Linear Applications and Matrices

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I) Linear applications

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  $\to$ A linear application allows the passage from a Vector Space to an other one.

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- For every $\mathbb{K}$-Vector Spaces E, the application **identity** $Id : E \to E$ defined as $\forall u \in E,\ Id(u) = u$ is linear.
- The application **differentiation** $f : \mathbb{K}[X] \to \mathbb{K}[X]$ defined as $\forall P \in \mathbb{K}[X]$, $f(P) = P'$ is linear.

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Determine which of the following applications is linear :

1) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\forall u = (x_1, x_2, x_3) \in \mathbb{R}^3, f(u) = (2x_1 + x_2, x_2 - x_3)$,

2) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\forall u = (x_1, x_2, x_3) \in \mathbb{R}^3, f(u) = (2x_1 x_2, x_2 + 3x_1)$,

3) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $\forall u = (x_1, x_2, x_3) \in \mathbb{R}^3, f(u) = (2x_1 + x_2 + x_3 + 5, x_2 - 3x_1)$,

4) $f : \mathbb{R}^3 \to \mathbb{R}$, $\forall u = (x_1, x_2, x_3) \in \mathbb{R}^3, f(u) = -x_1 + 2x_2 + 5x_3$.

II) Linear applications in $\mathbb{R}^n$

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Where the matrix A is called the **matrix of the linear application**.

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is the rotation of angle θ, and its matrix is $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

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- Similarly, the orthogonal projection over the (xz)-plane, resp. the (yz)-plane, is of matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, resp. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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$$f : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

is the reflection with respect to (xy)-plane, and its matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

- Similarly, the reflection with respect to the (xz)-plane, resp. the (yz)-plane, is of matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, resp. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

III) Properties of Linear Applications

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Examples :

- The linear application identity $\forall u \in E, Id_E(u) = u$ is an endomorphism.
- The linear application $\forall u = (x, y) \in \mathbb{R}^2, f(u) = (y, x)$ is an endomorphism.

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Proposition 2

Let E, E' be two $\mathbb{K}$-Vector Spaces of finite dimension and $f : E \to E'$ a linear application. Then, we have

$$\boxed{rank(f) \leq min(dimE, dimE')}$$

Exercise :

1) Determine the image and kernel of the following function f :

a) f is the function defined from $\mathbb{R}^3$ to $\mathbb{R}^3$ as :

$$f(x,y,z) = (x',y',z'), \text{ where } \begin{cases} x' = x + y - z \\ y' = 2x + y - 3z \\ z' = 3x + 2y - 4z \end{cases}$$

b) f is the function differentiation defined from $\mathbb{R}_n[X]$ to $\mathbb{R}_n[X]$, with $n \in \mathbb{N}^*$, as $f(P) = P'$.

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b) f is the function differentiation defined from $\mathbb{R}_n[X]$ to $\mathbb{R}_n[X]$, with $n \in \mathbb{N}^*$, as $f(P) = P'$.

2) Let f be the function defined from $\mathbb{R}^4$ to $\mathbb{R}^3$ as :

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Determine the rank of f and the dimension of the kernel of f.

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2. if f is surjective and S is a generating set of E, then the set $\{f(v_i)\}_{i=1\ldots p}$ is a generating set of E'.
3. if f is bijective then the image of a basis of E by f is a basis of E'.

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$$(1) \Leftrightarrow (2) \Leftrightarrow (3)$$

Exercise :

1) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x - y, x + y)$. Show that f is bijective.

2) Let f be the function defined from $\mathbb{R}_3[x]$ to $\mathbb{R}^4$ as :

$$\forall P \in \mathbb{R}_3[x],\ f(P) = \big(P(0), P(1), P(2), P(3)\big)$$

Show that f is an isomorphism.

IV) Matrices and Linear Applications

1) Rank of a set of vectors

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Example :

Determine the rank of the set $\{v_1, v_2, v_3\}$ in the Vector Space $\mathbb{R}^4$, with

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3) Rank of a matrix

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We define the rank of a matrix as the rank of the set of its column vectors.

Example :

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- The rank is then the number of non-zero columns.

Example :

1) Determine the rank of the set $\{v_1, v_2, v_3, v_4, v_5\}$ in the Vector Space $\mathbb{R}^4$, with

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 3 \\ 2 \\ -1 \\ -3 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 3 \\ 5 \\ 0 \\ -1 \end{pmatrix} \qquad v_5 = \begin{pmatrix} 3 \\ 8 \\ 1 \\ 1 \end{pmatrix}$$

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2) Consider the set $S = \{v_1, v_2, v_3, v_4\}$ in the Vector Space $\mathbb{R}^3$, with

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad v_4 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

Show that the set S generates $\mathbb{R}^3$.

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- rank(A) = rank(A^T).

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- $\mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(f)$ is the matrix whose columns are the images by f of the vectors of $\mathcal{B}$ expressed in $\mathcal{B}'$.

Example :

Determine the matrix of the linear application $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined as

$$f(x, y, z) = (x + y - z, x - 2y + 3z)$$

a) in the canonical bases $\mathcal{B}, \mathcal{B}'$,

b) in the bases $\mathcal{B}_0 = \{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$ and $\mathcal{B}'_0 = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$.

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$$\boxed{\text{Mat}_{\mathcal{B},\mathcal{B}''}(g \circ f) = \text{Mat}_{\mathcal{B}',\mathcal{B}''}(g) \times \text{Mat}_{\mathcal{B},\mathcal{B}'}(f)}$$

Exercise :

Let the two linear applications $f : E \to F$ and $g : F \to G$, with $E = \mathbb{R}^2$, $F = \mathbb{R}^3$ and $G = \mathbb{R}^2$.

Let $\mathcal{B} = (e_1, e_2)$ a basis of E, $\mathcal{B}' = (f_1, f_2, f_3)$ a basis of F and $\mathcal{B}'' = (g_1, g_2)$ a basis of G.

We consider furthermore that

$$A = \mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(f) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \mathrm{Mat}_{\mathcal{B}',\mathcal{B}''}(g) = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 1 & 2 \end{pmatrix}$$

Compute the matrix $C = \mathrm{Mat}_{\mathcal{B},\mathcal{B}''}(g \circ f)$ by means of two different methods.

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- The endomorphism rotation of angle θ centered at the origin $(r_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta - y \cos\theta))$ is of matrix
$$\text{Mat}_{\mathcal{B}}(r_\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

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Example :

Let r_θ be the rotation of angle θ in $\mathbb{R}^2$ centered at the origin. Determine the matrix of r_θ^p.

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Theorem : Characterisation of the matrix of an isomorphism

Let E, F be two $\mathbb{K}$-Vector Spaces of same finite dimension and $f : E \to F$ a linear application. Let $\mathcal{B}$ a basis of E, $\mathcal{B}'$ a basis of F and $A =\mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(f)$. Then

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Example :

Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of angle $\pi/6$ centered at the origin and s the reflection with respect to the line $y = x$. Determine the matrix of $(s \circ r)^{-1}$ in the canonical basis.

V) Change of basis

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$$Y = \mathrm{Mat}_{\mathcal{B}'}(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}_{\mathcal{B}'}$$
- Let $f : E \to F$ be a linear application and $A = \mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(f)$. If $y = f(x)$, then
$$\boxed{Y = A \times X} \quad \text{and} \quad \boxed{\mathrm{Mat}_{\mathcal{B}'}(y) = \mathrm{Mat}_{\mathcal{B},\mathcal{B}'}(f) \times \mathrm{Mat}_{\mathcal{B}}(x)}$$

Exercise :

Let E be a Vector Space of dimension 3 with a basis $\mathcal{B} = (e_1, e_2, e_3)$. Let f be an endomorphism of E of matrix

$$\mathrm{A}{=}\mathrm{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Determine the kernel and the image of f.

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2) Transformation (or transition) matrix

Definition

Let E be a Vector Space of finite dimension n and $\mathcal{B}$, $\mathcal{B}'$ two different bases of E. Then we call the **transformation (or transition) matrix** from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$, denoted $T_{\mathcal{B},\mathcal{B}'}$, the square matrix of size $n \times n$ whose columns are composed of the coordinates of the vectors of $\mathcal{B}'$ expressed in the basis $\mathcal{B}$.

Example :

Determine the transformation matrix in $\mathbb{R}^2$ from the basis $\mathcal{B} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$ to the basis $\mathcal{B}' = \{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}\}$.

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2. $T_{\mathcal{B},\mathcal{B}'}$ is invertible and its inverse matrix is $T_{\mathcal{B}',\mathcal{B}}$: $\boxed{T_{\mathcal{B}',\mathcal{B}} = \left(T_{\mathcal{B},\mathcal{B}'}\right)^{-1}}$,

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4. Let $x \in E$ with $X = \text{Mat}_{\mathcal{B}}(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{B}}$ and $X' = \text{Mat}_{\mathcal{B}'}(x) = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}_{\mathcal{B}'}$,

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   then
   $$\boxed{X = T_{\mathcal{B},\mathcal{B}'} \times X'}$$

3) Formula for basis change

Theorem

Let $f \in \mathcal{L}(E, F)$, $\mathcal{B}_E = \{e_1, \dots, e_n\}$, $\mathcal{B}'_E = \{e'_1, \dots, e'_n\}$ two bases of E and $\mathcal{B}_F = \{f_1, \dots, f_n\}$, $\mathcal{B}'_F = \{f'_1, \dots, f'_n\}$ two bases of F.

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Exercise :

Let the two bases $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\}$ and $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

Let f be the linear application of matrix in the basis $\mathcal{B}_1$

$$\text{A}=\text{Mat}_{\mathcal{B}_1}(f) = \begin{pmatrix} 1 & 0 & -6 \\ -2 & 2 & -7 \\ 0 & 0 & 3 \end{pmatrix}$$

Determine the matrix of f in the basis $\mathcal{B}_2$.

4) Matrix similarity

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Example :

Give two similar matrices for the linear application $f : (x, y) \mapsto (x - y, x + y)$.

END

References :