Назаров Рустам

Матан ИДЗ-2

М3232 Вар. 13

№1

$$\int_0^{+\infty}\frac{1-a^2x^2}{1+a^2x^2}\cdot\frac{\sin(\frac{1}{x})}{x}dx =$$

$a>0$

$t=\frac{1}{x}\quad dx=-\frac{dt}{t^2}$

$$= \int_0^{+\infty}\frac{1-\frac{a^2}{t^2}}{1+\frac{a^2}{t^2}}\cdot\frac{\sin(t)}{\frac{1}{t}}\cdot\left(-\frac{dt}{t^2}\right) = -\int_0^{+\infty}\frac{t^2-a^2}{t^2+a^2}\cdot\frac{\sin(t)}{t}dt =$$

$$= -\left(\underbrace{\int_0^{+\infty}\frac{t^2}{t^2+a^2}\frac{\sin t}{t}dt}_{\int_0^{+\infty}\frac{\sin t}{t}dt-\int_0^{+\infty}\frac{a^2}{t^2+a^2}\frac{\sin t}{t}dt} - \int_0^{+\infty}\frac{a^2}{t^2+a^2}\cdot\frac{\sin t}{t}dt\right) =$$

$$= -\left(\underbrace{\int_0^{+\infty}\frac{\sin t}{t}dt}_{\boxed{\frac{\pi}{2}}} - 2\underbrace{\int_0^{+\infty}\frac{a^2}{t^2+a^2}\cdot\frac{\sin(t)}{t}}_{\boxed{\frac{\pi}{2}e^{-a}}\,*}\right) =$$

$$= -\left(\frac{\pi}{2}-2\cdot\frac{\pi}{2}e^{-a}\right) = \boxed{\pi e^{-a}-\frac{\pi}{a}}$$

$$*\ \int_0^{+\infty}\frac{a^2}{t^2+a^2}\cdot\frac{\sin t}{t}dt = \frac{\pi}{2}e^{-a}$$

Преобразование Лапласа

$$\int_0^{+\infty}f(t)g(t)dt = L\{f(t)\}(s)\cdot L\{g(t)\}(s)$$

$f(t)$: $L\left(\frac{\sin t}{t}\right)(s) = \operatorname{arctg}\left(\frac{1}{s}\right)$

$g(t)$: $L\left(\frac{a^2}{t^2+a^2}\right)(s) = \frac{\pi}{2}e^{-as}$

$$\int_0^{+\infty}\frac{a^2}{t^2+a^2}\frac{\sin t}{t}dt = \operatorname{arctg}\left(\frac{1}{s}\right)\cdot\frac{\pi}{2}\cdot e^{-as} =$$

$s\to 0\quad \operatorname{arctg}\left(\frac{1}{0}\right)=\frac{\pi}{2}$ (см. интеграл Лапласа)

$$= \boxed{\frac{\pi}{2}\cdot e^{-a}}$$

№2

$$\int_0^{+\infty}\frac{\sqrt{x}\,dx}{1+x^3} = \left|\begin{matrix}x=t^{\frac{1}{3}}\\ dx=\frac{1}{3}t^{-2/3}dt\end{matrix}\right| = \int_0^{+\infty}\frac{\sqrt[6]{t}\,dt}{3\sqrt[3]{t^2}(1+t)} = \frac{1}{3}\int_0^{+\infty}\frac{t^{-\frac{1}{2}}}{1+t}dt = \Rightarrow x=\frac{1}{2}\ y=\frac{1}{2}$$

$$B(x,y)=\int_0^{+\infty}\frac{t^{x-1}dt}{(1+t)^{x+y}}$$

$$= \frac{1}{3}B\left(\frac{1}{2};\frac{1}{2}\right) = \frac{1}{3}\frac{\overbrace{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}^{=\sqrt{\pi}}}{\underbrace{\Gamma(1)}_{1}} = \frac{1}{3}\cdot\sqrt{\pi}\cdot\sqrt{\pi} = \boxed{\frac{\pi}{3}}$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

№3

$$f(t)=\begin{cases}\cos(2t)-1 & t\\ 0 & t\notin[-\pi,\pi]\end{cases}$$

$\int_{-\pi}^{\pi}$ нечетная $=0$!

четная, четная, нечетная

$$F[f][\xi]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)e^{-i\xi t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\underbrace{(\cos(2t)-1)}_{-2\sin^2 t}\underbrace{e^{-i\xi t}}_{\cos\xi t-i\sin\xi t}dt = -\frac{2}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin^2 t\,(\cos\xi t-i\sin\xi t)dt =$$

$$= -\frac{2}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sin^2 t\cdot\cos\xi t\,dt = -\frac{2}{\sqrt{2\pi}}\cdot\left(-\frac{4\sin(\pi\xi)}{\xi^3-4\xi}\right) = \boxed{\frac{4\sqrt{2}\sin(\pi\xi)}{\sqrt{\pi}\cdot\xi\cdot(\xi^2-4)}}$$

Прямое преобразов. Фурье

Обратное преобразование Фурье

$\int_{-\infty}^{+\infty}$ нечет $=0$

$$F^{-1}[F[f]][t]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{4\sqrt{2}}{\sqrt{\pi}}\cdot\frac{\sin(\pi\xi)}{\xi(\xi^2-4)}\underbrace{e^{i\xi t}}_{\cos\xi t+i\sin\xi t}d\xi = \frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\pi\xi)}{\xi(\xi^2-4)}\cos\xi t\,d\xi+\frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\pi\xi)}{\xi(\xi^2-4)}\cdot i\sin\xi t\,d\xi = \boxed{\frac{4}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\pi\xi)\cdot\cos(\xi t)}{\xi(\xi^2-4)}d\xi}$$