University of California Santa Barbara

Homework #2

Introduction to EE

ECE 3

Instructor: Christos Thrampoulidis

Worked Solutions

First & Last Name: ____________________

Perm Number: ____________________

For instructor use:

Question:	1	2	3	4	Total
Points:	8	13	19	15	55
Score:					

Instructions:

- Write your name in the space provided above.
- Please consult the Syllabus for HW grading and late-submission policies.
- Please first solve the problem in draft paper and present your **complete** final answer in *clean* form on the space provided. **Answer all of the questions in the spaces provided. Answers are expected to be succinct but complete.** Please only use extra space (attach to the end of your submission) if absolutely necessary.
- The HW is **due Tuesday October 22 9:00am sharp**.
- **The 4th problem here is the same as Problem 4 of HW 1.** Please upload your notebook to Gauchospace.
- The HW set includes **Programming Assignments** marked as **(P.A.)**. Create a new Jupyter notebook and write code for each one of them. Execute the code to obtain the desired results (e.g., plot of the signals). Upload the notebook **including the code and its output** on GauchoSpace.
- **Return a paper copy of your HW to the homework box in HF. Upload your P.A.s to Gauchospace.**

1. **Problem 1 [Warm up].**

   (a) (4 points) Convert the following complex numbers to *polar form*. Also, plot each one of them as a vector in the same complex plane. Briefly describe your work.

   1. $z = 0 + j2$
   2. $z = -3 - j4$
   3. $z = -j1$
   4. $z = 5\cos(3\pi/4) + j\sin(3\pi/4)$

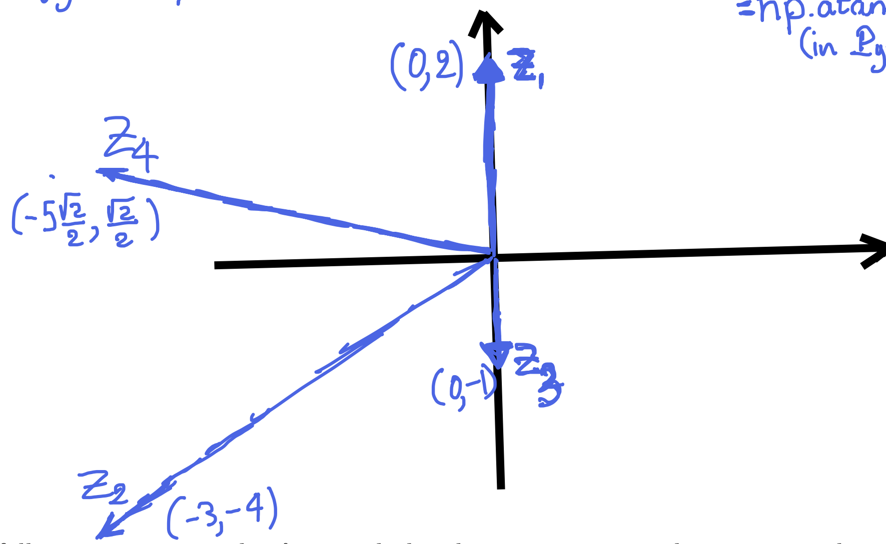
Recall that for $z=x+jy$, the polar form is $z=re^{j\theta}$ where $\begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\arctan(\frac{x}{y}) \\ \quad =\text{np.atan2}(y,x) \\ \quad \text{(in Python)} \end{cases}$

1. $z = 2e^{j\frac{\pi}{2}}$

2. $z = 5e^{-j2.2143}$

3. $z = 1e^{-j\frac{\pi}{2}}$

4. $z = \sqrt{13}\, e^{j2.9442}$



   (b) (4 points) Convert the following in *rectangular form* and plot them as vectors in the same complex plane. Briefly describe your work.

   1. $z = \sqrt{2}e^{j\frac{3\pi}{4}}$
   2. $z = 3e^{-j\frac{\pi}{2}}$
   3. $z = 7e^{-j7\pi}$
   4. $z = 3e^{j\frac{7\pi}{2}}$

Recall that for $z=re^{j\theta}$, the rectangular form is $z=x+jy$ with $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$

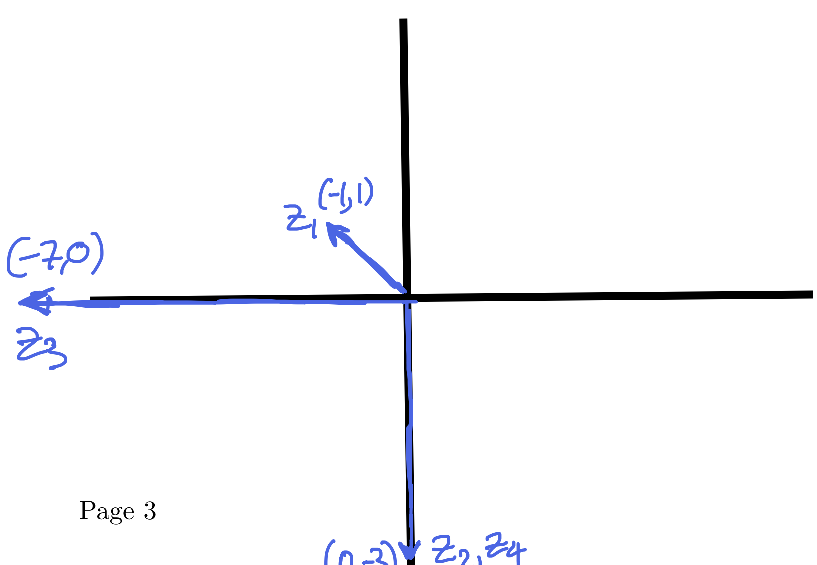
1. $z_1 = -1+j$

2. $z_2 = 0-j3$

3. $z_3 = 7e^{-j7\pi} = 7e^{-j8\pi+j\pi} =$
   $= 7e^{+j\pi} = -7$

4. $z_4 = 3e^{j\frac{7\pi}{2}} = 3\cdot e^{j(4\pi-\frac{\pi}{2})}$
   $= 3e^{-j\frac{\pi}{2}} = 0-j3$

recall $e^{j4\pi} = (e^{j2\pi})^2 = 1^2 = 1$



2. **Problem 2 [Getting comfy].**

(a) (2 points) Add the following pairs of complex numbers. Express the final result in polar form. Show your work.

1. $z_1 = 3$ and $z_2 = 4e^{j\frac{\pi}{4}}$
2. $z_1 = \cos(\pi/4) - j\sin(\pi/4)$ and $z_2 = \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}$

1. $z_1+z_2 = 3+4e^{j\frac{\pi}{4}} = 3+\left(4\cdot\frac{\sqrt{2}}{2}+j4\frac{\sqrt{2}}{2}\right) = \left(3+2\sqrt{2}\right)+j\,2\sqrt{2} = 6.4785\,e^{j0.4518}$

2. $z_1+z_2 = \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\right) =$

$= \left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)+j\left(-\frac{\sqrt{2}}{2}+\frac{1}{2}\right) = \frac{\sqrt{2}+1}{2} + j\,\frac{1-\sqrt{2}}{2} = \sqrt{\frac{3}{2}}\,e^{-j0.1699}$

(b) (2 points) Multiply the following pairs of complex numbers. Express the final result in polar form. Show your work.

1. $z_1 = 3$ and $z_2 = 4e^{j\frac{\pi}{4}}$
2. $z_1 = \cos(\pi/4) - j\sin(\pi/4)$ and $z_2 = \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}$

1. $z_1\cdot z_2 = 3\cdot 4e^{j\frac{\pi}{4}} = 12\cdot e^{j\frac{\pi}{4}}$

2. $z_1\cdot z_2 = \left(e^{-j\frac{\pi}{4}}\right)\cdot\frac{\sqrt{2}}{2}e^{j\frac{\pi}{2}} = \frac{\sqrt{2}}{2}$

(c) (3 points) For any complex number z, prove that the following three identities are true. Recall that z^* is the conjugate of z.

1. $\mathrm{Re(z)} = \frac{(z+z^*)}{2}$
2. $\mathrm{Im(z)} = \frac{(z-z^*)}{2j}$
3. $|z| = \sqrt{zz^*}$

Let $z = x+jy$

1. $Re(z) = x$

$$\frac{z+z^*}{2} = \frac{(x+jy)+(x-jy)}{2} = x \Rightarrow \frac{z+z^*}{2} = Re(z)$$

2. $Im(z) = y$

$$\frac{z-z^*}{2j} = \frac{1}{2j}\left(x+jy-(x-jy)\right) = \frac{1}{2j}2jy = y \Rightarrow \frac{z-z^*}{2j} = Im(z)$$

3. $|z| = \sqrt{x^2+y^2}$

$$zz^* = (x+jy)(x-jy) = x^2 + \cancel{jyx} - \cancel{jyx} - j^2 y = \Rightarrow |z| = \sqrt{zz^*}$$
$$= x^2+y^2$$

(d) (4 points) Consider a complex number $z = re^{j\theta}$ and let z^* be its complex conjugate. Prove that $\frac{1}{z^*} = \frac{1}{r}e^{j\theta}$. Plot an example for $z = 1+j$. For the same example, also plot $1/z$ and z^*.

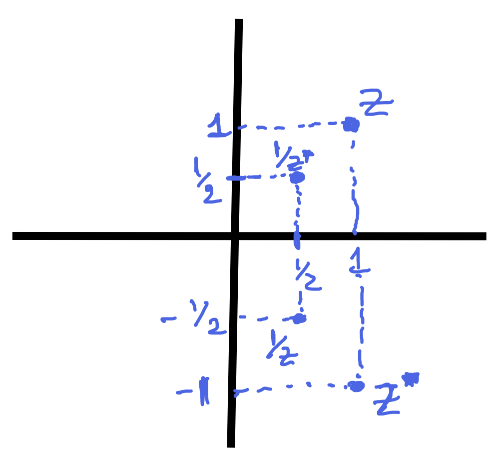
$z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}$

Thus, $\frac{1}{z^*} = \frac{1}{re^{-j\theta}} = \frac{1}{r}e^{-(-j\theta)} = \frac{1}{r}e^{j\theta}$

For $z = 1+j \underset{(polar)}{\Rightarrow} z = \sqrt{2}e^{j\frac{\pi}{4}} \Rightarrow \frac{1}{z^*} = \frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}} =$

$\frac{1}{z} = \frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}$

$z^* = \sqrt{2}e^{-j\frac{\pi}{4}}$



(e) (2 points) Starting from Euler's formula for the complex exponential, prove the following identities:

1. $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$ and $\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$
2. $e^{j\pi} + 1 = 0$

Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

1. $\cos\theta = Re(e^{j\theta}) \underset{=}{(c)} \frac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin\theta = Im(e^{j\theta}) \underset{=}{(c)} \frac{e^{j\theta}-e^{-j\theta}}{2}$

2. Apply Euler's formula for $\theta = \pi$.

3. **Problem 3 [Spectrum: the Complex version].**

(a) (4 points) Consider the signal

$$s(t) = 10 + 20\cos(200\pi t + \frac{\pi}{4}) + 10\cos(500\pi t). \quad (1)$$

1. Using Euler's formula, express the signal as a sum of *complex exponential signals.*
2. Is the signal $s(t)$ periodic? If so, what is the *fundamental period*?
3. Plot the spectrum of this signal versus f in Hz. Use the "complex" version of the line spectrum as we saw it in class.

1. $s(t) = 10 + 20\left(\frac{e^{j(200\pi t+\frac{\pi}{4})} + e^{-j(200\pi t+\frac{\pi}{4})}}{2}\right) + 10\left(\frac{e^{j(500\pi t)} + e^{-j500\pi t}}{2}\right)$

$= 10 + 10e^{j\frac{\pi}{4}}e^{j200\pi t} + 10e^{-j\frac{\pi}{4}}e^{-j200\pi t} + 5e^{j500\pi t} + 5e^{-j500\pi t}$

2. The signal is periodic with period $T_0 = \frac{1}{50}$ s.

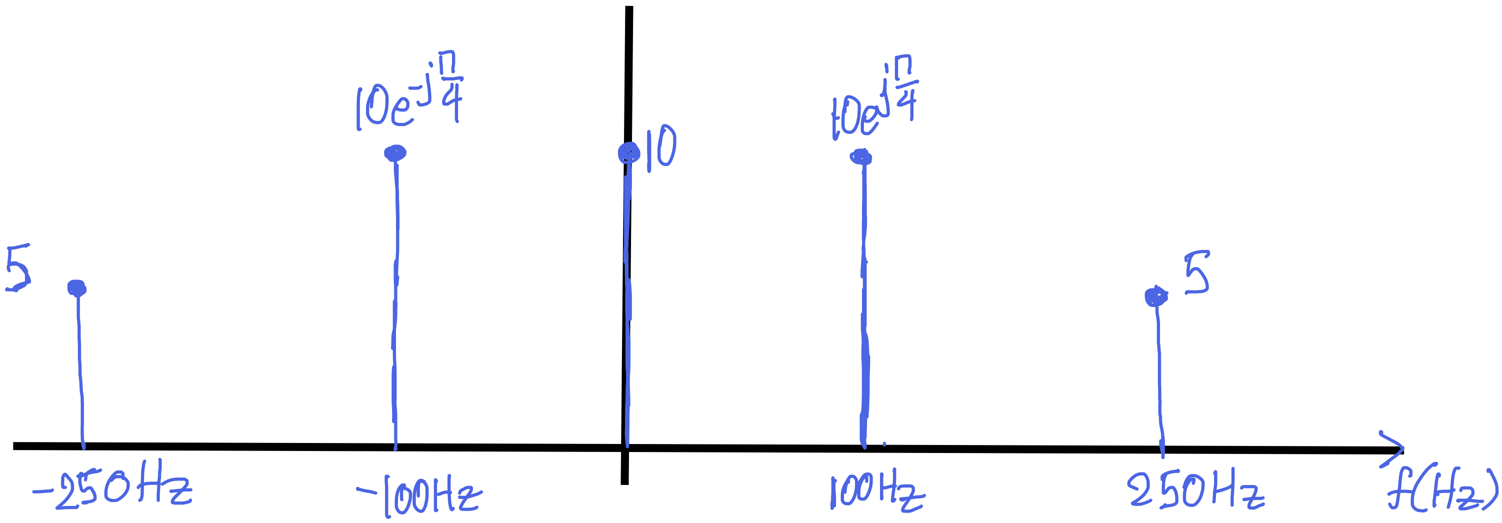
Proof 1: By inspection of the formula of $s(t)$, the signal is composed of the 2^{nd} & 5^{th} harmonics of the fundamental freq. $f_0 = 50$ Hz

Indeed: $200\pi = 2\pi \cdot 100 = 2\pi \cdot f_0 \cdot 2$

$500\pi = 2\pi \cdot 250 = 2\pi \cdot f_0 \cdot 5$

Proof 2 (harder): Verify that $s(t+\frac{1}{50}) = s(t)$ for all t.

3. We use the "complex" or two-sided spectrum.
We get the spectrum immediately from our answer in 1.:



In the file "HW2_Problem3.ipynb" you will find code that calls the fft algorithm (specifically *numpy.fft.fft*) in order to plot a frequency representation of (a sampled version of) $s(t)$. Clearly, in our computers, we can only generate a discretized version of $s(t)$. We do this by sampling the continuous signal in (1) with a sampling frequency $f_s = 5000$ Hz, i.e., 5000 samples per second.

(b) (3 points) Execute the code. Comment on the differences between the frequency plot that the code generates and the line spectrum that you sketched in Part (a)(iii).

See last page of Lecture notes "Lecture 06".

(c) (1 point) Despite the differences, can you still tell which are the dominant frequencies of the signals by looking at the magnitude of the fft coefficients? Which are the dominant frequencies?

Yes! The dominant freqs are those frequencies with largest magnitude in freq. domain (or else the "peaks")

The dominant freqs are 0, 100 and 250 Hz (and their negatives)

(d) (3 points) Change the sampling frequency to $f_s = 700$ Hz.

1. Does this change the frequency plot?
2. What is now the length of the array returned by the fft algorithm?
3. Report the changes (if any) that you observe in the frequency representation.

See the Jupyter notebook "Play with FFT" in Lab 04.

(e) (1 point) Can you still tell the dominant frequencies correctly from the frequency plot?

Yes!

(f) (3 points) Repeat Parts (d) and (e) for $f_s = 400$ Hz. What do you observe? Can you still tell the dominant frequencies correctly from the frequency plot?

We can't tell the dominant freqs any more.
Instead aliasing occurs (we will discuss more on this)

(g) (2 points) Play with different values of the sampling frequency f_s Hz and observe how the frequency plot changes. Can you find the minimum value of f_s for which the dominant frequencies in the corresponding plot coincide with the true dominant frequencies of $s(t)$?

The minimum value of f_s for which aliasing does not occur is $f_s^{min} = 500 Hz$

(h) (2 points) Let $f_{\max}$ be the *maximum* frequency in the continuous-time signal $s(t)$. Formally, $f_{\max}$ is the minimum positive frequency such that the frequency representation of $s(t)$ contains no frequencies larger than $f_{\max}$. Calculate $f_{\max}$. Also, how is your answer in Part (g) related to $f_{\max}$?

By the spectrum plot in part (a), the signal has no frequencies larger than $f_{max} = 250 Hz$

Notice that $f_{max} = \frac{f_s^{min}}{2}$

As we will see this is predicted by the Sampling Theorem.

4. (15 points) **Problem 4 [Find the note]. (P.A.)** Download the files "HW_1_find_the_note.ipynb" and "piano.txt" from GauchoSpace. Open the notebook "HW_1_find_the_note.ipynb" and go through the material. Fill in all the blanks in the code cells (indicated by three dots ...) and write your own lines of code wherever required. When appropriate, answer the questions posed (e.g., "which note was this?"). **Answer everything on the notebook and upload your work on GauchoSpace.**

——— *End of HW #1* ———