Math, Multivariable Calculus

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2.1 Vector Valued functions in multiple variables

Calculus of single variable is a study of $y = f(x)$

$$y = f(x), \text{ where } (x \in \mathbb{R}) \rightarrow (\text{f(x)} \in \mathbb{R}) \tag{1}$$
$$\tag{2}$$

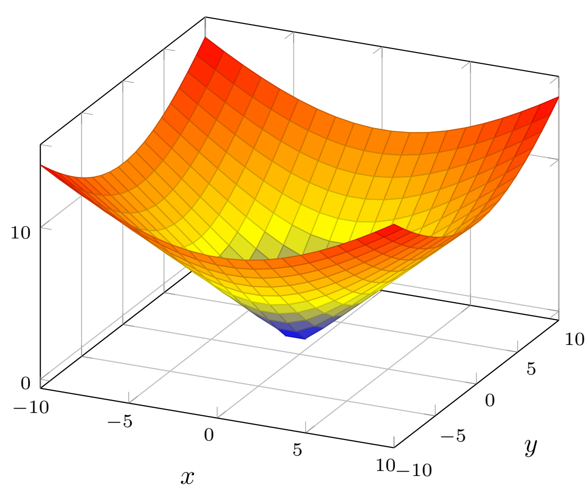
However, when we are dealing with multi-variable calculus, we have: an n variable vector valued function (if $m > 1$)

$$F = \begin{cases} (x_1, \ldots x_n) & \rightarrow F(x_1, \ldots x_n) \\ \in & \in \\ \mathbb{R}^{\text{n}} & \longrightarrow \quad \mathbb{R}^{\text{m}} \end{cases} \tag{3}$$

where $F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n),\ F_2(x_1, \ldots, x_n) \ldots F_m(x_1, \ldots, x_n))$
The inputs and ouputs are Vectors, for example

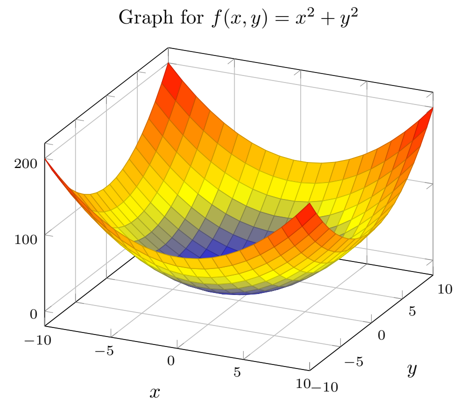
$$d(x, y) = \sqrt{x^2 + y^2}, (2variables) \tag{4}$$
$$(x, y)\mathbb{R}^2 \rightarrow (\sqrt{x^2 + y^2})\mathbb{R} \tag{5}$$
$$distance = \sqrt{x^2 + y^2} \tag{6}$$

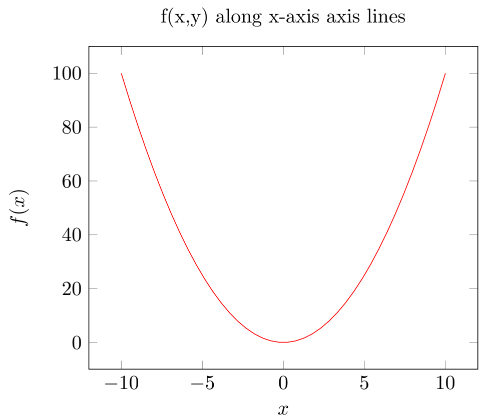
Graph for $f(x, y) = \sqrt{(}x^2 + y^2)$



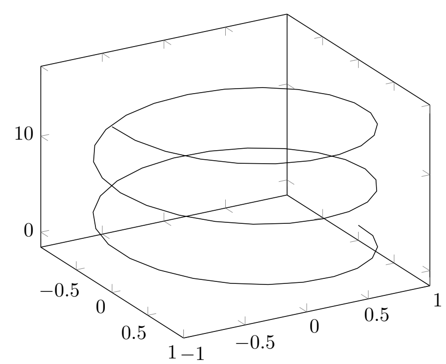
The graph should look like a cone

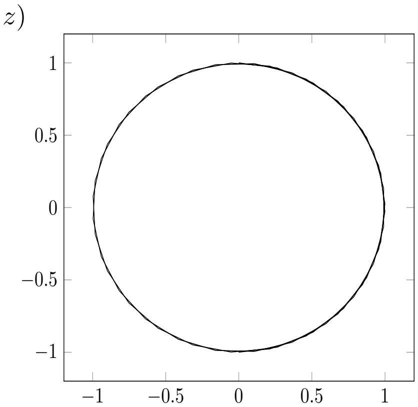
Similarly, we can have a function that takes all the x, y and z can calculate that distance. That would however be a four dimensional plot, which we cannot plot.
For a similar function, $z = x^2 + y^2$





For another function $f(z) = (\cos z, \sin z)$



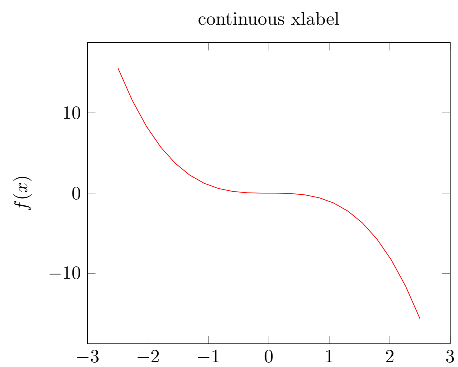


2.2 Limits and continuity

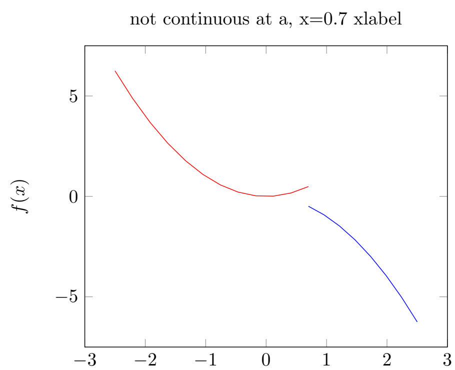
Definition:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \tag{7}$$

if $f(x,y)$ goes to L as (x,y) approaches (a,b)



$\lim_{(x \to a^+)} f(x) = \lim_{(x \to a^-)} f(x)$,



$\lim_{(x \to a^+)} g(x)$ DNE

Similarly, for $(x, y) \to (a, b)$ we need to look at all possible paths. If you can find 2 oaths to approach (a,b) such limits along these 2 paths are different, then the function is not continuous. For example:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2} = \text{DNE(do not exists) since:} \tag{8}$$

$$(x, 0) \to (0, 0),\ \lim = \lim_{x \to 0} = \frac{0}{x^2} = 0 \tag{9}$$

$$(x, x) \to (0, 0),\ \lim = \lim_{y \to 0} = \frac{2x * x}{x^2 + x^2} = 1 \tag{10}$$

$$1 \neq 0 \text{ hence D.N.E.}$$

Remember to take a look at all directions
Definition: $f(x, y)$ continuous at (a, b) if:

1. $f(x, y)$ defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} = f(a, b)$

How to tell if a function is continuous? $f(x, y, z) = e^{x-z} \sin(x + yz^2)$

1. (a) $f(x, y, z) = x$
   (b) $f(x, y, z) = y$
   (c) $f(x, y, z) = zallcontinuous$
2. $f(x, y, z) = x - z$
3. $f(x, y, z) = e^{x-z}(compositionofcontin.funciscontin.)$
4. $g(x, y, z) = \sin(x + yz^2)$ continuous similarly

Test: $\lim_{(x,y,z)\to(0,1,-1)} = e^1 \sin(1)$, hence continuous. A special case is when you have $\frac{f(x,y)}{g(x,y)}$, it will be continuous at (a,b) as long as $g(a, b) \neq 0$

For a vector function $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v} \ldots F_m(\vec{v})$, where $\vec{v}$ is a vector or $\mathrm{I\!R^n}, (\mathrm{x_1}, \mathrm{x_2} \ldots \mathrm{x_n})$, is a continuous function at vector $\vec{n}$ if all its component functions are continuous at $\vec{n}$
Example:

$$f(x, y, z) = (e^{x+y},\ z,\ \cos(x^2 - z)) \text{ continuous everywhere} \tag{11}$$

$$\text{since} \begin{cases} e^{x+y} \text{ continuous} \\ z \text{ continuous} \\ \cos(x^2 - z) \text{ continuous} \end{cases} \tag{12}$$

2.3 Partial Derivatives

The derivative of a function is the rate of change along the x axis. The partial derivatives of $F(x, y)$ would be the rate of change of F as one of the axis is fixed (along one axis).

$$\frac{dF(x,y)}{dx} = F_x(x,y) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x,y)}{\Delta x} \tag{13}$$

$$\frac{dF(x,y)}{dy} = F_y(x,y) = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x,y)}{\Delta y} \tag{14}$$

In order to compute the derivative, we treat all other variables as constants.
Example:

$$f(x, y, z) = xe^{xy} - \sin(y^2 + z^2) \tag{15}$$

$$\frac{df}{dx} = e^{xy} + yxe^{xy} - 0 = e^{xy} + yxe^{xy} \tag{16}$$

$$\frac{df}{dy} = x^2 e^{xy} - \cos(y^2 + z^2)(2y)\frac{df}{dz} = 0 - \cos(y^2 + z^2)(2z) \tag{17}$$

Take a point (a, b) for example

1. $\frac{df}{dx}$ = gradient of tangent at that point along x axis

2. $\frac{df}{dy}$ = gradient of tangent at that point along y axis

3. These two direction can span the tangent plane of f at that point
   (a) Tangent Vector
   (b) along x direction $= (1, 0, \frac{df}{dx}(a, b)) = \vec{v}$
   (c) along y direction $= (0, 1, \frac{df}{dy}(a, b)) = \vec{w}$
   (d) Using these vectors we can find the plane's equation

Example, find the tangent plane of $f(x,y)$ at (1,1):

$$f(x,y) = 1 - x^2 - 2y^2 \tag{18}$$
$$f(1,1) = z = -2, \text{ point at } (1,1,-2) \tag{19}$$
$$\frac{df}{dx} = (1,0,-2x|_{(1,1)}) = (1,0,-2) \tag{20}$$
$$\frac{df}{dy} = (1,0,-4y|_{(1,1)}) = (0,1,-4) \tag{21}$$
$$(1,0,-2) \times (0,1,-4) = (2,4,1) \tag{22}$$
$$2x + 4y + z = 4 \text{ a linear approximation of } f(x,y) \text{ at } (1,1) \tag{23}$$

Definition of a vector function:

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathrm{I\!R}^{\mathrm{m}} \tag{24}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{25}$$

The function F transforms$\mathrm{I\!R}^{\mathrm{n}} \rightarrow \mathrm{I\!R}^{\mathrm{m}}$, DF $\mathrm{I\!R}^{\mathrm{n}} \rightarrow \mathrm{I\!R}^{\mathrm{m}}$ is a linear map, a linear approximation of F.
Example:

$$F(x,y,z) = (e^{x+yz}, x^2+1, \sin{(y+z)}, 4y) \tag{26}$$
$$DF(x,y,z) = \begin{pmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y+z) & \cos(y+z) \\ 0 & 4 & 0 \end{pmatrix} DF(1,1,2) = \tag{27}$$
$$DF(1,1,2) = \begin{pmatrix} e^3 & 2e^3 & e^3 \\ 2 & 0 & 0 \\ 0 & \cos(3) & \cos(3) \\ 0 & 4 & 0 \end{pmatrix} \tag{28}$$

Matrix of numbers, a linear map

$$F(x,y,z) = (e^{x+yz}, x^2+1, \sin{(y+z)}, 4y) \tag{29}$$

2.4 Properties of Derivatives DF

Given two vector function F and G: $\mathrm{I\!R}^{\mathrm{n}} \rightarrow \mathrm{I\!R}^{\mathrm{m}}$ both differentiable at some point $\vec{x}$

$$D(f \pm G) = D(F) \pm D(G) \tag{30}$$
$$C \in \mathrm{I\!R}, \mathrm{D(cF) = cD(F)} \tag{31}$$

suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ a function in n variables, product rule still holds

$$D(f \cdot g) = D(F) \cdot g + f \cdot D(G) \tag{32}$$

where $D(f \cdot g)$ is a n^{th} dimension vector, D(f or g) is a vector function and f or g is a scalar function

$$D(\frac{f}{g}) = \frac{g \cdot D(F) - f \cdot D(G)}{g^2}, g(\vec{x}) \neq 0 \tag{33}$$

Given two vector function $\vec{v}(t)$ and $\vec{w}(t)$: $\mathbb{R} \to \mathbb{R}^n$ so $t \to \vec{v}(t)$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t) \text{scalar function} \tag{34}$$
$$f'(t) = \vec{v}(t)' \cdot \vec{w} + \vec{w}(t)' \cdot \vec{v} \tag{35}$$
$$u(t) = \vec{v}(t) \times \vec{w}(t) \text{vector function} \tag{36}$$
$$u'(t) = \vec{v}(t)' \times \vec{w}(t) + \vec{w}(t)' \times \vec{v}(t) \tag{37}$$
$$\vec{v}(t) = \langle v_1(t), v_2(t), \ldots v_n(t) \rangle \tag{38}$$

Chain rule, suppose $f : \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $g \cdot f : \mathbb{R}^m \to \mathbb{R}^p$, $\vec{x} \in \mathbb{R}^m$

$$D(g \circ f)(\vec{x})(\text{size } p \times m) = DG(F(\vec{x}))(\text{size } p \times n) \cdot DF\vec{x}(\text{size } n \times m) \tag{39}$$
$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{40}$$

Example:

$$f(x, y) = (x^3 + y, e^{xy}, 2 + xy)\mathbb{R}^2 \to \mathbb{R}^3 \tag{41}$$
$$g(y, v, w) = (y^2 + v, uv + w^3)\mathbb{R}^3 \to \mathbb{R}^2 \tag{42}$$
$$g(f(x, y)) = g \circ f = ((x^3 + y)^2 + e^{xy},\ (x^3 + y)e^{xy} + (2 + xy)^3) \tag{43}$$

$$D(g \circ f) = \begin{pmatrix} 2(x^3 + y)(3x^2) + ye^{xy} & 2(x^3 + y) + xe^{xy} \\ \begin{pmatrix} 3x^2 e^x y + (x^3 + y)ye^{xy} \\ +3(2 + xy)^2 y \end{pmatrix} & \begin{pmatrix} e^{xy} + (x^3 + y)xe^{xy} \\ +3(2 + xy)^2 x \end{pmatrix} \end{pmatrix} \tag{44}$$

$$DG = \begin{pmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{pmatrix} \tag{45}$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{46}$$

$$DG = \begin{pmatrix} 2(x^3 + y) & 1 & 0 \\ e^{xy} & x^3 + y & 3(2 + xy)^2 \end{pmatrix} \times \begin{pmatrix} 3x & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{pmatrix} \tag{47}$$

Check that the equality holds.

2.5 Directional derivatives

Recall the Jacobian matrix, in the following context, the function F is not necessarily defined on the entire $\mathbb{R}^{\mathrm{n}}$, but rather a subset $\subseteq \mathbb{R}^{\mathrm{n}}$

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathbb{R}^{\mathrm{m}} \tag{48}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{49}$$

Examples: 1)

$$f(x, y, z) = x^2 y + \ln z : \mathbb{R}^3 \to \mathbb{R}^1 \tag{50}$$

$$Df = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz})$$

$$= (2x, 1, \frac{1}{z}) \tag{51}$$

2)

$$\vec{v}(t) = (\cos(t), \sin(t)); \text{ vector function} \tag{52}$$

$$D\vec{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = (-\sin(t), \cos(t))^T \tag{53}$$

$$D\vec{v}(t) \cdot \vec{v}(t) = -\sin(t)\cos(t) + \cos(t)\sin(t) = 0 \tag{54}$$

$$D\vec{v}(t) \perp \vec{v}(t); \text{ hence a tangent vector} \tag{55}$$

3)

$$g(t) = e^t \text{ scalar function} \tag{56}$$

$$g(t) \cdot \vec{v}(t) = e^t(\cos(t), \sin(t)) = (e^t \cos(t), e^t \sin(t)) \tag{57}$$

$$D(g(t) \cdot \vec{v}(t)) = \begin{bmatrix} e^t \cos(t) - e^t \sin(t) \\ e^t \cos(t) + e^t \sin(t) \end{bmatrix} \tag{58}$$

4)

$$\vec{v}(t) = (t, \sin(t), \cos(t)), \vec{w}(t) = (3t, 0, 2) \tag{59}$$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t)$$

$$= 3t^2 + 0 + 2\cos(t); \text{ scalar function} \tag{60}$$

$$D(f(t)) = 6t - 2\sin(t) \tag{61}$$

$$\vec{w} \cdot D\vec{v} + \vec{v} \cdot D\vec{w} = \begin{bmatrix} 3t \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ +\cos(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} t \\ \sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \tag{62}$$

$$= 6t - 2\sin(t) \tag{63}$$

2.6 Partial derivative along a direction

Definition: Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), given a unit vector $\vec{u} = (u1, u2) \in \mathbb{R}^2$ then the directional derivatives of F along $\vec{u}$ is:

$$D_{\vec{u}}F(a,b) = \left[\frac{d}{dt}f((a,b)+t\vec{u})\right]_{t=0} \tag{64}$$
$$= \lim_{h\to 0}\frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \tag{65}$$

is the rate of change of F along the direction of $\vec{u}$
Two special cases would be when u are along the typical x or y direction, which would mean the class derivative along x-axis $\frac{df}{dx}$ or the y-axis $\frac{df}{dy}$
Theorum $\vec{u} = (u_1, u_2)$ unit vector,

$$D_{\vec{u}}f(x,y) = (\frac{df}{dx}, \frac{df}{dy})\dot{\vec{u}} = (u_1, u_2) \tag{66}$$
$$\text{since } \vec{u} = u_1(1,0) + u_2(0,1) \tag{67}$$
$$D_{\vec{u}}f = u_1\frac{df}{dx} + \frac{df}{dy} \tag{68}$$

Proof:

$$D_{\vec{u}}f = \lim_{h\to 0}\frac{F(a+hu_1, b+hu_2) - F(a,b)}{h} \tag{69}$$
$$= \lim_{h\to 0}\frac{(F(a+hu_1, b+hu_2) - F(a, b+hu_2))}{h} \tag{70}$$
$$+ \lim_{h\to 0}\frac{F(a, b+hu_2) - F(a,b)}{h} \tag{71}$$
$$= \lim_{u_1 h\to 0}\frac{[F(a+hu_1, b+hu_2) - F(a, b+hu_2)]\cdot u_1}{hu_1} \tag{72}$$
$$+ \lim_{h\to 0}\frac{F(a, b+hu_2) - F(a,b)}{h} \tag{73}$$
$$= \frac{df}{dx}(a, b+0)\cdot u_1 + \frac{df}{dy}\cdot u_s \tag{74}$$

Example: Find the directional Deriv of $f(x, y) = x^2 + 3xy$ along the direction of $(3,4)$ at the point $p = (2,-1)$

$$\vec{u} = \frac{(3,4)}{\sqrt{(3^2+4^2)}} = (\frac{3}{5}, \frac{4}{5}) \tag{75}$$

$$D_{\vec{u}} f(2,-1) = (2x+3y, 3x)|_{(2,-1)} \cdot (\frac{3}{5}, \frac{4}{5}) \tag{76}$$

$$= (1,6) \cdot (\frac{3}{5}, \frac{4}{5}) \tag{77}$$

$$= \frac{27}{5} \tag{78}$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = (\frac{df}{dx}, df dy) \cdot \vec{u} \tag{79}$$

Definition: Given $F : \mathbb{R}^n \to \mathbb{R}$, then define the gradient.

$$\nabla f = D_{\vec{u}} f = (\frac{df}{x_1}, \frac{df}{x_2}, \ldots, \frac{df}{x_n}) \tag{80}$$

Hence the directional derivative of a function along a direction is:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \tag{81}$$

The consequence of this:
1)

$$D_{\vec{u}} f \text{ max when } \vec{u} /\!\!/ \nabla f = (\frac{df}{dx}, \frac{df}{dy}) \tag{82}$$

since

$$\nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos(\theta) \tag{83}$$

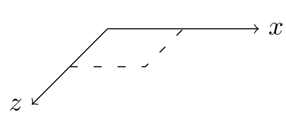
$$= |\nabla f| \cdot 1 \cdot \cos(\theta) \theta \;=\; ang(\nabla f, \vec{u}) \tag{84}$$

When maximized, $/\!\!/$, $\theta = 0$

$$\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{\frac{df}{dx}^2 + \frac{df}{dy}^2} \tag{85}$$

2)

$$\text{When } \vec{u} \perp \nabla f, D_{\vec{u}} f = 0 \tag{86}$$



Example: $f(x,y) = e^{-(x^2+y^2)}$, find the direction along this function which has the largest rate of increase/decrease at point $(1,1)$

Since at maximum, $\vec{u} /\!/ \nabla f$

$$\nabla f = (-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)})|_{(1,1)} \tag{87}$$
$$= (-2e^{-2}, -2e^{-2}) \tag{88}$$
$$\because \quad \vec{u} /\!/ \nabla f /\!/ (1,1) \tag{89}$$
$$\therefore \quad \vec{u} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \tag{90}$$

Largest rate of increase at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
Largest rate of decrease at $-(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
The gradient ∇f tells us the direction along which F changes the most.

2.7 Summary

Given a function $f(x, y)$	**n-th dim analogue**
Define Partial derivatives(deriv) $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f(x_1, x_2, \dots x_n)$
Define gradient $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f : \mathbb{R}^n \to \mathbb{R}$
Directional deriv: given **unit** vector $\vec{u} = (u_1, u_2)$	$\frac{df}{dx_1} \cdots \frac{df}{dx_n}$
$D_{\vec{u}} f = \lim_{h \to 0} \frac{f((a,b)+h\vec{u}) - f(a,b}{h}$	$\nabla f = (\uparrow)$
if$(\vec{u} = (1, 0))$, $D_{\vec{u}} f = \frac{df}{dx}$	$\vec{u} f \in \mathbb{R}^n$
if$(\vec{u} = (0, 1))$, $D_{\vec{u}} f = \frac{df}{dy}$	$D_{\vec{u}} f = ?$
Theorem:$D_{\vec{u}} = \nabla f \cdot \vec{u} = u_1 \frac{df}{dx} + u_2 \frac{df}{dy}$	generalizations?
Remark: We can define $D_{\vec{u}}$ for any $\vec{v} \in \mathbb{R}^2$	
in the exact same way	
$D_{\vec{u}} = \lvert\vec{v}\rvert D_{\frac{\vec{u}}{\lvert\vec{v}\rvert}} f$	

To each unit vector $\vec{u}$, we have a tangent vector associated to it:

$$(u1, u2, D_{\vec{u}} f) = u_1(1, 0, \frac{df}{dx}) + u_2(0, 1, \frac{df}{dy}) \tag{91}$$

Two special cases: $(1, 0, \frac{df}{dx})(0, 1, \frac{df}{dy})$
The two tangent vector span a tangent plane at $(a, b, f(a, b))$

2.8 Gradients and level sets

Recall

$$\nabla f = (\frac{df}{dx}, \frac{df}{dy}) \tag{92}$$
$$\because D_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos(\theta) \tag{93}$$

$D_{\vec{u}}$ is the rate of change along $\vec{u}$

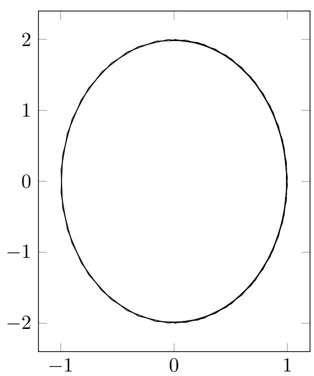
if $u /\!/ \nabla f$, $(\theta = 0)$ then $D_{\vec{u}}F$ max

$$|D_{\vec{u}}F| = \sqrt{(\frac{df}{dx})^2 + (\frac{df}{dy})^2} \tag{94}$$

In other words, along the direction of ∇F, the function F changes the most.
Definition: A level set of F(x,y) is:

$$(x, y) \in \mathbb{R}^2 : \text{f(x, y) = c} \tag{95}$$

Example:



$$f(x, y) = x^2 + \frac{y^2}{4} \tag{96}$$

$$\{f(x, y) = 1\} \text{ (Level set)} \tag{97}$$

$$= \{(x, y) | x^2 + \frac{y^2}{4} = 1\} \tag{98}$$

Theorem: Given $f(x, y)$ differentiable, then the gradient $\nabla f \perp$ the level set.
We can prove it by using the chain rule: let $\gamma(t)$ be a parametric function of my level set $\{f(x, y) = 1\}$, $f(\gamma(t)) = 1$ and differentiate both sides.

$$0 = D(f(\gamma(t))) = Df(\gamma(t))D(\gamma(t)) \tag{99}$$

$$= \nabla F(\gamma(t))_{1\times 2} \cdot (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{100}$$

$$\text{Hence } \Rightarrow \nabla F(\gamma(t))_{1\times 2} \perp (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{101}$$

Example: Look at a surface $\sin(xy) - 2\cos(yz) = 0$ Find the tangent plane of this surface at $(\frac{\pi}{2}, 1, \frac{\pi}{3})$. The idea is that consider the surface as a level set of $F(x, y, z) = \sin(xy) - 2\cos(yz), \{F = 0\}$
Using the Theorem: ∇F is hence perpendicular to this surface, giving us the normal direction of our tangent plane.

$$\nabla F = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}) = (y\cos(xy), x\cos(xy) + 2z\sin(yz), 2y\sin(tz)) \tag{102}$$

$$= (\cos(\frac{\pi}{2}), \frac{\pi}{2}\cos(\frac{\pi}{2}) + \frac{2\pi}{3}\sin(\frac{\pi}{3}), 2\sin(\frac{\pi}{3})) \tag{103}$$

$$= (0, \frac{\sqrt{3}\pi}{3}, \sqrt{3}) \tag{104}$$

$$0 = 0 \cdot (x - \frac{\pi}{2}) + \frac{\sqrt{3}\pi}{3}(y - 1) + \sqrt{3}(z - \frac{\pi}{3}) \tag{105}$$

Where the last equation above is the equation of the tangent plane.

2.9 Parametric Curves

...are vector functions in 1 variable.
Definition: A parametric curve in $\mathbb{R}^n$ is a vector valued function where $\vec{c}(t) : R^1 \to \mathbb{R}^n$
All the component functions of $\vec{c}(t)$ are functions in 1 variable.
Example:
1) Draws a Circle

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \tag{106}$$

2) Draws a eclipse at plane z = 3

$$\vec{c}(t) = (a\cos(t), b\sin(t), 3), t \in [0, 2\pi] \tag{107}$$

3) Eclipse spiraling up

$$\vec{c}(t) = (a\cos(t), b\sin(t), t), t \in [0, 2\pi] \tag{108}$$

Definition: Orientation of $\vec{c}(t)$, the direction corresponding to increasing t-value is called the positive orientation of $\vec{c}(t)$. The opposite direction is the negative orientation.
Example: Positive orientation = Clockwise

$$\vec{c}(t) = (\cos(t), -\sin(t), t), t \in [0, 2\pi] \tag{109}$$

Derivative of $\vec{c}(t)$
If $\vec{c}(t) = (x(t), y(t), z(t))$, $\vec{c'}(t) = (x'(t), y'(t), z'(t))$

1. $\vec{c}(t)$ is a vector function
2. $\vec{c'}(t)$ is a tangent to curve $\vec{c}(t)$

Terminology: $\vec{c'}(t)$ is the velocity of $\vec{c}(t)$, and $|\vec{c'}(t)|$ is the speed of $\vec{c}(t)$. The speed of the curve $\vec{c}(t) = (\cos(t), \sin(t))$ for example, is 1.

What is the length of curve $\vec{c}(t) = (\cos(t), \sin(t), t)$ between (0 to 2π)Given $\vec{c}(t)$ differentiable between range a and b, length of the curve would be

$$l = \int_a^b |\vec{c'}(t)| dt \tag{110}$$

$$|\vec{c'}(t)| = \sqrt{\sin(t)^2 + \cos(t)^2 + 1} = \sqrt{2} \tag{111}$$

$$l = \int_0^{2\pi} \sqrt{2} dt \tag{112}$$

Think of length as a function in t

$$l(t) = \int_a^t |\vec{c'}(x)| dx \tag{113}$$

If curve $\vec{c}(t)$ differentiable and $\vec{c}(t) \neq 0$,

1. $|\vec{c'}(t)| > 0$

2. $l(t)$ strictly increasing.

3. l is one to one function so has inverse.

$$l[a,b] \to \mathbb{R}^1, \text{ l has an inverse.} \tag{114}$$
$$t \to l(t), \text{ think of our parameter t} \tag{115}$$
$$\tag{116}$$

t can now be expressed as a function in terms of length, $\vec{c}(t) = \vec{c}(t(l)) = \vec{c'}(l)$

Proof:
In our last example of calculating length, we can find an inverse for $l(t) = \sqrt{2}t$ resulting $t = \frac{l}{\sqrt{2}}$, by substitution we can reparameterize $\vec{c}(t)$ into $\vec{c}(l)$

$$c(l) = \vec{c}(t(l)) \tag{117}$$
$$\frac{d\vec{c}(l)}{dl} = \frac{\vec{c}(t(l))}{dt} \cdot \frac{dt}{dl} \tag{118}$$
$$= \vec{c'}(t) \cdot \frac{1}{|\vec{c'}(t)|} \tag{119}$$
$$|\frac{d\vec{c}(l)}{dl}| = |\vec{c'}(t)| \cdot \frac{1}{|\vec{c'}(t)|} = 1 \tag{120}$$

2.10 Acceleration

Acceleration is the second derivatives of length, hence the definition of second derivative can be expressed as below:

Definition: Given $\vec{c}(t)$, $\vec{a}(t) = \vec{c''}(t)$ the acceleration of $\vec{c}$.

Example:
1) Ellipse

$$\vec{c_1}(t) = (\cos(t), 2\sin(t)), t \in [0, 2\pi] \tag{121}$$
$$\vec{c_1'}(t) = (-\sin(t), 2\cos(t)) \tag{122}$$
$$\vec{c_1''}(t) = (-\cos(t), -2\sin(t)) \tag{123}$$

2) Different para of the same Ellipse

$$\vec{c_2}(t) = (\cos(t^2), 2\sin(t^2)), t \in [0, \sqrt{2\pi}] \tag{124}$$
$$\vec{c_2'}(t) = 2t(-\sin(t^2), 2\cos(t^2)) \tag{125}$$
$$\vec{c_2''}(t) = (-2\cos(t^2) - 4t^2 cos(t^2), 4cos(t^2) - 8t^2\sin(t^2)) \tag{126}$$

Facts:

1. We can have different parameterations of a curve
2. Velocity, acceleration depend on the para we choose in general
3. For different parameterization $\vec{c_1}$, $\vec{c_2}$ of the same curve, $\vec{c_1'}(t) /\!/ \vec{c_2'}(t)$
4. "Unit Velocity" $\frac{\vec{c_1'}(t)}{|\vec{c_1'}(t)|} = \pm\frac{\vec{c_2'}(t)}{|\vec{c_2'}(t)|}$ are equal, the $\pm$ depends on the orientations of the $\vec{c_1}$ and $\vec{c_2}$.

Definition:Given $\vec{c}(t)$, assume $|\vec{c'}(t)| \neq 0$, defined the unit tangent vector $\vec{T}(t) = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$. If $\vec{c_1}(t)$ and $\vec{c_2}(t)$ are two different parameter of a curve with the same orientation, their unit tangent vector equals.

2.11 Curvature

Idea: Curvature is a measure of how sharply a curve is bending at a point, it is the "rate of change" of directions along a curve. So a straight line has no curvature, and a slightly curved line has a small curvature.

Definition: Quantitatively, if $\vec{c}(t)$ is parametered by length, (namely, $|\vec{c}(t)| = 1$), then we define the curvature at $l = l_0$ to be $k(l_0) = |\vec{c''}(l_0)| = |\frac{d^2\vec{c}(l)}{dl^2}|_{l=l_0}$

Remark: IF $\vec{c}(t)$ not parametered by length, then $|\vec{c''}(t)|$ is not the curvature.

Explanation: a curve is like a highway and a parameterized function $\vec{c}(t)$ is like someone driving along this highway giving his position vector at a given time. Parameterized by length would be driving at a constant/normalized speed. A curvature is how sharp a turn is, measured by looking at how quickly the driver need to turn their steering wheel.

Generally, $|\vec{c'}(l)| = 1 \Rightarrow \vec{c'}(l)$ unit tangent, so $\vec{c''}(t)$ is curvature.

Example:
1)Find parameter by length

$$\vec{c}(t) = \vec{a} + t\vec{v} \tag{127}$$

$$\text{Constant vectors: } \vec{a} = (1,2,3), \vec{v} = (0,2,1)$$

$$\text{Length function } l(t) = \int_0^t |\vec{c'}(x)|dx \tag{128}$$

$$= \int_0^t |\vec{v}|dx, \quad \vec{v} = (0,2,1) \tag{129}$$

$$l = l(t) = \sqrt{5}t \tag{130}$$

$$t = t(l) = \frac{l}{\sqrt{5}} \tag{131}$$

Curvature would be zero since it is a line, its second derivative would be a zero.

2) Find the curvature for a circle

$$\begin{aligned}
\vec{c}(t) =& (r\cos(t), r\sin(t)), r > 0 \text{ circle } x^2 + y^2 = r^2 && (132)\\
\text{length function } \ l(t) =& \int_0^t |\vec{c'}(x)| dx = rt && (133)\\
t(l) =& \frac{l}{t} && (134)\\
\vec{c}(l) =& (r\cos(\frac{l}{r}), r\sin(\frac{l}{r})) && (135)\\
\vec{c''}(l) =& (-\frac{\cos(l/r)}{r}, -\frac{\sin(l/r)}{r}) && (136)\\
k(l) =& |\vec{c''}(l)| && (137)\\
=& \sqrt{(-\frac{\cos(l/r)}{r})^2 + (-\frac{\sin(l/r)}{r})^2} = \frac{1}{r} && (138)
\end{aligned}$$

Curvature for circle of radius $r = \frac{1}{r}$

Second Definition of Curvature:
If $\vec{c}(t)$ any parameteric curve, $\vec{c'}(t) \neq 0$, recall the unit tangent vector $\vec{T} = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$, then define curvature $k(t) = \frac{|\vec{T'}(t)|}{|\vec{c'}(t)|}$
Theorem: Definition 1 and 2 of curvature are the same.

$$\begin{aligned}
\vec{c'}(t) = \frac{d\vec{c}(t(l))}{dt} && (139)\\
=& \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} && (140)\\
=& \frac{\vec{c'}(t)}{|\vec{c}(t)|} = \vec{T} && (141)\\
k(l(t)) = |\vec{c''}(t)| =& |\frac{d^2\vec{c}}{dl^2}| = |\frac{d\vec{T}(t(l))}{dl}| && (142)\\
=& \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} && (143)\\
=& |T'(t)| \cdot \frac{1}{|\vec{c'}(t)|} = k(t) && (144)
\end{aligned}$$

Using the second definition to find the curvature of the parabola $y = x^2$

$$\begin{aligned}
\vec{c} =&(t, t^2) && (145)\\
\vec{c'}(t) =&(1, 2t) && (146)\\
\vec{T}(t) =&\frac{(1,2t)}{\sqrt{1+4t^2}} && (147)\\
\vec{T'}(t) =&\frac{(0,2)}{\sqrt{1+4t^2}} + (1,2t)\cdot((a+4t^2)^{-\frac{1}{2}})' && (148)\\
=&\frac{(-4t,2)}{(1+4t^2)^{\frac{3}{2}}} && (149)\\
|T'(t)| =&\frac{\sqrt{4+16t^2}}{(1+4t^2)^{\frac{3}{2}}} && (150)\\
=&\frac{2}{1+4t^2} && (151)\\
K(t) =&|T'(t)|\cdot\frac{1}{|c'(t)|} && (152)\\
=&\frac{2}{(1+4t^2)}\cdot\frac{1}{\sqrt{1+4t^2}} && (153)\\
=&2\cdot(1+4t^2)^{-\frac{3}{2}} && (154)
\end{aligned}$$

As x approaches ∞, the curvature approaches 0, $y = x^2$ became more like a straight line.

Osculating Plane and Circle

1. Derivative — Tangent line
2. Curvature — Osculating Circle

Observations:
1) If $\vec{c}(l)$ parameterized by length, then

$$\begin{aligned}
\vec{c'}(l) \perp \vec{c''}(l) && (155)\\
\because (\vec{c'}(l)\cdot\vec{c'}(l))' =&(|\vec{c}(l)|^2)' = (1)' && (156)\\
\vec{c'}(l)\cdot\vec{c''}(l) + \vec{c'}(l)\cdot\vec{c''}(l) =&0 && (157)\\
\Rightarrow \vec{c'}(l)\cdot\vec{c''}(l) =&0 && (158)
\end{aligned}$$

so second derivative is the normal and the first derivative is the tangent.

2) If $\vec{c}$(t) is any parameterized curve, its unit tangent vector T and T' is

$$\begin{aligned}
\vec{T}(t) =& \frac{\vec{c'}(t)}{|\vec{c'}(t)|} && (159)\\
\because (\vec{T}(t) \cdot \vec{T}(t))' =& (|\vec{T}(t)|^2)' = (1)' && (160)\\
\Rightarrow \vec{T}(t) \cdot \vec{T}'(t) =& 0 && (161)\\
\therefore \vec{T}(l) \perp& \vec{T}'(l) && (162)
\end{aligned}$$

Definition: Osculating Plane: Given $\vec{c}(t)$, $\vec{c}(t) \neq 0$, then the osculating plane is the plane spanned by $\vec{T}(t)$ and $\vec{T}'(t)$

Recall that we computed for a circle with radius r its curvature $k = \frac{1}{r}$, therefore, if have a $\vec{c}(t)$ with $k(t_0) = k$, then at this point, the curve $\vec{c}(t)$ is "modeled" by a circle of radius $\frac{1}{k}$

Definition: Osculating Circle: Given $\vec{c}(t), \vec{c'}(t) \neq 0$, then the osculating circle of $\vec{c}$ at $t = t_0$ is a circle on the osculating plane at $t = t_0$ and has radius $\frac{1}{k(t_0)}$, and the tangent to the curve. (The best approximation of the curve at that point by the circle)

Example: Find the equation of the osculating plane at $\vec{c}(\pi) = (-a, 0, b\pi)$ of $\vec{c}(t) = (a\cos(t), a\sin(t), bt)$, $t \geq 0$; $a, b > 0$,

$$\begin{aligned}
\vec{T} =& \frac{\vec{c'}(t)}{|\vec{c'}(t)|} = \frac{(-a\sin(t), a\cos(t), b)}{\sqrt{a^2+b^2}} && (163)\\
\vec{T'} =& \frac{(-a\cos(t), -a\sin(t), 0)}{\sqrt{a^2+b^2}} && (164)\\
\text{Osc.Plane} =& \vec{T} \times \vec{T'} && (165)\\
=& \frac{1}{a^2+b^2}(ab\sin(t), -ab\cos(t), a^2)|_{t=\pi} && (166)\\
=& \frac{1}{a^2+b^2}(0, ab, a^2) && (167)
\end{aligned}$$

calculating plane at given point

$$a^2 b\pi = 0x + aby + a^2 z \qquad (168)$$

2.12 Higher order derivatives and Taylor expansions

Recall partial derivatives of $F(x,y)$ by x and y denoted by F_x and F_y, second order derivative = F_{xx} and F_{yy}, first x then y F_{xy} and first y then x F_{yx}, vise versa.

Theorem: If $F(x,y)$ differentiable and F_{xy}, F_{yx} continuous, $F_{xy} = F_{yx}$, hence the order can be switched

Example: $f(x,y,z) = x^2e^{y^2-x} + x^2yz^2 - cos(x^2+y^2)$, f_{xzzy} =? switch order, try finding f_{zzxy} instead

$$df/dz = f_z = 2zx^2y \Rightarrow fzz = 2x^2y \Rightarrow f_{zzxy} = 4x \tag{169}$$

2.13 Taylor Expansions

One Variable: $y = f(x)$, $f'(x) =$, tangent line of graph of the function, its linear approximation at point $x = x_0$, equation of tangent line is $y = f(x_0) + f'(x_0)(x - x_0)$. Higher order $f'(x) \to$ higher degree approximation.

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt \tag{170}$$

$$\int u'v = uv - \int v'u,\ u = t, v = f'(t) \tag{171}$$

$$= tf'(t)|_{x_0}^{x} - \int_{x_0}^{x} tdf'(t) \text{ by parts} \tag{172}$$

$$= [xf'(x) - x_0f'(x_0)] - \int_{x_0}^{x} tf''(t)dt \tag{173}$$

$$= (xf'x - xf'(x_0)) + (xf'x_0 - x_0f'(x_0)) - \int_{x_0}^{x} tf''(t)dt \tag{174}$$

$$= (x - x_0)f'(x_0) + x(f'(x) - f'(x_0)) - x\int_{x_0}^{x} tf''(t)dt \tag{175}$$

$$= f'(x_0)(x - x_0) + x\int_{x_0}^{x} f''(t)dt - \int_{x_0}^{x} tf''(t)dt \tag{176}$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} (x - t)f''(t)dt \tag{177}$$

$$\tag{178}$$

This is the 1 st Taylor expansion where $f(x_0) + f'(x_0)(x - x_0)$ is the 1st order approximation, a tangent line, and $\int_{x_0}^{x}(x-t)f''(t)dt$ is the 1st order remainder. error/diff. to linear approx.

We can continue to get a higher order Taylor expansion using integration by parts by setting $v = f''(t)$ and $u' = (x - t)$ which would yield:

$$\begin{aligned} f(x) =& f(x_0) + f'(x_0)(x - x_0) \\ &+ f''(x_0) \cdot \frac{(x - x_0)^2}{2} \\ &+ \int_{x_0}^{x} (x - t)f'''(t) \cdot \frac{(x - t)^2}{2} dt \end{aligned} \tag{179}$$

Theorem: if $f(x)$ differentiable, then $\frac{d^n(f(x))}{dx^n}$ exists for all n. Hence the n-th order of the Taylor expansion would be:

$$\begin{aligned} f(x) =& f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)}{2!} \dots \\ &+ f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \int_{x_0}^{x} (x-t) f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt \end{aligned} \tag{180}$$

Example:
Find the 2nd order Taylor expansion at $x = 0$ of $f(x) = x + e^{-2x}$ USe that to approximate the value $f(0.075) = 0.075 + e^{-0.15}$ and estimate the error.

$$\begin{aligned} f'(x) =& (1 - 2e^{2x}), f''(x) = 4e^{-2x} \\ f(x) =& f(0) + f'(0)(x-0) \\ &+ \frac{f''(0)(x-0)^2}{2} \\ &+ \int_0^x f'''(t)\frac{(x-t)^2}{2} dt \\ f(0.075) \approx& 1 - (0.075) + 2(0.075)^2 \end{aligned} \tag{181}$$

Error given y the remainder $R(x)$, a bound for the error

$$\text{error} \le |R(x)| \le \left| \int_0^x -8e^{2t} \cdot \frac{(x-t)^2}{2} dt \right| \tag{182}$$

$$\le 4x^2(x-0) = 4x^3 = (4 \cdot (0.075)^3) \tag{183}$$

For 2 dimension, the second order expansion of $F(x, y)$ at (x_0, y_0) is:

$$\begin{aligned} F(x,y) =& F(x_0, y_0) \\ &+ \frac{1}{1!}\left[\frac{\sigma F}{\sigma x}(x_0, y_0)(x - x_0) + \frac{\sigma F}{\sigma y}(x_0, y_0)(y - y_0)\right] \\ &+ \frac{1}{2!}\left[\frac{\sigma^2 F}{\sigma x^2}(x_0, y_0)(x - x_0) + \frac{\sigma^2 F}{\sigma y^2}(x_0, y_0)(y - y_0) \right.\\ &\left. + 2\frac{\sigma^2 F}{\sigma x \sigma y}\right] + R_2(x, y) \end{aligned} \tag{184}$$

Example: Find the second Taylor expansion of $f(x, y) = sin(x+y)+cos(x-3y)$

at (0,0)

$$\begin{aligned}
f_x =& \cos(x+y) - \sin(x-3y) && (185)\\
f_y =& \cos(x+y) + 3\sin(x-3y) && (186)\\
f_{xx} =& -\sin(x+y) - \cos(x-3y) && (187)\\
f_{xy} =& -\sin(x+y) + 3\cos(x-3y) && (188)\\
f_{yy} =& -\sin(x+y) - 9\cos(x-3y) && (189)\\
f(x,y) =& f(0,0) + f_x(0,0)x + f_y(0,0)y && (190)\\
&+\frac{1}{2}\left[f_{xx}x^2 + f_{yy}y^2 + 2f_{x,y}xy\right] + R_2(x,y) && (191)\\
=& 1 + x + y + \frac{1}{2}(-x^2 - 9y^2 + 6xy) + R_2(x,y) && (192)
\end{aligned}$$

Second Approximation, Taylor Polynomial.

For extreme values of F(x,y): Recall that given y = f(x), differentiable, n-th Taylor expansion at $x = x_0$

$$\begin{aligned}
f(x) =& f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\cdot\frac{(x-x0)^2}{2} \dots && (193)\\
&+ f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} \text{ Taylor polynomial} && (194)\\
&+ \int_{x_0}^{x} f^(n+1)(t)\frac{(x-t)^n}{n!}dt\ R_n(x)\text{remainder} && (195)
\end{aligned}$$

Taylor Expansion for $F(x,y)$ at (x_0,y_0)

$$\begin{aligned}
F(x,y) =& F(x_0,y_0) + F_x(x_0,y_0)x + F_y(x_0,y_0)y && (196)\\
&+ F_{xx}(x_0,y_0)\frac{x^2}{2} + F_{xy}(x_0,y_0)\frac{xy}{2} && (197)\\
&+ F_{yx}(x_0,y_0)\frac{yx}{2} + F_{yy}(x_0,y_0)\frac{y^2}{2} && (198)\\
&+ R_2(x,y) && (199)
\end{aligned}$$

Recall: for $y = f(x)$, critical points, $f'(x_0) = 0$ are candidates for local max and local min.

1. $f''(x_0) > 0$ local min
2. $f''(x_0) < 0$ local max
3. $f''(x_0) = 0$ inconclusive

if x_0 is critical point, first derivative term $f'(x_0) = 0$ removed from polynomial. The two variable are F(x,y). Definition F(x,y) differentiable, then say (x_0, y_0) is a critical point if $\nabla F(x_0, y_0) = 0$.

Recall equation of tangent plane

$$z = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(y - y_0) \tag{200}$$

Tangent plane is horizontal. Definition: for $F(x, y)$, define the Hessian matrix.

$$\begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} (x_0, y_0) \tag{201}$$

Define the discrimanant, (x_0, y_0) is critical point.

$$D(x_0, y_0) = det \begin{pmatrix} F_{xx}(x_0, y_0) & F_{xy}(x_0, y_0) \\ F_{yx}(x_0, y_0) & F_{yy}(x_0, y_0) \end{pmatrix} (x_0, y_0) = F_{xx}F_{yy} - F_{xy}F_{yx} \tag{202}$$

Theorem (2nd derivative test): Suppose (x_0, y_0) is a critical point, then

1. If $D(x_0, y_0) > 0$ and $F_{xx} > 0$ or $(F_{yy} > 0)$, then (x_0, y_0) is a local minimum
2. If $D(x_0, y_0) > 0$ and $F_{xx} < 0$ or $(F_{yy} < 0)$, then (x_0, y_0) is a local maximum
3. if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.
4. if $D(x_0, y_0) = 0$, then it is inconclusive.

Find and classify all critical points of $F(x, y) = x^3 - 3x^2 - 3y^2 + 3xy^2$

$$\begin{align} F_x =& 3x^2 - 6x + 3y^2 = 0 \tag{203} \\ F_y =& -6y + 6xy = 0 \tag{204} \\ \Rightarrow\ y =& 0 \ or \ x = 1 \text{ giving two critical point} \tag{205} \\ y = 0, \Rightarrow & (0, 0), (2, 0) \tag{206} \\ x = 1, \Rightarrow & (1, 1), (1, -1) \tag{207} \end{align}$$

Use Second derivative to test for the four critical points

$$\begin{align} D(x, y) =& F_{xx}F_{yy} - F_{xy}^2 \tag{208} \\ D(1, 1) =& \Rightarrow \text{ is a saddle point} \tag{209} \\ D(1, -1) =& \Rightarrow \text{ is a saddle point} \tag{210} \\ D(0, 0) =& \Rightarrow \text{ is local max} \tag{211} \\ D(2, 0) =& \Rightarrow \text{ is local min} \tag{212} \end{align}$$

Why is this true?
From the second Taylor expansion, at a critical point, the dominant part:

$$F_{xx}(x_0, y_0)(x - x_0)^2 + 2F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + F_{yy}(y - y_0)^2 \tag{213}$$

Homogeneous degree 2 polynomial in 2 variables:

$$P(x,y) = ax^2 + 2Cxy + by^2 \tag{214}$$

$$= a(x^2 + \frac{2c}{a}xy + \frac{b}{a}y^2) \tag{215}$$

$$= a((x - \frac{c}{a}y)^2 + \frac{ab - c^2}{a^2}y^2) \tag{216}$$

$$= a((x - \frac{c}{a}y)^2 + \frac{ab - c^2}{a^2}y^2) \tag{217}$$

$$\Delta = ab - c^2 > 0, \text{then} \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{ab-c^2}}{a}y \end{cases} \tag{218}$$

$$= a((x')^2 + (y')^2) \tag{219}$$

$$\Rightarrow a > 0 \text{ local min, } a < 0 \text{ local max} \tag{220}$$

$$\text{If } \Delta < 0 \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{c^2-ab}}{a}y \end{cases} \tag{221}$$

$$\text{then } P(x,y) = P(x', y') = a((x')^2 - (y')^2) \tag{222}$$

$$a > 0 \text{ saddle point} \tag{223}$$

Apply this to $F_{xx}(x_0, y_0)(x - x_0)^2 + 2F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + F_{yy}(y - y_0)^2$

$$a = F_{xx}(x_0, y_0)(x - x_0)^2 \tag{224}$$

$$b = F_{yy}(x_0, y_0)(x - x_0)^2 \tag{225}$$

$$2c = 2F_{xy}(x_0, y_0)(x - x_0)^2 \tag{226}$$

$$ab - c^2 = D(x,y) \tag{227}$$