WARDAYA COLLEGE - TOP UNIVERSITIES CLASS

TEST 3

READ THESE INSTRUCTIONS FIRST

1. Write your name on all the work you hand in
2. Answer **all** the questions
3. The total number of marks for this paper is 100

1. The number of Wardaya Pandas due to birth and death in a protected bamboo environment is modelled by the differential equation $\frac{dS}{dt} = kS(k-S)$ where S is the number of pandas time t years, and k is a constant. Given that k > S, show that the differential equation has general solution $S = \frac{kC}{e^{-k^2t} + C}$ where C is an arbitrary constant

   Hence, find the limiting value of the number of Wardaya Pandas in the bamboo environment, and explain the signicance of k
2. Show that the differential equation $3xy^2\frac{dy}{dx} + y^3 - 2x = 0$ may be reduced by means of the substitution $u = xy^3$ to $\frac{du}{dx} = 2x$

   Solve the differential equation, expressing y in terms of x.

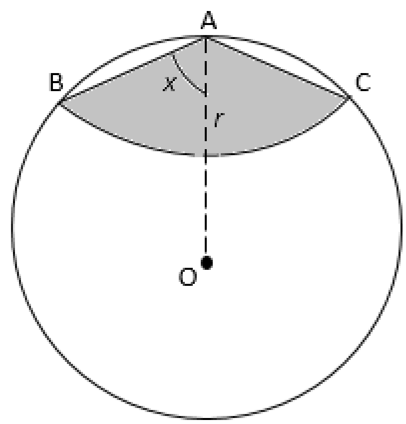
   Sketch the family of solution curves of the differential equation
3. If $y = \sqrt{2}\, tan^{-1}\left(\frac{x\sqrt{2}}{x+3}\right)$, show that $(x^2 + 2x + 3)\frac{dy}{dx} = 2$

   By further differentiation of the above result. Find the Maclaurin's series expansion for y up to and including the term in x^2

   Hence, find the first three non zero terms in the expansion of $\frac{1}{x^2+2x+3}$
4. It is given that $y = \sqrt[4]{2 - e^{2x}}$
   (i) Show that $y\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = 2y\frac{dy}{dx}$
   (ii) Hence, find the Maclaurin's series for y, up to and including the term in x^2
   (iii) Deduce that, for small x : $e^{2x}(2 - e^{2x})^{-3/4} \approx 1 + \frac{7}{2}x$
5. It is given that $I = \int_0^{0.3}(1 + 3x^2)^{-2}dx$
   (i) Use the trapezium rule with 3 intervals to find an approximation to I, giving the answer correct to 3 decimal places
   (ii) For small values of x, $(1 + 3x^2)^{-2} \approx 1 + ax^2 + bx^4$. Find the values of the constant a and b

Hence, by evaluating $\int_0^{0.3}(1 + ax^2 + bx^4)dx$, find a second approximation to I, giving the answer correct to 3 decimal places

6.



In the diagram, *A* is a point on the circumference of a circle with center *O* and radius *r*. A circular arc with center *A* meets the circumference at *B* and *C*. The angle OAB is equal to *x* radians. The shaded region is bounded by *AB*, *AC* and the circular arc with center *A* joining *B* and *C*. The perimeter of the shaded region is equal to half the circumference of the circle.

(i) Show that $x = \cos^{-1}\left(\frac{\pi}{4+4x}\right)$

(ii) Verify by calculation that *x* lies between 1 and 1.5

(iii) Use the iterative formula

$$x_{n+1} = \cos^{-1}\left(\frac{\pi}{4 + 4x_n}\right)$$

To determine the value of *x* correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

7. The equation $x = \left(\frac{10}{e^{2x}-1}\right)$ has one positive real root, denoted by α.

(i) Show that α lies between x = 1 and x =2.

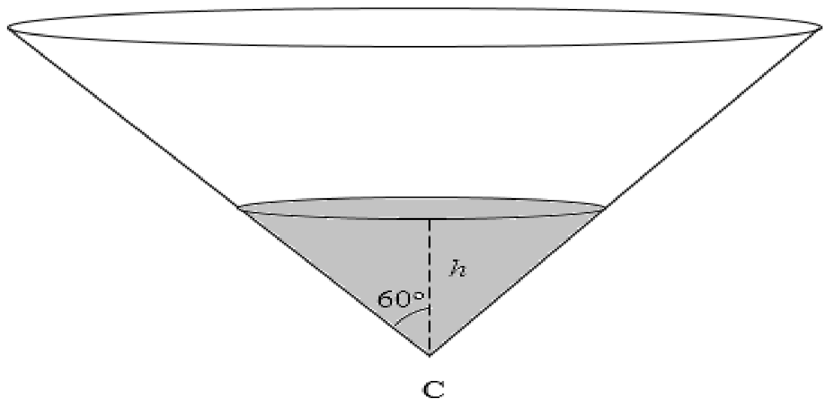
(ii) Show that if a sequence of positive values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\text{In}\left(1 + \frac{10}{x_n}\right)$$

Converges, then it converges to α

(iii) Use this iterative formula to determine α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

8.



A tank containing water is in the form of a cone with vertex *C*. The axis is vertical and the semi vertical angle 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is *H*. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where *h* is the depth of water at time *t*. The tank becomes empty when $t = 60$.

(i) Show that *h* and *t* satisfy a differential equation if the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}}$$

Where *A* is a positive constant.

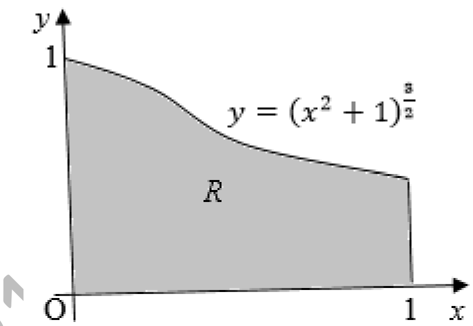
(ii) Solve the differential equation given in the part (i) and obtain an expression for t in terms of *h* and *H*

(iii) Find the time at which the depth reaches $\frac{1}{2}H$

[The volume C of a cone of vertical height h and base radius r id given by $V = \frac{1}{3}\pi r^2 h$.]

9.

The diagram shows the region R bounded by the axes, the curve $y = (x^2+1)^{-\frac{3}{2}}$ and the love x = 1. Use the trapezium rule, with ordinates at x = 0, x = $\frac{1}{2}$, and x = 1, to estimate the value of



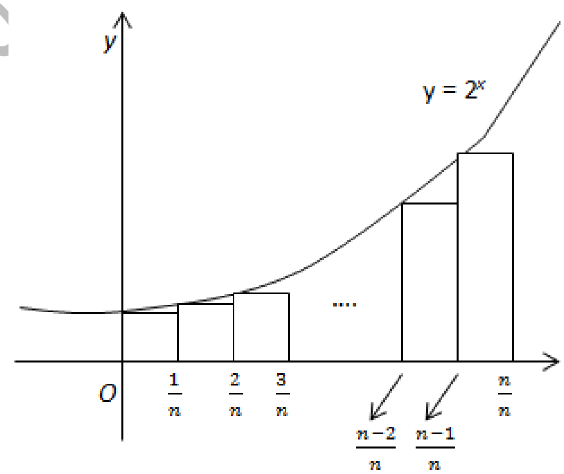
$$\int_0^1 (x^2+1)^{-\frac{3}{2}}\, dx,$$

Giving your answer correct to 2 significant figures.

10.

i. Prove that $1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$

ii. The graph of $y = 2^x$, for $0 \le x \le 1$, is shown in the diagram. Rectangles, each of width 1/*n*, are drawn under the curve . Show that the total area A of all *n* rectangles is given by $\frac{1}{n}\left(\frac{1}{2^{1/n}-1}\right)$.



Hence find the exact value of the limit of A as $n \to \infty$

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