University of California Santa Barbara

Homework #3

Introduction to EE

ECE 3

Instructor: Christos Thrampoulidis

Worked Solutions

First & Last Name: ______________________________

Perm Number: ____________________

For instructor use:

Question:	1	2	3	Total
Points:	16	12	12	40
Score:				

Instructions:

- Write your name in the space provided above.
- Please consult the Syllabus for HW grading and late-submission policies.
- Please first solve the problem in draft paper and present your **complete** final answer in *clean* form on the space provided. **Answer all of the questions in the spaces provided. Answers are expected to be succinct but complete.** Please only use extra space (attach to the end of your submission) if absolutely necessary.
- The HW is **due Thursday November 14 9:00am sharp**.
- The HW set includes **Programming Assignments** marked as **(P.A.)**. Create a new Jupyter notebook and write code for each one of them. Execute the code to obtain the desired results (e.g., plot of the signals). Upload the notebook **including the code and its output** on GauchoSpace. **Attach a printed copy of your code.**
- **Return a paper copy of your HW to the homework box in HF.**
- **The returned copy should include a printout of your code for the Programming assignments. Also upload your P.A.s to GauchoSpace.**

1. **Problem 1 [Aliases].** Consider the following three sinusoidal analog signals:

$$x_1(t) = \cos(8\pi t),$$
$$x_2(t) = \cos(\mathbf{12}\pi t),$$
$$x_3(t) = \cos(28\pi t).$$

(a) (1 point) What are the frequencies (in Hz) of the three signals?

Recall $\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi}$

So, the three freqs in (Hz) are $f_1 = \frac{8\pi}{2\pi} = 4$ Hz

$f_2 = \frac{12\pi}{2\pi} = 6$ Hz

$f_3 = \frac{28\pi}{2\pi} = 14$ Hz

(b) (5 points) We sample the three analog signals at a sampling rate of 10Hz. Write down mathematical expressions for the three discrete signals. Prove that the three discrete signals have *the same* sample value for *any* given time index n.

Let $x(t)$ be the continuous time signal and f_s the sampling rate

Then the DT signal is $x[n] = x(n/f_s)$, $n \in \mathbb{Z}$

Here, $f_s = 10$. Thus,

- $x_1[n] = x_1\left(\frac{n}{10}\right) = \cos\left(\frac{4\pi}{5} n\right)$
- $x_2[n] = x_2\left(\frac{n}{10}\right) = \cos\left(\frac{6\pi}{5} n\right)$
- $x_3[n] = x_3\left(\frac{n}{10}\right) = \cos\left(\frac{14\pi}{5} n\right)$

Denote $\hat{\omega} = \frac{4\pi}{5}$

$\frac{6\pi}{5} = 2\pi - \frac{4\pi}{5}$

$n \in \mathbb{Z}$ and cos periodic & even

Note that $x_2[n] = \cos\left(\frac{6\pi}{5} n\right) = \cos\left((2\pi - \hat{\omega})n\right) = \cos(2\pi n - \hat{\omega} n) =$

$= \cos(\hat{\omega} n) = x_1[n]$

$x_3[n] = \cos\left(\frac{14\pi}{5} n\right) = \cos\left((2\pi + \hat{\omega})n\right) = \cos(2\pi n + \hat{\omega} n) = \cos(\hat{\omega} n) = x_1[n]$

In other words the discrete time frequencies $\hat{\omega}_2 = \frac{6\pi}{5}$ and $\hat{\omega}_3 = \frac{14\pi}{5}$ are <u>aliases</u> of the discrete time freq $\hat{\omega}_1 = \hat{\omega} = \frac{4\pi}{5}$ of the first signal.

(c) (8 points) **(P.A.)** Use programming to numerically verify your conclusion in part (c). Specifically, plot **in the same** figure the three continuous signals $x_1(t)$, $x_2(t)$ and $x_3(t)$ for $t \in [0, 1]$ s. Use a high sampling frequency of 1000 Hz to emulate the continuous nature of the signals.

In the same figure plot the three discrete signals of part (b) (corresponding to sampling frequency 10 Hz). To plot the values of the discrete signals at time instants nT_s, where T_s is the sampling period, use the function "matplotlib.pyplot.scatter()" (instead of "matplotlib.pyplot.plot()" used thus far) of the matplotlib.pyplot package.

Attach a copy of the resulting plot below. **Also, print and attach a copy of your code.**

(d) (2 points) According to the Sampling Theorem, which of the three analog signals of Part (a) can be perfectly reconstructed from its samples at a sampling rate of 10Hz? Why?

According to the sampling theorem perfect reconstruction is possible if $f_s > 2 \cdot f_{max}$ where f_{max} is the maximum frequency in the spectrum of the signal.

- For $x_1[n] \rightarrow f_{max} = 4\,Hz \Rightarrow f_s = 10 > 2 \cdot 4 = 2 \cdot f_{max} \Rightarrow$ perfect reconstruction ✓

  $x_2[n] \rightarrow f_{max} = 6\,Hz \Rightarrow 10 < 2 \cdot 6 = 12 \Rightarrow$ ✗

  $x_3[n] \rightarrow f_{max} = 14\,Hz \Rightarrow 10 < 28 \Rightarrow$ ✗

2. **Problem 2 [Lost].** We have seen in class (also in Problem 1 above) that sampling a sinusoidal signal at a lower frequency can result in the original frequency of the sinusoid to be confused with a lower one. We called this phenomenon aliasing. When sampling a signal that is composed of multiple frequencies then aliasing can result in some of the frequency information of the original signal to "lost". This is illustrated in this problem.

Consider an analog signal given by a weighted sum of four sinusoidal signals as follows

$$x(t) = 5\cos(30\pi t) + 7\cos(80\pi t) + 3\cos(230\pi t) - 4\cos(320\pi t).$$

(a) (2 points) Which frequencies are present in the signal? What is its fundamental frequency?

The frequencies that are present in the signal are $f_1 = 15\text{Hz}$, $f_2 = 40\text{Hz}$, $f_3 = 115\text{Hz}$ and $f_4 = 160\text{Hz}$

Notice that f_1, f_2, f_3 and f_4 are integer multiples of 5

In fact, $\gcd(15, 40, 115, 160) = 5$

So the fundamental frequency is 5Hz

(b) (3 points) Write a mathematical expression for the discrete signal $x[n]$ obtained by sampling $x(t)$ at a rate of 100Hz.

$$X[n] = x\left(\frac{n}{f_s}\right) = 5\cos\left(\frac{3}{10}\pi \cdot n\right) + 7\cos\left(\frac{4}{5}\pi \cdot n\right) + 3 \cdot \cos\left(\frac{23}{10}\pi \cdot n\right) - 4 \cdot \cos\left(\frac{16}{5}\pi \cdot n\right)$$

(c) (3 points) Show that the expression of $x[n]$ in Part (c) simplifies into a weighted sum of **two** (instead of four) discrete sinusoidal signals.

$$x[n] = x\left(\frac{n}{f_s}\right) = 5\cos\left(\frac{3}{10}\pi\cdot n\right) + 7\cos\left(\frac{4}{5}\pi\cdot n\right) + 3\cdot\underbrace{\cos\left(\frac{23}{10}\pi\cdot n\right)} - 4\cdot\underbrace{\cos\left(\frac{16}{5}\pi\cdot n\right)}$$

$$= \cos\left(\frac{3}{10}\pi\cdot n + 2\pi\cdot n\right)$$
$$= \cos\left(\frac{3}{10}\pi\cdot n\right)$$

$$= \cos\left(4\pi\cdot n - \frac{4}{5}\pi\cdot n\right)$$
$$= \cos\left(\frac{4}{5}\pi n\right)$$

$$= 8\cdot\cos\left(\frac{3}{10}\pi n\right) + 3\cdot\cos\left(\frac{4}{5}\pi n\right)$$

(d) (4 points) Compute a second analog signal $s(t)$ that is different from $x(t)$ but when sampled at 100Hz results in the same discrete signal.

Let $s(t) = 8\cos(30\pi t) + 3\cos(80\pi t)$

If we sample $s(t)$ at a rate $f_s = 100$Hz, we get the following signal:

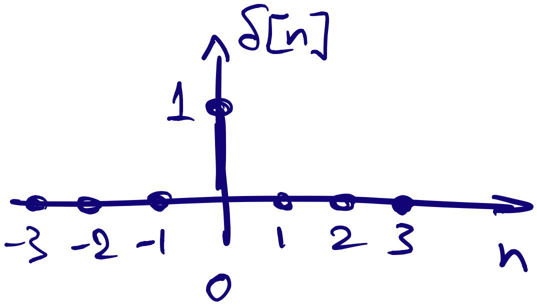
$$s[n] = s\left(\frac{n}{f_s}\right) = 8\cos\left(\frac{3}{10}\pi\cdot n\right) + 3\cdot\cos\left(\frac{4}{5}\pi n\right) \overset{\text{(Part (c))}}{=} x[n]$$

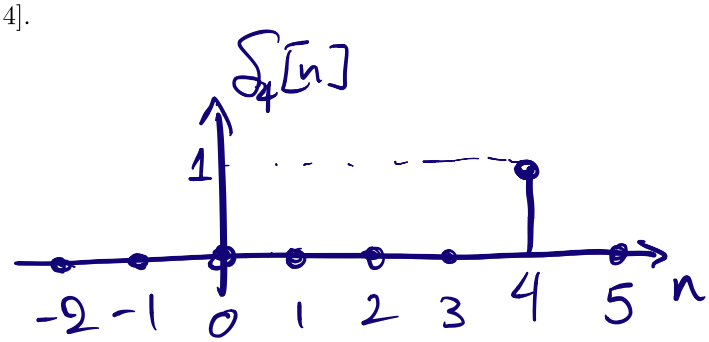
3. **Problem 3 [Impulse].**

One of the most important digital signals is the so-called **unit impulse sequence**, which is a discrete time function whose sample is equal to zero for all values of the time index n except $n = 0$, where it has unity value, that is,

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

(a) (2 points) Draw a graphical representation of the signal $\delta[n]$. The horizontal axis should indicate the time-index value $n = \ldots, -2, -1, 0, 1, 2, \ldots$ and the vertical axis the corresponding signal value. Also, plot the **delayed** signal $\delta_4[n] = \delta[n-4]$.





Unit impulse sequences are useful because any discrete signal can be written as a weighted linear combination of **shifted** impulse sequences. The **time-shifting** operation is used to shift the sequence $\delta[n]$ by a fixed *integer* value $\ell \in \mathbb{Z}$ to generate another sequence $\delta_\ell[n]$ defined as follows:

$$\delta_\ell[n] = \delta[n-\ell], \quad \forall n \in \mathbb{Z}.$$

The following three parts show the claim above. Let $s[n]$ be an arbitrary discrete-time signal.

(b) (2 points) Prove that $s[n]\delta[n-\ell] = s[\ell]\delta[n-\ell]$, for all integers $n \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$.

By definition of the unit impulse response signal:

$$s[n]\delta[n-\ell] = \begin{cases} s[\ell] & , n=\ell \\ 0 & , \text{otherwise} \end{cases}$$

Similarly,

$$s[\ell]\delta[n-\ell] = \begin{cases} s[\ell] & , n=\ell \\ 0 & , \text{otherwise} \end{cases}$$

Thus, $s[n]\delta[n-\ell] = s[\ell]\delta[n-\ell]$, as desired.

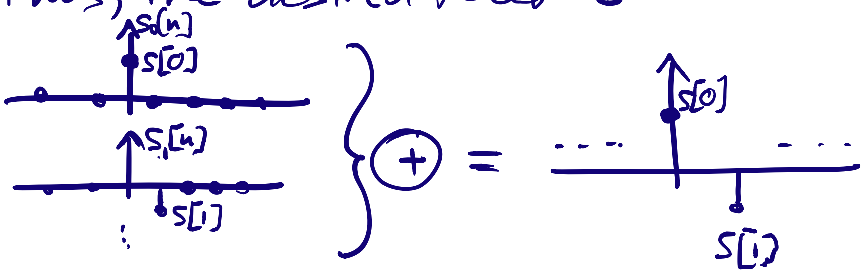
(c) (2 points) Prove that for all $n \in \mathbb{Z}$:

$$s[n] = \sum_{\ell \in \mathbb{Z}} s[n]\delta[n-\ell].$$

Define the signals $s_\ell[n] \triangleq s[n]\delta[n-\ell], \forall n$

By Part (b): $s_\ell[n] = \begin{cases} s[\ell], & n=\ell \\ 0, & \text{otherwise} \end{cases}$

Thus, the desired follows immediately, i.e. $s[n] = \sum_{\ell\in\mathbb{Z}} s_\ell[n] =$



$= \sum_{\ell\in\mathbb{Z}} s[n]\delta[n-\ell]$

(d) (2 points) Combine Parts (b) and (c) to show that for all $n \in \mathbb{Z}$:

$$s[n] = \sum_{\ell \in \mathbb{Z}} s[\ell]\delta[n-\ell].$$

In words, $s[n]$ is a weighted linear combination of shifted impulse sequences, where $s[\ell]\delta[n-\ell]$ is a unit impulse sequence of weight $s[\ell]$ located at $n = \ell$.

By Part (b): $s_\ell[n] = s[\ell]\delta[n-\ell]$

By Part (c): $s[n] = \sum_{n\in\mathbb{Z}} s_\ell[n]$

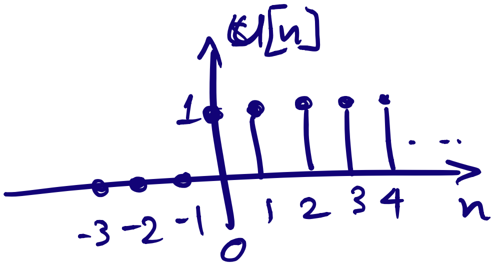
$\Longrightarrow s[n] = \sum_{\ell\in\mathbb{Z}} s[\ell]\delta[n-\ell]$

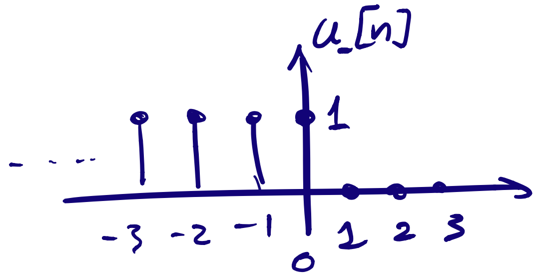
Albeit simple, the representation in Part (d) is very important in SP as it allows us to determine the expression of the output signal of digital systems of certain types (so called **Linear Time Invariant (LTI)** systems) for an arbitrary input sequence, provided we know the expression for the output signal for a unit impulse sequence as the input.

Another useful, basic discrete signal is the **unit step sequence** denoted by $u[n]$. It is a signal defined by

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

(e) (2 points) Give graphical representations of the signal $u[n]$ and of its **time reversed** signal defined as $u_-[n] = u[-n]$.

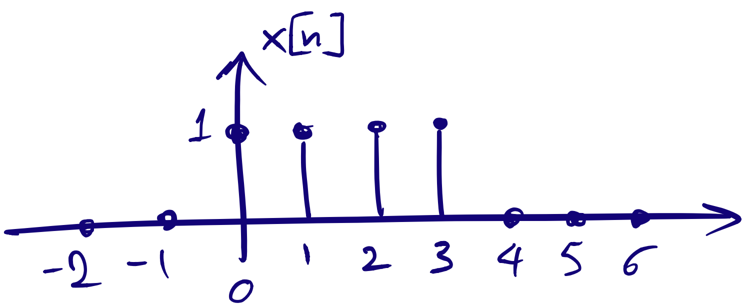




(f) (2 points) Determine and plot the discrete signal defined as $x[n] = u[n] - u[n-5]$.

Clearly,

$$x[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq 4 \\ 0, & \text{else} \end{cases}$$



——— *End of HW #1* ———