Math, Multivariable Calculus

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2.1 Vector Valued functions in multiple variables

Calculus of single variable is a study of $y = f(x)$

$$y = f(x), \text{ where } (x \in \mathbb{R}) \to (\text{f(x)} \in \mathbb{R}) \tag{1}$$

$$\tag{2}$$

However, when we are dealing with multi-variable calculus, we have: an n variable vector valued function (if $m > 1$)

$$F = \begin{cases} (x_1, \ldots x_n) & \to F(x_1, \ldots x_n) \\ \quad \in & \qquad \quad \in \\ \quad \mathbb{R}^{\mathrm{n}} & \longrightarrow \quad \mathbb{R}^{\mathrm{m}} \end{cases} \tag{3}$$

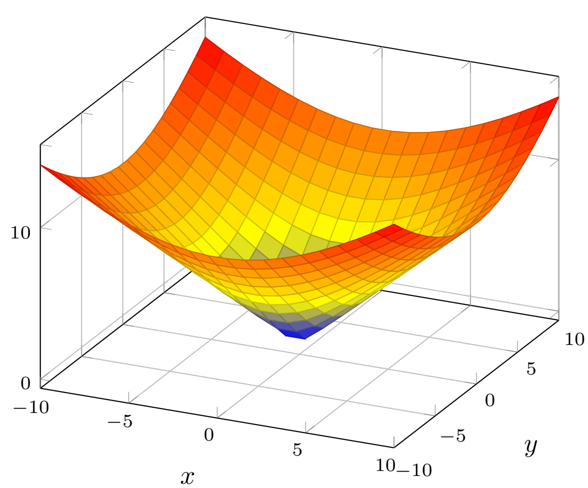
where $F(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n),\ F_2(x_1, \ldots, x_n) \ldots F_m(x_1, \ldots, x_n))$
The inputs and ouputs are Vectors, for example

$$d(x, y) = \sqrt{x^2 + y^2}, (2variables) \tag{4}$$

$$(x, y)\mathbb{R}^2 \to (\sqrt{x^2 + y^2})\mathbb{R} \tag{5}$$

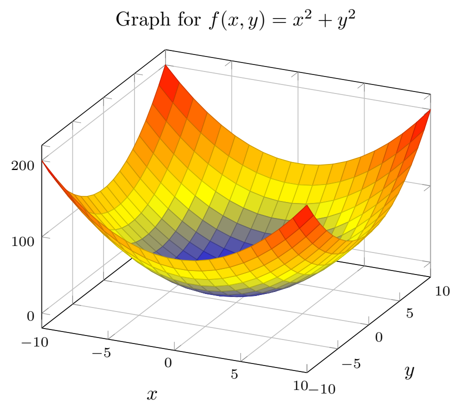
$$distance = \sqrt{x^2 + y^2} \tag{6}$$

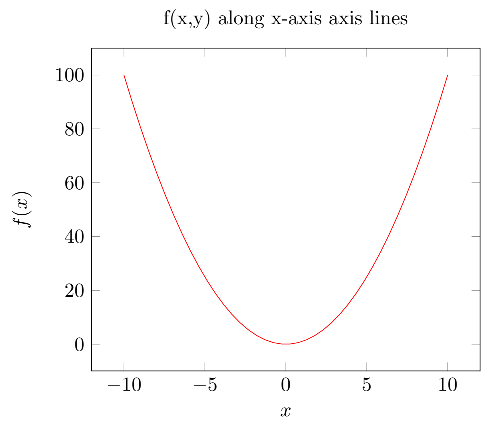
Graph for $f(x, y) = \sqrt{(}x^2 + y^2)$



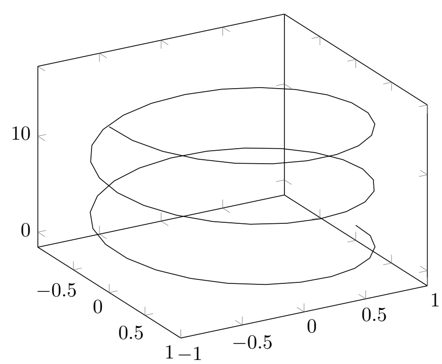
The graph should look like a cone

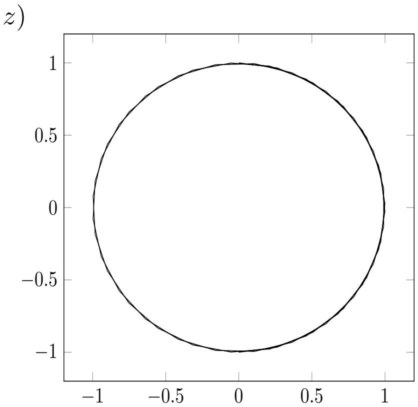
Similarly, we can have a function that takes all the x, y and z can calculate that distance. That would however be a four dimensional plot, which we cannot plot.
For a similar function, $z = x^2 + y^2$





For another function $f(z) = (\cos z, \sin z)$



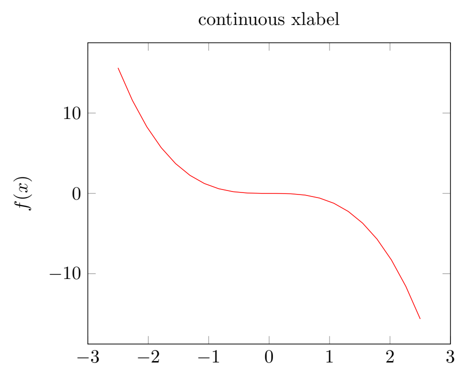


2.2 Limits and continuity

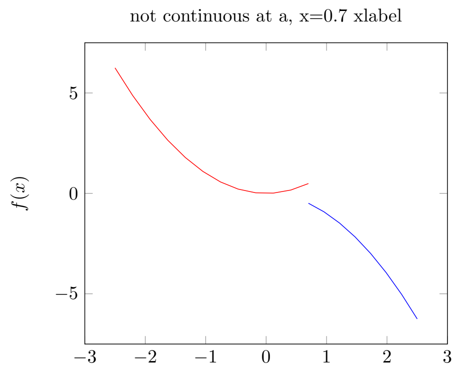
Definition:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \tag{7}$$

if $f(x, y)$ goes to L as (x, y) approaches (a, b)



$\lim_{(x \to a^+)} f(x) = \lim_{(x \to a^-)} f(x)$,



$\lim_{(x \to a^+)} g(x)$ DNE

Similarly, for $(x, y) \to (a, b)$ we need to look at all possible paths. If you can find 2 oaths to approach (a,b) such limits along these 2 paths are different, then the function is not continuous. For example:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \text{DNE(do not exists) since:} \tag{8}$$

$$(x, 0) \to (0, 0),\ \lim = \lim_{x\to 0} = \frac{0}{x^2} = 0 \tag{9}$$

$$(x, x) \to (0, 0),\ \lim = \lim_{y\to 0} = \frac{2x * x}{x^2 + x^2} = 1 \tag{10}$$

$$1 \neq 0 \text{ hence D.N.E.}$$

Remember to take a look at all directions
Definition: $f(x, y)$ continuous at (a, b) if:

1. $f(x, y)$ defined at (a, b)
2. $\lim_{(x,y)\to(a,b)} = f(a, b)$

How to tell if a function is continuous? $f(x, y, z) = e^{x-z} \sin(x + yz^2)$

1. (a) $f(x, y, z) = x$
   (b) $f(x, y, z) = y$
   (c) $f(x, y, z) = zallcontinuous$
2. $f(x, y, z) = x - z$
3. $f(x, y, z) = e^{x-z}(compositionofcontin.funciscontin.)$
4. $g(x, y, z) = \sin(x + yz^2)$ continuous similarly

Test: $\lim_{(x,y,z)\to(0,1,-1)} = e^1 \sin(1)$, hence continuous. A special case is when you have $\frac{f(x,y)}{g(x,y)}$, it will be continuous at (a,b) as long as $g(a, b) \neq 0$

For a vector function $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v} \dots F_m(\vec{v}))$, where $\vec{v}$ is a vector or $\mathrm{I\!R^n}, (\mathrm{x_1, x_2 \dots x_n})$, is a continuous function at vector $\vec{n}$ if all its component functions are continuous at $\vec{n}$
Example:

$$f(x, y, z) = (e^{x+y},\ z,\ \cos(x^2 - z)) \text{ continuous everywhere} \tag{11}$$

$$\text{since} \begin{cases} e^{x+y} \text{ continuous} \\ z \text{ continuous} \\ \cos(x^2 - z) \text{ continuous} \end{cases} \tag{12}$$

2.3 Partial Derivatives

The derivative of a function is the rate of change along the x axis. The partial derivatives of $F(x, y)$ would be the rate of change of F as one of the axis is fixed (along one axis).

$$\frac{dF(x, y)}{dx} = F_x(x, y) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} \tag{13}$$

$$\frac{dF(x, y)}{dy} = F_y(x, y) = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} \tag{14}$$

In order to compute the derivative, we treat all other variables as constants.
Example:

$$f(x, y, z) = xe^{xy} - \sin(y^2 + z^2) \tag{15}$$

$$\frac{df}{dx} = e^{xy} + yxe^{xy} - 0 = e^{xy} + yxe^{xy} \tag{16}$$

$$\frac{df}{dy} = x^2 e^{xy} - \cos(y^2 + z^2)(2y)\frac{df}{dz} = 0 - \cos(y^2 + z^2)(2z) \tag{17}$$

Take a point (a, b) for example

1. $\frac{df}{dx}$ = gradient of tangent at that point along x axis

2. $\frac{df}{dy}$ = gradient of tangent at that point along y axis

3. These two direction can span the tangent plane of f at that point
   (a) Tangent Vector
   (b) along x direction $= (1, 0, \frac{df}{dx}(a, b)) = \vec{v}$
   (c) along y direction $= (0, 1, \frac{df}{dy}(a, b)) = \vec{w}$
   (d) Using these vectors we can find the plane's equation

Example, find the tangent plane of $f(x,y)$ at (1,1):

$$f(x,y) = 1 - x^2 - 2y^2 \tag{18}$$
$$f(1,1) = z = -2, \text{ point at } (1,1,-2) \tag{19}$$
$$\frac{df}{dx} = (1,0,-2x|_{(1,1)}) = (1,0,-2) \tag{20}$$
$$\frac{df}{dy} = (1,0,-4y|_{(1,1)}) = (0,1,-4) \tag{21}$$
$$(1,0,-2) \times (0,1,-4) = (2,4,1) \tag{22}$$
$$2x + 4y + z = 4 \text{ a linear approximation of } f(x,y) \text{ at } (1,1) \tag{23}$$

Definition of a vector function:

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathrm{I\!R}^{\mathrm{m}} \tag{24}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{25}$$

The function F transforms$\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$, DF $\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ is a linear map, a linear approximation of F.
Example:

$$F(x,y,z) = (e^{x+yz}, x^2 + 1, \sin(y+z), 4y) \tag{26}$$
$$DF(x,y,z) = \begin{pmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y+z) & \cos(y+z) \\ 0 & 4 & 0 \end{pmatrix} DF(1,1,2) = \tag{27}$$
$$DF(1,1,2) = \begin{pmatrix} e^3 & 2e^3 & e^3 \\ 2 & 0 & 0 \\ 0 & \cos(3) & \cos(3) \\ 0 & 4 & 0 \end{pmatrix} \tag{28}$$

Matrix of numbers, a linear map

$$F(x,y,z) = (e^{x+yz}, x^2 + 1, \sin(y+z), 4y) \tag{29}$$

2.4 Properties of Derivatives DF

Given two vector function F and G: $\mathrm{I\!R}^{\mathrm{n}} \to \mathrm{I\!R}^{\mathrm{m}}$ both differentiable at some point $\vec{x}$

$$D(f \pm G) = D(F) \pm D(G) \tag{30}$$
$$C \in \mathrm{I\!R}, \mathrm{D(cF) = cD(F)} \tag{31}$$

suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ a function in n variables, product rule still holds

$$D(f \cdot g) = D(F) \cdot g + f \cdot D(G) \tag{32}$$

where $D(f \cdot g)$ is a n^{th} dimension vector, D(f or g) is a vector function and f or g is a scalar function

$$D(\frac{f}{g}) = \frac{g \cdot D(F) - f \cdot D(G)}{g^2}, g(\vec{x}) \neq 0 \tag{33}$$

Given two vector function $\vec{v}(t)$ and $\vec{w}(t)$: $\mathbb{R} \to \mathbb{R}^n$ so $t \to \vec{v}(t)$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t) \text{scalar function} \tag{34}$$
$$f'(t) = \vec{v}(t)' \cdot \vec{w} + \vec{w}(t)' \cdot \vec{v} \tag{35}$$
$$u(t) = \vec{v}(t) \times \vec{w}(t) \text{vector function} \tag{36}$$
$$u'(t) = \vec{v}(t)' \times \vec{w}(t) + \vec{w}(t)' \times \vec{v}(t) \tag{37}$$
$$\vec{v}(t) = \langle v_1(t), v_2(t), \ldots v_n(t) \rangle \tag{38}$$

Chain rule, suppose $f : \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $g \cdot f : \mathbb{R}^m \to \mathbb{R}^p$, $\vec{x} \in \mathbb{R}^m$

$$D(g \circ f)(\vec{x})(\text{size } p \times m) = DG(F(\vec{x}))(\text{size } p \times n) \cdot DF\vec{x}(\text{size } n \times m) \tag{39}$$
$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{40}$$

Example:

$$f(x, y) = (x^3 + y, e^{xy}, 2 + xy)\mathbb{R}^2 \to \mathbb{R}^3 \tag{41}$$
$$g(y, v, w) = (y^2 + v, uv + w^3)\mathbb{R}^3 \to \mathbb{R}^2 \tag{42}$$
$$g(f(x, y)) = g \circ f = ((x^3 + y)^2 + e^{xy},\ (x^3 + y)e^{xy} + (2 + xy)^3) \tag{43}$$

$$D(g \circ f) = \begin{pmatrix} 2(x^3 + y)(3x^2) + ye^{xy} & 2(x^3 + y) + xe^{xy} \\ \begin{pmatrix} 3x^2 e^x y + (x^3 + y)ye^{xy} \\ +3(2 + xy)^2 y \end{pmatrix} & \begin{pmatrix} e^{xy} + (x^3 + y)xe^{xy} \\ +3(2 + xy)^2 x \end{pmatrix} \end{pmatrix} \tag{44}$$

$$DG = \begin{pmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{pmatrix} \tag{45}$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \tag{46}$$

$$DG = \begin{pmatrix} 2(x^3 + y) & 1 & 0 \\ e^{xy} & x^3 + y & 3(2 + xy)^2 \end{pmatrix} \times \begin{pmatrix} 3x & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{pmatrix} \tag{47}$$

Check that the equality holds.

2.5 Directional derivatives

Recall the Jacobian matrix, in the following context, the function F is not necessarily defined on the entire $\mathrm{I\!R^n}$, but rather a subset $\subseteq \mathrm{I\!R^n}$

$$F(x_1, x_2 \ldots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \ldots F_m(\vec{x})) \in \mathrm{I\!R^m} \tag{48}$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F(Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \cdots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \cdots & \frac{dF_2}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \cdots & \frac{dF_m}{dx_n} \end{pmatrix} \tag{49}$$

Examples: 1)

$$f(x, y, z) = x^2 y + \ln z : \mathrm{I\!R}^3 \to \mathrm{I\!R}^1 \tag{50}$$

$$Df = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz})$$

$$= (2x, 1, \frac{1}{z}) \tag{51}$$

2)

$$\vec{v}(t) = (\cos(t), \sin(t)); \text{ vector function} \tag{52}$$

$$D\vec{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = (-\sin(t), \cos(t))^T \tag{53}$$

$$D\vec{v}(t) \cdot \vec{v}(t) = -\sin(t)\cos(t) + \cos(t)\sin(t) = 0 \tag{54}$$

$$D\vec{v}(t) \perp \vec{v}(t); \text{ hence a tangent vector} \tag{55}$$

3)

$$g(t) = e^t \text{ scalar function} \tag{56}$$

$$g(t) \cdot \vec{v}(t) = e^t(\cos(t), \sin(t)) = (e^t\cos(t), e^t\sin(t)) \tag{57}$$

$$D(g(t) \cdot \vec{v}(t)) = \begin{bmatrix} e^t\cos(t) - e^t\sin(t) \\ e^t\cos(t) + e^t\sin(t) \end{bmatrix} \tag{58}$$

4)

$$\vec{v}(t) = (t, \sin(t), \cos(t)), \vec{w}(t) = (3t, 0, 2) \tag{59}$$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t)$$

$$= 3t^2 + 0 + 2\cos(t); \text{ scalar function} \tag{60}$$

$$D(f(t)) = 6t - 2\sin(t) \tag{61}$$

$$\vec{w} \cdot D\vec{v} + \vec{v} \cdot D\vec{w} = \begin{bmatrix} 3t \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ +\cos(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} t \\ \sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \tag{62}$$

$$= 6t - 2\sin(t) \tag{63}$$

2.6 Partial derivative along a direction

Definition: Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b), given a unit vector $\vec{u} = (u1, u2) \in \mathbb{R}^2$ then the directional derivatives of F along $\vec{u}$ is:

$$D_{\vec{u}}F(a,b) = \left[\frac{d}{dt}f((a,b)+t\vec{u})\right]_{t=0} \tag{64}$$

$$= \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \tag{65}$$

is the rate of change of F along the direction of $\vec{u}$
Two special cases would be when u are along the typical x or y direction, which would mean the class derivative along x-axis $\frac{df}{dx}$ or the y-axis $\frac{df}{dy}$
Theorum $\vec{u} = (u_1, u_2)$ unit vector,

$$D_{\vec{u}}f(x,y) = (\frac{df}{dx}, \frac{df}{dy})\dot{\vec{u}} = (u_1, u_2) \tag{66}$$

$$\text{since } \vec{u} = u_1(1,0) + u_2(0,1) \tag{67}$$

$$D_{\vec{u}}f = u_1\frac{df}{dx} + \frac{df}{dy} \tag{68}$$

Proof:

$$D_{\vec{u}}f = \lim_{h\to 0} \frac{F(a+hu_1, b+hu_2) - F(a,b)}{h} \tag{69}$$

$$= \lim_{h\to 0} \frac{(F(a+hu_1, b+hu_2) - F(a, b+hu_2))}{h} \tag{70}$$

$$+ \lim_{h\to 0} \frac{F(a, b+hu_2) - F(a,b)}{h} \tag{71}$$

$$= \lim_{u_1 h\to 0} \frac{[F(a+hu_1, b+hu_2) - F(a, b+hu_2)] \cdot u_1}{hu_1} \tag{72}$$

$$+ \lim_{h\to 0} \frac{F(a, b+hu_2) - F(a,b)}{h} \tag{73}$$

$$= \frac{df}{dx}(a, b+0) \cdot u_1 + \frac{df}{dy} \cdot u_s \tag{74}$$

Example: Find the directional Deriv of $f(x,y) = x^2 + 3xy$ along the direction of $(3,4)$ at the point $p=(2,-1)$

$$\begin{aligned}
\vec{u} =& \frac{(3,4)}{\sqrt{(3^2+4^2)}} = (\frac{3}{5}, \frac{4}{5}) && (75)\\
D_{\vec{u}}f(2,-1) =& (2x+3y, 3x)|_{(2,-1)} \cdot (\frac{3}{5}, \frac{4}{5}) && (76)\\
=& (1,6) \cdot (\frac{3}{5}, \frac{4}{5}) && (77)\\
=& \frac{27}{5} && (78)
\end{aligned}$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = (\frac{df}{dx}, df dy) \cdot \vec{u} \tag{79}$$

Definition: Given $F: \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$, then define the gradient.

$$\nabla f = D_{\vec{u}}f = (\frac{df}{x_1}, \frac{df}{x_2}, \ldots, \frac{df}{x_n}) \tag{80}$$

Hence the directional derivative of a function along a direction is:

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} \tag{81}$$

The consequence of this:
1)

$$D_{\vec{u}}f \text{ max when } \vec{u} /\!/ \nabla f = (\frac{df}{dx}, \frac{df}{dy}) \tag{82}$$

since

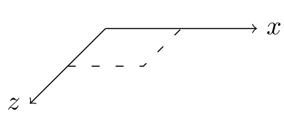
$$\begin{aligned}
\nabla f \cdot \vec{u} =& |\nabla f||\vec{u}| \cos(\theta) && (83)\\
=& |\nabla f| \cdot 1 \cdot \cos(\theta)\theta \ = \ ang(\nabla f, \vec{u}) && (84)
\end{aligned}$$

When maximized, $/\!/$, $\theta = 0$

$$\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{\frac{df}{dx}^2 + \frac{df}{dy}^2} \tag{85}$$

2)

$$\text{When } \vec{u} \perp \nabla f, D_{\vec{u}}f = 0 \tag{86}$$



Example:$f(x,y) = e^{-(x^2+y^2)}$, find the direction along this function which has the largest rate of increase/decrease at point $(1,1)$

Since at maximum, $\vec{u} /\!/ \nabla f$

$$
\begin{aligned}
\nabla f =& (-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)})|_{(1,1)} && (87)\\
=& (-2e^{-2}, -2e^{-2}) && (88)\\
\because \quad & \vec{u} /\!/ \nabla f /\!/ (1,1) && (89)\\
\therefore \quad & \vec{u} = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) && (90)
\end{aligned}
$$

Largest rate of increase at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
Largest rate of decrease at $-(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
The gradient ∇f tells us the direction along which F changes the most.

2.7 Summary

Given a function $f(x, y)$	**n-th dim analogue**
Define Partial derivatives(deriv) $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f(x_1, x_2, \dots x_n)$
Define gradient $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f : \mathbb{R}^n \to \mathbb{R}$
Directional deriv: given **unit** vector $\vec{u} = (u_1, u_2)$	$\frac{df}{dx_1} \cdots \frac{df}{dx_n}$
$D_{\vec{u} f = \lim_{h \to 0} \frac{f((a,b)+h\vec{u}) - f(a,b}{h}}$	$\nabla f = (\uparrow)$
if$(\vec{u} = (1,0))$, $D_{\vec{u}} f = \frac{df}{dx}$	$\vec{u} f \in \mathbb{R}^n$
if$(\vec{u} = (0,1))$, $D_{\vec{u}} f = \frac{df}{dy}$	$D_{\vec{u}} f =?$
Theorem:$D_{\vec{u}} = \nabla f \cdot \vec{u} = u_1 \frac{df}{dx} + u_2 \frac{df}{dy}$	generalizations?
Remark: We can define $D_{\vec{u}}$ for any $\vec{v} \in \mathbb{R}^2$	
in the exact same way	
$D_{\vec{u}} = \lvert\vec{v}\rvert D_{\frac{\vec{u}}{\lvert\vec{v}\rvert}} f$	

To each unit vector $\vec{u}$, we have a tangent vector associated to it:

$$
(u1, u2, D_{\vec{u}} f) = u_1(1, 0, \frac{df}{dx}) + u_2(0, 1, \frac{df}{dy}) \tag{91}
$$

Two special cases: $(1, 0, \frac{df}{dx})(0, 1, \frac{df}{dy})$
The two tangent vector span a tangent plane at $(a, b, f(a, b))$

2.8 Gradients and level sets

Recall

$$
\begin{aligned}
\nabla f =& (\frac{df}{dx}, \frac{df}{dy}) && (92)\\
\because D_{\vec{u}} =& \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos(\theta) && (93)
\end{aligned}
$$

$D_{\vec{u}}$ is the rate of change along $\vec{u}$

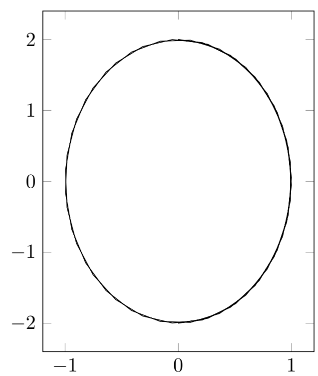
if $u /\!/ \nabla f$, $(\theta = 0)$ then $D_{\vec{u}}F$ max

$$|D_{\vec{u}}F| = \sqrt{(\frac{df}{dx})^2 + (\frac{df}{dy})^2} \tag{94}$$

In other words, along the direction of ∇F, the function F changes the most.
Definition: A level set of F(x,y) is:

$$(x, y) \in \mathbb{R}^2 : \text{f(x, y) = c} \tag{95}$$

Example:



$$f(x, y) = x^2 + \frac{y^2}{4} \tag{96}$$

$$\{f(x, y) = 1\} \text{ (Level set)} \tag{97}$$

$$= \{(x, y) | x^2 + \frac{y^2}{4} = 1\} \tag{98}$$

Theorem: Given $f(x, y)$ differentiable, then the gradient $\nabla f \perp$ the level set.
We can prove it by using the chain rule: let $\gamma(t)$ be a parametric function of my level set $\{f(x, y) = 1\}$, $f(\gamma(t)) = 1$ and differentiate both sides.

$$0 = D(f(\gamma(t))) = Df(\gamma(t))D(\gamma(t)) \tag{99}$$

$$= \nabla F(\gamma(t))_{1\times 2} \cdot (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{100}$$

$$\text{Hence } \Rightarrow \nabla F(\gamma(t))_{1\times 2} \perp (\gamma_1'(t), \gamma_2'(t))_{2\times 1} \tag{101}$$

Example: Look at a surface $\sin(xy) - 2\cos(yz) = 0$ Find the tangent plane of this surface at $(\frac{\pi}{2}, 1, \frac{\pi}{3})$. The idea is that consider the surface as a level set of $F(x, y, z) = \sin(xy) - 2\cos(yz), \{F = 0\}$
Using the Theorem: ∇F is hence perpendicular to this surface, giving us the normal direction of our tangent plane.

$$\nabla F = (\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}) = (y\cos(xy), x\cos(xy) + 2z\sin(yz), 2y\sin(tz)) \tag{102}$$

$$= (\cos(\frac{\pi}{2}), \frac{\pi}{2}\cos(\frac{\pi}{2}) + \frac{2\pi}{3}\sin(\frac{\pi}{3}), 2\sin(\frac{\pi}{3})) \tag{103}$$

$$= (0, \frac{\sqrt{3}\pi}{3}, \sqrt{3}) \tag{104}$$

$$0 = 0 \cdot (x - \frac{\pi}{2}) + \frac{\sqrt{3}\pi}{3}(y - 1) + \sqrt{3}(z - \frac{\pi}{3}) \tag{105}$$

Where the last equation above is the equation of the tangent plane.

2.9 Parametric Curves

...are vector functions in 1 variable.
Definition: A parametric curve in $\mathbb{R}^n$ is a vector valued function where $\vec{c}(t) : R^1 \to \mathbb{R}^n$
All the component functions of $\vec{c}(t)$ are functions in 1 variable.
Example:
1) Draws a Circle

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \tag{106}$$

2) Draws a eclipse at plane z = 3

$$\vec{c}(t) = (a\cos(t), b\sin(t), 3), t \in [0, 2\pi] \tag{107}$$

3) Eclipse spiraling up

$$\vec{c}(t) = (a\cos(t), b\sin(t), t), t \in [0, 2\pi] \tag{108}$$

Definition: Orientation of $\vec{c}(t)$, the direction corresponding to increasing t-value is called the positive orientation of $\vec{c}(t)$. The opposite direction is the negative orientation.
Example: Positive orientation = Clockwise

$$\vec{c}(t) = (\cos(t), -\sin(t), t), t \in [0, 2\pi] \tag{109}$$

Derivative of $\vec{c}(t)$
If $\vec{c}(t) = (x(t), y(t), z(t))$, $\vec{c'}(t) = (x'(t), y'(t), z'(t))$

1. $\vec{c}(t)$ is a vector function
2. $\vec{c'}(t)$ is a tangent to curve $\vec{c}(t)$

Terminology: $\vec{c'}(t)$ is the velocity of $\vec{c}(t)$, and $|\vec{c'}(t)|$ is the speed of $\vec{c}(t)$. The speed of the curve $\vec{c}(t) = (\cos(t), \sin(t))$ for example, is 1.

What is the length of curve $\vec{c}(t) = (\cos(t), \sin(t), t)$ between (0 to 2π)Given $\vec{c}(t)$ differentiable between range a and b, length of the curve would be

$$l = \int_a^b |\vec{c'}(t)| dt \tag{110}$$

$$|\vec{c'}(t)| = \sqrt{\sin(t)^2 + \cos(t)^2 + 1} = \sqrt{2} \tag{111}$$

$$l = \int_0^{2\pi} \sqrt{2} dt \tag{112}$$

Think of length as a function in t

$$l(t) = \int_a^t |\vec{c'}(x)| dx \tag{113}$$

If curve $\vec{c}(t)$ differentiable and $\vec{c}(t) \neq 0$,

1. $|\vec{c'}(t)| > 0$
2. $l(t)$ strictly increasing.
3. l is one to one function so has inverse.

$$l[a,b] \rightarrow \mathbb{R}^1, \text{ l has an inverse.} \tag{114}$$
$$t \rightarrow l(t), \text{ think of our parameter t} \tag{115}$$
$$\tag{116}$$

t can now be expressed as a function in terms of length, $\vec{c}(t) = \vec{c}(t(l)) = \vec{c'}(l)$

Proof:
In our last example of calculating length, we can find an inverse for $l(t) = \sqrt{2}t$ resulting $t = \frac{l}{\sqrt{2}}$, by substitution we can reparameterize $\vec{c}(t)$ into $\vec{c}(l)$

$$c(l) = \vec{c}(t(l)) \tag{117}$$
$$\frac{d\vec{c}(l)}{dl} = \frac{\vec{c}(t(l))}{dt} \cdot \frac{dt}{dl} \tag{118}$$
$$= \vec{c'}(t) \cdot \frac{1}{|\vec{c'}(t)|} \tag{119}$$
$$|\frac{d\vec{c}(l)}{dl}| = |\vec{c'}(t)| \cdot \frac{1}{|\vec{c'}(t)|} = 1 \tag{120}$$

2.10 Acceleration

Acceleration is the second derivatives of length, hence the definition of second derivative can be expressed as below:

Definition: Given $\vec{c}(t)$, $\vec{a}(t) = \vec{c''}(t)$ the acceleration of $\vec{c}$.

Example:
1) Ellipse

$$\vec{c_1}(t) = (\cos(t), 2\sin(t)), t \in [0, 2\pi] \tag{121}$$
$$\vec{c_1'}(t) = (-\sin(t), 2\cos(t)) \tag{122}$$
$$\vec{c_1''}(t) = (-\cos(t), -2\sin(t)) \tag{123}$$

2) Different para of the same Ellipse

$$\vec{c_2}(t) = (\cos(t^2), 2\sin(t^2)), t \in [0, \sqrt{2\pi}] \tag{124}$$
$$\vec{c_2'}(t) = 2t(-\sin(t^2), 2\cos(t^2)) \tag{125}$$
$$\vec{c_2''}(t) = (-2\cos(t^2) - 4t^2 cos(t^2), 4cos(t^2) - 8t^2\sin(t^2)) \tag{126}$$

Facts:

1. We can have different parameterations of a curve
2. Velocity, acceleration depend on the para we choose in general
3. For different parameterization $\vec{c_1}$, $\vec{c_2}$ of the same curve, $\vec{c_1'}(t) /\!/ \vec{c_2'}(t)$
4. "Unit Velocity" $\frac{\vec{c_1'}(t)}{|\vec{c_1'}(t)|} = \pm\frac{\vec{c_2'}(t)}{|\vec{c_2'}(t)|}$ are equal, the $\pm$ depends on the orientations of the $\vec{c_1}$ and $\vec{c_2}$.

Definition:Given $\vec{c}(t)$, assume $|\vec{c'}(t)| \neq 0$, defined the unit tangent vector $\vec{T}(t) = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$. If $\vec{c_1}(t)$ and $\vec{c_2}(t)$ are two different parameter of a curve with the same orientation, their unit tangent vector equals.

2.11 Curvature

Idea: Curvature is a measure of how sharply a curve is bending at a point, it is the "rate of change" of directions along a curve. So a straight line has no curvature, and a slightly curved line has a small curvature.

Definition: Quantitatively, if $\vec{c}(t)$ is parametered by length, (namely, $|\vec{c}(t)| = 1$), then we define the curvature at $l = l_0$ to be $k(l_0) = |\vec{c''}(l_0)| = |\frac{d^2\vec{c}(l)}{dl^2}|_{l=l_0}$

Remark: IF $\vec{c}(t)$ not parametered by length, then $|\vec{c''}(t)|$ is not the curvature.

Explanation: a curve is like a highway and a parameterized function $\vec{c}(t)$ is like someone driving along this highway giving his position vector at a given time. Parameterized by length would be driving at a constant/normalized speed. A curvature is how sharp a turn is, measured by looking at how quickly the driver need to turn their steering wheel.

Generally, $|\vec{c'}(l)| = 1 \Rightarrow \vec{c'}(l)$ unit tangent, so $\vec{c''}(t)$ is curvature.

Example:
1)Find parameter by length

$$\vec{c}(t) = \vec{a} + t\vec{v} \tag{127}$$

$$\text{Constant vectors: } \vec{a} = (1,2,3), \vec{v} = (0,2,1)$$

$$\text{Length function } l(t) = \int_0^t |\vec{c'}(x)| dx \tag{128}$$

$$= \int_0^t |\vec{v}| dx, \quad \vec{v} = (0,2,1) \tag{129}$$

$$l = l(t) = \sqrt{5}t \tag{130}$$

$$t = t(l) = \frac{l}{\sqrt{5}} \tag{131}$$

Curvature would be zero since it is a line, its second derivative would be a zero.

2) Find the curvature for a circle

$$\vec{c}(t) = (r\cos(t), r\sin(t)), r > 0 \text{ circle } x^2 + y^2 = r^2 \tag{132}$$

$$\text{length function } \; l(t) = \int_0^t |\vec{c'}(x)| dx = rt \tag{133}$$

$$t(l) = \frac{l}{t} \tag{134}$$

$$\vec{c}(l) = (r\cos(\frac{l}{r}), r\sin(\frac{l}{r})) \tag{135}$$

$$\vec{c''}(l) = (-\frac{\cos(l/r)}{r}, -\frac{\sin(l/r)}{r}) \tag{136}$$

$$k(l) = |\vec{c''}(l)| \tag{137}$$

$$= \sqrt{(-\frac{\cos(l/r)}{r})^2 + (-\frac{\sin(l/r)}{r})^2} = \frac{1}{r} \tag{138}$$

Curvature for circle of radius $r = \frac{1}{r}$

Second Definition of Curvature:
If $\vec{c}(t)$ any parameteric curve, $\vec{c'}(t) \neq 0$, recall the unit tangent vector $\vec{T} = \frac{\vec{c'}(t)}{|\vec{c'}(t)|}$, then define curvature $k(t) = \frac{|\vec{T'}(t)|}{|\vec{c'}(t)|}$
Theorem: Definition 1 and 2 of curvature are the same.

$$\vec{c'}(t) = \frac{d\vec{c}(t(l))}{dt} \tag{139}$$

$$= \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} \tag{140}$$

$$= \frac{\vec{c'}(t)}{|\vec{c}(t)|} = \vec{T} \tag{141}$$

$$k(l(t)) = |\vec{c''}(t)| = |\frac{d^2\vec{c}}{dl^2}| = |\frac{d\vec{T}(t(l))}{dl}| \tag{142}$$

$$= \frac{d\vec{c}(t)}{dt} \cdot \frac{dt}{dl} \tag{143}$$

$$= |T'(t)| \cdot \frac{1}{|\vec{c'}(t)|} = k(t) \tag{144}$$

Using the second definition to find the curvature of the parabola $y = x^2$

$$\begin{aligned}
\vec{c} =& (t, t^2) && (145)\\
\vec{c'}(t) =& (1, 2t) && (146)\\
\vec{T}(t) =& \frac{(1, 2t)}{\sqrt{1+4t^2}} && (147)\\
\vec{T'}(t) =& \frac{(0, 2)}{\sqrt{1+4t^2}} + (1, 2t) \cdot ((a+4t^2)^{-\frac{1}{2}})' && (148)\\
=& \frac{(-4t, 2)}{(1+4t^2)^{\frac{3}{2}}} && (149)\\
|T'(t)| =& \frac{\sqrt{4+16t^2}}{(1+4t^2)^{\frac{3}{2}}} && (150)\\
=& \frac{2}{1+4t^2} && (151)\\
K(t) =& |T'(t)| \cdot \frac{1}{|c'(t)|} && (152)\\
=& \frac{2}{(1+4t^2)} \cdot \frac{1}{\sqrt{1+4t^2}} && (153)\\
=& 2 \cdot (1+4t^2)^{-\frac{3}{2}} && (154)
\end{aligned}$$

As x approaches ∞, the curvature approaches 0, $y = x^2$ became more like a straight line.

Osculating Plane and Circle

1. Derivative — Tangent line
2. Curvature — Osculating Circle

Observations:
1) If $\vec{c}(l)$ parameterized by length, then

$$\begin{aligned}
\vec{c'}(l) \perp& \vec{c''}(l) && (155)\\
\because (\vec{c'}(l) \cdot \vec{c'}(l))' =& (|\vec{c}(l)|^2)' = (1)' && (156)\\
\vec{c'}(l) \cdot \vec{c''}(l) + \vec{c'}(l) \cdot \vec{c''}(l) =& 0 && (157)\\
\Rightarrow \vec{c'}(l) \cdot \vec{c''}(l) =& 0 && (158)
\end{aligned}$$

so second derivative is the normal and the first derivative is the tangent.

2) If $\vec{c}$(t) is any parameterized curve, its unit tangent vector T and T' is

$$\vec{T}(t) = \frac{\vec{c'}(t)}{|\vec{c'}(t)|} \tag{159}$$

$$\because (\vec{T}(t) \cdot \vec{T}(t))' = (|\vec{T}(t)|^2)' = (1)' \tag{160}$$

$$\Rightarrow \vec{T}(t) \cdot \vec{T}'(t) = 0 \tag{161}$$

$$\therefore \vec{T}(l) \perp \vec{T}'(l) \tag{162}$$

Definition: Osculating Plane: Given $\vec{c}(t)$, $\vec{c}(t) \neq 0$, then the osculating plane is the plane spanned by $\vec{T}(t)$ and $\vec{T}'(t)$

Recall that we computed for a circle with radius r its curvature $k = \frac{1}{r}$, therefore, if have a $\vec{c}(t)$ with $k(t_0) = k$, then at this point, the curve $\vec{c}(t)$ is "modeled" by a circle of radius $\frac{1}{k}$

Definition: Osculating Circle: Given $\vec{c}(t), \vec{c'}(t) \neq 0$, then the osculating circle of $\vec{c}$ at $t = t_0$ is a circle on the osculating plane at $t = t_0$ and has radius $\frac{1}{k(t_0)}$, and the tangent to the curve. (The best approximation of the curve at that point by the circle)

Example: Find the equation of the osculating plane at $\vec{c}(\pi) = (-a, 0, b\pi)$ of $\vec{c}(t) = (a\cos(t), a\sin(t), bt)$, $t \geq 0$; $a, b > 0$,

$$\vec{T} = \frac{\vec{c'}(t)}{|\vec{c'}(t)|} = \frac{(-a\sin(t), a\cos(t), b)}{\sqrt{a^2+b^2}} \tag{163}$$

$$\vec{T'} = \frac{(-a\cos(t), -a\sin(t), 0)}{\sqrt{a^2+b^2}} \tag{164}$$

$$\text{Osc.Plane} = \vec{T} \times \vec{T'} \tag{165}$$

$$= \frac{1}{a^2+b^2}(ab\sin(t), -ab\cos(t), a^2)|_{t=\pi} \tag{166}$$

$$= \frac{1}{a^2+b^2}(0, ab, a^2) \tag{167}$$

calculating plane at given point

$$a^2 b\pi = 0x + aby + a^2 z \tag{168}$$

2.12 Higher order derivatives and Taylor expansions

Recall partial derivatives of $F(x, y)$ by x and y denoted by F_x and F_y, second order derivative = F_{xx} and F_{yy}, first x then y F_{xy} and first y then x F_{yx}, vise versa.

Theorem: If $F(x, y)$ differentiable and F_{xy}, F_{yx} continuous, $F_{xy} = F_{yx}$, hence the order can be switched

Example: $f(x,y,z) = x^2 e^{y^2-x} + x^2yz^2 - cos(x^2+y^2)$, f_{xzzy} =? switch order, try finding f_{zzxy} instead

$$df/dz = f_z = 2zx^2y \Rightarrow fzz = 2x^2y \Rightarrow f_{zzxy} = 4x \tag{169}$$

2.13 Taylor Expansions

One Variable: $y = f(x)$, $f'(x) =$, tangent line of graph of the function, its linear approximation at point $x = x_0$, equation of tangent line is $y = f(x_0) + f'(x_0)(x - x_0)$. Higher order $f'(x) \to$ higher degree approximation.

$$\begin{aligned}
f(x) - f(x_0) = & \int_{x_0}^{x} f'(t)dt && (170)\\
\int u'v = & uv - \int v'u,\ u = t, v = f'(t) && (171)\\
= & tf'(t)|_{x_0}^{x} - \int_{x_0}^{x} tdf'(t) \text{ by parts} && (172)\\
= & [xf'(x) - x_0 f'(x_0)] - \int_{x_0}^{x} tf''(t)dt && (173)\\
= & (xf'x - xf'(x_0)) + (xf'x_0 - x_0 f'(x_0)) - \int_{x_0}^{x} tf''(t)dt && (174)\\
= & (x - x_0)f'(x_0) + x(f'(x) - f'(x_0)) - x\int_{x_0}^{x} tf''(t)dt && (175)\\
= & f'(x_0)(x - x_0) + x\int_{x_0}^{x} f''(t)dt - \int_{x_0}^{x} tf''(t)dt && (176)\\
f(x) = & f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} (x - t)f''(t)dt && (177)\\
& && (178)
\end{aligned}$$

This is the 1 st Taylor expansion where $f(x_0) + f'(x_0)(x - x_0)$ is the 1st order approximation, a tangent line, and $\int_{x_0}^{x}(x-t)f''(t)dt$ is the 1st order remainder. error/diff. to linear approx.

We can continue to get a higher order Taylor expansion using integration by parts by setting $v = f''(t)$ and $u' = (x - t)$ which would yield:

$$\begin{aligned}
f(x) = & f(x_0) + f'(x_0)(x - x_0) \qquad (179)\\
& + f''(x_0) \cdot \frac{(x - x_0)^2}{2}\\
& + \int_{x_0}^{x} (x - t)f'''(t) \cdot \frac{(x - t)^2}{2} dt
\end{aligned}$$

Theorem: if $f(x)$ differentiable, then $\frac{d^n(f(x))}{dx^n}$ exists for all n. Hence the n-th order of the Taylor expansion would be:

$$\begin{aligned} f(x) =& f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)}{2!} \dots \\ &+ f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \int_{x_0}^{x} (x-t) f^{(n+1)}(t) \cdot \frac{(x-t)^n}{n!} dt \end{aligned} \tag{180}$$

Example:
Find the 2nd order Taylor expansion at $x = 0$ of $f(x) = x + e^{-2x}$ USe that to approximate the value $f(0.075) = 0.075 + e^{-0.15}$ and estimate the error.

$$\begin{aligned} f'(x) =& (1 - 2e^{2x}), f''(x) = 4e^{-2x} \\ f(x) =& f(0) + f'(0)(x-0) \\ &+ \frac{f''(0)(x-0)^2}{2} \\ &+ \int_0^x f'''(t)\frac{(x-t)^2}{2} dt \\ f(0.075) \approx& 1 - (0.075) + 2(0.075)^2 \end{aligned} \tag{181}$$

Error given y the remainder $R(x)$, a bound for the error

$$\text{error} \leq |R(x)| \leq \left| \int_0^x -8e^{2t} \cdot \frac{(x-t)^2}{2} dt \right| \tag{182}$$

$$\leq 4x^2(x-0) = 4x^3 = (4 \cdot (0.075)^3) \tag{183}$$

For 2 dimension, the second order expansion of $F(x, y)$ at (x_0, y_0) is:

$$\begin{aligned} F(x,y) =& F(x_0, y_0) \\ &+ \frac{1}{1!}\left[\frac{\sigma F}{\sigma x}(x_0, y_0)(x - x_0) + \frac{\sigma F}{\sigma y}(x_0, y_0)(y - y_0)\right] \\ &+ \frac{1}{2!}\left[\frac{\sigma^2 F}{\sigma x^2}(x_0, y_0)(x - x_0) + \frac{\sigma^2 F}{\sigma y^2}(x_0, y_0)(y - y_0) \right. \\ &+ \left. 2\frac{\sigma^2 F}{\sigma x \sigma y}\right] + R_2(x,y) \end{aligned} \tag{184}$$

Example: Find the second Taylor expansion of $f(x, y) = sin(x+y)+cos(x-3y)$

at (0,0)

$$f_x = \cos(x+y) - \sin(x-3y) \tag{185}$$
$$f_y = \cos(x+y) + 3\sin(x-3y) \tag{186}$$
$$f_{xx} = -\sin(x+y) - \cos(x-3y) \tag{187}$$
$$f_{xy} = -\sin(x+y) + 3\cos(x-3y) \tag{188}$$
$$f_{yy} = -\sin(x+y) - 9\cos(x-3y) \tag{189}$$
$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y \tag{190}$$
$$+\frac{1}{2}\left[f_{xx}x^2 + f_{yy}y^2 + 2f_{x,y}xy\right] + R_2(x,y) \tag{191}$$
$$=1 + x + y + \frac{1}{2}(-x^2 - 9y^2 + 6xy) + R_2(x,y) \tag{192}$$

Second Approximation, Taylor Polynomial.

For extreme values of F(x,y): Recall that given y = f(x), differentiable, n-th Taylor expansion at $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \cdot \frac{(x - x0)^2}{2} \ \ldots \tag{193}$$
$$+ f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} \text{ Taylor polynomial} \tag{194}$$
$$+ \int_{x_0}^{x} f^(n+1)(t)\frac{(x-t)^n}{n!} dt \ R_n(x)\text{remainder} \tag{195}$$

Taylor Expansion for $F(x,y)$ at (x_0, y_0)

$$F(x,y) = F(x_0,y_0) + F_x(x_0,y_0)x + F_y(x_0,y_0)y \tag{196}$$
$$+ F_{xx}(x_0,y_0)\frac{x^2}{2} + F_{xy}(x_0,y_0)\frac{xy}{2} \tag{197}$$
$$+ F_{yx}(x_0,y_0)\frac{yx}{2} + F_{yy}(x_0,y_0)\frac{y^2}{2} \tag{198}$$
$$+ R_2(x,y) \tag{199}$$

Recall: for $y = f(x)$, critical points, $f'(x_0) = 0$ are candidates for local max and local min.

1. $f''(x_0) > 0$ local min
2. $f''(x_0) < 0$ local max
3. $f''(x_0) = 0$ inconclusive

if x_0 is critical point, first derivative term $f'(x_0) = 0$ removed from polynomial. The two variable are F(x,y). Definition F(x,y) differentiable, then say (x_0, y_0) is a critical point if $\nabla F(x_0, y_0) = 0$.

Recall equation of tangent plane

$$z = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(y - y_0) \tag{200}$$

Tangent plane is horizontal. Definition: for $F(x, y)$, define the Hessian matrix.

$$\begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix} (x_0, y_0) \tag{201}$$

Define the discrimanant, (x_0, y_0) is critical point.

$$D(x_0, y_0) = det \begin{pmatrix} F_{xx}(x_0, y_0) & F_{xy}(x_0, y_0) \\ F_{yx}(x_0, y_0) & F_{yy}(x_0, y_0) \end{pmatrix} (x_0, y_0) = F_{xx}F_{yy} - F_{xy}F_{yx} \tag{202}$$

Theorem (2nd derivative test): Suppose (x_0, y_0) is a critical point, then

1. If $D(x_0, y_0) > 0$ and $F_{xx} > 0$ or $(F_{yy} > 0)$, then (x_0, y_0) is a local minimum
2. If $D(x_0, y_0) > 0$ and $F_{xx} < 0$ or $(F_{yy} < 0)$, then (x_0, y_0) is a local maximum
3. if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.
4. if $D(x_0, y_0) = 0$, then it is inconclusive.

Find and classify all critical points of $F(x, y) = x^3 - 3x^2 - 3y^2 + 3xy^2$

$$\begin{aligned} F_x =& 3x^2 - 6x + 3y^2 = 0 && (203) \\ F_y =& -6y + 6xy = 0 && (204) \\ \Rightarrow \ y =& 0 \ or \ x = 1 \text{ giving two critical point} && (205) \\ y = 0, \Rightarrow& (0, 0), (2, 0) && (206) \\ x = 1, \Rightarrow& (1, 1), (1, -1) && (207) \end{aligned}$$

Use Second derivative to test for the four critical points

$$\begin{aligned} D(x, y) =& F_{xx}F_{yy} - F_{xy}^2 && (208) \\ D(1, 1) =& \Rightarrow \ \text{is a saddle point} && (209) \\ D(1, -1) =& \Rightarrow \ \text{is a saddle point} && (210) \\ D(0, 0) =& \Rightarrow \ \text{is local max} && (211) \\ D(2, 0) =& \Rightarrow \ \text{is local min} && (212) \end{aligned}$$

Why is this true?
From the second Taylor expansion, at a critical point, the dominant part:

$$F_{xx}(x_0, y_0)(x - x_0)^2 + 2F_{xy}(x_0, y_0)(x - x_0)(y - y_0) + F_{yy}(y - y_0)^2 \tag{213}$$

Homogeneous degree 2 polynomial in 2 variables:

$$\begin{aligned}
P(x,y) =& ax^2 + 2Cxy + by^2 && (214)\\
=& a(x^2 + \frac{2c}{a}xy + \frac{b}{a}y^2) && (215)\\
=& a((x - \frac{c}{a}y)^2 + \frac{ab-c^2}{a^2}y^2) && (216)\\
=& a((x - \frac{c}{a}y)^2 + \frac{ab-c^2}{a^2}y^2) && (217)\\
\Delta =& ab - c^2 > 0, \text{then} \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{ab-c^2}}{a}y \end{cases} && (218)\\
=& a((x')^2 + (y')^2) && (219)\\
& \Rightarrow a > 0 \text{ local min, } a < 0 \text{ local max} && (220)\\
\text{If } \Delta <& 0 \begin{cases} x' = x - \frac{c}{a}y \\ y' = x - \frac{\sqrt{c^2-ab}}{a}y \end{cases} && (221)\\
\text{then } P(x,y) =& P(x',y') = a((x')^2 - (y')^2) && (222)\\
a >& 0 \text{ saddle point} && (223)
\end{aligned}$$

Apply this to $F_{xx}(x_0,y_0)(x-x_0)^2 + 2F_{xy}(x_0,y_0)(x-x_0)(y-y_0) + F_{yy}(y-y_0)^2$

$$\begin{aligned}
a =& F_{xx}(x_0,y_0)(x-x_0)^2 && (224)\\
b =& F_{yy}(x_0,y_0)(x-x_0)^2 && (225)\\
2c =& 2F_{xy}(x_0,y_0)(x-x_0)^2 && (226)\\
ab - c^2 =& D(x,y) && (227)
\end{aligned}$$

Global Extreme Values for F(x,y).
Recall local max/min:

1. Find critical points $\nabla F(x_0,y_0) = \vec{0}$
2. Use second derivatives to test the critical points.

For Global Extreme values:

Example:

1) Find global min and max of $f(x,y) = x^3 - 3xy + 3y^2$ defined on the rectangle $D = \{(x,y)|0 \le x \le 1, 0 \le y \le 2$

First, find local critical point

$$f_x = 3x^2 - 3y = 0 \tag{228}$$
$$f_y = -3x + 6y = 0 \tag{229}$$
$$\Rightarrow x = 2y \tag{230}$$
$$\text{plug in to } f_x: \ 0 = 3(4y^2 - y) \tag{231}$$
$$0 = y(4y - 1) \tag{232}$$
$$\Rightarrow y = 0 \text{ or } y = \frac{1}{4} \tag{233}$$
$$\therefore (0,0) \ \& \ (0.5, 0.25) \text{ two critical point} \tag{234}$$
$$\text{Hessian Matrix } D(x,y) = det \begin{bmatrix} fxx & fxy \\ fxy & fyy \end{bmatrix} \tag{235}$$
$$= f_{xx} - f_{xy}^2 = 36x - 9 = 9(4x - 1) \tag{236}$$
$$\Rightarrow D(0,0) = -9 < 0, \text{ saddle, neither max/min} \tag{237}$$
$$\Rightarrow D(0.5, 0.25) = 9 > 0, \ f_{xx}(0.5, 0.25) = 3 > 0 \tag{238}$$
$$\tag{239}$$

Need to compare with boundary values.

$$\text{Side 1: } \{x = 0, 0 \le y \le 2\} \tag{240}$$
$$f(x,y) = x^3 - 3xy + 3y^2 | x = 0 \tag{241}$$
$$f(0,y) = 3y^2 \Rightarrow \text{ max 12 min 0} \tag{242}$$
$$\text{Side 2: } \{y = 0, 0 \le x \le 1\} \tag{243}$$
$$f(x,0) = x^3, \text{ max 1 min 0} \tag{244}$$
$$\text{Side 3: } \{x = 1, 0 \le y \le 2\} \tag{245}$$
$$f(1,y) = 1 - 3y + 3y^2, \text{ max 7 min 0.25} \tag{246}$$
$$\text{Side 4: } \{y = 2, 0 \le x \le 1\} \tag{247}$$
$$f(x,2) = x^3 - 6x + 12, \text{ max 12 min 7} \tag{248}$$

Hence global max 12 at (0,2) and -1/16 at (0.5,0.25)
2)Find for the function $f(x,y) = 2x^2 + 2y^2 - x + y$ on disk $D = \{(x,y) : x^2 + y^2 \le$

1} its global Max and min/min

$$f_x = 4x - 1, f_y = 4y + 1, \tag{249}$$
$$f_x = 0 \text{ and } f_y = 0 \to (x = 1/4, y = -1/4) \tag{250}$$
$$f_{xx} = 4, f_{yy} = 4, f_{xy} = 0 \tag{251}$$
$$D(x,y) = 16 - 0^2 = 16 > 0, f_{xx} = 4 > 0 \tag{252}$$
$$\text{see } (x = 1/4, y = -1/4) \text{ is local min} \tag{253}$$
$$\text{Parameterize the bounding (unit circle)}$$
$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \tag{254}$$
$$\text{plugin } f(\cos(t), \sin(t)) = 2 - \cos(t) + \sin(t),\ t \in [0, 2\pi] \tag{255}$$
$$f'(t) = \sin(t) + \cos(t) = 0 \tag{256}$$
$$-1 = \tan(t) \tag{257}$$
$$\therefore \text{critical points} t = \frac{3\pi}{4}, t = \frac{7\pi}{4} \tag{258}$$
$$f(t) = 2 - cos(t) + sin(t) \text{ at } 0\&2\pi = 1\&1 \tag{259}$$
$$f(\frac{3\pi}{4}) = 2 - \cos(\frac{3\pi}{4}) + \sin(\frac{3\pi}{4}) = 2 + \sqrt{2} \tag{260}$$
$$f(\frac{7\pi}{4}) = 2 - \sqrt{2} \text{ min on boundary} \tag{261}$$

Recall have local min at (0.25,-0.25)
Global max at $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $2 + \sqrt{2}$
Global max at $(\frac{1}{4}, -\frac{1}{4})$, $-\frac{1}{4}$
Optimization with constraint — Lagrangian multiplier.

Given f(x,y), find max/min with a constraint g(x,y) = k, k = a fixed number. The langrangian multiplier method would be:

$$\text{Step 1} \begin{cases} \nabla f = \lambda \nabla g, \lambda = \text{constant} \\ g(x,y) = k \end{cases} \tag{262}$$

solve this system of equation to get (x,y)
Step 2, plug in (x,y) into f(x,y)

Example: $f(x, y) = 4xy$, find max/min with condition $x^2 + y^2 = 4$

$$g(x, y) = x^2 + y^2, g(x, y) = 4 \tag{263}$$

$$f\&g = \begin{cases} \nabla f = & (2y, 2x) = \lambda \cdot \nabla g = (2x, 2y) \\ g(x, y) = & 4 = x^2 + y^2 \end{cases} \tag{264}$$

$$\therefore 2y = \lambda 2x \to \lambda = \frac{y}{x} \tag{265}$$

$$2x = \lambda 2y \to \lambda = \frac{x}{y} \tag{266}$$

$$\therefore x^2 = y^2 \& g(x, y) : x^2 + y^2 = 4, x^2 = y^2 = 2 \tag{267}$$

$$x, y = \pm\sqrt{2}, \text{4 solutions, max 8, min -8} \tag{268}$$

Why does it work? at max $\nabla g /\!/ \nabla f$

2.14 Vector Field and Flow

Definition: A vector field $\vec{F} : \mathbb{R}^m \to \mathbb{R}^m$

1. Example 1: $\vec{F}(x, y) = (x, y), \mathbb{R}^2 \to \mathbb{R}^2$
2. Example 2: $\vec{F}(x, y) = (-y, x)\mathbb{R}^2 \to \mathbb{R}^2$
3. Example 3: $\vec{F}(x, y, z) = (x^3, xy, e^{xyz})\mathbb{R}^3 \to \mathbb{R}^3$

Definition: (Flow/Integral Curve) Given $\vec{F} : \mathbb{R}^m \to \mathbb{R}^m$, a flow of $\vec{F}$ is curve $\vec{C}(t) : \mathbb{R}^1 \to \mathbb{R}^m$ such that $\vec{C'}(t) = \vec{F}(\vec{C}(t))$

Example:

$$\vec{F}(x, y) = (-y, x) \tag{269}$$

Verify that the curve $\vec{C}(t) = (cos(t), sin(t))$, is a flow/integral curve of $\vec{F}$ by checking if $\vec{C'}(t) = \vec{F}(\vec{C}(t))$

$$\vec{C'}(t) = (-sin(t), cos(t)) \tag{270}$$

$$\vec{C'}(t) \equiv \vec{F}(\vec{C}(t)) \tag{271}$$

Remark: Finding flows of a vector field is the focus of differential equations.
An important class of example of vector field is given by ∇F where $F : \mathbb{R}^m \to \mathbb{R}^1$, $\nabla F = (\frac{\sigma F}{\sigma x}, \frac{\sigma F}{\sigma y}, \frac{\sigma F}{\sigma z}) : \mathbb{R}^3 \to \mathbb{R}^3$
Looking at ∇F tells us something about F

1. ∇F is the direction along which F changes most dramatically.
2. $\nabla F \perp$ level sets $(F(x, y, z) = c)$.

Definition (Divergence and Curl): Given $\vec{F}\mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$. Define the divergence of $\vec{F}$

1. div $\vec{F} = (\frac{\sigma F}{\sigma x}, \frac{\sigma F}{\sigma y}, \frac{\sigma F}{\sigma z})$

2. div $\vec{F} = \mathbb{R}^3 \to \mathbb{R}^1$ a scalar function

Define the curl of $\vec{F}$

1. Curl $\vec{F} = \begin{bmatrix} \sigma/\sigma x \\ \sigma/\sigma y \\ \sigma/\sigma z \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

2. div $\vec{F} = \mathbb{R}^3 \to \mathbb{R}^3$ a vector function

Easier way to memorize these 2 notations:∇ is a vector of differential operators $< \frac{\sigma}{\sigma x}, \frac{\sigma}{\sigma y}, \frac{\sigma}{\sigma z} >$
Hence ∇F means the gradient of F. So $div\vec{F}$ is the dot product between ∇ and $\vec{F}$: $div\vec{F} = \nabla \cdot \vec{F}$
Hence ∇F means the gradient of F. So $curl\vec{F}$ is the cross product between ∇ and $\vec{F}$: $curl\vec{F} = \nabla \times \vec{F}$
Similar for two dimension curves, except for the curl, we add a zero to the third dimension making the original function a function plot on a plane in the three dimensional plot. $Curl\vec{F} = curl(F_1, F_2, 0)$, and the result will be $(0, 0, \frac{\sigma F_2}{\sigma x} - \frac{\sigma F_1}{\sigma y})$, and the scalar curl for $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $\frac{\sigma F_2}{x} - \frac{\sigma F_1}{y}$, a scalar function.

Meanings of divergence and curl:
Divergence of $\vec{F}$ at (x_0, y_0) is measuring the change of density at this point. Change of volume of flows

$$
\begin{aligned}
&= F_1(x + \Delta x, y)\Delta y - F_1(x, y)\Delta x && (272)\\
&\quad + \frac{F_2(x, y + \Delta y)\delta x - F_2(x, y)\Delta x}{\Delta x \Delta y} && (273)\\
&\approx \frac{\sigma F_1}{\sigma x} + \frac{\sigma F_2}{\sigma y} div\vec{F} && (274)
\end{aligned}
$$

Curl of $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$, scalar curl $= \frac{\sigma F_2}{x} - \frac{\sigma F_1}{y}$. is the measurement of the tendency of spin at a point. The Jacobian matrix $D\vec{F}$ capture the instant change of $\vec{F}$ In particular, $D\vec{F}$ capture the infinitely minimal tendency of spin at the point.
A bit about Linear Algebra
If The Jacobian matrix $D\vec{F}$ is symmetric, i.e.$D\vec{F} = D\vec{F}^T$, i.e. $\frac{\sigma F_2}{sigmax} = \frac{\sigma F_1}{sigmay}$. Then $D\vec{F}$ introduces no rotation. Curl measures how far away $D\vec{F}$ from being symmetric, the more asymmetric, the more $D\vec{F}$ rotates a basis.
Curl in third dimension:

1. $Curl\vec{F}$ = the axis of rotation of $D\vec{F}$.

2. $|Curl\vec{F}|$ = rate of rotation of $D\vec{F}$ the instant rate of rotation of $\vec{F}$ at the point.

Relationship between ∇, div, $curl$

$$\mathbb{R}^1 - (\nabla) \to \mathbb{R}^3 - (\text{curl}) \to \mathbb{R}^3 - (\text{div}) \to \mathbb{R}^1 \tag{275}$$

$\mathbb{R}^1 - (\nabla) \to \mathbb{R}^3$ are both tangent of functions.

$$curl(\nabla f) = \vec{0} \tag{276}$$

$$div(curl\vec{F}) = 0 \tag{277}$$

2.15 Integration along a path

Definition: A path (parametric curve) $\vec{c} : [a, b] \to \mathbb{R}^{\text{m}}$ is C' if $\vec{C}(t)$ exists and is continuous
Definition: A path $\vec{c} : [a, b] \to \mathbb{R}^{\text{m}}$ is piece-wise C' if can divide $[a, b]$ into finitely many sub-integrals. $[a_1 = a_1, a_2]$, $[a_2, a_3] \ldots$, $[a_{n-1}, a_n = b]$ such that $\vec{C}on[a_i, a_{i+1}]$ is C'.
Convention:From now on, we assume a path $\vec{C}(t)$ is always piece-wise C'.
Definition: If $\vec{C}(t)$ is a piece-wise C' path, we said that

1. $\vec{C}(t)$ is simple if it does not intersect itself or equivalently, $\vec{c} : [a, b] \to \mathbb{R}^{\text{m}}$ is an 1 to 1.
2. $\vec{C}(t)$ is simple closed path/curve if there exists $\vec{C}(a) = \vec{C}(b)$ and does not intersect itself except for the 2 end points.

Definition: for $\vec{C} : [a, b] \to \mathbb{R}^{\text{m}}$, $f(x_1 \ldots x_m) : \mathbb{R}^{\text{m}} \to \mathbb{R}^1$ then integral of f along $\vec{C}$ is denoted by:

$$\int_C f ds = \int_a^b f\vec{C}(t) \cdot |\vec{C}'(t)| dt \tag{278}$$

Example: Finding the integral of $f(x, y, z) = xyz$ along the path $\vec{C}(t) = (-\sin(t), \sqrt{2}\cos(t), sin(t)), t \in [0, \pi/2]$

$$\int_c f ds = \int_0^{\pi/2} f(\vec{C}(t)) \cdot |\vec{C}'(t)| dt \tag{279}$$

$$= \ldots \tag{280}$$

$$= -2 \int_0^{\pi/2} \sin^2(t) \cos(t) dt \tag{281}$$

$$= -\frac{2}{3} \tag{282}$$

Theorem: Path integral of a function does not depend on the parametrization of the path.

Namely, $\vec{C_1}(t)$ and $\vec{C_2}(t)$ are 2 different parameter of the same path, $\int_{C_1} f ds = \int_{C_2} f ds$

The integration of vector field along a path is similar:
Definition: For a path $\vec{c} : [a,b] \to \mathbb{R}^{\mathrm{m}}$ and a vector function $\vec{F} : \mathbb{R}^{\mathrm{m}} \to \mathbb{R}^{\mathrm{m}}$ the integral of $\vec{F}$ along $\vec{C}$ is

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) dt \tag{283}$$

This can be used in occasion like calculating the work done by a force field.
Example:$\vec{F}(x,y,z) = (-y, x, 1)$ force field, and there is a particle moving along a three dimension path $\vec{C}(t)$ and similarly, when the path is perpendicular, work done(hence the integral) would also be zero.

1) for $\vec{C}(t) = (t,t,t),\ t \in [0,1]$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_0^1 (-t, t, 1) \cdot (1,1,1) dt \tag{284}$$

$$=1\ (J) \tag{285}$$

2) for $\vec{C}(t) = (\cos(t), \sin(t), t),\ t \in [0, 2\pi]$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_0^2 \pi(-\sin(t), \cos(t), 1) \cdot (-\sin(t), \cos(t), 1) dt \tag{286}$$

$$=4\pi\ (J) \tag{287}$$

Properties of $\int_{\vec{C}} \vec{F} d\vec{s}$

$$\int_{\vec{C}} \vec{F} d\vec{s} = \int_a^b \vec{F}(\vec{C}(t)) \cdot \vec{C}'(t) dt \tag{288}$$

$$= \int_a^b |\vec{F}(\vec{C}(t))| \cdot |\vec{C}'(t)| \cos(\theta t) dt \tag{289}$$

The angle $\theta(t)$ is the angle formed by $\vec{F}(C(t))$, and $\vec{C}'(t)$ is always< 90 degree.
If $\vec{F}(c(t)) \perp C'(t)$

$$\int_{\vec{C}} \vec{F} d\vec{s} = 0 \tag{290}$$

If $\vec{C_1}(t)$ and $\vec{C_2}(t)$ are different parametrization of the same path with the same orientation, then

$$\int_{\vec{C_1}} \vec{F} d\vec{s} = \int_{\vec{C_2}} \vec{F} d\vec{s} \tag{291}$$

If opposite orientation

$$\int_{\vec{C_1}} \vec{F} d\vec{s} = -\int_{\vec{C_2}} \vec{F} d\vec{s} \tag{292}$$