

Math, Multivariable Calculus

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1 Vectors

1.1 Vector Calculations

2 Vector functions

2.1 Vector Valued functions in multiple variables

Calculus of single variable is a study of $y = f(x)$

$$y = f(x), \text{ where } (x \in \mathbb{R}) \rightarrow (f(x) \in \mathbb{R}) \quad (1)$$

(2)

However, when we are dealing with multi-variable calculus, we have: an n variable vector valued function (if $m > 1$)

$$F = \begin{cases} (x_1, \dots, x_n) & \rightarrow F(x_1, \dots, x_n) \\ \in & \in \\ \mathbb{R}^n & \longrightarrow \mathbb{R}^m \end{cases} \quad (3)$$

where $F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n) \dots F_m(x_1, \dots, x_n))$

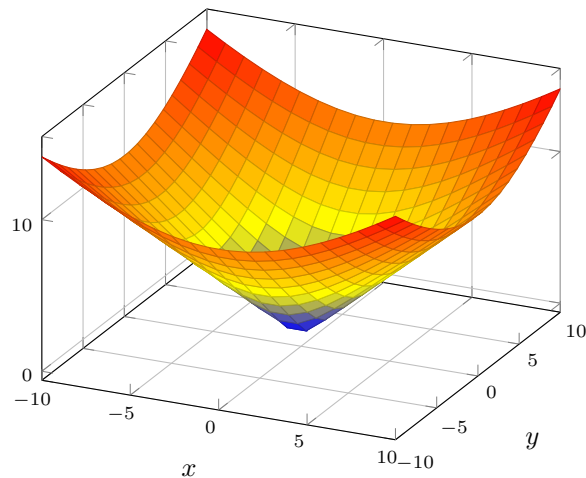
The inputs and outputs are Vectors, for example

$$d(x, y) = \sqrt{x^2 + y^2}, (2 \text{ variables}) \quad (4)$$

$$(x, y) \mathbb{R}^2 \rightarrow (\sqrt{x^2 + y^2}) \mathbb{R} \quad (5)$$

$$\text{distance} = \sqrt{x^2 + y^2} \quad (6)$$

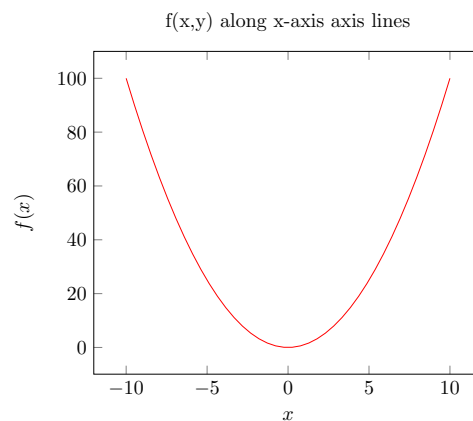
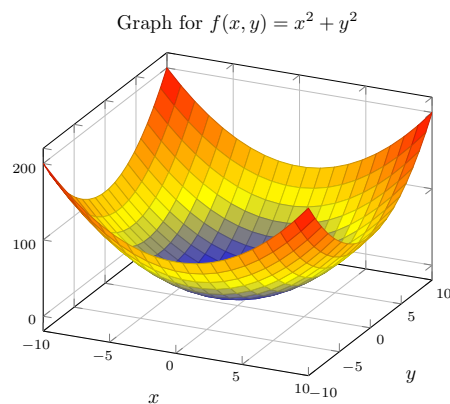
Graph for $f(x, y) = \sqrt{(x^2 + y^2)}$



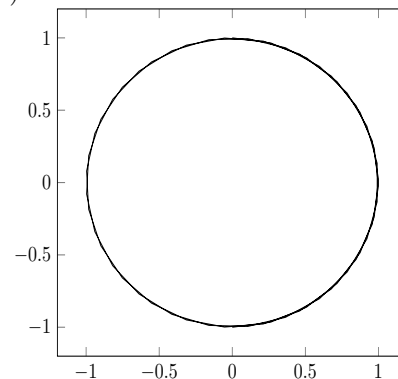
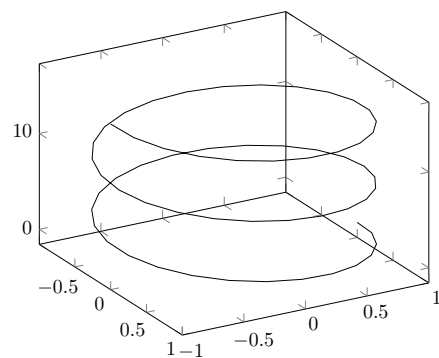
The graph should look like a cone

Similarly, we can have a function that takes all the x , y and z can calculate that distance. That would however be a four dimensional plot, which we cannot plot.

For a similar function, $z = x^2 + y^2$



For another function $f(z) = (\cos z, \sin z)$

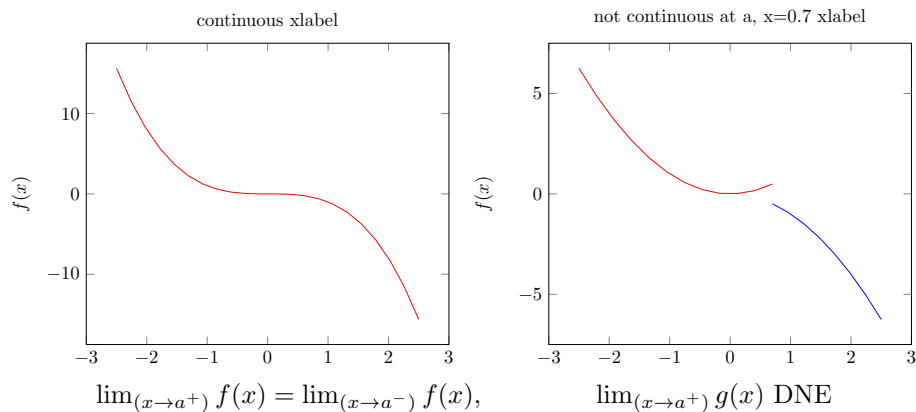


2.2 Limits and continuity

Definition:

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L \quad (7)$$

if $f(x, y)$ goes to L as (x, y) approaches (a, b)



Similarly, for $(x, y) \rightarrow (a, b)$ we need to look at all possible paths. If you can find 2 oaths to approach (a, b) such limits along these 2 paths are different, then the function is not continuous. For example:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2 + y^2} = \text{DNE}(\text{do not exists}) \text{ since:} \quad (8)$$

$$(x, 0) \rightarrow (0, 0), \lim = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0 \quad (9)$$

$$(x, x) \rightarrow (0, 0), \lim = \lim_{y \rightarrow 0} \frac{2x * x}{x^2 + x^2} = 1 \quad (10)$$

$1 \neq 0$ hence D.N.E.

Remember to take a look at all directions

Definition: $f(x, y)$ continuous at (a, b) if:

1. $f(x, y)$ defined at (a, b)
2. $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

How to tell if a function is continuous? $f(x, y, z) = e^{x-z} \sin(x + yz^2)$

1. (a) $f(x, y, z) = x$
 (b) $f(x, y, z) = y$
 (c) $f(x, y, z) = \text{zallcontinuous}$
2. $f(x, y, z) = x - z$
3. $f(x, y, z) = e^{x-z}(\text{composition of contin. func is contin.})$
4. $g(x, y, z) = \sin(x + yz^2)$ continuous similarly

Test: $\lim_{(x,y,z) \rightarrow (0,1,-1)} = e^1 \sin(1)$, hence continuous. A special case is when you have $\frac{f(x,y)}{g(x,y)}$, it will be continuous at (a, b) as long as $g(a, b) \neq 0$

For a vector function $F(\vec{v}) = (F_1(\vec{v}), F_2(\vec{v}) \dots F_m(\vec{v}))$, where \vec{v} is a vector or $\mathbb{R}^n, (x_1, x_2 \dots x_n)$, is a continuous function at vector \vec{n} if all its component functions are continuous at \vec{n}

Example:

$$f(x, y, z) = (e^{x+y}, z, \cos(x^2 - z)) \text{ continuous everywhere} \quad (11)$$

$$\text{since } \begin{cases} e^{x+y} \text{ continuous} \\ z \text{ continuous} \\ \cos(x^2 - z) \text{ continuous} \end{cases} \quad (12)$$

2.3 Partial Derivatives

The derivative of a function is the rate of change along the x axis. The partial derivatives of $F(x, y)$ would be the rate of change of F as one of the axis is fixed (along one axis).

$$\frac{dF(x, y)}{dx} = F_x(x, y) = \lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x, y) - F(x, y)}{\Delta x} \quad (13)$$

$$\frac{dF(x, y)}{dy} = F_y(x, y) = \lim_{\Delta y \rightarrow 0} \frac{F(x, y + \Delta y) - F(x, y)}{\Delta y} \quad (14)$$

In order to compute the derivative, we treat all other variables as constants.

Example:

$$f(x, y, z) = xe^{xy} - \sin(y^2 + z^2) \quad (15)$$

$$\frac{df}{dx} = e^{xy} + yxe^{xy} - 0 = e^{xy} + yxe^{xy} \quad (16)$$

$$\frac{df}{dy} = x^2e^{xy} - \cos(y^2 + z^2)(2y) \frac{df}{dz} = 0 - \cos(y^2 + z^2)(2z) \quad (17)$$

Take a point (a, b) for example

1. $\frac{df}{dx}$ = gradient of tangent at that point along x axis
2. $\frac{df}{dy}$ = gradient of tangent at that point along y axis
3. These two direction can span the tangent plane of f at that point
 - (a) Tangent Vector
 - (b) along x direction = $(1, 0, \frac{df}{dx}(a, b)) = \vec{v}$
 - (c) along y direction = $(0, 1, \frac{df}{dy}(a, b)) = \vec{w}$
 - (d) Using these vectors we can find the plane's equation

Example, find the tangent plane of $f(x, y)$ at $(1, 1)$:

$$f(x, y) = 1 - x^2 - 2y^2 \quad (18)$$

$$f(1, 1) = z = -2, \text{ point at } (1, 1, -2) \quad (19)$$

$$\frac{df}{dx} = (1, 0, -2x)|_{(1,1)} = (1, 0, -2) \quad (20)$$

$$\frac{df}{dy} = (1, 0, -4y)|_{(1,1)} = (0, 1, -4) \quad (21)$$

$$(1, 0, -2) \times (0, 1, -4) = (2, 4, 1) \quad (22)$$

$$2x + 4y + z = 4 \text{ a linear approximation of } f(x, y) \text{ at } (1, 1) \quad (23)$$

Definition of a vector function:

$$F(x_1, x_2, \dots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \dots F_m(\vec{x})) \in \mathbb{R}^m \quad (24)$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F (Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \dots & \dots & \dots & \dots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \dots & \frac{dF_m}{dx_n} \end{pmatrix} \quad (25)$$

The function F transforms $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $DF: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map, a linear approximation of F .

Example:

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin(y + z), 4y) \quad (26)$$

$$DF(x, y, z) = \begin{pmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y + z) & \cos(y + z) \\ 0 & 4 & 0 \end{pmatrix} DF(1, 1, 2) = \quad (27)$$

$$DF(1, 1, 2) = \begin{pmatrix} e^3 & 2e^3 & e^3 \\ 2 & 0 & 0 \\ 0 & \cos(3) & \cos(3) \\ 0 & 4 & 0 \end{pmatrix} \quad (28)$$

Matrix of numbers, a linear map

$$F(x, y, z) = (e^{x+yz}, x^2 + 1, \sin(y + z), 4y) \quad (29)$$

2.4 Properties of Derivatives DF

Given two vector function F and $G: \mathbb{R}^n \rightarrow \mathbb{R}^m$ both differentiable at some point \vec{x}

$$D(f \pm G) = D(F) \pm D(G) \quad (30)$$

$$C \in \mathbb{R}, D(cF) = cD(F) \quad (31)$$

suppose $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ a function in n variables, product rule still holds

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g) \quad (32)$$

where $D(f \cdot g)$ is a n^{th} dimension vector, $D(f \text{ or } g)$ is a vector function and f or g is a scalar function

$$D\left(\frac{f}{g}\right) = \frac{g \cdot D(f) - f \cdot D(g)}{g^2}, g(\vec{x}) \neq 0 \quad (33)$$

Given two vector function $\vec{v}(t)$ and $\vec{w}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ so $t \rightarrow \vec{v}(t)$

$$f(t) = \vec{v}(t) \cdot \vec{w}(t) \text{ scalar function} \quad (34)$$

$$f'(t) = \vec{v}(t)' \cdot \vec{w} + \vec{w}(t)' \cdot \vec{v} \quad (35)$$

$$u(t) = \vec{v}(t) \times \vec{w}(t) \text{ vector function} \quad (36)$$

$$u'(t) = \vec{v}(t)' \times \vec{w}(t) + \vec{w}(t)' \times \vec{v}(t) \quad (37)$$

$$\vec{v}(t) = \langle v_1(t), v_2(t), \dots, v_n(t) \rangle \quad (38)$$

Chain rule, suppose $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$, $g \circ f : \mathbb{R}^m \rightarrow \mathbb{R}^p$, $\vec{x} \in \mathbb{R}^m$

$$D(g \circ f)(\vec{x}) (\text{size } p \times m) = DG(F(\vec{x})) (\text{size } p \times n) \cdot DF\vec{x} (\text{size } n \times m) \quad (39)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad (40)$$

Example:

$$f(x, y) = (x^3 + y, e^{xy}, 2 + xy) : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (41)$$

$$g(y, v, w) = (y^2 + v, uv + w^3) : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad (42)$$

$$g(f(x, y)) = g \circ f = ((x^3 + y)^2 + e^{xy}, (x^3 + y)e^{xy} + (2 + xy)^3) \quad (43)$$

$$D(g \circ f) = \begin{pmatrix} 2(x^3 + y)(3x^2) + ye^{xy} & 2(x^3 + y) + xe^{xy} \\ 3x^2e^{xy} + (x^3 + y)ye^{xy} & e^{xy} + (x^3 + y)xe^{xy} \\ +3(2 + xy)^2y & +3(2 + xy)^2x \end{pmatrix} \quad (44)$$

$$DG = \begin{pmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{pmatrix} \quad (45)$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad (46)$$

$$DG = \begin{pmatrix} 2(x^3 + y) & 1 & 0 \\ e^{xy} & x^3 + y & 3(2 + xy)^2 \end{pmatrix} \times \begin{pmatrix} 3x & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{pmatrix} \quad (47)$$

Check that the equality holds.

2.5 Directional derivatives

Recall the Jacobian matrix, in the following context, the function F is not necessarily defined on the entire \mathbb{R}^n , but rather a subset $\subseteq \mathbb{R}^n$

$$F(x_1, x_2, \dots, x_n) = F(\vec{x}) = (F_1(\vec{x}) \dots F_m(\vec{x})) \in \mathbb{R}^m \quad (48)$$

Where each F_i is a component function. The derivative the size $(m \times n)$ matrix of F (Jacobian matrix) denoted by $DF(\vec{x})$

$$DF(\vec{x}) = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} & \dots & \frac{dF_1}{dx_n} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} & \dots & \frac{dF_2}{dx_n} \\ \dots & \dots & \dots & \dots \\ \frac{dF_m}{dx_1} & \frac{dF_m}{dx_2} & \dots & \frac{dF_m}{dx_n} \end{pmatrix} \quad (49)$$

Examples: 1)

$$f(x, y, z) = x^2y + \ln z : \mathbb{R}^3 \rightarrow \mathbb{R}^1 \quad (50)$$

$$\begin{aligned} Df &= \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right) \\ &= (2x, 1, \frac{1}{z}) \end{aligned} \quad (51)$$

2)

$$\vec{v}(t) = (\cos(t), \sin(t)); \text{ vector function} \quad (52)$$

$$D\vec{v}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = (-\sin(t), \cos(t))^T \quad (53)$$

$$D\vec{v}(t) \cdot \vec{v}(t) = -\sin(t) \cos(t) + \cos(t) \sin(t) = 0 \quad (54)$$

$$D\vec{v}(t) \perp \vec{v}(t); \text{ hence a tangent vector} \quad (55)$$

3)

$$g(t) = e^t \text{ scalar function} \quad (56)$$

$$g(t) \cdot \vec{v}(t) = e^t(\cos(t), \sin(t)) = (e^t \cos(t), e^t \sin(t)) \quad (57)$$

$$D(g(t) \cdot \vec{v}(t)) = \begin{bmatrix} e^t \cos(t) - e^t \sin(t) \\ e^t \cos(t) + e^t \sin(t) \end{bmatrix} \quad (58)$$

4)

$$\vec{v}(t) = (t, \sin(t), \cos(t)), \vec{w}(t) = (3t, 0, 2) \quad (59)$$

$$\begin{aligned} f(t) &= \vec{v}(t) \cdot \vec{w}(t) \\ &= 3t^2 + 0 + 2 \cos(t); \text{ scalar function} \end{aligned} \quad (60)$$

$$D(f(t)) = 6t - 2 \sin(t) \quad (61)$$

$$\vec{w} \cdot D\vec{v} + \vec{v} \cdot D\vec{w} = \begin{bmatrix} 3t \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ +\cos(t) \\ -\sin(t) \end{bmatrix} + \begin{bmatrix} t \\ \sin(t) \\ \cos(t) \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad (62)$$

$$= 6t - 2 \sin(t) \quad (63)$$

2.6 Partial derivative along a direction

Definition: Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (a, b) , given a unit vector $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$ then the directional derivatives of F along \vec{u} is:

$$D_{\vec{u}}F(a, b) = \left[\frac{d}{dt} f((a, b) + t\vec{u}) \right]_{t=0} \quad (64)$$

$$= \lim_{h \rightarrow 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \quad (65)$$

is the rate of change of F along the direction of \vec{u}

Two special cases would be when u are along the typical x or y direction, which would mean the class derivative along x -axis $\frac{df}{dx}$ or the y -axis $\frac{df}{dy}$

Theorem $\vec{u} = (u_1, u_2)$ unit vector,

$$D_{\vec{u}}f(x, y) = \left(\frac{df}{dx}, \frac{df}{dy} \right) \cdot \vec{u} = (u_1, u_2) \quad (66)$$

$$\text{since } \vec{u} = u_1(1, 0) + u_2(0, 1) \quad (67)$$

$$D_{\vec{u}}f = u_1 \frac{df}{dx} + \frac{df}{dy} \quad (68)$$

Proof:

$$D_{\vec{u}}f = \lim_{h \rightarrow 0} \frac{F(a + hu_1, b + hu_2) - F(a, b)}{h} \quad (69)$$

$$= \lim_{h \rightarrow 0} \frac{(F(a + hu_1, b + hu_2) - F(a, b + hu_2))}{h} \quad (70)$$

$$+ \lim_{h \rightarrow 0} \frac{F(a, b + hu_2) - F(a, b)}{h} \quad (71)$$

$$= \lim_{u_1 h \rightarrow 0} \frac{[F(a + hu_1, b + hu_2) - F(a, b + hu_2)] \cdot u_1}{hu_1} \quad (72)$$

$$+ \lim_{h \rightarrow 0} \frac{F(a, b + hu_2) - F(a, b)}{h} \quad (73)$$

$$= \frac{df}{dx}(a, b + 0) \cdot u_1 + \frac{df}{dy} \cdot u_s \quad (74)$$

Example: Find the directional Deriv of $f(x, y) = x^2 + 3xy$ along the direction

of $(3, 4)$ at the point $p = (2, -1)$

$$\vec{u} = \frac{(3, 4)}{\sqrt{(3^2 + 4^2)}} = \left(\frac{3}{5}, \frac{4}{5}\right) \quad (75)$$

$$D_{\vec{u}}f(2, -1) = (2x + 3y, 3x)|_{(2, -1)} \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \quad (76)$$

$$= (1, 6) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) \quad (77)$$

$$= \frac{27}{5} \quad (78)$$

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \left(\frac{df}{dx}, df dy\right) \cdot \vec{u} \quad (79)$$

Definition: Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$, then define the gradient.

$$\nabla f = D_{\vec{u}}f = \left(\frac{df}{x_1}, \frac{df}{x_2}, \dots, \frac{df}{x_n}\right) \quad (80)$$

Hence the directional derivative of a function along a direction is:

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} \quad (81)$$

The consequence of this:

1)

$$D_{\vec{u}}f \text{ max when } \vec{u} // \nabla f = \left(\frac{df}{dx}, \frac{df}{dy}\right) \quad (82)$$

since

$$\nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos(\theta) \quad (83)$$

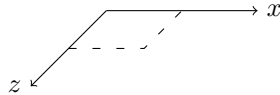
$$= |\nabla f| \cdot 1 \cdot \cos(\theta) \theta = \text{ang}(\nabla f, \vec{u}) \quad (84)$$

When maximized, $//, \theta = 0$

$$\nabla f \cdot \vec{u} = |\nabla f| = \sqrt{\frac{df^2}{dx} + \frac{df^2}{dy}} \quad (85)$$

2)

$$\text{When } \vec{u} \perp \nabla f, D_{\vec{u}}f = 0 \quad (86)$$



Example: $f(x, y) = e^{-(x^2 + y^2)}$, find the direction along this function which has the largest rate of increase/decrease at point $(1, 1)$

Since at maximum, $\vec{u} \parallel \nabla f$

$$\nabla f = (-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)})|_{(1,1)} \quad (87)$$

$$= (-2e^{-2}, -2e^{-2}) \quad (88)$$

$$\therefore \vec{u} \parallel \nabla f \parallel (1, 1) \quad (89)$$

$$\therefore \vec{u} = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \quad (90)$$

Largest rate of increase at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Largest rate of decrease at $-\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

The gradient ∇f tells us the direction along which F changes the most.

2.7 Summary

Given a function $f(x, y)$	n-th dim analogue
Define Partial derivatives (deriv) $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f(x_1, x_2, \dots, x_n)$
Define gradient $\frac{df}{dx}, \frac{df}{dy}$ or (f_x, f_y)	$f : \mathbb{R}^n \rightarrow \mathbb{R}$
Directional deriv: given unit vector $\vec{u} = (u_1, u_2)$	$\frac{df}{dx_1} \dots \frac{df}{dx_n}$
$D_{\vec{u}}f = \lim_{h \rightarrow 0} \frac{f((a,b)+h\vec{u}) - f(a,b)}{h}$	$\nabla f = (\uparrow)$
if $(\vec{u} = (1, 0))$, $D_{\vec{u}}f = \frac{df}{dx}$	$\vec{u}f \in \mathbb{R}^n$
if $(\vec{u} = (0, 1))$, $D_{\vec{u}}f = \frac{df}{dy}$	$D_{\vec{u}}f = ?$
Theorem: $D_{\vec{u}} = \nabla f \cdot \vec{u} = u_1 \frac{df}{dx} + u_2 \frac{df}{dy}$	generalizations?
Remark: We can define $D_{\vec{v}}$ for any $\vec{v} \in \mathbb{R}^2$ in the exact same way $D_{\vec{v}} = \vec{v} D_{\frac{\vec{v}}{ \vec{v} }} f$	

To each unit vector \vec{u} , we have a tangent vector associated to it:

$$(u_1, u_2, D_{\vec{u}}f) = u_1(1, 0, \frac{df}{dx}) + u_2(0, 1, \frac{df}{dy}) \quad (91)$$

Two special cases: $(1, 0, \frac{df}{dx})(0, 1, \frac{df}{dy})$

The two tangent vector span a tangent plane at $(a, b, f(a, b))$

2.8 Gradients and level sets

Recall

$$\nabla f = \left(\frac{df}{dx}, \frac{df}{dy}\right) \quad (92)$$

$$\therefore D_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos(\theta) \quad (93)$$

$D_{\vec{u}}$ is the rate of change along \vec{u}

if $u \parallel \nabla f$, ($\theta = 0$) then $D_{\vec{u}}F$ max

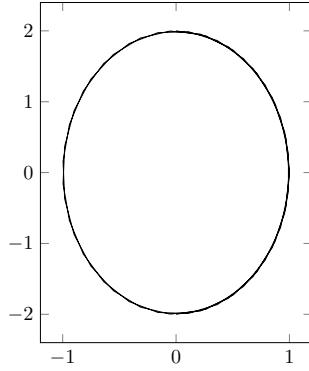
$$|D_{\vec{u}}F| = \sqrt{\left(\frac{df}{dx}\right)^2 + \left(\frac{df}{dy}\right)^2} \quad (94)$$

In other words, along the direction of ∇F , the function F changes the most.

Definition: A level set of $F(x,y)$ is:

$$(x, y) \in \mathbb{R}^2 : f(x, y) = c \quad (95)$$

Example:



$$f(x, y) = x^2 + \frac{y^2}{4} \quad (96)$$

$$\{f(x, y) = 1\} \text{ (Level set)} \quad (97)$$

$$= \{(x, y) | x^2 + \frac{y^2}{4} = 1\} \quad (98)$$

Theorem: Given $f(x, y)$ differentiable, then the gradient $\nabla f \perp$ the level set.

We can prove it by using the chain rule: let $\gamma(t)$ be a parametric function of my level set $\{f(x, y) = 1\}$, $f(\gamma(t)) = 1$ and differentiate both sides.

$$0 = D(f(\gamma(t))) = Df(\gamma(t))D(\gamma(t)) \quad (99)$$

$$= \nabla F(\gamma(t))_{1 \times 2} \cdot (\gamma'_1(t), \gamma'_2(t))_{2 \times 1} \quad (100)$$

$$\text{Hence } \Rightarrow \nabla F(\gamma(t))_{1 \times 2} \perp (\gamma'_1(t), \gamma'_2(t))_{2 \times 1} \quad (101)$$

Example: Look at a surface $\sin(xy) - 2\cos(yz) = 0$ Find the tangent plane of this surface at $(\frac{\pi}{2}, 1, \frac{\pi}{3})$. The idea is that consider the surface as a level set of $F(x, y, z) = \sin(xy) - 2\cos(yz)$, $\{F = 0\}$

Using the Theorem: ∇F is hence perpendicular to this surface, giving us the normal direction of our tangent plane.

$$\nabla F = \left(\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right) = (y \cos(xy), x \cos(xy) + 2z \sin(yz), 2y \sin(tz)) \quad (102)$$

$$= \left(\cos\left(\frac{\pi}{2}\right), \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \frac{2\pi}{3} \sin\left(\frac{\pi}{3}\right), 2 \sin\left(\frac{\pi}{3}\right)\right) \quad (103)$$

$$= \left(0, \frac{\sqrt{3}\pi}{3}, \sqrt{3}\right) \quad (104)$$

$$0 = 0 \cdot \left(x - \frac{\pi}{2}\right) + \frac{\sqrt{3}\pi}{3}(y - 1) + \sqrt{3}\left(z - \frac{\pi}{3}\right) \quad (105)$$

Where the last equation above is the equation of the tangent plane.

2.9 Parametric Curves

... are vector functions in 1 variable.

Definition: A parametric curve in \mathbb{R}^n is a vector valued function where $\vec{c}(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^n$

All the component functions of $\vec{c}(t)$ are functions in 1 variable.

Example:

1) Draws a Circle

$$\vec{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \quad (106)$$

2) Draws a ellipse at plane $z = 3$

$$\vec{c}(t) = (a \cos(t), b \sin(t), 3), t \in [0, 2\pi] \quad (107)$$

3) Ellipse spiraling up

$$\vec{c}(t) = (a \cos(t), b \sin(t), t), t \in [0, 2\pi] \quad (108)$$

Definition: Orientation of $\vec{c}(t)$, the direction corresponding to increasing t-value is called the positive orientation of $\vec{c}(t)$. The opposite direction is the negative orientation.

Example: Positive orientation = Clockwise

$$\vec{c}(t) = (\cos(t), -\sin(t), t), t \in [0, 2\pi] \quad (109)$$

Derivative of $\vec{c}(t)$

If $\vec{c}(t) = (x(t), y(t), z(t))$, $\vec{c}'(t) = (x'(t), y'(t), z'(t))$

1. $\vec{c}(t)$ is a vector function
2. $\vec{c}'(t)$ is a tangent to curve $\vec{c}(t)$

Terminology: $\vec{c}'(t)$ is the velocity of $\vec{c}(t)$, and $|\vec{c}'(t)|$ is the speed of $\vec{c}(t)$. The speed of the curve $\vec{c}(t) = (\cos(t), \sin(t))$ for example, is 1.

What is the length of curve $\vec{c}(t) = (\cos(t), \sin(t), t)$ between (0 to 2π) Given $\vec{c}(t)$ differentiable between range a and b, length of the curve would be

$$l = \int_a^b |\vec{c}'(t)| dt \quad (110)$$

$$|\vec{c}'(t)| = \sqrt{\sin(t)^2 + \cos(t)^2 + 1} = \sqrt{2} \quad (111)$$

$$l = \int_0^{2\pi} \sqrt{2} dt \quad (112)$$

Think of length as a function in t

$$l(t) = \int_a^t |\vec{c}'(x)| dx \quad (113)$$

If curve $\vec{c}(t)$ differentiable and $\vec{c}'(t) \neq 0$,

1. $|\vec{c}'(t)| > 0$
2. $l(t)$ strictly increasing.
3. l is one to one function so has inverse.

$$l[a, b] \rightarrow \mathbb{R}^1, l \text{ has an inverse.} \quad (114)$$

$$t \rightarrow l(t), \text{ think of our parameter } t \quad (115)$$

$$(116)$$

t can now be expressed as a function in terms of length, $\vec{c}(t) = \vec{c}(l(t)) = \vec{c}(l)$

Proof:

In our last example of calculating length, we can find an inverse for $l(t) = \sqrt{2}t$ resulting $t = \frac{l}{\sqrt{2}}$, by substitution we can reparameterize $\vec{c}(t)$ into $\vec{c}(l)$

$$c(l) = \vec{c}(l(t)) \quad (117)$$

$$\frac{d\vec{c}(l)}{dl} = \frac{\vec{c}(t(l))}{dt} \cdot \frac{dt}{dl} \quad (118)$$

$$= \vec{c}'(t) \cdot \frac{1}{|\vec{c}'(t)|} \quad (119)$$

$$|\frac{d\vec{c}(l)}{dl}| = |\vec{c}'(t)| \cdot \frac{1}{|\vec{c}'(t)|} = 1 \quad (120)$$