

# Math, Multivariable Calculus

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May 18, 2020

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# 1 Vectors

## 1.1 Vector Calculations

# 2 Vector functions

## 2.1 Vector Valued functions in multiple variables

Calculus of single variable is a study of  $y = f(x)$

$$y = f(x), \text{ where } (x \in \mathbb{R}) \rightarrow (f(x) \in \mathbb{R}) \quad (1)$$

(2)

However, when we are dealing with multi-variable calculus, we have: an  $n$  variable vector valued function (if  $m > 1$ )

$$F = \begin{cases} (x_1, \dots, x_n) & \rightarrow F(x_1, \dots, x_n) \\ \in \mathbb{R}^n & \longrightarrow \in \mathbb{R}^m \end{cases} \quad (3)$$

where  $F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n) \dots F_m(x_1, \dots, x_n))$

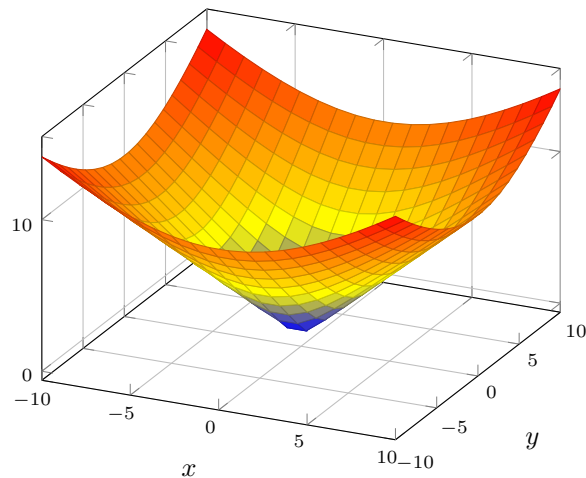
The inputs and outputs are Vectors, for example

$$d(x, y) = \sqrt{x^2 + y^2}, (2 \text{ variables}) \quad (4)$$

$$(x, y) \mathbb{R}^2 \rightarrow (\sqrt{x^2 + y^2}) \mathbb{R} \quad (5)$$

$$\text{distance} = \sqrt{x^2 + y^2} \quad (6)$$

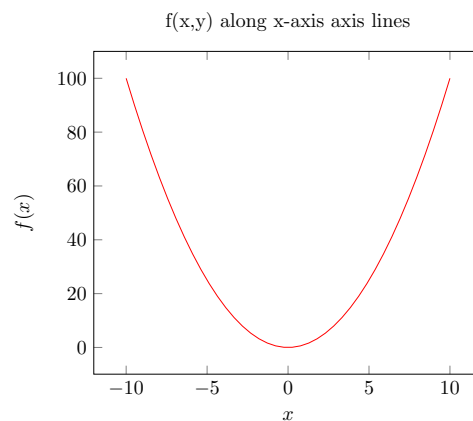
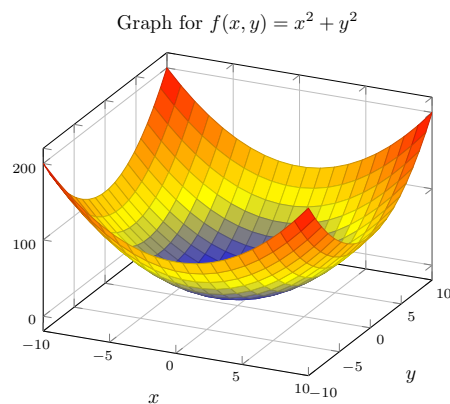
Graph for  $f(x, y) = \sqrt{(x^2 + y^2)}$



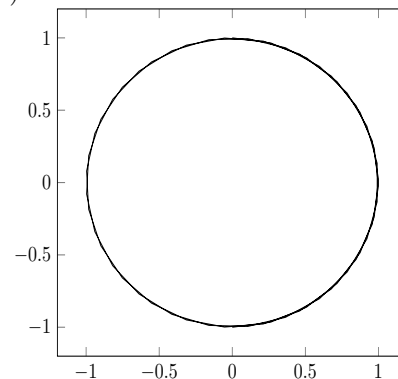
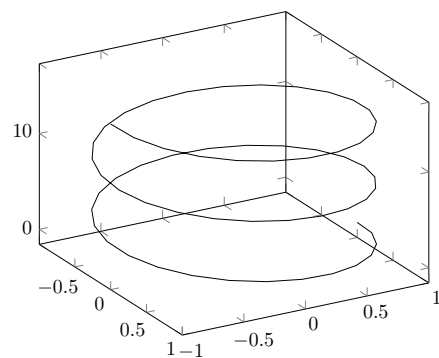
The graph should look like a cone

Similarly, we can have a function that takes all the  $x$ ,  $y$  and  $z$  can calculate that distance. That would however be a four dimensional plot, which we cannot plot.

For a similar function,  $z = x^2 + y^2$



For another function  $f(z) = (\cos z, \sin z)$

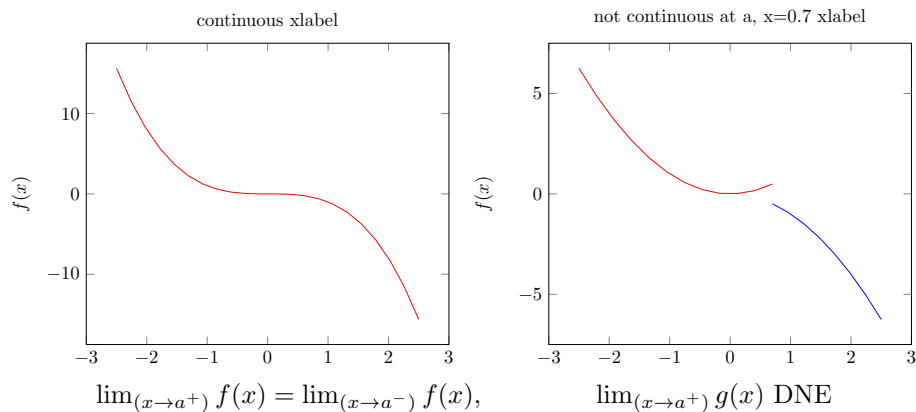


## 2.2 Limits and continuity

Definition:

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \quad (7)$$

if  $f(x, y)$  goes to  $L$  as  $(x, y)$  approaches  $(a, b)$



Similarly, for  $(x, y) \rightarrow (a, b)$  we need to look at all possible paths. If you can find 2 oaths to approach  $(a, b)$  such limits along these 2 paths are different, then the function is not continuous. For example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2} = \text{DNE}(\text{do not exists}) \text{ since:} \quad (8)$$

$$(x, 0) \rightarrow (0, 0), \lim = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0 \quad (9)$$

$$(x, x) \rightarrow (0, 0), \lim = \lim_{y \rightarrow 0} \frac{2x * x}{x^2 + x^2} = 1 \quad (10)$$

$1 \neq 0$  hence D.N.E.

Remember to take a look at all directions

Definition:  $f(x, y)$  continuous at  $(a, b)$  if:

1.  $f(x, y)$  defined at  $(a, b)$
2.  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

How to tell if a function is continuous?  $f(x, y, z) = e^{x-z} \sin(x + yz^2)$

1. (a)  $f(x, y, z) = x$   
(b)  $f(x, y, z) = y$   
(c)  $f(x, y, z) = \text{zallcontinuous}$
2.  $f(x, y, z) = x - z$
3.  $f(x, y, z) = e^{x-z}(\text{composition of contin. func is contin.})$
4.  $g(x, y, z) = \sin(x + yz^2)$  continuous similarly

Test:  $\lim_{(x,y,z) \rightarrow (0,1,-1)} = e^1 \sin(1)$ , hence continuous. A special case is when you have  $\frac{f(x,y)}{g(x,y)}$ , it will be continuous at  $(a, b)$  as long as  $g(a, b) \neq 0$

For a vector function  $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v}) \dots F_m(\vec{v}))$ , where  $\vec{v}$  is a vector or  $\mathbb{R}^n, (x_1, x_2 \dots x_n)$ , is a continuous function at vector  $\vec{n}$  if all its component functions are continuous at  $\vec{n}$

Example:

$$f(x, y, z) = (e^{x+y}, z, \cos(x^2 - z)) \text{ continuous everywhere} \quad (11)$$

$$\text{since } \begin{cases} e^{x+y} \text{ continuous} \\ z \text{ continuous} \\ \cos(x^2 - z) \text{ continuous} \end{cases} \quad (12)$$

### 2.3 Partial Derivatives

The derivative of a function is the rate of change along the x axis. The partial derivatives of  $F(x, y)$  would be the rate of change of F as one of the axis is fixed (along one axis).

$$\frac{dF(x, y)}{dx} = F_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} \quad (13)$$

$$\frac{dF(x, y)}{dy} = F_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} \quad (14)$$

In order to compute the derivative, we treat all other variables as constants.

Example:

$$f(x, y, z) = xe^{xy} - \sin(y^2 + z^2) \quad (15)$$

$$\frac{df}{dx} = e^{xy} + yxe^{xy} - 0 = e^{xy} + yxe^{xy} \quad (16)$$

$$\frac{df}{dy} = x^2e^{xy} - \cos(y^2 + z^2)(2y) \frac{df}{dz} = 0 - \cos(y^2 + z^2)(2z) \quad (17)$$

Take a point  $(a, b)$  for example

1.  $\frac{df}{dx}$  = gradient of tangent at that point along x axis
2.  $\frac{df}{dy}$  = gradient of tangent at that point along y axis
3. These two direction can span the tangent plane of f at that point
  - (a) Tangent Vector
  - (b) along x direction =  $(1, 0, \frac{df}{dx}(a, b)) = \vec{v}$
  - (c) along y direction =  $(0, 1, \frac{df}{dy}(a, b)) = \vec{w}$
  - (d) Using these vectors we can find the plane's equation

Example, find the tangent plane of  $f(x, y)$  at  $(1, 1)$ :

$$f(x, y) = 1 - x^2 - 2y^2 \quad (18)$$

$$f(1, 1) = z = -2, \text{ point at } (1, 1, -2) \quad (19)$$

$$\frac{df}{dx} = (1, 0, -2x)|_{(1,1)} = (1, 0, -2) \quad (20)$$

$$\frac{df}{dy} = (1, 0, -4y)|_{(1,1)} = (0, 1, -4) \quad (21)$$

$$(1, 0, -2) \times (0, 1, -4) = (2, 4, 1) \quad (22)$$

$$2x + 4y + z = 4 \text{ a linear approximation of } f(x, y) \text{ at } (1, 1) \quad (23)$$

Definition of a vector function:

$$F(x_1, x_2, \dots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \dots F_m(\vec{x})) \in \mathbb{R}^m \quad (24)$$

Where each  $F_i$  is a component function. The derivative the size  $(m \times n)$  matrix of  $F$  (Jacobian matrix) denoted by  $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \dots & \dots & \dots & \dots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \dots & \frac{dF_m}{dx_n} \end{pmatrix} \quad (25)$$

The function  $F$  transforms  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $DF: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear map, a linear approximation of  $F$ .

Example:

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin(y + z), 4y) \quad (26)$$

$$DF(x, y, z) = \begin{pmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y + z) & \cos(y + z) \\ 0 & 4 & 0 \end{pmatrix} DF(1, 1, 2) = \quad (27)$$

$$DF(1, 1, 2) = \begin{pmatrix} e^3 & 2e^3 & e^3 \\ 2 & 0 & 0 \\ 0 & \cos(3) & \cos(3) \\ 0 & 4 & 0 \end{pmatrix} \quad (28)$$

Matrix of numbers, a linear map

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin(y + z), 4y) \quad (29)$$

## 2.4 Properties of Derivatives DF

Given two vector function  $F$  and  $G: \mathbb{R}^n \rightarrow \mathbb{R}^m$  both differentiable at some point  $\vec{x}$

$$D(f \pm G) = D(F) \pm D(G) \quad (30)$$

$$C \in \mathbb{R}, D(cF) = cD(F) \quad (31)$$

suppose  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  a function in  $n$  variables, product rule still holds

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \quad (32)$$

where  $D(f \cdot g)$  is a  $n^{th}$  dimension vector,  $D(f \text{ or } g)$  is a vector function and  $f$  or  $g$  is a scalar function

$$D\left(\frac{f}{g}\right) = \frac{g \cdot D(f) - f \cdot D(g)}{g^2}, g(\vec{x}) \neq 0 \quad (33)$$

Given two vector function  $\vec{v}(t)$  and  $\vec{w}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  so  $t \rightarrow \vec{v}(t)$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t) \text{ scalar function} \quad (34)$$

$$f'(t) = \vec{v}(t)' \cdot \vec{w} + \vec{w}(t)' \cdot \vec{v} \quad (35)$$

$$u(t) = \vec{v}(t) \times \vec{w}(t) \text{ vector function} \quad (36)$$

$$u'(t) = \vec{v}(t)' \times \vec{w}(t) + \vec{w}(t)' \times \vec{v}(t) \quad (37)$$

$$\vec{v}(t) = \langle v_1(t), v_2(t), \dots, v_n(t) \rangle \quad (38)$$

Chain rule, suppose  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ ,  $g \circ f : \mathbb{R}^m \rightarrow \mathbb{R}^p$ ,  $\vec{x} \in \mathbb{R}^m$

$$D(g \circ f)(\vec{x}) (\text{size } p \times m) = DG(F(\vec{x})) (\text{size } p \times n) \cdot DF\vec{x} (\text{size } n \times m) \quad (39)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad (40)$$

Example:

$$f(x, y) = (x^3 + y, e^{xy}, 2 + xy) : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (41)$$

$$g(y, v, w) = (y^2 + v, uv + w^3) : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (42)$$

$$g(f(x, y)) = g \circ f = ((x^3 + y)^2 + e^{xy}, (x^3 + y)e^{xy} + (2 + xy)^3) \quad (43)$$

$$D(g \circ f) = \begin{pmatrix} 2(x^3 + y)(3x^2) + ye^{xy} & 2(x^3 + y) + xe^{xy} \\ 3x^2e^{xy} + (x^3 + y)ye^{xy} & e^{xy} + (x^3 + y)xe^{xy} \\ +3(2 + xy)^2y & +3(2 + xy)^2x \end{pmatrix} \quad (44)$$

$$DG = \begin{pmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{pmatrix} \quad (45)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad (46)$$

$$DG = \begin{pmatrix} 2(x^3 + y) & 1 & 0 \\ e^{xy} & x^3 + y & 3(2 + xy)^2 \end{pmatrix} \times \begin{pmatrix} 3x & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{pmatrix} \quad (47)$$

Check that the equality holds.



## 2.5 Directional derivatives

Recall the Jacobian matrix, in the following context, the function  $F$  is not necessarily defined on the entire  $\mathbb{R}^n$ , but rather a subset  $\subseteq \mathbb{R}^n$

$$F(x_1, x_2, \dots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \dots F_m(\vec{x})) \in \mathbb{R}^m \quad (48)$$

Where each  $F_i$  is a component function. The derivative the size  $(m \times n)$  matrix of  $F$  (Jacobian matrix) denoted by  $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \dots & \dots & \dots & \dots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \dots & \frac{dF_m}{dx_n} \end{pmatrix} \quad (49)$$

Examples: 1)

$$f(x, y, z) = x^2y + \ln z : \mathbb{R}^3 \rightarrow \mathbb{R}^1 \quad (50)$$

$$\begin{aligned} Df &= \left( \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right) \\ &= (2x, 1, \frac{1}{z}) \end{aligned} \quad (51)$$

2)

$$\vec{v}(t) = (\cos(t), \sin(t)); \text{ vector function} \quad (52)$$

$$D\vec{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = (-\sin(t), \cos(t))^T \quad (53)$$

$$D\vec{v}(t) \cdot \vec{v}(t) = -\sin(t) \cos(t) + \cos(t) \sin(t) = 0 \quad (54)$$

$$D\vec{v}(t) \perp \vec{v}(t); \text{ hence a tangent vector} \quad (55)$$

3)

$$g(t) = e^t \text{ scalar function} \quad (56)$$

$$g(t) \cdot \vec{v}(t) = e^t(\cos(t), \sin(t)) = (e^t \cos(t), e^t \sin(t)) \quad (57)$$

$$D(g(t) \cdot \vec{v}(t)) = \begin{bmatrix} e^t \cos(t) - e^t \sin(t) \\ e^t \cos(t) + e^t \sin(t) \end{bmatrix} \quad (58)$$

4)

$$\vec{v}(t) = (t, \sin(t), \cos(t)), \vec{w}(t) = (3t, 0, 2) \quad (59)$$

$$\begin{aligned} f(t) &= \vec{v}(t) \cdot \vec{w}(t) \\ &= 3t^2 + 0 + 2 \cos(t); \text{ scalar function} \end{aligned} \quad (60)$$

$$D(f(t)) = 6t - 2 \sin(t) \quad (61)$$

$$\vec{w} \cdot D\vec{v} + \vec{v} \cdot D\vec{w} = \begin{bmatrix} 3t \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ +\cos(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} t \\ \sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad (62)$$

$$= 6t - 2 \sin(t) \quad (63)$$

## 2.6 Partial derivative along a direction

Definition: Given a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is differentiable at  $(a, b)$ , given a unit vector  $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$  then the directional derivatives of  $F$  along  $\vec{u}$  is:

$$D_{\vec{u}}F(a, b) = \left[ \frac{d}{dt} f((a, b) + t\vec{u}) \right]_{t=0} \quad (64)$$

$$= \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \quad (65)$$

is the rate of change of  $F$  along the direction of  $\vec{u}$

Two special cases would be when  $u$  are along the typical  $x$  or  $y$  direction, which would mean the class derivative along  $x$ -axis  $\frac{df}{dx}$  or the  $y$ -axis  $\frac{df}{dy}$

Theorem  $\vec{u} = (u_1, u_2)$  unit vector,

$$D_{\vec{u}}f(x, y) = \left( \frac{df}{dx}, \frac{df}{dy} \right) \cdot \vec{u} = (u_1, u_2) \quad (66)$$

$$\text{since } \vec{u} = u_1(1, 0) + u_2(0, 1) \quad (67)$$

$$D_{\vec{u}}f = u_1 \frac{df}{dx} + \frac{df}{dy} \quad (68)$$

Proof:

$$D_{\vec{u}}f = \lim_{h \rightarrow 0} \frac{F(a + hu_1, b + hu_2) - F(a, b)}{h} \quad (69)$$

$$= \lim_{h \rightarrow 0} \frac{(F(a + hu_1, b + hu_2) - F(a, b + hu_2))}{h} \quad (70)$$

$$+ \lim_{h \rightarrow 0} \frac{F(a, b + hu_2) - F(a, b)}{h} \quad (71)$$

$$= \lim_{u_1 h \rightarrow 0} \frac{[F(a + hu_1, b + hu_2) - F(a, b + hu_2)] \cdot u_1}{hu_1} \quad (72)$$

$$+ \lim_{h \rightarrow 0} \frac{F(a, b + hu_2) - F(a, b)}{h} \quad (73)$$

$$= \frac{df}{dx}(a, b + 0) \cdot u_1 + \frac{df}{dy} \cdot u_s \quad (74)$$

Example: Find the directional Deriv of  $f(x, y) = x^2 + 3xy$  along the direction

of  $(3, 4)$  at the point  $p = (2, -1)$

$$\vec{u} = \frac{(3, 4)}{\sqrt{(3^2 + 4^2)}} = \left(\frac{3}{5}, \frac{4}{5}\right) \quad (75)$$

$$D_{\vec{u}}f(2, -1) = (2x + 3y, 3x)|_{(2, -1)} \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \quad (76)$$

$$= (1, 6) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \quad (77)$$

$$= \frac{27}{5} \quad (78)$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \left(\frac{df}{dx}, df dy\right) \cdot \vec{u} \quad (79)$$

Definition: Given  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ , then define the gradient.

$$\nabla f = D_{\vec{u}}f = \left(\frac{df}{x_1}, \frac{df}{x_2}, \dots, \frac{df}{x_n}\right) \quad (80)$$

Hence the directional derivative of a function along a direction is:

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} \quad (81)$$

The consequence of this:

1)

$$D_{\vec{u}}f \text{ max when } \vec{u} // \nabla f = \left(\frac{df}{dx}, \frac{df}{dy}\right) \quad (82)$$

since

$$\nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos(\theta) \quad (83)$$

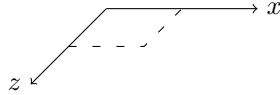
$$= |\nabla f| \cdot 1 \cdot \cos(\theta) \theta = \text{ang}(\nabla f, \vec{u}) \quad (84)$$

When maximized,  $//, \theta = 0$

$$\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{\frac{df^2}{dx} + \frac{df^2}{dy}} \quad (85)$$

2)

$$\text{When } \vec{u} \perp \nabla f, D_{\vec{u}}f = 0 \quad (86)$$



Example:  $f(x, y) = e^{-(x^2 + y^2)}$ , find the direction along this function which has the largest rate of increase/decrease at point  $(1, 1)$

Since at maximum,  $\vec{u} // \nabla f$

$$\nabla f = (-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)})|_{(1,1)} \quad (87)$$

$$= (-2e^{-2}, -2e^{-2}) \quad (88)$$

$$\therefore \vec{u} // \nabla f // (1, 1) \quad (89)$$

$$\therefore \vec{u} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad (90)$$

Largest rate of increase at  $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Largest rate of decrease at  $-\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

The gradient  $\nabla f$  tells us the direction along which F changes the most.

## 2.7 Summary

Given a function $f(x, y)$	n-th dim analogue
Define Partial derivatives (deriv) $\frac{df}{dx}, \frac{df}{dy}$ or $(f_x, f_y)$	$f(x_1, x_2, \dots, x_n)$
Define gradient $\frac{df}{dx}, \frac{df}{dy}$ or $(f_x, f_y)$	$f : \mathbb{R}^n \rightarrow \mathbb{R}$
Directional deriv: given <b>unit</b> vector $\vec{u} = (u_1, u_2)$	$\frac{df}{dx_1} \dots \frac{df}{dx_n}$
$D_{\vec{u}}f = \lim_{h \rightarrow 0} \frac{f((a,b)+h\vec{u}) - f(a,b)}{h}$	$\nabla f = (\uparrow)$
if $(\vec{u} = (1, 0))$ , $D_{\vec{u}}f = \frac{df}{dx}$	$\vec{u}f \in \mathbb{R}^n$
if $(\vec{u} = (0, 1))$ , $D_{\vec{u}}f = \frac{df}{dy}$	$D_{\vec{u}}f = ?$
Theorem: $D_{\vec{u}} = \nabla f \cdot \vec{u} = u_1 \frac{df}{dx} + u_2 \frac{df}{dy}$	generalizations?
Remark: We can define $D_{\vec{v}}$ for any $\vec{v} \in \mathbb{R}^2$ in the exact same way	
$D_{\vec{u}} =  \vec{v}  D_{\frac{\vec{v}}{ \vec{v} }} f$	

To each unit vector  $\vec{u}$ , we have a tangent vector associated to it:

$$(u_1, u_2, D_{\vec{u}}f) = u_1(1, 0, \frac{df}{dx}) + u_2(0, 1, \frac{df}{dy}) \quad (91)$$

Two special cases:  $(1, 0, \frac{df}{dx})(0, 1, \frac{df}{dy})$

The two tangent vector span a tangent plane at  $(a, b, f(a, b))$

## 2.8 Gradients and level sets

Recall

$$\nabla f = \left(\frac{df}{dx}, \frac{df}{dy}\right) \quad (92)$$

$$\therefore D_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos(\theta) \quad (93)$$

$D_{\vec{u}}$  is the rate of change along  $\vec{u}$

if  $u \parallel \nabla f$ , ( $\theta = 0$ ) then  $D_{\vec{u}}F$  max

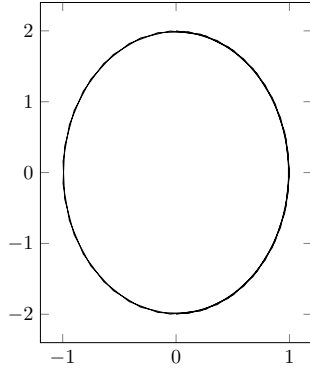
$$|D_{\vec{u}}F| = \sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2} \quad (94)$$

In other words, along the direction of  $\nabla F$ , the function  $F$  changes the most.

Definition: A level set of  $F(x,y)$  is:

$$(x, y) \in \mathbb{R}^2 : f(x, y) = c \quad (95)$$

Example:



$$f(x, y) = x^2 + \frac{y^2}{4} \quad (96)$$

$$\{f(x, y) = 1\} \text{ (Level set)} \quad (97)$$

$$= \{(x, y) | x^2 + \frac{y^2}{4} = 1\} \quad (98)$$

Theorem: Given  $f(x, y)$  differentiable, then the gradient  $\nabla f \perp$  the level set.

We can prove it by using the chain rule: let  $\gamma(t)$  be a parametric function of my level set  $\{f(x, y) = 1\}$ ,  $f(\gamma(t)) = 1$  and differentiate both sides.

$$0 = D(f(\gamma(t))) = Df(\gamma(t))D(\gamma(t)) \quad (99)$$

$$= \nabla F(\gamma(t))_{1 \times 2} \cdot (\gamma'_1(t), \gamma'_2(t))_{2 \times 1} \quad (100)$$

$$\text{Hence } \Rightarrow \nabla F(\gamma(t))_{1 \times 2} \perp (\gamma'_1(t), \gamma'_2(t))_{2 \times 1} \quad (101)$$

Example: Look at a surface  $\sin(xy) - 2\cos(yz) = 0$  Find the tangent plane of this surface at  $(\frac{\pi}{2}, 1, \frac{\pi}{3})$ . The idea is that consider the surface as a level set of  $F(x, y, z) = \sin(xy) - 2\cos(yz)$ ,  $\{F = 0\}$

Using the Theorem:  $\nabla F$  is hence perpendicular to this surface, giving us the normal direction of our tangent plane.

$$\nabla F = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right) = (y \cos(xy), x \cos(xy) + 2z \sin(yz), 2y \sin(tz)) \quad (102)$$

$$= \left(\cos\left(\frac{\pi}{2}\right), \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \frac{2\pi}{3} \sin\left(\frac{\pi}{3}\right), 2 \sin\left(\frac{\pi}{3}\right)\right) \quad (103)$$

$$= \left(0, \frac{\sqrt{3}\pi}{3}, \sqrt{3}\right) \quad (104)$$

$$0 = 0 \cdot \left(x - \frac{\pi}{2}\right) + \frac{\sqrt{3}\pi}{3}(y - 1) + \sqrt{3}\left(z - \frac{\pi}{3}\right) \quad (105)$$

Where the last equation above is the equation of the tangent plane.

## 2.9 Parametric Curves

... are vector functions in 1 variable.

Definition: A parametric curve in  $\mathbb{R}^n$  is a vector valued function where  $\vec{c}(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^n$

All the component functions of  $\vec{c}(t)$  are functions in 1 variable.

Example:

1) Draws a Circle

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \quad (106)$$

2) Draws a ellipse at plane  $z = 3$

$$\vec{c}(t) = (a \cos(t), b \sin(t), 3), t \in [0, 2\pi] \quad (107)$$

3) Ellipse spiraling up

$$\vec{c}(t) = (a \cos(t), b \sin(t), t), t \in [0, 2\pi] \quad (108)$$

Definition: Orientation of  $\vec{c}(t)$ , the direction corresponding to increasing t-value is called the positive orientation of  $\vec{c}(t)$ . The opposite direction is the negative orientation.

Example: Positive orientation = Clockwise

$$\vec{c}(t) = (\cos(t), -\sin(t), t), t \in [0, 2\pi] \quad (109)$$

Derivative of  $\vec{c}(t)$

If  $\vec{c}(t) = (x(t), y(t), z(t))$ ,  $\vec{c}'(t) = (x'(t), y'(t), z'(t))$

1.  $\vec{c}(t)$  is a vector function
2.  $\vec{c}'(t)$  is a tangent to curve  $\vec{c}(t)$

Terminology:  $\vec{c}'(t)$  is the velocity of  $\vec{c}(t)$ , and  $|\vec{c}'(t)|$  is the speed of  $\vec{c}(t)$ . The speed of the curve  $\vec{c}(t) = (\cos(t), \sin(t))$  for example, is 1.

What is the length of curve  $\vec{c}(t) = (\cos(t), \sin(t), t)$  between (0 to  $2\pi$ ) Given  $\vec{c}(t)$  differentiable between range a and b, length of the curve would be

$$l = \int_a^b |\vec{c}'(t)| dt \quad (110)$$

$$|\vec{c}'(t)| = \sqrt{\sin(t)^2 + \cos(t)^2 + 1} = \sqrt{2} \quad (111)$$

$$l = \int_0^{2\pi} \sqrt{2} dt \quad (112)$$

Think of length as a function in t

$$l(t) = \int_a^t |\vec{c}'(x)| dx \quad (113)$$

If curve  $\vec{c}(t)$  differentiable and  $\vec{c}'(t) \neq 0$ ,

1.  $|\vec{c}'(t)| > 0$
2.  $l(t)$  strictly increasing.
3.  $l$  is one to one function so has inverse.

$$l[a, b] \rightarrow \mathbb{R}^1, l \text{ has an inverse.} \quad (114)$$

$$t \rightarrow l(t), \text{ think of our parameter } t \quad (115)$$

$$(116)$$

$t$  can now be expressed as a function in terms of length,  $\vec{c}(t) = \vec{c}(l(t)) = \vec{c}(l)$

Proof:

In our last example of calculating length, we can find an inverse for  $l(t) = \sqrt{2}t$  resulting  $t = \frac{l}{\sqrt{2}}$ , by substitution we can reparameterize  $\vec{c}(t)$  into  $\vec{c}(l)$

$$\vec{c}(l) = \vec{c}(l(t)) \quad (117)$$

$$\frac{d\vec{c}(l)}{dl} = \frac{\vec{c}'(l(t))}{dt} \cdot \frac{dt}{dl} \quad (118)$$

$$= \vec{c}'(t) \cdot \frac{1}{|\vec{c}'(t)|} \quad (119)$$

$$|\frac{d\vec{c}(l)}{dl}| = |\vec{c}'(t)| \cdot \frac{1}{|\vec{c}'(t)|} = 1 \quad (120)$$

## 2.10 Acceleration

Acceleration is the second derivatives of length, hence the definition of second derivative can be expressed as below:

Definition: Given  $\vec{c}(t)$ ,  $\vec{a}(t) = \vec{c}''(t)$  the acceleration of  $\vec{c}$ .

Example:

1) Ellipse

$$\vec{c}_1(t) = (\cos(t), 2 \sin(t)), t \in [0, 2\pi] \quad (121)$$

$$\vec{c}'_1(t) = (-\sin(t), 2 \cos(t)) \quad (122)$$

$$\vec{c}''_1(t) = (-\cos(t), -2 \sin(t)) \quad (123)$$

2) Different para of the same Ellipse

$$\vec{c}_2(t) = (\cos(t^2), 2 \sin(t^2)), t \in [0, \sqrt{2\pi}] \quad (124)$$

$$\vec{c}'_2(t) = 2t(-\sin(t^2), 2 \cos(t^2)) \quad (125)$$

$$\vec{c}''_2(t) = (-2 \cos(t^2) - 4t^2 \cos(t^2), 4 \cos(t^2) - 8t^2 \sin(t^2)) \quad (126)$$

Facts:

1. We can have different parameterations of a curve
2. Velocity, acceleration depend on the para we choose in general
3. For different parameterization  $\vec{c}_1, \vec{c}_2$  of the same curve,  $\vec{c}_1'(t) // \vec{c}_2'(t)$
4. "Unit Velocity"  $\frac{\vec{c}_1'(t)}{|\vec{c}_1'(t)|} = \pm \frac{\vec{c}_2'(t)}{|\vec{c}_2'(t)|}$  are equal, the  $\pm$  depends on the orientations of the  $\vec{c}_1$  and  $\vec{c}_2$ .

Definition: Given  $\vec{c}(t)$ , assume  $|\vec{c}'(t)| \neq 0$ , defined the unit tangent vector  $\vec{T}(t) = \frac{\vec{c}'(t)}{|\vec{c}'(t)|}$ . If  $\vec{c}_1(t)$  and  $\vec{c}_2(t)$  are two different parameter of a curve with the same orientation, their unit tangent vector equals.

## 2.11 Curvature

Idea: Curvature is a measure of how sharply a curve is bending at a point, it is the "rate of change" of directions along a curve. So a straight line has no curvature, and a slightly curved line has a small curvature.

Definition: Quantitatively, if  $\vec{c}(t)$  is parametered by length, (namely,  $|\vec{c}(t)| = 1$ ), then we define the curvature at  $l = l_0$  to be  $k(l_0) = |\vec{c}''(l_0)| = \left| \frac{d^2 \vec{c}(l)}{dl^2} \right|_{l=l_0}$

Remark: IF  $\vec{c}(t)$  not parametered by length, then  $|\vec{c}''(t)|$  is not the curvature.

Explanation: a curve is like a highway and a parameterized function  $\vec{c}(t)$  is like someone driving along this highway giving his position vector at a given time. Parameterized by length would be driving at a constant/normalized speed. A curvature is how sharp a turn is, measured by looking at how quickly the driver need to turn their steering wheel.

Generally,  $|\vec{c}'(l)| = 1 \Rightarrow \vec{c}'(l)$  unit tangent, so  $\vec{c}''(t)$  is curvature.

Example:

1) Find parameter by length

$$\vec{c}(t) = \vec{a} + t\vec{v} \quad (127)$$

$$\text{Constant vectors: } \vec{a} = (1, 2, 3), \vec{v} = (0, 2, 1)$$

$$\text{Length function } l(t) = \int_0^t |\vec{c}'(x)| dx \quad (128)$$

$$= \int_0^t |\vec{v}| dx, \quad \vec{v} = (0, 2, 1) \quad (129)$$

$$l = l(t) = \sqrt{5}t \quad (130)$$

$$t = t(l) = \frac{l}{\sqrt{5}} \quad (131)$$



Curvature would be zero since it is a line, its second derivative would be a zero.

2) Find the curvature for a circle

$$\vec{c}(t) = (r \cos(t), r \sin(t)), r > 0 \text{ circle } x^2 + y^2 = r^2 \quad (132)$$

$$\text{length function } l(t) = \int_0^t |\vec{c}'(x)| dx = rt \quad (133)$$

$$t(l) = \frac{l}{r} \quad (134)$$

$$\vec{c}(l) = (r \cos(\frac{l}{r}), r \sin(\frac{l}{r})) \quad (135)$$

$$\vec{c}''(l) = (-\frac{\cos(l/r)}{r}, -\frac{\sin(l/r)}{r}) \quad (136)$$

$$k(l) = |\vec{c}''(l)| \quad (137)$$

$$= \sqrt{(-\frac{\cos(l/r)}{r})^2 + (-\frac{\sin(l/r)}{r})^2} = \frac{1}{r} \quad (138)$$

Curvature for circle of radius  $r = \frac{1}{r}$

Second Definition of Curvature:

If  $\vec{c}(t)$  any parametric curve,  $\vec{c}'(t) \neq 0$ , recall the unit tangent vector  $\vec{T} = \frac{\vec{c}'(t)}{|\vec{c}'(t)|}$ ,

then define curvature  $k(t) = \frac{|\vec{T}'(t)|}{|\vec{c}'(t)|}$

Theorem: Definition 1 and 2 of curvature are the same.

$$\vec{c}'(t) = \frac{d\vec{c}(t(l))}{dt} \quad (139)$$

$$= \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} \quad (140)$$

$$= \frac{\vec{c}'(t)}{|\vec{c}'(t)|} = \vec{T} \quad (141)$$

$$k(l(t)) = |\vec{c}''(t)| = \left| \frac{d^2\vec{c}}{dl^2} \right| = \left| \frac{d\vec{T}(t(l))}{dl} \right| \quad (142)$$

$$= \frac{d\vec{c}'(t)}{dt} \cdot \frac{dt}{dl} \quad (143)$$

$$= |\vec{T}'(t)| \cdot \frac{1}{|\vec{c}'(t)|} = k(t) \quad (144)$$

Using the second definition to find the curvature of the parabola  $y = x^2$

$$\vec{c} = (t, t^2) \quad (145)$$

$$\vec{c}'(t) = (1, 2t) \quad (146)$$

$$\vec{T}(t) = \frac{(1, 2t)}{\sqrt{1 + 4t^2}} \quad (147)$$

$$\vec{T}'(t) = \frac{(0, 2)}{\sqrt{1 + 4t^2}} + (1, 2t) \cdot ((a + 4t^2)^{-\frac{1}{2}})' \quad (148)$$

$$= \frac{(-4t, 2)}{(1 + 4t^2)^{\frac{3}{2}}} \quad (149)$$

$$|T'(t)| = \frac{\sqrt{4 + 16t^2}}{(1 + 4t^2)^{\frac{3}{2}}} \quad (150)$$

$$= \frac{2}{1 + 4t^2} \quad (151)$$

$$K(t) = |T'(t)| \cdot \frac{1}{|c'(t)|} \quad (152)$$

$$= \frac{2}{(1 + 4t^2)} \cdot \frac{1}{\sqrt{1 + 4t^2}} \quad (153)$$

$$= 2 \cdot (1 + 4t^2)^{-\frac{3}{2}} \quad (154)$$

As  $x$  approaches  $\infty$ , the curvature approaches 0,  $y = x^2$  became more like a straight line.

#### Osculating Plane and Circle

1. Derivative — Tangent line
2. Curvature — Osculating Circle

Observations:

1) If  $\vec{c}(l)$  parameterized by length, then

$$\vec{c}'(l) \perp \vec{c}''(l) \quad (155)$$

$$\because (\vec{c}'(l) \cdot \vec{c}'(l))' = (|\vec{c}'(l)|^2)' = (1)' \quad (156)$$

$$\vec{c}'(l) \cdot \vec{c}''(l) + \vec{c}'(l) \cdot \vec{c}''(l) = 0 \quad (157)$$

$$\Rightarrow \vec{c}'(l) \cdot \vec{c}''(l) = 0 \quad (158)$$

so second derivative is the normal and the first derivative is the tangent.

2) If  $\vec{c}(t)$  is any parameterized curve, its unit tangent vector  $\vec{T}$  and  $\vec{T}'$  is

$$\vec{T}(t) = \frac{\vec{c}'(t)}{|\vec{c}'(t)|} \quad (159)$$

$$\because (\vec{T}(t) \cdot \vec{T}(t))' = (|\vec{T}(t)|^2)' = (1)' \quad (160)$$

$$\Rightarrow \vec{T}(t) \cdot \vec{T}'(t) = 0 \quad (161)$$

$$\therefore \vec{T}(t) \perp \vec{T}'(t) \quad (162)$$

Definition: Osculating Plane: Given  $\vec{c}(t)$ ,  $\vec{c}'(t) \neq 0$ , then the osculating plane is the plane spanned by  $\vec{T}(t)$  and  $\vec{T}'(t)$

Recall that we computed for a circle with radius  $r$  its curvature  $k = \frac{1}{r}$ , therefore, if have a  $\vec{c}(t)$  with  $k(t_0) = k$ , then at this point, the curve  $\vec{c}(t)$  is "modeled" by a circle of radius  $\frac{1}{k}$

Definition: Osculating Circle: Given  $\vec{c}(t)$ ,  $\vec{c}'(t) \neq 0$ , then the osculating circle of  $\vec{c}$  at  $t = t_0$  is a circle on the osculating plane at  $t = t_0$  and has radius  $\frac{1}{k(t_0)}$ , and the tangent to the curve. (The best approximation of the curve at that point by the circle)

Example: Find the equation of the osculating plane at  $\vec{c}(\pi) = (-a, 0, b\pi)$  of  $\vec{c}(t) = (a \cos(t), a \sin(t), bt)$ ,  $t \geq 0$ ;  $a, b > 0$ ,

$$\vec{T} = \frac{\vec{c}'(t)}{|\vec{c}'(t)|} = \frac{(-a \sin(t), a \cos(t), b)}{\sqrt{a^2 + b^2}} \quad (163)$$

$$\vec{T}' = \frac{(-a \cos(t), -a \sin(t), 0)}{\sqrt{a^2 + b^2}} \quad (164)$$

$$\text{Osc.Plane} = \vec{T} \times \vec{T}' \quad (165)$$

$$= \frac{1}{a^2 + b^2} (ab \sin(t), -ab \cos(t), a^2)|_{t=\pi} \quad (166)$$

$$= \frac{1}{a^2 + b^2} (0, ab, a^2) \quad (167)$$

calculating plane at given point

$$a^2 b \pi = 0x + aby + a^2 z \quad (168)$$

## 2.12 Higher order derivatives and Taylor expansions

Recall partial derivatives of  $F(x, y)$  by  $x$  and  $y$  denoted by  $F_x$  and  $F_y$ , second order derivative  $= F_{xx}$  and  $F_{yy}$ , first  $x$  then  $y$   $F_{xy}$  and first  $y$  then  $x$   $F_{yx}$ , vise versa.

Theorem: If  $F(x, y)$  differentiable and  $F_{xy}$ ,  $F_{yx}$  continuous,  $F_{xy} = F_{yx}$ , hence the order can be switched

Example:  $f(x, y, z) = x^2 e^{y^2 - x} + x^2 y z^2 - \cos(x^2 + y^2)$ ,  $f_{xzzz} = ?$  switch order, try finding  $f_{zzxy}$  instead

$$df/dz = f_z = 2zx^2y \Rightarrow f_{zz} = 2x^2y \Rightarrow f_{zzxy} = 4x \quad (169)$$

## 2.13 Taylor Expansions

One Variable:  $y = f(x)$ ,  $f'(x)$  =, tangent line of graph of the function, its linear approximation at point  $x = x_0$ , equation of tangent line is  $y = f(x_0) + f'(x_0)(x - x_0)$ . Higher order  $f'(x) \rightarrow$  higher degree approximation.

$$f(x) - f(x_0) = \int_{x_0}^x f'(t) dt \quad (170)$$

$$\int u'v = uv - \int v'u, \quad u = t, v = f'(t) \quad (171)$$

$$= tf'(t)|_{x_0}^x - \int_{x_0}^x t df'(t) \text{ by parts} \quad (172)$$

$$= [xf'(x) - x_0f'(x_0)] - \int_{x_0}^x tf''(t) dt \quad (173)$$

$$= (xf'(x) - x_0f'(x_0)) + (x_0f'(x_0) - x_0f'(x_0)) - \int_{x_0}^x tf''(t) dt \quad (174)$$

$$= (x - x_0)f'(x_0) + x(f'(x) - f'(x_0)) - x \int_{x_0}^x tf''(t) dt \quad (175)$$

$$= f'(x_0)(x - x_0) + x \int_{x_0}^x f''(t) dt - \int_{x_0}^x tf''(t) dt \quad (176)$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x - t)f''(t) dt \quad (177)$$

$$(178)$$

This is the 1 st Taylor expansion where  $f(x_0) + f'(x_0)(x - x_0)$  is the 1st order approximation, a tangent line, and  $\int_{x_0}^x (x - t)f''(t) dt$  is the 1st order remainder. error/diff. to linear approx.

We can continue to get a higher order Taylor expansion using integration by parts by setting  $v = f''(t)$  and  $u' = (x - t)$  which would yield:

$$\begin{aligned} f(x) = & f(x_0) + f'(x_0)(x - x_0) \\ & + f''(x_0) \cdot \frac{(x - x_0)^2}{2} \\ & + \int_{x_0}^x (x - t)f'''(t) \cdot \frac{(x - t)^2}{2} dt \end{aligned} \quad (179)$$

Theorem: if  $f(x)$  differentiable, then  $\frac{d^n(f(x))}{dx^n}$  exists for all  $n$ . Hence the  $n$ -th order of the Taylor expansion would be:

$$f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} \dots + f^{(n)}(x_0) \frac{(x-x_0)^n}{n!} + \int_{x_0}^x (x-t) f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt \quad (180)$$

Example:

Find the 2nd order Taylor expansion at  $x = 0$  of  $f(x) = x + e^{-2x}$  Use that to approximate the value  $f(0.075) = 0.075 + e^{-0.15}$  and estimate the error.

$$\begin{aligned} f'(x) &= (1 - 2e^{2x}), f''(x) = 4e^{-2x} \\ f(x) &= f(0) + f'(0)(x-0) \\ &\quad + \frac{f''(0)(x-0)^2}{2} \\ &\quad + \int_0^x f'''(t) \frac{(x-t)^2}{2} dt \\ f(0.075) &\approx 1 - (0.075) + 2(0.075)^2 \end{aligned} \quad (181)$$

Error given by the remainder  $R(x)$ , a bound for the error

$$\text{error} \leq |R(x)| \leq \left| \int_0^x -8e^{2t} \cdot \frac{(x-t)^2}{2} dt \right| \quad (182)$$

$$\leq 4x^2(x-0) = 4x^3 = (4 \cdot (0.075)^3) \quad (183)$$

For 2 dimension, the second order expansion of  $F(x, y)$  at  $(x_0, y_0)$  is:

$$\begin{aligned} F(x, y) &= F(x_0, y_0) \\ &\quad + \frac{1}{1!} \left[ \frac{\sigma F}{\sigma x}(x_0, y_0)(x-x_0) + \frac{\sigma F}{\sigma y}(x_0, y_0)(y-y_0) \right] \\ &\quad + \frac{1}{2!} \left[ \frac{\sigma^2 F}{\sigma x^2}(x_0, y_0)(x-x_0)^2 + \frac{\sigma^2 F}{\sigma y^2}(x_0, y_0)(y-y_0)^2 \right. \\ &\quad \left. + 2 \frac{\sigma^2 F}{\sigma x \sigma y} \right] + R_2(x, y) \end{aligned} \quad (184)$$

Example: Find the second Taylor expansion of  $f(x, y) = \sin(x+y) + \cos(x-3y)$

at (0,0)

$$f_x = \cos(x+y) - \sin(x-3y) \quad (185)$$

$$f_y = \cos(x+y) + 3\sin(x-3y) \quad (186)$$

$$f_{xx} = -\sin(x+y) - \cos(x-3y) \quad (187)$$

$$f_{xy} = -\sin(x+y) + 3\cos(x-3y) \quad (188)$$

$$f_{yy} = -\sin(x+y) - 9\cos(x-3y) \quad (189)$$

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y \quad (190)$$

$$+ \frac{1}{2} [f_{xx}x^2 + f_{yy}y^2 + 2f_{xy}xy] + R_2(x,y) \quad (191)$$

$$= 1 + x + y + \frac{1}{2}(-x^2 - 9y^2 + 6xy) + R_2(x,y) \quad (192)$$

Second Approximation, Taylor Polynomial.

For extreme values of F(x,y): Recall that given y = f(x), differentiable, n-th Taylor expansion at  $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \cdot \frac{(x - x_0)^2}{2} \dots \quad (193)$$

$$+ f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} \text{ Taylor polynomial} \quad (194)$$

$$+ \int_{x_0}^x f^{(n+1)}(t) \frac{(x - t)^n}{n!} dt \text{ } R_n(x) \text{ remainder} \quad (195)$$

Taylor Expansion for  $F(x,y)$  at  $(x_0, y_0)$

$$F(x,y) = F(x_0, y_0) + F_x(x_0, y_0)x + F_y(x_0, y_0)y \quad (196)$$

$$+ F_{xx}(x_0, y_0) \frac{x^2}{2} + F_{xy}(x_0, y_0) \frac{xy}{2} \quad (197)$$

$$+ F_{yx}(x_0, y_0) \frac{yx}{2} + F_{yy}(x_0, y_0) \frac{y^2}{2} \quad (198)$$

$$+ R_2(x,y) \quad (199)$$

Recall: for  $y = f(x)$ , critical points,  $f'(x_0) = 0$  are candidates for local max and local min.

1.  $f''(x_0) > 0$  local min
2.  $f''(x_0) < 0$  local max
3.  $f''(x_0) = 0$  inconclusive

if  $x_0$  is critical point, first derivative term  $f'(x_0) = 0$  removed from polynomial. The two variable are F(x,y). Definition F(x,y) differentiable, then say  $(x_0, y_0)$  is a critical point if  $\nabla F(x_0, y_0) = 0$ .

Recall equation of tangent plane

$$z = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(y - y_0) \quad (200)$$

Tangent plane is horizontal. Definition: for  $F(x, y)$ , define the Hessian matrix.

$$\begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} (x_0, y_0) \quad (201)$$

Define the discriminant,  $(x_0, y_0)$  is critical point.

$$D(x_0, y_0) = \det \begin{pmatrix} F_{xx}(x_0, y_0) & F_{xy}(x_0, y_0) \\ F_{yx}(x_0, y_0) & F_{yy}(x_0, y_0) \end{pmatrix} (x_0, y_0) = F_{xx}F_{yy} - F_{xy}F_{yx} \quad (202)$$

Theorem (2nd derivative test): Suppose  $(x_0, y_0)$  is a critical point, then

1. If  $D(x_0, y_0) > 0$  and  $F_{xx} > 0$  or  $(F_{yy} > 0)$ , then  $(x_0, y_0)$  is a local minimum
2. If  $D(x_0, y_0) > 0$  and  $F_{xx} < 0$  or  $(F_{yy} < 0)$ , then  $(x_0, y_0)$  is a local maximum
3. if  $D(x_0, y_0) < 0$ , then  $(x_0, y_0)$  is a saddle point.
4. if  $D(x_0, y_0) = 0$ , then it is inconclusive.

Find and classify all critical points of  $F(x, y) = x^3 - 3x^2 - 3y^2 + 3xy^2$

$$F_x = 3x^2 - 6x + 3y^2 = 0 \quad (203)$$

$$F_y = -6y + 6xy = 0 \quad (204)$$

$$\Rightarrow y = 0 \text{ or } x = 1 \text{ giving two critical point} \quad (205)$$

$$y = 0, \Rightarrow (0, 0), (2, 0) \quad (206)$$

$$x = 1, \Rightarrow (1, 1), (1, -1) \quad (207)$$

Use Second derivative to test for the four critical points

$$D(x, y) = F_{xx}F_{yy} - F_{xy}^2 \quad (208)$$

$$D(1, 1) = \Rightarrow \text{is a saddle point} \quad (209)$$

$$D(1, -1) = \Rightarrow \text{is a saddle point} \quad (210)$$

$$D(0, 0) = \Rightarrow \text{is local max} \quad (211)$$

$$D(2, 0) = \Rightarrow \text{is local min} \quad (212)$$

Why is this true?

From the second Taylor expansion, at a critical point, the dominant part:

$$F_{xx}(x_0, y_0)(x - x_0)^2 + 2F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + F_{yy}(y - y_0)^2 \quad (213)$$

Homogeneous degree 2 polynomial in 2 variables:

$$P(x, y) = ax^2 + 2Cxy + by^2 \quad (214)$$

$$= a(x^2 + \frac{2c}{a}xy + \frac{b}{a}y^2) \quad (215)$$

$$= a((x - \frac{c}{a}y)^2 + \frac{ab - c^2}{a^2}y^2) \quad (216)$$

$$= a((x - \frac{c}{a}y)^2 + \frac{ab - c^2}{a^2}y^2) \quad (217)$$

$$\Delta = ab - c^2 > 0, \text{ then } \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{ab-c^2}}{a}y \end{cases} \quad (218)$$

$$= a((x')^2 + (y')^2) \quad (219)$$

$$\Rightarrow a > 0 \text{ local min, } a < 0 \text{ local max} \quad (220)$$

$$\text{If } \Delta < 0 \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{c^2-ab}}{a}y \end{cases} \quad (221)$$

$$\text{then } P(x, y) = P(x', y') = a((x')^2 - (y')^2) \quad (222)$$

$$a > 0 \text{ saddle point} \quad (223)$$

Apply this to  $F_{xx}(x_0, y_0)(x - x_0)^2 + 2F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + F_{yy}(y - y_0)^2$

$$a = F_{xx}(x_0, y_0)(x - x_0)^2 \quad (224)$$

$$b = F_{yy}(x_0, y_0)(x - x_0)^2 \quad (225)$$

$$2c = 2F_{xy}(x_0, y_0)(x - x_0)^2 \quad (226)$$

$$ab - c^2 = D(x, y) \quad (227)$$

Global Extreme Values for F(x,y).

Recall local max/min:

1. Find critical points  $\nabla F(x_0, y_0) = \vec{0}$
2. Use second derivatives to test the critical points.

For Global Extreme values:

Example:

1) Find global min and max of  $f(x, y) = x^3 - 3xy + 3y^2$  defined on the rectangle  $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 2\}$



First, find local critical point

$$f_x = 3x^2 - 3y = 0 \quad (228)$$

$$f_y = -3x + 6y = 0 \quad (229)$$

$$\Rightarrow x = 2y \quad (230)$$

$$\text{plug in to } f_x : 0 = 3(4y^2 - y) \quad (231)$$

$$0 = y(4y - 1) \quad (232)$$

$$\Rightarrow y = 0 \text{ or } y = \frac{1}{4} \quad (233)$$

$$\therefore (0, 0) \text{ \& } (0.5, 0.25) \text{ two critical point} \quad (234)$$

$$\text{Hessian Matrix } D(x, y) = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \quad (235)$$

$$= f_{xx} - f_{xy}^2 = 36x - 9 = 9(4x - 1) \quad (236)$$

$$\Rightarrow D(0, 0) = -9 < 0, \text{ saddle, neither max/min} \quad (237)$$

$$\Rightarrow D(0.5, 0.25) = 9 > 0, \quad f_{xx}(0.5, 0.25) = 3 > 0 \quad (238)$$

$$(239)$$

Need to compare with boundary values.

$$\text{Side 1: } \{x = 0, 0 \leq y \leq 2\} \quad (240)$$

$$f(x, y) = x^3 - 3xy + 3y^2 |_{x=0} \quad (241)$$

$$f(0, y) = 3y^2 \Rightarrow \max 12 \min 0 \quad (242)$$

$$\text{Side 2: } \{y = 0, 0 \leq x \leq 1\} \quad (243)$$

$$f(x, 0) = x^3, \max 1 \min 0 \quad (244)$$

$$\text{Side 3: } \{x = 1, 0 \leq y \leq 2\} \quad (245)$$

$$f(1, y) = 1 - 3y + 3y^2, \max 7 \min 0.25 \quad (246)$$

$$\text{Side 4: } \{y = 2, 0 \leq x \leq 1\} \quad (247)$$

$$f(x, 2) = x^3 - 6x + 12, \max 12 \min 7 \quad (248)$$

Hence global max 12 at (0,2) and -1/16 at (0.5,0.25)

2) Find for the function  $f(x, y) = 2x^2 + 2y^2 - x + y$  on disk  $D = \{(x, y) : x^2 + y^2 \leq$

1} its global Max and min/min

$$f_x = 4x - 1, f_y = 4y + 1, \quad (249)$$

$$f_x = 0 \text{ and } f_y = 0 \rightarrow (x = 1/4, y = -1/4) \quad (250)$$

$$f_{xx} = 4, f_{yy} = 4, f_{xy} = 0 \quad (251)$$

$$D(x, y) = 16 - 0^2 = 16 > 0, f_{xx} = 4 > 0 \quad (252)$$

$$\text{see } (x = 1/4, y = -1/4) \text{ is local min} \quad (253)$$

Parameterize the bounding (unit circle)

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \quad (254)$$

$$\text{plugin } f(\cos(t), \sin(t)) = 2 - \cos(t) + \sin(t), t \in [0, 2\pi] \quad (255)$$

$$f'(t) = \sin(t) + \cos(t) = 0 \quad (256)$$

$$-1 = \tan(t) \quad (257)$$

$$\therefore \text{critical points } t = \frac{3\pi}{4}, t = \frac{7\pi}{4} \quad (258)$$

$$f(t) = 2 - \cos(t) + \sin(t) \text{ at } 0 \& 2\pi = 1 \& 1 \quad (259)$$

$$f\left(\frac{3\pi}{4}\right) = 2 - \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right) = 2 + \sqrt{2} \quad (260)$$

$$f\left(\frac{7\pi}{4}\right) = 2 - \sqrt{2} \text{ min on boundary} \quad (261)$$

Recall have local min at (0.25, -0.25)

Global max at  $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ ,  $2 + \sqrt{2}$

Global max at  $(\frac{1}{4}, -\frac{1}{4})$ ,  $-\frac{1}{4}$

Optimization with constraint — Lagrangian multiplier.

Given  $f(x, y)$ , find max/min with a constraint  $g(x, y) = k$ ,  $k = \text{a fixed number}$ .

The langrangian multiplier method would be:

$$\text{Step 1 } \begin{cases} \nabla f = \lambda \nabla g, \lambda = \text{constant} \\ g(x, y) = k \end{cases} \quad (262)$$

solve this system of equation to get  $(x, y)$

Step 2, plug in  $(x, y)$  into  $f(x, y)$

Example:  $f(x, y) = 4xy$ , find max/min with condition  $x^2 + y^2 = 4$

$$g(x, y) = x^2 + y^2, g(x, y) = 4 \quad (263)$$

$$f \& g = \begin{cases} \nabla f = & (2y, 2x) = \lambda \cdot \nabla g = (2x, 2y) \\ g(x, y) = & 4 = x^2 + y^2 \end{cases} \quad (264)$$

$$\therefore 2y = \lambda 2x \rightarrow \lambda = \frac{y}{x} \quad (265)$$

$$2x = \lambda 2y \rightarrow \lambda = \frac{x}{y} \quad (266)$$

$$\therefore x^2 = y^2 \& g(x, y) : x^2 + y^2 = 4, x^2 = y^2 = 2 \quad (267)$$

$$x, y = \pm \sqrt{2}, 4 \text{ solutions, max } 8, \text{ min } -8 \quad (268)$$

Why does it work? at max  $\nabla g // \nabla f$

## 2.14 Vector Field and Flow

Definition: A vector field  $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$

1. Example 1:  $\vec{F}(x, y) = (x, y), \mathbb{R}^2 \rightarrow \mathbb{R}^2$
2. Example 2:  $\vec{F}(x, y) = (-y, x), \mathbb{R}^2 \rightarrow \mathbb{R}^2$
3. Example 3:  $\vec{F}(x, y, z) = (x^3, xy, e^{xyz}), \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Definition: (Flow/Integral Curve) Given  $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , a flow of  $\vec{F}$  is curve  $\vec{C}(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^m$  such that  $\vec{C}'(t) = \vec{F}(\vec{C}(t))$

Example:

$$\vec{F}(x, y) = (-y, x) \quad (269)$$

Verify that the curve  $\vec{C}(t) = (\cos(t), \sin(t))$ , is a flow/integral curve of  $\vec{F}$  by checking if  $\vec{C}'(t) = \vec{F}(\vec{C}(t))$

$$\vec{C}'(t) = (-\sin(t), \cos(t)) \quad (270)$$

$$\vec{C}'(t) \equiv \vec{F}(\vec{C}(t)) \quad (271)$$

Remark: Finding flows of a vector field is the focus of differential equations.

An important class of example of vector field is given by  $\nabla F$  where  $F : \mathbb{R}^m \rightarrow \mathbb{R}^1$ ,  $\nabla F = (\frac{\sigma F}{\sigma x}, \frac{\sigma F}{\sigma y}, \frac{\sigma F}{\sigma z}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Looking at  $\nabla F$  tells us something about F

1.  $\nabla F$  is the direction along which F changes most dramatically.
2.  $\nabla F \perp$  level sets ( $F(x, y, z) = c$ ).

Definition (Divergence and Curl): Given  $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $\vec{F} = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ . Define the divergence of  $\vec{F}$

1.  $\text{div } \vec{F} = (\frac{\sigma F}{\sigma x}, \frac{\sigma F}{\sigma y}, \frac{\sigma F}{\sigma z})$
2.  $\text{div } \vec{F} = \mathbb{R}^3 \rightarrow \mathbb{R}^1$  a scalar function

Define the curl of  $\vec{F}$

1.  $\text{Curl } \vec{F} = \begin{bmatrix} \sigma/\sigma x \\ \sigma/\sigma y \\ \sigma/\sigma z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$
2.  $\text{div } \vec{F} = \mathbb{R}^3 \rightarrow \mathbb{R}^3$  a vector function

Easier way to memorize these 2 notations:  $\nabla$  is a vector of differential operators  $\langle \frac{\sigma}{\sigma x}, \frac{\sigma}{\sigma y}, \frac{\sigma}{\sigma z} \rangle$

Hence  $\nabla F$  means the gradient of F. So  $\text{div } \vec{F}$  is the dot product between  $\nabla$  and  $\vec{F}$ :  $\text{div } \vec{F} = \nabla \cdot \vec{F}$

Hence  $\nabla F$  means the gradient of F. So  $\text{curl } \vec{F}$  is the cross product between  $\nabla$  and  $\vec{F}$ :  $\text{curl } \vec{F} = \nabla \times \vec{F}$

Similar for two dimension curves, except for the curl, we add a zero to the third dimension making the original function a function plot on a plane in the three dimensional plot.  $\text{Curl } \vec{F} = \text{curl}(F_1, F_2, 0)$ , and the result will be  $(0, 0, \frac{\sigma F_2}{\sigma x} - \frac{\sigma F_1}{\sigma y})$ , and the scalar curl for  $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined as  $\frac{\sigma F_2}{\sigma x} - \frac{\sigma F_1}{\sigma y}$ , a scalar function.

Meanings of divergence and curl:

Divergence of  $\vec{F}$  at  $(x_0, y_0)$  is measuring the change of density at this point.  
Change of volume of flows

$$= F_1(x + \Delta x, y)\Delta y - F_1(x, y)\Delta x \quad (272)$$

$$+ \frac{F_2(x, y + \Delta y)\Delta x - F_2(x, y)\Delta y}{\Delta x \Delta y} \quad (273)$$

$$\approx \frac{\sigma F_1}{\sigma x} + \frac{\sigma F_2}{\sigma y} \text{div } \vec{F} \quad (274)$$

Curl of  $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , scalar curl =  $\frac{\sigma F_2}{\sigma x} - \frac{\sigma F_1}{\sigma y}$ . is the measurement of the tendency of spin at a point. The Jacobian matrix  $D\vec{F}$  capture the instant change of  $\vec{F}$  In particular,  $D\vec{F}$  capture the infinitely minimal tendency of spin at the point.

A bit about Linear Algebra

If The Jacobian matrix  $D\vec{F}$  is symmetric, i.e.  $D\vec{F} = D\vec{F}^T$ , i.e.  $\frac{\sigma F_2}{\sigma x} = \frac{\sigma F_1}{\sigma y}$ .

Then  $D\vec{F}$  introduces no rotation. Curl measures how far away  $D\vec{F}$  from being symmetric, the more asymmetric, the more  $D\vec{F}$  rotates a basis.

Curl in third dimension:

1.  $\text{Curl } \vec{F}$  = the axis of rotation of  $D\vec{F}$ .
2.  $|\text{Curl } \vec{F}|$  = rate of rotation of  $D\vec{F}$  the instant rate of rotation of  $\vec{F}$  at the point.

Relationship between  $\nabla$ ,  $div$ ,  $curl$

$$\mathbb{R}^1 - (\nabla) \rightarrow \mathbb{R}^3 - (curl) \rightarrow \mathbb{R}^3 - (div) \rightarrow \mathbb{R}^1 \quad (275)$$

$\mathbb{R}^1 - (\nabla) \rightarrow \mathbb{R}^3$  are both tangent of functions.

$$curl(\nabla f) = \vec{0} \quad (276)$$

$$div(curl \vec{F}) = 0 \quad (277)$$

## 2.15 Integration along a path

Definition: A path (parametric curve)  $\vec{c}: [a, b] \rightarrow \mathbb{R}^m$  is  $C'$  if  $\vec{C}(t)$  exists and is continuous

Definition: A path  $\vec{c}: [a, b] \rightarrow \mathbb{R}^m$  is piece-wise  $C'$  if can divide  $[a, b]$  into finitely many sub-integrals.  $[a_1 = a, a_2], [a_2, a_3] \dots, [a_{n-1}, a_n = b]$  such that  $\vec{C}$  on  $[a_i, a_{i+1}]$  is  $C'$ .

Convention: From now on, we assume a path  $\vec{C}(t)$  is always piece-wise  $C'$ .

Definition: If  $\vec{C}(t)$  is a piece-wise  $C'$  path, we said that

1.  $\vec{C}(t)$  is simple if it does not intersect itself or equivalently,  $\vec{c}: [a, b] \rightarrow \mathbb{R}^m$  is an 1 to 1.
2.  $\vec{C}(t)$  is simple closed path/curve if there exists  $\vec{C}(a) = \vec{C}(b)$  and does not intersect itself except for the 2 end points.

Definition: for  $\vec{C}: [a, b] \rightarrow \mathbb{R}^m$ ,  $f(x_1 \dots x_m): \mathbb{R}^m \rightarrow \mathbb{R}^1$  then integral of  $f$  along  $\vec{C}$  is denoted by:

$$\int_C f ds = \int_a^b f(\vec{C}(t)) \cdot |\vec{C}'(t)| dt \quad (278)$$

Example: Finding the integral of  $f(x, y, z) = xyz$  along the path  $\vec{C}(t) = (-\sin(t), \sqrt{2}\cos(t), \sin(t))$ ,  $t \in [0, \pi/2]$

$$\int_c f ds = \int_0^{\pi/2} f(\vec{C}(t)) \cdot |\vec{C}'(t)| dt \quad (279)$$

$$= \dots \quad (280)$$

$$= -2 \int_0^{\pi/2} \sin^2(t) \cos(t) dt \quad (281)$$

$$= -\frac{2}{3} \quad (282)$$

Theorem: Path integral of a function does not depend on the parametrization of the path.

Namely,  $\vec{C}_1(t)$  and  $\vec{C}_2(t)$  are 2 different parameter of the same path,  $\int_{C_1} f ds = \int_{C_2} f ds$

The integration of vector field along a path is similar:

Definition: For a path  $\vec{C}: [a, b] \rightarrow \mathbb{R}^m$  and a vector function  $\vec{F}: \mathbb{R}^m \rightarrow \mathbb{R}^m$  the integral of  $\vec{F}$  along  $\vec{C}$  is

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) dt \quad (283)$$

This can be used in occasion like calculating the work done by a force field.

Example:  $\vec{F}(x, y, z) = (-y, x, 1)$  force field, and there is a particle moving along a three dimension path  $\vec{C}(t)$  and similarly, when the path is perpendicular, work done(hence the integral) would also be zero.

1) for  $\vec{C}(t) = (t, t, t)$ ,  $t \in [0, 1]$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_0^1 (-t, t, 1) \cdot (1, 1, 1) dt \quad (284)$$

$$= 1 \text{ (J)} \quad (285)$$

2) for  $\vec{C}(t) = (\cos(t), \sin(t), t)$ ,  $t \in [0, 2\pi]$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_0^{2\pi} \pi(-\sin(t), \cos(t), 1) \cdot (-\sin(t), \cos(t), 1) dt \quad (286)$$

$$= 4\pi \text{ (J)} \quad (287)$$

Properties of  $\int_{\vec{C}} \vec{F} d\vec{s}$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) dt \quad (288)$$

$$= \int_a^b |\vec{F}(\vec{C}(t))| \cdot |\vec{C}'(t)| \cos(\theta t) dt \quad (289)$$

The angle  $\theta(t)$  is the angle formed by  $\vec{F}(\vec{C}(t))$ , and  $\vec{C}'(t)$  is always  $< 90$  degree. If  $\vec{F}(\vec{C}(t)) \perp \vec{C}'(t)$

$$\int_{\vec{C}} \vec{F} d\vec{s} = 0 \quad (290)$$

If  $\vec{C}_1(t)$  and  $\vec{C}_2(t)$  are different parametrization of the same path with the same orientation, then

$$\int_{\vec{C}_1} \vec{F} d\vec{s} = \int_{\vec{C}_2} \vec{F} d\vec{s} \quad (291)$$

If opposite orientation

$$\int_{\vec{C}_1} \vec{F} d\vec{s} = - \int_{\vec{C}_2} \vec{F} d\vec{s} \quad (292)$$

Definition of Circulation:  $\vec{C} : [a, b] \rightarrow \mathbb{R}^m$  a simple closed path, for a vector field  $F$ , the integration of this vector field along this simple closed path is called the circulation of  $\vec{F}$  along  $C$ .

For example, along a magnetic field, and such that  $C$  is a closed curve, the Ampere's law states that:  $\int_C \vec{B} d\vec{s} = \mu I$  where  $I$  is the current.

Example: Take for example, the integration of the vector field  $F_1(\vec{x}, y) = (-y, x)$  a simple closed path  $\vec{C}(t) = (\cos(t), \sin(t))$ ,  $t \in [0, 2\pi]$ , the circulation would be

$$\vec{F}_1 = \int_{\vec{C}} \vec{F}_1 d\vec{s} \quad (293)$$

$$= \int_0^{2\pi} \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt = 2\pi \quad (294)$$

## 2.16 Gradient Vector Field

Recall the two different kinds of integrals along a path:

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) dt, \quad \int_C f ds = \int_a^b f(\vec{C}(t)) |\vec{C}'(t)| dt$$

Definition:  $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is called a gradient (conservative) vector field if there exists a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  such that  $\vec{F} = \nabla f$  There are vector fields that are not gradient, recall that  $Curl(\nabla f) = 0$ :

$$Curl(\nabla f) = Curl\left(\frac{\sigma f}{\sigma x}, \frac{\sigma f}{\sigma y}\right) \quad (295)$$

$$= \frac{\sigma(\sigma f / \sigma y)}{\sigma x} - \frac{\sigma(\sigma f / \sigma x)}{\sigma y} \quad (296)$$

$$= 0 \quad (297)$$

Hence the curl test: if  $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $\text{curl} \vec{F} \neq 0$ , then  $\vec{F}$  cannot be a gradient curve.

Theorem:: Suppose  $\vec{F} = \nabla f$ ,  $\vec{C} : [a, b] \rightarrow \mathbb{R}^m$  a path, then

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_{\vec{C}} \nabla f d\vec{s} = f(\vec{C}(b)) - f(\vec{C}(a)) \quad (298)$$

Example: find the integration of a vector field  $F = (2x, 2e^z, 2ye^z)$  along a curve  $abc$   $\vec{C}(a) = [0, 0, 0]$ ,  $\vec{C}(b) = [0, -1, 1]$ ,  $\vec{C}(c) = [1, -1, 0]$

$$\text{Observed that: } \vec{F} = \nabla(x^2 + 2ye^z) = \nabla f \quad (299)$$

Hence from theorem:

$$\int_{\vec{C}} \vec{F} d\vec{s} = f(1, -1, 0) - f(0, 0, 0) = -1 \quad (300)$$

Observations, for two different closed path with the same end points:

$$\int_{\vec{C}_1} \nabla f ds = \int_{\vec{C}_2} \nabla f ds \quad (301)$$

equivalently, for a simple closed curve:

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_{\vec{C}} \nabla f d\vec{s} = f(\vec{C}(b)) - f(\vec{C}(a)) = 0 \quad (302)$$

Theorem: Hence given a vector field  $\vec{F}$ , the following statements about it are equivalent:

1.  $\vec{F} = \nabla f$
2.  $\int_{\vec{C}_1} \nabla f ds = 0$
3. For any two paths  $c_1$   $c_2$  with the same end points,  $\int_{\vec{C}_1} \vec{F} ds = \int_{\vec{C}_2} \vec{F} ds$

The curl of  $\nabla f = 0$  is a necessary condition for gradient vector field, but is this true such that when  $\text{curl} \vec{F} = 0$ ,  $\vec{F} = \nabla f$ ? no

Example:  $\vec{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z)$  (its  $\text{Curl} = 0$ ) defined on  $u = \mathbb{R}^3$  (is not simple connected), but if we take a simple closed curve  $\vec{C}(t) = (\cos(t), \sin(t), 0)$   $t \in [0, 2\pi]$ , and  $\int_{\vec{C}} \vec{F} d\vec{s} \neq 0$ , hence not a gradient vector field by theorem.

Definition: Simply connected domains,  $n$  belongs to  $\mathbb{R}^n$  is said to be simply connected if any closed path can be deformed continuously to a point,  $u$  is simply connected. For example, a disk is simply connected and a ring is not as it cannot be shrunk to a simple point.

Theorem: if  $\vec{F} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  vector field defined on a simple connected domain and  $\text{curl} \vec{F} = 0$ , then  $\vec{F}$  is a gradient vector field.

Proof: supposed that  $u = \text{Disk}$  and  $\vec{F}$  defined on  $u$ ,  $\text{curl} \vec{F} = 0$ , then we claim there exists a function  $f(x,y)$  defined on  $u$  such that  $\nabla f = \vec{F}$

$$\vec{F} = (F_1(x, y), F_2(x, y)), (a, b) \in u \quad (303)$$

$$f(a, b) = \int_{\vec{C}_{a,b}} \vec{F} d\vec{s}, \text{ where } C_{a,b} \text{ is a line } (0, 0) - - - (a, b) \quad (304)$$

$$= \int_0^1 (F_1(ta, tb), F_2(ta, tb)) \cdot (a, b) dt \left\{ C_{a,b}(t) = (ta, tb) t \in [0, 1] \right. \quad (305)$$

$$= f(a, b) \quad (306)$$

$$(307)$$



Claim:  $\nabla F = \vec{F}$ , check for  $\frac{\sigma f}{\sigma a} F_1$  :

$$\frac{\sigma f(a, b)}{\sigma a} = \frac{\sigma}{\sigma a} \int_0^1 (F_1(ta, tb), F_2(ta, tb)) \cdot (a, b) dt \quad (308)$$

$$= \int_0^1 (F_1(ta, tb) + a \frac{\sigma F_1}{\sigma x} \cdot \frac{\sigma x}{\sigma a} + b \frac{\sigma F_2}{\sigma x} \cdot \frac{\sigma x}{\sigma a}) dt, \quad x = ta, \quad \frac{\sigma x}{\sigma a} = t \quad (309)$$

$$Curl \vec{F} = \frac{\sigma F_2}{\sigma x} - \frac{\sigma F_1}{\sigma y} = 0 \quad (310)$$

$$= \int_0^1 (F_1(ta, tb) + at \frac{\sigma F_1}{\sigma x} + bt \frac{\sigma F_1}{\sigma x}) dt \quad (311)$$

$$x = at, \quad \frac{\sigma x}{\sigma t} = a, \quad y = bt, \quad \frac{\sigma y}{\sigma t} = b \quad (312)$$

$$= \int_0^1 (F_1(ta, tb) + t \frac{\sigma F_1}{\sigma x} \cdot \frac{\sigma x}{\sigma t} + t \frac{\sigma F_1}{\sigma y} \cdot \frac{\sigma y}{\sigma t}) dt \quad (313)$$

$$= \int_0^1 (F_1(a, b) + t \frac{df}{dt}) dt \quad (314)$$

$$= \int_0^1 \frac{d}{dt} (t F_1(ta, tb)) dt \quad (315)$$

$$= (t F_1(ta, tb)) \quad (316)$$

$$\frac{\sigma f}{\sigma a} = F_1(a, b) \quad (317)$$

$$(318)$$

Rest can be proved similarly.

Next for arbitrary domain u, assume that  $curl \vec{F} = 0$  and we want to show that the integral for any two path with the same end points  $\int_{\vec{C}_1} \vec{F} d\vec{s} = \int_{\vec{C}_2} \vec{F} d\vec{s}$

Idea: deform  $C_1$  to  $C_2$  such that any 2 adjacent curves can be covered by a series of small disks. Because for each disk, we shown  $\int F$  independent of paths.

Then

$$\int_{c_1} \vec{F} ds = \sum_i \int_{P_i} \vec{F} ds = \sum_i \int_{Q_i} \vec{F} ds = \int_{C_3} \vec{F} ds \quad (319)$$

Similarly:

$$\int_{c_1} \vec{F} ds = \int_{C_3} \vec{F} ds = \int_{C_n} \vec{F} ds = \int_{C_2} \vec{F} ds \quad (320)$$

so  $\int \vec{F}$  is indenependent of choice of paths, by theorem 1,  $\vec{F}$  is a gradient vector field.