TEST 3 TOP UNIVERSITIES CLASS WARDAYA COLLEGE

VECTOR, COMPLEX, MATHEMATICS INDUCTION

2 HOURS – DR. Anton Wardaya, M.Sc

1. The complex number z is given by $z = re^{i\theta}$, where $r > 0$ and $0 \leq \theta \leq \frac{1}{2}\pi$.
   a. Given that $w = (1 - i\sqrt{3})z$, find $|w|$ in terms of r and arg w in terms of θ.
   b. Given that r has a fixed value, draw an Argand diagram to show the locus of z as θ varies. On the same diagram, show the corresponding locus of w. You should identify the modulus and argument of the end-points of each locus.
   c. Given that $\arg\left(\frac{z^{10}}{w^2}\right) = \pi$, find θ.
2. Prove by the method of mathematical induction that
$$\sum_{r=1}^{n} r(2r^2 + 1) = \frac{1}{2}n(n+1)(n^2 + n + 1).$$
   a. It given that $f(r) = 2r^3 + 3r^2 + r + 24$. Show that $f(r) - f(f - 1) = ar^2$. For a constant a to be determined. Hence find a formula for $\sum_{r=1}^{n} r^2$, fully factorising your answer.
   b. Find $\sum_{r=1}^{n} f(r)$. (You should not simplify your answer).
3. Do not use a graphic calculator in answering this question.
   a. The roots of the equation $z^2 = -8i$ are z_1 and z_2 in cartesian form $x + iy$, showing your working.
   b. Hence, or otherwise, find in cartesian form the roots w_1 and w_2 of the equation $w^2 + 4w + (4 + 2i) = 0$
   c. Using a single Argand diagram, sketch the loci
      i. $|z - z_1| = |z - z_2|$
      ii. $|z - w_1| = |z - w_2|$
   d. Give a reason why there are no points wich lie on both of these loci.
4. The plane p passes thrugh the points with coordinates $(4, -1, -3)$, $(-2, -5, 2)$ and $(4, -3, -2)$.
   a. Find a cartesian equation of p
   The line l_1 has equation $\frac{x-1}{2} = \frac{y-2}{-4} = \frac{z+3}{1}$ and the line l_2 has equation $\frac{x+2}{1} = \frac{y-1}{5} = \frac{z-3}{k}$, where k is a constant. It is given that l_1 and l_2 intersect.
   b. Find the value of k.
   c. Show that l_1 lies in p and find the coordinates of the point at which l_2 intersects p.
   d. Find the acute angle between l_2 and p.
5. The complex number z_1 and z_2 are given by $1 + i\sqrt{3}$ and $-1 - i$ respectively.
   a. Express each of z_1 and z_1 in polar from $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < 0 \leq \pi$. Give r and θ in exact from.
   b. Find the complex conjugate of $\frac{z_1}{z_2}$ in exact polar form.
   c. On a single Argand diagram, sketch the loci
      i. $|z - z_1| = 2$
      ii. Arg $(z - z_2) = \frac{1}{4}\pi$
   d. Find where the locus $|z - z_1| = 2$ meets the positive real axis.

6. The line l has equation $\frac{x-10}{-3} = \frac{y+1}{6} = \frac{z+3}{9}$, and the plane p has equation $x - 2y - 3z = 0$.
   a. Show that l is perpendicular to p.
   b. Find the coordinates of the point of intersection of l and p.
   c. Show that the point A with coordinates $(-2,\ 23,\ 33)$ ies on l. Find the coordinates of the point B which is the mirror image of A in p.
   d. Find the area of triangle OAB, where O is the origin, giving your answer to the nearst whole nomber
7. .
   a. Solve the equation $z^7 - (1 + i) = 0$. Giving the roots in the form $re^{i\propto}$, where $r > 0$ and $-\pi < \alpha < \pi$.
   b. Show the roots on an Argand diagram.
   c. The roots represented by z_1 and z_2 are such that $0 < \arg(z_1) < \arg(z_2) < \frac{1}{2}\pi$. Explain why the locus of all points z such that $|z - z_1| = |z - z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its exact cartesian equation.
8. The planes p_1 and p_2 have equations $r \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 1$ and $r \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 2$ respectively, and meet in a line l.
   a. Find the acute angle between p_1 and p_2.
   b. Find a vector equation of l.
   c. The plane p_3 has equation $2x + y + 3z - 1 + k(-x + 2y + z - 2) = 0$. Explain why l lies in p_3 for any constant k. Hence or otherwise find a cartesian equation of the plane in wich both l and the point $(2,\ 3, 4)$ lie.
9. The lie l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2,\ 3,\ 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find;
   a. The coordinates of the point of intersection of l and p.
   b. The acute angle between l and p
   c. The prependicular distance from A to p.
10. A sequace $u_1, u_2, u_3, \ldots$ is such that $u_l = 1$ and $u_{n+1} = u_n - \frac{2n+1}{n^2(n+1)^2}$ for all$n \geq 1$.
    a. Use the method of mathematical induction to prove that $u_n = \frac{1}{n^2}$
    b. Hence find $\sum_{n=1}^{N} \frac{2n+1}{n^2(n+1)^2}$.
    c. Give a reason why the series in part (ii) is convergent and state the sum to infinity.
    d. Use your answer to part (ii) fo find $\sum_{n=2}^{N} \frac{2n-1}{n^2(n-1)}$.