Simulating Voting Systems

Russell Emerine

CMU-CS-25-129
August 2025

Computer Science Department
School of Computer Science
Carnegie Mellon University
Pittsburgh, PA 15213

Thesis Committee:
Daniel Sleator, Chair
Daniel Anderson
Fei Fang

Submitted in partial fulfillment of the requirements
for the degree of Masters in Computer Science.

Keywords: Voting Systems, Monte Carlo Simulation, Range Voting, Ranked Voting, Arrow's Theorem, Voting System Simulation

Abstract

Much of the past work on voting systems focuses on *ranked voting systems*, which have a number of limitations such as Arrow's Theorem [1]. In this paper we consider ranked voting systems as well as the less commonly used class of *rated voting systems*. The systems differ in that ranked voting systems only allow the voter to order the candidates, while in rated voting systems the voter can score each candidate independently. In 2000, Warren Smith [4] evaluated ranked and rated voting systems under a Monte Carlo simulation model of voter utilities and behaviors. We replicate Smith's results with a wider selection of voting systems, voter utility distributions, and polling models, and conclude different polling models lead to significantly different evaluations of the voting systems.

This project's source code is freely available online at `https://github.com/RussellEmerine/voting_simulation`.

Acknowledgments

Thanks to Danny for providing direction and guidance, Daniel and Fei for supporting the paper, and Chris and Tianyue for coming to listen to me.

Contents

1 Introduction

Many various voting systems have been proposed, used, and analyzed. Past research often evaluates properties of the voting systems, such as unexpected election outcomes or strange strategic voting behavior. Arrow's theorem [1], one of the most well-known results in the field, reveals a limitation that applies to all possible ranked voting systems, where votes can be expressed as an ordering of the candidates. Arrow's theorem does not apply to rated voting systems, where votes can give information on how much one candidate is preferred over the other. However, both these classes of voting systems have the limitation that they are susceptible to strategic voting.

Taking inspiration from Smith [4], we seek to evaluate voting systems under a statistical Monte Carlo simulation model, allowing ranked and rated systems and accounting for strategic voting. We define a utility model so that we may numerically evaluate the societal outcome of an election result. We list several voting systems and specify how honest and strategic voters behave for each. We describe the method used to generate utilities and polling data. We run Monte Carlo simulations to repeatedly produce utilities, votes, an election result, and an evaluation of that particular result. We consider those evaluations collectively in order to evaluate each voting system. We take this method of evaluation to be a good indicator of the performance of a voting system, and conclude with an analysis across voting systems and polling models.

2 The Utility Model

We represent and evaluate voting systems under the utility model of internal voter preferences. Each voter has a utility in $[0, 1]$ for each candidate. When the candidate is elected, each voter receives the corresponding utility. The performance of the whole system is then the sum of the utilities (or equivalently the average utility across all voters).

Traditional analyses of ranked voting systems only allow each voter to have internal preferences in the form of an ordering of the candidates, on the grounds that voters cannot intrinsically know how their utilities compare to other voters' utilities. However, if we allow ourselves to believe voters can produce a reasonable comparison of their own utilities to other voters' utilities, we can model situations that ranked voting systems cannot. There is no good way to create utility preferences from on ranked preferences, while it is easy to create a ranked preferences from utility preferences.

The following case illustrates the difference between the two Suppose 60% of voters prefer candidate c_1 over candidate c_2 only slightly, and 40% of voters prefer c_2 over c_1 very strongly. Under a ranked preference system, c_1 is considered the better candidate. Under a utility preference system, c_2 is considered the better candidate.

Furthermore, evaluating a candidate against a model of ranked internal voter preferences is as complex as (in fact, equivalent to) making a ranked voting system. The declaration of which candidate best matches ranked internal voter preferences is not obvious, and may become impossible, or at the very least interpretable under multiple standards, when Condorcet cycles are present (see the "Condorcet Least Reversal" 4.3.11 and "Condorcet Ranked Pairs" 4.3.12 voting systems). Meanwhile, evaluating the outcome using the utility model is straightforward.

Votes generally are intended to encode voter preferences in some reasonable way. This is usually "obvious"; for instance, the vote in range voting is intended to directly represent the utility, and the vote in ranked systems is intended to directly represent the ranking (which as mentioned can be easily created from utilities). Voters that vote according to this intended encoding are *honest*. We will see that in some cases, voters will expect a better result if their votes deviate from encoding their internal preferences in order to account for the expected behavior of other voters. These voters are *strategic*. We will have varying proportions of strategic and honest voters.

If voters cannot directly compare their utilities to other voters' utilities (or are not confident in the accuracy of their comparisons), we can account for that incongruity in the simulation if we so choose. Before deciding on votes, we linearly scale each voter's utilities so that the least favored candidate has utility 0 and the most favored candidate has utility 1. We cast votes using these scaled utilities. (The scaling process happens before votes are decided, so honest voters use their utilities from after scaling.) Then, we evaluate the elected candidate against the voters' original unscaled utilities. We run the simulation program with and without this rescaling operation.

The metric we analyze to evaluate voting systems is *regret*, which is the difference between the total utility of the maximum-utility candidate and the total utility of the elected candidate. This number is always positive, but its scale depends on the number of voters and the distributions of their utilities. We take care to note these parameters when comparing regret values between runs.

3 Polling Models

Strategic voting requires some knowledge of the expected behavior of other voters. This is presented in the form of a *poll order* produced from a *poll*.

A poll is some process in which data is collected from a sample of voters, and aggregated into some output form that is freely available to all voters before the election. A poll order is an ordering of the candidates provided as the output, that is intended to correspond roughly to an ordering of the candidates by likelihood to win the election. Since all voting strategies considered only need this ordering of candidates, our polls will output poll orders (rather than, say, just the most popular candidate, or a global rating of the candidates).

The actual likelihoods are not easy to calculate from a sample of voters due to the complexity of some voting systems, so we use some simplified models to create polling orders, described below.

Considered across all simulation runs, each candidate is equally likely to win the election, due to the symmetry in utility generation. One polling method is to directly apply this observation, which means the polling order can be arbitrarily chosen — in our simulations, chosen as a *random poll order*. Smith uses this method exclusively (ordering candidates by their IDs rather than randomly).

If the voters in the sample provide ratings in $[0, 1]$ for each candidate, we can order candidates based on their average ratings. We call this *range poll order* since the rating mechanic is the same as in range voting.

If the voters in the sample provide their most favored candidate, we can order candidates based on the number of voters that favor them. We call this *plurality poll order* since the single-choice mechanic is the same as in plurality voting (though maybe “first-choice poll order” would be a better name).

We assume that poll participants are honest, which is reasonable due to the nature of how real polls are conducted. (Strategic poll behavior is theoretically possible, but quickly leads to complications.)

In our simulations, “polls” use the whole population rather than a sample. This is to accurately capture population statistics, since the total number of voters used is small enough that sampling can easily lead to misleading deviations from the population statistics. Polls done on a sample of a larger total number of voters are expected to be representative of the population statistics. So long as the total number of voters in the simulation is large enough to accurately represent the expected voting patterns that might occur with even more voters, this practical decision makes our simulations no less accurate.

4 Voting Systems

A *voting system* is a system where a (usually large) number of *voters* cast *votes* of some kind, and the votes are aggregated in some process to select one of a (usually small, but greater than one) number of *candidates*.

Some formulations allow the output to be a set of candidates or an ordering of candidates. For simplicity, we will only consider systems that output a single winner. We will also focus on cases with three or more candidates, as with two candidates, all reasonable voting systems fall into either the ranked or rated behaviors as described in 2.

We first describe several types of voting systems, then list and describe the particular voting systems we consider in our simulation.

4.1 Ranked and Rated Voting Systems

4.1.1 Ranked Voting Systems

In a ranked voting system, vote information can be encoded into an ordering of the candidates. For instance, plurality voting uses the most favored candidate from each ordering, while Borda voting uses the whole ordering. Ranked voting systems traditionally output an ordering of candidates (which in this section we refer to as the "outcome"); to choose our single winner we can simply choose the most favored candidate in the outcome.

Arrow's theorem states that, if there are at least three candidates, no ranked voting system may satisfy all of the following properties [1]:

- Pareto efficiency: If every voter places c_i before c_j, then the outcome also places c_i before c_j. (This can be weakened to non-imposition, i.e. that for any c_i and c_j, there is some cast of votes such that the output places c_i before c_j [5].)
- Non-dictatorship: There is no voter whose ordering is always the same as the outcome.
- Independence of irrelevant alternatives: In two elections with the same number of voters where each pair of corresponding voters has the same relative ordering of c_i and c_j, the outcomes of both elections have the same relative ordering of c_i and c_j.

The fact that it is impossible to have all three of these very reasonable conditions is a limitation of ranked voting systems.

4.1.2 Rated Voting Systems

Votes need not be restricted to orderings of candidates. Rated voting systems allow a voter to provide independent scores for each candidate, which can be more informative than an ordering of the candidates. For instance, range voting allows a voter to give each candidate their own score in the range $[0, 1]$. thereby allowing a voter to express how much they prefer their most favored candidate over their second most favored candidate.

Since these are not ranked voting systems, Arrow's theorem does not apply, and it is completely possible to have Pareto efficiency, non-dictatorship, and independence of irrelevant alternatives. However, they are still subject to some restrictions, notably Gibbard's theorem, that no deterministic process of collective decision may satisfy all of the following properties [2]:

- The process has more than two possible outcomes.
- There is no voter who singlehandedly determines the outcome.
- The game-theoretically optimal vote for a voter will not depend on the voter's beliefs of what other voters will vote.

When applied to voting systems, this implies that non-dictatorial voting systems with three or more possible outcomes (ranked, rated, or otherwise) require some kind of strategic voting that is not completely honest.

> *An aside about determinism:*
> Most voting systems do not have a good way of handling ties without using randomness or allowing multiple winners. However, when the number of voters is reasonably large, the chance of a tie is negligible. We will consider voting processes that are deterministic when there are no ties "good enough." In practice, when ties occur in the simulations, we break them arbitrarily.

4.1.3 Other Voting Systems

There are a few "obviously bad" voting systems that we consider, such as "random winner" and "worst candidate". These are only useful as a frame of reference, and are not expected to have any of the properties discussed for ranked and rated systems.

4.2 COAF and Non-COAF Systems

4.2.1 COAF Systems

Smith's specification of compact set based, one-vote, additive, fair voting systems describes many common ranked and rated voting systems [4]. In a COAF system with C candidates there is a compact set $S \subseteq \mathbb{R}^C$ of allowed votes, where S is symmetric across permutations of candidates (fair). Each voter chooses one vote in S to submit. Then, the votes are added, and the candidate with the greatest sum (or equivalently, average, referred to as "score" for the following proof) is selected as the winner. For instance, plurality voting is when $S = \{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, \dots), \dots\}$, and range voting is when $S = [0, 1]^C$.

Smith provides a proof of the optimality of the "moving average" strategy. This strategy generates the game theoretically optimal vote in any COAF system when given poll data in the form of a predicted ordering of candidates by expected score. (This ordering is equivalent to ordering the candidates by likelihood of winning. Such a poll can be produced by a random sample of voters, see 3). Consider a COAF system with vote set S and without loss of generality label the candidates as $c_1, c_2, \dots c_C$ in poll order. Let $U_1, U_2, \dots U_C$ be the utilities of the voter for each candidate, under the utility model discussed in 2. The vote is generated as follows:

- Let the set X_0 representing the set of potential votes start as $X_0 = S$.
- If $U_1 > U_2$, let X_1 be the subset of X_0 that maximizes the 1st component. Otherwise, let X_1 be the subset of X_0 that minimizes the 1st component.
- For each candidate c_i in poll order starting from c_2, if $U_i > \frac{1}{i-1}\sum_{j=1}^{i-1} U_j$, let X_i be the subset of X_{i-1} that maximizes the ith component. Otherwise, let X_i be the subset of X_{i-1} that minimizes the ith component.

The final set X_C will consist of exactly one vector, which will be the vote.

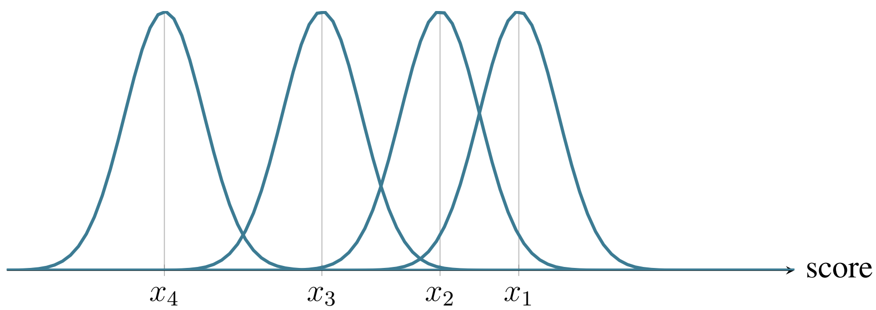
Consider an example with plurality voting. If $U_1 > U_2$, then X_1 is the singleton set with the vote for c_1, and $X_2 = X_1$ as X_1 is already a singleton. If $U_1 < U_2$, then X_1 is the set of votes for any candidate other than c_1, and X_2 is the singleton set with the vote for c_2 (comparing $U_2 > \frac{1}{2-1}\sum_{j=1}^{2-1} U_j = U_1$). Sets $X_3, X_4, \dots X_C$ are all equal to X_2 as X_2 is a singleton. To summarize, a strategic voter in plurality voting will vote for the more favored of the two frontrunners.

Consider another example with range voting. Set the c_1's score to 0 or 1 according to the comparison of U_1 to U_2. Then, proceed in poll order and set each score to 0 or 1 according to the comparison against the moving average as specified. The resulting vote has 0 or 1 at every component.

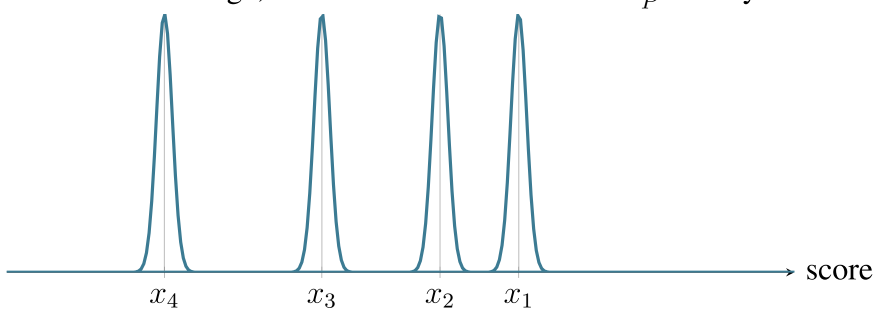
We provide a more formal specification and proof of the restrictions for X_1 and X_2.

Let us say there are C candidates, V votes, and P pollees, where V and P are reasonably large. Let us say that across all votes, the ith component of the vote vector lies in a distribution with mean μ_i and variance σ_i^2. μ_i is the actual score the candidate will receive. σ_i^2 has no direct effect on the election result but is useful for strategic analysis.

In this strategy, the only information voters have on the candidates is the polling data and their own personal associated utilities. To model this, we will assume a voter's belief of the distribution of a candidate c_i's score μ_i follows a normal distribution with the same variance as would be expected from the sample distribution, which by the central limit theorem is $\frac{\sigma_i^2}{P}$. These belief distributions have means we will call $x_1, x_2, \dots x_C$ — the numerical values of these are not important, but they are known to be in the polling order, as shown in this plot of the believed distributions of μ_4, μ_3, μ_2, and μ_1:



Since the P is large, we can assume the variance $\frac{\sigma_i^2}{P}$ is very small:



The voter's vote can only change the outcome of the election if the values of the actual two highest scores are an election-determining "near tie", within $\frac{1}{V}$ of each other. Otherwise, the vote cannot change the outcome of the election — this case can be considered a fixed component of the expected utility, and so can be ignored for reasoning about the optimal vote. The following reasoning will assume there is some near tie.

Given that there is a near tie between c_1 and c_2, there is a high chance that the value of the tie at $\mu_1 \approx \mu_2$ is somewhere close to the range $[x_2, x_1]$, and that every other μ_i is close to its respective x_i. Any near tie meeting these conditions is an election-determining near tie.

Meanwhile, given that there is a near tie between c_i and c_j where $i < j$ and $(i, j) \neq (1, 2)$, it is only an election-winning tie if all c_k for $k < i$ have $\mu_k < \mu_i$.

If $i \neq 1$, this can only happen either in the case that the tie occurs outside of its likely range of $[x_j, x_i]$ or in the case that every μ_k is outside of its likely range near x_k. It can be determined that this probability makes the election-winning near tie unlikely compared to the near tie between c_1 and c_2, so long as the $x_i - x_j$ is comparable to $x_1 - x_2$.

If $i = 1$, then the probability that c_1 and c_j tie is already significantly smaller than the probability that c_1 and c_2 tie, as x_1 and x_j are farther apart.

This means that the case of an election-determining near tie between c_1 and c_2 dominates the cases that the vote can affect. Optimizing for this case determines X_1 and X_2 as specified in the moving average strategy.

After narrowing down the potential votes to X_2 by considering when c_1 is a member of the near tie, there may still be many possible votes to make. It is already a very small chance for the optimization to X_2 to affect the outcome of the election, and it is even less likely for any remaining optimization to do so — and so the restriction to X_2 is already "good enough" for most purposes (this will also be useful for non-COAF systems). For the sake of implementation, we narrow down the possible votes from X_2 to a single vote by continuing to apply the moving average strategy as described.

4.2.2 Non-COAF Systems

Ranked and rated systems can also be non-COAF. For instance, the "plurality with runoff" system (see 4.3.7) re-scores two candidates based on the condition that they are the frontrunners by plurality — this type of conditional cannot be modeled in an additive system.

The strategic voter behavior for non-COAF systems can be complex. For instance, strategic voting in a three-candidate plurality with runoff system may require some voters with the same preferences to vote in different ways, as a split voting bloc [3]. However, as discussed in the previous section, it is very unlikely that any candidate other than the two poll frontrunners will have a chance to affect the election. We simply ensure the strategic vote optimizes for near ties between the two poll frontrunners and, under that restriction, use honest behavior for determining a unique vote. For the various ranked non-COAF systems, this is done by placing the frontrunners first and last, then keeping all other candidates in the same order.

4.3 A List of Voting Systems Considered

4.3.1 Random Winner

A candidate is chosen uniformly at random to be the winner. Useful as a frame of reference.

4.3.2 Random Dictator

A voter is chosen uniformly at random to be a dictator. The candidate that the dictator likes the most is the winner. Useful as a frame of reference.

4.3.3 Worst Candidate

The candidate that gives the worst possible sum of utilities is the winner. Useful as a frame of reference.

4.3.4 Range Voting

The most general COAF voting system, where $S = [0, 1]^C$.

Honest voters simply submit their utilities (possibly scaled as discussed in 2).

We evaluate two strategic voting behaviors for range voting. One is the moving average strategy. The other is an alternate strategy that sets a threshold at the average of the utilities of the two frontrunners, and votes 0 for candidates below the threshold and 1 for candidates above the threshold.

Note that strategic voters will always score candidates at 0 or 1, while honest voters might use other values.

4.3.5 Approval Voting

A COAF voting system where $S = \{0, 1\}^C$.

Honest voters use the average utility between all candidates as a threshold, and submit 0 for candidates below the threshold and 1 for candidates above the threshold.

Strategic voters in range voting always score candidates at 0 or 1, so their strategies work just as well in approval voting. We evaluate the same two strategies for approval voting as we do for range voting.

The only difference between approval voting and range voting is the behavior of honest voters, where range voting allows for describing one's preferences in more detail.

4.3.6 Plurality Voting

A COAF voting system where $S = \{(1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, \ldots), \ldots\}$. In other words, each voter votes for one candidate, and the candidate with the most votes wins. This is perhaps the most commonly known voting system.

Honest voters submit their most favored candidate.

Strategic voters submit their most favored candidate among the two frontrunners from the poll ordering.

4.3.7 Plurality with Runoff Voting

A non-COAF system where voters first make a plurality vote (the "first stage"), then decide between the two frontrunners by a two-candidate ranked vote (the "second stage").

For the first stage, honest voters sumbit their most favored candidate.

For the first stage, strategic voters submit their most favored candidate among the two frontrunners from the poll ordering.

For the second stage, honest and strategic voters submit their most favored candidate from the two allowed candidates.

This often is presented as a two-stage election, but can also be done with one vote by allowing $C \times 2^{\binom{C}{2}}$ possible votes, consisting of a single candidate for the plurality stage, then the most favored candidate of each pair, to be used depending on the outcome of the plurality stage. (This detail does not matter for analysis or our simulation but may be useful for other implementations of simulations.)

4.3.8 Bullet Voting

("Bullet voting" seems to be Smith's terminology. We use it for consistency.)

A COAF system where $S = \{(0, 1, 1, \ldots), (1, 0, 1, \ldots), (1, 1, 0, \ldots), \ldots\}$ (or equivalently $\{(-1, 0, 0, \ldots), (0, -1, 0, \ldots), (0, 0, -1, \ldots), \ldots\}$). In other words, each voter votes *against* one candidate, and the candidate with the *fewest* votes wins.

Honest voters vote against their least favored candidate.

Strategic voters vote against their least favored candidate out of the two frontrunners from the poll ordering.

4.3.9 Borda Voting

A COAF system where S is the permutations of $(0, 1, 2, \ldots, C-1)$ (or equivalently ($\frac{0}{C-1}$, $\frac{1}{C-1}$, $\ldots$, $\frac{C-1}{C-1}$)). In other words, each voter ranks all candidates, each candidate receives a linearly weighted score, and the candidate with the highest average score wins.

Honest voters rank the candidates honestly.

Strategic voters rank the candidates by the moving average strategy, in particular placing the more favored of the two frontrunners first in the ranking and the less favored of the two frontrunners last in the ranking.

4.3.10 Dabagh Voting

Also known as "Point-and-a-half" voting.

A COAF system where each vector in S has one candidate scored at 1, one candidate scored at 0.5, and all other candidates scored at 0. In other words, each voter ranks all candidates, each candidate receives a score of 1 for all voters putting them first and 0.5 for all voters putting them second, and the candidate with the highest average score wins.

Honest voters rank the candidates honestly, thus giving the score of 1 to their most favored candidate and the score of 0.5 to their second most favored candidate.

Strategic voters rank the candidates by the moving average strategy, in particular giving the score of 1 to the more favored of the two frontrunners, giving a score of 0 to the less favored of the two frontrunners, and giving the score of 0.5 to some other candidate.

4.3.11 Condorcet Least Reversal Voting

A non-COAF ranked system based on defeating Condorcet cycles. We consider the graph with candidates as vertices and pairwise margins of defeat as edges. For example, if 200 voters place c_1 before c_2 and 50 voters place c_2 before c_1, then there is an edge from c_2 to c_1 with a weight of 150. If there are any candidates with no outgoing edges (i.e. no pairwise defeats), this candidate is the *Condorcet winner*, and wins the election. Otherwise there is a cycle in the graph called a *Condorcet cycle*. Find the set of edges of least summed weight such that flipping them identifies a Condorcet winner; that candidate wins the election.

(This specific formulation is generalizable to multiple winners or rank output. However, for the single-winner case this is in practice a matter of finding the smallest sums of weights of each vertex's outgoing edges.)

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

4.3.12 Condorcet Ranked Pairs Voting

A non-COAF ranked system based on defeating Condorcet cycles. As before, we consider the graph with candidates as vertices and pairwise margins of defeat as edges. Consider each pair of candidates in order of largest to smallest margin of victory. If the pair does not form a Condorcet cycle, then add an edge between the two candidates; otherwise ignore it. At the end there will be a unique candidate that wins all pairwise comparisons under consideration.

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

4.3.13 Single Transferable Vote (Fewest First)

Also known as (a form of) instant runoff voting.

A non-COAF ranked system where the candidate with the fewest first-place rankings is repeatedly removed until there is one candidate remaining.

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

4.3.14 Single Transferable Vote (Most Last)

Also known as (a form of) instant runoff voting.

A non-COAF ranked system where the candidate with the most last-place rankings is repeatedly removed until there is one candidate remaining.

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

This version in particular is rather susceptible to strategic voting, since the strategic voting patterns described will often eliminate *both* frontrunners.

4.3.15 Copeland

A non-COAF ranked system based on defeating Condorcet cycles.

The candidate that wins against the greatest number of other individual candidates wins the election. Since ties are common no matter the size of the voter count (e.g. if c_1 and c_2 both win against 3 of their 4 opponents), some tiebreaker is necessary. Borda count is a common tiebreaker, and is what we use.

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

4.3.16 Bucklin

If there is a candidate c_i with more than half of the voters placing c_i in first, then c_i wins. If not, then if there is a candidate c_i with more than half of the voters placing c_i in first or second, then c_i wins. This repeats for ranks 1 through k for each k. If there are ties (i.e. if incrementing k creates multiple potential winners at once), they are broken by the count of voters placing in ranks 1 through k.

Honest voters rank the candidates honestly.

Strategic voters rank the more favored of the two frontrunners first, the less favored of the two frontrunners last, and the rest of the candidates honestly.

4.3.17 STAR

An acronym for “score then automatic runoff”.

A non-COAF rated system where each voter submits a rating [1] of all the candidates. Then, the two candidates with the highest rating (call them c_1 and c_2) are put in a runoff. If more voters score c_1 higher than c_2, then c_1 wins; otherwise c_2 wins.

Honest voters rate the candidates honestly.

Strategic voters rate the more favored of the two frontrunners at 1, the less favored of the two frontrunners at 0, and the rest of the candidates honestly.

[1] Wikipedia specifies that the ratings are integers in $\{0, 1, 2, 3, 4, 5\}$, but I instead just use reals in $[0, 1]$ as with standard range voting.

5 Utility Distributions

Smith's original simulations use two methods to randomly generate utilities. The first is for each voter to have a uniformly random utility in $[0,1]$ for each candidate. The second is an *issue-based* method, where there are I issues, and each candidate and voter has a uniformly random *stance* on each issue in $[-1,1]$ (so that a stance vector is in $[-1,1]^I$). The voter then has a utility for the candidate equal to $\frac{\vec{v}\cdot\vec{c}+I}{2I}$, the dot product of the stance vectors normalized to $[0,1]$.

We expand upon this by allowing normal distributions, as well as bimodal distributions produced by mixing two normal distributions with different means. Either can be used to generate utilities directly or to generate stances. These distributions are truncated to the appropriate bounds, i.e. $[0,1]$ for utilities and $[-1,1]$ for stances. The normal distributions have reasonably small standard deviations so that truncation does not have a large effect on the distribution.

Note that this utility generation process is in fact symmetric across voters and candidates, as assumed in the previous sections.

6 Simulation

The process in the simulation for one set of parameters (i.e. voting system, voter count, candidate count, ratio of honest/strategic voters, utility distribution generation method), is to repeatedly execute the following sequence:

- Generate utility distributions by the chosen method.
- Create votes using the utilities according to the voting system and honest or strategic behavior.
- Determine the winner of the election by that set of votes in the given voting system.
- Calculate the regret of the winner against the highest-total-utility candidate.

We then plot the regrets so that they can be evaluated against each other.

The source code is freely available online at `https://github.com/RussellEmerine/voting_simulation`.

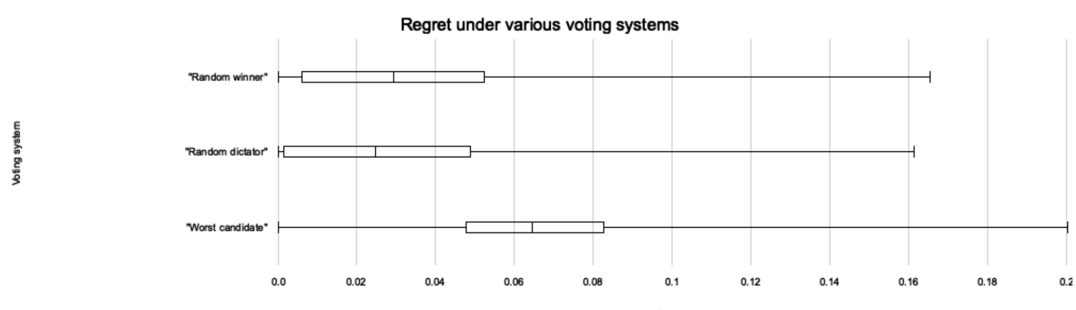
7 Analysis

The following plots are box plots displaying the quartiles of the data gathered. This gives a general idea of the distribution over all trials. We also record the average regret over all runs in a text file, but do not display it on the plot.

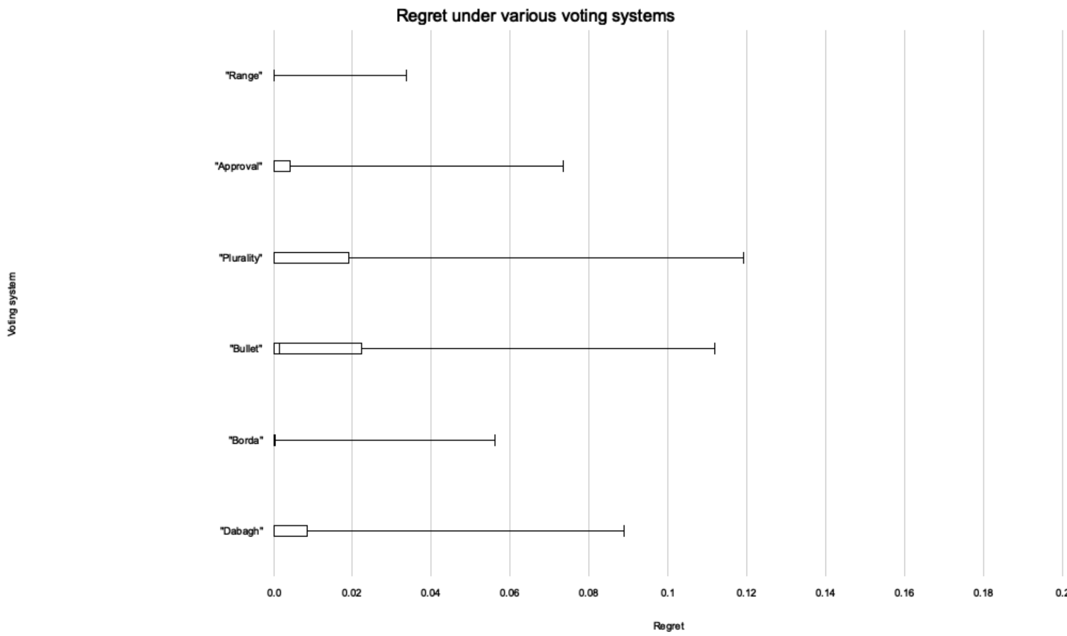
Take as an example the following simulation parameters:

- 100 voters
- 5 candidates
- 10000 trials
- All honest voters
- Utility renormalization
- Random Poll Order

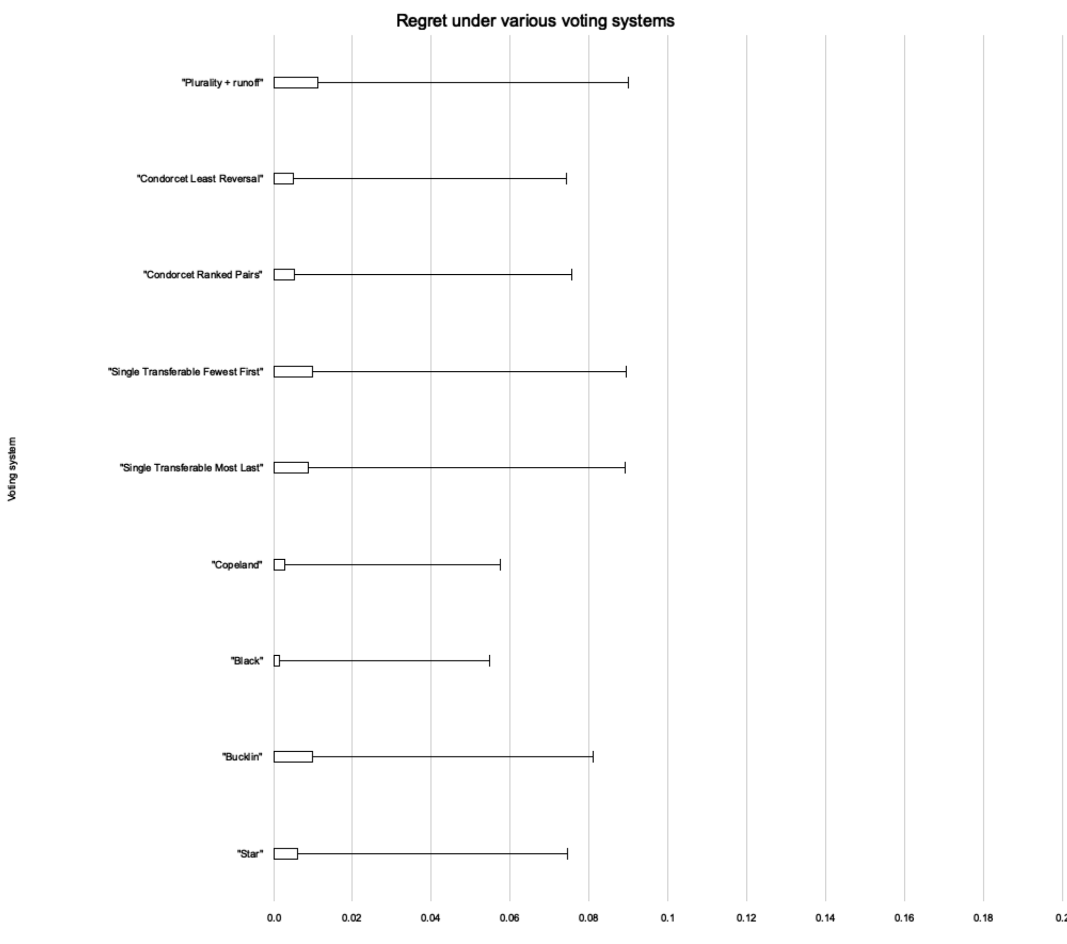
The plots include the "obviously bad" frame of reference voting systems,



the COAF systems,



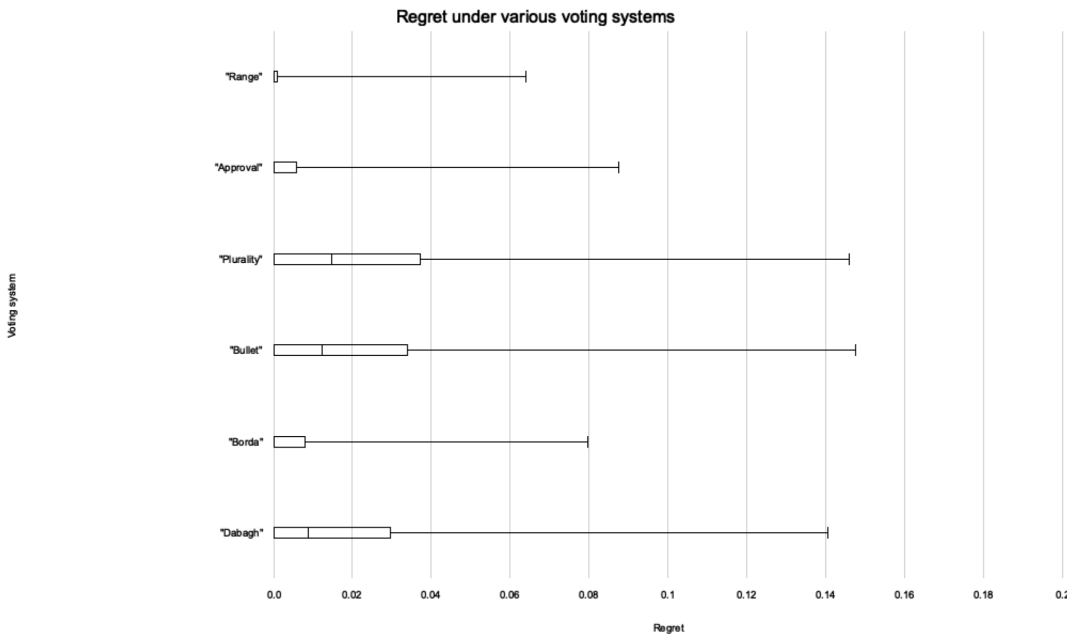
and the non-COAF systems.



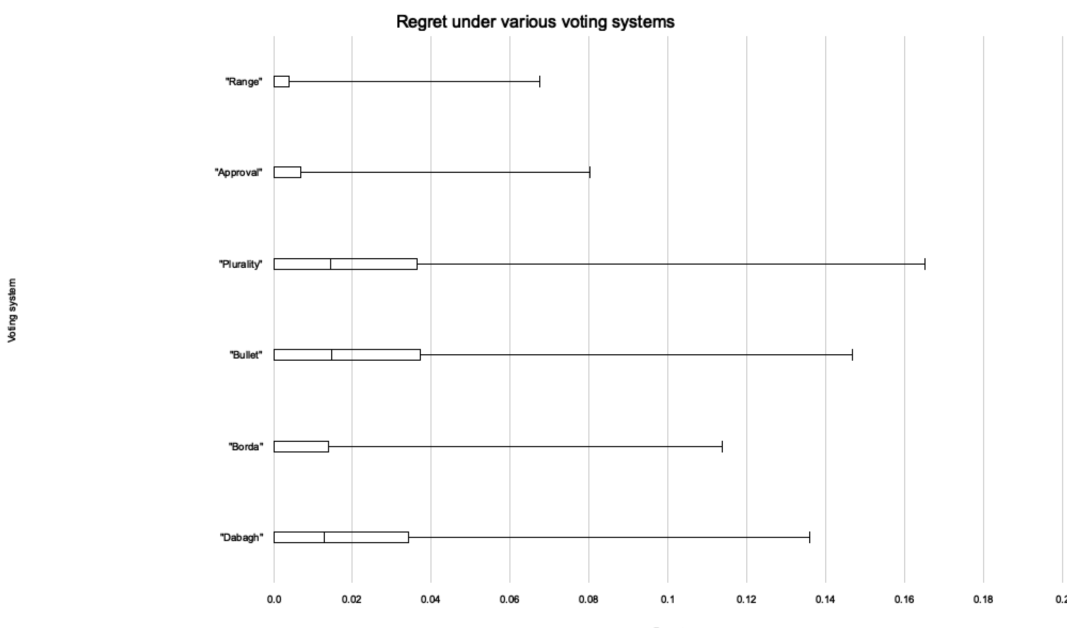
One important feature to note is that in almost all voting systems, more than half of all elections elect the best candidate. Another is that, as expected, honest range voting almost always elects the best candidate, since it directly represents utilities (and only fails when renormalization changes the best average utility).

Observe the COAF systems with an increasing ratio of strategic voters.

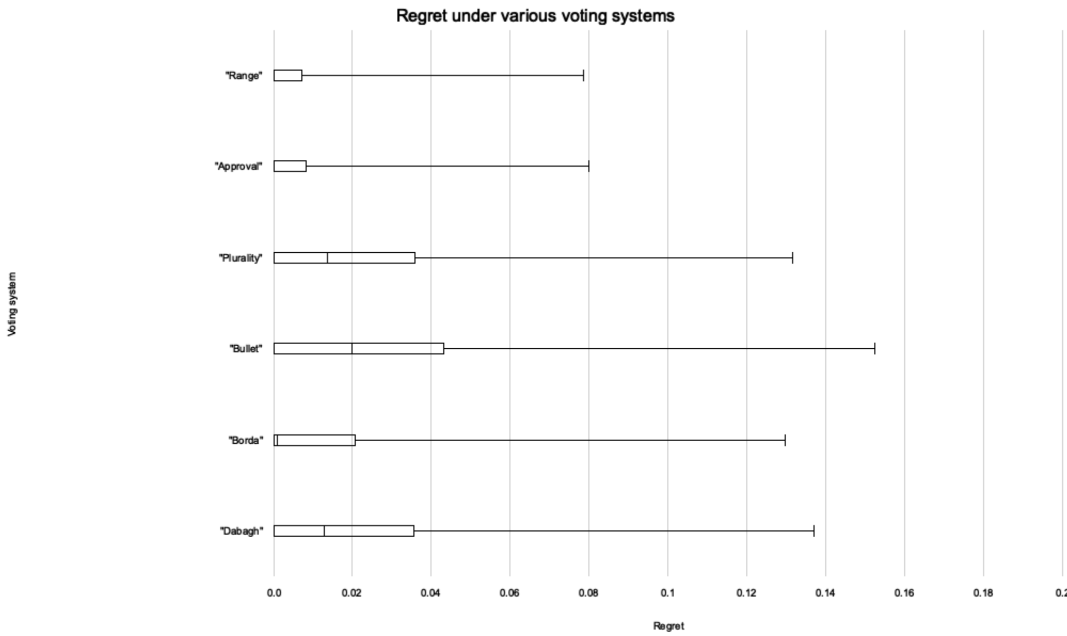
75% honest voters:



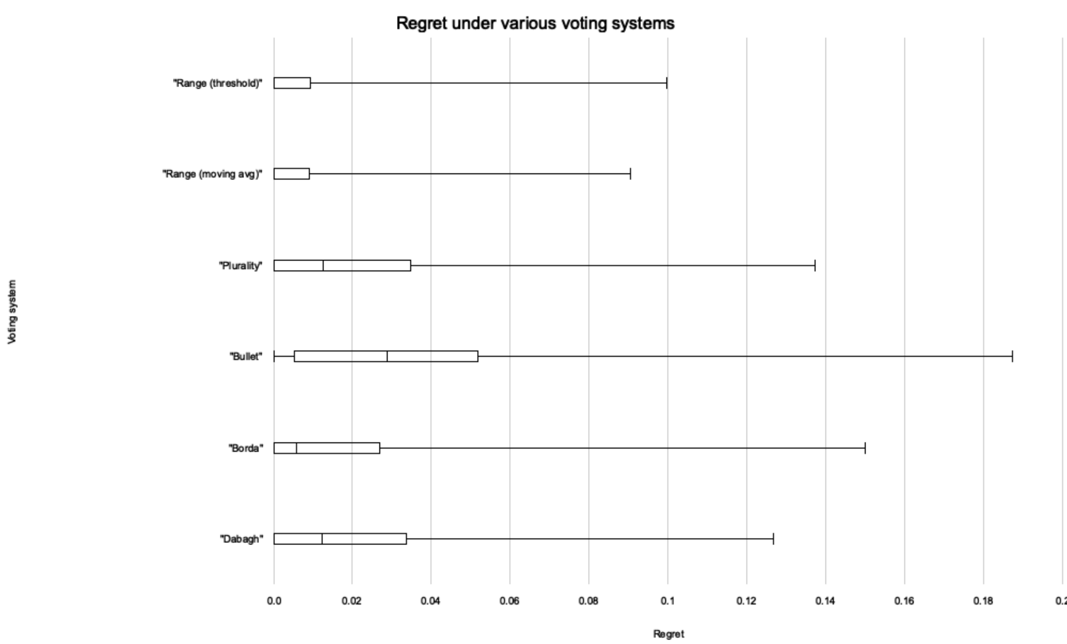
50% honest voters:



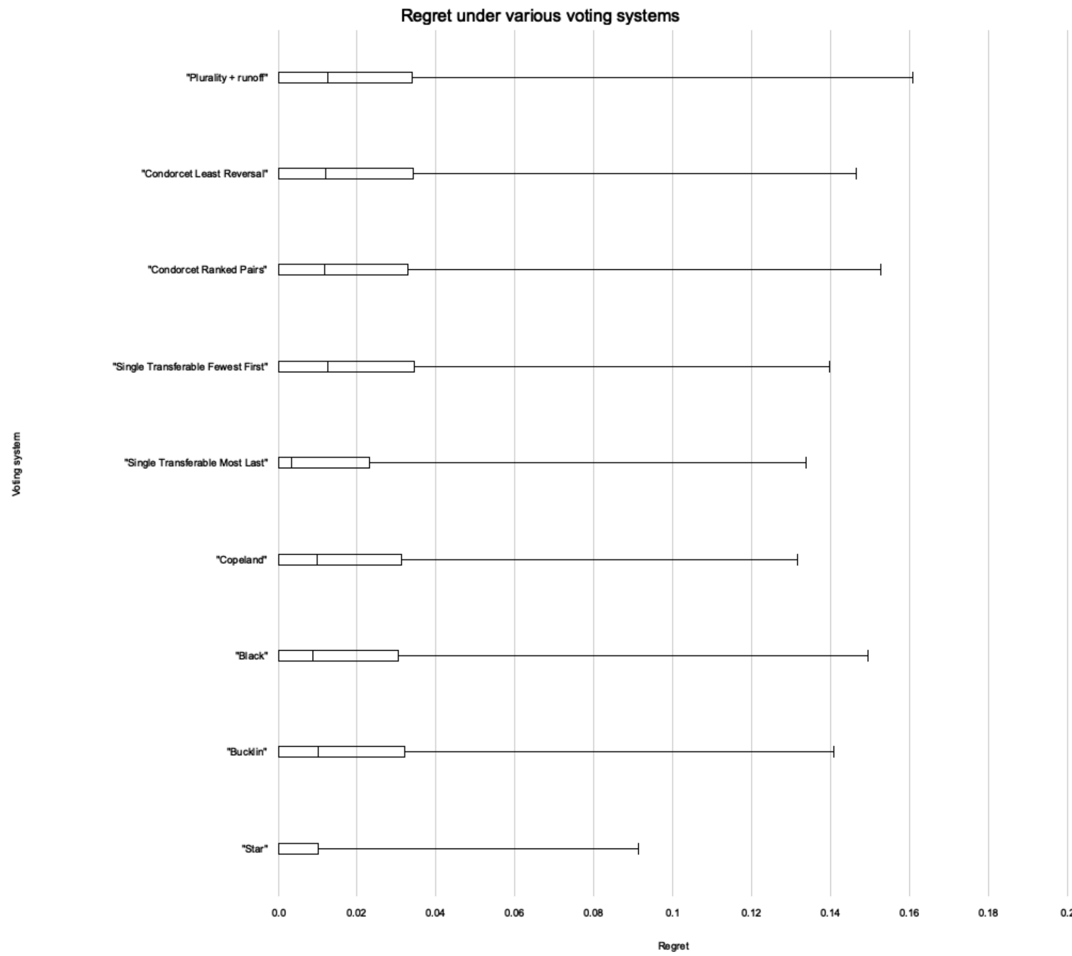
25% honest voters:



All strategic voters:



And all strategic voters for non-COAF systems, for comparison:

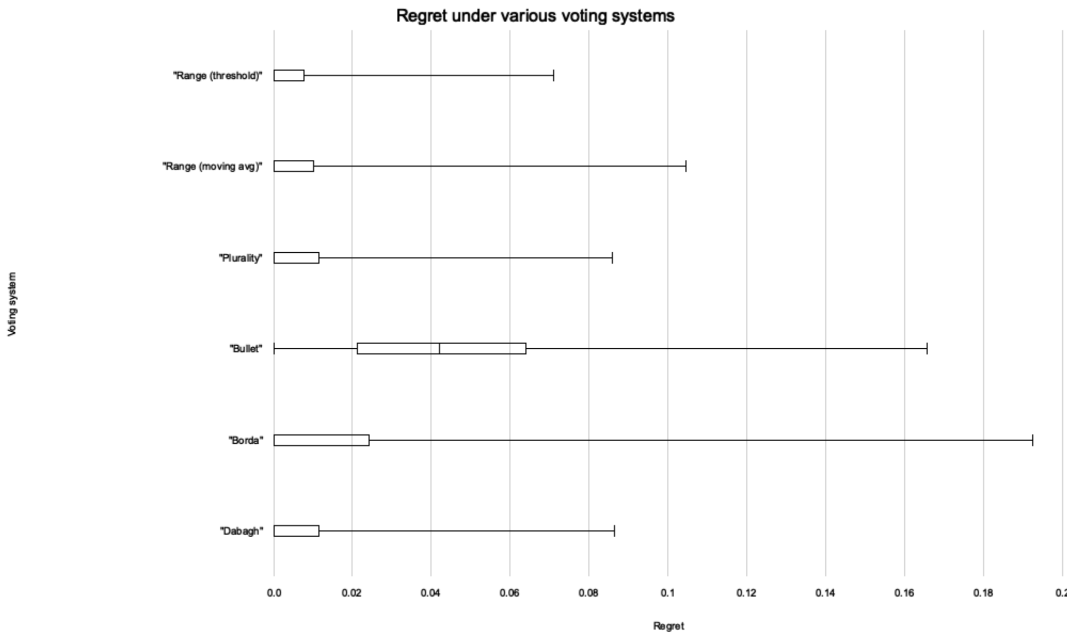


In these graphs, all voting systems generally have more regret for larger proportions of strategic voters. Range voting maintains its position as the voting system with the least regret. The average regrets (not shown on the plots) have the same ratios between each other as in Smith's simulations, and the plots visually reflect the similarity as well. This therefore replicates Smith's initial results.

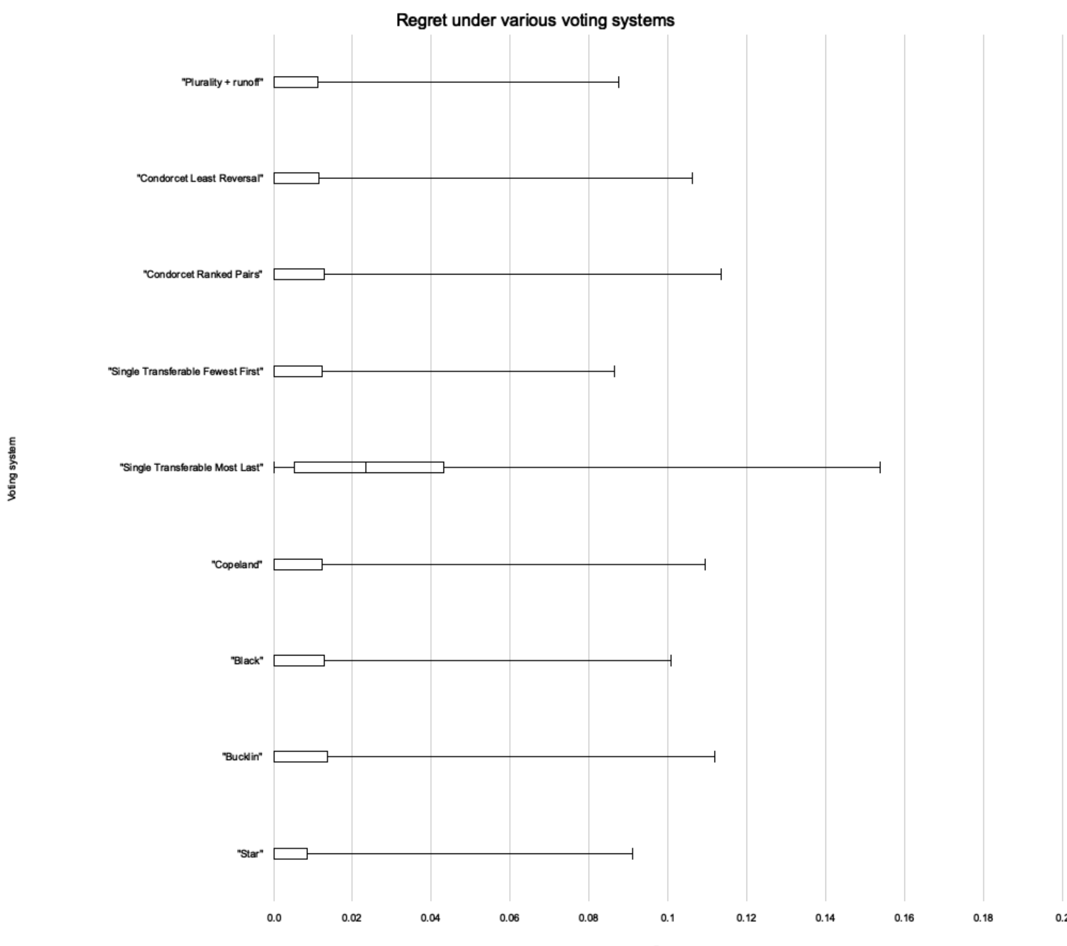
Smith does not consider STAR voting, since it was invented after the 2000 paper. This plot shows that STAR voting is comparable to range voting for strategic voters (but the earlier plot shows that it has greater regret for honest voters).

However, we see drastically different behavior for strategic voters in other poll orders. Consider plurality poll order.

All strategic voters for COAF systems with plurality poll order:



All strategic voters for non-COAF systems with plurality poll order:

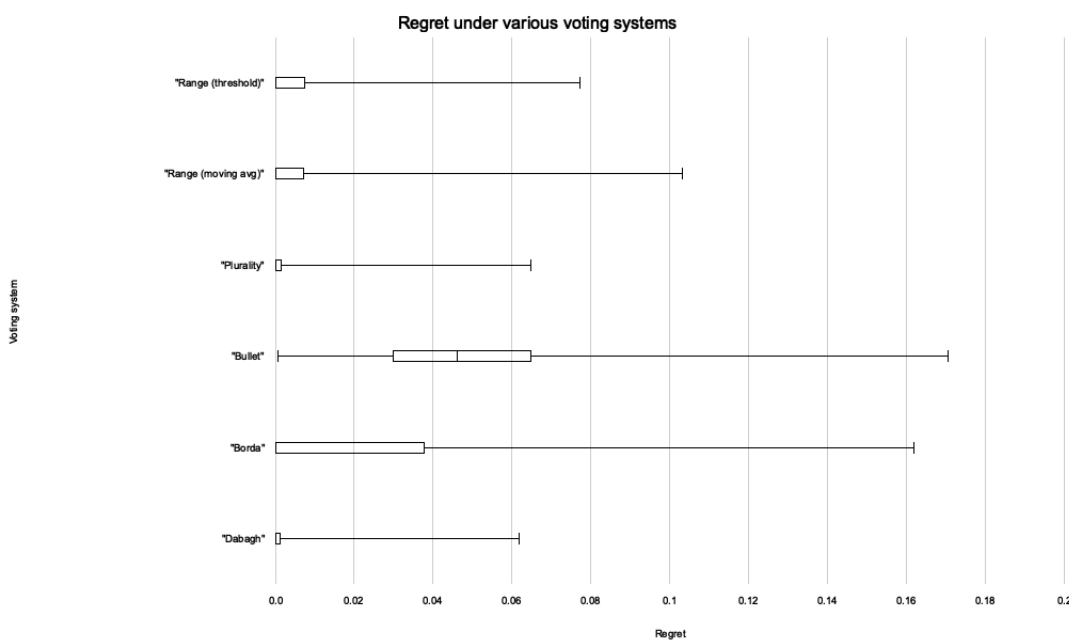


In the case of plurality poll order (looking at the means as well as the plots), range voting still does slightly better than all other voting systems. However, the regrets are generally all closer.

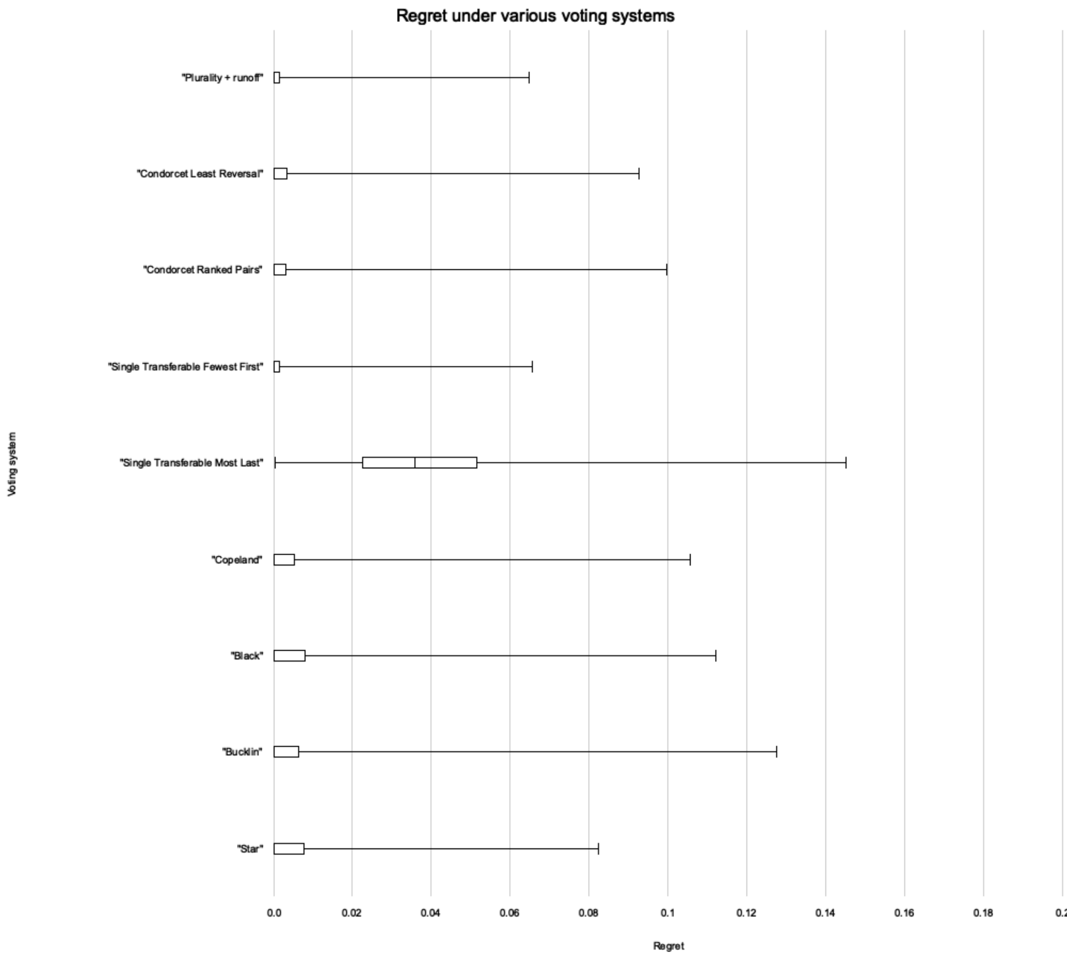
The least-regret system is range voting with the threshold strategy, followed by STAR voting, followed by range voting with the moving average strategy. It is unclear if there is any particular reason for the threshold strategy to have lower regret than the moving average strategy.

Consider range poll order.

All strategic voters for COAF systems with range poll order:



All strategic voters for non-COAF systems with range poll order:



In the case of range poll order, range voting does *worse* than plurality voting. This is because the poll frontrunners in the range poll order are in fact the overall best two candidates by utility, and since every strategic voter in the plurality voting system votes for one of the two frontrunners, it is guaranteed that one of the best two candidates is elected.

This effect applies more strongly as the proportion of strategic voters is set higher. Plurality voting has more regret than range voting with 50% honest voters, but has less with 75% honest voters. The same effect lowers the regret to a similar value to that of plurality voting for the similar plurality with runoff, Dabagh, and fewest-first single transferable vote systems.

The above analysis was all done with standard uniform utility generation and utility renormalization. Results with other methods of utility generation and without utility renormalization are similar up to linear scaling. A cursory analysis of some of them is available at 9.

8 Conclusions

We successfully replicate Smith's results, but only with random poll order. We find that with other poll order models, the performance of different voting systems changes significantly. Range voting has low regret but is close to other voting systems when using plurality poll order. Plurality voting has lower regret than range voting when using range poll order with a high proportion of strategic voters (due to fairly straightforward properties of the polls and strategies).

The simulations used in this paper are, of course, highly simplified as compared to real-life elections. Polling methods in particular face a number of practical considerations such as response rates and sample selection. Further work may be done to make a closer examination of the nature of simulated polls and elections and how they relate to real polls and elections.

Further work may also be done to explore more ways of simulating polling. We may also consider other factors, such as voting with incomplete information about utilities, or even more voting systems and utility generation methods.

9 Appendix A: Other Utility Generation Methods

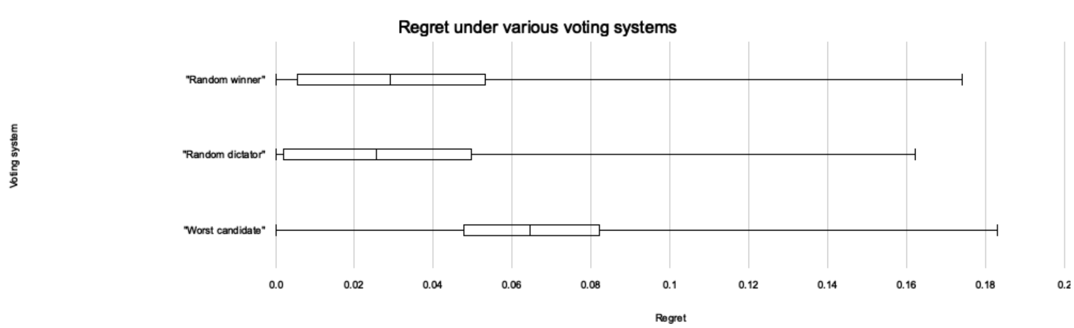
This section includes some plots that are qualitatively similar to those discussed in 7, and are included to demonstrate that similarity. A complete collection of all the plots is available in the Github repository.

The following plots, like the plots in 7, have

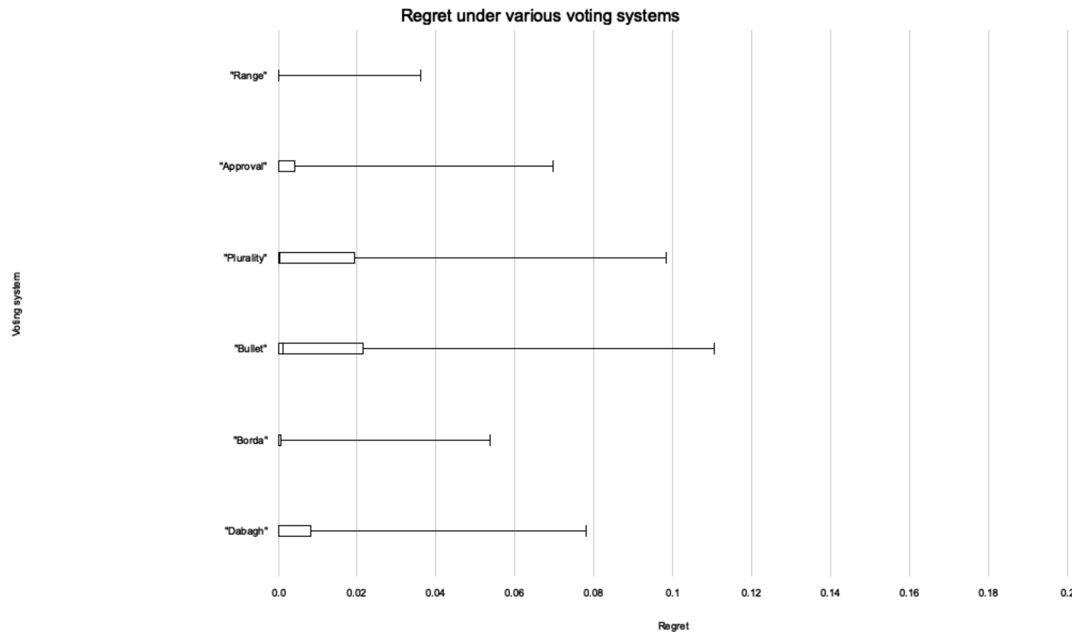
- 100 voters
- 5 candidates
- 10000 trials

The following are for standard uniform with plurality poll order, with renormalization. (These are already in 7, but are included again for convenience of comparison.)

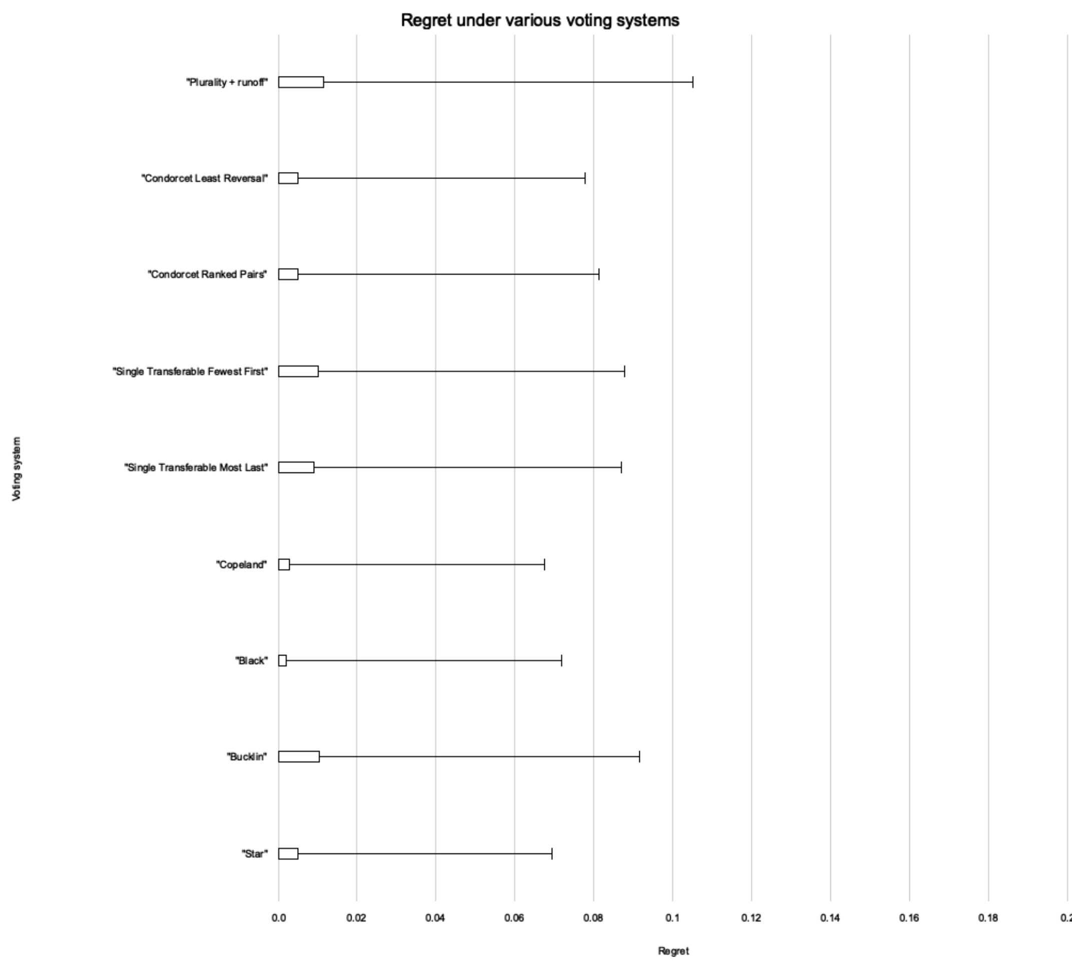
Frame of reference:



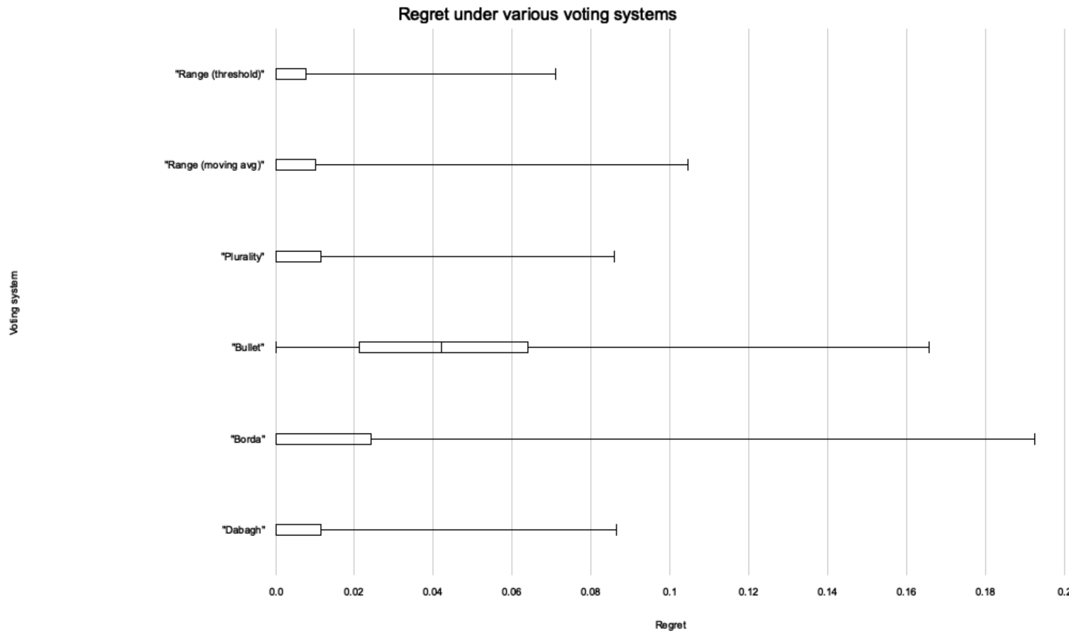
Honest COAF:



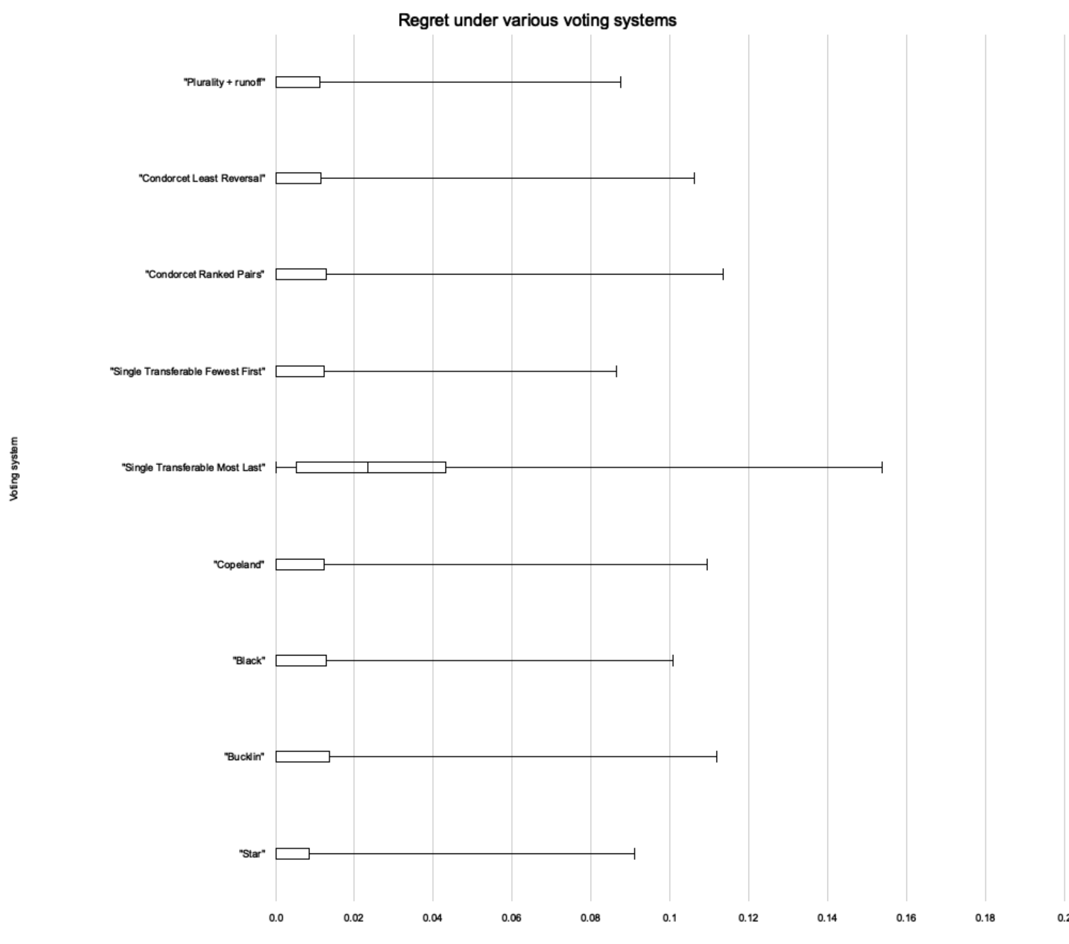
Honest non-COAF:



Strategic COAF:

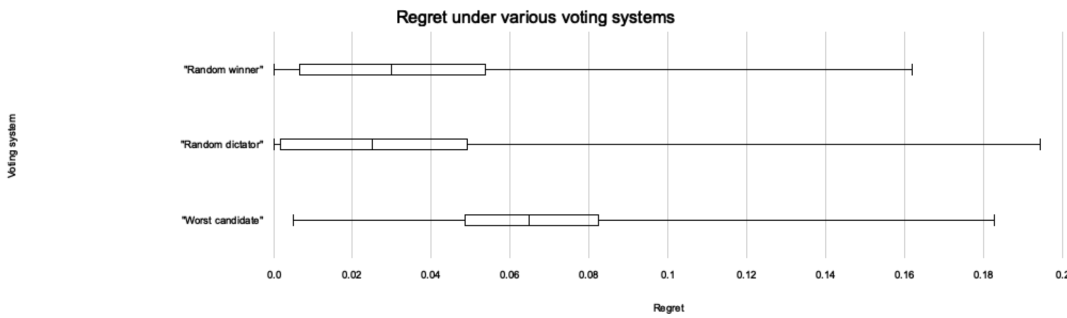


Strategic non-COAF:

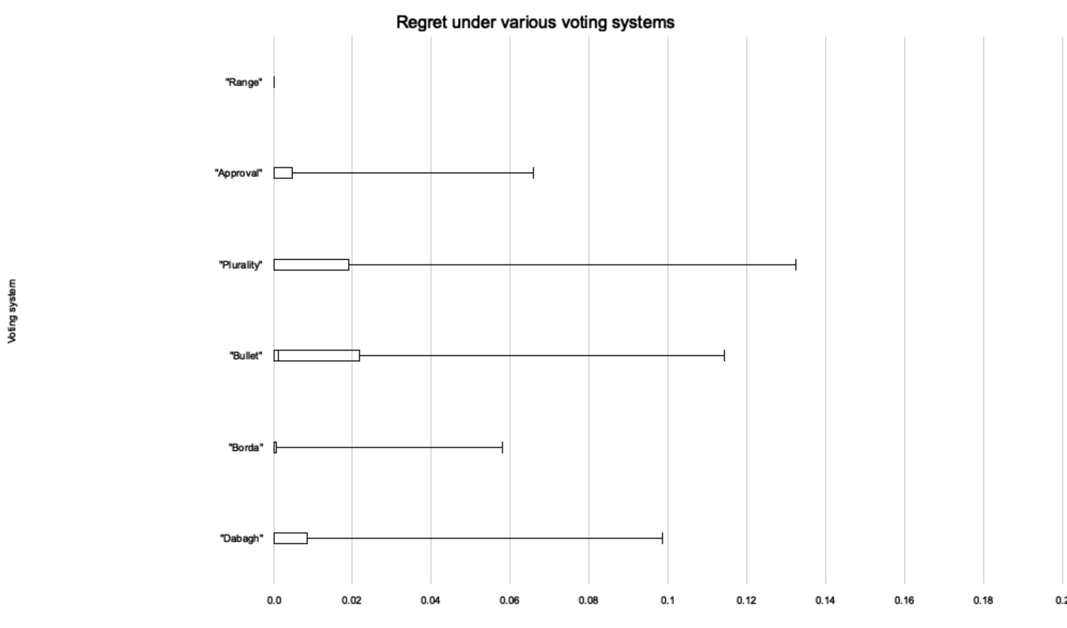


The following are the same plots, but without renormalization.

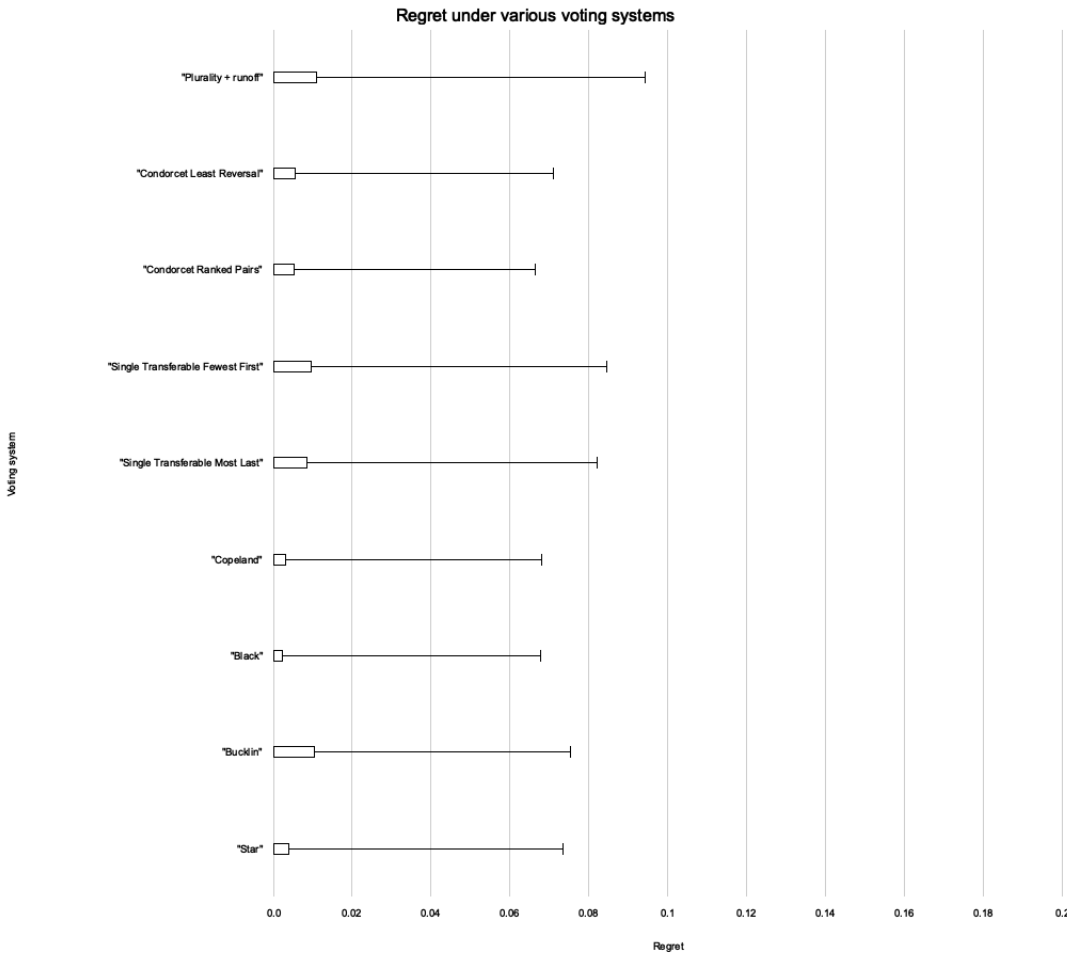
Frame of reference:



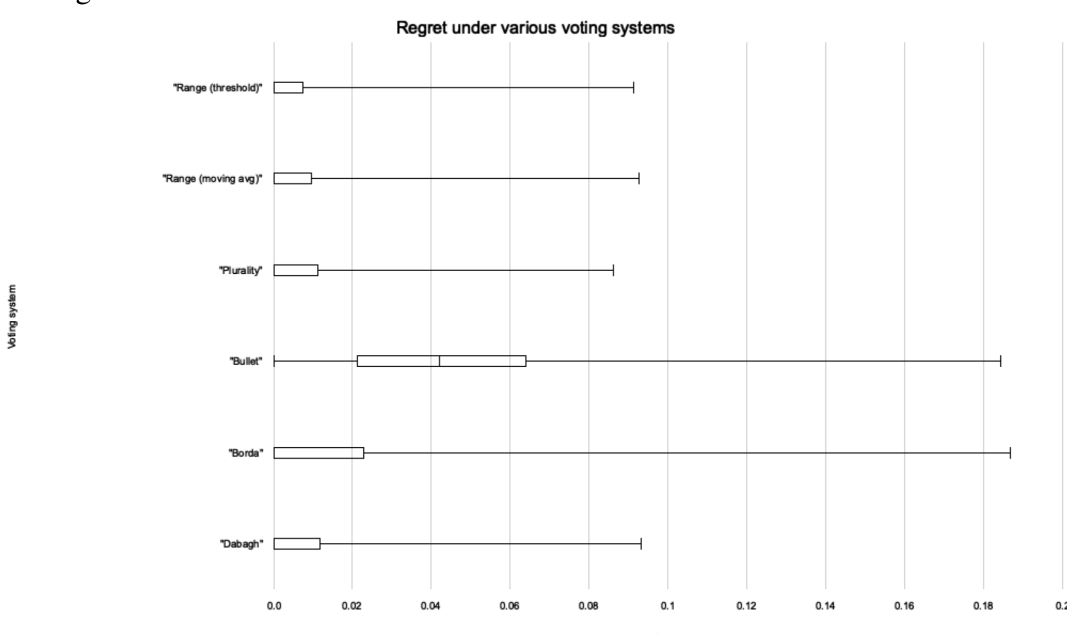
Honest COAF:



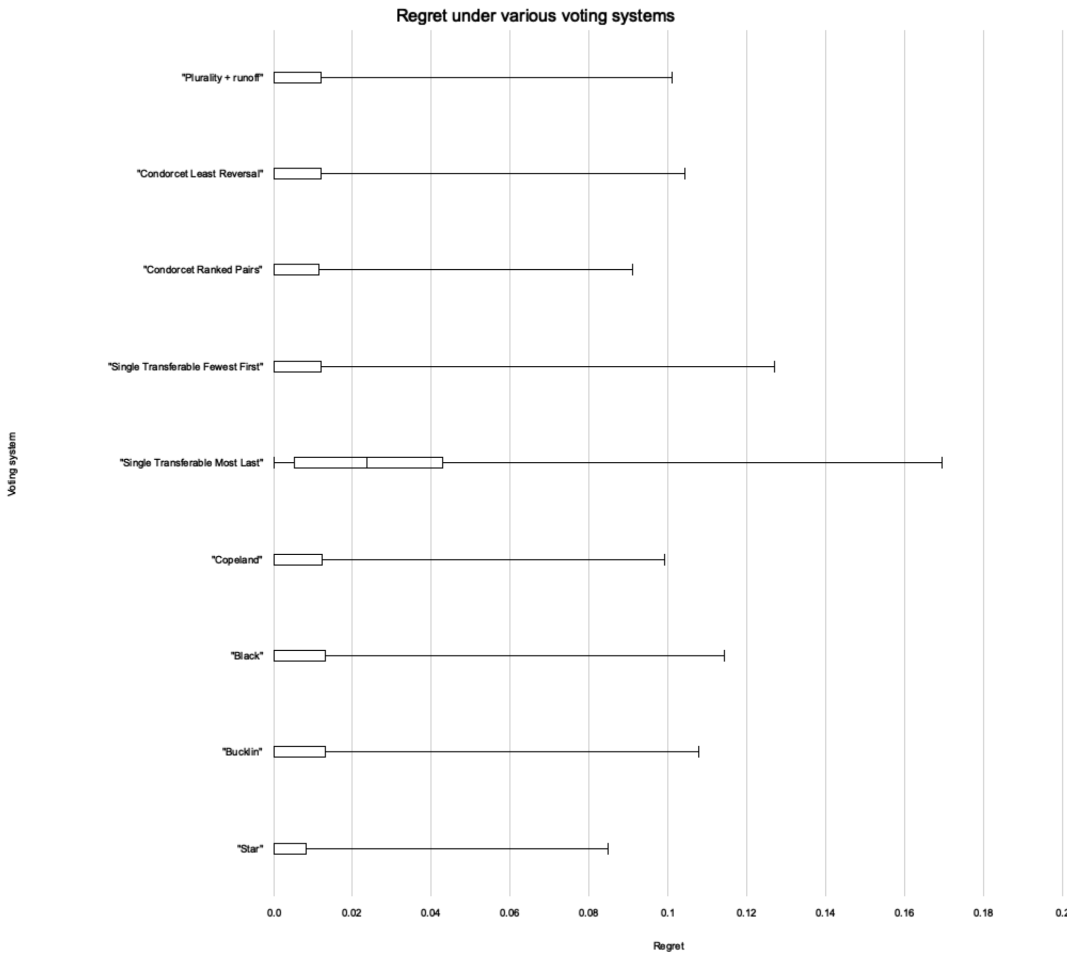
Honest non-COAF:



Strategic COAF:



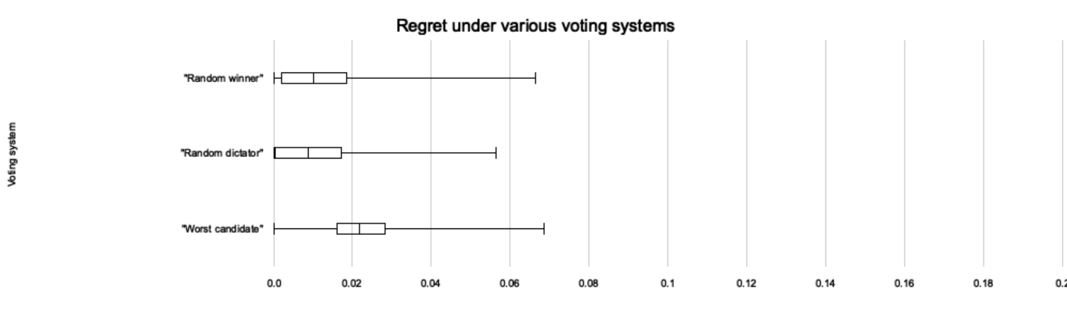
Strategic non-COAF:



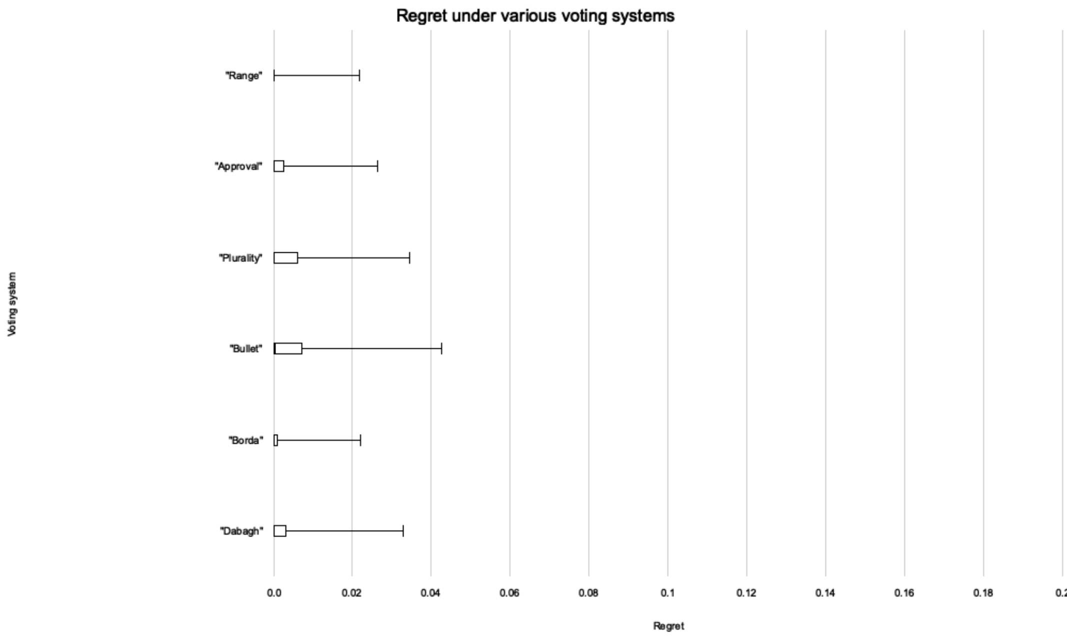
The only major difference between these sets of plots is that honest range voting without renormalization always has regret of exactly 0, since the range voting score is exactly the same as the average utility of the candidate. Other than that, renormalization does not make any major change on election results.

The following are the same plots for the normal distribution for utility generation (with renormalization again):

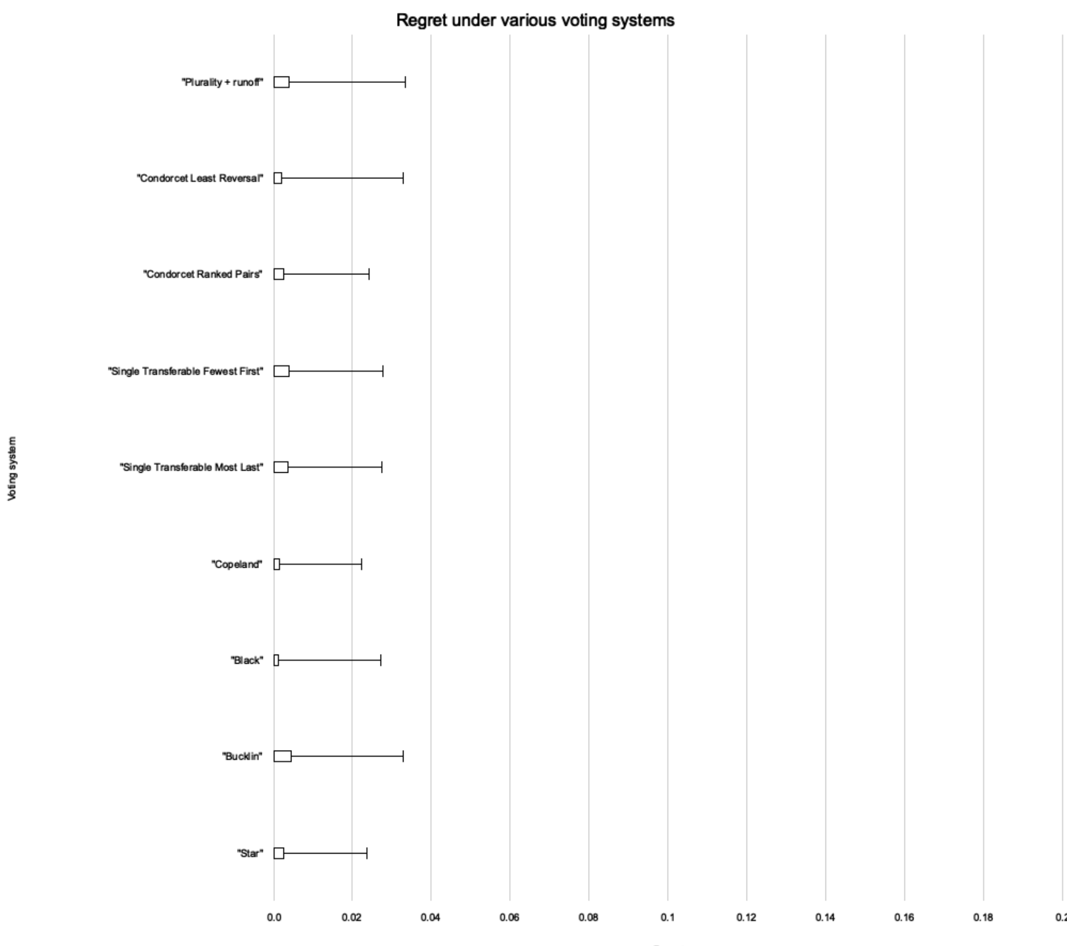
Frame of reference:



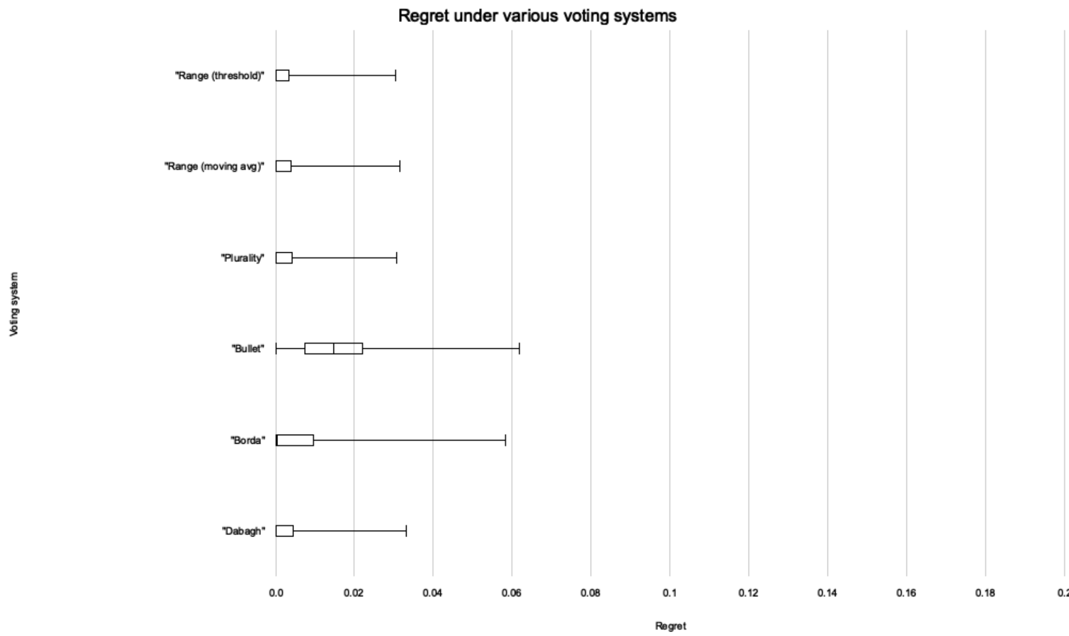
Honest COAF:



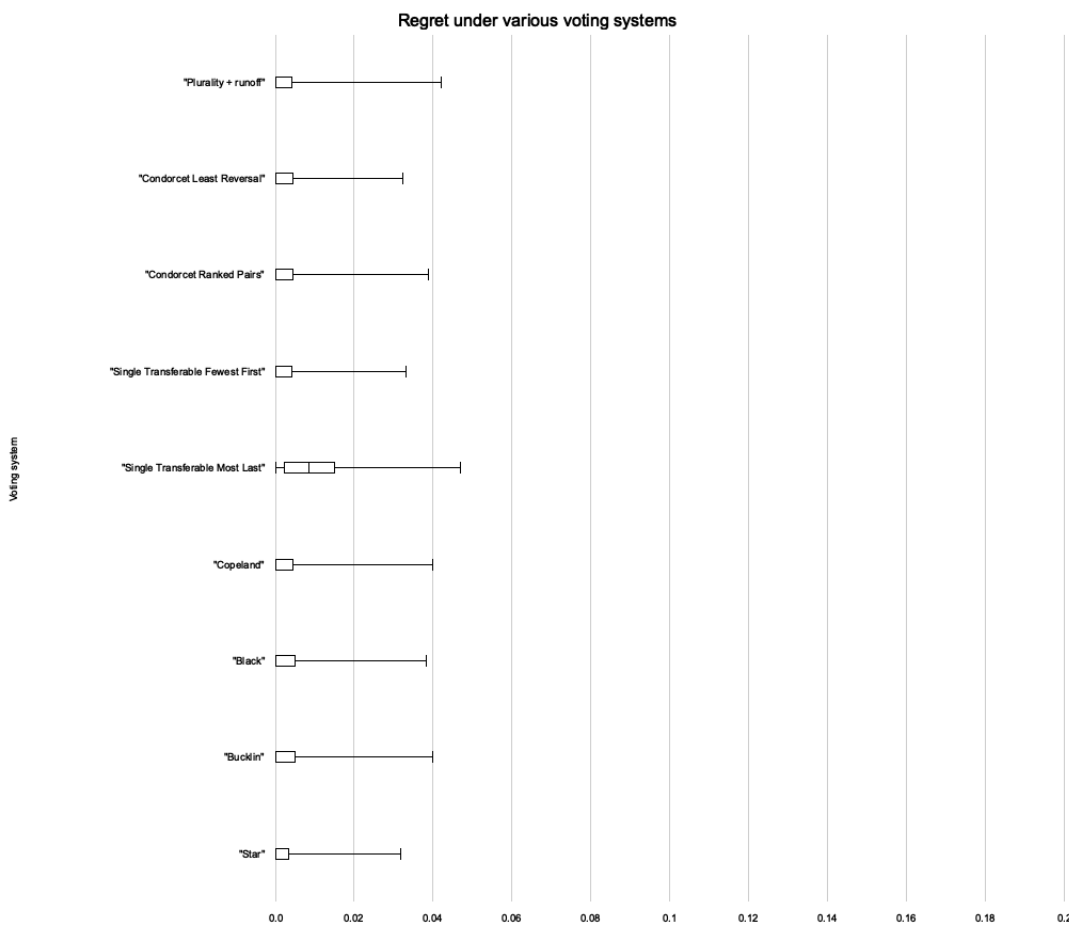
Honest non-COAF:



Strategic COAF:



Strategic non-COAF:



All of the regrets are smaller since the utilities are closer in value. However, the curvature of the normal distribution seems to not significantly affect the election results. Perhaps the only interesting observation to make is that despite Borda using an innate linear scale to score candidates, it still performs similarly (relatively to other voting systems in the same context), even when the utilities are generated with nonlinear values.

Similar statements can be made about the bimodal and issue-based utility generation methods.

Bibliography

[1] Kenneth J. Arrow. A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4):328–346, Aug 1950. doi: 10.1086/256963. (document), 1, 4.1.1

[2] Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4): 587–601, 1973. ISSN 00129682, 14680262. URL `http://www.jstor.org/stable/1914083`. 4.1.2

[3] Emerson Niou. Strategic voting under plurality and runoff rules. *Journal of Theoretical Politics - J THEOR POLIT*, 13:209–227, 04 2001. doi: 10.1177/0951692801013002004. 4.2.2

[4] Smith. Range voting. 2000. URL `https://rangevoting.org/WarrenSmithPages/homepage/rangevote.pdf`. (document), 1, 4.2.1

[5] Robert Wilson. Social choice theory without the Pareto Principle. *Journal of Economic Theory*, 5(3):478–486, 1972. ISSN 0022-0531. doi: 10.1016/0022-0531(72)90051-8. URL `https://www.sciencedirect.com/science/article/pii/0022053172900518`. 4.1.1