NUS
National University of Singapore

UNIVERSITY ENTRANCE EXAMINATION

MATHEMATICS ('A' LEVEL EQUIVALENT)

Time allowed : 2 hours

<u>**INSTRUCTIONS TO CANDIDATES**</u>

1. This examination paper has TWO (2) sections – A and B, and comprises **FIFTEEN (15)** printed pages.
2. Attempt all sections.
3. Answer all questions in section A. Indicate your answers on the answer paper provided. Each question carries 2 marks. Marks will not be deducted for wrong answers.
4. Answer FOUR (4) questions from Section B with not more than THREE (3) from any one option. Write your answers on the answer paper provided. Begin each question on a fresh sheet of paper. Write the question number clearly. Each question carries 15 marks.
5. A non-programmable scientific calculator may be used. However, candidates should lay out systematically the various steps in the calculation.
6. At the end of the examination, attach the cover paper on top of your answer script. Complete the information required on the cover page and tie the papers together with the string provided. The colour of the cover paper for this examination is **GREEN**.
7. Do not take any paper, including the question paper and unused answer paper, out of the examination hall.

SECTION A (40 Marks)

Answer all questions in this section. Each question carries 2 marks.

1. Suppose $-7 \le x \le -3$ and $2 \le y \le 9$. Then the largest value of $(y-6)^2 + x^2$ is

   (A) 18

   (B) 25

   (C) 58

   (D) 65

   (E) none of the above

2. The derivative of $\dfrac{\ln 3x}{e^{3x}}$ with respect to x is

   (A) $e^{-3x}\big(\frac{1}{x} - \ln 3x\big)$

   (B) $e^{-3x}\big(\frac{3}{x} - 3\ln 3x\big)$

   (C) $e^{-3x}\big(\frac{1}{x} - 3\ln 3x\big)$

   (D) $e^{-3x}\big(\frac{1}{3x} - 3\ln 3x\big)$

   (E) none of the above

3. The line $y = 2x + 1$ is reflected about the line $y = 2$. The equation of the reflected line is

   (A) $y = -2x + 2$

   (B) $y = -2x + 3$

   (C) $y = 2x + 2$

   (D) $y = 2x + 3$

   (E) none of the above

4. The function $g(x) = x^3 + mx + p$ is such that the equation $g(x) = x$ has solutions $x = 2$ and $x = 3$. The value of p is

   (A) -30

   (B) -18

   (C) 18

   (D) 30

   (E) none of the above

5. The derivative of $\sin(\sin x)$ with respect to x is

   (A) $\cos(\sin x)\sin x$

   (B) $\cos(\sin x)\cos x$

   (C) $\cos(\cos x)\sin x$

   (D) $\cos(\cos x)\cos x$

   (E) none of the above

6. The integral

$$\int \frac{e}{\sqrt{ex+1}}\,dx$$

   equals

   (A) $\sqrt{ex+1} + C$

   (B) $2\sqrt{ex+1} + C$

   (C) $e\sqrt{ex+1} + C$

   (D) $2e\sqrt{ex+1} + C$

   (E) none of the above

7. A man throws three ordinary fair dice and observes the number on the top face of each die. The probability that all the three numbers are different is

(A) $\frac{60}{216}$

(B) $\frac{120}{216}$

(C) $\frac{210}{216}$

(D) $\frac{215}{216}$

(E) none of the above

8. The minimum value of the function $f(x) = (2\sin x - 5)^2 + 5$ is

(A) 5

(B) 10

(C) 14

(D) 30

(E) none of the above

9. The sum of 40 consecutive even numbers is 2280. The smallest value of these numbers is

(A) 20

(B) 22

(C) 24

(D) 26

(E) none of the above

10. Which option corresponds to the partial fraction decomposition of the rational function $\dfrac{13}{-6x^2+5x+6}$?

(A) $-\dfrac{2}{2x-3}+\dfrac{3}{3x+2}$

(B) $\dfrac{2}{2x-3}+\dfrac{3}{3x+2}$

(C) $-\dfrac{2}{2x-3}-\dfrac{3}{3x+2}$

(D) $\dfrac{2}{2x-3}-\dfrac{3}{3x+2}$

(E) none of the above

11. Five cards each have a single digit written on them. The digits are $1, 1, 3, 4, 5$ respectively. The number of different 3-digit numbers that can be formed by placing three of the cards side by side is

(A) 12

(B) 24

(C) 33

(D) 60

(E) none of the above

12. Suppose $p > 0$ and $\log_p 9 + \log_p n = 0$. Then the value of n is

(A) -9

(B) $-\frac{1}{9}$

(C) $\frac{1}{9}$

(D) 9

(E) none of the above

13. From a class of 6 boys and 4 girls, a group of 3 children is to be selected. The number of possible groups if at least 1 boy is selected is

(A) 36

(B) 80

(C) 116

(D) 120

(E) none of the above

14. The line $y + 9x = k$ is a tangent to the curve $y = 9x^2 + 5$. The value of k is

(A) $-\frac{11}{4}$

(B) $-\frac{9}{4}$

(C) $\frac{9}{4}$

(D) $\frac{11}{4}$

(E) none of the above

15. The position vectors of the points A, B and C, relative to the origin O, are $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 13\mathbf{j}$ respectively. The value of m for which $\overrightarrow{OA} + m\overrightarrow{OB}$ is perpendicular to $\overrightarrow{OC}$ is

(A) $-\frac{8}{3}$

(B) $-\frac{3}{8}$

(C) $\frac{3}{8}$

(D) $\frac{8}{3}$

(E) none of the above

16. A geometric progression has first term a and common ratio $\frac{1}{\sqrt{2}}$. The sum to infinity of the progression is

(A) $a(\sqrt{2}+2)$

(B) $a(\sqrt{2}-2)$

(C) $a(-\sqrt{2}+2)$

(D) $a(-\sqrt{2}-2)$

(E) none of the above

17. The maximum value of the function $f(x) = (15-x)(x-17)$ is

(A) -15×17

(B) 15×17

(C) -1

(D) 1

(E) none of the above

18. The derivative of $\dfrac{\sin(x^2)}{x^2}$ with respect to x is

(A) $\dfrac{x\cos(x^2) - 2\sin(x^2)}{x^3}$

(B) $\dfrac{2x^2\cos(x^2) - 2\sin(x^2)}{x^3}$

(C) $\dfrac{x^2\cos(x^2) - 2\sin(x^2)}{x^3}$

(D) $\dfrac{-2x^2\cos(x^2) - 2\sin(x^2)}{x^3}$

(E) none of the above

19. Suppose $\lg x = a$ and $\lg y = b$. Then $\lg \sqrt{\dfrac{100x^2}{y^3}}$ in terms of a and b is

(A) $a - 1 + \frac{3}{2}b$

(B) $a - 1 - \frac{3}{2}b$

(C) $a + 1 + \frac{3}{2}b$

(D) $a + 1 - \frac{3}{2}b$

(E) none of the above

20. Which of the following is the result of completing the square of the expression $-5x^2 + 3x - 1$?

(A) $-5\left(x - \dfrac{3}{5}\right)^2 + \dfrac{4}{5}$

(B) $-5\left(x - \dfrac{3}{5}\right)^2 - \dfrac{14}{5}$

(C) $-5\left(x - \dfrac{3}{10}\right)^2 - \dfrac{11}{20}$

(D) $-5\left(x - \dfrac{3}{10}\right)^2 - \dfrac{29}{20}$

(E) none of the above

SECTION B (60 Marks)

Answer FOUR (4) questions with not more than THREE (3) from any one option.

Option (a) - Pure Mathematics

21(a). Evaluate $\displaystyle\int_{-3}^{1} \frac{x+5}{x^2+6x+25}\,dx$. [5 Marks]

21(b). Evaluate $\displaystyle\int_{1}^{2} \frac{1}{x^2}\ln(2x)\,dx$. [5 Marks]

21(c). By means of the substitution $x = \cos 2\theta$, or otherwise, find the exact value of the integral $\displaystyle\int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\,dx$. [5 Marks]

22(a). Prove by induction, or otherwise, that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2}\Big(1 - \frac{1}{2n+1}\Big).$$ [5 Marks]

22(b). Solve the inequality $\dfrac{4}{x+1} < x - 2$.

Hence, or otherwise, solve the inequality $\dfrac{4}{|x|+1} < x - 2$. [5 Marks]

22(c). Solve the simultaneous equations

$$3z + w = 1$$

$$(1+i)z + iw = -3 + 2i,$$

where $i = \sqrt{-1}$. [5 Marks]

23(a). By using the substitution $y = ux$, find the general solution of the differential equation

$$x\frac{dy}{dx} = y + 2x^2 - 1,$$

expressing y in terms of x.

Find the particular solution which has a stationary point on the positive x-axis. Sketch this particular solution. [8 Marks]

23(b). Find the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = 4\sin 2x,$$

expressing y in terms of x. [7 Marks]

24(a). Relative to an origin O, the lines L_1 and L_2 have equations

$$\mathbf{r} = (7\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \quad \text{and} \quad \mathbf{r} = (\mathbf{i} + \mathbf{j} - 5\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{k})$$

respectively, where λ and μ are parameters.

(i) Find the acute angle between L_1 and L_2 giving your answer correct to the nearest $0.1°$. [2 Marks]

Let P be a point on L_1 and Q be a point on L_2.

(ii) Find, in terms of λ and μ, the vector $\overrightarrow{PQ}$. [4 Marks]

(iii) Find the values of λ and μ for which $\overrightarrow{PQ}$ is perpendicular to both L_1 and L_2. [4 Marks]

24(b). The line L_3 with equation $\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ passes through the point $A(5, -2, 3)$. The line L_4 which is parallel to the vector $-\mathbf{j} + \mathbf{k}$ intersects L_3 at the point $B(3, -1, 2)$. Find the length of projection of AB onto the line L_4, leaving your answer in surd form. [5 Marks]

Option (b) - Particle Mechanics

[In this secion, take the acceleration due to gravity to be 9.8 ms^{-2}, unless otherwise stated. Give non-exact numerical answers correct to three significant figures, unless otherwise specified.]

25. Two trains, A and B, travelling on parallel rails in the same direction with uniform accelerations f ms^{-2} and $\frac{3f}{2}$ ms^{-2} respectively, simultaneously pass a signal box P with speeds u ms^{-1} and $\frac{u}{2}$ ms^{-1} respectively. The trains are once again level when they pass the next signal box Q, and both trains A and B immediately brake with uniform decelerations f ms^{-2} and F ms^{-2} respectively, to stop at the terminal station R.

   (i) For the stretch of the journey between P and Q, write expressions for the distance of each train from P, t seconds after the trains pass P. Show that the greatest distance A gets ahead of B is $\frac{u^2}{4f}$ metres. [4 Marks]

   (ii) Find the speeds of the trains as they pass the signal box Q. Calculate the distance PQ in terms of u and f. [5 Marks]

   (iii) Show that $F : f = 49 : 36$. [3 Marks]

   (iv) Show that train B reaches the terminal station $\frac{18u}{7f}$ seconds after passing through Q. [3 Marks]

26(a). Two particles of masses m kg and $2m$ kg are moving in the same straight line towards each other with speeds u ms^{-1} and $2u$ ms^{-1} respectively. When they collide, the impulse acting on each particle has magnitude $3mu$. Show that the loss in kinetic energy is $\frac{9mu^2}{4}$. [8 Marks]

26(b). A ball is thrown from a point P on a cliff of height h metres above the seashore. It strikes the shore at a point Q, where PQ is inclined at an angle α below the horizontal. If the angle of projection is also α, show that the speed of projection is $\frac{\sqrt{gh}}{2\sin\alpha}$ ms^{-1}, where g is the acceleration due to gravity. Prove also that the ball strikes the shore at an angle $\tan^{-1}(3\tan\alpha)$ with the horizontal. [7 Marks]

27. A pile-driver of mass 1200 kg falls freely from a height 5 metres and strikes without rebounding a pile of mass 800 kg. After the blow, the pile and the driver move on together. If the pile is driven a distance of 0.3 metres into the ground, find

(i) the speed at which the pile starts to move into the ground, [7 Marks]

(ii) the average resistance of the ground to penetration, [5 Marks]

(iii) the time for which the pile is in motion. [3 Marks]

28(a). The engine of a car of mass 900 kg can produce a maximum power of 32 kW. The magnitude of the resistance to motion of the car may be assumed to be proportional to the square of its speed. When the car is moving on a horizontal road, it has a maximum speed of $40\,\mathrm{ms}^{-1}$. Find the angle of inclination to the horizontal of the steepest hill that the car can ascend at a constant speed of $16\,\mathrm{ms}^{-1}$, giving your answer correct to the nearest $0.1°$. [7 Marks]

28(b). A particle of mass 1 kg moves on the positive x-axis under the action of a force of magnitude $\dfrac{1}{2x^2}$ N, where x metres is the distance of the particle from the origin O. The force is directed towards O. The particle is projected from the point $x = 1$ with initial speed $2\,\mathrm{ms}^{-1}$, in the direction of increasing x. When the particle is x metres from O, it speed is $v\,\mathrm{ms}^{-1}$. Find v in terms of x. Determine whether the particle ever returns to the point $x = 1$, justifying your answer. [8 Marks]

Option (c) - Probability and Statistics

[In this secion, probabilities should be expressed as either fractions in lowest terms or decimals with three significant figures.]

29(a). The probability of a football team winning any match is $\frac{1}{2}$ and the probability of losing any match is $\frac{1}{3}$. Two points are obtained for a win, one point for a draw and no points for a defeat. If the team plays four matches, find the probability that the team

(i) loses all four matches, [2 Marks]

(ii) loses exactly two matches, [2 Marks]

(iii) obtains exactly two points. [4 Marks]

29(b). Everyday, Robin has a choice of 2 routes to get to school. The probability that he gets to school without being delayed is $\dfrac{43}{50}$. The probability that he chooses the first route is $\dfrac{3}{5}$. If he gets to school without being delayed, the probability that the first route is chosen is $\dfrac{27}{43}$. Find the probability that he gets to school without being delayed if

(i) he chooses the first route, [4 Marks]

(ii) he chooses the second route. [3 Marks]

30. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} k(x+1), & -1 < x < 1, \\ \frac{k}{x}, & 1 \leq x \leq e, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Show that $k = \dfrac{1}{3}$. Find $E(X)$ and $Var(X)$. [4 Marks]

(ii) Find the cumulative distribution function $F(x)$ of X. [3 Marks]

(iii) Find the exact value of b such that $P(X \leq b) = \dfrac{3}{4}$. [2 Marks]

(iv) Let $Y = \dfrac{X}{X+1}$. Find the exact value of $E(Y)$. [3 Marks]

(v) Using a suitable approximation, find the probability that the mean of 100 independent observations of X is greater than 1. [3 Marks]

31. On average a Coastguard station receives one distress call every two days. A "bad" week is a week in which 5 or more distress calls are received.

(i) Find the probability that a week is a bad week. [5 Marks]

(ii) Find the probability that, in 8 randomly chosen weeks, at least 2 are bad weeks. [5 Marks]

(iii) Find the probability that, in 80 randomly chosen weeks, at least 30 are bad weeks. [5 Marks]

32. A machine is used to bag coal. The mass of coal delivered per bag may be assumed to be normally distributed with mean 65 kg and standard deviation 1.2 kg.

(i) Suppose 100 filled bags are chosen at random. Use a suitable approximation to calculate the probability that at least 60 of them contain 65 kg or more. [5 Marks]

(ii) Suppose two filled bags are chosen at random. Calculate the probabilities that

(a) each bag contains at least 67 kg, [2 Marks]

(b) the combined mass of coal in the two bags is less than 132 kg, [3 Marks]

(c) one bag contains at least 1 kg more than the other bag, [3 Marks]

(d) one bag weighs more than 68 kg and the other bag weighs less than 68 kg. [2 Marks]

END OF PAPER