Math, Multivariable Calculus

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2.1 Vector Valued functions in multiple variables

Calculus of single variable is a study of $y = f(x)$

$$y = f(x), \text{ where } (x \in \mathbb{R}) \to (\text{f(x)} \in \mathbb{R}) \tag{1}$$
$$\tag{2}$$

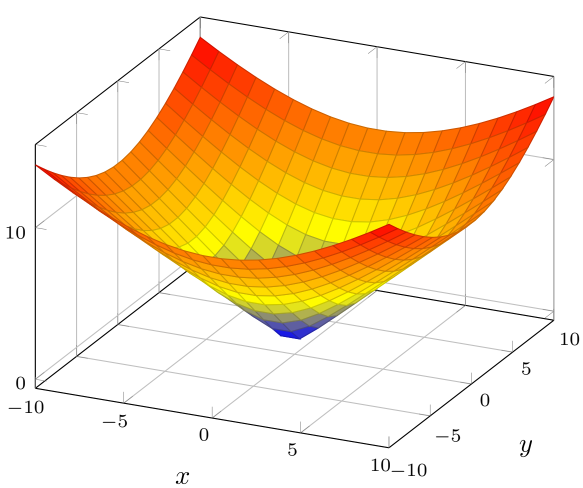
However, when we are dealing with multi-variable calculus, we have: an n variable vector valued function (if $m > 1$)

$$F = \begin{cases} (x_1, \ldots x_n) & \to F(x_1, \ldots x_n) \\ \quad\in & \qquad\quad \in \\ \quad\mathbb{R}^{\text{n}} & \longrightarrow \qquad \mathbb{R}^{\text{m}} \end{cases} \tag{3}$$

where $F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n),\ F_2(x_1, \ldots, x_n) \ldots F_m(x_1, \ldots, x_n))$
The inputs and ouputs are Vectors, for example

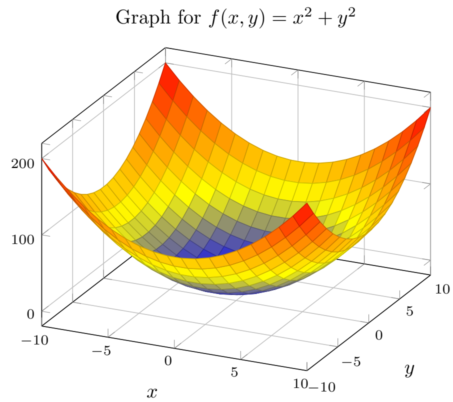
$$d(x, y) = \sqrt{x^2 + y^2}, (2 variables) \tag{4}$$
$$(x, y)\mathbb{R}^2 \to (\sqrt{x^2 + y^2})\mathbb{R} \tag{5}$$
$$distance = \sqrt{x^2 + y^2} \tag{6}$$

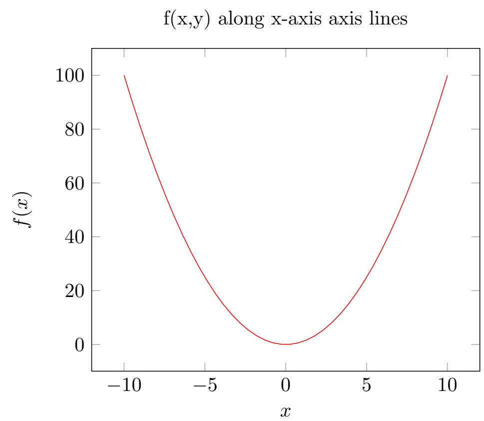
Graph for $f(x, y) = \sqrt{(}x^2 + y^2)$



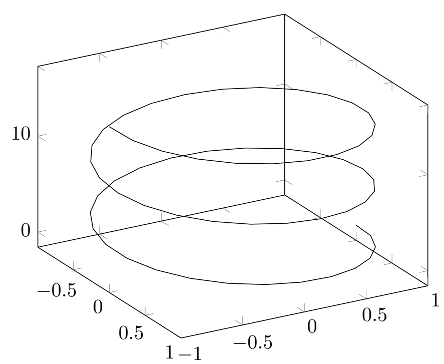
The graph should look like a cone

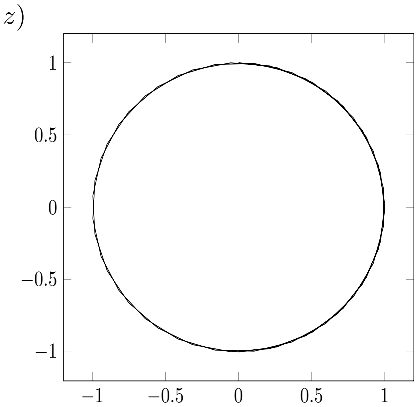
Similarly, we can have a function that takes all the x, y and z can calculate that distance. That would however be a four dimensional plot, which we cannot plot.
For a similar function, $z = x^2 + y^2$





For another function $f(z) = (\cos z, \sin z)$



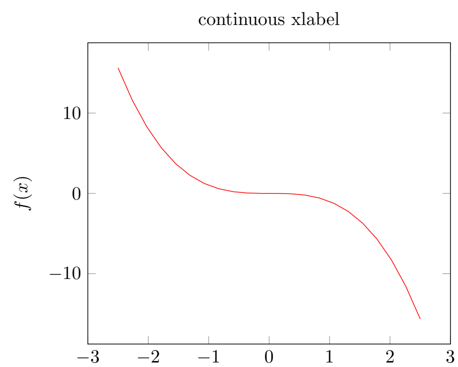


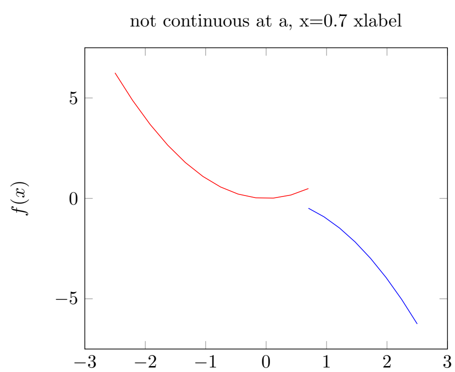
2.2 Limits and continuity

Definition:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{7}$$

if $f(x, y)$ goes to L as (x, y) approaches (a, b)





$\lim_{(x\to a^+)} f(x) = \lim_{(x\to a^-)} f(x)$, $\lim_{(x\to a^+)} g(x)$ DNE

Similarly, for $(x,y) \to (a,b)$ we need to look at all possible paths. If you can find 2 oaths to approach (a,b) such limits along these 2 paths are different, then the function is not continuous. For example:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \text{DNE(do not exists) since:} \tag{8}$$

$$(x,0) \to (0,0),\ \lim = \lim_{x\to 0} = \frac{0}{x^2} = 0 \tag{9}$$

$$(x,x) \to (0,0),\ \lim = \lim_{y\to 0} = \frac{2x*x}{x^2+x^2} = 1 \tag{10}$$

$$1 \neq 0 \text{ hence D.N.E.}$$

Remember to take a look at all directions
Definition: $f(x,y)$ continuous at (a,b) if:

1. $f(x,y)$ defined at (a,b)
2. $\lim_{(x,y)\to(a,b)} = f(a,b)$

How to tell if a function is continuous? $f(x,y,z) = e^{x-z}\sin(x+yz^2)$

1. (a) $f(x,y,z) = x$
   (b) $f(x,y,z) = y$
   (c) $f(x,y,z) = z all continuous$
2. $f(x,y,z) = x - z$
3. $f(x,y,z) = e^{x-z}(composition of contin. func is contin.)$
4. $g(x,y,z) = \sin(x+yz^2)$ continuous similarly

Test: $\lim_{(x,y,z)\to(0,1,-1)} = e^1 \sin(1)$, hence continuous. A special case is when you have $\frac{f(x,y)}{g(x,y)}$, it will be continuous at (a,b) as long as $g(a,b) \neq 0$

For a vector function $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v} \ldots F_m(\vec{v})$, where $\vec{v}$ is a vector or $\mathrm{I\!R^n}, (\mathrm{x_1}, \mathrm{x_2} \ldots \mathrm{x_n})$, is a continuous function at vector $\vec{n}$ if all its component functions are continuous at $\vec{n}$
Example:

$$f(x, y, z) = (e^{x+y},\ z,\ \cos(x^2 - z)) \text{ continuous everywhere} \tag{11}$$

$$\text{since} \begin{cases} e^{x+y} \text{ continuous} \\ z \text{ continuous} \\ \cos(x^2 - z) \text{ continuous} \end{cases} \tag{12}$$

2.3 Partial Derivatives

The derivative of a function is the rate of change along the x axis. The partial derivatives of $F(x, y)$ would be the rate of change of F as one of the axis is fixed (along one axis).

$$\frac{dF(x, y)}{dx} = F_x(x, y) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} \tag{13}$$

$$\frac{dF(x, y)}{dy} = F_y(x, y) = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} \tag{14}$$

In order to compute the derivative, we treat all other variables as constants.
Example:

$$f(x, y, z) = xe^{xy} - \sin(y^2 + z^2) \tag{15}$$

$$\frac{df}{dx} = e^{xy} + yxe^{xy} - 0 = e^{xy} + yxe^{xy} \tag{16}$$

$$\frac{df}{dy} = x^2 e^{xy} - \cos(y^2 + z^2)(2y)\frac{df}{dz} = 0 - \cos(y^2 + z^2)(2z) \tag{17}$$

Take a point (a, b) for example

1. $\frac{df}{dx}$ = gradient of tangent at that point along x axis

2. $\frac{df}{dy}$ = gradient of tangent at that point along y axis

3. These two direction can span the tangent plane of f at that point
   (a) Tangent Vector
   (b) along x direction $= (1, 0, \frac{df}{dx}(a, b)) = \vec{v}$
   (c) along y direction $= (0, 1, \frac{df}{dy}(a, b)) = \vec{w}$
   (d) Using these vectors we can find the plane's equation

Example, find the tangent plane of $f(x, y)$ at (1,1):

$$f(x,y) = 1 - x^2 - 2y^2 \tag{18}$$
$$f(1,1) = z = -2, \text{ point at } (1,1,-2) \tag{19}$$
$$\frac{df}{dx} = (1, 0, -2x|_{(1,1)}) = (1, 0, -2) \tag{20}$$
$$\frac{df}{dy} = (1, 0, -4y|_{(1,1)}) = (0, 1, -4) \tag{21}$$
$$(1, 0, -2) \times (0, 1, -4) = (2, 4, 1) \tag{22}$$
$$2x + 4y + z = 4 \text{ a linear approximation of } f(x, y) \text{ at } (1, 1) \tag{23}$$

Definition of a vector function:

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathrm{I\!R}^{\mathrm{m}} \tag{24}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{25}$$

The function F transforms$\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$, DF $\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ is a linear map, a linear approximation of F.
Example:

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin{(y + z)}, 4y) \tag{26}$$
$$DF(x, y, z) = \begin{pmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y+z) & \cos(y+z) \\ 0 & 4 & 0 \end{pmatrix} DF(1,1,2) = \tag{27}$$
$$DF(1,1,2) = \begin{pmatrix} e^3 & 2e^3 & e^3 \\ 2 & 0 & 0 \\ 0 & \cos(3) & \cos(3) \\ 0 & 4 & 0 \end{pmatrix} \tag{28}$$

Matrix of numbers, a linear map

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin{(y + z)}, 4y) \tag{29}$$

2.4 Properties of Derivatives DF

Given two vector function F and G: $\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ both differentiable at some point $\vec{x}$

$$D(f \pm G) = D(F) \pm D(G) \tag{30}$$
$$C \in \mathrm{I\!R}, \mathrm{D(cF) = cD(F)} \tag{31}$$

suppose $f, g : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ a function in n variables, product rule still holds

$$D(f \cdot g) = D(F) \cdot g + f \cdot D(G) \tag{32}$$

where $D(f \cdot g)$ is a n^{th} dimension vector, D(f or g) is a vector function and f or g is a scalar function

$$D(\frac{f}{g}) = \frac{g \cdot D(F) - f \cdot D(G)}{g^2}, g(\vec{x}) \neq 0 \tag{33}$$

Given two vector function $\vec{v}(t)$ and $\vec{w}(t)$: $\mathbb{R} \to \mathbb{R}^{\mathrm{n}}$ so $t \to \vec{v}(t)$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t)\text{scalar function} \tag{34}$$
$$f'(t) = \vec{v}(t)' \cdot \vec{w} + \vec{w}(t)' \cdot \vec{v} \tag{35}$$
$$u(t) = \vec{v}(t) \times \vec{w}(t)\text{vector function} \tag{36}$$
$$u'(t) = \vec{v}(t)' \times \vec{w}(t) + \vec{w}(t)' \times \vec{v}(t) \tag{37}$$
$$\vec{v}(t) = \langle v_1(t), v_2(t), \ldots v_n(t) \rangle \tag{38}$$

Chain rule, suppose $f : \mathbb{R}^{\mathrm{m}} \to \mathbb{R}^{\mathrm{n}}$, $g : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{p}}$, $g \cdot f : \mathbb{R}^{\mathrm{m}} \to \mathbb{R}^{\mathrm{p}}$, $\vec{x} \in \mathbb{R}^{\mathrm{m}}$

$$D(g \circ f)(\vec{x})(\text{size } p \times m) = DG(F(\vec{x}))(\text{size } p \times n) \cdot DF\vec{x}(\text{size } n \times m) \tag{39}$$
$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{40}$$

Example:

$$f(x, y) = (x^3 + y, e^{xy}, 2 + xy)\mathbb{R}^2 \to \mathbb{R}^3 \tag{41}$$
$$g(y, v, w) = (y^2 + v, uv + w^3)\mathbb{R}^3 \to \mathbb{R}^2 \tag{42}$$
$$g(f(x, y)) = g \circ f = ((x^3 + y)^2 + e^{xy},\ (x^3 + y)e^{xy} + (2 + xy)^3) \tag{43}$$

$$D(g \circ f) = \begin{pmatrix} 2(x^3 + y)(3x^2) + ye^{xy} & 2(x^3 + y) + xe^{xy} \\ \begin{pmatrix} 3x^2e^xy + (x^3 + y)ye^{xy} \\ +3(2 + xy)^2y \end{pmatrix} & \begin{pmatrix} e^{xy} + (x^3 + y)xe^{xy} \\ +3(2 + xy)^2x \end{pmatrix} \end{pmatrix} \tag{44}$$

$$DG = \begin{pmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{pmatrix} \tag{45}$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{46}$$

$$DG = \begin{pmatrix} 2(x^3 + y) & 1 & 0 \\ e^{xy} & x^3 + y & 3(2 + xy)^2 \end{pmatrix} \times \begin{pmatrix} 3x & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{pmatrix} \tag{47}$$

Check that the equality holds.

2.5 Directional derivatives

Recall the Jacobian matrix, in the following context, the function F is not necessarily defined on the entire $\mathbb{R}^n$, but rather a subset $\subseteq \mathbb{R}^n$

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathbb{R}^m \tag{48}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{49}$$

Examples: 1)

$$f(x, y, z) = x^2 y + \ln z : \mathbb{R}^3 \to \mathbb{R}^1 \tag{50}$$

$$Df = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz})$$
$$= (2x, 1, \frac{1}{z}) \tag{51}$$

2)

$$\vec{v}(t) = (\cos(t), \sin(t));\ \text{vector function} \tag{52}$$

$$D\vec{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = (-\sin(t), \cos(t))^T \tag{53}$$

$$D\vec{v}(t) \cdot \vec{v}(t) = -\sin(t)\cos(t) + \cos(t)\sin(t) = 0 \tag{54}$$

$$D\vec{v}(t) \perp \vec{v}(t);\ \text{hence a tangent vector} \tag{55}$$

3)

$$g(t) = e^t \text{ scalar function} \tag{56}$$

$$g(t) \cdot \vec{v}(t) = e^t(\cos(t), \sin(t)) = (e^t \cos(t), e^t \sin(t)) \tag{57}$$

$$D(g(t) \cdot \vec{v}(t)) = \begin{bmatrix} e^t \cos(t) - e^t \sin(t) \\ e^t \cos(t) + e^t \sin(t) \end{bmatrix} \tag{58}$$

4)

$$\vec{v}(t) = (t, \sin(t), \cos(t)), \vec{w}(t) = (3t, 0, 2) \tag{59}$$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t)$$
$$= 3t^2 + 0 + 2\cos(t);\ \text{scalar function} \tag{60}$$

$$D(f(t)) = 6t - 2\sin(t) \tag{61}$$

$$\vec{w} \cdot D\vec{v} + \vec{v} \cdot D\vec{w} = \begin{bmatrix} 3t \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ +\cos(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} t \\ \sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \tag{62}$$

$$= 6t - 2\sin(t) \tag{63}$$

2.6 Partial derivative along a direction

Definition: Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), given a unit vector $\vec{u} = (u1, u2) \in \mathbb{R}^2$ then the directional derivatives of F along $\vec{u}$ is:

$$D_{\vec{u}}F(a,b) = \left[\frac{d}{dt} f((a,b) + t\vec{u})\right]_{t=0} \tag{64}$$

$$= \lim_{h\to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h} \tag{65}$$

is the rate of change of F along the direction of $\vec{u}$
Two special cases would be when u are along the typical x or y direction, which would mean the class derivative along x-axis $\frac{df}{dx}$ or the y-axis $\frac{df}{dy}$
Theorum $\vec{u} = (u_1, u_2)$ unit vector,

$$D_{\vec{u}}f(x,y) = (\frac{df}{dx}, \frac{df}{dy})\dot{\vec{u}} = (u_1, u_2) \tag{66}$$

$$\text{since } \vec{u} = u_1(1,0) + u_2(0,1) \tag{67}$$

$$D_{\vec{u}}f = u_1\frac{df}{dx} + \frac{df}{dy} \tag{68}$$

Proof:

$$D_{\vec{u}}f = \lim_{h\to 0} \frac{F(a + hu_1, b + hu_2) - F(a,b)}{h} \tag{69}$$

$$= \lim_{h\to 0} \frac{(F(a + hu_1, b + hu_2) - F(a, b + hu_2))}{h} \tag{70}$$

$$+ \lim_{h\to 0} \frac{F(a, b + hu_2) - F(a,b)}{h} \tag{71}$$

$$= \lim_{u_1 h\to 0} \frac{[F(a + hu_1, b + hu_2) - F(a, b + hu_2)] \cdot u_1}{hu_1} \tag{72}$$

$$+ \lim_{h\to 0} \frac{F(a, b + hu_2) - F(a,b)}{h} \tag{73}$$

$$= \frac{df}{dx}(a, b+0) \cdot u_1 + \frac{df}{dy} \cdot u_s \tag{74}$$

Example: Find the directional Deriv of $f(x,y) = x^2 + 3xy$ along the direction of $(3,4)$ at the point $p=(2,-1)$

$$\begin{aligned} \vec{u} =& \frac{(3,4)}{\sqrt{(3^2+4^2)}} = (\frac{3}{5},\frac{4}{5}) && (75) \\ D_{\vec{u}}f(2,-1) =& (2x+3y,3x)|_{(2,-1)} \cdot (\frac{3}{5},\frac{4}{5}) && (76) \\ =& (1,6)\cdot(\frac{3}{5},\frac{4}{5}) && (77) \\ =& \frac{27}{5} && (78) \end{aligned}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = (\frac{df}{dx}, df dy)\cdot \vec{u} \quad (79)$$

Definition: Given $F:\mathbb{R}^{\mathrm{n}} \to \mathbb{R}$, then define the gradient.

$$\nabla f = D_{\vec{u}}f = (\frac{df}{x_1},\frac{df}{x_2},\ldots,\frac{df}{x_n}) \quad (80)$$

Hence the directional derivative of a function along a direction is:

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} \quad (81)$$

The consequence of this:
1)

$$D_{\vec{u}}f \text{ max when } \vec{u} /\!/ \nabla f = (\frac{df}{dx},\frac{df}{dy}) \quad (82)$$

since

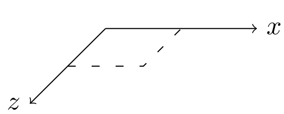
$$\begin{aligned} \nabla f \cdot \vec{u} =& |\nabla f||\vec{u}|\cos(\theta) && (83) \\ =& |\nabla f|\cdot 1 \cdot \cos(\theta)\theta \ = \ ang(\nabla f, \vec{u}) && (84) \end{aligned}$$

When maximized, $/\!/$, $\theta = 0$

$$\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{\frac{df}{dx}^2 + \frac{df}{dy}^2} \quad (85)$$

2)

$$\text{When } \vec{u} \perp \nabla f, D_{\vec{u}}f = 0 \quad (86)$$



Example: $f(x,y) = e^{-(x^2+y^2)}$, find the direction along this function which has the largest rate of increase/decrease at point $(1,1)$

Since at maximum, $\vec{u} /\!/ \nabla f$

$$\nabla f =(-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)})|_{(1,1)} \tag{87}$$
$$=(-2e^{-2}, -2e^{-2}) \tag{88}$$
$$\because \quad \vec{u} /\!/ \nabla f /\!/ (1,1) \tag{89}$$
$$\therefore \quad \vec{u} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \tag{90}$$

Largest rate of increase at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
Largest rate of decrease at $-(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
The gradient ∇f tells us the direction along which F changes the most.

2.7 Summary

Given a function $f(x, y)$	**n-th dim analogue**
Define Partial derivatives(deriv) $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f(x_1, x_2, \ldots x_n)$
Define gradient $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f : \mathbb{R}^n \to \mathbb{R}$
Directional deriv: given **unit** vector $\vec{u} = (u_1, u_2)$	$\frac{df}{dx_1} \cdots \frac{df}{dx_n}$
$D_{\vec{u}f = \lim_{h\to 0} \frac{f((a,b)+h\vec{u})-f(a,b}{h}}$	$\nabla f = (\uparrow)$
if$(\vec{u} = (1, 0))$, $D_{\vec{u}} f = \frac{df}{dx}$	$\vec{u} f \in \mathbb{R}^n$
if$(\vec{u} = (0, 1))$, $D_{\vec{u}} f = \frac{df}{dy}$	$D_{\vec{u}} f = ?$
Theorem:$D_{\vec{u}} = \nabla f \cdot \vec{u} = u_1 \frac{df}{dx} + u_2 \frac{df}{dy}$	generalizations?
Remark: We can define $D_{\vec{u}}$ for any $\vec{v} \in \mathbb{R}^2$	
in the exact same way	
$D_{\vec{u}} = \lvert\vec{v}\rvert D_{\frac{\vec{u}}{\lvert\vec{v}\rvert}} f$	

To each unit vector $\vec{u}$, we have a tangent vector associated to it:

$$(u1, u2, D_{\vec{u}} f) =u_1(1, 0, \frac{df}{dx}) + u_2(0, 1, \frac{df}{dy}) \tag{91}$$

Two special cases: $(1, 0, \frac{df}{dx})(0, 1, \frac{df}{dy})$
The two tangent vector span a tangent plane at $(a, b, f(a, b))$

2.8 Gradients and level sets

Recall

$$\nabla f =(\frac{df}{dx}, \frac{df}{dy}) \tag{92}$$
$$\because D_{\vec{u}} =\nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos(\theta) \tag{93}$$

$D_{\vec{u}}$ is the rate of change along $\vec{u}$

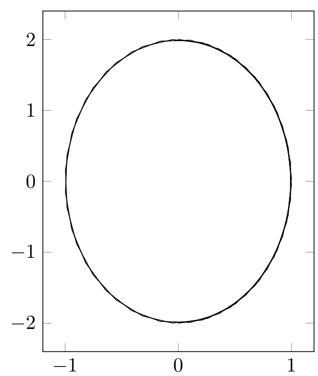
if $u /\!/ \nabla f$, $(\theta = 0)$ then $D_{\vec{u}}F$ max

$$|D_{\vec{u}}F| = \sqrt{(\frac{df}{dx})^2 + (\frac{df}{dy})^2} \tag{94}$$

In other words, along the direction of ∇F, the function F changes the most.
Definition: A level set of F(x,y) is:

$$(x, y) \in \mathbb{R}^2 : \mathrm{f(x, y) = c} \tag{95}$$

Example:



$$f(x, y) = x^2 + \frac{y^2}{4} \tag{96}$$

$$\{f(x, y) = 1\} \text{ (Level set)} \tag{97}$$

$$= \{(x, y) | x^2 + \frac{y^2}{4} = 1\} \tag{98}$$

Theorem: Given $f(x, y)$ differentiable, then the gradient $\nabla f \perp$ the level set.
We can prove it by using the chain rule: let $\gamma(t)$ be a parametric function of my level set $\{f(x, y) = 1\}$, $f(\gamma(t)) = 1$ and differentiate both sides.

$$0 = D(f(\gamma(t))) = Df(\gamma(t))D(\gamma(t)) \tag{99}$$

$$= \nabla F(\gamma(t))_{1\times 2} \cdot (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{100}$$

$$\text{Hence } \Rightarrow \nabla F(\gamma(t))_{1\times 2} \perp (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{101}$$

Example: Look at a surface $\sin(xy) - 2\cos(yz) = 0$ Find the tangent plane of this surface at $(\frac{\pi}{2}, 1, \frac{\pi}{3})$. The idea is that consider the surface as a level set of $F(x, y, z) = \sin(xy) - 2\cos(yz), \{F = 0\}$
Using the Theorem: ∇F is hence perpendicular to this surface, giving us the normal direction of our tangent plane.

$$\nabla F = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}) = (y\cos(xy), x\cos(xy) + 2z\sin(yz), 2y\sin(tz)) \tag{102}$$

$$= (\cos(\frac{\pi}{2}), \frac{\pi}{2}\cos(\frac{\pi}{2}) + \frac{2\pi}{3}\sin(\frac{\pi}{3}), 2\sin(\frac{\pi}{3})) \tag{103}$$

$$= (0, \frac{\sqrt{3}\pi}{3}, \sqrt{3}) \tag{104}$$

$$0 = 0 \cdot (x - \frac{\pi}{2}) + \frac{\sqrt{3}\pi}{3}(y - 1) + \sqrt{3}(z - \frac{\pi}{3}) \tag{105}$$

Where the last equation above is the equation of the tangent plane.

2.9 Parametric Curves

...are vector functions in 1 variable.
Definition: A parametric curve in $\mathbb{R}^n$ is a vector valued function where $\vec{c}(t) : R^1 \to \mathbb{R}^n$
All the component functions of $\vec{c}(t)$ are functions in 1 variable.
Example:
1) Draws a Circle

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \tag{106}$$

2) Draws a eclipse at plane z = 3

$$\vec{c}(t) = (a\cos(t), b\sin(t), 3), t \in [0, 2\pi] \tag{107}$$

3) Eclipse spiraling up

$$\vec{c}(t) = (a\cos(t), b\sin(t), t), t \in [0, 2\pi] \tag{108}$$

Definition: Orientation of $\vec{c}(t)$, the direction corresponding to increasing t-value is called the positive orientation of $\vec{c}(t)$. The opposite direction is the negative orientation.
Example: Positive orientation = Clockwise

$$\vec{c}(t) = (\cos(t), -\sin(t), t), t \in [0, 2\pi] \tag{109}$$

Derivative of $\vec{c}(t)$
If $\vec{c}(t) = (x(t), y(t), z(t))$, $\vec{c'}(t) = (x'(t), y'(t), z'(t))$

1. $\vec{c}(t)$ is a vector function
2. $\vec{c'}(t)$ is a tangent to curve $\vec{c}(t)$

Terminology: $\vec{c'}(t)$ is the velocity of $\vec{c}(t)$, and $|\vec{c'}(t)|$ is the speed of $\vec{c}(t)$. The speed of the curve $\vec{c}(t) = (\cos(t), \sin(t))$ for example, is 1.

What is the length of curve $\vec{c}(t) = (\cos(t), \sin(t), t)$ between (0 to 2π)Given $\vec{c}(t)$ differentiable between range a and b, length of the curve would be

$$l = \int_a^b |\vec{c'}(t)| dt \tag{110}$$

$$|\vec{c'}(t)| = \sqrt{\sin(t)^2 + \cos(t)^2 + 1} = \sqrt{2} \tag{111}$$

$$l = \int_0^{2\pi} \sqrt{2} dt \tag{112}$$

Think of length as a function in t

$$l(t) = \int_a^t |\vec{c'}(x)| dx \tag{113}$$

If curve $\vec{c}(t)$ differentiable and $\vec{c}(t) \neq 0$,

1. $|\vec{c'}(t)| > 0$

2. $l(t)$ strictly increasing.

3. l is one to one function so has inverse.

$$
\begin{aligned}
l[a,b] \to& \mathbb{R}^1, \text{ l has an inverse.} && (114)\\
t \to& l(t), \text{ think of our parameter t} && (115)\\
& && (116)
\end{aligned}
$$

t can now be expressed as a function in terms of length, $\vec{c}(t) = \vec{c}(t(l)) = \vec{c'}(l)$

Proof:
In our last example of calculating length, we can find an inverse for $l(t) = \sqrt{2}t$ resulting $t = \frac{l}{\sqrt{2}}$, by substitution we can reparameterize $\vec{c}(t)$ into $\vec{c}(l)$

$$
\begin{aligned}
c(l) =& \vec{c}(t(l)) && (117)\\
\frac{d\vec{c}(l)}{dl} =& \frac{\vec{c}(t(l))}{dt} \cdot \frac{dt}{dl} && (118)\\
=& \vec{c'}(t) \cdot \frac{1}{|\vec{c'}(t)|} && (119)\\
|\frac{d\vec{c}(l)}{dl}| =& |\vec{c'}(t)| \cdot \frac{1}{|\vec{c'}(t)|} = 1 && (120)
\end{aligned}
$$

2.10 Acceleration

Acceleration is the second derivatives of length, hence the definition of second derivative can be expressed as below:

Definition: Given $\vec{c}(t)$, $\vec{a}(t) = \vec{c''}(t)$ the acceleration of $\vec{c}$.

Example:
1) Ellipse

$$
\begin{aligned}
\vec{c_1}(t) =& (\cos(t), 2\sin(t)), t \in [0, 2\pi] && (121)\\
\vec{c_1'}(t) =& (-\sin(t), 2\cos(t)) && (122)\\
\vec{c_1''}(t) =& (-\cos(t), -2\sin(t)) && (123)
\end{aligned}
$$

2) Different para of the same Ellipse

$$
\begin{aligned}
\vec{c_2}(t) =& (\cos(t^2), 2\sin(t^2)), t \in [0, \sqrt{2\pi}] && (124)\\
\vec{c_2'}(t) =& 2t(-\sin(t^2), 2\cos(t^2)) && (125)\\
\vec{c_2''}(t) =& (-2\cos(t^2) - 4t^2 cos(t^2), 4cos(t^2) - 8t^2\sin(t^2)) && (126)
\end{aligned}
$$

Facts:

1. We can have different parameterations of a curve
2. Velocity, acceleration depend on the para we choose in general
3. For different parameterization $\vec{c_1}$, $\vec{c_2}$ of the same curve, $\vec{c_1'}(t) /\!/ \vec{c_2'}(t)$
4. "Unit Velocity" $\frac{\vec{c_1'}(t)}{|\vec{c_1'}(t)|} = \pm \frac{\vec{c_2'}(t)}{|\vec{c_2'}(t)|}$ are equal, the $\pm$ depends on the orientations of the $\vec{c_1}$ and $\vec{c_2}$.

Definition:Given $\vec{c}(t)$, assume $|\vec{c'}(t)| \neq 0$, defined the unit tangent vector $\vec{T}(t) = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$. If $\vec{c_1}(t)$ and $\vec{c_2}(t)$ are two different parameter of a curve with the same orientation, their unit tangent vector equals.

2.11 Curvature

Idea: Curvature is a measure of how sharply a curve is bending at a point, it is the "rate of change" of directions along a curve. So a straight line has no curvature, and a slightly curved line has a small curvature.

Definition: Quantitatively, if $\vec{c}(t)$ is parametered by length, (namely, $|\vec{c}(t)| = 1$), then we define the curvature at $l = l_0$ to be $k(l_0) = |\vec{c''}(l_0)| = |\frac{d^2\vec{c}(l)}{dl^2}|_{l=l_0}$

Remark: IF $\vec{c}(t)$ not parametered by length, then $|\vec{c''}(t)|$ is not the curvature.

Explanation: a curve is like a highway and a parameterized function $\vec{c}(t)$ is like someone driving along this highway giving his position vector at a given time. Parameterized by length would be driving at a constant/normalized speed. A curvature is how sharp a turn is, measured by looking at how quickly the driver need to turn their steering wheel.

Generally, $|\vec{c'}(l)| = 1 \Rightarrow \vec{c'}(l)$ unit tangent, so $\vec{c''}(t)$ is curvature.

Example:
1)Find parameter by length

$$\vec{c}(t) = \vec{a} + t\vec{v} \tag{127}$$

$$\text{Constant vectors: } \vec{a} = (1,2,3), \vec{v} = (0,2,1)$$

$$\text{Length function } l(t) = \int_0^t |\vec{c'}(x)| dx \tag{128}$$

$$= \int_0^t |\vec{v}| dx, \quad \vec{v} = (0,2,1) \tag{129}$$

$$l = l(t) = \sqrt{5}t \tag{130}$$

$$t = t(l) = \frac{l}{\sqrt{5}} \tag{131}$$

Curvature would be zero since it is a line, its second derivative would be a zero.

2) Find the curvature for a circle

$$\vec{c}(t) = (r\cos(t), r\sin(t)), r > 0 \text{ circle } x^2 + y^2 = r^2 \tag{132}$$

$$\text{length function } \; l(t) = \int_0^t |\vec{c'}(x)| dx = rt \tag{133}$$

$$t(l) = \frac{l}{t} \tag{134}$$

$$\vec{c}(l) = (r\cos(\frac{l}{r}), r\sin(\frac{l}{r})) \tag{135}$$

$$\vec{c''}(l) = (-\frac{\cos(l/r)}{r}, -\frac{\sin(l/r)}{r}) \tag{136}$$

$$k(l) = |\vec{c''}(l)| \tag{137}$$

$$= \sqrt{(-\frac{\cos(l/r)}{r})^2 + (-\frac{\sin(l/r)}{r})^2} = \frac{1}{r} \tag{138}$$

Curvature for circle of radius $r = \frac{1}{r}$

Second Definition of Curvature:
If $\vec{c}(t)$ any parameteric curve, $\vec{c'}(t) \neq 0$, recall the unit tangent vector $\vec{T} = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$, then define curvature $k(t) = \frac{|\vec{T'}(t)|}{|\vec{c'}(t)|}$
Theorem: Definition 1 and 2 of curvature are the same.

$$\vec{c'}(t) = \frac{d\vec{c}(t(l))}{dt} \tag{139}$$

$$= \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} \tag{140}$$

$$= \frac{\vec{c'}(t)}{|\vec{c}(t)|} = \vec{T} \tag{141}$$

$$k(l(t)) = |\vec{c''}(t)| = |\frac{d^2\vec{c}}{dl^2}| = |\frac{d\vec{T}(t(l))}{dl}| \tag{142}$$

$$= \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} \tag{143}$$

$$= |T'(t)| \cdot \frac{}{|\vec{c'}(t)|} = k(t) \tag{144}$$