UNIVERSITY ENTRANCE EXAMINATION 2017

MATHEMATICS ('A' LEVEL EQUIVALENT)

Duration: 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper has TWO (2) sections − A and B, and comprises SIXTEEN (16) printed pages.
2. Attempt all sections.
3. Answer all questions in Section A. Indicate your answers on the answer paper provided. Each question carries 2 marks. Marks will not be deducted for wrong answers.
4. Answer FOUR (4) questions from Section B with not more than THREE (3) from any one option. Write your answers on the answer paper provided. Begin each question on a fresh sheet of paper. Write the question number clearly. Each question carries 15 marks.
5. A non-programmable scientific calculator may be used. However, candidates should lay out systematically the various steps in the calculation.
6. At the end of the examination, attach the cover paper on top of your answer script. Complete the information required on the cover page and tie the papers together with the string provided. The colour of the cover paper for this examination is GREEN.
7. Do not take any paper, including the question paper and unused answer paper, out of the examination hall.

SECTION A (40 Marks)

Answer all questions in this section. Each question carries 2 marks.

1. There are 11 girls and 22 boys in a class. Two students are chosen at random. The probability that the two students chosen are of the sex is

   (A) $\frac{20}{176}$

   (B) $\frac{44}{176}$

   (C) $\frac{99}{176}$

   (D) $\frac{156}{176}$

   (E) none of the above

2. The number of ways to choose a pair of distinct numbers a and b from the set $\{1, 2, \ldots, 49\}$ such that $\left|a - b\right| \leq 3$ is

   (A) 141

   (B) 144

   (C) 147

   (D) 150

   (E) none of the above

3. The derivative of $\ln \sqrt{2x}$ with respect to x is

   (A) $\dfrac{1}{2\sqrt{2x}}$

   (B) $\dfrac{1}{\sqrt{2x}}$

   (C) $\dfrac{1}{2x}$

   (D) $\dfrac{1}{x}$

   (E) none of the above

4. The maximum value of the function $f(x) = (3\sin x - 2)^2 - 2$ is

   (A) 2

   (B) 7

   (C) 23

   (D) 25

   (E) none of the above

5. The function $f(x) = ax + b$ is such that $f(3) = 5$ and $f^{-1}(4) = 1$. The value of a is

   (A) -1

   (B) $-\frac{1}{2}$

   (C) $\frac{1}{2}$

   (D) 1

   (E) none of the above

6. Let $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{AC} = w\mathbf{i} - 5\mathbf{j}$. Suppose C lies on AB produced. Then the value of w is

   (A) -10

   (B) $-\frac{5}{2}$

   (C) $\frac{5}{2}$

   (D) 10

   (E) none of the above

7. The derivative of $\dfrac{\ln(7-x)}{e^{7x}}$ with respect to x is

(A) $-e^{-7x}\Big(\dfrac{1}{7-x} - \ln(7-x)\Big)$

(B) $-e^{-7x}\Big(\dfrac{7}{7-x} - 7\ln(7-x)\Big)$

(C) $-e^{-7x}\Big(\dfrac{1}{7-x} + \ln(7-x)\Big)$

(D) $-e^{-7x}\Big(\dfrac{1}{7-x} + 7\ln(7-x)\Big)$

(E) none of the above

8. Suppose the line $py + qx + 20 = 0$ is parallel to the line $5y = 2x + 35$. Then the value of $\dfrac{p}{q}$ is

(A) $\dfrac{2}{5}$

(B) $\dfrac{5}{2}$

(C) $-\dfrac{2}{5}$

(D) $-\dfrac{5}{2}$

(E) none of the above

9. The integral

$$\int \frac{\pi}{\sqrt{2\pi x + 3}}\, dx$$

equals

(A) $\sqrt{2\pi x + 3} + C$

(B) $2\sqrt{2\pi x + 3} + C$

(C) $\pi\sqrt{2\pi x + 3} + C$

(D) $2\pi\sqrt{2\pi x + 3} + C$

(E) none of the above

10. Suppose $-7 \le x \le 9$ and $-6 \le y \le 5$. Then the largest value of $(y-3)^2 + (x-4)^2$ is

(A) 101

(B) 136

(C) 200

(D) 225

(E) none of the above

11. The derivative of $\cos(\sqrt{2x})$ with respect to x is

(A) $-\sqrt{2}\sin(\sqrt{2x})$

(B) $-\sqrt{2x}\sin(\sqrt{2x})$

(C) $-\frac{1}{\sqrt{2x}}\sin(\sqrt{2x})$

(D) $-\frac{\sqrt{2}}{\sqrt{x}}\sin(\sqrt{2x})$

(E) none of the above

12. Which of the following is the result of completing the square of the expression $-5x^2 + 3x - 1$?

(A) $-5\left(x - \dfrac{3}{10}\right)^2 - \dfrac{14}{5}$

(B) $-5\left(x - \dfrac{3}{10}\right)^2 - \dfrac{29}{20}$

(C) $-5\left(x - \dfrac{3}{10}\right)^2 - \dfrac{11}{20}$

(D) $-5\left(x - \dfrac{3}{10}\right)^2 + \dfrac{4}{5}$

(E) none of the above

13. Two men P and Q start at the same point and travel in opposite directions by car. The speed at which P travels is $4\,\text{kmh}^{-1}$ less than that of Q. Suppose after 5 hours, they are $580\,\text{km}$ apart. Then the speed at which P travels is

(A) 44

(B) 48

(C) 52

(D) 56

(E) none of the above

14. The equation of a curve is $y = x^3 - 3x^2 + 1$. The value of c for which the line $y = c + \frac{1}{3}x$ is a normal to the curve is

(A) -4

(B) -2

(C) 2

(D) 4

(E) none of the above

15. Which option corresponds to the partial fraction decomposition of the rational function $\dfrac{-13}{6x^2 - 5x - 6}$?

(A) $\dfrac{2}{2x-3} + \dfrac{3}{3x+2}$

(B) $\dfrac{2}{2x-3} - \dfrac{3}{3x+2}$

(C) $-\dfrac{2}{2x-3} + \dfrac{3}{3x+2}$

(D) $-\dfrac{2}{2x-3} - \dfrac{3}{3x+2}$

(E) none of the above

16. The maximum value of the function $f(x) = (14 - x)(x - 16) - 15$ is

(A) -14

(B) -16

(C) $-(14 \times 16) - 15$

(D) $(14 \times 16) - 15$

(E) none of the above

17. The line $x + y = 6$ intersects the curve $\dfrac{1}{x-1} = \dfrac{3}{y} + \dfrac{1}{4}$ at the points A and B. The equation of the perpendicular bisector of the line AB is

(A) $y = x - 17$

(B) $y = x - 11$

(C) $y = x + 11$

(D) $y = x + 17$

(E) none of the above

18. The derivative of $\dfrac{\sin(e^{-x})}{e^x}$ with respect to x is

(A) $\dfrac{-\cos(e^{-x}) - e^x \sin(e^{-x})}{e^{2x}}$

(B) $\dfrac{e^x \sin(e^{-x}) - \cos(e^{-x})}{e^{2x}}$

(C) $\dfrac{e^x \cos(e^{-x}) - \sin(e^{-x})}{e^{2x}}$

(D) $\dfrac{\cos(e^{-x}) - e^x \sin(e^{-x})}{e^{2x}}$

(E) none of the above

19. Suppose $\lg x = a$ and $\lg y = b$. Then $\lg \sqrt[3]{\dfrac{x^6}{1000y}}$ in terms of a and b is

(A) $2a - 1 + \frac{1}{3}b$

(B) $2a - 1 - \frac{1}{3}b$

(C) $2a + 1 + \frac{1}{3}b$

(D) $2a + 1 - \frac{1}{3}b$

(E) none of the above

20. The line $y = 2x + 1$ is reflected about the line $x = 3$. The equation of the reflected line is

(A) $y = -2x + 3$

(B) $y = -2x + 5$

(C) $y = -2x + 10$

(D) $y = -2x + 13$

(E) none of the above

SECTION B (60 Marks)

Answer FOUR (4) questions with not more than THREE (3) from any one option.

Option (a) - Pure Mathematics

21(a). Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{2x^2+x}{(1-x)(2x^2+1)}\,dx.$ [5 Marks]

21(b). Evaluate $\displaystyle\int_1^e (\ln x)^2\,dx.$ [5 Marks]

21(c). By means of the substitution $t = \sin x$, or otherwise, find the exact value of the integral $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{4\cos x}{3+\cos^2 x}\,dx.$ [5 Marks]

22(a). Prove by mathematical induction that, for every positive integer n,

$$\sum_{r=1}^{n}(r+1)2^r = n(2^{n+1}).$$ [5 Marks]

22(b). Solve the inequality $\dfrac{x}{x^2-4} \leq 0.$

Hence, or otherwise, solve the inequality $\dfrac{1}{|x|+2} \leq \dfrac{1}{2-|x|}.$ [5 Marks]

22(c). Let the complex number $z = a + ib$, where a and b are real numbers. Suppose $\dfrac{1}{e^{iz}} = 2 + i$. Find the **exact** values of a and b. [5 Marks]

23(a). Solve the differential equation

$$(1+x)\frac{dy}{dx} - xy = xe^{-x},$$

given that $y = 1$ when $x = 0$. [8 Marks]

23(b). By using the substitution $y = \frac{v}{x}$, find the general solution of the differential equation

$$x(x-2)\frac{dy}{dx} = y(x+2) - 4,$$

expressing y in terms of x. [7 Marks]

24. Relative to the origin O, points A and B have position vectors $\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. The position vector of the point P is $\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

(i) Find the vector equation of line AB. [3 Marks]

(ii) The point C on the line AB is such that $AC : CB = 2 : 3$. Find the position vector of the point C. [2 Marks]

(iii) The point F lies on the line AB. Suppose $\overrightarrow{PF}$ is perpendicular to the line AB. Find the position vector of the point F. [5 Marks]

(iv) Find the perpendicular distance from the point P to the line AB, giving your answer in surd form. [2 Marks]

(v) The point Q lies on the line PF produced and is such that F is the mid-point of PQ. Find the position vector of Q. [3 Marks]

Option (b) - Particle Mechanics

[In this secion, take the acceleration due to gravity to be 9.8 ms^{-2}, unless otherwise stated. Give non-exact numerical answers correct to three significant figures, unless otherwise specified.]

25. A light elastic string has natural L metres and modulus of elasticity $2mg$ N, where g is the acceleration due to gravity. One end of the string is attached to a fixed point A. A particle P of mass m kg is attached to the other end of the string.

    (i) When the particle P hangs vertically below A in equilibrium, find the length of the string in terms of L. [3 Marks]

    When the particle P moves with constant speed in a horizontal circle with centre C, where C is at a distance L metres vertically below A, the angular velocity of P is ω radians per second and the string AP is inclined at an angle θ to AC.

    (ii) Show that $L\omega^2 = g$. [4 Marks]

    (iii) Find the angle θ. Find also the tension in the string, giving your answer in terms of m and g. [5 Marks]

    (iv) Find the time taken for P to move once around the circle, giving your answer in terms of π, L and g. [3 Marks]

26. [Take the acceleration due to gravity to be $10\,\text{ms}^{-2}$ in this question.]

A railway engine of mass 20 000 kg is working at a rate of 300 kW against resistance to motion of $(1000 + kv)\,\text{N}$, where k is a constant. If the engine can maintain a speed of $75\,\text{ms}^{-1}$ on level ground, find the value of k. [3 Marks]

Assuming the power remains unchanged.

(i) Find the acceleration of the engine when it is travelling at $50\,\text{ms}^{-1}$ on level ground. [3 Marks]

Travelling on the level track, the train reaches the bottom of a slope of inclination α, where $\sin\alpha = \dfrac{1}{125}$. It then begins to climb the slope.

(ii) Show that when the train is travelling at speed $v\,\text{ms}^{-1}$, the acceleration $\dfrac{dv}{dt}$, is given by $\dfrac{dv}{dt} = \dfrac{7500 - 65v - v^2}{500v}$. [4 Marks]

(iii) Find the maximum speed that the engine can maintain on this incline. [2 Marks]

(iv) If the train is travelling at $10\,\text{ms}^{-1}$ when it is at the bottom of the slope, find the time it takes to reach a speed of $40\,\text{ms}^{-1}$ up this slope. [3 Marks]

27. [Take the acceleration due to gravity to be $10\,\mathrm{ms^{-2}}$ in this question.]

A particle is projected from a point O which is 25 metres above horizontal ground. Its velocity of projection is $U\,\mathrm{ms^{-1}}$ at an angle α above the horizontal. The particle moves freely under gravity and hits the ground at the point A. Taking the horizontal and vertical through O as the $x-$axis and $y-$axis respectively, derive the equation of trajectory. [3 Marks]

At a height of 5 metres above O, the speed of the particle is found to be $20\,\mathrm{ms^{-1}}$ and the direction of motion is 60° to the horizontal. Find U and $\tan\alpha$. Hence, show that the equation of trajectory can be reduced to

$$y = 20 - \frac{1}{20}(x-20)^2. \qquad \text{[6 Marks]}$$

Find the greatest height of the particle above the ground. [2 Marks]

Find the horizontal distance between O and A, the direction of motion of the particle just before it hits the ground at A and the time taken from O to A.

[4 Marks]

28. A light elastic string of natural length L metres and modulus of elasticity $mg\,\mathrm{N}$, where g is the acceleration due to gravity, has one end A attached to a fixed point on a rough plane which is inclined to the horizontal at an angle α. The other end B is attached to a particle of mass $m\,\mathrm{kg}$ which rests on the plane, with AB along a line of greatest slope of the plane and B is lower than A. The coefficient of friction between the particle and the plane is $\mu\ (< \tan\alpha)$. The particle is released from rest when $AB = L$ metres. Show that the particle moves down the plane a distance $2L(\sin\alpha - \mu\cos\alpha)$ metres before coming to rest. [5 Marks]

Show that if $u > \frac{1}{3}\tan\alpha$, there is no further motion, but that if $\mu < \frac{1}{3}\tan\alpha$, the particle moves a distance $2L(\sin\alpha - 3\mu\cos\alpha)$ metres up the plane before coming to instantaneous rest again. [10 Marks]

Option (c) - Probability and Statistics

[In this secion, probabilities should be expressed as either fractions in lowest terms or decimals with three significant figures.]

29(a). In a group of 50 students, there are 20 boys and 30 girls. It is known that $\frac{1}{3}$ of the girls have perfect vision and $\frac{2}{5}$ of the boys have perfect vision. One student is chosen randomly from the group. Let A represents the event that the student chosen is a boy. Let B be the event that the student chosen has perfect vision.

(i) Find $P(A' \cap B)$, where A' denotes the complement of A. [1 Marks]

(ii) Find $P(A \cup B')$, where B' denotes the complement of B. [2 Marks]

(iii) Find $P(A|B)$. [2 Marks]

(iv) If four students are chosen randomly from this group, find the probability that there are 2 boys and 2 girls, of whom exactly 2 girls and 1 boy have perfect vision. [3 Marks]

29(b). A bag initially contains 1 red ball and 2 blue balls. A trial consists of selecting a ball at random, noting the colour, and replacing it together with an additional ball of the same colour. Suppose three trials are made.

(i) Find the probability that at least one blue ball is drawn. [2 Marks]

(ii) Find the probability that exactly one blue ball is drawn. [2 Marks]

(iii) Given that all the three balls drawn are of the same colour, find the probability that they are all red. [3 Marks]

30. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} \frac{3}{28}(1+x^2), & -2 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Sketch the graph of $f(x)$ and hence state the value of $E(X)$. [3 Marks]

(ii) Find the exact value of $E(X^2)$ and $E\left(\dfrac{1}{1+X^2}\right)$. [5 Marks]

(iii) Find the cumulative distribution function of X. [4 Marks]

(iv) Find the value a such that $P(\left|X\right| < a) = \dfrac{2}{7}$. [3 Marks]

31. A dealer has a stock of 6 similar television sets which he rents out to customers on a monthly basis. It is known from past experience of the dealer that the monthly demand for the television sets has a Poisson distribution with mean 3.56.

(i) Find the probability that in any month at least two sets are not rented out. [3 Marks]

(ii) Calculate the probability that in any month the demand is not fully met. [3 Marks]

(iii) Find the probability that exactly one set not rented out in exactly two out of four months. [4 Marks]

If one of the television sets is temporarily withdrawn for repairs, find the expected number of television sets rented out in a month. [5 Marks]

32. The weight of a pig is normally distributed with mean 375 g and standard deviation 22 g. The weight of a rabbit is normally distributed with mean 425 g and standard deviation 25 g.

(i) Find the probability that a randomly chosen pig weighs between 360 g and 390 g. [3 Marks]

(ii) Five pigs and four rabbits are randomly chosen. Find the probability that the pigs weigh more than the rabbits. [4 Marks]

(iii) Find the probability that the average weight of a sample of 10 rabbits will exceed the average weight of a sample of 8 pigs by at least 39 g. [4 Marks]

(iv) If 50 pigs are chosen at random, what is the probability that at most 21 will weigh between 360 g and 390 g? [4 Marks]

END OF PAPER