Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

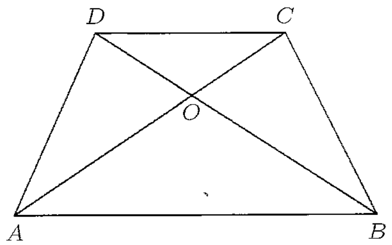
(Junior Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. In the following figure, $AB \parallel DC$, $AB = b$, $CD = a$ and $a < b$. Let S be the area of the trapezium $ABCD$. Suppose the area of $\triangle BOC$ is $2S/9$. Find the value of a/b.



2. Equilateral triangles ABE and BCF are erected externally on the sides AB and BC of a parallelogram $ABCD$. Prove that $\triangle DEF$ is equilateral.

3. Let n be a positive integer and d be the greatest common divisor of n^2+1 and $(n+1)^2+1$. Find all the possible values of d. Justify your answer.

4. The difference between the product and the sum of two different integers is equal to the sum of their GCD (greatest common divisor) and LCM (least common multiple). Find all these pairs of numbers. Justify your answer.

5. For any positive integer n, let $f(n)$ denote the nth positive nonsquare integer, i.e., $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 6$, etc. Prove that

$$f(n) = n + \{\sqrt{n}\}$$

where $\{x\}$ denotes the integer closest to x. (For example, $\{\sqrt{1}\} = 1$, $\{\sqrt{2}\} = 1$, $\{\sqrt{3}\} = 2$, $\{\sqrt{4}\} = 2$.)

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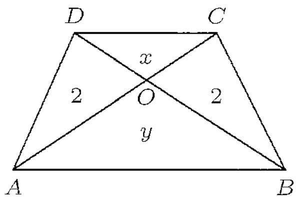
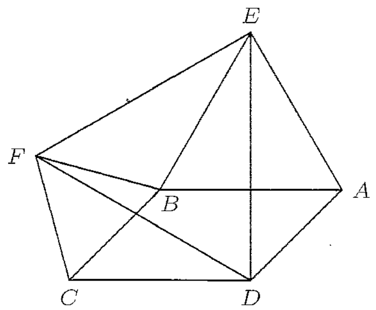
1. Without loss of generality, let $S = 9$. Then $[BOC] = 2$. Since $[ABD] = [ABC]$, we have $[AOD] = [BOC] = 2$. Let $[DOC] = x$ and $[AOB] = y$. Then $x/2 = 2/y$, i.e., $xy = 4$. Also $x + y = 5$. Thus $x(5 - x) = 4$. Solving, we get $x = 1$ and $y = 4$. Since $\triangle DOC \sim \triangle BOA$, we have $x/y = a^2/b^2$. Thus $a/b = 1/2$.



2. We have

$$\begin{aligned}\angle EBF &= 240^\circ - \angle ABC = 240^\circ - (180^\circ - \angle BCD) \\ &= 60^\circ + \angle BCD = \angle DCF\end{aligned}$$

Also $FB = FC$ and $BE = BA = CD$. Thus $\triangle FBE \cong \triangle FCD$. Therefore $FE = FD$. Similarly $\triangle EAD \cong \triangle DCF$. Therefore $ED = DF$. Thus $\triangle DEF$ is equilateral.



3. For $n = 1$ and $n = 2$, the gcd are 1 and 5, respectively. Any common divisor d of $n^2 + 1$ and $(n + 1)^2 + 1$ divides their difference, $2n + 1$. Hence d divides $4(n^2 + 1) - (2n + 1)(2n - 1) = 5$. Thus the possible values are 1 and 5.

4. Let the integers be x and y and assume that $x > y$. First we note that x and y are both nonzero and that their GCD and LCM are both positive by definition. Let M be the GCD, then $|x| = Ma$ and $|y| = Mb$, where a and b are coprime integers. Thus the LCM of x and y is Mab. If $y = 1$, then $M = 1$ and it's easily checked that there is no solution. When $y > 1$, $xy > x + y$. Thus $xy - (x + y) = M + Mab$. After substituting for x and y and simplifying, we have

$$ab(M-1) = 1 + a + b \quad \Rightarrow \quad M = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \quad \Rightarrow \quad 1 < M \leq 4.$$

If $M = 2$, then $ab - a - b = 1$, i.e., $(a-1)(b-1) = 2$. Thus $a = 3$, $b = 2$ or $x = 6$, $y = 4$. Similarly, when $M = 3$, we get $2ab - a - b = 1$. Multiplying throughout by 2 and then factorize, we get $(2a-1)(2b-1) = 3$ which gives $x = 6$ and $y = 3$. When $M = 4$, we get $x = y = 4$ which is rejected as x and y are distinct.
Next we consider the case $x > 0 > y$. Then $x = Ma$ and $y = -Mb$. Using similar arguments, we get $x + y - xy = M + Mab$. Thus $M = 1 + \frac{1}{ab} + \frac{1}{a} - \frac{1}{b}$ which yields $1 \leq M \leq 2$. When $M = 1$, we get $a = 1 + b$. Thus the solutions are $b = t$, $a = 1 + t$ or $x = 1 + t$, $y = -t$, where $t \in \mathbb{N}$. When $M = 2$, the equation simplifies to $(a-1)(b+1) = 0$. Thus we get $a = 1$ and b arbitrary as the only solution. The solutions are $x = 2$, $y = -2t$, where $t \in \mathbb{N}$.
Finally, we consider the case $0 > x > y$. Here $x = -Ma$, $y = -Mb$ and $M^2ab + Ma + Mb = M + Mab$. Since $M2ab \geq Mab$ and $Ma + Mb > M$, there is no solution.
Thus the solutions are $(6,3)$, $(6,4)$, $(1+t, -t)$ and $(2, -2t)$ where $t \in \mathbb{N}$.

5. If $f(n) = n + k$, then there are exactly k square numbers less than $f(n)$. Thus $k^2 < f(n) < (k+1)^2$. Now we show that $k = \{\sqrt{n}\}$. We have

$$k^2 + 1 \leq f(n) = n + k \leq (k+1)^2 - 1.$$

Hence
$$\left(k - \frac{1}{2}\right)^2 + \frac{3}{4} = k^2 - k + 1 \leq n \leq k^2 + k = \left(k + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Therefore
$$k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2},$$

so that $\{\sqrt{n}\} = k$.