Integral Exercise Answer Key

NTU – NUS PREPARATORY

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1.

$$\int \frac{dx}{1+\cos x} \to \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx = \int \frac{1-\cos x}{1-\cos^2 x}dx$$

$$\int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$$

Separate the terms

$$= \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}dx$$

$$\int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}dx \to \int \csc^2 x\, dx - \int \cot x \csc x\, dx$$

$$\int \csc^2 x\, dx - \int \cot x \csc x\, dx = -\cot x + \csc x + C$$

2.

$$\int \tan x \sec^4 x\, dx \to \int \tan x \sec x \sec^3 x\, dx$$

Let:

$$u = \sec x\, ;\, du = \tan x \sec x\, dx$$

$$\int \tan x \sec x \sec^3 x\, dx \to \int u^3\, du = \frac{1}{4}u^4 + C \to \frac{1}{4}\sec^4 x + C$$

$$\int \tan x \sec^4 x\, dx = \frac{1}{4}\sec^4 x + C$$

3.

$$\int \frac{dx}{x^2-4x+13} \to \int \frac{dx}{x^2-4x+4+9} = \int \frac{dx}{(x-2)^2+9}$$

$$\int \frac{dx}{(x-2)^2+9} = \frac{1}{9}\int \frac{dx}{\left(\frac{x-2}{3}\right)^2+1}$$

Let:

$$u = \frac{x-2}{3};\, du = \frac{1}{3}dx$$

$$\frac{1}{9}\int \frac{dx}{\left(\frac{x-2}{3}\right)^2+1} \to \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C$$

$$\int \frac{dx}{x^2-4x+13} = \frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C$$

4.

$$\int \cos(\ln x)\, dx$$

Let:

$$y = \ln x\, ;\, dy = \frac{1}{x}dx \to e^y dy = dx$$

$$\int cos(\ln x)\, dx \to \int e^y \cos y\, dy$$

Use Partial Integral method to solve the Integral

$$\int u\, dv = uv - \int v\, du$$

$e^y = u \to e^y dy = du\, ;\, \cos y\, dy = dv \to \sin y = v$

$$\int e^y \cos y\, dy = e^y \sin y - \int e^y \sin y\, dy$$

$$\int e^y \sin y\, dy = e^y(-\cos y) - \int e^y\,(-\cos y)dy$$

$$\int e^y \cos y\, dy = e^y \sin y - \int e^y \sin y\, dy$$

$$\int e^y \cos y\, dy = e^y \sin y + e^y \cos y - \int e^y \cos y\, dy$$

$$2\int e^y \cos y\, dy = e^y \sin y + e^y \cos y$$

$$\int e^y \cos y\, dy = \frac{1}{2} e^y(\sin y + \cos y)$$

$$\int \cos(\ln x)\, dx = \frac{1}{2} x(\sin(\ln x) + \cos(\ln x)) + C$$

5.

$$\int \tan^2 x \sec^4 x\, dx \to \int \tan^2 x \sec^2 x \sec^2 x\, dx$$

Let:

$u = \tan x \to du = \sec^2 x\, dx$

$$\int u^2 \sec^2 x\, du \to \int u^2(\tan^2 x + 1)\, du$$

$$\int u^2(u^2 + 1)\, du = \int (u^4 + u^2)\, du$$

$$\int (u^4 + u^2)\, du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$$

$$\int \tan^2 x \sec^4 x\, dx = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$$

6.

$$\int \frac{e^x}{e^{2x} + 3e^x + 2} dx$$

Let:

$u = e^x \to du = e^x dx$

$$\int \frac{du}{u^2 + 3u + 2}$$

Use Partial Fraction method to solve the Integral

$$\frac{1}{u^2+3u+2}=\frac{A}{u+2}+\frac{B}{u+1}$$

$1=A(u+1)+B(u+2)\rightarrow 1=(A+B)u+A+2B$

Coefficient:

$u^1\rightarrow A+B=0$

$u^0\rightarrow A+2B=1$

$B=1; A=-1$

$$\frac{1}{u^2+3u+2}=\frac{-1}{u+2}+\frac{1}{u+1}$$

$$\int\frac{du}{u^2+3u+2}=\int-\frac{du}{u+2}+\int\frac{du}{u+1}$$

$$\int-\frac{du}{u+2}+\int\frac{du}{u+1}=-\ln|u+2|+\ln|u+1|$$

$-\ln|u+2|+\ln|u+1|\rightarrow-\ln|e^x+2|+\ln|e^x+1|$

$$\int\frac{e^x}{e^{2x}+3e^x+2}dx=\ln\left|\frac{e^x+1}{e^x+2}\right|+C$$

7.

$$\int\frac{\cos x}{\sin^2 x+\sin x-6}dx\rightarrow\int\frac{\cos x}{(\sin x+3)(\sin x-2)}dx$$

Let:

$u=\sin x+3\rightarrow du=\cos x\,dx$

$$\int\frac{du}{u(u-5)}$$

Use Partial Fraction method to solve the Integral

$$\frac{1}{u(u-5)}=\frac{A}{u}+\frac{B}{u-5}$$

$1=A(u-5)+Bu\rightarrow 1=(A+B)u-5A$

Coefficient:

$u^1\rightarrow A+B=0$

$u^0\rightarrow -5A=1$

$A=-\frac{1}{5}; B=\frac{1}{5}$

$$\frac{1}{u(u-5)}=\frac{1}{5}\left(\frac{1}{u-5}-\frac{1}{u}\right)$$

$$\int\frac{du}{u(u-5)}=\int\frac{1}{5}\left(\frac{1}{u-5}-\frac{1}{u}\right)du$$

$$\int\frac{1}{5}\left(\frac{1}{u-5}-\frac{1}{u}\right)du=\frac{1}{5}\left[\int\frac{du}{u-5}-\int\frac{du}{u}\right]$$

$$\frac{1}{5}\left[\int \frac{du}{u-5} - \int \frac{du}{u}\right] = \frac{1}{5}(ln|u-5| - ln|u|)$$

$$\int \frac{\cos x}{\sin^2 x + \sin x - 6}dx = \frac{1}{5}(ln|\sin x - 2| - ln|\sin x + 3|) + C$$

8.

$$\int \frac{(x+1)^2 \tan^{-1} 3x + 9x^3 + x}{(9x^2+1)(x+1)^2}dx$$

Separate the terms.

$$= \int \frac{(x+1)^2 \tan^{-1} 3x}{(9x^2+1)(x+1)^2}dx + \int \frac{9x^3 + x}{(9x^2+1)(x+1)^2}dx$$

$$I_1 = \int \frac{(x+1)^2 \tan^{-1} 3x}{(9x^2+1)(x+1)^2}dx = \int \frac{\tan^{-1} 3x}{9x^2+1}dx$$

Let:

$$u = \tan^{-1} 3x \to du = \frac{3}{9x^2+1}dx$$

$$\int \frac{\tan^{-1} 3x}{9x^2+1}dx \to \int \frac{1}{3}u\, du = \frac{1}{6}u^2 + C_1$$

$$I_1 = \int \frac{(x+1)^2 \tan^{-1} 3x}{(9x^2+1)(x+1)^2}dx = \frac{1}{6}(\tan^{-1} 3x)^2 + C_1$$

$$I_2 = \int \frac{9x^3 + x}{(9x^2+1)(x+1)^2}dx = \int \frac{x}{(x+1)^2}dx$$

$$\int \frac{x}{(x+1)^2}dx = \int \frac{x+1-1}{(x+1)^2}dx \to \int \frac{dx}{1+x} - \int \frac{dx}{(1+x)^2}$$

$$\int \frac{dx}{1+x} - \int \frac{dx}{(1+x)^2} = ln|1+x| + (1+x)^{-1} + C_2$$

$$I_2 = \int \frac{9x^3 + x}{(9x^2+1)(x+1)^2}dx = ln|1+x| + (1+x)^{-1} + C_2$$

$$\int \frac{(x+1)^2 \tan^{-1} 3x + 9x^3 + x}{(9x^2+1)(x+1)^2}dx = I_1 + I_2$$

$$\int \frac{(x+1)^2 \tan^{-1} 3x + 9x^3 + x}{(9x^2+1)(x+1)^2}dx = \frac{1}{6}(\tan^{-1} 3x)^2 + ln|1+x| + (1+x)^{-1} + C_3$$

9.

$$\int \frac{2x^4}{x^3 - x^2 + x - 1}dx \to \int \frac{2x^4 - 2 + 2}{(x^2+1)(x-1)}dx$$

$$\int \frac{2x^4 - 2 + 2}{(x^2+1)(x-1)}dx = \int \frac{2(x^4-1)}{(x^2+1)(x-1)}dx + \int \frac{2}{(x^2+1)(x-1)}dx$$

$$I_1 = \int \frac{2(x^2+1)(x+1)(x-1)}{(x^2+1)(x-1)}dx \to \int 2(x+1)\, dx$$

$$I_1 = \int 2(x+1)\, dx = x^2 + 2x + C_1$$

$$I_2 = \int \frac{2}{(x^2+1)(x-1)} dx$$

Use the Partial Fraction method to solve the 2nd Integral

$$\frac{2}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$2 = Ax^2 + A + Bx^2 + (C-B)x - C$$

Coefficient:

$x^2 \rightarrow A + B = 0$

$x^1 \rightarrow C - B = 0$

$x^0 \rightarrow A - C = 2$

$B = -1, C = -1, A = 1$

$$\frac{2}{(x^2+1)(x-1)} = \frac{1}{x-1} - \frac{x+1}{x^2+1}$$

$$\int \frac{2}{(x^2+1)(x-1)} dx = \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1} dx$$

$$\int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1} dx = ln|x-1| - \int \frac{x}{x^2+1} dx - \int \frac{dx}{x^2+1}$$

$$ln|x-1| - \int \frac{x}{x^2+1} dx - \int \frac{dx}{x^2+1} = ln|x-1| - \frac{1}{2} ln|x^2-1| - tan^{-1} x + C_2$$

$$\int \frac{2x^4}{x^3 - x^2 + x - 1} dx = I_1 + I_2$$

$$\int \frac{2x^4}{x^3 - x^2 + x - 1} dx = x^2 + 2x + ln|x-1| - \frac{1}{2} ln|x^2-1| - tan^{-1} x + C_3$$

10.

$$\int ln(x^2+x) dx = \int (ln\, x + ln(x+1)) dx$$

$$\int (ln\, x + ln(x+1)) dx = \int ln\, x\, dx + \int ln(x+1)\, dx$$

Use Partial Integral method to find the integral of $ln\, x$.

$$\int ln\, x\, dx$$

Let:

$$u = ln\, x\, ; dx = dv \rightarrow du = \frac{dx}{x} ; x = v$$

$$\int ln\, x\, dx = x\, ln\, x - dx \rightarrow \int ln\, x\, dx = x\, ln\, x - x$$

Hence:

$$\int ln\, x\, dx + \int ln(x+1)\, dx = x\, ln\, x - x + (x+1)\, ln(x+1) - x - 1$$

$$\int ln(x^2+x)dx = x\,ln(x^2+x) + ln(x+1) - 2x - 1 + C$$

11.

$$\int \frac{dx}{x^2\sqrt{x^2+4}}$$

Let:

$$x = 2\tan\theta \to dx = 2\sec^2\theta\, d\theta$$

$$\int \frac{\sec^2\theta}{2\tan^2\theta\sqrt{4\tan^2\theta+4}}d\theta = \int \frac{\sec^2\theta}{4\tan^2\theta\sec\theta}d\theta$$

$$\int \frac{\sec^2\theta}{4\tan^2\theta\sec\theta}d\theta = \int \frac{\sec\theta}{4\tan^2\theta}d\theta$$

$$\int \frac{\sec\theta}{4\tan^2\theta}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta$$

Let:

$$\mu = \sin\theta \to d\mu = \cos\theta\, d\theta$$

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta \to \frac{1}{4}\int \frac{d\mu}{\mu^2}$$

$$\int \frac{d\mu}{\mu^2} = -\frac{1}{\mu} + C \to -\csc\theta + C$$

$$\frac{x}{2} = \tan\theta \to \theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\csc\left(\tan^{-1}\left(\frac{x}{2}\right)\right) + C$$

12.

$$\int \frac{dx}{\cos x + 2\sin x + 2}$$

Use the ***Weierstrass Substitution****.*

$$t = \tan\frac{x}{2} \to dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$$

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \to dx = \frac{2}{1+t^2}dt$$

$$\int \frac{\frac{2}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{4t}{1+t^2} + 2}dt = \int \frac{2}{t^2+4t+3}dt$$

$$\int \frac{2}{t^2+4t+3}dt = 2\int \frac{dt}{(t+3)(t+1)}$$

$$\frac{1}{(t+3)(t+1)} = \frac{A}{t+3} + \frac{B}{t+1}$$

$$1 = A(t+1) + B(t+3)$$

$$A + 3B = 1 \ldots (1)$$

$A + B = 0 \dots (2)$

$B = \frac{1}{2}; A = -\frac{1}{2}$

$$\frac{1}{(t+3)(t+1)} = \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t+3}\right)$$

$$2\int \frac{dt}{(t+3)(t+1)} = \int \left(\frac{1}{t+1} - \frac{1}{t+3}\right) dt$$

$$\int \left(\frac{1}{t+1} - \frac{1}{t+3}\right) dt = ln\left|\frac{t+1}{t+3}\right|$$

$$\int \frac{dx}{\cos x + 2\sin x + 2} = ln\left|\frac{tan\left(\frac{x}{2}\right) + 1}{tan\left(\frac{x}{2}\right) + 3}\right| + C$$

13.

$$\int \frac{2x^3 - 1}{x^4 + x} dx = \frac{1}{2}\int \frac{4x^3 - 2}{x^4 + x} dx$$

$$\frac{1}{2}\int \frac{4x^3 - 2}{x^4 + x} dx = \frac{1}{2}\int \frac{4x^3 + 1 - 3}{x^4 + x} dx$$

Separate the Integral.

$$\frac{1}{2}\int \frac{4x^3 + 1 - 3}{x^4 + x} dx = \frac{1}{2}\int \frac{4x^3 + 1}{x^4 + x} dx - \frac{3}{2}\int \frac{dx}{x^4 + x}$$

$$I_1 = \frac{1}{2}\int \frac{4x^3 + 1}{x^4 + x} dx$$

Let:

$$u = x^4 + x \rightarrow du = (4x^3 + 1)dx$$

$$\frac{1}{2}\int \frac{4x^3 + 1}{x^4 + x} dx \rightarrow \frac{1}{2}\int \frac{du}{u}$$

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2} ln\,|u| + C_1 \rightarrow \frac{1}{2} ln\,|x^4 + x| + C_1$$

$$I_2 = -\frac{3}{2}\int \frac{dx}{x^4 + x}$$

$$\int \frac{dx}{x^4 + x} = \int \frac{dx}{x^4\left(1 + \frac{1}{x^3}\right)}$$

Let:

$$u = \frac{1}{x^3} + 1 \rightarrow -\frac{du}{3} = \frac{dx}{x^4}$$

$$\int \frac{dx}{x^4\left(1 + \frac{1}{x^3}\right)} \rightarrow -\frac{1}{3}\int \frac{du}{u}$$

$$-\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3} ln|u| + C_2 \rightarrow -\frac{1}{3} ln\left|\frac{1}{3}x^3 + 1\right| + C_2$$

Hence:

$I_2 = \frac{1}{2} ln \left|\frac{1}{3}x^3 + 1\right| + C_2$

$\int \frac{2x^3 - 1}{x^4 + x} dx = I_1 + I_2$

$\int \frac{2x^3 - 1}{x^4 + x} dx = \frac{1}{2} ln\,|x^4 + x| + \frac{1}{2} ln \left|\frac{1}{3}x^3 + 1\right| + C_3$

14.

$\int \frac{\cos 4x + 1}{\cot x - \tan x} dx$

Trigonometric Formula:

$\cos 2\theta = 2\cos^2\theta - 1$

$\int \frac{\cos 4x + 1}{\cot x - \tan x} dx = \int \frac{2\cos^2 2x}{\cot x - \tan x} dx$

$\int \frac{2\cos^2 2x}{\cot x - \tan x} dx \to \int \frac{2\cos^2 2x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}} dx$

$\int \frac{2\cos^2 2x \sin x \cos x}{\cos^2 x - \sin^2 x} dx$

Trigonometric Formula:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\int \frac{2\cos^2 2x \sin x \cos x}{\cos^2 x - \sin^2 x} dx = \int \frac{\cos^2 2x \sin 2x}{\cos 2x} dx$

$\int \frac{\cos^2 2x \sin 2x}{\cos 2x} dx \to \int \cos 2x \sin 2x\, dx = \frac{1}{2}\int \sin 4x\, dx$

$\frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + C$

15.

$\int \frac{e^x + \cos x}{e^x + \sin x} dx$

Let:

$u = e^x + \sin x \to du = (e^x + \cos x)\, dx$

$\int \frac{e^x + \cos x}{e^x + \sin x} dx \to \int \frac{du}{u} = ln\,|u| + C$

$\int \frac{e^x + \cos x}{e^x + \sin x} dx = ln\,|e^x + \sin x\,| + C$

16.

$\int \frac{dx}{x^2 \sqrt[4]{(x^4 + 1)^3}}$

Let:

$$u = (x^4+1)^{\frac{1}{4}} \to du = x^3(x^4+1)^{-\frac{3}{4}}dx$$

$$u^4 - 1 = x^4 \to x = \sqrt[4]{u^4-1}$$

$$\int \frac{dx}{x^2\sqrt[4]{(x^4+1)^3}} \to \int \frac{x^3dx}{x^5\sqrt[4]{(x^4+1)^3}}$$

$$\int \frac{x^3dx}{x^5\sqrt[4]{(x^4+1)^3}} \to \int \frac{du}{(u^4-1)^{\frac{5}{4}}}$$

$$\int \frac{du}{(u^4-1)^{\frac{5}{4}}} = \int \frac{du}{u^5\left(1-\frac{1}{u^4}\right)^{\frac{5}{4}}}$$

Let:

$$\rho = 1 - \frac{1}{u^4} \to d\rho = \frac{4}{u^5}du$$

$$\int \frac{du}{u^5\left(1-\frac{1}{u^4}\right)^{\frac{5}{4}}} \to \int \frac{d\rho}{\rho^{\frac{5}{4}}} = \frac{1}{1+\frac{5}{4}}\rho^{\frac{5}{4}+1} + C(\rho)$$

$$\frac{1}{1+\frac{5}{4}}\rho^{\frac{5}{4}+1} + C_n = \frac{4}{9}\rho^{\frac{9}{4}} + C(\rho)$$

$$\frac{4}{9}\rho^{\frac{9}{4}} + C(\rho) \to \frac{4}{9}\left(1-\frac{1}{u^4}\right)^{\frac{9}{4}} + C(u)$$

$$\frac{4}{9}\left(1-\frac{1}{u^4}\right)^{\frac{9}{4}} + C(u) \to \frac{4}{9}\left(1-\frac{1}{x^4+1}\right)^{\frac{9}{4}} + C$$

$$\int \frac{dx}{x^2\sqrt[4]{(x^4+1)^3}} = \frac{4}{9}\left(1-\frac{1}{x^4+1}\right)^{\frac{9}{4}} + C$$

17.

$$\int \frac{2x}{1+\cos 2x}dx$$

Trigonometric Formula:

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\int \frac{2x}{1+\cos 2x}dx = \int \frac{x}{\cos^2 x}dx$$

Use Partial Integral to solve the Integral.

$$u = x \to du = dx$$

$$\frac{dx}{\cos^2 x} = dv \to \tan x = v$$

$$\int \frac{x}{\cos^2 x}dx = x\tan x - \int \tan x\, dx$$

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$$

Let:

$u = \cos x \to du = -\sin x\, dx$

$$\int \frac{\sin x}{\cos x} dx \to -\int u^{-1}\, du = -\ln|u| + C$$

$$\int \tan x\, dx = -\ln|\cos x| + C$$

Thus,

$$\int \frac{x}{\cos^2 x} dx = x \tan x + \ln|\cos x| + C$$

18.

$$\int_{-2\pi^2}^{2\pi^2} \frac{\sin(x^{10^n})}{x^{10n-1}} dx, n \in \mathbb{Z}$$

$$\sin x = x - \frac{x^3}{3!} + \cdots + o(x^n)$$

Where $o(x^n)$ is the Lagrange remainder, note that if you want to use the Taylor approximation/expansion to solve some integral problem, first you need to prove that the Lagrange remainder of the function is equal to 0 or at least the error of the approximation is approaching 0 $[o(x^n) \to 0]$

$$\sin x^{10^n} = x^{10^n} - \frac{x^{3\cdot 10^n}}{3!} + \cdots + o(x^n)$$

Under assumption that the Lagrange remainder is equal to 0, we may use the Maclaurin approximation to solve the integral.

$$\int_{-2\pi^2}^{2\pi^2} \frac{\sin(x^{10^n})}{x^{10n-1}} dx = \int_{-2\pi^2}^{2\pi^2} \frac{x^{10^n} - \frac{x^{3\cdot 10^n}}{3!} + \cdots + o(x^n)}{x^{10n-1}} dx$$

$k \cdot 10^n - 10n - 1, n \in \mathbb{Z}$ *always yield an odd number, hence:*

$$\int_{-2\pi^2}^{2\pi^2} \frac{\sin(x^{10^n})}{x^{10n-1}} dx = 0$$

19.

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} \to \int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + 1} dx$$

Let:

$u = e^x \to du = e^x dx$

$$\int_{-\infty}^{\infty} \frac{e^x}{e^{2x} + 1} dx \to \int_{0}^{\infty} \frac{du}{u^2 + 1} = -\frac{\pi}{2}$$

20.

$$\int \frac{x}{\sqrt{2 - x}} dx$$

Let:

$u = 2 - x \rightarrow du = -dx$

$$\int \frac{x}{\sqrt{2-x}}\,dx \rightarrow \int \frac{u-2}{\sqrt{u}}\,du$$

$$\int \frac{u-2}{\sqrt{u}}\,du = \int \sqrt{u}\,du - \int \frac{2}{\sqrt{u}}\,du$$

$$\int \sqrt{u}\,du - \int \frac{2}{\sqrt{u}}\,du = \frac{2}{3}u^{\frac{3}{2}} - 4\sqrt{u} + C$$

$$\int \frac{x}{\sqrt{2-x}}\,dx = \frac{2}{3}(2-x)^{\frac{3}{2}} - 4\sqrt{2-x} + C$$