CN'K'V'$ is an isosceles trapezoid, and the same goes for $CN'M'U'$. Let L' be the intersection of CW' and $U'V'$. Apply the Law of Sines in triangles $V'L'W'$ and $CK'W'$ and we get

$$\frac{L'V'}{\sin L'W'K'} = \frac{L'W'}{\sin L'V'W'}$$

$$\frac{p}{\sin L'W'K'} = \frac{CW'}{\sin V'K'T'}$$

Dividing these two yields

$$\frac{L'V'}{p} = \frac{L'W'}{CW'}\frac{\sin V'K'T'}{\sin L'V'W'} = \frac{L'W'}{CW'}\frac{s}{p-s}$$

Similarly, we get that

$$\frac{L'U'}{p} = \frac{L'W'}{CW'}\frac{t}{p-t}$$

and by dividing these two equations we get

$$\frac{L'V'}{L'U'} = \frac{p-t}{p-s}\cdot\frac{s}{t}$$

Now,

$$CV' = 2s\cos\frac{CT'S'}{2} == 2s\sin\frac{T'}{2} = 2s\sqrt{\frac{p(p-t)}{sc}}$$

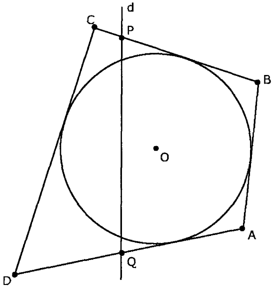
and similarly $CU' = 2t\sqrt{\frac{p(p-s)}{tc}}$. This means that

$$\frac{CU'^2}{CV'^2} = \frac{t}{s}\cdot\frac{p-s}{p-t} = \frac{L'U'}{L'V'}$$

which is exactly the condition for CL' to be isogonal to the median CN' in the triangle $CU'V'$. Therefore, $\angle V'CW' = \angle N'CU' = \angle CU'W'$ because $CN'M'U'$ is an isosceles trapezoid. But this means that CV' is tangent to $(CU'W')$ and our problem is thus proven.

27. E Assume that our line d does not pass through O (the center of the inscribed circle) and let the region which contains O be $ABPQ$ with P on BC and Q on AD on the sides of the quadrilateral (that region could also be a triangle or a pentagon, but the proof is perfectly similar then). If S, p and r are the area semiperimeter and inradius of the original quadrilateral, we would have

$$\frac{S}{2} = [ABPQ] > [OAQ] + [OAB] + [OBQ] = \frac{r}{2}(AQ + AB + BQ) = r\left(p - \frac{PQ}{2}\right)$$



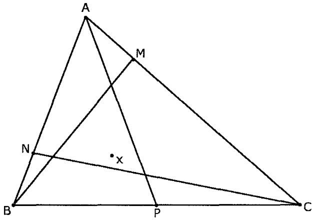
Similarly, we would have

$$\frac{S}{2} = [CDPQ] < [ODQ] + [ODC] + [OCP] = \frac{r}{2}(DQ + DC + CP) = r\left(p - \frac{PQ}{2}\right)$$

Obviously, we cannot have both the above relations at the same time, so we must have O on the line d.

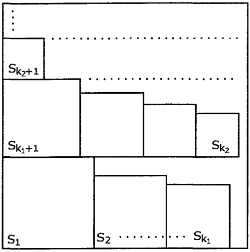
28. *M* We will show that the area of the middle triangle equals the sum of the areas of the other 3. Let the barycentric coordinates of X be (α, β, γ) and then we will have $\frac{PC}{BC} = \beta \Rightarrow \frac{[APC]}{S} = \beta$, where S denotes the area of the triangle. So

$$\frac{[APC] + [CNB] + [BMA]}{S} = \alpha + \beta + \gamma = 1 \Rightarrow [APC] + [CNB] + [BMA] - S = 0$$



But the expression $[APC] + [CNB] + [BMA] - S = 0$ is precisely the sum of the areas of the small triangles minus that of the middle one. The problem is thus proven.

29. *M* We will try to pack our squares into the unit square by using the following algorithm: label the squares $S_1, ..., S_n$ in the descending order of their radii $r_1 \geq r_2 \geq ... \geq r_n$. We will arrange them in layers like in the figure, where we let $i_0 = 0$: suppose we have already arranged several layers containing squares $S_1, S_2, ..., S_{k_i}$ and we will construct a new layer as follows: place S_{k_i+1} right above square $S_{k_{i-1}+1}$ and keep aligning squares next to it until we can't place any other (because we would exit the unit square). Let $S_{k_{i+1}}$ be the last square you place in the current layer, and then repeat the algorithm until we run out of the squares.



Let $S_{k_{t+1}}$ be the last square we place, so the total height of our layers will be $r_1 + r_{k_1+1} + ... + r_{k_t}$, which we will assume for the purpose of contradiction to be > 1 (if it were at most 1, we could pack our layers inside the square). But we will have for all i

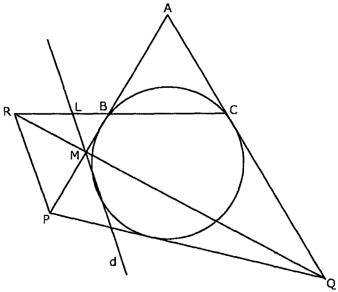
$$r_{k_i+2}^2 + ... + r_{k_{i+1}+1}^2 \geq r_{k_{i+1}+1}(r_{k_i+2} + ... + r_{k_{i+1}+1}) > r_{k_{i+1}+1}(1 - r_{k_i+1}) \geq r_{k_{i+1}+1}(1 - r_1) \quad (1)$$

where we used the fact that the r_j's form a descending sequence and that $r_{k_i+2} + ... + r_{k_{i+1}+1} > 1 - r_{k_i+1}$ (because if this inequality weren't so, we could put $P_{k_{i+1}+1}$ on the same level with P_{k_i+1}). But by summing (1) for all i's, we would get on the LHS the total sum of our squares except for P_1, so

$$\frac{1}{2} - r_1^2 \geq \sum_{i \geq 0} r_{k_{i+1}+1}(1 - r_1) > (1 - r_1)^2$$

by our assumption that $r_1 + r_{k_1+1} + ... + r_{k_t+1} > 1$. So we will have $0 > 2r_1^2 - 2r_1 + \frac{1}{2} = \left(\sqrt{2}r_1 - \frac{1}{\sqrt{2}}\right)^2$, which is impossible. Our packing therefore works.

30. M Consider the line d, tangent to the circle in N and parallel to AC. Suppose it intersects RC in L and RQ in M; the point L is fixed, and we will show that M is that point which is on all the lines RQ. For this, we need to show that LM is constant no matter what points P and Q we choose.



Since $LM \parallel QC$ we will have

$$LM = CQ\frac{RL}{RC} = CQ\left(1 - \frac{LC}{RC}\right) \tag{1}$$

But the diameter NC is equal to $2r$ (where r is the radius of our circle), and we will also have $NC \perp AC$; therefore, $\angle LCN = 90^\circ - \angle BCA = \frac{\angle A}{2}$ which implies that $LC = \frac{2r}{\cos \angle LCN} = \frac{2r}{\cos \frac{A}{2}}$. But the triangles ABC and PBR are similar, and therefore PBR will be isosceles (like ABC is) and $RB = BC\frac{BP}{AB} \Rightarrow RC = BC\frac{AP}{AB}$. Equation (1) thus becomes

$$LM = CQ\left(1 - \frac{2r \cdot AB}{BC \cdot AP \cos \frac{A}{2}}\right) \tag{2}$$

Now, we will have $AB = r\cot\frac{A}{2}$ and $BC = 2r\cos\frac{A}{2}$. By letting a, p, q be the side-lengths of the triangle APQ, $2p_0$ be the perimeter of the triangle and S the area ($r = \frac{S}{p_0}$), equation (2) becomes

$$LM = (p_0 - q)\left(1 - \frac{2r}{q\sin A}\right) = (p_0 - q)\left(1 - \frac{2Sp}{p_0 \cdot 2S}\right) = \frac{(p_0 - q)(p_0 - p)}{p_0}$$

But a well-known formula in any triangle is $(p_0 - q)(p_0 - p) = pq\sin^2\frac{A}{2}$, and therefore the above relation becomes

$$LM = \frac{pq}{p_0}\sin^2\frac{A}{2} = \frac{2S}{p_0\sin A}\sin^2\frac{A}{2} = \frac{2r}{\sin A}\sin^2\frac{A}{2}$$

which is constant, no matter what points P and Q we pick. All the lines RQ will therefore pass through the point M.

31. M Let $ABCD$ be our tetrahedron; we will denote by $\overrightarrow{AB}\cdot\overrightarrow{CD}$ the dot product of the two vectors and by $AB \cdot CD$ the product of the lengths of the two segments. Let $a = \overrightarrow{OA}, b = \overrightarrow{OB}, c = \overrightarrow{OC}, d = \overrightarrow{OD}$, where O is an arbitrary point and letting α be the common angle, we will have

$$(a-d)(b-c) = \pm AD \cdot BC \cos\alpha$$
$$(a-b)(c-d) = \pm AB \cdot DC \cos\alpha$$
$$(a-c)(d-b) = \pm AC \cdot BD \cos\alpha$$

Summing these up yields

$$0 = \cos\alpha(\pm AD \cdot BC \pm AB \cdot DC \pm AC \cdot BD) \tag{1}$$

But we will show $\pm CD \cdot BC + \pm AB \cdot DC + \pm AC \cdot BD \neq 0$ whatever the signs; if the signs were all equal, then this is obvious. Otherwise, we would need to have something of the form

$$AD \cdot BC = AB \cdot DC + AC \cdot BD$$

Take an inversion around A, and the images of the points B, C, D will not be collinear (otherwise, our four points would be on a circle). Then, we would have

$$B'C' < B'D' + C'D' \Rightarrow \frac{BC}{AB \cdot AC} < \frac{BD}{AB \cdot AD} + \frac{CD}{AC \cdot AD} \Rightarrow AD \cdot BC < AB \cdot CD + AC \cdot BC$$

So the bracket in the RHS of (1) will always be non-zero, so we must have $\alpha = \frac{\pi}{2}$.

32. M LEMMA In any triangle with the usual notations, we will have

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1 \tag{1}$$

It is a well-known trigonometric relation that the RHS is equal to $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$. Now, by multiplying together $\sin\frac{A}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}}$ and the corresponding expressions for B and C we get

$$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 4\frac{S^2}{abcp} = \frac{\frac{S}{p}}{\frac{abc}{4S}} = \frac{r}{R}$$

which yields (1).

Back to the problem, where we will consider the triangle ABP (denote its sides and angles by a, b, p, A, B, P), whose largest side is obviously AB. Then, first note that $\cos A + \cos B \geq 1$, which happens because $\cos A + \cos B \geq \frac{a\cos B + b\cos A}{p} = 1$. Now, take a Cartesian coordinate system which has AB as its x-axis and the points P, C, D above it. We will compute the absolute values of the slopes of the lines OI and CD and show that their product is 1, which will prove that the lines are perpendicular. The slope of OI will be

$$\frac{|r - R\cos P|}{\frac{|a-b|}{2}} = \frac{\left|\frac{r}{R} - \cos P\right|}{|\sin A - \sin B|} = \frac{\cos A + \cos B - 1}{|\sin A - \sin B|}$$

where we used Lemma 1 to evaluate $\frac{r}{R}$. But the slope of CD is

$$\frac{p|\sin B - \sin A|}{p(\cos B + \cos A - 1)}$$

and it is easy to see that the product of the two slopes is 1, as required.

33. M Consider coordinate axes parallel to the sides of our square. Divide the square into 22 even-spaced horizontal (horizontal meaning parallel to the x-axis and vertical will mean parallel to the y-axis) strips of width $\frac{1}{22}$. By the pigeonhole principle, one of these will contain (at least) 23 points $A_1, ..., A_{23}$ and let their x-coordinates be $x_1 \leq x_2 \leq ... \leq x_{23}$ and their y-coordinates $y_1, ..., y_{23}$. Obviously, since $x_{23} - x_1 \leq 1$, we will have either $x_{12} - x_1 \leq \frac{1}{2}$ or $x_{23} - x_{12} \leq \frac{1}{2}$. Assume that we have $x_{12} - x_1 \leq \frac{1}{2}$.

Now, because the height of the strip where $A_1, ..., A_{12}$ are located is $\frac{1}{22}$, it means that $|y_i - y_j| \leq \frac{1}{22}$ for all i, j, and therefore

$$A_1A_2 + ... + A_{11}A_{12} < |x_2 - x_1| + ... + |x_{12} - x_{11}| + |y_2 - y_1| +$$
$$+ |y_3 - y_2| + ... + |y_{12} - y_{11}| \leq \frac{1}{2} + \frac{11}{22} = 1.$$

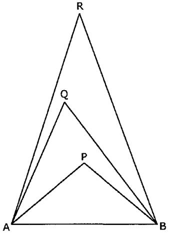
34. M We shall first prove two very simple and useful lemmas.

LEMMA 1 If $XYZT$ is a convex quadrilateral, then $XZ + YT > XY + ZT$. This follows easily if we let M be the intersection of the diagonals and sum up the two inequalities $XM + YM > XY$ and $ZM + TM > ZT$.

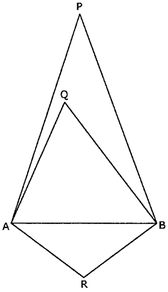
LEMMA 2 If X is inside the triangle EFG, then $EX + FX < EG + FG$. Indeed, let EX and FX intersect the sides GF and GE in S and T. Then, we will have $EG + GS > ES$ and $SF > FX - SX$; by summing them up, we will get the required inequality.

Now, if any two of the points P, Q, R form a convex quadrilateral together with A and B such that AB is not a diagonal (say, $ABPQ$ is convex in this order) then the problem becomes simple, as Lemma 1 tells us that $AP+BQ > AB+PQ$, and if we add to this the triangle inequalities $AQ+AR \geq QR$ and $BP+BR \geq PR$ then we are done. Therefore, we are only left with the case when any quadrilateral formed by any two of the P, Q, R together with A and B is either concave or has AB as a diagonal.

So we have two cases: either all of P, Q, R are on the same side of AB, or two of them are on one side and the third on the other. In the first case, there are no convex quadrilaterals with AB as a diagonal, so $ABPQ, ABQR, ABRP$ must all be concave. Assume that P is the closest of the three points to AB and R the farthest. Then, because the three quadrilaterals mentioned above are concave we must have Q inside ABR and P inside ABQ. In that case, Q will be in one of the triangles APR or BPR, and we will assume WLOG that it lies in APR. Lemma 2 tells us that $PQ+QR \leq AP+AR$, and summing this up with the triangle inequalities $PR \leq BP+BR$ and $AB \leq AQ+BQ$ will yield the desired result.

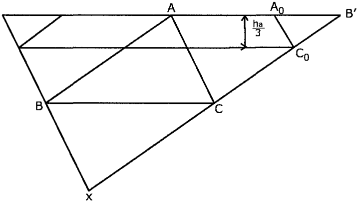


The second case is when P and Q are on the opposite side of AB as R is. In this case, because $ABPQ$ is convex, Q will be inside ABP (or the other way around), and it will lie in either APR or BPR. Assume it in APR and by doing the exact same thing as in the previous case, we will get the desired result.



35. D Let our pentagon be $ABCDE$. First of all, if the 5 points do not form a convex polygon, then consider their convex hull. One of the other points will be inside the hull, so it will be inside one of the triangles with vertices on the convex hull. So four of our points will be A, B, C, X such that X is inside the triangle ABC. Thus, the area of ABC is equal to the sum of the areas of XAB, XBC, XCA, and thus at least 6.

We are left with the case when our points form a convex pentagon. Let ABC be the triangle of maximum area among those with vertices in our points and assume $[ABC] < 3$. Then draw lines through the vertices of ABC parallel to the opposite sides, and let these lines determine a triangle $A'B'C'$. Because of the maximality of the area of ABC, D and E must lie inside the triangle $A'B'C'$. Moreover, they cannot lie inside the triangle ABC (as the pentagon is convex) so they can only lie in one of the three triangular regions $A'BC$, $AB'C$ and ABC'. Suppose first of all that they lie in different regions, say E lies in $AB'C$ and D lies in $AC'B$.



Consider the line A_0C_0 parallel to AC, as in the figure, where A_0C_0 is two thirds away from AC as B is. If E lies closer to AC that A_0C_0 the we will have

$$2 \leq [EAC] \leq \frac{2}{3}[ABC] < 2$$

which is impossible, so E must lie in the little shaded triangle. In the same way, D must lie in the shaded region inside the triangle ABC'. But then, since $B'A = C'A = a$ we will have $DE \leq 2a$ and the distance from A to DE is at most $\frac{h_a}{3}$ (since any circle centered at A with the radius $\frac{h_a}{3}$ always cuts DE for any D, E in those regions). So

$$2 \leq [ADE] \leq \frac{ah_a}{3} = \frac{2[ABC]}{3} \Rightarrow [ABC] \geq 3 \tag{1}$$

exactly what we needed. The astute reader will notice that the above argument does not hold when one of the angles B and C is obtuse. Then, a similar argument can easily be formulated to show that the triangle ADE always has area less than $\frac{2}{3}$ of that of ABC, which as in (1) gives us $[ABC] \geq 3$ (it is not hard to show that with the constraints placed on D and E, the area of ADE is largest when D and E are as close to BC as possible in their little shaded regions, and even then it will be less than $\frac{2}{3}[ABC]$).

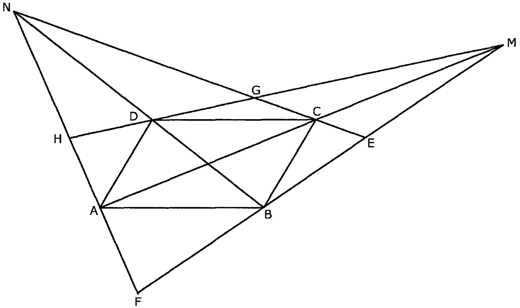
We still have to study the case when D and E are in the same region, say inside $A'BC$. Then, we will have

$$[ABC] = [XBC] \geq [DEBC] = [DEB] + [EBC] \geq 4$$

which finishes our proof.

36. M Let O be the intersection of the diagonals, E be the intersection of MB and NC, F that of MB and NA and G that of NC with MD. Let $\frac{NO}{OD} = n$ and $\frac{MO}{OC} = m$. Then, we will apply Menelaos' Theorem in triangle MBO with the line NCE:

$$\frac{ME}{EB}\frac{BN}{NO}\frac{OC}{CM} = 1 \Rightarrow \frac{ME}{EB} = \frac{n(m-1)}{n+1} \Rightarrow \frac{ME}{MB} = \frac{n(m-1)}{mn+1} \tag{1}$$



Similarly, applying it in the triangle MOB with the line NAF will give us

$$\frac{MF}{FB}\frac{BN}{NO}\frac{OA}{AM} = 1 \Rightarrow \frac{MF}{FB} = \frac{n(m+1)}{n+1} \Rightarrow \frac{MF}{MB} = \frac{n(m+1)}{mn-1} \tag{2}$$

By multiplying (1) and (2) we get $\dfrac{ME \cdot MF}{MB^2} = \dfrac{m^2n^2 - n^2}{m^2n^2 - 1}$. Doing the same thing to evaluate $\dfrac{MG \cdot MH}{MD^2}$ will give us this value to be equal to the same $\dfrac{m^2n^2 - n^2}{m^2n^2 - 1}$. So we will always have

$$\frac{ME \cdot MF}{MG \cdot MH} = \frac{MB^2}{MD^2} \tag{3}$$

Now, if $EFGH$ is cyclic, the LHS becomes 1 and then $MB = MD$. But then, MO will be a median in the isosceles triangle MBD and this would imply $MO \perp BD$, and so $ABCD$ is a rhombus. Conversely, if $ABCD$ is a rhombus, then $MB = MD$ and (3) will tell us that $ME \cdot MF = MG \cdot MH$, which implies that $EFGH$ is cyclic.

37. E Let our polygon be $A_1A_2 \ldots A_n$ and we will consider three categories of segments A_iA_j: the edges (of which there are n), the diagonals of the form A_iA_{i+2} (of which there are also n) and the other diagonals (of which we have $\dfrac{n(n-5)}{2}$).

Let T be the number of triangles of area 1, and $3T$ will be the number of segments A_iA_j involved in these triangles. But each segment from the first category (the edges) cannot be part of more than 2 triangles of area 1 (because if it were part of three, then the vertices of those at least three triangles would be at equal distance from our edge and thus on a line, contradicting convexity). Similarly, any diagonal of the form A_iA_{i+2} appears in at most 3 triangles, and any other diagonal in 4 of them. So this means that

$$3T \leq 2n + 3n + 4 \cdot \frac{n(n-5)}{2} = 2n^2 - 5n \Rightarrow T \leq \frac{n(2n-5)}{3}$$

which is exactly what we needed to prove.

38. *E* Let our square be $ABCD$ and suppose that all triangles formed by our 2003 points have areas greater than $\frac{1}{10}$. Now, take any two points X, Y inside the rectangle and suppose the line XY splits up A and B from C and D (it's the same thing if XY splits, say, A from the other three vertices). Then, break the figure up into the six triangles $ABX, AXY, AYD, CDX\ DXY, BXC$. We will prove by induction the following statement: for every $6 \leq n \leq 2003$ there will exist n of our points such that we can triangulate the square with respect to those n points (meaning that we can break up the square into disjoint triangles with vertices only among our chosen n points).

    For $n = 6$ it is exactly the construction above. Now, suppose that we have split the square up in such a way for points $A_1, ..., A_{n-1}$ and we wish to find n such points with the desired property. Take any other point X and it will lie in a triangle $A_iA_jA_k$ (since these triangles span the square). Then, just add the segments XA_i, XA_j and XA_k, thus adding two more triangles (actually, we add in 3 of them, but $A_iA_jA_k$ is no longer part of our triangulation). We thus pass from $2n-8$ to $2n-6$.

    So in the end, we break up our entire arrangement of 2003 points into 4000 triangles ($= 2 \cdot 2003 - 6$). Our assumption says that each of these will have area greater than $\frac{1}{10}$, so the whole square will have area greater than 400. But the side-length of the square is 20, so contradiction.

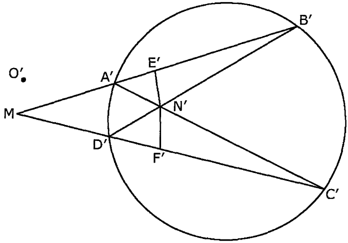
39. *M* First of all, note that if we have a triangle with one side 1, another > 1 and the third < 1, then the height that corresponds to the largest side is less than $\frac{\sqrt{3}}{2}$. This is easy to observe, because that height will equal 1 (meaning the length of the side of length 1) $\times$ the sine of the smallest angle. Since that angle is $< 60°$, the height will be $< \frac{\sqrt{3}}{2}$.

    Now, we know that in a tetrahedron we have $r = \dfrac{3V}{S_1 + S_2 + S_3 + S_4}$ where $S_1 \leq S_2 \leq S_3 \leq S_4$ are the areas of the sides (we can prove this assertion by noting that if we connect the incenter with the 4 vertices, we split our initial tetrahedron into 4 smaller ones, each of them of height r). Thus, we have $r \leq \dfrac{3V}{4S_1}$ and assume that A is the vertex opposed to the side of area S_1. Then $3V = h_a \cdot S_1$, and h_a will obviously be smaller than the height from A in triangle ABC. We proved before that that height is less than $\frac{\sqrt{3}}{2}$, so we will have

$$r \le \frac{3V}{4S_1} = \frac{h_a}{4} < \frac{\sqrt{3}}{8}$$

which ends our proof.

40. M Let us consider an inversion around M and let us denote by X' the inverse of any point X. The points A', B', C', D' will still be on a circle and the lines $A'B'$ and $C'D'$ will pass through M. The circles (ACM) and (BDM) will transform into the lines $A'C'$ and $B'D'$, which will meet in N'. What we need to prove is that the angle between the lines MN and ON, meaning the angle between MN' and the circle $(MO'N')$, is $90°$. This angle will be the angle $MO'N'$; the greatest problem we face is finding O', the image of O.



Assume that M is outside the segments $[AB]$ and $[CD]$ (if it lies inside, the proof is identical). But

$$OA = OB \Rightarrow \frac{O'A'}{MA'} = \frac{O'B'}{MB'} \Rightarrow \frac{O'A'}{O'B'} = \frac{MA'}{MB'}$$

and that means that O and M are on one of the Apollonian circles of the segment AB. But if we let $E' \in [A'B']$ such that $\frac{E'A'}{E'B'} = \frac{MA'}{MB'}$, then O' will be on the circle of diameter ME' (the circle of diameter ME' being that Apollonian circle). Similarly, O' will be on the circle of diameter MF', where F' is the analogous of E' on the segment $[D'C']$. And what we want to prove is that $\angle MO'N' = \frac{\pi}{2}$, meaning that O' is also on the circle of diameter MN'. So we need to show that the circles of diameters ME', MF', MN' have another common point beside M (and that point will be O'), which can only happen iff their centers are collinear. But the centers of these three circles may be obtained from E', F', N' by a dilation by a factor of $\frac{1}{2}$, so it is enough to prove that E', F', N' are collinear.

Let $\angle A'N'B' = \alpha$, $\angle A'N'E' = p$ and $\angle C'N'F' = q$ and we need to show that $p = q$. Since the function $\frac{\sin x}{\sin(\alpha - x)}$ is increasing, it will be enough to show that $\dfrac{\sin p}{\sin(\alpha - p)} = \dfrac{\sin q}{\sin(\alpha - q)}$. But the Law of Sines in triangle $A'N'E'$ and $E'N'B'$ gives us

$$\frac{\sin p}{\sin(\alpha - p)} = \frac{A'E'}{B'E'}\frac{B'N'}{A'N'} = \frac{A'M}{B'M}\frac{B'N'}{A'N'}$$

By writing the similar relation for q, it follows that all we need to do is to prove that

$$\frac{A'M}{B'M}\frac{B'N'}{A'N'} = \frac{C'M}{D'M}\frac{D'N}{C'N} \Leftrightarrow \frac{A'M}{C'M}\frac{D'M}{B'M} = \frac{A'N'}{C'N'}\frac{D'N'}{B'N'}$$

But this last relation is very easy to derive, just by expressing all those ratios from the Law of Sines in triangles $A'C'M, D'B'M, A'C'N', B'D'N'$ respectively, keeping in mind that A', B', C' and D' are on the same circle.

Chapter 2
NUMBER THEORY SOLUTIONS

1. D For all natural numbers $x = k^2 l$, where l is a square-free number, call l the non-square part of x. Now, note that $a_3 = 2 \cdot 97^2 - 1$; we will try to turn our recurrence into a simpler one:

$$a_{n+1} - a_n a_{n-1} = \sqrt{(a_n^2 - 1)(a_{n-1}^2 - 1)} \Rightarrow$$

$$\Rightarrow a_{n+1}^2 - 2a_n a_{n-1} a_{n+1} + a_n^2 a_{n-1}^2 = a_n^2 a_{n-1}^2 - a_n^2 - a_{n-1}^2 + 1 \Rightarrow$$

$$\Rightarrow a_{n+1}^2 + a_n^2 + a_{n-1}^2 = 2a_n a_{n-1} a_{n+1} + 1 \qquad (1)$$

Subtracting this relation for $m+1$ from itself for $m+2$ will give us

$$a_{m+3}^2 - a_m^2 = 2a_{m+1}a_{m+2}(a_{m+3} - a_m) \Rightarrow a_{m+3} + a_m = 2a_{m+1}a_{m+2} \qquad (2)$$

where we used that $a_{m+3} - a_m \neq 0$ (which is true, as from the hypothesis we note that the sequence is increasing). From the above, since $a_1, a_2, a_3 \in \mathbb{Z}$, it is simple to prove by induction that all the terms of the sequence are natural numbers. But writing (1) for $m = n + 2$ yields

$$a_{m+3}(a_{m+3} - 2a_{m+2}a_{m+1}) + a_{m+2}^2 + a_{m+1}^2 - 1 = 0 \Rightarrow$$

$$\Rightarrow a_{m+3}a_n + 1 = a_{m+2}^2 + a_{m+1}^2 \qquad (3)$$

Adding (2) and (3) gives us

$$(a_{m+3} + 1)(a_m + 1) = (a_{m+2} + a_{m+1})^2 \qquad (4)$$

So the product of $a_{m+3} + 1$ and $a_m + 1$ is always a square. Therefore, the non-square part of $a_m + 1$ is the same as the non-square part of $a_{m+3} + 1$. But the non-square parts of $a_1 + 1, a_2 + 1, a_3 + 1$ are 2, and therefore by induction the non-square part of any $a_n + 1$ will be 2. So $a_n + 1 = 2k_n^2$, and taking the square root of (4) gives us that

$$k_{m+3}k_m = k_{m+1}^2 + k_{m+2}^2 - 1 \tag{5}$$

But now subtract (2) from (3), and we will have the same remark as above for $a_m - 1$, that it has the same non-square part for all m. Therefore, $a_m - 1 = 6l_m^2$ for some l_m's (since the non-square part of $a_1 - 1, a_2 - 1, a_3 - 1$ is 6) and we will thus have that $6l_m^2 + 2 = 2k_m^2 \Rightarrow (k_m - 1)(k_m + 1) = 3l_m^2$. It is easy to prove by induction (from the relation (2)) that the a_n's are all of the form $4k + 1$. Therefore, k_n is odd, and thus the greatest common divisor of $k_m - 1$ and $k_m + 1$ will be 2. Thus, from $(k_m - 1)(k_m + 1) = 3l_m^2$ we get that one of $k_m - 1$ and $k_m + 1$ is of the form $2b^2$ and the other is of the form $6c^2$ (since the two numbers have no common divisors except for 2). But from (5) is easy to prove by induction that k_m is always of the form $6q + 1$, and thus $k_m + 1$ must be the one of the form $2b^2$.

Therefore, we will have $\sqrt{2 + \sqrt{2 + 2a_m}} = \sqrt{2 + 2k_m} = 2b$, and this is all we had to prove.

2. E Let $x = y + h$ and the relation becomes

$$3(y + h)^2 + y + h = 4y^2 + y \Rightarrow 6yh + 3h^2 + h = y^2 \Rightarrow$$

$$\Rightarrow y^2 - 6yh - (3h^2 + h) = 0$$

For this second degree equation in y to have an integer solution y, the discriminant must be a perfect square (so that square root of it is an integer). So we need to have $9h^2 + 3h^2 + h = 12h^2 + h$ a perfect square. But take any prime p that divides h and let k be the exponent of p in h's decomposition into primes. The exponent of p in the expression $12h^2$ is at least $2k$, so the exponent of p in $12h^2 + h$ will be exactly k. But that is exactly the exponent of p in a perfect square (since $12h^2 + h$ is a square), and that is why k must be even. But then, h itself is a square, and that is precisely what we had to prove.

3. M We will show that the ways in which n can be expressed in the form $P(2) = a_k 2^k + a_{k-1}2^{k-1} + ... + a_0$ with $a_i \in \{0, 1, 2, 3\}$ is equal to the number of ways in which n can be written as $2x + y$ with non-negative x and y. For that we shall construct a bijection between the set of P's with $P(2) = n$ and the set of ways to write $n = 2x + y$.

Indeed, for every writing of n as $2x + y$, let $x = b_i 2^i + ... + b_0$ be the representation of x in base 2 (so $b_x \in \{0, 1\}$ for every x) and $y = c_j 2^j + ... + c_0$ be the representation of y in base 2. Then, we will write n as $a_i 2^i + ... + a_0$, where $a_x = 2b_{x-1} + c_x$ for all x. It is easy to see that a_x is always in the set $\{0, 1, 2, 3\}$.

Conversely, if $n = a_i 2^i + ... + a_0$, then if $a_x = 3$ let $b_{x-1} = 1$ and $c_x = 1$, if $a_x = 2$ let $b_{x-1} = 1$ and $c_x = 0$, if $a_x = 1$ let $b_{x-1} = 0$ and $c_x = 1$ and if $a_x = 0$ let $b_{x-1} = 0$ and $c_x = 0$. Then, let $x = \sum b_i 2^i$ and $y = \sum c_j 2^j$ and we must have $n = 2x + y$. Since the two functions we just described between the polynomials such that $P(2) = n$ and the writings of n as $2x + y$ are inverse to each other, they will be bijections between these two sets of objects.

So the number of polynomials with coefficients in $\{0, 1, 2, 3\}$ that satisfy $P(2) = n$ equals the number of ways to write n as $2x + y$ with non-negative x, y. This last number can be calculated easily, for there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ non-negative x's such that $2x \leq n$, and for each of these x's there is a unique y such that $n = 2x + y$. So the number we are looking for is $\left\lfloor \frac{n}{2} \right\rfloor + 1$.

4. *E* LEMMA Every natural number $p \geq 6$ can be written as $a^2 + b^2 - c^2$, with $a, b, c > 0$.

First of all, note that any natural number greater than 1 that is not of the form $4k + 2$ can be written as $b^2 - c^2$, as we can always let that number be of the form xy with $x > y$ and $x \equiv_2 y$ and write it is $\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$. Now, for every natural number p, if it is not of the form $4k + 3$, write $p - 1$ as $b^2 - c^2$ and let $p = 1^2 + b^2 - c^2$. If p is of the form $4k + 3$, let $p - 4$ be $b^2 - c^2$ and $p = 2^2 + b^2 - c^2$.

Now, we will prove our problem by induction over n (for $n = 2$ we can just pick 8). Suppose it true for $n - 1$ and let us prove it for n by constructing a good number. Let x be writable as a sum of 2,3,...,$n - 1$ squares, and then the lemma tells us that x can be written as $a^2 + b^2 - c^2$. Then the number $x + c^2$ can be written both as a sum of $3, 4, ..., n$ squares (add the square c^2 to the representations of x as sums of squares), but also as a sum of 2 squares, as it is $a^2 + b^2$. Our problem is thus solved.

5. *M* First of all, we will prove that:

LEMMA 1 The sum of the quadratic residues modulo any $p \geq 5$ is congruent to 0 mod p.

Let that residues be $r_1, ..., r_k$ and $S = r_1 + ... + r_k$. But the numbers $4r_1, 4r_2, ..., 4r_k$ are also the residues (yet in another order), so their sum must be congruent with S. Therefore $S \equiv_p 4S \Rightarrow S \equiv_p 0$.

LEMMA 2 We have that

$$p^2 | (p-1)! \left(1 + \frac{1}{2} + ... + \frac{1}{p-1}\right)$$

The RHS of the above is

$$(p-1)!\left(1+\frac{1}{2}+...+\frac{1}{p-1}\right)=\sum_{i=1}^{\frac{p-1}{2}}(p-1)!\left(\frac{1}{i}+\frac{1}{p-i}\right)=$$

$$=p\sum_{i=1}^{\frac{p-1}{2}}(p-1)!\frac{1}{i(p-i)}$$

To prove our claim, it is therefore enough to show that

$$p|\sum_{i=1}^{\frac{p-1}{2}}(p-1)!\frac{1}{i(p-i)}\Leftrightarrow\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i(p-i)}\equiv_p 0\Leftrightarrow\sum_{i=1}^{\frac{p-1}{2}}\frac{1}{i^2}\equiv_p 0$$

where $\frac{1}{x}$ refers to the reciprocal of x in the field $\mathbb{Z}_p$. But the numbers $1^2, 2^2, ..., \left(\frac{p-1}{2}\right)^2$ are all the quadratic residues (since any two of them are non-congruent and there are $\frac{p-1}{2}$ of them). So the numbers $\frac{1}{1^2}, \frac{1}{2^2}, ..., \frac{1}{\left(\frac{p-1}{2}\right)^2}$ are also residues, and they will be all the residues. But their sum is 0 mod p, as was shown in Lemma 1, so Lemma 2 is true.

From now on, the problem becomes simpler. Lemma 2 tells us that

$$1+\frac{1}{2}+...+\frac{1}{p-1}=p^2\cdot\frac{x}{y}$$

where x, y are integers with y non-divisible by p. So

$$\frac{a}{b}=p^2\cdot\frac{x}{y}+\frac{1}{p}\Rightarrow\frac{ap-b}{bp}=\frac{p^2x}{y}\Rightarrow ap-b=\frac{p^3bx}{y} \tag{1}$$

But $p|b$, because it is the denominator of the fraction

$$\frac{1\cdot 2\cdot ...\cdot(p-1)+1\cdot 3\cdot 4\cdot ...\cdot p+...+2\cdot 3\cdot ...\cdot p}{1\cdot 2\cdot ...\cdot p}$$

whose numerator is non-divisible by p and whose denominator is a multiple of p. Therefore, (1) tells us that $p^4|ap-b$.

6. D Let $m=(b,d)$ and for any $0\le k\le 5$ and $0\le l\le m-1$, let us consider the rectangle P_{kl} given by $\frac{k}{6}<x<\frac{k+1}{6}$, $\frac{l}{m}<y<\frac{l+1}{m}$. Suppose the rectangle P_{00} contains α points of S. The points (x, y) in S which are located in any rectangle P_{kl} are in a one-to-one correspondence with the points in S that sit in P_{00} through the map $f(x,y)=\left(x-\frac{k}{6}, y-\frac{l}{m}\right)$, so there will be α points of S in every P_{kl}. That means that the number of points of S situated in the disjoint squares P_{kl} is $6m\alpha$, because we have $6m$ such squares.

But the total number of points in S will be this number plus the number of points of S which are not in any rectangle. But the points which are not contained in any rectangle either have the x coordinate $\frac{k}{6}$ or the y coordinate $\frac{l}{m}$. And it is easy to see that for a point such as this to be in S, it needs to have both coordinates of the given form (so $x = \frac{k}{6}$ and $y = \frac{l}{m}$). But the number of these points contained in the unit square is $5(m-1)$ and we will have that $2004 = 6m\alpha + 5(m-1) \Rightarrow 6m|2004 - 5(m-1)$.

The fact that m divides $2004 - 5m + 5$ tells us that m divides 2009. But $6|2004 - 5m + 5 \Rightarrow 6|m - 1$, so m must be of the form $6k + 1$; the only divisors of 2009 of that form are 1,7 and 49. Then, let $a = 6p$, $c = 6r$, $b = mq$ and $d = ms$ (where $(p, r) = (q, s) = 1$) and notice that the function $f(x, y) = (6x, my)$ maps all points from $S \bigcap P_{00}$ to the set S' of points in the unit square which satisfy $px + qy \in \mathbb{N}$ and $rx + sy \in \mathbb{N}$. Moreover, the correspondence is a bijection and so there are α points in S'.

But for any α, just pick $p = \alpha + 2$ and $q = r = s = 1$ and it is easy to note that there will be exactly α points in S'. Therefore, going back on our argument, the numbers $a = 6\alpha$, $c = 6$, $b = d = m$ will contain exactly 2004 points in their interior for every one of the values of m we found to work. Therefore, the possible values of M are $1, 7, 49$.

7. E We will prove that $(0,0)$ is the only solution. Indeed, suppose there were any other solutions (m, n) and we would need to have m and n of the same sign. Suppose them positive (as (m, n) solution implies $(-m, -n)$ solution) and we will have $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$.

But $(5 + 3\sqrt{2})^m = a + b\sqrt{2}$ and $(3 + 5\sqrt{2})^n = c + d\sqrt{2}$ for some natural numbers a, b, c, d. There is equality between the two powers if and only if $a = c$, and by expanding the powers with Newton's formula we will have

$$a = 5^m + 5^{m-2} \cdot 18\binom{m}{2} + \dots$$

$$c = 3^m + 3^{m-2} \cdot 50\binom{m}{2} + \dots$$

These must be equal, and it is obvious that 5 does not divide c and 3 does no divide a. But $a = c$ and therefore 5 cannot divide a, which can only happen if in the expansion of a will end with the term $5^0 \cdot 18^{\frac{m}{2}}\binom{m}{m}$. This implies that m is even and similarly n is even. Then, by extracting square root out of the relation $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$ we get that (m, n) solution implies $\left(\frac{m}{2}, \frac{n}{2}\right)$ solution. This means that we can always decrease our solutions by a factor of

2 and get valid solutions, which is impossible given that $m, n \in \mathbb{N}$ (this is a contradiction known as "infinite descent"). So the only solution will be $(0,0)$.

8. *M* LEMMA The exponent of any prime p in the prime decomposition of any factorial $x!$ is

$$\sum_{i=1}^{\infty} \left\lfloor \frac{x}{p^i} \right\rfloor \tag{1}$$

Obviously, the above sum is finite, because from a point onward the integer parts will be 0. Now, count a 1 for every multiple of p among the numbers $1, 2, ..., x$; count another 1 for every multiple of p^2 among these numbers, count another 1 for every multiple of p^3 and so on. What we get is exactly the sum in (1). But the number of times every number in $1, 2, ..., x$ has been counted is exactly p's exponent in it. So the sum in (1) will be the sum of the exponents of p in the representations of $1, 2, ..., x$, which is the exponent of p in $x!$

Back to the problem. The least common multiple (LCM) of several numbers is the product after all primes p of the greatest powers of that prime which divide any of our numbers. So for the two least common multiples to be equal we would need to have, for all primes p, that the highest power of p (call it a) which divides the numbers $1, 2, ..., n$ to equal the highest power of p which divides $\binom{n}{1}, ..., \binom{n}{n}$. But for all k we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

and let p^a be the highest power of p which divides any of the numbers $1, 2, ..., n$. According to the lemma, the exponent of p in (2) will be

$$\sum_{i=1}^{a} \left(\left\lfloor \frac{n}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{n-k}{p^i} \right\rfloor \right) \tag{3}$$

We took the sum only up to a because for greater indices the integer parts above will be 0. But $\lfloor b+c \rfloor - \lfloor b \rfloor - \lfloor c \rfloor$ will always be 0 or 1, so the sum in 2 will always be at most a. Therefore, in order for the exponents of p in the two LCM's to be equal, there must be at least one k such that the sum in (3) is a, which is saying that all the terms in (3) are 1 for that particular k.

We need to show that this happens for all p if and only if $n+1$ is prime. Suppose that it happens, and let p be a prime divisor of $n+1$, so $n = pq - 1$ for some q. If p is not the only divisor of $n+1$, then $p \leq n$ so the exponent of p in n will be nonzero. Then by assumption, for some k we need to have $\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{k}{p} \right\rfloor - \left\lfloor \frac{n-k}{p} \right\rfloor = 1$. If $k = pq' + s$ with $s < p$ we would need that

$$q-1-q'-\left\lfloor\frac{pq-1-pq'-s}{p}\right\rfloor=1\Rightarrow\left\lfloor\frac{-s-1}{p}\right\rfloor=-2$$

which can only happen if $\frac{s+1}{p}>1$. Of course, we took $s<p$, so this can never happen, and we have thus obtained a contradiction. So $n+1$ must be prime.

Conversely, suppose that $n+1$ is a prime. We need to show that for every prime p, there will exist a k such that all the terms in the sum (3) be 1. Just pick $k=p^a-1$ (where a is such that $p^a\le n<p^{a+1}$) and all we need to prove is that

$$\left\lfloor\frac{n}{p^i}\right\rfloor-\left\lfloor\frac{n+1-p^a}{p^i}\right\rfloor-\left\lfloor\frac{p^a-1}{p^i}\right\rfloor=1$$

for all i. The above relation is easy to prove since $i\le a$, and we will have

$$\left\lfloor\frac{n}{p^i}\right\rfloor-\left\lfloor\frac{n+1-p^a}{p^i}\right\rfloor-\left\lfloor\frac{p^a-1}{p^i}\right\rfloor=\left\lfloor\frac{n}{p^i}\right\rfloor-\left\lfloor\frac{n+1}{p^i}\right\rfloor+p^{a-i}-(p^{a-i}-1)=1$$

Our problem is thus solved.

9. M SOLUTION BY ADRIAN ZAHARIUC

First of all, it is easy to eliminate the case z even, because then x^2 and y^z are two perfect squares with difference 3. It is easy to note that the only solution in this case is $x=z=1$, $y=2$.

So we are left with the equation $x^2+3=y^z$ when z is of the form $4k+3$. Then, note that x cannot be odd because then $x^2+3=y^z$ will be of the form $8k+4$, so divisible by 4 and not by 8 (which means that it cannot be a power greater than 2). So $x=2x'$ is even and y will be odd. Moreover, since the LHS will be of the form $4k-1$, the RHS must be likewise and therefore y will be of the same form. We will have

$$4(x'^2+1)=y^z+1=(y+1)(1-y+y^2-y^3+\ldots+y^{z-1})$$

and taken modulo 4 the term $1-y+y^2-\ldots+y^{z-1}$ will be congruent to -1. But then it will have a prime divisor congruent to -1 modulo 4, and that prime divisor will divide x'^2+1. This is of course a contradiction, as -1 is a quadratic non-residue modulo any prime of the form $4k-1$.

10. M Is is easy to remark that the invisible points are those whose coordinates have common prime factors. The Chinese remainder lemma tells us that for any distinct primes p_{ij} $(1\le i,j\le n)$ there will be natural solutions to the systems $a\equiv_{p_{i1}p_{i2}\ldots p_{in}}-i$ (for $1\le i\le n$) and $b\equiv_{p_{1i}p_{2i}\ldots p_{ni}}-i$. This means that for any (i,j), the number p_{ij} will divide both $a+i$ and $b+j$. So $(a+i,b+j)$

will be invisible points for all $1 \le i, j \le n$, and thus the square with vertices $(a+1, b+1), (a+1, b+n), (a+n, b+n), (a+n, b+1)$ will be formed out of only invisible points.

11. *M* It is easy to prove by induction that all the terms of the sequence are natural numbers. But first of all, let us prove that:

LEMMA For any n and k we have $a_n | a_{nk}$.

First, let r and s be the roots of the characteristic equation $x^2 - ax + 1 = 0$ and we will have $r + s = a$ and $rs = 1$. The general solution of our second degree linear recurrence with the initial conditions $a_1 = 1$ and $a_2 = a$ will be $a_n = \dfrac{r^n - s^n}{r - s}$. But for any n and k we have

$$\frac{a_{nk}}{a_n} = \frac{r^{nk} - s^{nk}}{r^n - s^n} = r^{(k-1)n} + r^{(k-2)n}s^n + \ldots + s^{(k-1)n} =$$

$$= r^{(k-1)n} + r^{(k-3)n} + r^{(k-5)n} \ldots + s^{(k-3)n} + s^{(k-1)n} \tag{1}$$

because we know that $rs = 1$. But the quantity $r^x + s^x$ is an integer for all integer x, since it is given by the recurrence $r^{x+2} + s^{x+2} = (r^x + s^x)(r^2 + s^2) - (r^{x-2} + s^{x-2})$. That 4-th degree recurrence yields only integers, as $r^2 + s^2 = a^2 - 2$ and the first 4 terms of the sequence $r^x + s^x$ are integers (they are $2, a, a^2 - 2, a^3 - 3a$). So the expression in (1) is just a sum of such $r^x + s^x$, and therefore it is an integer. The lemma is thus proved.

But now we can show that if $(p, q) = 1$ then $(a_p, a_q) = 1$; this would be sufficient to proving the problem, as we can choose as our subsequence the subsequence given by prime indices. If $(p, q) = 1$ then we can find natural numbers k and l such that $pk = ql + 1$ (or $pk + 1 = ql$, but it is exactly the same thing). So a_p divides a_{pk} (because of the lemma) and a_q divides $a_{ql} = a_{pk-1}$. But two consecutive terms of our sequence are always co-prime, because if $d > 1$ were to divide a_x and a_{x-1}, then it would also divide (by the recurrence) a_{x-2}, $a_{x-3}, \ldots$, and so on, until it had to divide $a_1 = 1$ (which is impossible). Therefore a_{pk} and a_{ql} must be co-prime, so their divisors a_p and a_q must always be co-prime.

12. *D* First of all, we shall prove by induction on n a very useful lemma:

LEMMA Given non-zero integers $a_1, \ldots, a_n$ with the greatest common divisor 1, it is possible to write every integer uniquely as $k_1a_1 + k_2a_2 + \ldots + k_na_n$, where for $i \ge 2$ k_i ranges over a set of the form $\{0, 1, \ldots, \alpha_i - 1\}$ for some natural number α_i and k_1 ranges over all the integers.

This is rather well-known for $n = 2$, because given co-prime numbers a and b, any natural number p may be written uniquely as $ka + lb$ where $l \in \{0, 1, \ldots, a - 1\}$. Indeed, because a and b are co-prime, there will always be an l'

such that $1 \equiv_a l'b$ and so $p \equiv_a pl'b$. If we take l to be the residue of pl' modulo a we get $bl \equiv_a p$ and so p will be of the form $ka + lb$ for some integer k.

Now, assume the lemma true for $n-1$ and let us prove it for n. Let d be the common divisor of $a_1, ..., a_{n-1}$ and because of the induction hypothesis, any integral number will be uniquely writable as $k_1 \frac{a_1}{d} + ... + k_{n-1} \frac{a_{n-1}}{d}$, where $k_2, ..., k_{n-1}$ range over sets of the form $\{0, 1, ..., \beta - 1\}$ (not necessarily the same β for all k_i's). Then, any multiple of d may be written uniquely as $k_1 a_1 + ... + k_{n-1} a_{n-1}$. But d and a_n are co-prime, and as the base-case $n = 2$ tells us, every integer may be written as $kd + k_n a_n$, where k_n ranges over $\{0, 1, ..., d-1\}$. Yet as we said before, any number of the form kd can be written uniquely as $k_1 a_1 + ... + k_{n-1} a_{n-1}$. Therefore, any integer can be written as $k_1 a_1 + ... + k_n a_n$, where $k_2, ..., k_n$ range over sets of the form $\{0, ..., \alpha - 1\}$ for some values of α.

Therefore, the lemma is true. But what it tells us is that there to every integer $k_1 a_1 + ... + k_n d_n$ may be associated the unique point in the lattice $(k_1, k_2, ..., k_n)$, where k_1 ranges over the whole of $\mathbb{Z}$ and k_i ranges over $\{0, 1, ..., \alpha_i - 1\}$ for $i > 1$. So to have a function ϕ as required by the problem, it is enough to construct it such that the points associated to $\phi(k)$ and $\phi(k-1)$ are always adjacent in the lattice in $\mathbb{R}^n$.

But such a function may be constructed rather easily. The set of our points $(k_1, ..., k_n)$ with $k_i \in \{0, 1, ..., \alpha_i - 1\}$ for $i > 1$ is a box in n dimensions which is infinite in one direction (the direction of the coordinate k_1). It is the analogue in more dimensions of an infinite strip in $\mathbb{R}^2$ or of an infinitely long cube in $\mathbb{R}^3$. And it is not hard to place the points of this infinite box in a sequence which is one-to-one with $\mathbb{Z}$ such that two consecutive points are adjacent.

Indeed, it can be easily proven by induction over q that the lattice points of a box of the form $(l_1, ..., l_q)$ with l_i ranging over some $\{0, 1, ..., \beta_i - 1\}$ can be put in a sequence such that two consecutive points are adjacent in the lattice. Then, to construct our infinite sequence in our infinite box $(k_1, ..., k_n)$ just stick together the sequences obtained in the finite boxes of the form $(x, k_2, ..., k_n)$ for fixed x. That yields a sequence of the points in our infinite box such that two consecutive points are adjacent.

13. *M* First of all, it is easy to prove by induction that every x_n is natural and square-free. But then, let $p_1 < p_2 < ...$ be the primes, and associate to every n a number $f(n)$ whose binary representation is this: $f(n)$ has a 1 as its k-th digit from the right if and only if $p_k | x_n$. Thus, $f(n)$ completely determines x_n, as it tells us which primes divide it.

    But the recurrence is nothing more than $f(n+1) = f(n) + 1$, and as $f(0) = 0$ ($x_0 = 1$ has no prime divisors) then $f(2005) = 2005 = 11111010101$. x_{2005} must be therefore be

$$p_1p_3p_5p_7p_8p_9p_{10}p_{11} = 2 \cdot 5 \cdot 11 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 = 734653810$$

14. *M* Let us assume that the numbers $a_1, ..., a_k$ are non-zero, and then for any n we will have $a_1^n + ... + a_k^n$ a perfect square. But take any prime p greater than all the a_j's and let $d_1, ..., d_k$ be the orders of $a_1, ..., a_k$ modulo p. Take n a multiple of $d_1, ..., d_k$ and then the perfect square $a_1^n + ... + a_k^n$ will be congruent to $1 + 1 + ... + 1 = k$ modulo p. But this means that k is a quadratic residue modulo any prime p greater than all the a_j's.

   We will show that this implies that k is a perfect square; indeed, assume it is not, and then it will be writable as $l^2 2^e q_1 q_2 ... q_m$ where $q_1, ..., q_m$ are distinct primes different from 2, and e is 0 or 1. But considering the Legendre symbol gives us:

$$\left(\frac{2}{p}\right)^e \left(\frac{q_1}{p}\right) ... \left(\frac{q_m}{p}\right) = 1$$

   for all primes p large enough. Applying quadratic reciprocity yields

$$(-1)^{\frac{p^2-1}{8}e}(-1)^{\frac{p-1}{2}\left(\frac{q_1-1}{2}+...+\frac{q_m-1}{2}\right)} \left(\frac{p}{q_1}\right) ... \left(\frac{p}{q_m}\right) = 1 \qquad (1)$$

   But Dirichlet's Theorem tells us that in any arithmetic progression with the ratio co-prime with the first term we can find an infinity of prime numbers. So just consider the progression $8q_1q_2...q_m \cdot n + 2rq_2...q_m + 1$ where r is chosen such that $2rq_2...q_m + 1$ is a non-residue modulo q_1 (we can always do that, as $2q_2...q_m$ and q_1 are co-prime). But in this arithmetic progression we will find an infinity of primes p, and for those primes all the Legendre symbols in (1) will be 1, except for that of q_1, which will be -1. We thus find a contradiction, so k must be a perfect square. Therefore, the number of zeroes among $a_1, ..., a_{2005}$ will be $2005 - k$, and since k must be a perfect square, $2005 - k$ cannot be less than $2005 - 44^2 = 69$. Of course, this minimum number of zeroes can be attained, for example for $a_1 = a_2 = ... = a_{69} = 0$ and $a_{70} = a_{71} = ... = a_{2005} = 1$.

   We can also sidestep using Dirichlet's Theorem by obtaining a contradiction in this way: just pick any b of the form $8q_1q_2...q_m \cdot n + 2rq_2...q_m + 1$ and by taking its prime divisors to be $p_1, ..., p_m$ we will see that it is impossible for $k = l^2 2^e q_1...q_m$ to be a quadratic residue modulo all the p_i's. Of course, in choosing b we must be careful not to have any of the divisors of the a_i's among its prime divisors, but this can easily be done by using the Chinese Remainder Lemma.

15. *M* LEMMA For p prime and for all $i < p - 1$ we have that $0^i + 1^i + 2^i + ... + (p-1)^i \equiv_p 0$.

Proving this is simple, as we only need to take a primitive root of p (call it a), and the numbers $a, 2a, ..., (p-1)a$ will just be the numbers $1, 2, ..., p-1$ in some order (taken modulo p). So we will have

$$1^i + 2^i + ... + (p-1)^i \equiv_p a^i + (2a)^i + ... + ((p-1)a)^i \equiv$$
$$\equiv_p a^i(1^i + 2^i + ... + (p-1)^i) \Rightarrow (1^i + ... + (p-1)^i)(a^i - 1) \equiv_p 0 \Rightarrow$$
$$\Rightarrow 1^i + 2^i + ... + (p-1)^i \equiv_p 0$$

since a^i cannot be congruent to 1 mod p, as a is a primitive root. Let us now return to the problem.

Let us assume that $d < p-1$, and then we will have that $f(0)+f(1)+...+f(p-1) \equiv_p 0$. This happens because f can be written as a linear combination of X^i's with $i < p-1$, and we always have $0^i + 1^i + 2^i + ... + (p-1)^i \equiv_p 0$. But since all the $f(i)$'s are 0 or 1 modulo p, their sum will be between 0 and p modulo p. But it can be neither 0, nor p, since we have among them at least one 0 (namely $f(0)$) and at least a 1 (namely $f(1)$). Thus, we have reached a contradiction, as the sum $f(0)+f(1)+...+f(p-1)$ must be divisible by p. So $d \geq p-1$.

16. D Let us first consider the case $(n, 10) = 1$ and we will prove that for any a co-prime with 10 that has such a multiple and for any prime $p \neq 2, 5$, ap will also have a multiple with the desired property.

Let $\overline{a_k...a_1}$ be the multiple of a with the sum of the digits a. Since $p \neq 2, 5$ there will be an $d > k$ such that $10^d \equiv_p 1$. Then, the number

$$\overline{00...0a_k...a_1 00...0a_k...a_1...00...0a_k...a_1}$$

where the sequence $\overline{a_k...a_1}$ is repeated p times, and there are $d-k$ zeroes between any two such sequences, has the sum of the digits ap. But this number is equal to

$$\overline{a_k...a_1}(\overline{1...0100...0100...01}) = a(1 + 10^d + 10^{2d} + ... + 10^{(p-1)d}) \qquad (1)$$

where the number $\overline{1...0100...0100...01}$ has p 1's and $d-1$ zeroes between any two 1's. But the RHS of (1) is divisible by ap, since $10^d \equiv_p 1 \Rightarrow 1 + 10^d + 10^{2d} + ... + 10^{(p-1)d} \equiv_p 0$. So we have found a multiple of ap with the sum of the digits ap.

But since every numbers which is co-prime with 10 can be written as a product of primes, the above claim tells us that it is writable in such a way (proof by induction, because if one of its divisors is writable, it too will be). But now take an $n = 2^a 5^b m$ with $(m, 10) = 1$ and consider $c > a, b$. Then, let

a be a multiple of m with digit sum m, and consider the number obtained by placing 2^a5^b copies of the number $10^c m$ one after the other. The result will have the sum of its digits 2^a5^bm, and it will be divisible by m and by 10^c, which in turn is divisible by 2^a5^b. The problem is thus proven.

17. *E* We will try to cut out a rectangle with sidelengths $k_0, l_0 \in \mathbb{N}$ such that $k_0l_0 > m$ and $kl \neq m$ for all $k \leq k_0$ and $l \leq l_0$. First suppose that m is not a perfect square, nor writable as $a(a+1)$ and then let d be its greater divisor less than $\sqrt{m}$. Take $k_0 = d+1$ and $l_0 = \frac{m}{d} - 1$ and we will have $k_0l_0 = m + \frac{m}{d} - d - 1 > m$, because $\frac{m}{d} \geq d+2$ ($\frac{m}{d} - d = 0$ would imply that m is a perfect square and it being 1 would imply that m is of the form $d(d+1)$).

    But suppose there are k, l with $k \leq k_0$ and $l \leq l_0$ such that $kl = m$. If $k \leq k_0 - 1$ then $kl \leq d\left(\frac{m}{d} - 1\right)$ which cannot be equal to m. If $k = k_0$, then we need to have $(d+1)l = m$, so $d+1$ is also a divisor of m. But d is the greatest divisor of m less than $\sqrt{m}$, so then we would need to have $d+1 > \sqrt{m} > d$ and since d and $d+1$ both divide m, it would follow that $m = d(d+1)$ (which we assumed not true).

    Now, if $m = d^2$, take $k_0 = d+2$ and $l_0 = d-1$ ($k_0l_0 > m$ since $m > 12$) and suppose that $kl = m$. If $l \leq l_0 - 1$ then $m = d^2 = kl \leq (d+2)(d-2) = d^2 - 4$, which is impossible. So $l = l_0$ and then $d - 1 | d^2$, which is impossible.

    Finally, if $m = d(d+1)$ then pick $k_0 = d+3$ and $l_0 = (d-1)$, and we will have $(d+3)(d-1) = d^2 + 2d - 3 > d^2$. Suppose that we had $kl = m = d(d+1)$. Since $d-1$ is co-prime with d, $d-1$ cannot possibly divide $d(d+1)$, so $l \leq d-2$. But then $kl \leq (d+3)(d-2) = d^2 + d - 6 < m$, and thus we have proved that with these k_0 and l_0 we will not have any k, l whose product is m.

18. *D* LEMMA For any t there exists an n_0 such that for any $n \geq n_0$, if the fraction $1 + \ldots + \frac{1}{n}$ has the numerator a power of p, that exponent will be at least t. The proof of this fact is simple, as Chebysev's Theorem guarantees that there will be a prime q between $\frac{n}{2}$ and n. That prime appears in the denominator of $1 + \ldots + \frac{1}{n}$, but it won't appear in the numerator (it will appear in all terms in the numerator but one, so it won't divide the numerator). So q will divide the reduced denominator of the fraction $1 + \ldots + \frac{1}{n}$, and thus that reduced denominator will be greater than $\frac{n}{2}$. But since the entire expression is > 1, the numerator (which is a power of p) will be greater than $\frac{n}{2}$. As n grows to infinity, the exponent of p in the numerator will do the same.

    Back to the problem. Assume that there are only finitely many such numbers, and therefore for all $n \geq$ some N we will have that $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ has only one prime factor in the numerator. Then, for any prime $p > N$ we will prove by induction over k that the unique prime factor in the numerator of $E_k = 1 + \frac{1}{2} + ... + \frac{1}{p^k - 1}$ is p.

First of all, remark that the sum of rationals whose numerators in reduced form are divisible by p also has the numerator divisible by p. For $k = 1$, the expression E_k can be written as $\left(1 + \frac{1}{p-1}\right) + \left(2 + \frac{1}{p-2}\right) + ...$, which is a sum of numbers whose numerators are divisible by p. The numerator of the sum will also be divisible by p, and thus we have proved the base case of our induction. Now, assume the problem true for $k-1$ and let us prove it for k.

First, if $\frac{p^\alpha}{D}$ is the reduced form of $1 + \frac{1}{2} + ... + \frac{1}{p^{k-1} - 1}$, we can infer that $\alpha > 1$ by an argument similar to that in the Lemma (invoking Chebyshev's Theorem). But now, note that for all m the sums $S_m = \frac{1}{mp+1} + ... + \frac{1}{mp+p-1}$ all have the numerator divisible by p (for the same reason for which the numerator of $E(1)$ is divisible by p). Moreover, the sum $\frac{1}{p} + \frac{1}{2p} + ... + \frac{1}{p(p^{k-1}-1)}$ also has the numerator divisible by p (because of the induction hypothesis); summing up this expression with $S_0, S_1, ..., S_{p^{k-1}-1}$ will yield the desired result.

So now we have that $S(k) = 1 + \frac{1}{2} + ... + \frac{1}{p^k - 1}$ has the numerator a power of p. But as it was proved above, $\frac{1}{p^k - (p-1)} + \frac{1}{p^k - (p-2)} + ... + \frac{1}{p^k - 1}$ also has the numerator a power of p. So the difference of these two numbers, namely $T(k) = 1 + ... + \frac{1}{p^k - p}$ always has the numerator a power of p. But the Lemma tells us that for any n there will be a k large enough such that the exponents of p in the numerators of both $S(k)$ and $T(k)$ are greater than n. But then p^n will divide the exponent of the difference $S(k) - T(k) = \frac{1}{p^k - (p-1)} + ... + \frac{1}{p^k - 1}$.

This yields a contradiction, because bringing everything to a common denominator makes the numerator of $S(k) - T(k)$ a divisor of

$$p^k \cdot l + 1 \cdot ... \cdot (p-1) + 1 \cdot 3 \cdot ... \cdot (p-1) + ... + 2 \cdot 3 \cdot ... \cdot (p-1)$$

for some l. It is easy to see why p^n cannot divide this for any n and with k large enough.

19. M First of all, let us show that there are an infinity of n such that $S(3^n) \geq S(3^{n+1})$. Assume that it is not so, and that for $n \geq N$ we have $S(3^n) < S(3^{n+1})$; but $S(3^n)$ is always a multiple of 9 (well, for $n \geq 2$) and therefore we will have $S(3^{n+1}) \geq S(3^n) + 9$ for $n \geq N$ and by iterating this we get $S(3^n) \geq S(3^N) + 9(n - N)$. But the number 3^n has at most $1 + \log_{10} 3^n$ digits, so $S(3^n) \leq 9(1 + n \log_{10} 3)$. Of course, this would imply that for all n

$$9(1 + n \log_{10} 3) \geq 9(n - N) \Rightarrow n(\log_{10} 3 - 1) \geq -1 - N$$

Of course, this is impossible for all n large enough, so our assumption is false. Now, for the second part, assume that it is not true. Therefore, there will exist an N' such that for all $n \geq 6N'$ we have $S(2^n) < S(2^{n+1})$. For any number x we will have that $S(2x) = 2S(x) - 9m(x)$ where $m(x)$ represents the number of digits $5, 6, 7, 8, 9$ in the decimal representation of x. This is easy to prove by applying digit-by-digit addition of x with x, and noting that every digit doubles, but all digits ≥ 5 make us decrease 10 out of their double, yet add 1 back when we pass onto the next digit. So $S(2^{n+1}) \equiv_9 2S(2^n) \Rightarrow S(2^{n+6}) \equiv_9 2^6 S(2^n) \equiv_9 S(2^n)$. But then for all m, $S(2^{6m}) \equiv_9 S(0) = 1$ will be of the form $9k + 1$, $S(2^{6m+3}) \equiv_9 8$ will be of the form $9k + 8$ and $S(2^{6m+4})$ will be of the form $9k + 7$. Therefore, the difference between $S(2^{6(k+1)})$ and $S(2^{6k})$ (which will be a multiple of 9) cannot be only 9, as there would be no room between these two numbers to put in both $S(2^{6k+3}) \equiv_9 8$ and $S(2^{6k+4}) \equiv_9 7$.

So $S(2^{6(k+1)}) \geq S(2^{6k}) + 18$ and by iterating this we will have $S(2^{6k}) \geq 18(k - 6N')$ for all $k \geq N'$. On the other hand, $S(2^{6k})$ has at most $1 + 6k \log_{10} 2$ digits, so we will have

$$9(1 + 6k \log_{10} 2) \geq S(2^{6k}) \geq 18(k - 6N') \Rightarrow 2k(\log_{10} 8 - 1) \geq -4N'$$

which is impossible for large enough k. Our problem is thus solved.

20. *E* Obviously, we must have $p > n$. Now, $p | n^3 - 1 = (n-1)(n^2 + n + 1)$ implies that $p | n^2 + n + 1$, as p and $n - 1$ are co-prime (since $p > n - 1$). Let $p = kn + 1$ and then we will have $kn + 1 | n^2 + n + 1 | kn^2 + kn + k$, but also the obvious $kn + 1 | kn^2 + n$. By subtracting these two we will have $kn + 1 | kn + k - n$, but it will also be true that $kn + 1 | kn + 1$. Again, subtracting these two will yield $kn + 1 | k - n - 1$; it is obvious that the absolute value of the number $k - n - 1$ is less than $k + n$, which is at most $kn + 1$. Therefore, the only possibility that $kn + 1 | k - n - 1$ is that $k = n + 1$. In this case, $p = n^2 + n + 1$ and therefore $4p - 3 = (2n + 1)^2$, exactly what we needed to prove.

21. *M* We will show that 1 is the only such n. Indeed, suppose that there were other such n, and let p be the smallest prime which divides n (take $n = pk$). Therefore, $p - 1$ and k cannot have any common divisors, as $p - 1$ is smaller than any primes that divide k. Moreover, p cannot be 3 as $p^2 | 3^n + 1$, and it can't be 2 because n cannot be even. If n were even, then 4 would divide $9^{\frac{n}{2}} + 1 \equiv_4 2$.

    So we have $p | 3^{pk} + 1 \Rightarrow 3^{pk} \equiv_p -1$. But $a^p \equiv_p a$ (Fermat's theorem) and therefore $3^k \equiv_p -1 \Rightarrow 3^{2k} \equiv_p 1$. Let d be the order of 3 modulo p and thus $d | 2k$. Since $3^{p-1} \equiv_p 1$ we must have $d | p - 1$, but since $p - 1$ and k are co-prime, we would need to have $d|2$. If $d = 1$ we would have $p|3^1 - 1 = 2$, which is impossible since $p \neq 2$, and if $d = 2$ we would have $p|3^2 - 1 = 8$, which is impossible for the same reason.

Therefore, $n = 1$ is the only solution.

22. D Let us consider such an n and for any prime $p|n$ let $n = pk$; we will have

$$1^{pk} + ... + (pk-1)^{pk} \equiv_p 1$$

But $(lp+i)^{pk} \equiv_p i^{pk} \equiv_p i^k$ (from Fermat's Theorem) so the above becomes

$$k(1^k + ... + (p-1)^k) \equiv_p 1 \tag{1}$$

But the expression $1^k + 2^k + ... + (p-1)^k$ is 0 modulo p if $p-1$ does not divide k and -1 if $p-1$ divides k. This can be seen easily knowing that $\mathbb{Z}_p$ has a generator, i.e. there exists a primitive root g of $\mathbb{Z}_p$ and then the numbers $1, 2, ..., p-1$ are $1, g, g^2, ..., g^{p-2}$ in some order. Therefore,

$$1^k + 2^k + ... + (p-1)^k = 1 + g^k + g^{2k} + ... + g^{(p-2)k} = \frac{1 - g^{(p-1)k}}{1 - g^k}$$

which is 0 if the denominator is non-zero (meaning if $p-1$ does not divide k, as $p-1$ is the order of g) and -1 otherwise. Therefore, since in (1) we cannot have that sum equal to 0, we must have $p-1|k$ and then $k \equiv_p -1 \Rightarrow p|k+1$. This means that n is square free, since p cannot divide k.

Let $n = p_1p_2...p_m$ with $p_1 < p_2 < ... < p_m$ and we will use $p-1|k$ for every one of its prime divisors. We will have $p_1 - 1|p_2..p_m$, but since $p_2...p_m$ has no prime factors less than p_1 we must have $p_1 - 1 = 1 \Rightarrow p_1 = 2$. Then, $p_2 - 1|2p_3...p_m$, and $p_2 - 1$ can only have prime factors with 2 and since $p_2 > 2$ we need that $p_2 = 3$. Furthermore $p_3 - 1|6p_4...p_m$, and $p_3 - 1|6$ which implies $p_3 = 7$ and finally $p_4 - 1|42p_5...p_m \Rightarrow p_4 - 1|42 \Rightarrow p_4 = 43$. We cannot have a p_5, since there are no prime values of $p_5 > 43$ which satisfy $p_5 - 1|42 \cdot 43$.

So the above argument tells us that the only possible values of n are $1, 2, 6, 42, 42 \cdot 43$. And each of these works, since every one of the primes that comprise them divide the corresponding expressions.

23. M We will determine all natural k's such that there is a solution (a, b) to

$$\frac{a^2 + b^2 + ab}{ab - 1} = k \Leftrightarrow a^2 + b^2 - (k-1)ab = -k \tag{1}$$

Take the solution with the smallest value of $a + b$, and we may assume $b \leq a$. If (a, b) is a solution, then so is $((k-1)b - a, b)$, as it is easy to note from (1). But if (a, b) verifies (1), then $b^2 = -k + a(b(k-1) - a) < a(b(k-1) - a)$, so $b(k-1) - a > 0$. But then $b(k-1) - a + b \geq a + b$ (from the minimality of the solution (a, b)) so we will have that $b(k-1) \geq 2a$. But note that

$$b^2-(k-1)ab+a^2+k=0 \Rightarrow b=\frac{(k-1)a\pm\sqrt{a^2(k^2-2k-3)-4k}}{2}$$

and since $b \le a$ we must have the $-$ solution if $k>3$ (we'll see later what happens for $k=1,2,3$). Therefore,

$$b=\frac{(k-1)a-\sqrt{a^2(k^2-2k-3)-4k}}{2}\ge\frac{2a}{k-1}\Rightarrow$$
$$\Rightarrow a(k^2-2k-3)\ge(k-1)\sqrt{a^2(k^2-2k-3)-4k}\Rightarrow$$
$$\Rightarrow a^2(k^2-2k-3)^2\ge a^2(k^2-2k+1)(k^2-2k-3)-4k(k-1)^2\Rightarrow$$
$$\Rightarrow 4k(k-1)^2\ge 4a^2(k^2-2k-3) \tag{2}$$

But now look at (1). If $b=1$ it easily follows that $k=7$, while $b=2$ implies that $k=4$ or $k=7$. Now consider $b>2$ and then $ab-1\ge 2a \Rightarrow k=\frac{a^2+b^2+ab}{ab-1}\le\frac{3a^2}{2a}=\frac{3a}{2}$ and thus (2) becomes

$$k(k-1)^2\ge a^2(k^2-2k-3)\ge\frac{4k^2}{9}(k^2-2k-3)\Rightarrow$$
$$\Rightarrow 9(k-1)^2\ge 4k((k-1)^2-4)$$

It is easy to see that this last relation cannot hold for $k\ge 5$, so our only possibilities are $k=7$ (which could only happen if $b=1$) and $k\le 4$. But $k=\frac{a^2+b^2+ab}{ab-1}\ge\frac{3ab}{ab-1}>3$ and therefore $k>3$. So our only possibilities are $k=4$ (verified by $a=b=2$) and $k=7$ (verified for $a=2$, $b=1$).

24. *D* For any two rational numbers a and b which satisfy the hypothesis, the set of the numbers n for which a^n-b^n is an integer will be called the "associated set" of a and b. Let us first prove the problem in the case that the associated set has the greatest common divisor 1.

Thus, we will write a and b as $\frac{k}{s}$ and $\frac{l}{s}$, where k and l are integers and s is the least common denominator of the fractions a and b. Let us note that s and k (as well as s and l) are co-prime, since if s and k would have a common prime divisor p, then because $s^n|k^n-l^n$ for any n in the associated set, we would have $p|l$ and thus $p|s$, $p|k$ and $p|l$; this contradicts the fact that s is the least common denominator.

Suppose that $s\ne 1$ (meaning that a and b are not integers) and let p be a prime divisor of s. As it was said before, k and l are non-divisible by p.

This means that for every n in the associated set, we have $k^n\equiv_p l^n$ and thus $(kl^{-1})^n\equiv_p 1$. If the order of kl^{-1} modulo p is a number $d>1$, then this d must divide all the n in the associated set. But this is impossible, since we considered the gcd of the associated set 1. So this order is 1, and thus we have

$k \equiv_p l$. This means that $k = p^\alpha t + l$, where α is a positive integer and t is not divisible by p.

So we have, for each n,

$$p^n | (p^\alpha t + l)^n - l^n$$

and by expanding with Newton's formula we get

$$p^n | p^\alpha t l^{n-1} n + \sum_{i=2}^{n} p^{\alpha i} t^i l^{n-i} \binom{n}{i} \tag{1}$$

CASE 1: $p > 2$.

We will now show that the RHS has the exponent of p equal to $\alpha+\beta$, where β is the exponent of p in n.

Indeed, let us look at the exponent of p in the term $p^{\alpha i} t^i l^{n-i} \binom{n}{i}$. The number $\binom{n}{i}$ has the exponent at least $\beta - i + 2$ for $i \geq 3$, because $\binom{n}{i} = \frac{n}{i} \cdot \binom{n-1}{i-1}$ and the number $\frac{n}{i}$ has the exponent at least $\beta - i + 2$ (easy to see). We must concur that the exponent of the term $p^{\alpha i} t^i l^{n-i} \binom{n}{i}$ is at least $\alpha i + \beta - i + 2$, which is greater than $\alpha + \beta$ for $i \geq 3$. Also, the term $p^{2\alpha} t^2 l^{n-2} \binom{n}{2}$ has the exponent at least $2\alpha + \beta$. We see thus that all the terms in the expansion (1) have exponents greater than $\alpha+\beta$ except for the first term which has exactly $\alpha + \beta$ (remember that t and l are non-divisible by p).

This means that $p^n | p^{\alpha+\beta} j$, where j is non-divisible by p. This means that $n \leq \alpha + \beta \leq \alpha + \log_p n$. This is impossible for sufficiently large n. Contradiction.

CASE 2: $p = 2$

We have that $2^n | k^n - l^n$ for an infinity of n and with k and l odd. Thus, we have $2^n | (kl^{-1})^n - 1$ (where l^{-1} is the reciprocal of l modulo 2^n) and let e be the order of (kl^{-1}) modulo 2^n. Since $e | \varphi(2^n) = 2^{n-1}$, e must be of the form 2^d. But $e | n$ also.

Now, we have $2^n | k^{2^d} - l^{2^d}$. Since 4 cannot divide $k^{2^{d-1}} + l^{2^{d-1}}$ for $d \geq 2$ (those two numbers are congruent to 1 modulo 4) we must have $2^n | (k^{2^{d-1}} - l^{2^{d-1}})(k^{2^{d-1}} + l^{2^{d-1}}) \Rightarrow 2^{n-1} | (k^{2^{d-1}} - l^{2^{d-1}})$. By continuing this, we would ultimately obtain $2^{n-d+2} | k^2 - l^2$.

Thus, let α be the exponent of 2 in the prime decomposition of $k^2 - l^2$ (since $k \neq l$, the exponent is finite). We will have $n - d + 2 \leq \alpha$, but $2^d | n$ (since 2^d is the order of (kl^{-1}) modulo 2^n) and this yields $d \leq \log_2 n$. Of course this would produce $\alpha \geq n - \log_2 n + 2$, which is impossible for sufficiently large n. Contradiction.

We have therefore proved the problem in the case when the "associated set" of a and b has the gcd 1. Now, consider any two different numbers a and b and consider the gcd of their infinite "associated set" d. Thus, their associated set would be $\{dn_1 < dn_2 < \ldots < dn_k < \ldots\}$. Now take the numbers a^d and b^d whose associated set includes $\{n_1 < n_2 < \ldots < n_k < \ldots\}$. The associated set of a^d and b^d is infinite and has the gcd 1. From what has been stated above, this means that a^d and b^d are integers and from this it follows that a and b are integers.

25. *M* We will prove the statement by induction; it is trivial to verify it for the first 11 terms of the sequence, so assume that all the terms up to a_{n+5} $(n \geq 6)$ are integers and we will show that a_{n+6} will be.

First of all we shall prove that for any $m \leq n+5$, a_m is co-prime with all of $a_{m-1}, a_{m-2}, a_{m-3}, a_{m-4}$. Indeed, suppose that there existed a prime p such that it divides both a_m and a_{m-k} for $k \leq 4$, and take the smallest m for which this happens. A look at the first 6 terms shows that we must have $m \geq 6$. Then, $a_m a_{m-5} = a_{m-1}a_{m-4} + a_{m-2}a_{m-3}$ and whatever the value of k, p will have to divide the term in the RHS which does not contain a_{m-k}. One of the factors of that term and a_{m-k} itself will be both divisible by p, and this contradicts the minimality of m. So any a_m and a_{m-k} for $1 \leq k \leq 4$ must be co-prime.

What we need to do to prove the induction is to show that

$$a_{n+1} | a_{n+2}a_{n+5} + a_{n+4}a_{n+3} \Leftrightarrow a_{n+2}a_{n+5} \equiv -a_{n+3}a_{n+4} \tag{1}$$

where from now on all equivalences are taken modulo a_{n+1}. Taking the recurrence relations for $n-1, n-2, n-3, n-4, n-5$ modulo a_{n+2} yield to us $a_{n+5}a_n \equiv a_{n+2}a_{n+3}$, $a_{n+4}a_{n-1} \equiv a_n a_{n+3}$, $a_{n+3}a_{n-2} \equiv a_{n-1}a_{n+2}$, $a_{n+2}a_{n-3} \equiv a_n a_{n-1}$ and $a_n a_{n-3} \equiv -a_{n-1}a_{n-2}$.

By multiplying the first and the fourth of the above relations, and by dividing from them the product of the second, the third and the fifth (where division by a number means multiplying by its inverse modulo a_{n+1}, as we have proved that all those numbers are co-prime with a_{n+1}) we will obtain precisely the desired (1).

26. SOLUTION BY ANDREI STEFANESCU

M For any $k \in \mathbb{Z}$ we will consider the numbers $x_k = 1+6^{3k+1}$, $y_k = 1-6^{3k+1}$ and $z_k = -6^{2k+1}$. Note that the triplet (x_k, y_k, z_k) is a solution of the equation, as

$$(1+6^{3k+1})^3 + (1-6^{3k+1})^3 + (-6^{2k+1})^3 =$$

$$= 1 + 3 \cdot 6^{3k+1} + 3 \cdot 6^{6k+2} + 6^{9k+3} + 1 - 3 \cdot 6^{3k+1} + 3 \cdot 6^{6k+2} - 6^{9k+3} - 6^{6k+3} =$$

$$= 2 + 6^{6k+3} - 6^{6k+3} = 2$$

Since we can define this triplet for all k's, there will be an infinity of solutions.

27. Solution by Andrei Stefanescu

D First of all, we can exclude the cases $n = 1, 2$, as in these cases we will have the solutions $a = b = c = d = 4$ and $a = b = c = d = 2$. For each $n \geq 3$, we will write the solutions of our equation as (a, b, c, d) with $a \geq b \geq c \geq d$. Assume that we have such a solution for a certain n, and pick that solution with the minimal a (meaning that any other solution would have $a' \geq a$). Let $s = b + c + d$, and therefore $s \leq 3b \leq 3a$. The hypothesis thus becomes

$$(a+s)^2 = nabcd \Leftrightarrow a^2 - a(nbcd - 2s) + s^2 = 0 \qquad (1)$$

which can be perceived as a quadratic equation in a. Then, it will have two solutions, a and $\frac{s^2}{a}$ (because of (1), a will divide s^2). But because we have chose (a, b, c, d) to be the minimal solution, we would need that $a \leq \frac{s^2}{a} \Rightarrow a \leq s$. We will therefore have two cases:

If $bcd \geq s$, then because $a \geq \frac{s}{3}$, (1) tells us that

$$s^2 = a(n^2bcd - 2s - a) \geq \frac{s}{3}(n^2 s - 2s - s) = \frac{s^2(n^2-3)}{3}$$

by using $a \leq s$. This implies that $n^2 \leq 6$, but we assumed that $n \geq 3$.

If $bcd < s$, then because we have $s \leq 3b$, it follows that $cd < 3$ and therefore $d = 1$ and $c \in \{1, 2\}$. If $c = 2$, then $bcd < s$ becomes $2b < 1 + 2 + b$ and this implies $b < 3$. Since $b \geq c = 2$, we obtain that $b = 2$ and then going back to the hypothesis, it would yield $n = \frac{a+5}{2\sqrt{a}}$. Thus, a is a perfect square, $a \geq 2$ and $\sqrt{a}|5$, leading to $a = 25$ and $n = 3$. If $c = 1$, it follows that $\sqrt{ab} \in \mathbb{N}$ and $b \leq a \leq b + 2$ (since $a \leq s = b + 1 + 1$). But $\sqrt{a(a-1)}$ and $\sqrt{a(a-2)}$ can never be integers, hence $a = b$ and $n = \frac{2(a+1)}{a}$. Thus $a \in \{1,\ 2\}$; for $a = 1$ we obtain $n = 4$, and for $a = 2$ we get $n = 3$.

Therefore, all the n's for which we have solutions are $n \in \{1, 2, 3, 4\}$.

Chapter 3
COMBINATORICS SOLUTIONS

1. *M* Let $B = \bigcup_{i=1}^{k} A_i$ and $m = \#B$. Denote $B = \{b_1, ..., b_m\}$ and let x_i be the number of subsets A_j in which b_i appears. We obviously have that $x_i \geq 1$, because B is the union of all those subsets.

Consider an $m \times k$ table, in which the intersection of row i with column j contains 1 if $b_i \in A_j$ and 0 otherwise. The sum of all the numbers in our table is $\sum_1^m x_i$ (summing by rows) and it equals the sum of $\#A_j$ (summing by columns). We thus have the equality

$$S = \sum_1^m x_i = \sum_1^k \#A_j \geq \frac{nk}{2} \tag{1}$$

In our table, let us calculate the number of pairs of 1's situated on the same row. Summing by columns, this is $\sum_{i<j} \#A_i \bigcap A_j$, but summing by rows we get $\sum_1^m \binom{x_i}{2} = \frac{x_i^2 - x_i}{2}$. Therefore,

$$\frac{k(k-1)}{2}\frac{n}{4} \geq \sum_{i<j} |A_i \bigcap A_j| = \sum_1^m \frac{x_i^2 - x_i}{2}$$

But $\sum x_i^2 \geq \frac{\left(\sum x_i\right)^2}{m} = \frac{S^2}{m}$ (where $S = \sum x_i$) and thus the above inequality becomes

$$\frac{nk(k-1)}{8} \geq \frac{S^2 - mS}{2m} = \frac{S}{2m}(S - m)$$

But (1) tells us that $S \geq \frac{nk}{2} \geq n > m$ and we can bound S from below with the value obtained from (1). We therefore have the desired

$$\frac{nk(k-1)}{8} \geq \frac{nk}{4m}\left(\frac{nk}{2} - m\right) \Rightarrow m(k-1) \geq nk - 2m \Rightarrow$$

$$\Rightarrow m(k+1) \geq nk \Rightarrow m \geq \frac{k}{k+1}n$$

2. *M* We will solve this problem by constructing G', the dual graph of G. We will work in G' instead of G, and we have to prove that the minimum number of colors we need for the vertices of G' such that two neighbors are colored differently is 2 iff all the regions of G' have an even number of edges. The first implication is easy, because it is impossible for 2 colors to be enough if we have a region with an odd number of vertices (we cannot color an odd cycle with just 2 colors such that neighbors are always different).

   To prove the second implication, write the planar graph out as a tree (you can assume the graph connected, as if it weren't, you could just do the same thing for all the components one at a time). Thus, take a vertex and call it "level 0". Call all its neighbors "level 1", call all the neighbors of the level 1's (except for level 0) "level 2", call all the neighbors of level 2's (except for the previous levels) "level 3" and so on...

   Now, color all the even levels white and all the odd levels black. The only case when this coloring wouldn't work is if two white vertices (or 2 black vertices, but it's the same thing) were connected. So we would have a level-$2k$ vertex A connected to a level-$2l$ vertex B, and consider the smallest level vertex from which you can go down (by going down, I mean from a lower-numbered level to higher-numbered one) to both A and B. But then, the paths from that vertex to our two vertices plus the edge between them would generate an simple (without self-intersections) odd cycle C. But then C will contain a number of regions in it, and we already said that every region in G' has an even number of boundary edges. So if we sum the number of boundary edges for the regions inside C, we will get an even number of edges. But every edge inside C is counted twice, and every edge on the boundary of C once. So that would mean that C has an even number of edges. Contradiction.

3. *D* We will prove the following lemma by induction over n, keeping in mind that after k steps the ant always ends up on the k-th diagonal (the line $x + y = k$).

   LEMMA Let $k_n \leq n$ be the number of squares among the first n diagonals that will be poisoned immediately after the ant has reached the n-th diagonal (so the first n poisoned points). Then the ant can reach $n+1-k_n$ of the squares of the n-th diagonal in safety.

   We will prove this by induction, and note that it is obvious for $n = 1$. Assume it true for n and let's prove it for $n+1$. The induction hypothesis tells us that the ant can reach $n+1-k_n$ safe points on the n-th diagonal. But after the ant will step on the $n+1$-th diagonal, there will be at most $n+1-k_n$ poisoned points on that diagonal (because there are at least k_n among the first n diagonals). There are at least $n+2-k_n$ points on the $n+1$-th diagonal accessible to the ant from safe points on the n-th diagonal (it's easy to see that p points on a diagonal have at least $p+1$ neighbors altogether on the next diagonal). But from the definition of k_n, there are at most $k_{n+1}-k_n$ poisoned points on the $n+1$-th diagonal after the ant has reached the $n+1$-th diagonal. Therefore, since we can access $n+2-k_n$ points on the $n+1$-th diagonal, and at most $k_{n+1}-k_n$ are poisoned, then at least $n+2-k_{n+1}$ points on the $n+1$-th diagonal can be reached in safety. This proves our induction.

So we just showed that the ant can reach any diagonal (since $n+1-k_n > 0$ always), but we didn't yet prove the existence of an infinite path. Let A_n be the set of all the points on the n-th diagonal on which the ant can get to, and connect a point $X \in A_n$ to a point $Y \in A_{n-1}$ by a line if the ant can safely get from Y to X. We'll construct an infinite path (meaning a sequence k_i of points such that $k_i \in A_i$ and k_i is connected to k_{i-1} for all i) which will be the correct path for the ant, as follows:

For any m, we will try to find a $k_m \in A_m$ from which we can construct arbitrarily long paths. We'll assume that $k_0, ..., k_{n-1}$ have been built with this property and we will show that there exists such a k_n exists ($k_0 = (0,0)$ has this property, as as proved above). By assumption, we can build arbitrarily long paths from k_{n-1}, and these paths pass through a finite set (A_n). Therefore, a sequence of arbitrarily long paths from k_{n-1} will pass through some element of A_n, which we will pick as our k_n. By following the path $k_1, ..., k_n, ...$ the ant will always survive.

4. D Assume that there are no such three fields. First of all, note that no color can appear in more than 16 fields (otherwise, by the pigeonhole principle, one of the other colors would appear in less than 6 fields). Now, let us prove the following fact:

   LEMMA No row (or column) can contain more than 4 colors. Moreover, if it contains 4 colors it must be of the form $ABBCBBCBBD$, where A, B, C, D are 4 different colors.

   PROOF Consider a row, and take its smallest segment which contains at least 4 colors. Take its endpoints to be of colors A and D, and because that is the smallest segment to contain at least 4 colors, then $A \neq D$. Moreover, the colors A and D do not appear again in the interior of the segment, which must be paved by at least two other colors. The segment must have an even length (or otherwise the endpoints A and D would have a different color midway between them).

Now, if the interior of the segment has exactly 2 other colors B and C, it is easy to see (through a brief analysis of the cases) that the only possibility which doesn't permit three equidistant different fields is when the segment is the whole row, and of the form $ABBCBBCBBD$. If the interior of the segment contained $r > 2$ colors, then let one of them be X and color the other $r - 1 \geq 2$ colors in another color (say Y). The only possibility for A, B, X, Y to coexist peacefully is the above-mentioned pattern ($AXXYXXYXXB$ or $AYYXYYXYYB$), but when we split Y back into its original colors, we will always obtain a triplet of equidistant different fields.

Back to the problem. Since the number of times each color appears is at least 6, we notice that the number of rows which contain a color A + the number of columns which contain it can be no less than 5. Thus, if we sum this after all 15 colors, we get that total number of times when a color appears on a row + the total number of times when a color appears on a column is at least 75. On the other hand, this number is the sum after all the rows of how many colors are on that row + the sum after all the columns of how many colors are on that column. Because we have in total 20 rows and columns and because none of them can contain more than 4 colors, by the pigeonhole principle we have 15 rows or columns which contain 4 colors each. Let's assume we have at least 8 such rows (it's exactly the same if we have at least 8 such columns).

Call the color that appears 6 times on a certain 4-color row (or column) the "big color" of that row (or column). Call the other 3 the "little colors" of that row. No color can be a "big color" for more than two rows (otherwise we would have 18 instances of that color, and we said that each color appears at most 16 times). So among the "big colors" of the eight 4-color rows there are at least 4 colors. So the columns 2,3,5,6,8,9 each contain all these four "big colors", and thus these columns will be 4-color columns. But one of the eight 4-color rows will be one of the rows 2,3,5,6,8,9 and therefore its "big color" will be the "big color" of the 4-color columns 2,3,5,6,8,9. Therefore, that color will appear at least $6 \times 6 = 36$ times! Contradiction.

5. E The following configuration of 16 knights works, and we will show that we have no such configuration with more than 16. Now, we will split the squares of the 5×5 board into a number of subsets, depending on how many jump options a knight on that field has. So, we have squares from which we can make 2,3,4,6 or 8 possible jumps, and note that our configuration is that of the 3-squares and 4-squares (an x-square is a square of x possible jumps). So assume that there exists a configuration of at least 17 knights.

2	3	4	3	2
3	4	6	4	3
4	6	8	6	4
3	4	6	4	3
2	3	4	3	2

THE CONFIGURATION OF 16 KNIGHTS THAT WORKS IS THAT WITH KNIGHTS ON THE SHADED SQUARES

We will call two squares neighbors if a knight can jump between them. First of all, if we had a knight in one of the corners (the corners are 2-squares), then both the corner's neighbors have to contain knights. But those neighbors have in total 10 neighbors of their own, and at most 3 of these can be occupied by knights (the corner field being one of them). So we have at least 7 unoccupied squares. Now, we can have at most 8 squares blank (as we have $25 - 8 = 17$ knights), so at least 2 of the other corners must have knights. But then the neighbors of the 3 corners would just have too many neighbors which would have to be left blank, as it is easy to see.

So none of the corners are taken. Then, suppose the center has a knight, and because it has 8 neighbors, 6 of them would have to be left blank. $6 + 4 > 25 - 17 = 8$, which is the maximum number of blank squares that we can have. So the center is empty too, and therefore at most 3 of the 3,4 and 6-squares can still be blank. But any 6-square has four 4-square neighbors, so if a 6-square were in the configuration, at least two 4-squares would be missing. But any 4-square is missing, then its at least two 3-square neighbors must also be blank (because a 3-square only has three neighbors: the blank center and two 4-squares, and if one of the 4-squares does not have a knight, our 3-square can't have either). So in this case, we would also have under 17 total occupied squares.

Therefore, all the 6-squares must be blank and we are left with the 16-configuration we showed in the beginning of the solution. This is therefore the maximum number.

6. *D* Let a be the total number of the 4-pieces, a_{00} the number of tromiones with their vertex (the field which is adjacent to the two other fields in the tromino) at the intersection of an even column with an even line, a_{10} the number of tromnioes with the vertex on an odd line and an even column, a_{01} those on an even line and an odd column and a_{11} the number of those on an odd line and an odd column.

First of all, color all the odd lines white, and all the even lines black, such that there will be $2n+1$ more white fields than black ones. A 4-piece will always have as many whites as blacks, and the only pieces which have more white fields than black fields are the trominoes with their vertex on an odd line (and in the case of these tromnioes, the difference between whites and blacks is 1). So the number of tromnios with their vertex on an odd line is at least $2n+1$ (because the $2n+1$ difference between whites and blacks must be accounted for), and thus $a_{11}+a_{10} \geq 2n+1$.

Now consider the following coloring: color black all the fields with both coordinates even and white all the rest. We will have n^2 black fields and $3n^2+4n+1$ whites. Any 4-piece will cover three whites and 1 black, and any tromino with one of the coordinates of the vertex even will cover 2 whites and 1 black. All other tromnios will cover 3 whites. Now, the number of whites is

$$3n^2+4n+1 = 3a + 3a_{00} + 2(a_{11}+a_{10}+a_{01})$$

and the number of blacks is

$$n^2 = a + a_{11} + a_{10} + a_{01}$$

If we eliminate a from these we get

$$3a_{00} - (a_{11}+a_{10}+a_{01}) = 4n+1 \qquad (1)$$

But $a_{11}+a_{10} \geq 2n+1 \Rightarrow 4(a_{11}+a_{10}+a_{01}) \geq 8n+4$. Sum this up with (1), and we will get

$$3(a_{00}+a_{11}+a_{10}+a_{01}) \geq 12n+5 \Rightarrow a_{00}+a_{11}+a_{10}+a_{01} \geq 4n+2$$

So the number of trominoes is at least $4n+2$.

7. *M* We shall prove a lemma related to the inclusion-exclusion principle.

LEMMA For any sets $A_1, A_2, ..., A_n$ we have

$$\#\{A_1 \bigcup A_2 \bigcup ... \bigcup A_n\} \geq \#\{A_1\} + ... + \#\{A_n\} - \sum_{1 \leq i < j \leq n} \#\{A_i \bigcap A_j\} \qquad (1)$$

PROOF For any element $x \in \#\{A_1 \bigcup A_2 \bigcup ... \bigcup A_n\}$, suppose it is part of exactly k of the sets A_i. Then x's contribution to the LHS of (1) is 1, while its contribution to its RHS is $k - \frac{k(k-1)}{2}$. But $1 \geq k - \frac{k(k-1)}{2}$ (with equality iff $k \in \{1,2\}$), and by summing up all these inequalities for all x we get exactly (1). Moreover, we only have equality if $k \in \{1,2\}$ for all x's, so each x must be part of 1 or 2 of the sets in order to have equality.

Back to the problem. There can be no more than n pairs of P_i's that intersect, as each such intersection between two P_i's uniquely determines a set P_k. And since we have only n P_k's, we have at most n pairs of P_i's that intersect. Obviously, since all sets are distinct, two such sets intersect in only one element. So we will have

$$\sum_{1\le i<j\le n} \#\{A_i \bigcap A_j\} \le n$$

But then

$$n \ge \#\{A_1 \bigcup A_2 \bigcup ... \bigcup A_n\} \ge$$
$$\ge \#\{A_1\} + ... + \#\{A_n\} - \sum_{1\le i<j\le n} \#\{A_i \bigcap A_j\} \ge 2n - n = n \qquad (2)$$

So we thus need to have equality in (2), and this can only happen if each a_i is part of 1 or 2 of the sets P_j. But the total number of appearances of the a_i's is $2n$ (2 appearances for each set P_i), so each a_i must appear twice in the P_i's, exactly what we needed to prove.

8. D Assume that we have no such three numbers and let the six sets be A, B, C, D, E, F. By the pigeonhole principle, there will be one of these containing at least $\left\lceil \frac{2005}{6} \right\rceil = 335$ numbers. Let's assume that A contains them, and let 335 of A's elements be $a_1 < a_2 < ... < a_{335}$. The numbers $a_{335} - a_1, a_{335} - a_2, ..., a_{335} - a_{334}$ will thus have to be in one of the other five sets, and again by the pigeonhole principle, at least 67 of them will be in one set, say B. Let them be $b_1 < b_2 < ... < b_{67}$, and note that the numbers $b_{67} - b_1, ..., b_{67} - b_{66}$ cannot be in A or in B (because these numbers are both differences of two elements in A and differences of elements of B).

   So these 66 numbers must be in 4 sets. So one of them, say C, contains 17 numbers $c_1 < ... < c_{17}$. Similarly with the above argument, $c_{17} - c_1, ..., c_{17} - c_{16}$ cannot be in A, B or C. So they must be in 3 sets, so 6 of them will be in one of the sets, say D. Let these be $d_1 < ... < d_6$ and so $d_6 - d_1, ..., d_6 - d_5$ can only be in the sets E and F. Therefore, 3 of them (say $e_1 < e_2 < e_3$) will be in the set E. Therefore, $e_3 - e_2$ and $e_3 - e_1$ can only be in F, but then $e_3 - e_1$ can be in none of the sets. Contradiction, and so we found that we have 3 numbers in the same set such that $a + b = c$.

9. M We shall first prove a lemma that easily characterizes non-bipartite graphs.

   LEMMA A graph is not bipartite iff it has a cycle of odd length.

   PROOF It's easy to see why a graph which has a cycle of odd length cannot be bipartite. Now take a graph which is not bipartite and break it up into its connected components, and note that for the graph to not be bipartite, at least one of those components must be non-bipartite. Consider that component, and choose a vertex from it, which we will call "level 0". All its neighbors will be called "level 1". All the neighbors of the vertices in level 1 which are not in level 0 will be called "level 2". All the neighbors of those in level 2 which aren't in levels 0 or 1 will be called "level 3", and so on until we use up all our vertices (we are working with a connected component, so every vertex will appear eventually).

Now consider all the odd levels as a set, and all the even levels as the other. The graph would be bipartite if two vertices in any one of the sets wouldn't be connected. And if a point A in level x is connected to a point B in level $x+2k$, then they determine a cycle of odd length (because there would be a point Y in a level above A and B such that from Y you can go down to both A and B, and going from A up to Y, down to B and from there to A via the edge (A, B) determines a cycle of odd length). Our lemma is thus proven.

Back to the problem. Take an odd cycle C of minimum length (suppose that length is c), and note that it must be of length at most 5 (no triangles). If any two non-adjacent vertices of C would be connected, that that line breaks C up into an odd cycle and an even cycle, thus contradicting C's minimality. Now, any point $x \neq C$ is connected to at most 2 points in C. If it were connected to three of them (any two adjacent points of C cannot be both connected to x, as we have no triangles), then at least two of these (say A and B) would be a distance $k \neq 2$ apart from each other. So these A and B would be linked by a path of length 2 (through x), which contradicts the minimality of C.

But that means that the number of edges between C and the points outside C is at most $2(n-c)$ (each vertex not in C is connected to at most two of those in C). So there is a vertex in C which has at most $\frac{2(n-c)}{c}$ connections to the points outside c. But as it was said before, it is connected to only 2 of the points in C (its immediate neighbors), and thus its degree is at most $\frac{2n}{c} \leq \frac{2n}{5}$ (because $c \geq 5$).

10. *M* Let us assume that there is no such infinite monochrome sequence, and take a maximal white sequence $2k_1 < k_1 + k_2 < \ldots < 2k_n$ (by maximal I mean which cannot be extended any further) and a maximal black sequence $2l_1 < l_1 + l_2 < \ldots < 2l_m$. Assume that $k_n < l_m$.

Let us now consider all the white even natural numbers between $2k_n + 1$ and an arbitrary $2x$, and suppose their number is W. If one of these even white numbers (say $2k$) had $k + k_n$ white, then we could extend the maximal white sequence, which we assumed impossible. Thus, if $2k_n < 2k \leq 2x$ is white, then $2k < k + k_n \leq x + k$ is black. And therefore, the number b of black numbers between $2k_n + 1$ and $x + k_n$ is at least W.

In the same way, if B is the number of black even numbers between $2l_m + 1$ and $2x$, the number w of whites between $2l_m + 1$ and $x + l_m$ is at least B. But $B + W \geq x - l_m$ (the total number of even numbers between $2l_m + 1$ and $2x$) and $b + w \leq x - k_n$ (the total number of numbers between $2k_n + 1$ and $x + k_n$). So $0 \leq (b - W) + (w - B) \leq l_m - k_n$ for all x. This means that $b - W$ (for instance) is bounded for all x, and therefore there can be only a finite number of black squares which can't be written as $k_n + k$ for some white $2k$. So from a point onward, all black squares are of the form $k + k_n$ where $2k$ is white and similarly from a point onward all white squares will be of the form $l_m + l$ for some black $2l$.

So, we can say that for some N, then $k \geq N$ is black iff $2k - 2k_n$ is white and $l \geq N$ is white iff $2l - 2l_m \geq N$ is black. But for any k, $2k - 2k_n$ and $2k - 2l_m$ must be of the same color, or otherwise k would have to be both white and black. So if $l_m - k_n = a > 0$, and that means that the numbers $2x$ and $2x + 2a$ have the same color from a certain point onward. So the infinite arithmetic sequence $2x, 2x + 2a, ..., 2x + 2na + ...$ has the same color throughout. But it is easy to see that this progression also verifies the conditions in the hypothesis, so we proved our initial assumption to be wrong.

11. E Let the sides of the scale be A and B and the weights be in the order $c_1 \geq c_2 \geq ... \geq c_{n+1}$, and we have $c_1 + ... + c_{n+1} = 2n$. We will prove the following two lemmas, which will be sufficient for our proof.

    LEMMA 1 No weight is greater than the sum of all the weights that come after it. Indeed, if this were the case, we would have $c_k > c_{k+1} + ... + c_{n+1} \Rightarrow c_k \geq n - k + 2$. But $c_1 \geq c_2 \geq ... \geq c_k \geq n - k + 2$, so we would have

$$2n = c_1 + ... + c_{n+1} \geq k \cdot c_k + (c_{k+1} + ... + c_{n+1}) \geq k(n - k + 2) + n - k + 1$$

    which is easy to verify that is impossible.

    LEMMA 2 At every step, the difference between the readings of the two sides is at most the sum of the remaining weights.

    PROOF We can prove this by induction over each step. Initially, that difference is 0. Now, assume that it is true at some point, and we will prove the claim for the next weight that is placed. Assume that at a certain point $A \geq B$ and $A - B$ is at most the sum of the following weights. The next weight, c_i, will be placed on B. If $A \geq c_i + B$, then the difference $A - B - c_i$ will still be at most the sum of the remaining weights. If $A < B + c_i$ then $B + c_i - A < c_i$, which by Lemma 1 is at most the sum of the remaining weights.

    Now, the problem is simple, because the last weight must necessarily be a 1 (we have $n + 1$ weights of total weight $2n$), and Lemma 2 tells us that right before it the difference between the scales was at most 1. If the difference were 1, we will put our weight on the lower scale and in the end the scale would be in equilibrium. If the difference were 0, that means that the total sum of the weights is odd, which is impossible.

12. M The problem becomes quite simple if we apply Pick's theorem, which is used to calculate the area of polygons with lattice points as vertices. This theorem

says that the area of such a polygon (not necessarily convex) will be $q + \frac{p}{2} - 1$, where q is the number of lattice points in the polygon's interior, and p is the number of lattice points on its sides.

Thus, let our polygon be the broken line from A to C, the segment from C to $(n, n+1)$, the segment from $(n, n+1)$ to $(-1, n+1)$, the segment from $(-1, n+1)$ to $(-1, 0)$ and the segment from $(-1, 0)$ to $(0, 0)$ (we are choosing the polygon like this to have no self-intersections, even though we can apply Pick even with self-intersections). The area of this polygon is the area we are looking for plus $2n+1$.

But the number of interior points is 0 (since the broken line passes through every point in the square) and the number of points on the boundary is $(n+1)^2$ (the number of points of the square and its interior) plus $2n+3$ (the number of the points on the segments we are adding). Therefore,

$$S + 2n + 1 = \frac{(n+1)^2 + 2n + 3}{2} - 1 \Rightarrow S = \frac{n^2}{2}$$

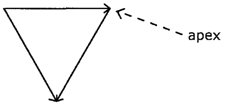
13. *E* We will prove the problem by induction over n. For $n = 1, 2$ the problem is trivial. Now, assume it true for $n-1$ and let us prove it for n. If n will ever be first in the sequence, it will be moved to the last place by the next move, and nothing will ever remove it from there (the only move that could invert the last number would be the inversion of all the numbers, but this can only happen when n is first in line). In this case, the other $n-1$ numbers would behave independently of n and according to the induction hypothesis, 1 will reach the top of the sequence sometime.

    If n never reaches the top, then the number k which is initially last will never be moved away from there (as we explained in the previous paragraph). Thus, we will give the number n the "role" of k, and note that the first $n-1$ numbers (meaning except for the number k) behave exactly like a sequence of $n-1$ numbers. Indeed, it does not matter that we have n instead of k, because we assumed anyway that n never reaches the top, so its exact value does not matter. So if $k \neq 1$, because of the induction hypothesis, 1 will eventually reach the top.

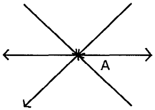
    But if we had $k = 1$ then n has the role of 1, and according to the induction hypothesis, n will initially reach the top, which we assumed not to be true.

14. *D* Draw an arrow on every edge from the smallest to the largest number, and note that every triangle XYZ will have only one vertex (call it the 'apex' of that triangle, and say it is X) for which one of XY and XZ points toward X and the other points away from it. It is easy to check that a triangle is positive if the arrow that points toward X is before the arrow that points away from

it in clockwise order, and negative otherwise. Suppose that we have at most 6 positive triangles.



But we have in total 24 triangles, and to each there is associated an apex (a certain point may be an apex to more triangles). It is easy to observe that in the case of an interior point, the difference between the number of negative triangles to which it is an apex and the number of positive triangles to which it is an apex is 0, whereas in the case of the boundary points, this difference is at most 1. So the difference between negative and positive triangles (which is at least 12) is due to these boundary points, which will cause a difference of at most 12 (because there are only 12 boundary points). So these inequalities must be equalities, and every boundary point must have one more negative triangle to which it is an apex than positive ones.



# of positive triangles to which A is apex =
# of negative triangles to which A is apex

But it is easy to check that this happens for all boundary points iff the edge between each of them and the next boundary point in clockwise order points away from it. This means that the boundary points and the edges between them form a cycle, which is obviously impossible, as we cannot have each number greater than the next. We must therefore have at least 7 positive triangles.

15. E Let us consider, from all the ways to split the set of our vertices into $k-1$ disjoint sets $A_1, ..., A_{k-1}$, that way which yields the smallest number of edges inside one of the sets. Now take a vertex from A_i; if it would have less neighbors in some A_j that it has in A_i, then we could move it to A_j and contradict the minimality of our arrangement. So the number of neighbors it has in A_j is $\geq$ the number of neighbors it has in A_i for all j. Summing this up for all j, we have that the degree of that vertex is $\geq (k-1)$ times the number of neighbors that vertex has in its own set. Summing this up for all vertices and dividing by 2 yields that the total number of edges is $\geq (k-1)$ times the number of edges

inside the same set. So we can erase those in-set edges, and the remaining graph will be multipartite, and thus it will have no k-clique.

16. *M* In a complete oriented graph, call well-ordered triangles those whose edges form a cycle, and call the other triangles badly-ordered. The minimum number of well-ordered triangles is 0, as we can construct such a graph as follows: place the numbers $1, 2, ..., n$ in the vertices of our n-graph and draw the edge between i and j from vertex i to vertex j. It's easy to see that in this case we have no well-ordered triangles.

Since the total number of triangles is always $\binom{n}{3}$, instead of finding the maximum number of well-ordered triangles, we will find the minimum number of badly-ordered triangles. Note that every badly-ordered triangle contains one and only one vertex from which two of its edges leave. Therefore, the number of badly-ordered vertices is equal to the number of pairs of edges that leave from the same vertex. If k_i is the out-degree of vertex i then the number we are looking for is $\sum_{i=1}^{n}\binom{k_i}{2}$. But keeping in mind that $k_1 + ... + k_n$ equals the total number of edges $\binom{n}{2}$, the number of badly-ordered triangles is

$$T = \sum \binom{k_i}{2} = \sum \frac{k_i(k_i-1)}{2} = \frac{\sum k_i^2}{2} - \frac{\sum k_i}{2} \tag{1}$$

But for any numbers $a_1, ..., a_n$ we have that

$$n(a_1^2 + ... + a_n^2) = (a_1 + ... + a_n)^2 + \sum_{i<j}(a_i - a_j)^2$$

Using this expression in (1), we have

$$T = \frac{\frac{(k_1 + ... + k_n)^2}{n} + \frac{\sum_{i<j}(k_i - k_j)^2}{n}}{2} - \frac{n(n-1)}{4} =$$

$$= \frac{n(n-1)^2}{8} - \frac{n(n-1)}{4} + \frac{\sum_{i<j}(k_i - k_j)^2}{2n}$$

So to find the minimum value of T is equivalent to determining the minimum of the non-negative sum $S = \sum_{i<j}(k_i - k_j)^2$. For odd n, we can make it 0 by taking $k_i = \frac{n-1}{2}$ for all i (this can be accomplished by drawing an arrow out of i into $i+1, i+2, ..., i + \frac{n-1}{2}$ for all i).

For even $n = 2a$ it is more complicated, but we can show that the sum is always greater than or equal to a^2. Indeed, consider the numbers $l_i = 2k_i - 2a + 1$

and l_i are odd integers that sum up to 0; then S will equal $\frac{1}{4}\sum_{i<j}(l_i - l_j)^2$. Assume that $l_1, ..., l_k < 0 < l_{k+1}, ..., l_{2a}$ and that $k \geq a$ ($k < a$ is perfectly similar); we will have $l_{k+1} + ... + l_{2a} = -l_1 - ... - l_k = M$ and obviously $M \geq a$. Therefore,

$$S \geq \frac{1}{4}\sum_{i=1}^{k}\sum_{j=k+1}^{2a}(l_j - l_i)^2 \geq \frac{1}{4}\frac{\left(\sum_{i=1}^{k}\sum_{j=k+1}^{2a}(l_j - l_i)\right)^2}{k(2a-k)} = \frac{1}{4}\frac{(2aM)^2}{k(2a-k)} \geq$$

$$\geq \frac{a^2M^2}{a^2} = M^2 \geq a^2$$

Now, the last thing we need to prove is that there actually exists such a graph for which $S = a^2$. For any $1 \leq i < j \leq a$ draw an arrow from i to j, from j to $2a+1-i$, from $2a+1-i$ to $2a+1-j$ and from $2a+1-j$ to i, and then draw arrows from i to $2a+1-i$ for all $1 \leq i \leq a$. It's easy to check that we have $S = a^2$. So the number of well-ordered triangles will be, for odd n, at most

$$\frac{n(n-1)(n-2)}{6} - \frac{n(n-1)^2}{8} + \frac{n(n-1)}{4} =$$
$$= \frac{n(n-1)(4n-8-3n+3+6)}{24} = \frac{n^3-n}{24}$$

and for even n at most

$$\frac{n(n-1)(n-2)}{6} - \frac{n(n-1)^2}{8} + \frac{n(n-1)}{4} - \frac{n}{8} =$$
$$= \frac{n(n-1)(4n-8-3n+3+6)-3n}{24} = \frac{n^3-4n}{24}$$

17. SOLUTION BY ADRIAN ZAHARIUC

M We will show that the minimum number of colors is m^2 for $n = 2m$ and $m(m+1)$ for $n = 2m+1$. For $n = 2m$, the example that we must have at least m^2 colors is when the natives are split into two groups of m and each of them is friends only with the ones in the other group. In this case, it is easy to see that each of the m^2 pairs of friends needs a colored bead different from all the others to mark their friendship. Therefore, we need at least m^2 colors, and we will prove by induction over m that they are enough.

The base case $m = 1$ is trivial. Now, assume that for $2m$ natives m^2 colors are enough and we will show that for $2m+2$ natives $(m+1)^2$ are enough. Take any two friends A and B and assign beads to the other $2m$ of them; by the induction hypothesis, m^2 colors will be enough. Now, assign to each native X beside A and B a bead of a different color, and assign that color to

A and B iff they are friends with X. Finally, give A and B and extra color only for themselves, and we have added in total only $2m+1$ beads. So we have accomplished our coloring with just $(m+1)^2$ beads, which proves our hypothesis.

The case $n = 2m+1$ is proved in the exact same way, just that the equality case is when we split up the natives into a group of m and a group of $m+1$ and make everyone be friends only with those of the other group.

18. *M* Let us assume that n is not a power of 2, and consider the polynomials $f(X) = X^{a_1} + X^{a_2} + \ldots + X^{a_n}$ and $g(X) = X^{b_1} + X^{b_2} + \ldots + X^{b_n}$. We have, by assumption, that $h = f - g \neq 0$ and that

$$f^2(X) - f(X^2) = 2\sum_{i<j} X^{a_i+a_j} = 2\sum_{i<j} X^{b_i+b_j} = g^2(X) - g(X^2) \Rightarrow$$
$$\Rightarrow h(X)(f(X) + g(X)) = h(X^2) \qquad (1)$$

We will prove by complete induction over i the fact that $h^{(i)}(1) = 0$ (the i-th formal derivative of h). Let $f(X) + g(X) = p(X)$ and of course $p(1) = 2n$. Moreover, $h(1) = 0$; assume that $h^{(i)}(1) = 0$ for i from 0 to some $k-1$ and let's prove it for k. By differentiating (1) k times and evaluating it in 1, all the lower order derivatives of h will vanish, and we will only be left with the k-th, so

$$h^{(k)}(1)p(1) = 2^k h^{(k)}(1)$$

Since $p(1) = 2n$ and we assumed n a non-power of 2, then this can only happen if $h^{(k)}(1) = 0$. Thus, the induction is complete, and we will have all the derivatives of h at the point 1 equal to 0. Of course, for polynomials this can only happen if h is zero throughout, which is impossible because $f \neq g$.

19. *D* Initially, on the i-th row and j-th column, we will find the number $10(i-1)+j$. After k transformations, suppose that $a_k(i,j)$ is the number situated at the intersection of the i-th row and j-th column. It is easy to see that the quantity $S_k = \sum_{i,j} a_k(i,j)(10(i-1)+j)$ is left invariant by all transformations. But the problem tells us that at some point, the numbers $a_k(i,j)$ are just a permutation of the numbers $10(i-1)+j$, and so by the rearrangement inequality we have $S_k \leq \sum_{i,j}(10(i-1)+j)^2$ for all k. But S_k is conserved, and its initial value is $\sum_{i,j}(10(i-1)+j)^2$, so the previous inequality is an equality, and therefore the numbers are in the initial order at that point.

20. *M* We shall exclude the cases when all the a_i's are greater than all the b_i's or that when all the b_i's are greater than all the a_i's, since they are trivial. That means that $a_1 < b_1$ and $a_n > b_n$, so by continuity there must be a certain k

such that $a_k < b_k$ and $a_{k+1} > b_{k+1}$. Therefore, $a_i < b_i$ for all i from 1 to k, and $a_i > b_i$ for all i from k to n. Therefore, our sum is

$$b_1 - a_1 + b_2 - a_2 + ... + b_k - a_k + a_{k+1} - b_{k+1} + ... + a_n - b_n =$$
$$= (a_1 + ... + a_n + b_1 + ... + b_n) - 2(a_1 + ... + a_k + b_{k+1} + ... + b_n) \quad (1)$$

But $a_1 + ... + a_n + b_1 + ... + b_n = n(2n+1)$, since these are just the numbers from 1 to $2n$. But the n numbers $a_1, ..., a_k, b_{k+1}, ..., b_n$ are, by our assumption and the hypothesis, smaller than all the n numbers $a_{k+1}, ..., a_n, b_1, ..., b_k$. Therefore, $a_1, ..., a_k, b_{k+1}, ..., b_n$ must be the numbers $1, 2, ..., n$ in some order and so twice their sum will be $n(n+1)$. Therefore, the sum in (1) will be just $n(2n+1) - n(n+1) = n^2$, exactly what we needed to prove.

21. *M* Suppose the triangles initially had hypothenuse 1 and let θ be one of their angles. It's not hard to prove that a cut splits a triangle of hypothenuse x into one of hypothenuse $x\cos\theta$ and one of hypothenuse $x\sin\theta$. So all the triangles we will obtain will have hypothenuse of the form $(\cos\theta)^m(\sin\theta)^n$, and we will model that triangle by placing a checker on the point (m, n) in the lattice plane. So initially, we have 4 checkers on the origin, and at every step a "cutting" means cutting a checker in 2 and placing each of the halves in the points immediately above and to the right.

   It is easy to see that any cut preserves $S = \sum_k \frac{1}{2^{i_k+j_k}}$, where (i_k, j_k) are the coordinates of the checker k. Because this sum is initially 4, it will always be equal to this. But at any stage, if all checkers are on different lattice points, then we will have at most $i+1$ checkers with the sum of the coordinates i, for any i. Therefore, the sum S will be

$$4 = S < \sum_{i=0}^{\infty} \frac{i+1}{2^i} = 4$$

   Contradiction, and thus we will always have two congruent triangles.

22. SOLUTION BY ADRIAN ZAHARIUC

   D Let us first inspect the case $|A_i| = m$ for all $i \in \{1, 2, ..., n\}$.

   LEMMA Let $m, n, k \in \mathrm{N}$ be such that $n > 2m$ and $A_1, A_2, ..., A_k \subset \{1, 2, ..., n\}$ so that $|A_i| = m$ and $A_i \bigcap A_j \neq \emptyset$, $\forall 1 < i \leq j < n$. Then we have

$$k \leq \binom{n-1}{m-1}$$

   Consider any permutation $\pi \in S_n$ of the elements of the set $\{1, 2, ..., n\}$. Let us write the numbers of this permutation on a circle which has a red dot, starting the writing of the numbers from the red dot, counterclockwise. Call an

m-block any succession of m numbers on our circle. We call an m-block good if its m elements are in fact the elements of one of the A_i.

For every permutation $\pi \in S_n$, denote by $N(\pi)$ the number of good m-blocks. Observe that $N(\pi) \leq m$ for all $\pi \in S_n$, because otherwise we would have two disjoint good m-blocks (which is impossible, as any two of the A_i's intersect).

Let us now count how many times does each of our A_i's appear as an m-block. The block can be placed on the circle in n ways, the elements of the block can be permuted in $m!$ ways, and the elements outside the block can be permuted in $(n-m)!$ ways; so in total, each set appears $n \cdot m! \cdot (n-m)!$ times as an m-block. Thus counting in two ways, we have that

$$k \cdot n \cdot m! \cdot (n-m)! = \sum_{\pi \in S_n} N(\pi) \leq m \cdot n! \Rightarrow k \leq \binom{n-1}{m-1}$$

Our problem is now proven, as the lemma tells us that there are at most $\binom{n-1}{m-1}$ sets with m elements and at most $\binom{n-1}{m-2}$ with $m-1$ elements, so we have at most $\binom{n-1}{m-1} + \binom{n-1}{m-2} = \binom{n}{m}$ sets in total. This upper bound can be reached by considering all the sets of m and $m-1$ elements which contain the element 1.

23. *M* The total number of colorings of our 2005 points with 2 colors is 2^{2005}. Similarly, the number of colorings which leave a certain 18-number arithmetic progression of the same color is $2^{2005-17}$ (all the points outside the progression can be colored in any way, and the progression can be colored either all white or all black). If m is the total number of 18-progressions, let A_i be the set of those colorings which preserve progression i monochrome. But we just showed before that $\#A_i = 2^{2005-17}$ and we will have that

$$\#A_1 \bigcup ... \bigcup A_m \leq \sum_{i=1}^{m} \#A_i = m \cdot 2^{2005-17} \tag{1}$$

But what is m? A progression $a, a+r, ..., a+17r$ is completely defined by a and r such that $a+17r \leq 2005$. For any a we have thus $\left\lfloor \frac{2005-a}{17} \right\rfloor \leq \frac{2005-a}{17}$ progressions which start with a. So the total number of progressions will be

$$m \leq \sum_{a=1}^{2005} \frac{2005-a}{17} = \frac{2005^2 - \dfrac{2005 \cdot 2006}{2}}{17} = \frac{2005 \cdot 2004}{34}$$

So equation (1) becomes

$$\#A_1 \bigcup \ldots \bigcup A_m \leq 2^{2005-17} \cdot \frac{2005 \cdot 2004}{34} < 2^{2005}$$

as an easy computation reveals. So not all colorings are in $A_1 \bigcup \ldots \bigcup A_m$ and thus at least one coloring will leave no 18-progression monochrome, as we needed to prove.

24. D Let us assume that a checker will eventually have the y-coordinate 5, and choose the origin such that the point where that checker reaches the 5-th level is $(0,5)$. Let $a = \frac{\sqrt{5}+1}{2}$, and a will satisfy the equation $a^2 = a + 1$. At any moment, assign to our distribution of checkers the sum $\sum_k a^{j_k - |i_k|}$, where (i_k, j_k) are the coordinates of checker k. It is easy to see that any upward jump in the lattice plane preserves this sum, whereas all the downward, leftward and rightward jumps decrease it. Therefore, in the end our sum will be at least a^5 (because we have a checker on $(0,5)$), but it will be at most equal to the initial value of the sum (when the entire lower half-plane was covered by checkers).

    Let us calculate the initial value of the sum, which we will call S. The sum after all the checkers on the line $y = j \leq 0$ will be

$$\sum_{i=-\infty}^{\infty} a^{j-|i|} = -a^j + 2\sum_{i=0}^{\infty} a^{j-i} = -a^j + 2a^j \sum \frac{1}{a^i} =$$

$$= a^j \left(\frac{2a}{a-1} - 1 \right) = a^j \frac{a+1}{a-1}$$

    The initial value of S will be the sum after all non-positive j's of the above expression, so because his sum is non-increasing we will have

$$a^5 \leq \sum_{j=0}^{-\infty} a^j \frac{a+1}{a-1} = \frac{a+1}{a-1} \frac{a}{a-1} \Leftrightarrow a^4(a-1)^2 \leq a+1 \Leftrightarrow$$

$$\Leftrightarrow a^4(a-1)^2 \leq a^2 \Leftrightarrow (a(a-1))^2 \leq \Leftrightarrow 1 \leq 1$$

    So, in order to have equality here, all the above inequalities must be equalities. Therefore, we cannot have any transformation that strictly decreases the sum, so all the jumps must be upward. This is obviously impossible, given our initial position.

25. D We will consider the oriented graph G with 2^{n-1} vertices, where in each vertex we place one of the numbers with $n-1$ digits in base 2, and we draw an arrow from a to b iff the last $n-2$ digits of a are the first $n-2$ digits of b. It is easy to see that the out-degree and in-degree of every vertex are both 2. Moreover, our graph is connected, as we can always

get from vertex $\overline{a_1...a_{n-1}}$ to $\overline{b_1...b_{n-1}}$ by the path $(\overline{a_1...a_{n-1}}, \overline{a_2...a_{n-1}b_1})$, $(\overline{a_2...a_{n-1}b_1}, \overline{a_3...a_{n-1}b_1b_2}),...,(\overline{a_{n-1}b_1...b_{n-2}}, \overline{b_1,...,b_{n-1}})$

An arrow between a and b will therefore represent an n-digit number, obtained by taking a's first digit, a and b's $n-2$ common digits, and b's last digit. Therefore, our problem is equivalent with finding an Euler cycle in G. We will therefore prove that every m-vertex connected graph with all out-degrees and all in-degrees 2 has an Euler cycle.

We will first prove the following lemma.

LEMMA Any graph with $\text{out}(v) = \text{in}(v)$ for all vertices v can be broken up into disjoint cycles (disjoint as in having no common edge, but allowing common vertices).

We will prove it by induction after the number of edges e. For $e = 0$ we have nothing to prove; assume it true for all numbers smaller that e and let's prove it for e. Take a vertex and create a path out of it. That path will intersect itself at some point, and thus we have a cycle. If we remove that cycle, we are still left with a graph that has $\text{out}(v) = \text{in}(v)$ for all vertices v and the induction is proven.

Now, consider our graph as broken up into k disjoint cycles, and we will rearrange that cycles such that the graph is broken up into $k-1$ disjoint cycles (where we don't restrict ourselves just to simple cycles; a vertex may appear may appear more than once in a cycle, but an edge may not). Take a cycle C. If all the edges are in C, it is our Euler cycle; if not, there is some cycle D which touches C (if not, then C would be isolated from the edges that are outside of it, and connectedness is violated). We can therefore unite C and D into one big graph in the obvious way: start from one of their common points, walk on all of C and return to that point, then walk on all of D and return to that point. What you have is just a bigger cycle.

So we will be able to put all our edges into one big cycle, which will be our desired Euler cycle.

26. *M* Call a set even if the sum of its elements is even, and odd otherwise. Let $d(n,k)$ be the difference between the number of even and odd subsets of $\{1,2,...,n\}$ with k elements. We will compute a recurrence for $d(n,k)$.

First of all, $d(2a+1, 2b+1) = -d(2a, 2b)$, because we can split up the number of $2b+1$-element sets of $\{1,2,...,2a+1\}$ into two groups: those which contain $2a+1$ and those which do not. In the group of those which do not contain $2a+1$, there is a bijection between the even subsets and the odd subsets (that bijection is $f(x_1,...,x_{2b+1}) = (2a+1-x_1,...,2a+1-x_{2b+1})$), and so in that group the difference between the number of even subsets and odd subsets is 0. So the only difference between even sets and odd sets arises from the sets which contain $2a+1$ and $2b$ other numbers from $\{1, 2, ..., 2a\}$. Hence the formula $d(2a+1, 2b+1) = -d(2a, 2b)$ (the minus sign arises because even-summed sets in $\{1, 2, ..., 2a+1\}$ become odd-summed sets in $\{1, 2, ..., 2a\}$ when we remove the element $2a+1$).

Similarly, using the same argument and bijection, we come up with the recurrences $d(2a+1, 2b) = d(2a, 2b)$ and $d(2a, 2b+1) = 0$. Computing a recurrence for $d(2a, 2b)$ is straightforward, as we again split the difference in "the difference in number between even and odd sets that contain $2a$" plus "the difference between even and odd sets that do not contain $2a$". We thus get $d(2a, 2b) = d(2a-1, 2b-1) + d(2a-1, 2b)$, which in light of our other recurrences becomes $d(2a, 2b) = -d(2a-2, 2b-2) + d(2a-2, 2b)$. If we let $f(a, b) = (-1)^b d(2a, 2b)$ then the recurrence becomes $f(a, b) = f(a-1, b) + f(a-1, b-1)$. It is easy to see that $\binom{a}{b}$ verifies both this relation and the initial values $f(k, 1)$ (which are easy to check). Therefore, we will have

$$d(2a, 2b) = (-1)^b \binom{a}{b} \Rightarrow d(2a+1, 2b+1) = (-1)^{b+1} \binom{a}{b} \Rightarrow$$

$$\Rightarrow d(2005, 77) = (-1)^{39} \binom{1002}{38} < 0$$

27. *M* Suppose there are no such three players. We will consider a graph with $3n$ vertices broken up into 3 groups A, B, C, each corresponding to a club. Draw an edge between two vertices if the corresponding chess players have played each other. Obviously, no two vertices from the same group will be connected, since we don't care about games between players of the same club. We will call a 3-cycle of length $3k$ a cycle with vertices $a_1, b_1, c_1, a_2, b_2, c_2, ..., a_k, b_k, c_k, a_1$ where every two consecutive vertices are connected and $a_i \in A$, $b_i \in B$ and $c_i \in C$.

We will show that such a cycle exists by the following argument: because every vertex has at least $n+1$ neighbors, it will have neighbors in both the other sets. Construct a cycle as follows: take a vertex a_1 from A, one of its neighbors b_1 from B, one of b_1's neighbors from C, namely C_1... at after a finite number of steps of this process, we will stumble upon a vertex previously used in the process. What we obtain there will be a $3k$-cycle.

Now, we will prove our problem by induction over n: $n = 1$ is trivial, and suppose it true for all $m < n$ and let us prove it for n. Take a $3k$-cycle Γ of minimum length, and suppose first that $k < n$. Now, any point outside Γ (say, $X \in A - \Gamma$) cannot be connected to more than k points inside Γ. Indeed, if x were connected to $k+1$ of he b_i's and c_j's, there would be an i such that b_i and c_i are both connected to x (contradicting our assumption that there are no triangles). So every point outside Γ is connected to at most k points inside Γ, so with at least $n-k+1$ points outside Γ. Applying the induction hypothesis to the points outside Γ, we will find a triangle.

But what if $k=n$, i.e. the entire graph could be under the form of such a cycle? Then b_1 cannot be connected to any c_j for $j \geq 2$, or we would get a 3-cycle of smaller length. Since b_1 is connected to $n+1$ others, those must be c_1 and all the a_i's. But c_1 must be connected to at least 1 of the a_i's (since it has $n+1$ neighbors), and that a_i together with b_1 and c_1 would form a triangle.

28. D We will construct a bijection between the set A of good $n-1$-paths and the set B of good n-paths with no even return. Given any $n-1$-path $p \in A$, transform it into a n-path from B like this: if p has no even returns, place a climb and a descent at the beginning of p; if p has even descents, place a climb at the beginning of p and a descent at the end of the last even descent. It is easy to see that this procedure yields a good n-path with no even return.

    The inverse of this transformation is the following: take a good n-path with no even returns and consider its first return (which will be odd). Remove the first climb of the path and the last descent of that first return, and we obtain a good $n-1$-path. It is rather easy to see that these two transformations are inverses to each other, so the sets A and B have the same cardinality. This is exactly what we needed to prove.

29. E We will show that it is possible to make any certain column 0, and this will be enough to prove our problem. Indeed, once a column is 0, no other transformations on the other lines and columns will affect it, so we can just make the columns 0 one by one.

    So consider our column, and consider the smallest sum the elements of that column can have after applying transformations such that all the entries are positive. Let $c_1, \ldots, c_n$ be this minimal configuration, and obviously we will have 1's among them (otherwise we could just decrease the column by 1 and contradict the minimality of our sum). Let $c_1 = \ldots = c_k = 1$ and all the others be > 1. Then, multiply the lines $1, 2, \ldots, k$ by 2 and decrease 1 from our column, transforming it into $(1, 1, \ldots, 1, c_{k+1} - 1, \ldots, c_n - 1)$. Now, this contradicts our minimality, unless all the c_i's are 1; if all the c_i's are 1, just decrease the column by 1 and we are done.

30. Solution by Adrian Zahariuc

    D We can assume that $a < b$. Let $A = \{0, a, a+b\}$ and $B = \{0, b, a+b\}$; thus, we will try to write $\mathbb{N}$ as a disjoint union of sets of the form $A + a_i$ and $B + b_j$, where $a_i, b_j \in \mathbb{N}$. To accomplish this, we will inductively construct sets X_n, where $X_0 = \emptyset$ and X_{n+1} is obtained from X_n by adding an $A + a_i$ or an $B + b_j$ disjoint from X_n; moreover, the set we add to X_n will contain the smallest number that was not in X_n. It is thus obvious that $X_0 \subset X_1 \subset X_2 \subset \ldots$ and $X_{n+1} - X_n$ is an (a, b)-adapted set. Therefore, we will be able to write $\mathbb{N}$ as a union of (a, b)-adapted sets $\bigcup_{n\geq 0}(X_{n+1} - X_n)$, because the sets are disjoint and every time we add a new set, we also add the smallest element that had been left out.

But let us show how to pass from X_n to X_{n+1}. For every n, let k_n be the smallest number not in X_n. Our algorithm is to add to X_n the set $A + k_n$, or if that isn't possible, the set $B + k_n$. Assume that we cannot perform this.

First of all, because $X_m \subset X_n$ for $m \leq n$, we will also have $k_m < k_n$ (strictly). But from the description of the algorithm, it follows that in X_n there are no sets of the form $A + x$ or $B + x$, with $x \geq k_n$ (if there were, they would have to be added at a certain point, so x would be an k_m for $m < n$; but this is impossible, as $x \geq k_n$). Therefore, the number $k_n + a + b$ is not in X_n and the only reason why neither of the sets $A + k_n$ and $B + k_n$ can be added is that the numbers $k_n + a$ and $k_n + b$ are already added.

So the number $k_n + b$ has already been added, and note that it is only part of the sets $A + (k_n + b)$, $B + (k_n + b)$, $A + (k_n + b - a)$, $A + (k_n - a)$, $B + (k_n - a)$. The first three sets could not have been added before (as it was said in the previous paragraph), so one of the last two must have been added sometime. If $A + (k_n - a)$ was added before, it would mean that k_n is already added, which is not by its definition. So the one which was added before is $B + (k_n - a)$. But why would we have ever chosen to add $B + (k_n - a)$ over $A + (k_n - a)$ (which, as was said, was never added) and from its description, our algorithm always prefers $A + k_m$ over $B + k_m$? We thus obtain a contradiction, so the algorithm must function indefinitely.

31. *M* Note that if the main diagonal is initially infected, the whole board will become so in time. Therefore, n cells are enough, and we will show that they are the minimum number. Indeed, suppose that we initially have k infected cells, and we will call an edge in our board infected if it is the boundary of an infected square (we consider as edges only segments of unit length)

Since we initially have k infected squares, we have at most $4k$ infected edges. With every new infection, we get at most two new infected edges (as the square that is to be infected has at least two already infected edges), and since we will make $n^2 - k$ new infections until we cover the board, the total number of infected edges will be at most $4k+2(n^2-k)$. But in the end all the edges must be infected, and there are $2n(n+1)$ of them, so $2n(n+1) \leq 4k+2(n^2-k) \Rightarrow n \leq k$, which is exactly what we needed to prove.

32. D Three triangles with a common vertex determine a 7-graph that works. We will now show that 7 is the maximum number, and that there are no such 8-vertex graphs. Suppose for the purpose of contradiction that there were one, G.

LEMMA Any 6-graph has either a triangle or 3 independent vertices (3 vertices with no edges between them). This can be proven easily, by taking a vertex and looking at how many neighbors it has. If it has at least 3 neighbors, then if any of them are connected we get a triangle, otherwise the neighbors are 3 independent vertices. If our vertex has at most 2 neighbors, then if any of its 3 non-neighbors are not connected we get 3 independent vertices, otherwise the 3 non-neighbors form a triangle.

The lemma is thus proved and we shall return to the problem. Every vertex $V \in G$ has at least 2 neighbors, because otherwise we could apply the lemma and get a triangle or 3 independent vertices among V's non-neighbors. It is easy to see that those 3 points and V contradict the hypothesis. Moreover, a vertex A cannot be connected to all other vertices; if it were and if any B were connected to two other vertices besides A, these four vertices would form an impossible 4-cycle. Otherwise, the degree of all vertices in the 7-vertex subgraph obtained by removing A is at most 1, and we can easily see that we will have 4 independent vertices.

Now, it is easy to see that in every graph with the degree of every vertex at least 2 we will have a cycle. Let us consider a minimal cycle with n vertices and note that $n \neq 4$ (4-cycles contradict our hypothesis). If $n = 3$ then let our triangle be ABC and note that any other vertex V must be connected to exactly one of A, B and C (otherwise $VABC$ would be a contradictory 4-subgraph) Thus, the other 5 vertices except for those of our triangle will be split into 3 groups: A's neighbors, B's neighbors and C's neighbors. Moreover, note that two vertices from two of these groups cannot be connected, otherwise those vertices and two of A, B, C would form an impossible 4-cycle. If two vertices in the same groups (say A's group) were connected, then these two vertices, A and a non-neighbor of A (we said that no vertex can be connected to all other vertices) would yield a contradiction. But then, all 5 vertices except for A, B, C are independent, which is impossible.

So the cycle of minimum length is not a triangle, and then $n \geq 5$. Then any two non-consecutive vertices in the cycle cannot be connected (otherwise, we contradict the minimality of our cycle). Note that a vertex that is not in our cycle cannot be connected to two vertices in our cycle; indeed, if that were the case we would obtain a triangle, a 4-cycle or a cycle smaller than the minimal one. Therefore, if $n = 5$, there will be two of the vertices outside our cycle which are not connected (because they can't form a triangle). So those two vertices (call them U and V) will be connected to at most two vertices in our pentagon; thus take any A and B of the other three vertices of the pentagon such that A and B are non-connected, and therefore A, B, U, V are independent. If $n \in \{6, 7\}$, take any vertex V outside our cycle and take the at least 5 vertices of the cycle to which it is not connected. There will be 3 independent vertices among them, and these 3 together with V will form a contradictory 4-subgraph. Finally, if $n = 8$ there will be 4 independent vertices in this octogon. The problem is therefore proved.

33. D Suppose that the dimensions of the initial rectangle are both non-integers. Then, call the direction of one of its sides the horizontal and the other direction the vertical; let b be the length of the big rectangle's horizontal side. Now, take all the rectangles of the subdivision and break their integral dimension into several rectangles with that dimension 1. Thus, all our small rectangles will have a dimension 1, and call a rectangle horizontal if the dimension of size 1 is horizontal, or vertical otherwise.

Let all the vertical rectangles whose lowest point is at a distance x from the bottom of the big rectangle form "level x". Obviously, there will be a finite number of such levels $x_1, ..., x_k$ where we consider levels with integer x, even if there are no rectangles in those levels. Let $S(x_k)$ be the sum of the horizontal dimensions of the rectangle in level x_k.

We will prove by induction after i the following statement: we have $\{S(i)\} = \{b\}$ and for every one of finitely many levels $i + h$ with $0 < h < 1$ we will have $\{S(i+h)\} = 0$. ($\{y\}$ means the fractional part of y). It is easy to note that the base case $i = 0$ is proven in the same way as the inductive step, so we will just show that if the theorem is true for $i - 1$ it is true for i.

Indeed, consider all the levels $i - 1 + j_k$ $(0 < j_k < 1)$ and all the levels $i + h_k$ $(0 < h_k < 1)$, and we know that $\{S(i-1)\} = \{b\}$ and $\{S(i - 1 + j_k)\} = 0$. Take the horizontal line situated at height i and look at the rectangles which are not below it and which it touches. That line will cut the entire rectangle over a length b, but it will cut horizontal rectangles over integer lengths, the levels $i - 1 + j_k$ over integer lengths (by the induction hypothesis) and the level i over the length $S(i)$. So $S(i)$ plus a number of integers yields b, and therefore $\{S(i)\} = \{b\}$. A similar argument will prove that $\{S(i + h_k)\} = 0$.

But this brings us to our final argument; if the vertical dimension of the big rectangle were not an integer, then the only level of vertical rectangles that can touch the top side will be a level of the form $i + h$, where $0 < h < 1$. Looking at the top part of the big rectangle, it cuts the whole thing over a length b, it cuts the horizontal rectangles over an integer length and the level

$i+h$ over an integer length (since $\{S(i+h)\}=0$). Therefore b is an integer, which contradicts our initial assumption. We must have at least one dimension of our big rectangle integral.

34. E Let us call the coordinates of a certain field of our matrix the pair (i,j), where i is its row and j its column. An independent set of 1's will be a collection of 1's all situated on different rows and columns. Let k be the maximum size of an independent set of 1's, and assume WLOG that we have an independent set of 1's on the first k rows and columns, i.e. there is a 1 at the intersection of row i with column i (for $1 \le i \le k$). Let m be the smallest number of rows and columns that cover our 1's.

    Because that set is maximal, there are no 1's with coordinates (i,j) with $i>k$ and $j>k$. Call one of the rows $1,2,...,k$ or one of the columns $1,2,...,k$ necessary iff there is a 1 on it outside the square $\{1 \le i \le k, 1 \le j \le k\}$. We can see that row i and column i cannot both be necessary, because if they were we could replace (i,i) with the two elements on column i and row i that are outside the square $\{1 \le i \le k, 1 \le j \le k\}$ and get a greater independent set of 1's. So there are at most k necessary rows and columns, and its easy to see now that all the 1's must be on them. Thus, we proved that $m \le k$ and showing that $m \ge k$ is simple (we cannot even cover the independent set of size k with less than k rows and columns, much less the whole table). So we must have $k=m$.

35. D We will prove the statement by induction over n. It is true for 1, as we have at least ten 1-digit sequences (since each every digit must appear once). So there are no such numbers for $n=1$, but the problem remains logically true, because any such number will be rational.

    Now, assume it true for n and let us prove it for $n+1$, where we have at most $n+9$ sequences of $n+1$ successive digits. If we had two $n+1$-sequences with the same first n digits, then the number of n-sequences of consecutive digits must be at most $n+8$ and we can apply induction to discover that x is rational. But if the opposite holds, i.e. to every n-sequence s_k there corresponded a unique digit right after it (call it d_k) then we can easily prove that x is rational.

    Indeed, as the decimal representation of our number is infinite, we will find a sequence $\overline{a_1...a_n}$ which appears twice. But let $\overline{b_1...b_m}$ be the digits between the two apparitions of this sequence, and that means that the digit that comes after $\overline{a_1...a_n}$ is b_1; so b_1 must appear after the second apparition of $\overline{a_1...a_n}$ as well. Then, the digit that comes after $\overline{a_2...a_nb_1}$ is b_2, so it must come after the second apparition of $\overline{a_2...a_nb_1}$ as well... Thus, our number will be periodic, as the sequence $\overline{a_1...a_nb_1...b_m}$ repeats to infinity and x will therefore be rational.

36. Solution by Andrei Stefanescu

M Since the game cannot end in a draw, one of the players must have a winning strategy (because if one of them does not have a winning strategy, it means that the other player can always stop him from winning by winning himself). We will show that the first player always has a winning strategy, and to do that assume for the purpose of contradiction that the second player has a winning strategy.

Then, it means that whatever k the first player (call him A) initially writes down, the second player (B) can win. But note that then A can just write down 1 and from this point the game is just as if B had started (since even in the case when B starts, the number 1 could never be written except on the first turn). Therefore, A has a winning strategy, and this yields a contradiction to our initial assumption. Thus, A always has a winning strategy.

37. Solution by Andrei Stefanescu

We will prove the statement by induction over n, where the case $n = 2$ is trivial to check. Assume it true for all $k < n$ and we will prove it for n; now, Turan's theorem tells us that our graph must have a triangle, whose vertices we will call A, B, C. Let k_A, k_B and k_C be the number of neighbors each of these points has outside the set $\{A, B, C\}$. If $k_A + k_B + k_C \leq 3n - 5$ then $(k_A + k_B) + (k_B + k_C) + (k_C + k_A) \leq 6n - 10$; by the pigeonhole principle, one of the brackets will be at most $2n - 4$, and we will assume $k_A + k_B$ to be that bracket. Then, by removing A and B, the resulting graph will have $2n - 2$ vertices and at least $(n-1)^2 + 1$ edges, so by the induction hypothesis it will have at least $n - 1$ triangles. Together with the triangle ABC, our initial graph will have at least n triangles.

If $k_A + k_B + k_C \geq 3n - 4$, then let n_i for $0 \leq i \leq 3$ be the number of vertices outside $\{A, B, C\}$ which have i neighbors in $\{A, B, C\}$. We will have $n_1 + 2n_2 + 3n_3 = k_A + k_B + k_C \geq 3n - 4$, and the number of triangles formed by vertices outside $\{A, B, C\}$ with the vertices $\{A, B, C\}$ will be at least $n_2 + 3n_3$. This happens because any vertex X which is connected to two of A, B, C will form a triangle with those two, while any X which is connected to all of A, B, C will form three triangles together with pairs of them. But $n_1 + n_2 \leq 2n - 3$ (since we have just $2n$ vertices in total), so $n_2 + 3n_3 \geq 3n - 4 - (n_1 + n_2) \geq n - 1$. Together with the triangle ABC, this yields in total at least n triangles.

Chapter 4
ALGEBRA
SOLUTIONS

1. *E* Let $P(X) = X^n + X^{n-1}a_{n-1} + ... + a_0 = (X - x_1)(X - x_2)...(X - x_n)$, where $x_1, ..., x_n$ are the roots of the polynomial, all in the interval $(0, 1)$. The formal derivative of P will be

$$P'(X) = nX^{n-1} + (n-1)a_{n-1}X^{n-2} + \sum_{i=0}^{n-2} ia_iX^{i-1} \tag{1}$$

But the derivative is also equal to

$$(X - x_1)...(X - x_{n-1}) + (X - x_1)...(X - x_{n-2})(X - x_n) +$$
$$+ ... + (X - x_2)...(X - x_n) \tag{2}$$

Therefore, evaluating (1) and (2) at $X = 1$ and equating them will yield

$$n + (n-1)a_{n-1} + S = (1 - x_1)...(1 - x_{n-1}) +$$
$$+(1 - x_1)...(1 - x_{n-2})(1 - x_n) + ... + (1 - x_2)...(1 - x_n) \tag{3}$$

where S is the sum we need to prove positive. But for all $y_1, ..., y_{n-1} \in (0, 1)$ we have

$$(1 - y_1)...(1 - y_{n-1}) > 1 - y_1 - ... - y_{n-1} \tag{4}$$

We can prove this by induction over n. Indeed, it is trivial for $n = 3$; suppose it true for k and let us prove it for $k + 1$. We will have

$$(1 - y_1)...(1 - y_{k-1}) > 1 - y_1 - ... - y_{k-1} \Rightarrow$$
$$\Rightarrow (1 - y_1)...(1 - y_{k-1})(1 - y_k) > (1 - y_1 - ... - y_{k-1})(1 - y_k) >$$
$$> 1 - y_1 - ... - y_k$$

which proves (4). Applying it to every product in the RHS of (3) now yields

$$n - (n-1)(x_1 + ... + x_n) + S > n - (n-1)(x_1 + ... + x_n) \Rightarrow S \geq 0$$

exactly what we needed to prove.

2. *M* We must have $\frac{a}{b} < \sqrt{2002} \Rightarrow a^2 < 2002 \cdot b^2$ and also

$$2002 < \frac{(\lambda + a^2)^2}{a^2b^2} \Rightarrow 2002a^2b^2 \leq a^4 + 2a^2\lambda + \lambda^2 \tag{1}$$

But now let $a^2 = 2002 \cdot b^2 - k$ and we will show that $k \geq 10$. Indeed, since $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ we would need to have $-k$ a quadratic residue modulo 7,11 and 13. A quick computation reveals that 10 is the smallest number which satisfies all these, so $a^2 \leq 2002 \cdot b^2 - 10$ and (1) becomes

$$10a^2 \leq a^2(2002b^2 - a^2) < 2a^2\lambda + \lambda^2 \Rightarrow 2a^2(5 - \lambda) \leq \lambda^2 \tag{2}$$

But this takes place for an infinity of fractions $\frac{a}{b}$, and since $\frac{a}{b}$ is very close to $\sqrt{2002}$ an infinity of values of a must be numerators for this infinite number of fractions. So if $\lambda < 5$, then the LHS of (2) can be arbitrarily large, while the RHS is finite. So $\lambda \geq 5$.

3. *M* Let us take two cases. First of all, suppose that there is a number i such that $a_i + a_{i+2} > \frac{1}{2}$. Then our sum will be

$$S = a_{i-1}a_i + a_ia_{i+1} + a_{i+1}a_{i+2} + a_{i+2}a_{i+3} + \ldots \tag{1}$$

First, note that all the numbers except for a_i and a_{i+2} must be less than $\frac{1}{2}$. But let us assume that $a_{i+3} \geq a_{i-1}$ (the other case is treated identically) and we will have $a_{i+3} + \ldots + a_n + a_1 + \ldots + a_{i-2} \geq a_{i+4} + \ldots + a_n + a_1 + \ldots + a_{i-1}$. Therefore, by bounding the a_j's by $\frac{1}{2}$ the "..." part in the relation (1) will be

$$a_{i+3}a_{i+4} + \ldots + a_{i-2}a_{i-1} \leq \frac{1}{2}(a_{i+4} + \ldots + a_{i-1}) \leq$$

$$\leq (a_i + a_{i+2})(a_{i+4} + \ldots + a_{i-1}) \leq a_i(a_{i+3} + \ldots + a_{i-2}) + a_{i+2}(a_{i+4} + \ldots + a_{i-1})$$

By applying this into (1), we get

$$S \leq (a_i + a_{i+2})(a_{i+1} + a_{i+3} + \ldots + a_{i-1}) = (a_i + a_{i+2})(1 - a_i - a_{i+2}) \leq \frac{1}{4}$$

because $x(1 - x) \leq \frac{1}{4}$ for all x between 0 and 1. Now, let us examine the second case, when for all i we have $a_i + a_{i+2} \leq \frac{1}{2}$. Multiplying this by a_{i+1} gives us

$$a_ia_{i+1} + a_{i+1}a_{i+2} \leq \frac{a_{i+1}}{2}$$

which summed up over all i yields $2S \leq \frac{1}{2} \Rightarrow S \leq \frac{1}{4}$.

4. M Since a, b, c are the sides of a triangle, then the cyclic sum

$$\sum(a+c-b)(a-b)^2 \geq 0$$

But this sum is

$$\sum(a+c-b)(a^2-2ab+b^2) =$$
$$= \sum \left(a^3+a^2c-a^2b-2a^2b-2abc+2ab^2+ab^2+b^2c-b^3\right) =$$
$$= \sum 4a^2c - \sum 2a^2b - 6abc \geq 0$$

By dividing this by $2abc$ we get

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{a}{c}+\frac{b}{a}+\frac{c}{b}+3$$

exactly what we had to prove.

5. D Since the sequences a_i and b_i are ordered in the same way, Chebyshev's inequality tells us that

$$(a_1^2+\ldots+a_n^2)(b_1^2+\ldots+b_n^2) \leq n(a_1^2b_1^2+\ldots+a_n^2b_n^2)$$

and if we denote a_ib_i by α_i, it will suffice to prove that

$$n(\alpha_1^2+\ldots+\alpha_n^2) \leq (\alpha_1+\ldots+\alpha_n)^2\frac{(\alpha_n+\alpha_1)^2}{4\alpha_1\alpha_n} \tag{1}$$

for $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. But note that if there were k, l with $a_1 < a_k < a_l < a_n$, we could enlarge a_l by some ε and decrease a_k by the same ε, keeping the RHS of (1) constant, but increasing the LHS. So for a certain value of the RHS, the LHS is maximum when all the numbers are equal to either a_1 or a_n, except for at most one of them.

If the LHS of (1) is maximum when k of our numbers are α_1 and l of them are α_n with $k+l=n$, (1) translates into

$$(k+l)(k\alpha_1^2+l\alpha_n^2) \leq (k\alpha_1+l\alpha_n)^2\frac{(\alpha_1+\alpha_n)^2}{4\alpha_1\alpha_n} \Leftrightarrow$$
$$\Leftrightarrow 4\alpha_1\alpha_n(k(k+l)\alpha_1^2+l(k+l)\alpha_n^2) \leq (k^2\alpha_1^2+2kl\alpha_1\alpha_n+l^2\alpha_n^2)(\alpha_1^2+2\alpha_1\alpha_n+\alpha_n^2) \Leftrightarrow$$
$$\Leftrightarrow 2k(k+l)\alpha_1^3\alpha_n+2l(k+l)\alpha_n^3\alpha_1 \leq k^2\alpha_1^4+(k^2+4kl+l^2)\alpha_1^2\alpha_n^2+l^2\alpha_n^4 \Leftrightarrow$$
$$\Leftrightarrow 0 \leq (k\alpha_1-(k+l)\alpha_1\alpha_n+l\alpha_n)^2$$

which is true. But what if the maximum is attained when k of our numbers are α_1, l of them are α_n and one is some e with $\alpha_1 < e < \alpha_n$? Then, (1) becomes

$$(k+l+1)(k\alpha_1^2+l\alpha_n^2+e^2)-(k\alpha_1+l\alpha_n+e)^2\frac{(\alpha_1+\alpha_n)^2}{4\alpha_1\alpha_n} \leq 0 \tag{2}$$

If $(k+l+1)-\dfrac{(\alpha_1+\alpha_n)^2}{4\alpha_1\alpha_n}\leq 0$, then (2) holds because of the obvious $(k\alpha_1^2+l\alpha_n^2+e^2)\leq(k\alpha_1+l\alpha_n+e)^2$. But if $(k+l+1)-\dfrac{(\alpha_1+\alpha_n)^2}{4\alpha_1\alpha_n}>0$, expression (2) will be a second degree polynomial in e with positive dominant coefficient. So it will be convex, and its value will be at most the maximum of the values at the points α_1 and α_n (since $\alpha_1<e<\alpha_n$). But we showed above that when all the numbers are α_1 or α_n the inequality holds, so our problem is proved.

6. *E* The AM-GM inequality gives us

$$2a^2+\frac{b}{2}\geq 2a\sqrt{b}$$

$$2b^2+\frac{a}{2}\geq 2b\sqrt{a}$$

$$2ab+\frac{b}{2}\geq 2b\sqrt{a}$$

$$2ab+\frac{a}{2}\geq 2a\sqrt{b}$$

By summing all these up, we get

$$2(a+b)^2+a+b\geq 4a\sqrt{b}+4b\sqrt{a}\Rightarrow\frac{(a+b)^2}{2}+\frac{a+b}{4}\geq a\sqrt{b}+b\sqrt{a}$$

7. Solution by Gabriel Kreindler

D If we let $x=f(y)$ in the given relation we obtain $f(0)=f(f(y))+(f(y))^2+f(f(y))-1$ so we must have $f(f(x))=\dfrac{a+1-(f(x))^2}{2}$ for all real x, where $a=f(0)$. Now put $f(x)$ instead of x in the hypothesis and we get

$$f(f(x)-f(y))=f(f(x))+f(x)\cdot f(y)+f(f(y))-1=a-\frac{1}{2}(f(x)-f(y))^2 \quad (1)$$

for all real x,y. Now we will try to prove that $f(x)-f(y)$ can take on any real value. Consider in the initial relation $y=y_0$ with $f(y_0)=b\neq 0$ (if such a number y_0 does not exist then f is identical zero, which is a contradiction) and we get

$$f(x_0-b)-f(x_0)=bx_0+\frac{a-1-b^2}{2}$$

But by varying x_0, the number $m=bx_0+\dfrac{a-1-b^2}{2}$ will take on any real value, so letting $x=x_0-b$ and $y=x_0$ in (1) we get $f(m)=a-\dfrac{m^2}{2}$, for any real m.

Replacing $f(x)$ by $a-\frac{x^2}{2}$ in the relation $f(f(x)) = \frac{a+1-f^2(x)}{2}$ we obtain $a = f(0) = 1$, and so the function $f(x) = 1 - \frac{x^2}{2}$ is our only possible solution. It is easy to see that it verifies the hypothesis.

8. E Proving that $aB + bC + cA \le k^2$ is equivalent to showing that

$$aB + bC + cA = a(k-b) + b(k-c) + c(k-a) \le k^2 \Leftrightarrow$$

$$\Leftrightarrow k(a+b+c) \le k^2 + ab + bc + ca$$

But this is equivalent to

$$k^2 - ka + bc \ge (b+c)(k-a) \Leftrightarrow bc \ge (k-a)(b+c-k)$$

If $b + c - k \le 0$, then the above is obviously true. If not, then we have

$$(k-b)(k-c) \ge 0 \Rightarrow bc \ge k(b+c-k) \ge (k-a)(b+c-k)$$

and thus our inequality holds.

9. Solution by Adrian Zahariuc

D Let us first apply the Cauchy-Schwartz inequality:

$$\left(\sum \frac{1}{a_i^2+x}\right)^2 \le \left(\sum \frac{1}{a_i}\right)\left(\sum \frac{a_i}{(a_i^2+x)^2}\right) \le \sum \frac{a_i}{(a_i^2+x)^2}$$

so it will be enough to prove that the RHS of this is at most the desired $\frac{1}{2}\frac{1}{a_1(a_1-1)+x}$. This is the tricky part, which we shall accomplish by summing up the inequalities

$$\frac{a_1}{(a_i^2+x)^2} \le \frac{1}{2}\frac{1}{a_i(a_i-1)+x} - \frac{1}{2}\frac{1}{a_{i+1}(a_{i+1}-1)+x}$$

for all i's. These inequalities are true, because

$$(a_i^2 - a_i + x)(a_i^2 + a_i + x) \le (a_i^2 + x)^2 \Rightarrow$$

$$\Rightarrow \frac{2a_i}{(a_i^2+x)^2} \le \frac{2a_i}{(a_i^2 - a_i + x)(a_i^2 + a_i + x)} =$$

$$= \frac{1}{a_i^2 - a_i + x} - \frac{1}{a_i^2 + a_i + x} \le \frac{1}{a_i^2 - a_i + x} - \frac{1}{a_{i+1}^2 - a_{i+1} + x}$$

since $a_i(a_i+1) \le a_{i+1}(a_{i+1}-1)$ (because the a_i's are an increasing sequence of natural numbers).

10. D Let $y_k = \dfrac{1}{n-1+x_k}$ and let us assume for the purpose of contradiction that $\sum y_k > 1$. But then

$$y_k = \frac{1}{n-1+x_n} \Rightarrow x_n = \frac{1-(n-1)y_k}{y_k} \tag{1}$$

Then let $1-(n-1)y_k = a_k$, and our assumption becomes $S = \sum a_k < 1$; but $x_n = \dfrac{(n-1)a_k}{1-a_k}$. It is easy to see that the y_k's must be positive, and therefore (1) tells us that they must be $< \dfrac{1}{n-1}$. This means that the a_k's are positive and less than 1. However, the hypothesis now becomes

$$(n-1)\sum \frac{a_k}{1-a_k} = \sum x_i = \sum \frac{1}{x_i} = \sum \frac{1-a_k}{a_k(n-1)} \tag{2}$$

But observe that

$$\sum \frac{1-a_k}{a_k} > \sum \frac{S-a_k}{a_k} = \sum_{j\neq k} \frac{a_j}{a_k} =$$
$$= \sum_k a_k \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{k-1}} + \frac{1}{a_{k+1}} + \ldots + \frac{1}{a_n}\right) \geq$$
$$\geq \sum a_k \frac{(n-1)^2}{a_1 + \ldots + a_{k-1} + a_{k+1} + \ldots + a_n} =$$
$$= (n-1)^2 \frac{a_k}{S-a_k} > (n-1)^2 \frac{a_k}{1-a_k}$$

This contradicts (2), so we cannot have $\sum \dfrac{1}{n-1+x_k} > 1$. Our problem is thus proven.

11. SOLUTION BY GABRIEL DOSPINESCU

D The Cauchy-Schwartz inequality gives us

$$\sum \frac{ax}{b+c} \leq \sqrt{\sum x^2 \sum \left(\frac{a}{b+c}\right)^2}$$

By adding 3 to both sides of this we get

$$\sum \frac{ax}{b+c} + 3 \leq \sqrt{\sum x^2 \cdot \sum \left(\frac{a}{b+c}\right)^2} + \sqrt{\sum xy \cdot \frac{3}{4}} + \sqrt{\sum xy \cdot \frac{3}{4}} \tag{1}$$

Apply Cauchy-Schwartz again to the last expression and we will get

$$\sqrt{\sum x^2 \sum \left(\frac{a}{b+c}\right)^2} + \sqrt{\sum xy \cdot \frac{3}{4}} + \sqrt{\sum xy \cdot \frac{3}{4}} \leq$$

$$\leq \sqrt{\left(\sum x^2+\sum xy+\sum xy\right)\left(\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{4}+\frac{3}{4}\right)} =$$

$$= (x+y+z)\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}} \qquad (2)$$

But it is easy to see that

$$\sqrt{\sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2}} \leq \sum\frac{a}{b+c} \Leftrightarrow$$

$$\Leftrightarrow \sum\left(\frac{a}{b+c}\right)^2+\frac{3}{2} \leq \sum\left(\frac{a}{b+c}\right)^2+2\sum\frac{ab}{(a+c)(b+c)} \Leftrightarrow$$

$$\Leftrightarrow \frac{3}{4} \leq \sum\frac{ab}{(a+c)(b+c)} \Leftrightarrow 3(a+b)(b+c)(c+a) \leq 4\sum ab(a+b) \Leftrightarrow$$

$$\Leftrightarrow 6abc \leq \sum a^2b+ab^2 \qquad (3)$$

which is true, because of the AM-GM inequality. So putting (1),(2) and (3) together, we get that

$$\sum\frac{ax}{b+c}+3 \leq (x+y+z)\sum\frac{a}{b+c} \Rightarrow \sum\frac{a(y+z)}{b+c} \geq 3$$

12. M The problem becomes much simpler is we use the concept of formal derivative of a polynomials. Let $P(X-3) = R(X) \in \mathbb{Z}[X]$ and suppose the polynomial which squared yields $(X^2+6X+10)P^2-1$ is $Q(X+3)$, where $Q \in \mathbb{Z}[X]$. The reason why we chose $Q(X+3)$ instead of Q and defined $R(X) = P(X-3)$ is because the hypothesis looks much nicer:

$$((X+3)^2+1)R^2(X+3) = Q^2(X+3)+1 \Rightarrow (X^2+1)R^2(X) = Q^2(X)+1 \qquad (1)$$

Of course, if the degree of Q is n, then the degree of R will be $n-1$. Differentiating (1) gives us

$$2XR^2+(X^2+1)2RR' = 2QQ' \qquad (2)$$

(2) tells us that R divides $2QQ'$, yet from (1) it is clear that R and Q have no common factors (not even in $\mathbb{C}[X]$). Therefore, R must divide $2Q'$, and since the degree of R is equal to the degree of Q' we will have $R = aQ'$ for some real number a. Equation (1) thus becomes $a^2(X^2+1)Q'^2 = Q^2+1$. Equating the leading coefficients in this relation gives us $a^2 = \frac{1}{n^2}$, and therefore we have

$$(X^2+1)Q'^2 = n^2(Q^2+1) \qquad (3)$$

But let $Q = a_nX^n + ... + a_0$ and assume that it has other non-zero coefficients than the leading one. Let X^k be the greatest power $k < n$ and with a non-zero coefficient and the coefficients of $n+k$ on both sides of (3) will be equal. But they will be $2nka_na_k = 2n^2a_na_k \Rightarrow n = k$ (since $a_n, a_k \neq 0$) which is impossible. Therefore, $Q = a_nX^n$, and in this case (3) immediately yields $n = 1$ and $a_1 = 1$. So the only solution for Q is $Q(X) = X$. But then $R = \pm 1$ and so the only solutions for P are $P(X) = \pm 1$.

13. M Letting $x = 0$ yields $f(f(y)) = y + f^2(0)$, and therefore we will have

$$f(x^2 + f(f(y))) = f(y) + f^2(x) \Rightarrow$$
$$\Rightarrow f(f(x^2 + y + f^2(0))) = f(f^2(x) + f(y)) \Rightarrow$$
$$\Rightarrow x^2 + y + 2f^2(0) = y + (f(f(x)))^2 = y + x^2 + 2xf^2(0) + f^4(0)$$

Since this holds for all x's, we must necessarily have $f(0) = 0$ and therefore $f(f(y)) = y$ and thus f is injective. But by making $y = 0$ in the hypothesis, we will have $f(x^2) = f^2(x)$. First of all, this tells us that f is positive for all positive values of x. Second of all, it tells us that $f(-x) = \pm f(x)$, and since f is injective we must have $f(-x) = -f(x)$. So by making $z = \sqrt{x}$ in the hypothesis and taking $f(y)$ instead of y, we will have

$$f(y + z) = f(y) + f^2(\sqrt{z}) = f(y) + f(z)$$

for all positive z. But since $f(x) = -f(-x)$, it is easy to show that the above relation holds for all y and z. So f is additive, but we also know that it is positive for positive values of the argument. So if $x > y$ then

$$f(x) - f(y) = f(x - y) \geq 0 \Rightarrow f(x) \geq f(y)$$

and therefore f is increasing. But any increasing additive function is of the form $f(x) = ax$, and we can easily show that. Let $a = f(1)$ and we will have $f(k) = f(1 + ... + 1) = kf(1) = ak$ for all integers k. For any rational $\frac{p}{q}$ we will have

$$f(p) = f\left(\frac{p}{q} + ... + \frac{p}{q}\right) = qf\left(\frac{p}{q}\right) \Rightarrow f\left(\frac{p}{q}\right) = a\frac{p}{q}$$

So we proved that $f(x) = ax$ holds for any rational x. But we also know that the function is increasing, and if we take two sequences of rational numbers $k_n \nearrow x$ (the rationals k_n are smaller than x and approach it) and $l_n \searrow x$ (the rationals l_n are greater than x and approach it) we will have

$$ak_n \leq f(x) \leq al_n$$

and since k_n and l_n tend to x we will have, for all x, $ax \leq f(x) \leq ax \Rightarrow f(x) = ax$. But we also have $f(f(x)) = x$, so $a^2 = 1$, and therefore the only two functions that can work are $f(x) = x$ and $f(x) = -x$. We can easily check that only $f(x) = x$ satisfies.

14. *M* We will show that the largest such k is 5. Indeed, any $k > 5$ is not good, because we can choose $a = b = 1$ and $c = 2$ (which are not sides of a triangle) and we will have $k \cdot 2 \cdot 1 \cdot 1 > 2^3 + 1^3 + 1^3 \Leftrightarrow 2k > 10$.

Now, let us prove that any $k \leq 5$ works for us. Take $0 < a \leq b \leq c$ which satisfy $kabc > a^3 + b^3 + c^3$ and assume that they are not side-lengths of a triangle. This could only happen if $c > a + b$ and then we would have

$$5ab \geq kab > c^2 + \frac{a^3+b^3}{c} \geq c^2 + \frac{(a+b)^3}{4c} \tag{1}$$

We used that

$$4(a^3+b^3) \geq (a+b)^3 \Leftrightarrow a^3+b^3 \geq a^2b + b^2a \Leftrightarrow (a^2-b^2)(a-b) \geq 0$$

which is true for all positive a, b. But for any positive α, the function $f(x) = x^2 + \dfrac{\alpha^3}{4x}$ is increasing for $x \geq \alpha$ (just differentiate it, or write down what it means for $f(x) > f(y)$). Because $c \geq a + b$ and by taking $\alpha = a + b$, we will have $f(c) \geq f(a+b) = \dfrac{5}{4}(a+b)^2$. Therefore, (1) becomes

$$5ab > \frac{5}{4}(a+b)^2 \Rightarrow 4ab > (a+b)^2$$

which is obviously impossible. We have thus proved that the greatest such k is indeed 5.

15. *D* LEMMA There exists a sequence on positive integers $1 = k_1 < k_2 < \ldots < k_n < \ldots$ such that

$$\sum_{i=2}^{n} \frac{a_{k_i} - a_{k_{i-1}}}{a_{k_i}} \tag{1}$$

is unbounded.

We can construct this sequence inductively such that for all i we have $\dfrac{a_{k_i} - a_{k_{i-1}}}{a_{k_i}} \geq \dfrac{2}{3}$. Indeed, suppose that we have constructed $k_1 < \ldots < k_{n-1}$ and since the sequence a_k is unbounded, we will always find a k_n such that

$$a_{k_n} \geq 3a_{k_{n-1}} \Rightarrow \frac{a_{k_n} - a_{k_{n-1}}}{a_{k_n}} \geq \frac{2}{3}$$

This sequence will yield all the terms in the series (1) greater than $\dfrac{2}{3}$, and therefore the sum will be unbounded.

But, back to the problem, for all i we will always have

$$\frac{a_{k_i+1}-a_{k_i}}{a_{k_i+1}}+\frac{a_{k_i+2}-a_{k_i+1}}{a_{k_i+2}}+\ldots+\frac{a_{k_{i+1}}-a_{k_{i+1}-1}}{a_{k_{i+1}}}\geq$$

$$\frac{a_{k_i+1}-a_{k_i}}{a_{k_{i+1}}}+\frac{a_{k_i+2}-a_{k_i+1}}{a_{k_{i+1}}}+\ldots+\frac{a_{k_{i+1}}-a_{k_{i+1}-1}}{a_{k_{i+1}}}\geq\frac{a_{k_{i+1}}-a_{k_i}}{a_{k_{i+1}}}$$

But by summing this up or all i, our series will be greater than series (1), which is unbounded. Our series is therefore unbounded.

16. M Note that any function of the form $f(x)=x+\frac{a}{x}+b$ satisfies, for all real a and b. We will prove that these are the only solutions, and to do that we will consider the function $g(x)=f(x)-x$. This function satisfies

$$xg\left(x+\frac{1}{y}\right)+yg(y)=yg\left(y+\frac{1}{x}\right)+xg(x)$$

Expressing it for $\frac{1}{y}$ instead of y will give us

$$xg(x+y)+\frac{1}{y}g\left(\frac{1}{y}\right)=\frac{1}{y}g\left(\frac{1}{x}+\frac{1}{y}\right)+xg(x)\Rightarrow$$

$$\Rightarrow xyg(x+y)-g\left(\frac{1}{x}+\frac{1}{y}\right)=xyg(x)-g\left(\frac{1}{y}\right) \tag{1}$$

Of course, since the LHS is symmetrical in x and y, we should obtain the same thing in the RHS if we switch x and y, so

$$xyg(x)-g\left(\frac{1}{y}\right)=xyg(y)-g\left(\frac{1}{x}\right)\Rightarrow xg(x)-g(1)=xg(1)-g\left(\frac{1}{x}\right)$$

We can now express $g\left(\frac{1}{x}\right)$ in terms of x and $g(x)$, so going back to the above relation will give us

$$xyg(x)-yg(1)-g(1)+yg(y)=xyg(y)-xg(1)-g(1)+xg(x) \tag{2}$$

If we now let $g(1)=a+b$ and $g(-1)=-a+b$, writing (2) for $y=-1$ gives us

$$-xg(x)+a+b-a-b+a-b=(a-b)x-x(a+b)-a-b+xg(x)\Rightarrow$$

$$\Rightarrow g(x)=\frac{a}{x}+b$$

Therefore, we will have $f(x)=x+\frac{a}{x}+b$, as desired.

17. E By making $x = y = 0$ we obtain $f(0) = 0$. By switching x and y in the hypothesis we get $f(x-y) = f(y-x)$ for all x and y, and therefore $f(-x) = f(x)$ for all x. But let $f(1) = a$ and we will prove by induction that for all positive integers n we have $f(n) = an^2$. We know it is true for 1 and assume it true for $1, 2, ..., n-1$ and we will prove it for n. Indeed, by taking $x = n-1$ and $y = 1$ we get

$$f(n) = 2f(n-1) - f(n-2) + 2a =$$
$$= a(2n^2 - 4n + 2 - n^2 + 4n - 4 + 2) = an^2$$

In the same way, for any x we can prove that $f(nx) = n^2 f(x)$, and by taking $x = \frac{m}{n}$ we get

$$f\left(\frac{m}{n}\right) = \frac{f(m)}{n^2} = a\left(\frac{m}{n}\right)^2$$

for all positive m, n. Since the function is even, for all rational numbers q we will have $f(q) = aq^2$. It is easy to see that these functions verify the hypothesis.

18. M Let $\frac{1}{x} = a$, $\frac{1}{y} = b$, $\frac{1}{z} = c$ and the hypothesis turns into $a + b + c = 1$. By dividing our relation by $\sqrt{xyz}$, we will have to prove that

$$\sum\sqrt{\frac{1}{yz} + \frac{1}{x}} \geq 1 + \sum\sqrt{\frac{1}{yz}} \Leftrightarrow \sum\sqrt{bc+a} \geq 1 + \sum\sqrt{bc} \Leftrightarrow$$

$$\Leftrightarrow \sum(\sqrt{bc+a} - \sqrt{bc}) \geq 1 \Leftrightarrow \sum\frac{a}{\sqrt{bc+a} + \sqrt{bc}} \geq 1 \tag{1}$$

But

$$\sqrt{bc+a} + \sqrt{bc} \leq 1 \Leftrightarrow \sqrt{bc+a} \leq 1 - \sqrt{bc} \Leftrightarrow$$
$$\Leftrightarrow bc + a \leq 1 + bc - 2\sqrt{bc} \Leftrightarrow 2\sqrt{bc} \leq 1 - a = b + c$$

and this means that

$$\sum\frac{a}{\sqrt{bc+a} + \sqrt{bc}} \geq \sum a = 1$$

and therefore (1) is true; because we came to this point by equivalences, the problem is proved.

19. D No, there doesn't exist such a set. Indeed, suppose there was an $M = \{a_1, ..., a_k\}$ with this property and let $F = \{f \in M[X]\}$, the set of polynomials with coefficients in M and with all the roots in M. According to our assumption, we have such polynomials of arbitrarily large degree.

Consider an arbitrary f in F, of the form

$$f = (X - a_1)^{t_1}(X - a_2)^{t_2}...(X - a_k)^{t_k} =$$

$$= (X^{t_1} - t_1X^{t_1-1}a_1 + \binom{t_1}{2}X^{t_1-2}a_1^2 + ...)...(X^{t_k} - t_kX^{t_k-1}a_k + \binom{t_k}{2}X^{t_k-2}a_k^2 + ...)$$

Let $n = t_1 + ... + t_k$ be the degree of the polynomial, and it is easy to see that the coefficient of X^{n-1} is $S = -\sum_i t_ia_i$, and the coefficient X^{n-2} is $T = \sum_i \binom{t_1}{2}a_i^2 + \sum_{i<j} t_it_ja_ia_j$. These are both in M, so they belong to a finite set of nonzero numbers. But

$$T = \sum_i \binom{t_i}{2}a_i^2 + \sum_{i<j} t_it_ja_ia_j = \frac{1}{2}\left(\sum_i t_i^2a_i^2 - \sum_i t_ia_i^2 + 2\sum_{i<j} t_it_ja_ia_j\right) =$$

$$= \frac{1}{2}\left(\sum_i t_ia_i\right)^2 - \frac{1}{2}\sum_i t_ia_i^2 = \frac{S^2}{2} - \frac{1}{2}\sum_i t_ia_i^2$$

Now let $m = \min a_i^2$ and obviously $m > 0$. Therefore,

$$T \le \frac{S^2}{2} - m(t_1 + ... + t_k)$$

But because we have such polynomials for arbitrarily large $n = t_1 + ... + t_k$ and $\frac{S^2}{2}$ can only take a finite number of values, then T can be arbitrarily small and thus cannot take only a finite number of values. We have reached a contradiction.

20. SOLUTION BY GABRIEL KREINDLER

M Let us denote the roots of the polynomial by z_1, z_2,..., z_n. We know that $|z_i| < 1$ for all i and that $P(z) = \prod_{i=1}^{n}(z - z_i)$. Suppose that for all $z \in \mathbb{C}$, $|z| = 1$, $|P(z)| < 1$, and let us consider $M = \max_{|z|=1}|P(z)|$. We know that this maximum is attained (the unit circle is closed and bounded, and therefore compact), so $M < 1$.

Furthermore, let us choose for an arbitrary $k \in \mathbb{N}$, the roots ε_0, ε_1,..., ε_{k-1} of the polynomial $z^k - 1$. We know that for any $0 \le i \le k-1$ $|P(\varepsilon_i)| \le M$, so

$$M^k \ge \prod_{i=0}^{k-1}|P(\varepsilon_i)| = \left|\prod_{i=0}^{k-1}\prod_{j=1}^{n}(\varepsilon_i - z_j)\right| = \left|\prod_{i=1}^{n}\left(\prod_{j=0}^{k-1}(z_i - \varepsilon_j)\right)\right| =$$

$$= \left|\prod_{i=1}^{n}(1 - z_i^k)\right| \ge \prod_{i=1}^{n}(1 - |z_i|^k)$$

Therefore, we have $M^k \geq \prod_{i=1}^{n}(1-|z_i|^k)$, but $\lim_{k\to\infty} M^k = 0$ and

$$\lim_{k\to\infty}\prod_{i=1}^{n}(1-|z_i|^k) = \prod_{i=1}^{n}\left(1-\lim_{k\to\infty}|z_i|^k\right) = 1$$

so by making k sufficiently large, we obtain a contradiction ($0 \geq 1$). Our assumption was thus wrong, so $M \geq 1$.

21. *E* We will employ brute force with this problem. Let $S = y+z$ and $P = yz$ and the first equation gives us $x = \frac{4-S}{1+P}$. Therefore, we need to have $S \leq 4$ and we must prove that

$$S + \frac{4-S}{1+P} \geq P + \frac{4S-S^2}{1+P} \Leftrightarrow 4+PS \geq P+P^2+4S-S^2 \Leftrightarrow$$

$$\Leftrightarrow (S-2)^2 \geq P(P+1-S) \tag{1}$$

If $P(P+1-S) \leq 0$ the above inequality holds and we are done. So let us assume it positive; but $P \leq \frac{S^2}{4}$ so

$$P(P+1-S) \leq \frac{S^2(S^2+4-4S)}{16} = (S-2)^2\frac{S^2}{16} \leq (S-2)^2$$

which proves (1) and the problem.

22. *D* We will prove by induction over n that a_n can be written as $2^{x_n}+2^{y_n}$ where x_n and y_n are integers. Moreover, we will always have $x_n = x_{n-1}$ and $y_n = y_{n-1}+1$ or $x_n = x_{n-1}+1$ and $y_n = y_{n-1}$. Indeed, it is true for $n = 1, 2$; let us assume the above statement true for all numbers smaller than n and let us prove it for n.

Suppose that $a_{n-2} = 2^x+2^y$ and $a_{n-1} = 2^{x+1}+2^y$ (the same thing happens if $a_{n-1} = 2^x+2^{y+1}$). Then if $a_n = 2a_{n-2}$ we will have $a_n = 2^{x+1}+2^{y+1}$; if $a_n = 3a_{n-1}-2a_{n-2}$ we will have $a_n = 2^{x+2}+2^y$. Either way, the proposition is verified, and therefore a_n will always be writable as $2^{x_n}+2^{y_n}$. The problem now becomes very simple if we realize that no number between 1600 and 2000 can be written in this way, since $2^{10}+2^9 = 1536 < 1600 < 2000 < 2048 = 2^{11}$.

23. *M* We will prove the statement by induction over $x_1+\ldots+x_n$ (the base case is trivial). Now, if there are no numbers having more than one digit, then our sum is at most $1+\frac{1}{2}+\ldots+\frac{1}{9}$, which a computation shows to be smaller than 3. Therefore, we can assume that the largest of our n numbers (call it x) has at least 2 digits. Let $x = 10a+d$, where d is its last digit, and because of the property, the number a cannot be among our numbers; consider all the numbers in our sequence whose representations begin with a and have one more digit,

and let these numbers be $x_k, x_l, \ldots$ (where our x is among these) and let all the other numbers in the sequence be $x_e, x_f, \ldots$.

It is easy to see that if we eliminate these numbers and replace them all with a, the property is preserved (no number is the beginning sequence of any other). According to the induction hypothesis, we will have

$$\frac{1}{x_e} + \frac{1}{x_f} + \ldots + \frac{1}{a} < 3 \tag{1}$$

But

$$\frac{1}{a} > \frac{1}{10a} + \frac{1}{10a+1} + \ldots + \frac{1}{10a+9} > \frac{1}{x_k} + \frac{1}{x_l} + \ldots$$

since the numbers $x_k, x_l, \ldots$ are among $10a$, $10a+1$,...,$10a+9$. But then, from (1) we get

$$\frac{1}{x_e} + \frac{1}{x_f} + \ldots. + \frac{1}{x_k} + \frac{1}{x_l} + \ldots < 3$$

which is exactly what we needed to prove.

24. Solution by Andrei Ungureanu

D We will actually prove that there is such a term among the first 1626 terms. So assume for the purpose of contradiction that for all $1 \le k \le 1626$ we have that $a_k \ge 0$. Then,

$$a_{n+1} = a_n - \frac{1}{a_n} \Rightarrow a_{n+1}^2 = a_n^2 + \frac{1}{a_n^2} - 2$$

and

$$a_{n+1} = a_n - \frac{1}{a_n} \Rightarrow a_{n+1} < a_n \Rightarrow a_m < a_n$$

for all $m > n$. By successively applying the recurrence in the hypothesis, for all m and n we will have that

$$a_{m+n} = a_n - \frac{1}{a_n} - \frac{1}{a_{n+1}} - \ldots - \frac{1}{a_{m+n-1}} < a_n - \frac{1}{a_n} - \frac{1}{a_n} - \ldots - \frac{1}{a_n} = a_n - \frac{m}{a_n}$$

But if $a_k < \sqrt{28}$, then by applying the above relation we would have $a_{k+28} < a_k - \frac{28}{a_k} = \frac{a_k^2 - 28}{a_k} < 0$, and if $k \le 1598$ this would contradict the assumption. Therefore, $a_k \ge \sqrt{28}$ for all $1 \le k \le 1598$.

But we know that $a_{n+1}^2 = a_n^2 + \frac{1}{a_n^2} - 2 \le a_n^2 + \frac{1}{28} - 2 \le a_n^2 - \frac{55}{28}$. By successively applying this inequality for $n = 1, \ldots, 1598$ and summing all the inequalities up, we have that:

$$a_{1598}^2 < a_1^2 - 1597 \cdot \frac{55}{28} \Rightarrow a_{1598}^2 \le 56^2 - 3136.96 < 0$$

which is obviously impossible. We thus have a contradiction, and there will exist a k with $1 \le k \le 1626$ such that $a_k < 0$.

25. M Let us assume that $f = ax^2 + bx + c$ and $g = a'x^2 + b'x + c'$ and that the leading coefficients a and a' are positive (without loss of generality). Take $k = \left\lfloor \frac{a'}{a} \right\rfloor$ and then whenever $f(x) \in \mathbb{Z}$, we will also have $h = g - kf \in \mathbb{Z}$. Let $h = \alpha x^2 + \beta x + \gamma$ and note that $0 \le \alpha < a$.

Now, for n large enough, the number $x_n = \dfrac{-b + \sqrt{b^2 - 4ac + 4an}}{2a}$ will be real. Note that

$$x_{n+1} - x_n = \frac{\sqrt{b^2 - 4ac + 4an + 4a} - \sqrt{b^2 - 4ac + 4an}}{2a} =$$
$$= \frac{2}{\sqrt{b^2 - 4ac + 4an} + \sqrt{b^2 - 4ac + 4an + 4}}$$

which is approximately equal to $\dfrac{1}{\sqrt{b^2 - 4ac + 4an}}$ for large enough n. But $h(x_{n+1})$ and $h(x_n)$ are integers (because $f(x_n) = n$), and since from a point onward h will be increasing, we will have $|h(x_{n+1}) - h(x_n)| \ge 1$. But

$$1 \le |h(x_{n+1}) - h(x_n)| = (x_{n+1} - x_n)(\alpha(x_{n+1} + x_n) + \beta)$$

But as n tends to infinity, $\alpha(x_{n+1} + x_n) + \beta$ becomes approximately equal to $2\alpha \cdot \dfrac{\sqrt{b^2 - 4ac + 4an}}{2a}$, so the above inequality become

$$1 \le \frac{1}{\sqrt{b^2 - 4ac + 4an}} \cdot \frac{\alpha}{a} \cdot \sqrt{b^2 - 4ac + 4an} \Rightarrow a \le \alpha$$

This contradicts our assumption that $\alpha < a$, so we can only have $\alpha = 0$. But then the above argument also shows that the second coefficient of h is 0, so h must be an integer constant. And thus, we proved that $g = kf + c$, where k and c are integers.

26. M First of all, by taking $y = -1$ we get $f(-1) = f(x) + f(-x) + f(-1) \Rightarrow f(-x) = -f(x)$, so f is an odd function. But the hypothesis is

$$f(xy + x + y) = f(xy) + f(y) + f(x) \tag{1}$$

and by taking $-y$ instead of y in the hypothesis we will have

$$f(-xy - y + x) = f(-xy) + f(-y) + f(x) \tag{2}$$

By adding (1) and (2) together we get

$$f(xy + x + y) + f(x - y - xy) = 2f(x) \tag{3}$$

and note that for all a and b such that $a + b + 2 \neq 0$ we can find x and y such that $xy + x + y = a$ and $x - y - xy = b$. Indeed, just take $x = \frac{a+b}{2}$ and $y = \frac{a-b}{a+b+2}$, and therefore (3) tells us that for all such a and b we will have that $f(a) + f(b) = 2f\left(\frac{a+b}{2}\right)$. It is easy to note that this holds even if $a + b + 2 \neq 0$, because then we just switch a and b in the previous argument and get a valid result.

But we can apply this back into the hypothesis, and we have that

$$f(xy + x + y) = f(xy) + 2f\left(\frac{x+y}{2}\right)$$

Denote $xy = P$ and $x + y = 2S$ and note that for every $S^2 \geq P$ we have x and y such that $xy = P$ and $x + y = 2S$. So if $S^2 \geq P$ we will always have that $f(P + 2S) = f(P) + 2f(S)$. By making $P = 0$ we get that $f(2S) = 2f(S)$ for all S, and then $f(a) + f(b) = 2f\left(\frac{a+b}{2}\right)$ becomes $f(a) + f(b) = f(a + b)$, exactly what we had to prove.

27. *M* Yes, there exist such functions, and to construct one, let us first discover some of their properties. First of all, note that our function is one-to-one (injective), because $f(y) = f(z)$ implies $f(xf(y)) = f(xf(z)) \Rightarrow \frac{f(x)}{y} = \frac{f(x)}{z} \Rightarrow y = z$. But make $x = y = 1$ and we will get $f(f(1)) = f(1)$, and since f is injective, this yields $f(1) = 1$. By making $x = 1$ we get

$$f(f(y)) = \frac{1}{y} \Rightarrow f(f(f(y))) = f\left(\frac{1}{y}\right) = \frac{1}{f(y)} \tag{1}$$

From $f(f(y)) = \frac{1}{y}$ we also realize that the function is unto (surjective). Now, replace in the hypothesis y by $f\left(\frac{1}{y}\right)$ and (1) tells us that we will have

$$f\left(xf\left(f\left(\frac{1}{y}\right)\right)\right) = \frac{f(x)}{f\left(\frac{1}{y}\right)} \Rightarrow f(xy) = f(x)f(y) \tag{2}$$

so the function is multiplicative. Moreover, if our function is multiplicative and $f(f(y)) = \frac{1}{y}$, it is easy to see that it verifies the hypothesis. Let $p_1 < \ldots < p_n < \ldots$ be the prime numbers, and assign $f(p_{2i-1}) = p_{2i} \Rightarrow f(p_{2i}) = \frac{1}{p_{2i-1}}$. We must, because of (1), have $f\left(\frac{1}{p_{2i-1}}\right) = \frac{1}{p_{2i}}$ and $f\left(\frac{1}{p_{2i}}\right) = p_{2i-1}$. But any rational q has a unique writing as $q = p_{i_1}^{x_1} \ldots p_{i_k}^{x_k}$ where the p_i's are different primes and the x_j's are non-zero integers. Therefore, we can define $f(q) = f(p_{i_1})^{x_1}...f(p_{i_k})^{x_k}$ and it is easy to see that this function verifies (1) and (2).

28. D Let us define the function

$$f_{n,\alpha}(a_1,...,a_{n-1},x) = \frac{4}{x} - \frac{\alpha}{a_1+...+a_{n-1}+x} + 4\left(\sum_{i=1}^{n-1}\frac{1}{a_i}\right) - \sum_{i=1}^{n-1}\frac{i}{a_1+...+a_i}$$

By considering it as a function of just the variable x (so assuming $a_1,...,a_{n-1}$ fixed), we can show (by differentiating, for example) that its minimum is attained when $x = \frac{2(a_1+...+a_{n-1})}{\sqrt{\alpha}-2}$ in the case $\alpha \geq 4$. Therefore, we always have

$$f_{n,\alpha}(a_1,...,a_{n-1},x) \geq$$

$$\geq \frac{2\sqrt{\alpha}-4}{a_1+...+a_{n-1}} - \frac{\alpha}{(a_1+...+a_{n-1})\left(1+\frac{2}{\sqrt{\alpha}-2}\right)} +$$

$$+4\left(\sum_{i=1}^{n-1}\frac{1}{a_i}\right) - \sum_{i=1}^{n-1}\frac{i}{a_1+...+a_i} =$$

$$= \frac{4}{a_{n-1}} - \frac{n+\alpha+3-4\sqrt{\alpha}}{a_1+...+a_{n-1}} + 4\left(\sum_{i=1}^{n-2}\frac{1}{a_i}\right) - \sum_{i=1}^{n-2}\frac{i}{a_1+...+a_i} =$$

$$= f_{n-1,n+\alpha+3-4\sqrt{\alpha}}(a_1,...,a_{n-1}) = f_{n-1,n-1+(\sqrt{\alpha}-2)^2}(a_1,...,a_{n-1})$$

Since this happens for all x, it will also happen for $x = a_n$ and we will therefore have

$$f_{n,\alpha}(a_1,...,a_n) \geq f_{n-1,n-1+(\sqrt{\alpha}-2)^2}(a_1,...,a_{n-1}) \quad (1)$$

and what we need to prove is that $f_{n,n}(a_1,...,a_n) \geq 0$. If we define $\alpha_n = n$ and $\alpha_{k-1} = k-1+(\sqrt{\alpha}_k-2)^2$, note that $\alpha_1,...,\alpha_n$ are all non-negative and taking their square root makes sense. Moreover, $\alpha_k \geq k$. But by iterating (1) we will have that

$$f_{n,\alpha_n}(a_1,...,a_n) \geq f_{n-1,\alpha_{n-1}}(a_1,...,a_{n-1}) \geq ... \geq f_{3,\alpha_3}(a_1,a_2,a_3)$$

We had to stop at $k = 3$ because we determined that the minimum is what we found it to be only for $\alpha \geq 4$. So what we need to prove is that

$$f_{3,\alpha_3}(a_1,a_2,a_3) \geq 0 \Leftrightarrow \frac{4}{a_1} + \frac{4}{a_2} + \frac{4}{a_3} - \frac{1}{a_1} - \frac{1}{a_1+a_2} - \frac{\alpha_3}{a_1+a_2+a_3} \geq 0 \quad (1)$$

Yet we can prove by induction that $\alpha_{n-i} \le (n-i-1)^2$ for all $0 \le i \le n-3$. Indeed, this is verified for $i = 0$, and if we have that $\alpha_{n-i+1} \le (n-i)^2$ then

$$\alpha_{n-i} = n - i + (\sqrt{a_{n-i+1}} - 2)^2 \le n - i + (n - i - 2)^2 =$$
$$= (n-i)^2 - 3(n-i) + 4 \le (n-i-1)^2$$

for all $n - i \ge 3$. Then we have $\alpha_3 \le 4$, and from here it is easy to see that (1) must hold.

29. E We will prove that the only such functions are constant functions (which obviously verify). Take an arbitrary n and construct the sequence m_i as follows: let m_0 be any natural number, and let $m_{i+1} = m_i^2 + n^2 > m_i$. It is easy to see from the hypothesis that $f(m_{i+1})$ is contained between $f(m_i)$ and $f(n)$, and therefore between $f(m_0)$ and $f(n)$. So the $f(m_i)$'s take on values between two fixed integers ($f(n)$ and $f(m_0)$), and thus there will be a subsequence of m_i (call it m_{i_k}, on which f will be constant. So $f(m_{i_k}) = a$ for any k.

    Now, assume that f is not constant, and then there will exist a n such that $f(n) = a + r$ with $r \ne 0$. But for any k,

$$(m_{i_k} + n) f(m_{i_k}^2 + n^2) = m_{i_k} f(n) + n f(m_{i_k}) \Rightarrow m_{i_k} + n | m_{i_k}(a + r) + na \Rightarrow$$
$$\Rightarrow m_{i_k} + n | (a + r)(m_{i_k} + n) - rn \Rightarrow m_{i_k} + n | rn$$

    But this last statement is impossible for all k, since an entire sequence of increasing positive integers cannot all divide one number. Therefore, f is constant throughout.

30. M First of all, we will prove the problem when S is the set of all even indices and we will show afterward that the whole problem reduces itself to this case. Therefore, let $S = \{2, 4, .., 2k\}$ and let $T_i = x_1 + ... + x_i$ (where $T_0 = 0$); what we have to prove is that

$$(a_2 + a_4 + ... + a_{2k})^2 \le \sum_{1 \le i \le j \le 2k} (a_i + ... + a_j)^2$$

    Actually, the RHS can have even more terms, if $n = 2k + 1$, but they are all non-negative so the inequality will remain true. Written in terms of the T_i, the above statement becomes

$$(T_2 - T_1 + T_4 - T_3 + ... + T_{2k} - T_{2k-1})^2 \le \sum_{i \le j} (T_j - T_{i-1})^2 \Leftrightarrow$$
$$\Leftrightarrow (T_2 + ... + T_{2k})^2 + (T_1 + ... + T_{2k-1})^2 - 2(T_2 + ... + T_{2k})(T_1 + ... + T_{2k-1}) \le$$
$$\le 2k(T_1^2 + ... + T_{2k^2}) - 2\sum_{i<j} T_i T_j \Leftrightarrow$$

$$\Leftrightarrow T_1^2 + ... + T_{2k}^2 + 2\sum_{i<j} T_{2i}T_{2j} + 2\sum_{i<j} T_{2i-1}T_{2j-1} - 2\sum_{i,j} T_{2i}T_{2j-1} \leq$$
$$\leq 2k(T_1^2 + ... + T_{2k^2}) - 2\sum_{i<j} T_iT_j \Leftrightarrow$$
$$\Leftrightarrow 4\sum_{i<j} T_{2i}T_{2j} + 4\sum_{i<j} T_{2i-1}T_{2j-1} \leq (2k-1)(T_1 + ... + T_{2k})^2$$

which is true, since $4\sum_{i<j} T_{2i}T_{2j} \leq 2(k-1)(T_2^2 + ... + T_{2k}^2)$ and $4\sum_{i<j} T_{2i-1}T_{2j-1} \leq 2(k-1)(T_1^2 + ... + T_{2k-1}^2)$. Now, we need to get from this special case to the general case. We will assume that S is composed of blocks of consecutive integers which we will denote by $B_2, B_4, ..., B_{2k}$. But between blocks B_{2i} and B_{2i+2} there will exist consecutive indices not in S, and let B_{2i+1} be that block. Let B_1 and B_{2k+1} be the blocks before B_2 and after B_{2k} respectively.

If b_i will be the sum of the numbers in block B_i, then the sum in LHS will be $(b_2 + b_4 + ... + b_{2k})^2$. From the particular case, we know that this is at most the sum of squares of the form $(b_i + ... + b_j)^2$. But it is easy to see that all these terms are among the terms in the RHS of what we have to prove (alongside other non-negative squares). Therefore, the inequality is proven.

31. E Let the roots of the polynomial be $-a_1, ..., -a_n$. Since all the coefficients are non-negative, f will always be positive for positive values of x, so it does not change sign on $[0, \infty)$. Therefore, all its roots will be negative, and $f(2) = (2 + a_1)(2 + a_2)...(2 + a_n)$, where the a_i's are positive numbers whose product is 1. But

$$f(2) = \sum_{k=0}^{n} \left(2^{n-k} \sum_{1 \leq i_1 < ... < i_k \leq n} a_{i_1}a_{i_2}...a_{i_k} \right)$$

By applying the $AM - GM$ inequality to the numbers $a_{i_1}a_{i_2}...a_{i_k}$ we get that $\sum_{1 \leq i_1 < ... < i_k \leq n} a_{i_1}a_{i_2}...a_{i_k} \geq \binom{n}{k}$ and therefore we will have

$$f(2) \geq \sum_{k=0}^{n} 2^{n-k}\binom{n}{k} = 3^n$$

32. M We will prove by induction over k the following statement: for any such sequence given by $a_0 = 0$, $a_1 = 2\alpha + 1$ and $a_n = (4x + 2)a_{n-1} + (2y + 1)a_{n-2}$ (where $\alpha, x, y \in \mathbb{Z}$), and for all n between 1 and 2^k we have that $\frac{a_n}{n}$ is an irreducible fraction with both numerator and denominator odd. It is trivial for $k = 1$ and now, let us assume it true for $k - 1$ and prove it for k.

First of all, $a_n \equiv_2 a_{n-2}$ so it is easy to prove by induction that for all even n, a_n is even and for all odd n, a_n is odd. Therefore, we have solved our problem for odd values of n and now we must prove it for even values. Since for all $1 \leq m \leq 2^{k-1}$ the number a_{2m} is even, let $a_{2m} = 2b_m$. Now, we want to find a recurrence for the sequence b_m; if r, s are the roots of the equation $z^2 - (4x+2)z - (2y+1) = 0$, then a_n is of the form $c_1 r_1^n + c_2 r_2^n$. Therefore,

$$b_m = \frac{a_{2m}}{2} = \frac{c_1}{2} r_1^{2m} + \frac{c_2}{2} r_2^{2m}$$

and b_m will be given by a second-degree linear recurrence, the roots of whose characteristic equation are r_1^2 and r_2^2. That recurrence will be

$$\begin{aligned} b_m &= (r_1^2 + r_2^2) b_{m-1} + r_1^2 r_2^2 b_{m-2} = \\ &= ((4x+2)^2 - (4y+2)) b_{m-1} - (2y+1)^2 b_{m-2} \end{aligned}$$

and $b_0 = 0$, $b_1 = \frac{a_2}{2} = (2x+1)(2\alpha+1)$. Thus, b_m will be a sequence of the same form and we can apply the induction hypothesis to it. Therefore, for all m between 1 and 2^{k-1}, $\frac{b_m}{m}$ will have both numerator and denominator odd. For any $2m$ between 1 and 2^k we will have $\frac{a_{2m}}{2m} = \frac{b_m}{m}$, a fraction with odd numerator and denominator. Since we already proved this statement for odd n, it is true for all n between 1 and 2^k, and the induction is complete.

33. D We can easily solve this problem if we can just break up the expressions $a^3+b^3+c^3-1$ and $a^5+b^5+c^5-1$ into simpler terms. Let us take the polynomial in the variables X and Y $f(X,Y) = (1-X-Y)^3+X^3+Y^3-1$. Note that $f(X,1) = f(1,Y) = f(X,-X) = 0$, and this means that f is divisible by the polynomials $(X-1)$, $(Y-1)$ and $(X+Y)$. Therefore, $(X-1)(Y-1)(X+Y) | f(X,Y)$, but since the degree of f is 3 we must have $f(X,Y) = \alpha(X-1)(Y-1)(X+Y)$. The constant α can be determined by the coefficient of X in f, which is -3. Therefore, we have $f(X,Y) = -3(X-1)(Y-1)(X+Y)$, and by letting $X = b, Y = c$ we will have

$$\begin{aligned} a^3 + b^3 + c^3 - 1 &= (1-b-c)^3 + b^3 + c^3 = \\ &= -3(b-1)(c-1)(b+c) = 3(a-1)(b-1)(c-1) \qquad (1) \end{aligned}$$

Now, to do the same thing for $g(X,Y) = (1-X-Y)^5 + X^5 + Y^5 - 1$. For the exact same reason, the polynomials $X-1$, $Y-1$ and $X+Y$ will divide g, and because g has degree 5 it will be equal to $(X-1)(Y-1)(X+Y)$ times some polynomial of degree two. So we will have

$$g(X,Y) = (X-1)(Y-1)(X+Y)(\alpha X^2 + \beta Y^2 + \gamma XY + \delta X + \epsilon Y + \zeta) \qquad (2)$$

and we need only to find those coefficients. But the coefficient of X in g is -5, while in the decomposition of (2) it is ζ (X can obtained in (2) only by taking -1 from the first and second parenthesis, X from the third, and ζ

from the fourth). Therefore $\zeta = -5$; now, identify the coefficients of X^2 on both sides and we will get $10 = -\zeta + \delta \Rightarrow \delta = 5$ and similarly $\epsilon = 5$. But identify the coefficients of X^4Y and get $\alpha = -5$ and similarly $\beta = -5$. Finally, by identifying X^3Y^2 we get $-10 = \gamma + \alpha$ and so $\gamma = -5$. Our decomposition is thus

$$\begin{aligned}&(1-X-Y)^5 + X^5 + Y^5 - 1 = \\ &= (X-1)(Y-1)(X+Y)(-5X^2-5Y^2-5XY+5X+5Y-5) = \\ &= -5(X-1)(Y-1)(X+Y)(X^2+Y^2+XY-X-Y+1)\end{aligned}$$

Therefore, letting $X = b$, $Y = c$ and $X + Y = 1 - a$ we will have $a^5 + b^5 + c^5 - 1 = 5(a-1)(b-1)(c-1)(b^2+c^2+bc-b-c+1)$, and what we need to prove is that

$$30|(a-1)(b-1)(c-1)| \leq 45|(a-1)(b-1)(c-1)||b^2+c^2+bc-b-c+1| \quad (3)$$

But for all b, c, by denoting $b + c = S$ and $bc = P$ we will have $|P| \leq \frac{S^2}{4}$ and therefore

$$\begin{aligned}&b^2+c^2+bc-b-c+1 = S^2 - S + 1 - P \geq \\ &\geq \frac{3S^2}{4} - S + \frac{1}{3} + \frac{2}{3} = \left(\frac{S\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\right)^2 + \frac{2}{3} \geq \frac{2}{3}\end{aligned}$$

By using this inequality, (3) is obviously true, and our problem is thus solved.

34. Solution by Andrei Ungureanu

D We will show that the answer is no, so assume for the purpose of contradiction that there is such an f. Then, $f(2) = 2$ as it can be seen from letting $x = 1$, and let $2 + a = \sup|f(x)|$, where $a \geq 0$ (we know that $\sup \geq 2$, since $f(2) = 2$ and that it is finite since the function is bounded). Assume that $a > 0$, and then pick x such that $|f(x)| = 2 + a - \varepsilon$, where $\varepsilon < \frac{a^2}{2(2+a)}$ (this is possible, as $2 + a$ is the sup of $|f|$) and plugging $\frac{1}{x}$ in the relation gives us

$$f\left(x^2 + \frac{1}{x}\right) \geq -2 - a + (2 + a - \varepsilon)^2 = 2 + 3a + a^2 - 2\varepsilon(2+a) + \varepsilon^2 > 2 + a$$

which is impossible; so $\sup|f|$ cannot be greater than 2, and therefore it must be 2. But, then letting $x = \frac{1}{2}$ yields

$$f\left(\frac{9}{2}\right) \geq f\left(\frac{1}{2}\right) + 4 \geq 2$$

and since $f\left(\frac{9}{2}\right) \le 2$ the above can only happen when $f\left(\frac{1}{2}\right) = -2$. Letting $x = 2$ gives us

$$f\left(\frac{9}{4}\right) = 6$$

which is impossible. Therefore, there are no such functions.

35. *M* First of all, note that f is one-to-one, because $f(n) = f(p)$ obviously implies $n = p$ (from the hypothesis, setting $m = 1$). Now, we will have

$$f(f(m + f(n))) = f(n + f(m + 2005)) = m + 2005 + f(n + 2005)$$

and for any $p > f(n)$, by letting $m = p - f(n)$, we will have $f(f(p)) = f(n+2005) - f(n) + p + 2005$. But for any m and n, let $p > f(m)$ and $p > f(n)$ and then $f(f(p)) = f(n+2005) - f(n) + p + 2005 = f(m+2005) - f(m) + p + 2005$. Therefore, for any m and n we will have $f(m+2005) - f(m) = f(n+2005) - f(n)$, so this quantity will always be a constant c. Then, we have $f(n+2005) - f(n) = c$ for all n and $f(f(p)) = p + 2005 + c$ for large enough p.

But $f(n + 2005) - f(n) = c$, so if we let $f(i) = \alpha_i$ for all $1 \le i \le 2005$ we will have $f(2005k + i) = kc + \alpha_i$ for any k. Of course, the α_i's are different modulo c (f is one-to-one), so the image of our function will consist of 2005 distinct arithmetic progressions of ratio c (each of them generated by one of the α_i's).

But by letting $m = 1$ in the hypothesis we have $f(1+f(n)) = n+f(1+2005)$, so all but a finite number of positive integers are in the image of f (which will be 'almost' all of $\mathbb{N}$). Therefore, we must have at least c disjoint progressions of ratio c to cover the image of f, and then $2005 \ge c$. But if $2005 > c$, then two of the 2005 numbers α_i's will have the same remainder when divided by c, which contradicts injectivity (since two of the progressions would overlap). Therefore, we must have $c = 2005$ and $f(2005k + i) = 2005k + \alpha_i$; moreover, $f(f(p)) = p + 2 \cdot 2005$ for sufficiently large p. Therefore, for any n, $n + 2005k$ will be large enough for a certain k and $f(f(n+2005k)) = n+2005k+2\cdot 2005$ implies $f(f(n)) = n + 2 \cdot 2005$ for all n. So by replacing n by $f(n)$ in the hypothesis we get

$$f(m + f(f(n))) = f(n) + f(m + 2005) \Rightarrow$$
$$\Rightarrow f(m + n + 2 \cdot 2005) = f(n) + f(m + 2005) \Rightarrow$$
$$\Rightarrow f(m + n) = f(m) + f(n) - 2005 \Rightarrow f(m + 1) = f(m) + f(1) - 2005$$

From here it is very easy to prove by induction that $f(m) = (m-1)(f(1) - 2005) + f(1)$. But since the function takes on all values but a finite number of them, we must have $f(1) - 2005 = 1$ and then our function will be $f(m) = m + 2005$.

36. E What we need to prove is equivalent to

$$\frac{1-x}{1+x+x^2+x^3} + \frac{1-y}{1+y+y^2+y^3} + \frac{1-z}{1+z+z^2+z^3} \geq 0 \tag{1}$$

We will use Chebysev's inequality, because the sequences $1-x, 1-y, 1-z$ and $\frac{1}{1+x+x^2+x^3}, \frac{1}{1+y+y^2+y^3}, \frac{1}{1+z+z^2+z^3}$ are ordered in the same way. The inequality tells us that

$$\frac{1-x}{1+x+x^2+x^3} + \frac{1-y}{1+y+y^2+y^3} + \frac{1-z}{1+z+z^2+z^3} \geq$$
$$\geq (1-x+1-y+1-z) \cdot$$
$$\cdot \left(\frac{1}{1+x+x^2+x^3} + \frac{1}{1+y+y^2+y^3} + \frac{1}{1+z+z^2+z^3} \right) = 0$$

37. M Letting $x = y = 0$ we get $f(0) = 0$. and $y = 0$ gives us $f(x^2) = xf(x)$. Letting $x = 0$ yields $f(-y^2) = -yf(y) = -f(y^2)$ and therefore our function is odd. Denote $x - y = a$ and $x + y = b$, and therefore for all $a, b \in \mathbb{R}$ we will have

$$f(ab) = a\left(f\left(\frac{a+b}{2}\right) + f\left(\frac{b-a}{2}\right)\right) \Rightarrow \frac{f(ab)}{a} - f\left(\frac{b-a}{2}\right) = f\left(\frac{a+b}{2}\right)$$

But this same relation with a and b switched gives us

$$\frac{f(ab)}{a} - f\left(\frac{b-a}{2}\right) = \frac{f(ab)}{b} - f\left(\frac{a-b}{2}\right) \Rightarrow$$
$$\Rightarrow f(ab)\left(\frac{1}{a} - \frac{1}{b}\right) = f\left(\frac{b-a}{2}\right) - f\left(\frac{a-b}{2}\right) \Rightarrow$$
$$f(ab)\frac{b-a}{ab} = 2f\left(\frac{b-a}{2}\right) \Rightarrow \frac{f(ab)}{ab} = \frac{f\left(\frac{b-a}{2}\right)}{\frac{b-a}{2}} \tag{1}$$

for all $0 \neq a \neq b \neq 0$ (because we divided by a, b and $b - a$). But for what positive x and y can we find such a and b with $ab = x$ and $\frac{b-a}{2} = y$? We need to have $b = \frac{x}{a}$ and

$$\frac{x}{a} - a = 2y \Rightarrow a^2 + 2ay - x = 0$$

and this always has a real solution. Therefore, for all positive x and y we will have $\frac{f(x)}{x} = \frac{f(y)}{y}$. So this expression is constant, and therefore for all positive x we will have $f(x) = ax$. Because the function is odd, this will hold for all x and this will be our solution (which obviously satisfies).

38. SOLUTION BY ANDREI STEFANESCU

D We call any polynomial $P(X) = a_nX^n + ... + a_0$ symmetrical iff $a_i = a_{n-i}$ for all i. We will use the following lemma:

LEMMA If P is a polynomial which can be written as $P = Q \cdot R$, where Q and R are symmetrical polynomials, then P is symmetrical too.

Let $Q(X) = b_sX^s + ... + b_0$ and $R(X) = c_tX^t + ... + c_0$. We have that $a_i = \sum_{j=0}^{i} b_j c_{i-j}$ and $a_{n-i} = \sum_{j=0}^{i} b_{s-j} c_{t-(i-j)}$. Since Q and R are symmetrical, it follows that $b_j = b_{s-j}$ and $c_{i-j} = c_{t-(i-j)}$. Hence $a_i = a_{n-i}$ for any i, and thus P is symmetrical.

Let $f(X) = X^n + X^{n-1} ... + 1$ and assume that $f = g \cdot h$, where g and h are nonconstant polynomials with nonnegative coefficients. Let $g(X) = b_sX^s + ... + b_0$ and $h(X) = c_tX^t + ... + c_0$. Since $f = g \cdot h$, all the complex roots of g and h have absolute value 1. Also, if $g(\epsilon) = 0$, then $g(\overline{\epsilon}) = g\left(\frac{1}{\epsilon}\right) = 0$. It follows that both g and h are a product of symmetric terms of the form $(X - \epsilon)\left(X - \frac{1}{\epsilon}\right) = X^2 - 2\mathrm{Re}(\epsilon) + 1$ or $X + 1$, hence by lemma g and h are symmetrical.

If all the coefficients of g and h are in $\{0, 1\}$ we are done. Otherwise, let k be the least number such that $\{b_k, c_k\} \not\subseteq \{0, 1\}$. Since $b_s = c_t = 1$, it follows that $b_0 = c_0 = 1$, thus $k \geq 1$. As g is symmetrical, $b_k = b_{s-k}$. Computing the coefficient of X^s gives us $1 = b_sc_0 + \ldots + b_{s-k}c_k + \ldots + b_0c_s$. Since $b_s = c_0 = 1$ and all the terms of the sum are nonnegative, we obtain $b_{s-k}c_k = b_kc_k = 0$, thus one of b_k and c_k (say c_k) is 0. Computing the coefficient of X^k, we have $1 = b_kc_0 + \ldots + b_0c_k$. As $\{b_i, c_i\} \subseteq \{0, 1\}$ for $i < k$ and $c_k = 0$, all the terms but b_kc_0 must be in $\{0, 1\}$. It follows that $b_kc_0 = b_k$ must be in $\{0, 1\}$, which leads to a contradiction with the definition of k.

APPENDIX 1: USEFUL FACTS

1. APOLLONIAN CIRCLES Given a segment AB, and a positive number k, the locus of the points M such that $\frac{MA}{MB} = k$ is a circle with the center on AB, called one of the Apollonian circles of AB. More precisely, let E and F be the points inside and respectively outside the segment AB such that $\frac{EA}{EB} = \frac{FA}{FB} = k$. We will show that our Apollonian circle is the circle of diameter EF.

   To do that, note that because $\frac{EA}{EB} = \frac{MA}{MB} = \frac{FA}{FB}$, then ME and MF must be the two bisectors of the angle A in the triangle ABC. But the two bisectors are always perpendicular, so $\angle EMF = \frac{\pi}{2}$ and therefore M lies on the circle of diameter EF. Proving the converse, that every point on that circle satisfies $\frac{MA}{MB} = k$, is just a computation with angles and the Law of Sines.
2. BARYCENTRIC COORDINATES Given a triangle ABC, we can define the barycentric coordinates of a point X inside it as the triplet (x_a, x_b, x_c) with $x_a = \frac{d_a}{h_a}$ and similar definitions for b and c. Here, d_a is the distance from X to BC and h_a is the altitude from A. Note that $x_a = \frac{[XBC]}{[ABC]}$ and because the areas $[XAB], [XBC]$ and $[XAC]$ sum up to $[ABC]$, we will always have $x_a + x_b + x_c = 1$.

   So it turns out that every point inside the triangle only depends on two of these coordinates, the other being implied by the fact that the sum of all three is 1. An important fact is that given X, Y, Z inside the triangle ABC (determined by coordinates $(x_a, x_b, x_c), (y_a, y_b, y_c)$ and (z_a, z_b, z_c) respectively), X, Y, Z are collinear only if the coordinates of one of them is a convex sum of those of the other two. The fact that the coordinates of Y to be a convex sum of those of X and Z means that there exists an α between 0 and 1 such that

$$y_a = \alpha x_a + (1 - \alpha) z_a$$

$$y_b = \alpha x_b + (1 - \alpha) z_b$$

$$y_c = \alpha x_c + (1 - \alpha) z_c$$

   Of course, since the coordinates always sum up to 1, it is enough to verify two of these equations and the third one will be implied.

   We can also define barycentric coordinates for points outside the triangle, just that we take $x_a = -\frac{d_a}{h_a}$ if X is on the opposite side of BC as A is. It is easy to verify that the sum of the barycentric coordinates is also 1 in this case.

The barycentric coordinates of some of the most important points inside the triangle are: the centroid G has $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the incenter I has $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$, the circumcenter O has $\left(\frac{\cos A}{2\sin B\sin C}, \frac{\cos B}{2\sin A\sin C}, \frac{\cos C}{2\sin A\sin B}\right)$ and the orthocenter H has coordinates $(\cot B\cot C, \cot A\cot C, \cot A\cot B)$.

3. CHEBYSHEV'S THEOREM There exists a prime number between x and $2x$ for any real $x \geq 1$.
4. CHINESE REMAINDER LEMMA This very important (and somehow underrated) theorem in Number Theory states that given any pairwise co-prime integers $a_1, ..., a_n$ and any integers $b_1, ..., b_n$ the system of equations $x \equiv_{a_i} b_i$, $1 \leq i \leq n$ has a unique solution for $0 \leq x \leq a_1a_2...a_n$. Of course, since x solution implies $x + ka_1a_2...a_n$ solution for any k, then our solutions form an arithmetic progression of ratio k.

   The problem can also be extended to the case when $a_1, ..., a_n$ are not pairwise co-prime, but then the existence of solutions is subject to certain conditions.
5. CONGRUENCES AND $\mathbb{Z}_n$ We say that $x \equiv_n y$ (and read it "x congruent to y modulo n") iff $n|x - y$, which is the same as saying that x and y have the same remainder when divided by n. Now, we can define the set $\mathbb{Z}_n$ of remainders modulo n as the set $\{0, 1, 2, ..., n-1\}$ such that when we take the sum or the product of two numbers in $\mathbb{Z}_n$, the result is still in $\mathbb{Z}_n$. So, if $a, b \in \mathbb{Z}_n$ we will define their sum and product as the remainder modulo n of the sum and product of the corresponding integers a and b. For example, in $\mathbb{Z}_6$ (the remainders modulo 6) we will have $2 + 5 = 1$ and $2 \cdot 5 = 4$. Thus, even if they are closely related, it is important to make a distinction between the element $a \in \mathbb{Z}_n$ and the integer $a \in \mathbb{Z}$.

   The set $\mathbb{Z}_n$ together with the sum and product as defined above inherits many of its properties from $\mathbb{Z}$, including commutativity, identity elements (0 in the identity element of addition, because $x+0 = 0+x = x$ and 1 is the identity for multiplication, because $1 \cdot x = x \cdot 1 = x$) and inverse elements for addition $x + (-x) = (-x) + x = 0$. But the truly interesting case happens when n is prime, and in that case $\mathbb{Z}_n$ has the extra property that the multiplication has an inverse element; that is, for every $x \in \mathbb{Z}_n$ different from 0 there is some y such that $xy = yx = 1$. That y is an element of $\mathbb{Z}_n$ and it is called x^{-1}.
6. DIAMETER OF A POINT SET Given a set of points M, a diameter is a segment AB with $A, B \in M$ which is longer or equal to any other such segment

7. DILATION The dilation of center P and of ratio k is a geometric transformation which transforms any point A into an A' such that $\overrightarrow{PA} = k\overrightarrow{PA}$. This is a very useful and natural transformation which transforms any set of points into a set of points of the same size, but larger (or smaller if $k < 1$). Thus, circles are transformed into circles, triangles into triangles and any line into another line parallel to it. Dilation also preservers angles.

   LEMMA An important fact about dilation that is used in this book is that it two non-congruent triangles have pairwise parallel sides, then the triangles can be obtained one from the other through a dilation. This statement is easily proved, by taking the triangles to be ABC and $A'B'C'$ and M be $AA' \bigcap B'B'$; of course, if this intersection does not exist, then just pick M to be the intersection of BB' and CC'. If that intersection didn't exist either, then we would have $AA' \parallel BB' \parallel CC'$, which implies that $AA'B'B$, $BB'C'C$ and $CC'A'A$ are parallelograms. This contradicts our hypothesis, as it would give $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ (contradicting the fact that the triangles are non-congruent).

   So if M is the intersection of AA' and BB', take the dilation around M that sends A into A'. The line AB will be sent into a line parallel to it which passes through A'; but that line is $A'B'$, and since M, B, B' are collinear, the only possible image of B is B'. But then, the lines AC and BC are mapped into lines parallel through them which pass through A' and B' respectively. But these lines are $A'C'$ and $B'C'$, so the image of the intersection of AC and BC must be the intersection of their images. In other words, the image of C is C' and thus we obtained $A'B'C'$ from ABC through a dilation around M.
8. DIRICHLET'S THEOREM This theorem states that in any arithmetic progression with the ratio and the first term co-prime there will be an infinity of prime numbers. Though seemingly simple, the proof of this statement is quite non-elementary and thus we don't recommend using it in competitions, except as a last resort.

   We use the theorem to give quite a simple proof of the fact that any number which is quadratic residue modulo all primes must be a perfect square. This fact also has elementary solutions, but which are quite more complicated than the one using Dirichlet's Theorem.
9. EULER CYCLES AND HAMILTONIAN CYCLES A Hamiltonian cycle in a graph is a simple cycle (so without self-intersections) that contains all the vertices of the graph. An Euler cycle is a cycle that contains all the edges of a graph exactly once (where some vertices may appear more than once in our graph). It can be shown that a graph contains an Euler cycle iff all its vertices have even degree (the necessity is quite easy to see).

Moreover, we can define Hamiltonian and Euler cycles in directed graphs as well, where they are cycles with respect to the orientation (meaning that we only move in the direction of the arrows).

10. FERMAT'S LITTLE THEOREM AND EULER'S THEOREM Probably the most important theorem in elementary number theory is Fermat's Little Theorem, which claims that $a^{p-1} \equiv_p 1$ for any integer a and every prime p such that $(a,p) = 1$. The theorem admits a generalization called Euler's Theorem, which claims that $a^{\varphi(n)} \equiv_n 1$ for any co-prime integers a and n. The exponent $\varphi(n)$ is called the Euler function, and is the number of positive integers smaller than n and co-prime to it. If the decomposition of n into prime factors is $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)...\left(1 - \frac{1}{p_k}\right)$$

The proofs of both these theorems and the proof for the formula of Euler's function can be found in any elementary number theory book or course.

11. FORMAL DERIVATIVE OF A POLYNOMIAL Though this notion comes from analysis, one can understand it even within the range of elementary mathematics. Indeed, given any polynomial

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + ... + a_0$$

we define its formal derivative as

$$P'(X) = n a_n X^{n-1} + (n-1) a_{n-1} X^{n-2} + ... + a_1$$

The way we obtained this polynomial is by taking every term of the form cX^i of P and turning it into $i \cdot cX^{i-1}$ and turning any constant term cX^0 into 0. It is easy to verify that the formal derivative satisfies the properties $(P \pm Q)' = P' \pm Q'$ and the product rule $(PQ)' = P'Q + Q'P$. We can also define $P^{(i)}$ as P differentiated successively i times; of course $P^{(0)} = P$ (P differentiated 0 times is just P itself).

The importance of the derivative P' of a polynomial is that if $P(X) = a(X - x_1)(X - x_2)...(X - x_n)$ then

$$P'(X) = a \sum_{i=1}^{n} \frac{(X - x_1)(X - x_2)...(X - x_n)}{X - x_i}$$

By using this property, one can prove by induction over i that a certain x_1 is a root of P of multiplicity i (meaning that the term $(X - x_1)$ appears i times in the factorization of P) iff $P(x_1) = 0$, $P^{(1)}(x_1) = 0$,..., $P^{(i-1)}(x_1) = 0$ and $P^{(i)}(x_1) \neq 0$.

12. FORMULAS IN A TRIANGLE In a triangle of side-lengths a, b, c and area, circumradius, inradius and semiperimeter S, R, r, p, the following identities hold:

$$r = \frac{S}{p}, S = \frac{abc}{4R}, S = \sqrt{p(p-a)(p-b)(p-c)}$$
$$a^2 = b^2 + c^2 - 2bc\cos A \text{ (Law of Cosines)},$$
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ (Law of Sines)}$$
$$\cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}}, \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$$
$$m_a = \frac{2b^2 + 2c^2 - a^2}{4}, l_a = \frac{2\sqrt{bcp(p-a)}}{b+c}$$

where m_a is the length of the median from A and l_a that of the bisector.

13. GRAPH The graph is one of the basic concepts of combinatorics, and is basically just a finite set of points (called vertices) and a set of edges joining certain pairs of this points. A graph can be drawn in a plane, but its actual physical representation bears no importance. The only thing that matters in a graph are the set of vertices and which pairs of them are connected by edges. We will begin by introducing the basic language of graph theory.

The degree of an edge X in a graph G is just the number of neighbors of X, where a neighbor of X is any vertex of the graph which is connected to it by an edge. An easy exercise is that the sum of the degrees of all vertices in a graph equals the number of edges in the graph.

A path in a graph G is a succession of vertices $v_0, v_1, \ldots, v_k$ such that any two consecutive vertices are connected. If $v_0 = v_k$ the path is called a cycle. A graph in which any two vertices are the endpoints of a path is called connected.

A complete graph with n vertices (called K_n) is one which has all pairs of its vertices connected by edges. A subgraph of a graph is a subset of the set of vertices together with all the edges between those vertices. A k-clique of a graph G is a complete subgraph with k vertices of G.

A bipartite graph $K_{a,b}$ is one whose vertices can be split into two disjoint parts A and B with $\#A = a$ and $\#B = b$, such that all the edges in the graph go from an element from A to an element for B (meaning that there are no edges inside the sets A and B).

An directed graph is just a graph where every edge is given a sense of direction (usually represented by an arrow). Thus, in an oriented graph an arrow from vertex X to vertex Y is not considered the same as an arrow from vertex Y to vertex X. The outdegree of a certain vertex is the number of arrows that point away from it, while the indegree is the number of arrows that point toward it.

14. I.E. "That is"
15. IFF "If and only If"
16. INF The infimum of a function $f(x)$ (denoted by $\inf f$) is the greatest real number which is less than of equal to all the values of f. If there is no such real number, we consider the supremum to be $-\infty$.
17. INVERSION The inversion around a point P in the plane is a geometric transformation that sends a point $A \neq P$ in a point A' (called the inverse of A) such that $\overline{PA} = \overline{PA'} \cdot \frac{k}{PA'^2}$ for a fixed $k \neq 0$. Though its definition might seem a bid unnatural, it is a very useful transformation. We list here several of its properties:

    1)It is an involution, meaning that if A' is the inverse of A, then A is the inverse of A'. The same thing can be said about sets of points, such as geometric shapes.

    2)It turns circles that pass through P into lines and lines that don't pass through P and lines that don't pass through P into circles that pass through P. Given a circle that passes through P, it will be mapped into a line which is perpendicular to the diameter from P.

    3)It keeps lines that pass through P invariant and it turns circles that don't pass through P into circles that don't pass through P.

    4)If A' and B' are the inverses of A and B, then $A'B' = \frac{k^2 AB}{OA \cdot OB}$.

    5)It is a conformal transformation, meaning that it preserves the angles between curves. The angle between any two curves (like circles, lines etc.) at a certain point is the angle between their tangents at that point.
18. ISOGONAL Given a triangle ABC and a cevian AM, its isogonal is a cevian AN such that $\angle MAB = \angle NAC$. Note that isogonal cevians AM and AN are symmetric with respect to the bisector of the angle A. The following fact about isogonals is used in one of the problems, and it can be easily proven by applying the Law of Sines in the triangles ABM, ACM, ABN, ACN:

    LEMMA If M and N are points on the segment BC of a triangle ABC, then AM and AN are isogonal iff

$$\frac{BM}{CM}\frac{BN}{CN} = \frac{AB^2}{AC^2}$$

19. LHS "Left Hand Side" of an equation, inequality, identity etc.
20. LINEAR RECURRENCE RELATIONS Consider a sequence $(x_n)_{n \in \mathbb{N}_0}$ which satisfies the recurrence $x_{n+k} = a_{k-1}x_{n+k-1} + a_{k-2}x_{n+k-2} + \ldots + a_0 x_n$ (called a k-th order linear recurrence). We can generally determine the form of x_n in terms of n, the first k terms $x_0, \ldots, x_{k-1}$ and the numbers $a_0, \ldots, a_{k-1}$. More precisely, if $r_1, \ldots, r_k$ are the roots of the polynomial $X^k - a_{k-1}X^{k-1} - a_{k-2}X^{k-2} - \ldots - a_0$

(called the characteristic polynomial of the recurrence) and if all the roots are distinct, then

$$x_n = c_1 r_1^n + c_2 r_2^n + \dots + c_k r_k^n$$

where the constants $c_1, \dots, c_k$ are given by the initial values $x_0, \dots, x_{k-1}$. It is easy to check that the above formula satisfies the recurrence.

However, a problem arises when the roots are not all distinct. Then, the formula is a bit more complicated, as it also includes powers of n. But for a second degree linear recurrence $x_{n+2} = a_1 x^{n+1} + a_0 x^n$ with r as the only root of the characteristic polynomial $X^2 - a_1 X - a_0$, the form of x_n will be

$$x_n = c_1 r^n + n c_2 r^n$$

where the constants c_1 and c_2 are determined by the terms x_0 and x_1.

21. Orders and Primitive Roots in $\mathbb{Z}_n$ Given an element $a \in \mathbb{Z}_n$ with $(a, n) = 1$, call the order of a the smallest non-zero integer d such that $a^d \equiv_n 1$. Such a number always exists, as Euler's Theorem guarantees that $a^{\varphi(n)} \equiv_n 1$. Probably the most important thing about orders is the following fact:

    Lemma If d is the order of a modulo n and e is such that $a^e \equiv_n 1$, then $d|e$.

    Indeed, assume that d does not divide e, and therefore $e = kd + r$ for $0 < r < d-1$. Then, from the definition of the order we will have $a^{dk} \equiv_n (a^d)^k \equiv_n 1$, but also $a^e \equiv_n a^{dk+r} \equiv_n 1$. So we will have $a^r \equiv_n 1$, but this would mean that r is smaller than the order d (which contradicts the minimality of d). Therefore $r = 0$ and $d|e$.

    Euler's Theorem tells us that a corollary of this lemma is that $d|\varphi(n)$, because $a^{\varphi(n)}$ is always $\equiv_n 1$. Here arises the concept of a primitive root: g is called a primitive root of n if the order of g modulo n is $\varphi(n)$ (so the largest possible). There is a theorem that says that all numbers of the form p^α and $2p^\alpha$ for prime p have primitive roots.

    In the case when $n = p$ is prime, a primitive root g has the order $\varphi(p) = p - 1$. Therefore, it is easy to note that the numbers $1, g, g^2, \dots, g^{p-2}$ are all different modulo p (as $p - 1$ is the smallest exponent to which we can raise g to reach 1). But then, the residues $1, g, g^2, \dots, g^{p-2}$ will be the same as the residues $1, 2, \dots, p - 1$ (in some order).

22. Pick's Theorem Given any polygon (not necessarily concave) with vertices in lattice points, its area is $p + \frac{q}{2} - 1$, where p is the number of interior points and q is the number of points on the boundary of the polygon (including vertices).

    This can be proven by triangulating the polygon with respect to all its inside lattice points, and noting that the triangles which appear have no lattice points inside or on their edges. The basic exercise concerning lattices is to show that any such triangle (with vertices in lattice points and no lattice points on the boundary or inside) has area exactly $\frac{1}{2}$. So if we show that there are $2p + q - 2$ triangles in our triangulation, each of them will have area $\frac{n}{2}$ and Pick's Theorem will be proven.

But we can show that the total number of triangles in a polygon with q vertices (we can consider the lattice points on the edges as 'vertices') and triangulated with respect to p interior lattice points is $2p + q - 2$ by induction over $2p + q$. This is rather simple to do, because note that upon removing a triangle from the polygon, the quantity $2p + q$ in the remaining polygon drops by 1.

23. PLANAR GRAPH A planar graph is a graph that can be drawn in the plane without any intersections between its edges (except in vertices, of course). It is now well-known that the planar graphs are just the graphs that do not contain any sub-graph of the form K_5 or $K_{3,3}$, but the proof of this fact goes is both very difficult and beyond elementary graph theory.

    A planar graph is like a map in the plane, and therefore the notion of "region" is well-defined, as the vertices and edges of the graph break the plane up into a number of regions. If v is the number of vertices, e that of edges and r that of the regions of a planar graph (including the infinite region), the Euler's formula $v + r = e + 2$ holds.

    The chromatic number of a planar graph is just the minimum number of colors needed to color its regions such that no two adjacent regions (regions that have a common side) are colored in the same way. The four-color problem, solved only recently, tells us that every planar graph has chromatic number at most 4, meaning that 4 colors are always enough to color any plane map (such that any two adjacent regions have different colors).

    A very important concept in planar graphs is that of the dual graph. Take a planar graph G, place a vertex V inside every region R of G (including the infinite one) and through each edge E of the graph connect the two vertices V and V' which you have constructed in the regions that touch the edge E (thus, since two regions in the graph can have more common edges, two vertices in the dual graph can have more edges between them). The resulting graph G' will also be planar, and every region of G corresponds to a vertex of G' (since there lies one vertex of G' in every region of G) and every vertex of G corresponds to a region of G' (since every region of G' contains only one vertex of G). Moreover, the degree of every vertex of G equals the number of edges of the corresponding region on G' and vice-versa. Note that the planar graph of the planar graph of G is G itself.

24. POLYNOMIALS IN SEVERAL VARIABLES Just like a 'regular' polynomial is a sum of terms of the form a_iX^i for some variable X, we can define a polynomial in several variables $X_1, ..., X_k$ as a sum of terms of the form $a_{i_1 i_2 ... i_k} X_1^{i_1} X_2^{i_2} ... X_k^{i_k}$, where $i_1, ..., i_k$ are natural numbers. We call $i_1 + ... + i_k$ the degree of the expression $aX_1^{i_1} X_2^{i_2} ... X_k^{i_k}$. For example,

$$P(X, Y, Z) = X^3Y - XYZ^2 + 4Y^2 + 56Z$$

is a polynomial in the variables X, Y, Z. The degree of such a polynomial is the highest degree of its terms (our example has degree 4, for instance). However, we can also perceive a polynomial in the variables $X_1, ..., X_k$ as a polynomial in one of the variables (say X_i); in the case of our example, seeing P as a polynomial in Z would be

$$P(X, Y, Z) = P_{X,Y}(Z) = -XY \cdot Z^2 + 56 \cdot Z + (X^3Y + 4Y^2)$$

Note that the coefficients of this polynomial in Z are polynomials in the other two variables. Moreover, the degree of P when seen as a polynomial of all variables does not have to be equal to the degree when seen as a polynomial of only one variable. Many things can be said about these polynomials, but we will prove one fact which is used in one of the problems (this fact can easily be generalized for polynomials of more variables quite easily).

LEMMA If the polynomial $P(X, Y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} X^i Y^j$ has the property that for all x and y that satisfy $x - ay - b = 0$ we have $P(x, y) = 0$, then $P(X, Y) = Q(X, Y)(X - aY - b)$ for some polynomial Q.

Note that in the case of only one variable, the problem will be the very well-known "if $P(X)$ is 0 for all x such that $x - a = 0$, then $P(X) = Q(X)(X - a)$". As in the case of one variable, we will solve our lemma using an analogue of division with remainder. That is, we can always write our polynomial as

$$P(X, Y) = X^k R_k(Y) + X^{k-1} R_{k-1}(Y) + ... + X R_1(Y) + R_0(Y)$$

where the $R_i(Y)$'s are polynomials in Y. Then, it is easy to prove by induction over k that this can always be written as

$$P(X, Y) = (X - aY - b)(X^{k-1} S_{k-1}(Y) + ... + X S_1(Y) + S_0(Y)) + R(Y)$$

where $S_{k-1}, ..., S_0, R$ are polynomials in R. But viewing the expression $(X^{k-1} S_{k-1}(Y) + ... + X S_1(Y) + S_0(Y))$ as a polynomial in X and Y, we can write

$$P(X,Y) = (X - ay - b)Q(X,Y) + R(Y) \tag{1}$$

But now we can finally apply our hypothesis, that P is 0 for all $x = ay + b$. So for any y, use $X = ay + b$ and $Y = y$ in (1), and we will get $R(y) = 0$. Since this holds for all y and R is just a polynomial in one variable, we concur that $R = 0$ and thus $P(X,Y) = (X - aY - b)Q(X,Y)$.

25. QUADRATIC RESIDUES AND LEGENDRE SYMBOLS Given a prime number $p > 2$ (the condition $p > 2$ is important for the establishment of properties in the next paragraph) and an integer a, we call a a quadratic residue modulo p iff there exists some k such that $a \equiv_p k^2$. Note that any multiple of p is always a quadratic residue modulo p, and any a with $(a,p) = 1$ is a quadratic residue iff $a^{\frac{p-1}{2}} \equiv_p 1$ (this makes sense somehow, since for a co-prime with p the number $a^{\frac{p-1}{2}}$ is always either congruent to 1 or to -1 modulo p; this is a direct consequence of Fermat's little theorem) We often use the term quadratic residues only to the elements of $\mathbb{Z}_p$, since it is easy to see that x is a quadratic residue iff the remainder of x upon division by p is a quadratic residue.

   Thus, we can define the Legendre symbol $\left(\frac{a}{p}\right)$ as 0 if $p|a$, 1 if a is a quadratic residue modulo p (with $(a,p) = 1$) and -1 of it is not a quadratic residue (it is a non-residue). Therefore, the observation in the previous paragraph tells us that $\left(\frac{a}{p}\right) \equiv_p a^{\frac{p-1}{2}}$. The following are the basic properties of Legendre symbols, which can be used to compute the Legendre symbol of any number with respect to every prime.

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$$

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right) \quad \text{for any integers } a, b$$

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)(-1)^{\frac{p-1}{2}\frac{q-1}{2}} \quad \text{for any primes } p, q$$

   The last property is called Gauss's Quadratic Reciprocity Theorem.

26. REARRANGEMENT INEQUALITY Given two sequences $a_1 < a_2 < ... < a_n$ and $b_1 < b_2 < ... < b_n$, prove that for any permutation σ of $\{1, 2, ..., n\}$ we have

$$a_1 b_n + a_2 b_{n-1} + ... + a_n b_1 \le a_1 b_{\sigma_1} + a_2 b_{\sigma_2} + ... + a_n b_{\sigma_n} \le$$

$$\le a_1 b_1 + a_2 b_2 ... + a_n b_n \tag{1}$$

Moreover, by summing up these inequalities for the n permutations $\sigma_1,\sigma_2,...,\sigma_n$ defined by $\sigma_i(j) = j+i$ or $j+i-n$ (whichever of those two numbers lies in $\{1, 2, ..., n\}$) we get Chebyshev's inequality

$$a_1b_n + a_2b_{n-1} + ... + a_nb_1 \le \frac{1}{n}(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) \le$$
$$\le a_1b_1 + a_2b_2 + ... + a_nb_n$$

We can prove the RHS of (1) by considering that permutation σ for which the expression $a_1b_{\sigma_1} + a_2b_{\sigma_2} + ... + a_nb_{\sigma_n}$ is maximum (proving the LHS is done in the same way, by choosing that permutation for which it is minimum). It is easy to see that the permutation σ can have no inversions (meaning pairs $i < j$ for which $\sigma(i) > \sigma(j)$), or else we could further increase the value of the expression by interchanging $\sigma(i)$ and $\sigma(j)$. Of course, the only permutation which has no inversions is the identity, so the given expression is maximum when it is $a_1b_1 + a_2b_2 + ... + a_nb_n$.

27. RHS "Right Hand Side" of an equation, inequality, identity etc.
28. SIMSON LINE Given a triangle ABC and a point M on its circumcircle, it can be easily proven that the projections of M on the sides of the triangle are collinear. The line they determine is called the Simson line of M with respect to ABC and is denoted by $l(M, ABC)$. The Simson line verifies a number of properties, of which we will use in the course of this book two:

    LEMMA 1 If M is on the circumcircle of ABC, then the Simson line $l(M, ABC)$ cuts MH in half (where H is the orthocenter of the triangle).

    PROOF Assume WLOG that M is situated on the small arc (AC). Let A' and C' be the projections of M onto BA and BC and let A'' and C'' be the intersections between MA' and HA and between MC' and HC respectively. Note that $MC''HA''$ is a parallelogram (parallel opposite sides), so the midpoint of MH is the same as the midpoint of $A''C''$. Let O be the intersection of $A'C'$ and $A''C''$.

    Applying the Law of Sines in triangles $A'OA''$ and $C'OC''$ yields

$$\frac{A''O}{\sin OA'M} = \frac{A'A''}{\sin A'OA''}$$
$$\frac{C''O}{\sin OC'M} = \frac{C'C''}{\sin C'OC''}$$

Dividing the two of them gives us ($\angle A'OA'' = \pi - \angle C'OC''$)

$$\frac{A''O}{C''O} = \frac{A'A''}{C'C''} \cdot \frac{\sin OA'M}{\sin OC'M} \tag{1}$$

That last ratio, is by the Law of Sines in triangle $A'MC'$, equal to $\frac{MC'}{MA'}$. It is easy to see that we have $\angle AA''A' = \angle CC''C = B$ and so the triangles

$AA'A''$ and $CC'C''$ are similar. Moreover, the triangles AMA' and CMC' are similar (because they are right-angled and $\angle A'AM = \pi - \angle BAM = \angle C'CM$). Thus, because of the above similarities, (1) becomes

$$\frac{A''O}{C''O} = \frac{AA'}{CC'}\frac{MC'}{MA'} = 1$$

so O is indeed the midpoint of $A''C''$, and thus $A'C'$ cuts $A''C''$ and MH in their midpoint.

LEMMA 2 If $ABCD$ are points on a circle, the 4 Simson lines of one with respect to the other three are concurrent.

PROOF Let a, b, c, d be the complex numbers that determine this point in the complex plane with our circle as the unit circle. It is a well-known fact that the orthocenter H of BCD is given by the complex number $b+c+d$ and so the midpoint of AH will be given by $\frac{a+b+c+d}{2}$. The previous lemma tells us that this point is on the Simson line $l(A, BCD)$. But the same argument tells us that it is also on $l(B, ACD)$, $l(C, ABD)$ and $l(D, ABC)$, which proves our claim.

29. SUP The supremum of a function $f(x)$ (denoted by $\sup f$) is the least real number which is greater than of equal to all the values of f. If there is no such real number, we consider the supremum to be ∞.
30. TRIANGULATION We say that a polygon is triangulated if we can break it up into triangles with disjoint interiors which only have vertices among the vertices of the polygon. It can be shown that the number of ways in which we can triangulate a convex polygon with $n+2$ sides is the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

    Another interesting thing is triangulating a certain polygon whose set of vertices is V with respect to a certain finite set A of interior points. This means that we must break up the polygon into triangles with disjoint interiors which only have vertices among the points of $V \bigcup A$, such that the points of $V \bigcup A$ are located only in the vertices of triangles of the triangulation (so not on the edges or in the interiors). It can be easily proven (by induction on $\#V \bigcup A$) that any polygon can be triangulated with respect to any finite set of interior points.
31. TURAN'S THEOREM This theorem tells us that the maximum number of edges in a graph with n vertices with no p-clique is obtained when our graph is the complete multipartite graph $K_{l,l,\ldots,l+1,\ldots,l+1}$ where $n = (p-1)l + r$ with $0 \le r < p-1$. That multipartite graph is a graph whose vertices are broken up into $p-1$ disjoint groups, each with l or $l+1$ vertices, such that a vertex is connected only to vertices from the other groups. This graph is practically

the optimum division of n vertices into $p-1$ independent components, and it's easy to see that it has no p-clique.

The case $p=3$ tells us that a necessary condition for not having triangles in our graph is having less than $\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor$ edges. Many nice proofs of this very important elementary theorem can be found in "Proofs from the Book", by Martin Aigner and Gunter M. Ziegler.

32. WLOG "Without Loss Of Generality"

APPENDIX 2: SOURCES OF PROBLEMS

(1) IMAR TESTS are a series of tests organized biannually by the Mathematics Institute of the Romanian Academy (IMAR). All students are invited, particularly those who are interested in taking the IMO Team Selection Tests in the following year.
(2) 77 DE TURE is a collection of 77 mathematics problems gathered from IMO team leaders from around the world. The Romanian IMO 2004 team used these problems as practice.
(3) MOSP The US IMO team Math Olympiad Summer Program.
(4) JMBO Junior Balkan Mathematica Olympiad.
(5) AMERICAN MATHEMATICAL MONTHLY or AMM is probably the most important American mathematics periodical.
(6) TEAM CONTEST These problems where given at a team contest organized in 2004 by several Romanian college students who were IMO veterans for their younger peers.
(7) MATHLINKS CONTEST is the contest organized yearly (or biannually) by the Mathlinks forum.
(8) ROMANIAN-HUNGARIAN TRAINING CAMP is the yearly common IMO training of the Romanian and Hungarian teams. The camp lasts for one week and is organized alternatively by the two countries.
(9) PUTNAM is the most important math competition organized in the USA for undergraduate students.

GIL

This is a text on elementary mathematics written by a very promising young mathematician who is at his first experience of this kind. The author, Andrei Neguț, now a student at Princeton, was, for years, the winner of high-school national and International Mathematical competitions.
This is an old dream of his, to gather and comment some of the most beautiful problems in mathematics he was thinking about during his intense work and learning.
The result is this wonderful book that contains beside a list of problems in classical fields of mathematics (algebra, geometry, combinatorics) that Andrei loved the most, a lot of original and sometimes even wonderful solutions. The text is well written. The explanations are complete and proving a deep intuition of the author.
I recommend the book to anyone interested professionally or as a hobby in problem solving. Students preparing for mathematical competitions will find a good training material.
I must confess that the feeling that the book represents the beginning of an important career in mathematics, was anytime present when reading the text.

Radu Gologan

University "Politehnica" Bucharest and
Institute of Mathematics
Coordinator of the Romanian Mathematical Olympiads professors."

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