Matrix Diagonalization

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I) Introduction to Diagonalization

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Example :

1) Show that the matrices $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$ are similar using the matrix $P = \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$.

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2) Deduce that A is diagonalizable.

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Show that the matrices $A = \begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not similar.

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Show that the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are not diagonalizable.

II) Diagonalization and linear applications

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A is diagonalizable $\Leftrightarrow$ there exists a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{K}^n$ such that
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$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

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Let $f \in \mathcal{L}(E)$ where E is a Vector Space of finite dimension. We say that f is diagonalizable if there exists a basis $\mathcal{B}$ of E composed of eigen-vectors of f such that $\mathrm{Mat}_{\mathcal{B}}(f)$ is diagonal.

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Example :

Let the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Find the matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

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1) Let the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the characteristic polynomial of A.

2) Let the matrix $A = \begin{pmatrix} 6 & 2 & 1 \\ -6 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$.

   a) Find the characteristic polynomial of A,
   b) Find the eigen-values and the eigen-vectors of A,
   c) Diagonalize A.

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We suppose furthermore that $\forall i \in \{1 \ldots n\}$, the vector e_i is associated to the eigen-value λ_i.

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$\Rightarrow$ If the scalars $\lambda_1, \ldots, \lambda_n$ are distinct, then the set $\{e_1, \ldots, e_n\}$ is linearly independent

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For each λ_i, let n_i be the dimension of the associated eigen-space $E_{\lambda_i} = Ker(f - \lambda_i Id_E)$, and let $\{e_{i1}, e_{i2}, \ldots, e_{in_i}\}$ be a basis of E_{λ_i}.

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Consequence

Let $A \in M_n(\mathbb{K})$. If the characteristic polynomial of A has n distinct roots of $\mathbb{K}$, then A is diagonalizable.

Exercise :

Show that the following matrix is diagonalizable :

$$A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

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$$\boxed{\dim(E_{\lambda_i}) = \alpha_i}$$

Exercise :

Show that the following matrix is diagonalizable :

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

END

References :

-*Pierre Cuillot, Cours Concis, IRMA.*