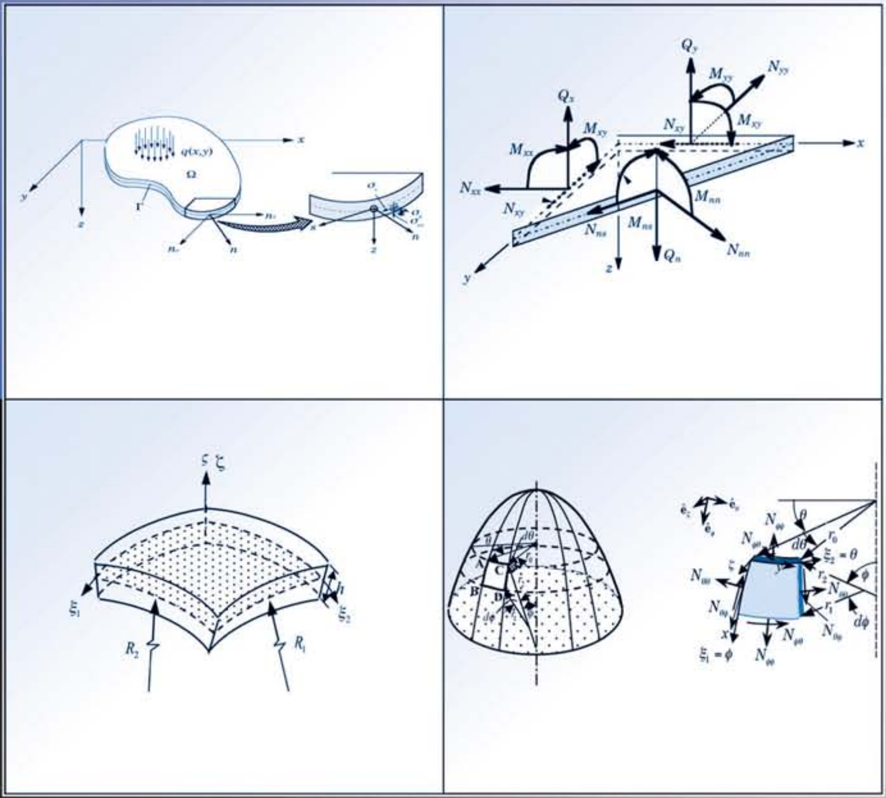
Theory and Analysis of Elastic Plates and Shells

Second Edition



J. N. Reddy

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“Whence all creation had its origin,
he, whether he fashioned it or whether he did not,
he, who surveys it all from highest heaven,
he knows—or maybe even he does not know.”

Rig Veda

Contents

Preface to the Second Edition

The objective of this second edition of *Theory and Analysis of Elastic Plates and Shells* remains the same – to present a complete and up-to-date treatment of classical as well as shear deformation plate and shell theories and their solutions by analytical and numerical methods. New material has been added in most chapters, along with some rearrangement of topics to improve the clarity of the overall presentation.

The first 10 chapters are the same as those in the first edition, with minor changes to the text. Section 2.3 on Castigliano's Theorems, Section 5.6 on axisymmetric buckling of circular plates, and Section 10.5 on relationships between the solutions of classical and shear deformation theories are new. Chapter 11 is entirely new and deals with theory and analysis of shells, while Chapter 12 is the same as the old Chapter 11, with the exception of a major new section on nonlinear finite element analysis of plates.

This edition of the book, like the first, is suitable as a textbook for a first course on theory and analysis of plates and shells in aerospace, civil, mechanical, and mechanics curricula. Due to the coverage of the linear and nonlinear finite element analysis, the book may be used as a reference for courses on finite element analysis. It can also be used as a reference by structural engineers and scientists working in industry and academia on plates and shell structures. An introductory course on mechanics of materials and elasticity should prove to be helpful, but not necessary, because a review of the basics is included in the first two chapters of the book.

A solutions manual is available from the publisher for those instructors who adopt the book as a textbook for a course.

J. N. Reddy
College Station, Texas

Preface

The objective of this book is to present a complete and up-to-date treatment of classical as well as shear deformation plate theories and their solutions by analytical and numerical methods. *Beams* and *plates* are common structural elements of most engineering structures, including aerospace, automotive, and civil engineering structures, and their study, both from theoretical and analysis points of view, is fundamental to the understanding of the behavior of such structures.

There exists a number of books on the theory of plates and most of them cover the classical Kirchhoff plate theory in detail and present the Navier solutions of the theory for rectangular plates. Much of the latest developments in shear deformation plate theories and their finite element models have not been compiled in a textbook form. The present book is aimed at filling this void in the literature.

The motivation that led to the writing of the present book has come from many years of the author's research in the development of shear deformation plate theories and their analysis by the finite element method, and also from the fact that there does not exist a book that contains a detailed coverage of shear deformation beam and plate theories, analytical solutions, and finite element models in one volume. The present book fulfills the need for a complete treatment of the classical and shear deformation theories of plates and their solution by analytical and numerical methods.

Some mathematical preliminaries, equations of elasticity, and virtual work principles and variational methods are reviewed in Chapters 1 and 2. A reader who has had a course in elasticity or energy and variational principles of mechanics may skip these chapters and go directly to Chapter 3, where a complete derivation of the equations of motion of the classical plate theory (CPT) is presented. Solutions for cylindrical bending, buckling, natural vibration, and transient response of plate strips are developed in Chapter 4. A detailed treatment of circular plates is undertaken in Chapter 5, and analytical and Rayleigh–Ritz solutions of axisymmetric and asymmetric bending are presented for various boundary conditions and loads. A brief discussion of natural vibrations of circular plates is also included here.

Chapter 6 is dedicated to the bending of rectangular plates with all edges simply supported, and the Navier and Rayleigh–Ritz solutions are presented. Bending of rectangular plates with general boundary conditions are treated in Chapter 7. The Lévy solutions are presented for rectangular plates with two parallel edges simply supported while the other two have arbitrary boundary conditions; the Rayleigh–Ritz solutions are presented for rectangular plates with arbitrary conditions. General buckling of rectangular plates under various boundary conditions is presented in Chapter 8. The Navier, Lévy, and Rayleigh–Ritz solutions are developed here. Chapter 9 is devoted to the dynamic analysis of rectangular plates, where solutions are developed for free vibration and transient response.

The first-order and third-order shear deformation plate theories are discussed in Chapter 10. Analytical solutions presented in these chapters are limited to rectangular plates with simply supported boundary conditions on all four edges (the Navier solution). Parametric effects of the material orthotropy and plate aspect ratio on bending deflections and stresses, buckling loads, and vibration frequencies are discussed. Finally, Chapter 11 deals with the linear finite element analysis of beams and plates. Finite element models based on both classical and first-order shear deformation plate theories are developed and numerical results are presented.

The book is suitable as a textbook for a first course on theory of plates in civil, aerospace, mechanical, and mechanics curricula. It can be used as a reference by engineers and scientists working in industry and academia. An introductory course on mechanics of materials and elasticity should prove to be helpful, but not necessary, because a review of the basics is included in the first two chapters of the book.

The author's research in the area of plates over the years has been supported through research grants from the Air Force Office of Scientific Research (AFOSR), the Army Research Office (ARO), and the Office of Naval Research (ONR). The support is gratefully acknowledged. The author also wishes to express his appreciation to Dr. Filis T. Kokkinos for his help with the illustrations in this book.

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J. N. Reddy is Distinguished Professor and the Holder of Oscar S. Wyatt Endowed Chair in the Department of Mechanical Engineering at Texas A&M University, College Station, Texas.

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Professor Reddy is the recipient of the *Walter L. Huber Civil Engineering Research Prize* of the American Society of Civil Engineers (ASCE), the *Worcester Reed Warner Medal* and the *Charles Russ Richards Memorial Award* of the American Society of Mechanical Engineers (ASME), the 1997 *Archie Higdon Distinguished Educator Award* from the American Society of Engineering Education (ASEE), the 1998 *Nathan M. Newmark Medal* from the American Society of Civil Engineers, the 2003 *Bush Excellence Award for Faculty in International Research* from Texas A&M University, the 2003 *Computational Solid Mechanics Award* from the U.S. Association of Computational Mechanics (USACM), and the 2000 *Excellence in the Field of Composites* and 2004 *Distinguished Research Award* from the American Society of Composites.

Professor Reddy is a Fellow of AIAA, ASCE, ASME, the American Academy of Mechanics (AAM), the American Society of Composites (ASC), the U.S. Association of Computational Mechanics (USACM), the International Association of Computational Mechanics (IACM), and the Aeronautical Society of India (ASI). Professor Reddy is the Editor-in-Chief of *Mechanics of Advanced Materials and Structures*, *International Journal of Computational Methods in Engineering Science and Mechanics,* and *International Journal of Structural Stability and Dynamics*, and he serves on the editorial boards of over two dozen other journals, including *International Journal of Non-Linear Mechanics*, *International Journal for Numerical Methods in Engineering, Computer Methods in Applied Mechanics and Engineering, Engineering Computations, Engineering Structures,* and *Applied Mechanics Reviews.*

1

Vectors, Tensors, and Equations of Elasticity

1.1 Introduction

The primary objective of this book is to study theories and analytical as well as numerical solutions of plate and shell structures, i.e., thin structural elements undergoing stretching and bending. The plate and shell theories are developed using certain assumed kinematics of deformation that facilitate writing the displacement field explicitly in terms of the thickness coordinate. Then the principle of virtual displacements and integration through the thickness are used to obtain the governing equations. The theories considered in this book are valid for *thin* and *moderately thick* plates and shells.

The governing equations of solid and structural mechanics can be derived by either vector mechanics or energy principles. In vector mechanics, better known as Newton's second law, the vector sum of forces and moments on a typical element of a structure is set to zero to obtain the equations of equilibrium or motion. In energy principles, such as the principle of virtual displacement or its derivative, the principle of minimum total potential energy, is used to obtain the governing equations. While both methods can give the same equations, the energy principles have the advantage of providing information on the form of the boundary conditions suitable for the problem. Energy principles also enable the development of refined theories of structural members that are difficult to formulate using vector mechanics. Finally, energy principles provide a natural means of determining numerical solutions of the governing equations. Hence, the energy approach is adopted in the present study to derive the governing equations of plates and shells.

In order to study theories of plates and shells, a good understanding of the basic equations of elasticity and the concepts of work done and energy stored is required. A study of these topics in turn requires familiarity with the notions of vectors, tensors, transformations of vector and tensor components, and matrices. Therefore, a brief review of vectors and tensors is presented first, followed by a review of the equations of elasticity. Readers familiar with these topics may skip the remaining portion of this chapter and go directly to Chapter 2, where the principles of virtual work and classical variational methods are discussed.

1.2 Vectors, Tensors, and Matrices

1.2.1 Preliminary Comments

All quantities appearing in analytical descriptions of physical laws can be classified into two categories: *scalars* and *nonscalars*. The scalars are given by a single real or complex number. Nonscalar quantities not only need a specified magnitude, but also additional information, such as direction and/or differentiability. Time, temperature, volume, and mass density are examples of scalars. Displacement, temperature gradient, force, moment, velocity, and acceleration are examples of nonscalars.

The term *vector* is used to imply a nonscalar that has magnitude and "direction" and obeys the parallelogram law of vector addition and rules of scalar multiplication. A vector in modern mathematical analysis is an abstraction of the elementary notion of a *physical vector*, and it is "an element from a linear vector space." While the definition of a vector in abstract analysis does not require it to have a magnitude, in nearly all cases of practical interest it does, in which case the vector is said to belong to a "normed vector space." In this book, we only need vectors from a special normed vector space − that is, physical vectors.

Not all nonscalar quantities are vectors. Some quantities require the specification of magnitude and two directions. For example, the specification of stress requires not only a force, but also an area upon which the force acts. Such quantities are called second-order tensors. Vector is a tensor of order one, while stress is a second-order tensor.

1.2.2 Components of Vectors and Tensors

In the analytical description of a physical phenomenon, a coordinate system in the chosen frame of reference is introduced and various physical quantities involved in the description are expressed in terms of measurements made in that system. The form of the equations thus depends upon the chosen coordinate system and may appear different in another type of coordinate system. The laws of nature, however, should be independent of the choice of a coordinate system, and we may seek to represent the law in a manner independent of a particular coordinate system. A way of doing this is provided by vector and tensor notation. When vector notation is used, a particular coordinate system need not be introduced. Consequently, use of vector notation in formulating natural laws leaves them *invariant* to coordinate transformations.

Often a specific coordinate system that facilitates the solution is chosen to express governing equations of a problem. Then the vector and tensor quantities appearing in the equations are expressed in terms of their components in that coordinate system. For example, a vector $\mathbf{A}$ in a three-dimensional space may be expressed in terms of its components a_1, a_2, and a_3 and *basis vectors* $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ as

$$\mathbf{A} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1.2.1}$$

If the basis vectors of a coordinate system are constants, i.e., with fixed lengths and directions, the coordinate system is called a *Cartesian coordinate system*. The general Cartesian system is oblique. When the Cartesian system is orthogonal, it is called *rectangular Cartesian*. When the basis vectors are of unit length and mutually orthogonal, they are called *orthonormal*. We denote an orthonormal Cartesian basis by

$$(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) \quad \text{or} \quad (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) \tag{1.2.2}$$

The Cartesian coordinates are denoted by

$$(x_1, x_2, x_3) \quad \text{or} \quad (x, y, z) \tag{1.2.3}$$

The familiar rectangular Cartesian coordinate system is shown in Figure 1.2.1. We shall always use a right-hand coordinate system.

A second-order tensor, also called a *dyad*, can be expressed in terms of its rectangular Cartesian system as

$$\begin{aligned}\mathbf{\Phi} = {} & \varphi_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \varphi_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + \varphi_{13}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 \\ & + \varphi_{21}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 + \varphi_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \varphi_{23}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 \\ & + \varphi_{31}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 + \varphi_{32}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 + \varphi_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\end{aligned} \tag{1.2.4}$$

Here we have selected a rectangular Cartesian basis to represent the second-order tensor $\mathbf{\Phi}$. The first- and second-order tensors (i.e., vectors and dyads) will be of greatest utility in the present study.

1.2.3 Summation Convention

It is convenient to abbreviate a summation of terms by understanding that a *once repeated* index means summation over all values of that index. For example, the component form of vector $\mathbf{A}$

$$\mathbf{A} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 \tag{1.2.5}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are basis vectors (not necessarily unit), can be expressed in the form

$$\mathbf{A} = \sum_{j=1}^{3} a^j\mathbf{e}_j = a^j\mathbf{e}_j \tag{1.2.6}$$

The repeated index is a *dummy index* in the sense that any other symbol that is not already used in that expression can be used:

$$\mathbf{A} = a^j\mathbf{e}_j = a^k\mathbf{e}_k = a^m\mathbf{e}_m \tag{1.2.7}$$

An index that is not repeated is called a *free index*, like j in $A_iB_iC_j$. The range of summation is always known in the context of the discussion. For example, in the present context, the range of j, k, and m is 1 to 3, because we are discussing vectors in a three-dimensional space.

A vector $\mathbf{A}$ and a second-order tensor $\mathbf{P}$ can be expressed in a short form using the summation convention

$$\mathbf{A} = A_i\hat{\mathbf{e}}_i, \quad \mathbf{P} = P_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{1.2.8}$$

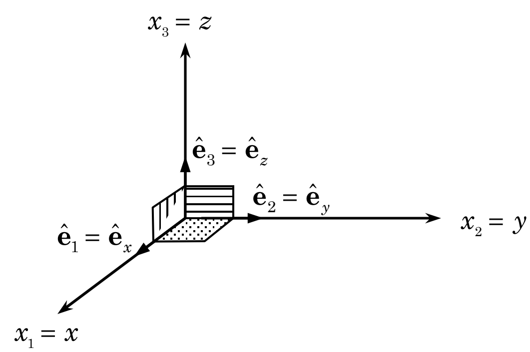


Figure 1.2.1 A rectangular Cartesian coordinate system, $(x_1, x_2, x_3) = (x, y, z)$; $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) = (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z)$ are the unit basis vectors.

and an nth-order tensor has the form

$$\mathbf{\Phi} = \varphi_{ijk\ell\ldots}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j\hat{\mathbf{e}}_k\hat{\mathbf{e}}_\ell \cdots \tag{1.2.9}$$

A unit second-order tensor $\mathbf{I}$ in a rectangular cartesian system is represented as

$$\mathbf{I} = \delta_{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j \tag{1.2.10}$$

where δ_{ij}, called the *Kronecker delta*, is defined as

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{1.2.11}$$

for any values of i and j. Note that $\delta_{ij} = \delta_{ji}$ and $\delta_{ij} \equiv \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$. In Eqs. (1.2.8)–(1.2.10) we have chosen a rectangular Cartesian basis to represent the tensors.

The *dot product* (or *scalar product*) and *cross product* (or *vector product*) of vectors can be defined in terms of their components with the help of Kronecker delta symbol and

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order and } i \neq j \neq k \\ -1, & \text{if } i, j, k \text{ are not in cyclic order and } i \neq j \neq k \\ 0, & \text{if any of } i, j, k \text{ are repeated} \end{cases} \tag{1.2.12}$$

The symbol ε_{ijk} is called the *alternating symbol permutation symbol*, or *alternating tensor*, since it is a Cartesian component of a third-order tensor.

In an orthonormal basis, the scalar and vector products can be expressed in the index form using the Kronecker delta and the alternating symbol

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \cdot (B_j\hat{\mathbf{e}}_j) = A_iB_j\delta_{ij} = A_iB_i \\ \mathbf{A} \times \mathbf{B} &= (A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j) = A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k \end{aligned} \tag{1.2.13}$$

Note that

$$\hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) = \varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}; \quad \varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} \tag{1.2.14}$$

The parenthesis around $\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k$ may be omitted since the expression makes no sense any other way. Thus, a cyclic permutation of the indices does not change the sign, while the interchange of any two indices will change the sign. Further, the Kronecker delta and the permutation symbol are related by the identity, known as the *ε-δ identity*

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{1.2.15}$$

Note that in the above expression i is a dummy index while j, k, m, and n are free indices.

The permutation symbol and the Kronecker delta prove to be very useful in proving vector identities and simplifying vector equations. The following example illustrates some of the uses of δ_{ij} and ε_{ijk}.

Example 1.2.1

We wish to simplify the vector operation $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$ in an alternate vector form and thereby establish a vector identity. We begin with

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (A_i\hat{\mathbf{e}}_i) \times (B_j\hat{\mathbf{e}}_j) \cdot (C_m\hat{\mathbf{e}}_m) \times (D_n\hat{\mathbf{e}}_n) \\
&= (A_iB_j\varepsilon_{ijk}\hat{\mathbf{e}}_k) \cdot (C_mD_n\varepsilon_{mnp}\hat{\mathbf{e}}_p) \\
&= A_iB_jC_mD_n\varepsilon_{ijk}\varepsilon_{mnp}\delta_{kp} \\
&= A_iB_jC_mD_n\varepsilon_{ijk}\varepsilon_{mnk} \\
&= A_iB_jC_mD_n(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \\
&= A_iB_jC_mD_n\delta_{im}\delta_{jn} - A_iB_jC_mD_n\delta_{in}\delta_{jm}
\end{aligned}$$

where we have used the ε-δ identity. Since $C_m\delta_{im} = C_i$, $A_i\delta_{im} = A_m$, and $A_iC_i = \mathbf{A} \cdot \mathbf{C}$, and so on, we can write

$$\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= A_iB_jC_iD_j - A_iB_jC_jD_i \\
&= A_iC_iB_jD_j - A_iB_jD_jC_i \\
&= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})
\end{aligned}$$

Although the above vector identity is established using an orthonormal basis, it holds in a general coordinate system.

1.2.4 The Del Operator

A position vector to an arbitrary point (x, y, z) or (x_1, x_2, x_3) in a body, measured from the origin, is given by (sometimes denoted by $\mathbf{x}$)

$$\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z = x_1\hat{\mathbf{e}}_1 + x_2\hat{\mathbf{e}}_2 + x_3\hat{\mathbf{e}}_3 \tag{1.2.16}$$

or, in summation notation, by

$$\mathbf{r} = x_j\hat{\mathbf{e}}_j \ (= \mathbf{x}) \tag{1.2.17}$$

Consider a scalar field ϕ which is a function of the position vector, $\phi = \phi(\mathbf{r})$. The differential change is given by

$$d\phi = \frac{\partial \phi}{\partial x_i}\, dx_i \tag{1.2.18}$$

The differentials dx_i are components of $d\mathbf{r}$, that is,

$$d\mathbf{r} = dx_i \hat{\mathbf{e}}_i$$

We would now like to write $d\phi$ in such a way that we account for the *direction* as well as the magnitude of $d\mathbf{r}$. Since $dx_i = d\mathbf{r} \cdot \hat{\mathbf{e}}_i$, we can write

$$d\phi = d\mathbf{r} \cdot \left(\hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i} \right)$$

Let us now denote the magnitude of $d\mathbf{r}$ by $ds \equiv |d\mathbf{r}|$. Then $\hat{\mathbf{e}} = d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$, and we have

$$\left(\frac{d\phi}{ds} \right)_{\hat{\mathbf{e}}} = \hat{\mathbf{e}} \cdot \left(\hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i} \right) \tag{1.2.19}$$

The derivative $(d\phi/ds)_{\hat{\mathbf{e}}}$ is called the *directional derivative of* ϕ. We see that *it is the rate of change of* ϕ *with respect to distance* and that *it depends on the direction* $\hat{\mathbf{e}}$ *in which the distance is taken.*

The vector that is scalar multiplied by $\hat{\mathbf{e}}$ can be obtained immediately whenever the scalar field is given. Because the magnitude of this vector is equal to the maximum value of the directional derivative, it is called the *gradient vector* and is denoted by grad ϕ:

$$\text{grad}\ \phi \equiv \hat{\mathbf{e}}_i \frac{\partial \phi}{\partial x_i} \tag{1.2.20}$$

We interpret grad ϕ as some operator operating on ϕ, that is, grad $\phi = \nabla\phi$. This operator is denoted, in a general coordinate system (q^1, q^2, q^3) with basis $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3)$, by

$$\nabla \equiv \mathbf{e}^1 \frac{\partial}{\partial q^1} + \mathbf{e}^2 \frac{\partial}{\partial q^2} + \mathbf{e}^3 \frac{\partial}{\partial q^3} = \mathbf{e}^i \frac{\partial}{\partial q^i} \tag{1.2.21}$$

and is called the del operator. In rectangular Cartesian system, we have

$$\nabla \equiv \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{1.2.22}$$

The del operator is a *vector differential* operator. However, it is important to note that the del operator has some of the properties of a vector, but it does not have them all because it is an operator. For example, $\nabla \cdot \mathbf{A}$ is a scalar, called the *divergence* of $\mathbf{A}$, whereas $\mathbf{A} \cdot \nabla$ is a scalar *differential operator*. Thus, the del operator does not commute in this sense. The operation $\nabla \times \mathbf{A}$ is called the *curl* of $\mathbf{A}$. The gradient of a vector is a second-order tensor. The scalar differential operator

$\nabla \cdot \nabla = \nabla^2$ is known as the *Laplacian* operator. The properties of the del operator are illustrated in Example 1.2.2.

Example 1.2.2

Here we illustrate the use of index notation and the del operator to rewrite the following vector expressions in alternative forms:

(a) grad $(\mathbf{r} \cdot \mathbf{A})$
(b) div $(\mathbf{r} \times \mathbf{A})$
(c) curl $(\mathbf{r} \times \mathbf{A})$

where $\mathbf{A} = \mathbf{A}(r)$ is a vector and $\mathbf{r}$ is the position vector with magnitude r.

For simplicity, we use the rectangular Cartesian basis to establish the identities.

(a) Expressing in components

$$\begin{aligned}
\nabla(\mathbf{r} \cdot \mathbf{A}) &= \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right)(x_j \hat{\mathbf{e}}_j \cdot A_k \hat{\mathbf{e}}_k) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j A_k \delta_{jk}) \\
&= \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j A_j) = \hat{\mathbf{e}}_i \left(\frac{\partial x_j}{\partial x_i} A_j + x_j \frac{\partial A_j}{\partial x_i}\right) \\
&= \hat{\mathbf{e}}_i (\delta_{ij} A_j + x_j A_{j,i}) = \hat{\mathbf{e}}_i A_i + \hat{\mathbf{e}}_i x_j A_{j,i}
\end{aligned}$$

The first expression is clearly equal to $\mathbf{A}$. A close examination of the second expression on the right side of the equality indicates that the basis vector goes with the differential operator because both have the same subscript i. Also x_j and A_j have the same subscript, implying that there is a dot product operation between them. Hence, we can write

$$\hat{\mathbf{e}}_i \frac{\partial A_j}{\partial x_i} x_j = (\nabla \mathbf{A}) \cdot \mathbf{r}$$

Thus, we have

$$\nabla(\mathbf{r} \cdot \mathbf{A}) = \mathbf{A} + (\nabla \mathbf{A}) \cdot \mathbf{r}$$

Note that $\nabla \mathbf{A}$ is a second-order tensor $F_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$ with components $F_{ij} = \partial A_j / \partial x_i$.

(b) Writing in components form

$$\begin{aligned}
\nabla \cdot (\mathbf{r} \times \mathbf{A}) &= \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right) \cdot (x_j \hat{\mathbf{e}}_j \times A_k \hat{\mathbf{e}}_k) = \hat{\mathbf{e}}_i \cdot (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) \frac{\partial}{\partial x_i}(x_j A_k) \\
&= \varepsilon_{ijk} \left(\frac{\partial x_j}{\partial x_i} A_k + x_j \frac{\partial A_k}{\partial x_i}\right) = \varepsilon_{ijk} (\delta_{ij} A_k + x_j A_{k,i}) \\
&= \varepsilon_{iik} A_k + \varepsilon_{ijk} x_j A_{k,i}
\end{aligned}$$

Clearly the first term is zero on account of the repeated subscript in the permutation symbol. Now consider the second term

$$\varepsilon_{ijk} x_j A_{k,i} = -A_{k,i} \varepsilon_{ikj} x_j = -(\nabla \times \mathbf{A}) \cdot \mathbf{r}$$

Thus, we have

$$\nabla \cdot (\mathbf{r} \times \mathbf{A}) = -(\nabla \times \mathbf{A}) \cdot \mathbf{r}$$

(c) Writing in cartesian component form

$$
\begin{aligned}
\nabla \times (\mathbf{r} \times \mathbf{A}) &= \left(\hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}\right) \times (x_j \hat{\mathbf{e}}_j \times A_k \hat{\mathbf{e}}_k) = \hat{\mathbf{e}}_i \times (\hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k) \frac{\partial}{\partial x_i} (x_j A_k) \\
&= \varepsilon_{jkm} \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_m \left(\frac{\partial x_j}{\partial x_i} A_k + x_j \frac{\partial A_k}{\partial x_i}\right) \\
&= \varepsilon_{jkm} \varepsilon_{imn} \hat{\mathbf{e}}_n \left(\delta_{ij} A_k + x_j A_{k,i}\right) \\
&= (\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn}) \left(\delta_{ij} A_k + x_j A_{k,i}\right) \hat{\mathbf{e}}_n \\
&= (A_n - 3A_n + x_n A_{i,i} - x_i A_{n,i}) \hat{\mathbf{e}}_n \\
&= -2\mathbf{A} + \mathbf{r}(\nabla \cdot \mathbf{A}) - (\mathbf{r} \cdot \nabla)\mathbf{A} \\
&= \mathbf{r} \ \mathrm{div}\mathbf{A} - 2\mathbf{A} - \mathbf{r} \cdot \mathrm{grad}\mathbf{A}
\end{aligned}
$$

where we have used the identities $\delta_{jn}\delta_{ij} = \delta_{in}$, $\delta_{ij}\delta_{ij} = \delta_{ii}$ and $\delta_{ii} = 3$.

In an orthogonal curvilinear coordinate system, the del operator can be expressed in terms of its orthonormal basis. Two commonly used orthogonal curvilinear coordinate systems are *cylindrical coordinates* and *spherical coordinates,* which are shown in Figure 1.2.2. Here we summarize the relationships between the rectangular Cartesian and the two orthogonal curvilinear coordinate systems and the forms of the del and Laplacian operators.

Cylindrical Coordinates (r, θ, z)

The rectangular Cartesian coordinates (x, y, z) are related to the cylindrical coordinates (r, θ, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z \tag{1.2.23}$$

and the inverse relations are

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z \tag{1.2.24}$$

The basis vectors are

$$\mathbf{e}_r = \hat{\mathbf{e}}_r, \qquad \mathbf{e}_\theta = r\hat{\mathbf{e}}_\theta, \qquad \mathbf{e}_z = \hat{\mathbf{e}}_z \tag{1.2.25}$$

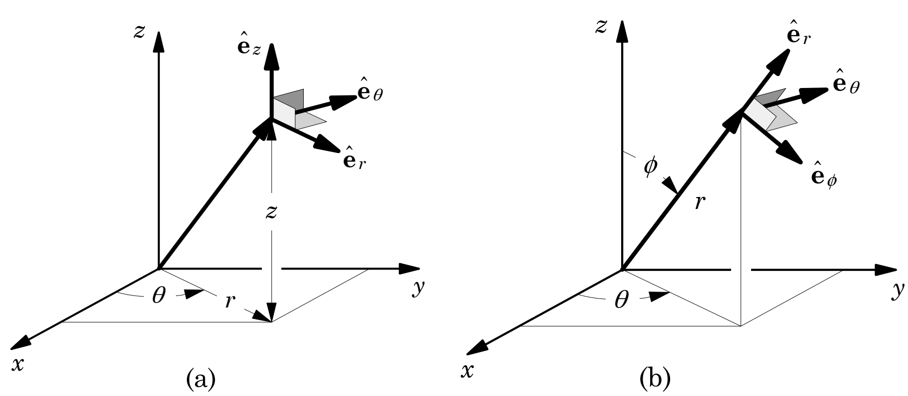


Figure 1.2.2 (a) Cylindrical coordinate system. (b) Spherical coordinate system.

In terms of the Cartesian basis, we have

$$
\begin{aligned}
\hat{\mathbf{e}}_r &= \cos\theta\ \hat{\mathbf{e}}_x + \sin\theta\ \hat{\mathbf{e}}_y \\
\hat{\mathbf{e}}_\theta &= -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y \\
\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z
\end{aligned}
\tag{1.2.26}
$$

The Cartesian basis in terms of the cylindrical orthonormal basis is

$$
\begin{aligned}
\hat{\mathbf{e}}_x &= \cos\theta\ \hat{\mathbf{e}}_r - \sin\theta\ \hat{\mathbf{e}}_\theta \\
\hat{\mathbf{e}}_y &= \sin\theta\ \hat{\mathbf{e}}_r + \cos\theta\ \hat{\mathbf{e}}_\theta \\
\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z
\end{aligned}
\tag{1.2.27}
$$

The position vector is given by

$$
\mathbf{r} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z \tag{1.2.28}
$$

The nonzero derivatives of the basis vectors are

$$
\begin{aligned}
\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} &= -\sin\theta\ \hat{\mathbf{e}}_x + \cos\theta\ \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_\theta \\
\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} &= -\cos\theta\ \hat{\mathbf{e}}_x - \sin\theta\ \hat{\mathbf{e}}_y = -\hat{\mathbf{e}}_r
\end{aligned}
\tag{1.2.29}
$$

The del (∇) and Laplacian ($\nabla^2 = \nabla \cdot \nabla$) operators are given by

$$
\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{1.2.30}
$$

$$
\nabla^2 = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + r\frac{\partial^2}{\partial z^2}\right] \tag{1.2.31}
$$

Then the gradient, divergence, and curl of a vector $\mathbf{u}$ in the cylindrical coordinate system can be obtained as (see Problems 1.10 and 1.11 at the end of the chapter)

$$
\begin{aligned}
\nabla \mathbf{u} = {} & \frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta \\
& + \frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r
\end{aligned}
\tag{1.2.32}
$$

$$
\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{1.2.33}
$$

$$
\nabla \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{e}}_z \tag{1.2.34}
$$

Spherical Coordinates (r, ϕ, θ)

The rectangular Cartesian coordinates (x, y, z) are related to the spherical coordinates (r, ϕ, θ) by

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned} \tag{1.2.35}$$

and the inverse relations are

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \phi &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{1.2.36}$$

Note that r in the spherical coordinates is not the same as that in the cylindrical coordinate system.

The basis vectors are

$$\mathbf{e}_r = \hat{\mathbf{e}}_r, \qquad \mathbf{e}_\phi = \hat{\mathbf{e}}_\phi, \qquad \mathbf{e}_\theta = r \sin\theta \; \hat{\mathbf{e}}_\theta \tag{1.2.37}$$

It is clear that the basis vectors are not unit vectors in general. In terms of the Cartesian basis, we have

$$\begin{aligned} \hat{\mathbf{e}}_r &= \sin\phi \cos\theta \; \hat{\mathbf{e}}_x + \sin\phi \sin\theta \; \hat{\mathbf{e}}_y + \cos\phi \; \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\phi &= \cos\phi \cos\theta \; \hat{\mathbf{e}}_x + \cos\phi \sin\theta \; \hat{\mathbf{e}}_y - \sin\phi \; \hat{\mathbf{e}}_z \\ \hat{\mathbf{e}}_\theta &= -\sin\theta \; \hat{\mathbf{e}}_x + \cos\theta \; \hat{\mathbf{e}}_y \end{aligned} \tag{1.2.38}$$

On the other hand, the Cartesian basis in terms of the spherical orthonormal basis is

$$\begin{aligned} \hat{\mathbf{e}}_x &= \sin\phi \cos\theta \hat{\mathbf{e}}_r + \cos\phi \cos\theta \; \hat{\mathbf{e}}_\phi - \sin\theta \; \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_y &= \sin\phi \sin\theta \; \hat{\mathbf{e}}_r + \cos\phi \sin\theta \; \hat{\mathbf{e}}_\phi + \cos\phi \; \hat{\mathbf{e}}_\theta \\ \hat{\mathbf{e}}_z &= \cos\phi \; \hat{\mathbf{e}}_r - \sin\phi \; \hat{\mathbf{e}}_\phi \end{aligned} \tag{1.2.39}$$

The position vector is given by

$$\mathbf{r} = r\hat{\mathbf{e}}_r \tag{1.2.40}$$

and the nonzero derivatives of the basis vectors are given by

$$\begin{aligned} &\frac{\partial \hat{\mathbf{e}}_r}{\partial \phi} = \hat{\mathbf{e}}_\phi, \qquad \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \sin\phi \; \hat{\mathbf{e}}_\phi, \qquad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} = -\hat{\mathbf{e}}_r \\ &\frac{\partial \hat{\mathbf{e}}_\phi}{\partial \theta} = \cos\phi \; \hat{\mathbf{e}}_\theta, \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\sin\phi \; \hat{\mathbf{e}}_r - \cos\phi \; \hat{\mathbf{e}}_\phi \end{aligned} \tag{1.2.41}$$

The del (∇) and Laplacian (∇^2) operators are given by

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\phi}{r}\frac{\partial}{\partial \phi} + \frac{1}{r\sin\phi}\hat{\mathbf{e}}_\theta \frac{\partial}{\partial \theta} \tag{1.2.42}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2} \tag{1.2.43}$$

The gradient, divergence, and curl of a vector $\mathbf{u}$ in the spherical coordinate system can be obtained as (see Problem 1.15)

$$\begin{aligned}
\nabla\mathbf{u} = {} & \frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_r + \frac{\partial u_\phi}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_\phi + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_\theta \\
& + \frac{1}{r}\left(\frac{\partial u_r}{\partial \phi} - u_\phi\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_r + \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\phi + \frac{1}{r}\frac{\partial u_\theta}{\partial \phi}\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta \\
& + \frac{1}{r\sin\phi}\left[\left(\frac{\partial u_r}{\partial \theta} - u_\theta\sin\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_r + \left(\frac{\partial u_\phi}{\partial \theta} - u_\theta\,\cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\phi\right. \\
& \left. + \left(\frac{\partial u_\theta}{\partial \theta} + u_r\sin\phi + u_\phi\cos\phi\right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\theta\right]
\end{aligned} \tag{1.2.44}$$

$$\nabla\cdot\mathbf{u} = 2\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(u_\phi\sin\phi)}{\partial \phi} + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta} \tag{1.2.45}$$

$$\begin{aligned}
\nabla\times\mathbf{u} = {} & \frac{1}{r\sin\phi}\left[\frac{\partial(u_\theta\,\sin\phi)}{\partial \phi} - \frac{\partial u_\phi}{\partial \theta}\right]\hat{\mathbf{e}}_r + \left[\frac{1}{r\sin\phi}\frac{\partial u_r}{\partial \theta} - \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\right]\hat{\mathbf{e}}_\phi \\
& + \frac{1}{r}\left[\frac{\partial(ru_\phi)}{\partial r} - \frac{\partial u_r}{\partial \phi}\right]\hat{\mathbf{e}}_\theta
\end{aligned} \tag{1.2.46}$$

1.2.5 Matrices and Cramer's Rule

A second-order tensor has nine components in any coordinate system. Equation (1.2.4) contains the Cartesian components of a tensor $\mathbf{\Phi}$. The form of the equation suggests writing down the scalars φ_{ij} (component in the ith row and jth column) in the rectangular array $[\Phi]$:

$$[\Phi] = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix}$$

The rectangular array of φ_{ij}'s is called a *matrix* when it satisfies certain properties. It is also common to use a boldface letter to denote a matrix (i.e., $[\Phi] = \mathbf{\Phi}$).

Two matrices are equal if and only if they are identical, i.e., all elements of the two arrays are the same. If a matrix has m rows and n columns, we will say that it is m by n ($m \times n$), the number of rows always being listed first. The element in the ith row and jth column of a matrix $[C]$ is generally denoted by c_{ij}, and we will sometimes write $[C] = [c_{ij}]$ to denote this. A square matrix is one that has the same number of rows as columns. The elements of a square matrix for which the row number and the column number are the same (i.e., c_{11}, $c_{22}, \cdots, c_{nn}$) are called *diagonal elements*. The sum of the diagonal elements is called the *trace* of the matrix. The line of elements where the row number and the column number are the same is called the *diagonal*. A square matrix is said to be a *diagonal matrix* if all of the off-diagonal elements are zero (i.e., $c_{ij} = 0$ for $i \neq j$). An *identity matrix*, denoted by $[I]$, is a diagonal matrix whose elements are all 1's. The elements of the identity matrix can be written as $[I] = [\delta_{ij}]$.

We wish to associate with an $n \times n$ matrix $[C]$ a scalar that, in some sense, measures the "size" of $[C]$ and indicates whether $[C]$ is singular or nonsingular. The *determinant* of the matrix $[C] = [c_{ij}]$ is defined to be the scalar det $[C] = |C|$ computed according to the rule (in this part of the chapter, the summation convention is not adopted)

$$\det[C] = |c_{ij}| = \sum_{i=1}^{n} (-1)^{i+1} c_{i1} |c_{i1}| \tag{1.2.47}$$

where $|c_{i1}|$ is the determinant of the $(n-1) \times (n-1)$ matrix that remains after deleting the ith row and the first column of $[C]$. For 1×1 matrices, the determinant is defined according to $|c_{11}| = c_{11}$. For convenience, we define the determinant of a *zeroth-order* matrix to be unity. In the above definition, special attention is given to the first column of the matrix $[C]$. We call it the expansion of $|C|$ according to the first column of $[C]$. One can expand $|C|$ according to any column or row:

$$|C| = \sum_{i=1}^{n} (-1)^{i+j} c_{ij} |c_{ij}| \tag{1.2.48}$$

where $|c_{ij}|$ is the determinant of the matrix obtained by deleting the ith row and jth column of matrix $[C]$.

For an $n \times n$ matrix $[C]$, the determinant of the $(n-1) \times (n-1)$ submatrix of $[C]$ obtained by deleting row i and column j of $[C]$ is called *minor* of c_{ij} and is denoted by $M_{ij}(C)$. The quantity $\mathrm{cof}_{ij}(C) \equiv (-1)^{i+j} M_{ij}(C)$ is called the *cofactor* of c_{ij}. The determinant of $[C]$ can be cast in terms of the minor and cofactor of c_{ij} for any value of j:

$$\det[C] = \sum_{i=1}^{n} c_{ij} \ \mathrm{cof}_{ij}(C) \tag{1.2.49}$$

The *adjunct* (also called *adjoint*) of a matrix $[C]$ is the transpose of the matrix obtained from $[C]$ by replacing each element by its cofactor. The adjunct of $[C]$ is denoted by Adj(C).

Let $[C]$ be an $n \times n$ square matrix with elements c_{ij}. The cofactor of c_{ij}, denoted here by C_{ij}, is related to the minor M_{ij} of c_{ij} by

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{1.2.50}$$

By definition, the determinant $|C|$ is given by

$$|C| = \sum_{k=1}^{n} c_{ik} C_{ik} = \sum_{k=1}^{n} c_{kj} C_{kj} \tag{1.2.51}$$

for any value of i and j $(i, j \leq n)$. Then we have

$$\sum_{k=1}^{n} c_{rk} C_{ik} = |C|\ \delta_{ri}, \qquad \sum_{k=1}^{n} c_{ks} C_{kj} = |C|\ \delta_{sj} \tag{1.2.52}$$

Using the above result, one can show that the solution to a set of n linear equations in n unknown quantities

$$\sum_{j=1}^{n} c_{ij} u_j = f_i, \qquad i = 1, 2, \ldots, n \tag{1.2.53}$$

in the case when the determinant of the matrix of coefficients is not zero ($|C| \neq 0$) is given by

$$u_j = \frac{1}{|C|} \sum_{i=1}^{n} C_{ij} b_i, \qquad j = 1, 2, \ldots, n \tag{1.2.54}$$

The result in Eq. (1.2.54) is known as *Cramer's rule.*

Example 1.2.3

Consider the system of equations

$$\begin{bmatrix} 6 & 3 & -1 \\ 1 & 4 & -3 \\ 2 & -2 & 5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ -5 \\ 8 \end{Bmatrix}$$

Using Cramer's rule, we obtain

$$u_1 = \frac{1}{|C|} \begin{vmatrix} f_1 & 3 & -1 \\ f_2 & 4 & -3 \\ f_3 & -2 & 5 \end{vmatrix}, \quad u_2 = \frac{1}{|C|} \begin{vmatrix} 6 & f_1 & -1 \\ 1 & f_2 & -3 \\ 2 & f_3 & 5 \end{vmatrix}, \quad u_3 = \frac{1}{|C|} \begin{vmatrix} 6 & 3 & f_1 \\ 1 & 4 & f_2 \\ 2 & -2 & f_3 \end{vmatrix}$$

where $|C|$ is the determinant of the coefficient matrix

$$\begin{aligned} |C| &= \sum_{i=1}^{3} (-1)^{i+1} c_{i1} |c_{i1}| \\ &= 6 \begin{vmatrix} 4 & -3 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 4 \\ 2 & -2 \end{vmatrix} \\ &= 6(20 - 6) - 3(5 + 6) - (-2 - 8) = 61 \end{aligned}$$

Hence, the solution becomes

$$\begin{aligned} u_1 &= \frac{1}{61} (14 f_1 - 13 f_2 - 5 f_3) = \frac{123}{61} \\ u_2 &= \frac{1}{61} (-11 f_1 + 32 f_2 + 17 f_3) = -\frac{101}{61} \\ u_3 &= \frac{1}{61} (-10 f_1 + 18 f_2 + 21 f_3) = \frac{8}{61} \end{aligned}$$

1.2.6 Transformations of Components

In structural analysis, one is required to refer all quantities used in the analytical description of a structure to a common coordinate system. Scalars, by definition, are independent of any coordinate system. While vectors and tensors are independent of a particular coordinate system, their components are not. The same vector, for example, can have different components in different coordinate systems. Any two sets of components of a vector and tensor can be related by writing one set of components in terms of the other. Such relationships are called transformations.

To establish the rules of the transformation of vector components, we consider barred $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and unbarred (x_1, x_2, x_3) coordinate systems that are related by the equations

$$\begin{aligned} x_1 &= x_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_2 &= x_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_3 &= x_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) \end{aligned} \tag{1.2.55}$$

The inverse relations are

$$\begin{aligned} \bar{x}_1 &= \bar{x}_1(x_1, x_2, x_3) \\ \bar{x}_2 &= \bar{x}_2(x_1, x_2, x_3) \\ \bar{x}_3 &= \bar{x}_3(x_1, x_2, x_3) \end{aligned} \tag{1.2.56}$$

The bases in the two systems are denoted $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\bar{\mathbf{e}}}_1, \hat{\bar{\mathbf{e}}}_2, \hat{\bar{\mathbf{e}}}_3)$, as shown in Figure 1.2.3. The relations between the bases of the barred and unbarred coordinate systems can be written as $[\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}}_i)\hat{\mathbf{e}}_i]$

$$\hat{\bar{\mathbf{e}}}_i = (\hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_k)\ \hat{\mathbf{e}}_k = a_{ik}\hat{\mathbf{e}}_k, \quad a_{ij} \equiv \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_j \cdot \hat{\bar{\mathbf{e}}}_i \tag{1.2.57}$$

where the first subscript of a_{ij} comes from the base vector of the barred system. In matrix notation, we can write the relations in Eq. (1.2.57) as

$$\begin{Bmatrix} \hat{\bar{\mathbf{e}}}_1 \\ \hat{\bar{\mathbf{e}}}_2 \\ \hat{\bar{\mathbf{e}}}_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix} \tag{1.2.58}$$

where $[A]$ denotes the 3×3 matrix array whose elements are the direction cosines, a_{ij}. We also have the inverse relation

$$\hat{\mathbf{e}}_j = (\hat{\mathbf{e}}_j \cdot \hat{\bar{\mathbf{e}}}_k)\ \hat{\bar{\mathbf{e}}}_k = a_{kj}\hat{\bar{\mathbf{e}}}_k \tag{1.2.59}$$

or, in matrix form

$$\begin{Bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{Bmatrix} \hat{\bar{\mathbf{e}}}_1 \\ \hat{\bar{\mathbf{e}}}_2 \\ \hat{\bar{\mathbf{e}}}_3 \end{Bmatrix} \tag{1.2.60}$$

From Eqs. (1.2.57) and (1.2.59), we note that

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij} \tag{1.2.61}$$

or

$$[A][A]^{\mathrm{T}} = [A]^{\mathrm{T}}[A] = [I] \tag{1.2.62}$$

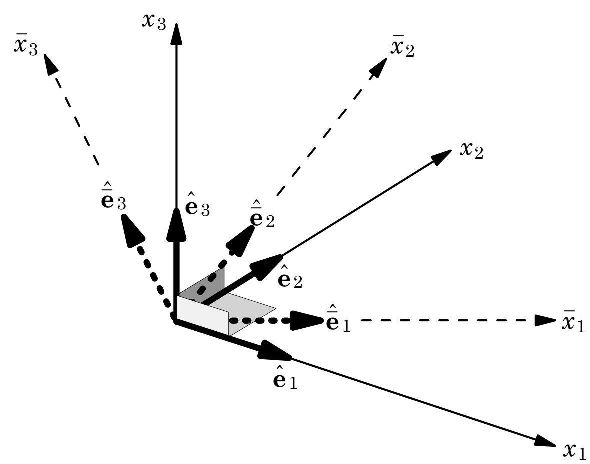


Figure 1.2.3 Unbarred and barred rectangular coordinate systems.

In other words, $[A]^{\mathrm{T}}$ is equal to its inverse. Such transformations are called *orthogonal transformations* and $[A]$ is called an orthogonal matrix.

The transformations (1.2.57) and (1.2.59) between two orthogonal sets of bases also hold for their respective coordinates:

$$\bar{x}_i = a_{ik} x_k \ , \qquad x_j = a_{kj} \bar{x}_k \tag{1.2.63}$$

Analogous to Eq. (1.2.63), the components of a vector $\mathbf{u}$ in the barred and unbarred coordinate systems are related by the expressions

$$\bar{u}_i = a_{ij} u_j \quad (\{\bar{u}\} = [A]\{u\}) \, , \quad u_i = a_{ji} \bar{u}_j \quad \Big(\{u\} = [A]^{\mathrm{T}}\{\bar{u}\}\Big) \tag{1.2.64}$$

A second-order tensor can be expressed in two different coordinate systems using the corresponding bases. In a barred system, we have

$$\mathbf{\Phi} = \bar{\varphi}_{mn} \hat{\bar{\mathbf{e}}}_m \hat{\bar{\mathbf{e}}}_n \tag{1.2.65}$$

and in an unbarred system, it is expressed as

$$\mathbf{\Phi} = \varphi_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \tag{1.2.66}$$

Using Eq. (1.2.59) for $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}_j$ in Eq. (1.2.66), we arrive at the equation

$$\mathbf{\Phi} = \varphi_{ij} a_{mi} a_{nj} \hat{\bar{\mathbf{e}}}_m \hat{\bar{\mathbf{e}}}_n \tag{1.2.67}$$

Comparing Eq. (1.2.67) with Eq. (1.2.65), we arrive at the relations

$$\bar{\varphi}_{mn} = \varphi_{ij} a_{mi} a_{nj} \tag{1.2.68}$$

or, in matrix form, we have

$$[\bar{\varphi}] = [A][\varphi][A]^{\mathrm{T}} \tag{1.2.69}$$

The inverse relation can be derived using the orthogonality property of $[A]$: $[A]^{-1} = [A]^{\mathrm{T}}$. Premultiply both sides of Eq. (1.2.69) with $[A]^{-1} = [A]^{\mathrm{T}}$ and postmultiply both sides of the resulting equation with $([A]^{\mathrm{T}})^{-1} = [A]$, and obtain the result

$$[\varphi] = [A]^{\mathrm{T}}[\bar{\varphi}][A] \tag{1.2.70}$$

Equations (1.2.69) and (1.2.70) are useful in transforming second-order tensors (e.g., stresses and strains) from one coordinate system to another coordinate system, as we shall see shortly.

Example 1.2.4

Suppose that $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is obtained from (x_1, x_2, x_3) by rotating the x_1x_2-plane counter-clockwise by an angle θ about the x_3-axis (see Figure 1.2.4). Then the two sets of coordinates are related by

$$\begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{1.2.71}$$

and the inverse relation is given by

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} \tag{1.2.72}$$

Clearly, we have $[A]^{-1} = [A]^{\mathrm{T}}$. The components $\bar{\varphi}_{ij}$ and φ_{ij} of a second-order tensor $\mathbf{\Phi}$ in the two coordinate systems, $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ and (x_1, x_2, x_3), respectively, are related by Eqs. (1.2.69) and (1.2.70). Equation (1.2.69) in the present case gives

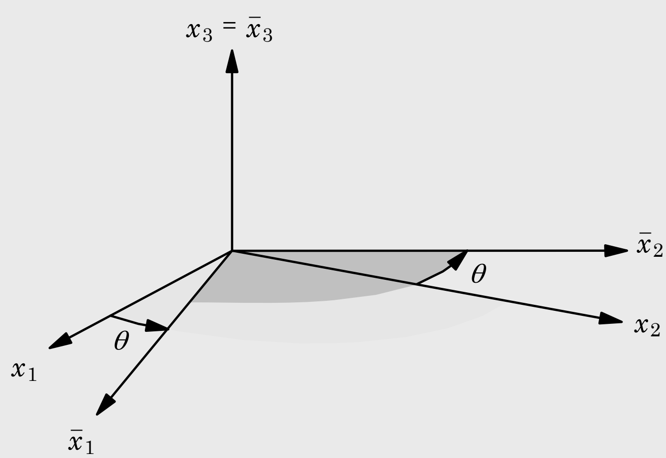


Figure 1.2.4 A special rotational transformation between unbarred and barred rectangular Cartesian coordinate systems.

$$
\begin{aligned}
\bar{\varphi}_{11} &= \varphi_{11}\cos^2\theta + \varphi_{22}\sin^2\theta + (\varphi_{12} + \varphi_{21})\cos\theta\sin\theta \\
\bar{\varphi}_{12} &= (\varphi_{22} - \varphi_{11})\cos\theta\sin\theta + \varphi_{12}\cos^2\theta - \varphi_{21}\sin^2\theta \\
\bar{\varphi}_{21} &= (\varphi_{22} - \varphi_{11})\cos\theta\sin\theta + \varphi_{21}\cos^2\theta - \varphi_{12}\sin^2\theta \\
\bar{\varphi}_{22} &= \varphi_{11}\sin^2\theta + \varphi_{22}\cos^2\theta - (\varphi_{12} + \varphi_{21})\cos\theta\sin\theta \\
\bar{\varphi}_{13} &= \varphi_{13}\cos\theta + \varphi_{23}\sin\theta \\
\bar{\varphi}_{23} &= \varphi_{23}\cos\theta - \varphi_{13}\sin\theta \\
\bar{\varphi}_{31} &= \varphi_{31}\cos\theta + \varphi_{32}\sin\theta \\
\bar{\varphi}_{32} &= \varphi_{32}\cos\theta - \varphi_{31}\sin\theta \\
\bar{\varphi}_{33} &= \varphi_{33}
\end{aligned}
\tag{1.2.73}
$$

The inverse relations are provided by Eq. (1.2.70):

$$
\begin{aligned}
\varphi_{11} &= \bar{\varphi}_{11}\cos^2\theta + \bar{\varphi}_{22}\sin^2\theta - (\varphi_{12} + \bar{\varphi}_{21})\cos\theta\sin\theta \\
\varphi_{12} &= (\bar{\varphi}_{11} - \bar{\varphi}_{22})\cos\theta\sin\theta + \bar{\varphi}_{12}\cos^2\theta - \bar{\varphi}_{21}\sin^2\theta \\
\varphi_{21} &= (\bar{\varphi}_{11} - \bar{\varphi}_{22})\cos\theta\sin\theta + \bar{\varphi}_{21}\cos^2\theta - \bar{\varphi}_{12}\sin^2\theta \\
\varphi_{22} &= \bar{\varphi}_{11}\sin^2\theta + \bar{\varphi}_{22}\cos^2\theta + (\bar{\varphi}_{12} + \bar{\varphi}_{21})\cos\theta\sin\theta \\
\varphi_{13} &= \bar{\varphi}_{13}\cos\theta - \bar{\varphi}_{23}\sin\theta \\
\varphi_{23} &= \bar{\varphi}_{23}\cos\theta + \bar{\varphi}_{13}\sin\theta \\
\varphi_{31} &= \bar{\varphi}_{31}\cos\theta - \bar{\varphi}_{32}\sin\theta \\
\varphi_{32} &= \bar{\varphi}_{32}\cos\theta + \bar{\varphi}_{31}\sin\theta \\
\varphi_{33} &= \bar{\varphi}_{33}
\end{aligned}
\tag{1.2.74}
$$

The transformation law (1.2.68) is often taken to be the *definition* of a second-order tensor. In other words, $\mathbf{\Phi}$ is a second-order tensor if and only if its components transform according to Eq. (1.2.68). In general, an nth-order tensor transforms according to the formula

$$\bar{\varphi}_{mnpq\ldots} = \varphi_{ijk\ell\ldots}a_{mi}a_{nj}a_{pk}a_{q\ell}\cdots \tag{1.2.75}$$

The *trace* of a second-order tensor is defined to be the double-dot product of the tensor with the unit tensor $\mathbf{I}$

$$\operatorname{tr}\mathbf{\Phi} = \mathbf{\Phi} : \mathbf{I} = \varphi_{ii} \tag{1.2.76}$$

The trace of a tensor is *invariant*, called the first principal invariant, and it is denoted by I_1; i.e., it is invariant under coordinate transformations ($\phi_{ii} = \bar{\phi}_{ii}$). The first, second, and third *principal invariants* of a second-order tensor are defined, in terms of the rectangular Cartesian components, as

$$I_1 = \varphi_{ii}, \quad I_2 = \frac{1}{2}\left(\varphi_{ii}\varphi_{jj} - \varphi_{ij}\varphi_{ji}\right), \quad I_3 = |\varphi| \tag{1.2.77}$$

1.3 Equations of an Elastic Body

1.3.1 Introduction

The objective of this section is to review the basic equations of an elastic body. The equations governing the motion and deformation of a solid body can be classified into four basic categories:

(1) Kinematics (strain-displacement equations)
(2) Kinetics (conservation of momenta)
(3) Thermodynamics (first and second laws of thermodynamics)
(4) Constitutive equations (stress–strain relations).

Kinematics is a study of the geometric changes or deformation in a body without the consideration of forces causing the deformation or the nature of the body. *Kinetics* is the study of the static or dynamic equilibrium of forces acting on a body. The thermodynamic principles are concerned with the conservation of energy and relations among heat, mechanical work, and thermodynamic properties of the body. The constitutive equations describe the constitutive behavior of the body and relate the dependent variables introduced in the kinetic description to those in the kinematic and thermodynamic descriptions. These equations are supplemented by appropriate boundary and initial conditions of the problem.

In the following sections, an overview of the kinematic, kinetic, and constitutive equations of an elastic body is presented. The thermodynamic principles are not reviewed, as we will account for thermal effects only through constitutive relations.

1.3.2 Kinematics

The term *deformation* of a body refers to relative displacements and changes in the geometry experienced by the body. In a rectangular Cartesian frame of reference $(X_1,\ X_2,\ X_3)$, every particle X in the body corresponds to a position $\mathbf{X} = (X_1,\ X_2,\ X_3)$. When the body is deformed, the particle X moves to a new position $\mathbf{x} = (x_1, x_2, x_3)$. If the displacement of every particle in the body is known, we can construct the deformed configuration κ from the reference (or undeformed) configuration κ_0 (Figure 1.3.1). In the Lagrangian description, the displacements are expressed in terms of the *material coordinates* $\mathbf{X}$

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \tag{1.3.1}$$

The deformation of a body can be measured in terms of the strain tensor $\mathbf{E}$, which is defined such that it gives the change in the square of the length of the material vector $d\mathbf{X}$

$$\begin{aligned} 2d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X} &= d\mathbf{x}\cdot d\mathbf{x} - d\mathbf{X}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot(\mathbf{I}+\nabla\mathbf{u})\cdot[d\mathbf{X}\cdot(\mathbf{I}+\nabla\mathbf{u})] - d\mathbf{X}\cdot d\mathbf{X} \\ &= d\mathbf{X}\cdot\left[(\mathbf{I}+\nabla\mathbf{u})\cdot(\mathbf{I}+\nabla\mathbf{u})^{\mathrm{T}} - \mathbf{I}\right]\cdot d\mathbf{X} \end{aligned} \tag{1.3.2}$$

where ∇ denotes the gradient operator with respect to the material coordinates $\mathbf{X}$ and $\mathbf{E}$ is known as the *Green–Lagrange strain tensor* and it is defined by

$$\mathbf{E} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} + \nabla\mathbf{u}\cdot(\nabla\mathbf{u})^{\mathrm{T}}\right] \tag{1.3.3}$$

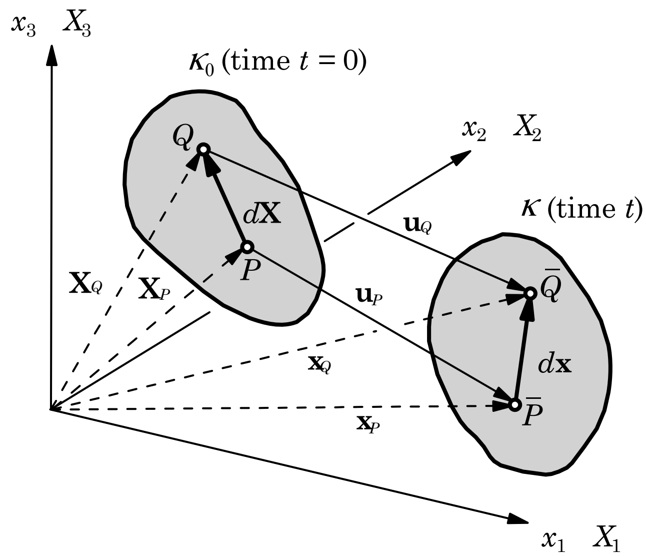


Figure 1.3.1 Undeformed and deformed configurations of a body.

Note that the Green–Lagrange strain tensor is symmetric, $\mathbf{E} = \mathbf{E}^{\mathrm{T}}$. The strain components defined in Eq. (1.3.3) are called finite strain components because no assumption concerning the smallness (compared to unity) of the strains is made.

The rectangular Cartesian component form of $\mathbf{E}$ is

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_k} + \frac{\partial u_k}{\partial X_j} + \frac{\partial u_m}{\partial X_j}\frac{\partial u_m}{\partial X_k}\right) \tag{1.3.4}$$

where summation on repeated subscripts is implied. In explicit form, we have

$$\begin{aligned}
E_{11} &= \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right]\\
E_{22} &= \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right]\\
E_{33} &= \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right]\\
E_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right)\\
E_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_3}\right)\\
E_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right)
\end{aligned} \tag{1.3.5}$$

The strain components in other coordinate systems can be derived from Eq. (1.3.3) by expressing the tensor $\mathbf{E}$ and del operator ∇ in that coordinate system. See Eqs. (1.2.30) and (1.2.42) for the definition of ∇ in cylindrical and spherical coordinates systems, respectively. For example, the Green–Lagrange strain tensor components in the cylindrical coordinate system are given by

$$
\begin{aligned}
E_{rr} &= \frac{\partial u_r}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right] \\
E_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)^2\right] \\
&\quad + \frac{1}{r^2}\left(2u_r\frac{\partial u_\theta}{\partial \theta} - 2u_\theta\frac{\partial u_r}{\partial \theta} + u_\theta^2 + u_r^2\right) \\
E_{zz} &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] \\
E_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{1}{2r}\left(\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \theta}\frac{\partial u_\theta}{\partial \theta}\right. \\
&\quad \left. + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta} + u_r\frac{\partial u_\theta}{\partial r} - u_\theta\frac{\partial u_r}{\partial r}\right) \\
E_{z\theta} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) + \frac{1}{2r}\left(\frac{\partial u_r}{\partial \theta}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial \theta}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial \theta}\frac{\partial u_z}{\partial z}\right. \\
&\quad \left. - u_\theta\frac{\partial u_r}{\partial z} + u_r\frac{\partial u_\theta}{\partial z}\right) \\
E_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right)
\end{aligned}
\tag{1.3.6}
$$

If the displacement gradients are so small, $|u_{i,j}| << 1$, that their squares and products are negligible compared to $|u_{i,j}|$ and the difference between $\frac{\partial u_i}{\partial x_j}$ and $\frac{\partial u_i}{\partial X_j}$ vanishes, then the Green–Lagrange strain tensor reduces to the *infinitesimal strain tensor*:

$$\varepsilon = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right] \tag{1.3.7}$$

or in Cartesian component form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1.3.8}$$

Suppose that we use the notation $x_1 = x$, $x_2 = y$, and $x_3 = z$, and let $(u_1, u_2, u_3) = (u, v, w)$ denote the displacements along (x, y, z). Then the infinitesimal strain components in Eq. (1.3.8) become

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}\,, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}\,, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{1.3.9}$$

$$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad 2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{1.3.10}$$

Example 1.3.1

The displacement field of a beam under the Euler–Bernoulli kinematic hypothesis (i.e., a transverse normal line before deformation remains straight, inextensible, and normal after deformation) can be expressed as (with $X = x$, $Y = y$, and $Z = z$)

$$u(x, y, z) = u_0(x) - z\frac{dw_0}{dx}, \;\; v = 0, \;\; w(x, y, z) = w_0(x) \tag{1.3.11}$$

Then the only nonzero, nonlinear strains are given by

$$\begin{aligned}
\varepsilon_{xx} &= \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right] \\
&= \frac{du_0}{dx} + \frac{1}{2}\left[\left(\frac{du_0}{dx}\right)^2 + \left(\frac{dw_0}{dx}\right)^2\right] - z\left(\frac{d^2w_0}{dx^2} + \frac{du_0}{dx}\frac{d^2w_0}{dx^2}\right) + \frac{z^2}{2}\left(\frac{d^2w_0}{dx^2}\right)^2 \\
\varepsilon_{zz} &= \frac{dw}{dz} + \frac{1}{2}\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dw}{dz}\right)^2\right] = \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 \\
2\varepsilon_{xz} &= \frac{du}{dz} + \frac{dw}{dx} + \frac{du}{dx}\frac{du}{dz} + \frac{dw}{dx}\frac{dw}{dz} = -\frac{du_0}{dx}\frac{dw_0}{dx} + z\frac{d^2w_0}{dx^2}\frac{dw_0}{dx}
\end{aligned} \tag{1.3.12}$$

If we assume that the following terms are negligible compared to du_0/dx,

$$\left(\frac{du_0}{dx}\right)^2, \quad h\frac{du_0}{dx}\frac{d^2w_0}{dx^2}, \quad h^2\left(\frac{d^2w_0}{dx^2}\right)^2, \quad \frac{du_0}{dx}\frac{dw_0}{dx}, \quad h\frac{dw_0}{dx}\frac{d^2w_0}{dx^2} \tag{1.3.13}$$

where h denotes the height of the beam, then the nonlinear strains in Eq. (1.3.12) can be simplified to

$$\varepsilon_{xx} = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 - z\frac{d^2w_0}{dx^2}, \quad \varepsilon_{zz} = \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2, \quad 2\varepsilon_{xz} = 0 \tag{1.3.14}$$

1.3.3 Compatibility Equations

By definition, the components of the strain tensor can be computed from a differentiable displacement field using Eq. (1.3.3). However, if the *six* components of strain tensor are given and if we are required to find the *three* displacement components, the strains given should be such that a unique solution to the six differential equations relating the strains and displacements exists. For example, consider the relations

$$\frac{\partial u}{\partial x} = x^2 + 2y, \;\; \frac{\partial u}{\partial y} = 3x + y^2$$

The two equations cannot be solved for u because they are inconsistent. The inconsistency can be verified by computing the second derivative $\partial^2 u/\partial x\partial y$ from the two relations. The first equation gives 2 and the second equation gives 3, which are unequal.

The existence of a unique solution to the six partial differential equations defined by the strain-displacement relations is guaranteed if the infinitesimal strain components satisfy certain integrability conditions, known in elasticity as the *compatibility conditions*. In Cartesian component form, these are

$$\varepsilon_{ikm}\varepsilon_{j\ell n}\,\varepsilon_{ij,k\ell} = 0 \tag{1.3.15}$$

or in vector form

$$\nabla \times (\nabla \times \varepsilon)^{\mathrm{T}} = \mathbf{0} \quad \text{(zero dyad)} \tag{1.3.16}$$

Note that m and n are the free subscripts. In the three-dimensional case, Eq. (1.3.15) gives 81 relations among the strain components. However, due to the symmetry of $\varepsilon_{ij,k\ell}$ with respect to i and j and k and ℓ, there are only six independent relations. The sufficiency of the statements (1.3.15) and (1.3.16) can be verified.

Equation (1.3.15) can be written as

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_m \partial x_n} + \frac{\partial^2 \varepsilon_{mn}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{im}}{\partial x_j \partial x_n} - \frac{\partial^2 \varepsilon_{jn}}{\partial x_i \partial x_m} = 0 \tag{1.3.17}$$

for any $i, j, m, n = 1, 2, 3$. For the two-dimensional case, Eq. (1.3.17) reduces to the following single compatibility equation

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \tag{1.3.18}$$

It should be noted that the strain compatibility equations are satisfied automatically when the strains are computed from a displacement field. Thus, one needs to verify the compatibility conditions only when the strains are computed from stresses that are in equilibrium.

Example 1.3.2

(a) Consider the following two-dimensional strain field:

$$\varepsilon_{11} = c_1 x_1 \left(x_1^2 + x_2^2\right), \ \varepsilon_{22} = \frac{1}{3} c_2 x_1^3, \ \varepsilon_{12} = c_3 x_1^2 x_2$$

where c_1, c_2, and c_3 are constants. We wish to check if the strain field is compatible. Using Eq. (1.3.18) we obtain

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 2c_1 x_1 + 2c_2 x_1 - 4c_3 x_1$$

Thus, the strain field is, in general, not compatible unless $c_1 = c_2 = c_3$.

(b) Given the strain tensor $\mathbf{E} = E_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + E_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$ in an axisymmetric body (i.e., E_{rr} and $E_{\theta\theta}$ are functions of r and z only), we wish to determine the compatibility conditions on E_{rr} and $E_{\theta\theta}$.

Using the vector form of compatibility conditions, Eq. (1.3.16), we obtain (see Problem 1.11)

$$\mathbf{F} \equiv \nabla \times \mathbf{E} = \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta + \frac{\partial E_{rr}}{\partial z} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r - \frac{\partial E_{\theta\theta}}{\partial z} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta$$

and

$$\begin{aligned}
\nabla \times (\mathbf{F})^{\mathrm{T}} = & \frac{\partial}{\partial r} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta) \, \hat{\mathbf{e}}_z \\
& - \frac{\partial^2 E_{\theta\theta}}{\partial r \partial z} (\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta) \, \hat{\mathbf{e}}_r + \frac{1}{r} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) \left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} \right) \hat{\mathbf{e}}_z \\
& + \frac{1}{r} \frac{\partial E_{rr}}{\partial z} \left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} \right) \hat{\mathbf{e}}_\theta + \frac{1}{r} \frac{\partial E_{rr}}{\partial z} (\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r) \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} \\
& - \frac{1}{r} \frac{\partial E_{\theta\theta}}{\partial z} \left(\hat{\mathbf{e}}_\theta \times \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} \right) \hat{\mathbf{e}}_r + \frac{\partial}{\partial z} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_\theta) \, \hat{\mathbf{e}}_z \\
& + \frac{\partial^2 E_{rr}}{\partial z^2} (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r) \, \hat{\mathbf{e}}_\theta - \frac{\partial^2 E_{\theta\theta}}{\partial z^2} (\hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r) \, \hat{\mathbf{e}}_r = \mathbf{0}
\end{aligned}$$

Noting that

$$\begin{gathered}
\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta, \quad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r \\
\hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_z, \quad \hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r, \quad \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta
\end{gathered} \tag{1.3.19}$$

and that a tensor is zero only when all its components are zero, we obtain

$$\begin{aligned}
\hat{\mathbf{e}}_z \hat{\mathbf{e}}_z : \quad & \frac{\partial}{\partial r} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) + \frac{1}{r} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) = 0 \\
\hat{\mathbf{e}}_z \hat{\mathbf{e}}_r : \quad & -\frac{\partial^2 E_{\theta\theta}}{\partial r \partial z} + \frac{1}{r} \frac{\partial}{\partial z} (E_{rr} - E_{\theta\theta}) = 0 \\
\hat{\mathbf{e}}_r \hat{\mathbf{e}}_z : \quad & -\frac{\partial}{\partial z} \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) = 0 \\
\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta : \quad & \frac{\partial^2 E_{rr}}{\partial z^2} = 0, \qquad \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r : \quad -\frac{\partial^2 E_{\theta\theta}}{\partial z^2} = 0
\end{aligned}$$

1.3.4 Stress Measures

Stress at a point is a measure of force per unit area. The force per unit area acting on an elemental area ds of the deformed body is called the *stress vector* acting on the element. The concept also applies to the surface created by slicing the deformed body with a plane. The stress vector on the area element Δs is the vector $\Delta \mathbf{f}/\Delta s$, which has the same direction as $\Delta \mathbf{f}$, but with a magnitude equal to $|\Delta \mathbf{f}|/\Delta s$. The stress vector at a point P on Δs is defined by

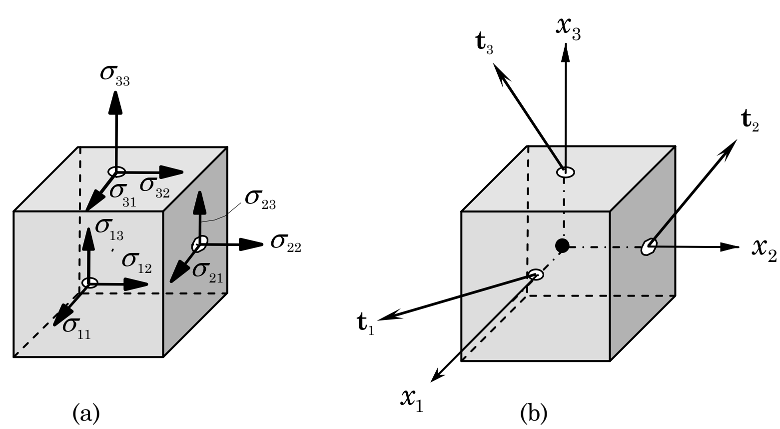
$$\mathbf{T} \equiv \lim_{\Delta s \to 0} \frac{\Delta \mathbf{f}}{\Delta s} \tag{1.3.20}$$

Equation (1.3.20) implies that the stress vector at a point depends on the direction and magnitude of the force vector and the surface area on which it acts. The surface area in turn depends on the orientation of the plane used to slice the body. For example, if a constant cross-section member is sliced with a plane perpendicular to its axis, the resulting surface area A_0 is the same as the area of cross section of the member. However, if it is sliced at the same point with a plane oriented at an angle $0^\circ < \theta < 90^\circ$ to the vertical axis, the surface area A obtained will be different: $A = A_0/\cos\theta$. Thus, the stress vector at a point depends on the force vector as well as the orientation of the surface on which it acts. For this reason, the stress vector acting on a plane with normal $\hat{\mathbf{n}}$ is denoted by $\mathbf{T}^{(\hat{\mathbf{n}})}$.

The state of stress at a point inside a body can be expressed in terms of stress vectors on three mutually perpendicular planes, say planes perpendicular to the rectangular coordinate axes. Let $\mathbf{T}^{(i)}$ denote the stress vector at a point P on a plane perpendicular to the x_i-axis [Figure 1.3.2(a)]. Each vector $\mathbf{T}^{(i)}$, $(i = 1, 2, 3)$ can be resolved into components along the coordinate lines

$$\mathbf{T}^{(i)} = \sigma_{ij}\hat{\mathbf{e}}_j \quad (i \ = \ 1, 2, 3) \tag{1.3.21}$$

where the symbol σ_{ij} denotes the component of the stress vector $\mathbf{T}^{(i)}$ at the point P along the x_j-direction, and $\hat{\mathbf{e}}_j$ is the unit base vector along the x_j-direction. There are a total of nine stress components σ_{ij}, $(i, j = 1, 2, 3)$ at the point P. It can be shown that the quantities σ_{ij} transform like the components of a second-order tensor called the stress tensor, σ. The stress components σ_{ij} are shown on three perpendicular planes in Figure 1.3.2(b). Note the cube that is used to indicate the stress components is a point cube, so that all nine components are acting at the point P.



$$\mathbf{t}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \sigma \text{ or } t_i^{(\hat{\mathbf{n}})} = n_j \sigma_{ji}$$

$$\mathbf{t}_i = \sigma_{ij}\hat{\mathbf{e}}_j, \ \sigma = \mathbf{t}_i\hat{\mathbf{e}}_i = \hat{\mathbf{e}}_i\sigma_{ij}\hat{\mathbf{e}}_j$$

Figure 1.3.2 Stress vectors. (a) Components of stress tensor. (b) Stress vector on coordinate planes.

Using the linear momentum principle or Newton's second law of motion, it can be shown that the stress vector $\mathbf{T}^{(\hat{\mathbf{n}})}$ on a plane with unit normal $\hat{\mathbf{n}}$ is related to the stress tensor σ by the relation

$$\mathbf{T}^{(\hat{\mathbf{n}})} = \hat{\mathbf{n}} \cdot \sigma = \sigma^T \cdot \hat{\mathbf{n}} \quad \text{or} \quad \{T^{(\hat{\mathbf{n}})}\} = [\sigma]^T \{n\} \tag{1.3.22}$$

Equation (1.3.22) is known as the *Cauchy stress formula* and σ is called the *Cauchy stress tensor*.

In the above discussion, it is understood that stress at a point in a deformed body is measured as the force per unit area in the deformed body. The area element Δs in the deformed body corresponds to an area element ΔS in the reference configuration. The stress vector $\mathbf{T}^{(\hat{\mathbf{n}})}$ is *measured* with respect to the area element in the deformed body. Stress vectors can be *referred* to either the reference coordinate system (X_1, X_2, X_3) or the spatial coordinate system (x_1, x_2, x_3). The stress that is measured and referred to the deformed configuration $\mathcal{C}$ is called the *Cauchy stress* or *true stress*, σ. The components σ_{ij} can be interpreted as the jth component of the force per unit area in the current configuration $\mathcal{C}$ acting on a surface segment whose outward normal at $\mathbf{x}$ is in the ith direction.

A suitable strain measure to use with the Cauchy stress tensor σ would be the infinitesimal strain tensor ε of Eq. (1.3.7). Since we defined strain tensor $\mathbf{E}$ as a function of the material point $\mathbf{X}$ in a reference state, the stress must also be expressed as a function of the material point. The stress measure that is used in nonlinear analysis of solid bodies is the second Piola–Kirchhoff stress tensor, which is measured in the deformed body but referred to the material coordinates, X_i. The strain measure to use with the second Piola–Kirchhoff stress is the Green–Lagrange strain tensor. For small deformation problems, the difference between the two measures of stress disappears. In the present study, we consider only small strain problems and, therefore, will not distinguish between various measures of stress, although it should be understood that σ denotes the second Piola–Kirchhoff stress tensor in the rest of the study. The principle of conservation of moment of momentum, in the absence of any distributed moments, leads to the symmetry of the stress tensor, $\sigma = \sigma^T$ $(\sigma_{ij} = \sigma_{ji})$. Thus, there are only six independent components of the Cauchy and second Piola–Kirchhoff stress tensors.

1.3.5 Equations of Motion

Consider a given mass of a material body $\mathcal{B}$ at any instant occupying a volume Ω bounded by a surface Γ. Suppose that the body is acted upon by external forces $\mathbf{T}$ (per unit surface area) and $\mathbf{f}$ (per unit mass). The principle of conservation of linear momentum states that the rate of change of the total linear momentum of a given continuous medium equals the vector sum of all the external forces acting on the body $\mathcal{B}$, which initially occupied a configuration $\mathcal{C}^0$, provided Newton's Third Law of action and reaction governs the internal forces. The principle leads to the following result:

$$\nabla \cdot \sigma + \rho \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{1.3.23}$$

where ∇ is the gradient operator with respect to the coordinates $\mathbf{X} \approx \mathbf{x}$ and ρ is the material density in the undeformed body. In writing Eq. (1.3.23), all variables are assumed to be functions of $\mathbf{X}$ and time t. For the static case, Eq. (1.3.23) reduces to the equilibrium equations

$$\nabla \cdot \sigma + \rho \mathbf{f} = \mathbf{0} \quad (\sigma_{ji,j} + \rho f_i = 0) \tag{1.3.24}$$

The equations of motion or equilibrium are supplemented by the boundary conditions on the displacement vector and/or stress vector:

$$\mathbf{u} = \hat{\mathbf{u}}, \quad \sigma \cdot \hat{\mathbf{n}} = \hat{\mathbf{T}} \tag{1.3.25}$$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{T}}$ are specified displacement and stress vectors, respectively.

In rectangular Cartesian coordinates (x, y, z), the equations of motion (1.3.23) can be expressed as

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y &= \rho \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z &= \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{1.3.26}$$

For coordinate systems other than the rectangular Cartesian systems, the equations of motion can be derived from the vector form (1.3.23) by expressing σ, $\mathbf{f}$, $\mathbf{u}$, and ∇ in the chosen coordinate system. For example, the equations of motion (1.3.23) in the cylindrical coordinate system (r, θ, z) are given by (see Problem 1.18)

$$\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho f_r &= \rho \frac{\partial^2 u_r}{\partial t^2} \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho f_\theta &= \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho f_z &= \rho \frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{1.3.27}$$

where u_r, u_θ, and u_z are, for example, the components of the displacement vector $\mathbf{u}$ in the cylindrical coordinate system.

Example 1.3.3

Here we illustrate the derivation of the equations of equilibrium of an elastic body using Newton's laws. Consider the static equilibrium of an infinitesimal parallelepiped of dimensions dx_1, dx_2, and dx_3 and with its sides parallel to the coordinate planes, as shown in Figure 1.3.3. The stresses acting on typical planes are shown in the figure. Each stress component on a surface perpendicular to the negative x_k-axis is assumed to increase from its value σ_{ij} to

$$\sigma_{ij} + \frac{\partial \sigma_{ij}}{\partial x_k} dx_k$$

on the plane perpendicular to the positive x_k-axis. For example, the forces acting on the plane perpendicular to the negative x_1-axis are $\sigma_{11} dx_2 dx_3$, $\sigma_{12} dx_2 dx_3$, and $\sigma_{13} dx_2 dx_3$, all in the opposite directions to the positive coordinates.

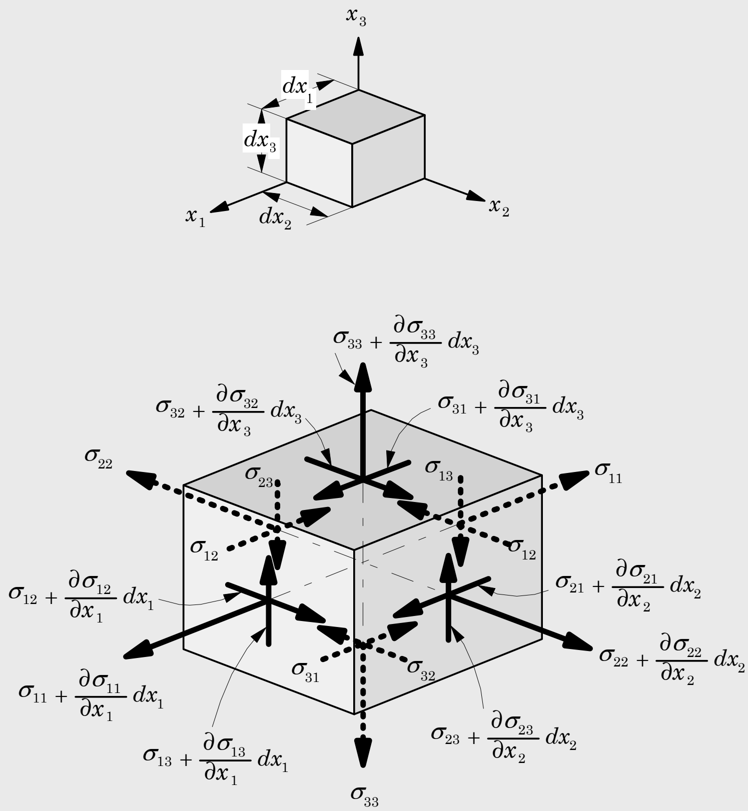


Figure 1.3.3 Equilibrating stress components on a parallelepiped.

Summing the forces along the x_1-axis, we obtain

$$-\sigma_{11}dx_2dx_3 - \sigma_{21}dx_1dx_3 - \sigma_{31}dx_1dx_2 + \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}dx_1\right)dx_2dx_3$$
$$+\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}dx_2\right)dx_1dx_3 + \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3}dx_3\right)dx_1dx_2 + \rho f_1 dx_1dx_2dx_3 = 0$$

or

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \rho f_1 = 0$$

Similarly, summation of forces in the x_2- and x_3-directions yield

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \rho f_2 = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho f_3 = 0$$

which agree with Eq. (1.3.24).

1.3.6 Constitutive Equations

The kinematic relations and the mechanical and thermodynamic principles are applicable to any continuum irrespective of its physical constitution. Here we consider equations characterizing the individual material and its reaction to applied loads. These equations are called the *constitutive equations*. The formulation of the constitutive equations for a given material is guided by certain rules (i.e., constitutive axioms). We will not discuss them here, but will review the linear constitutive relations for solids undergoing small deformations.

A material body is said to be *homogeneous* if the material properties are the same throughout the body. In a *heterogeneous* body, the material properties are a function of position. An *anisotropic* body is one that has different values of a material property in different directions at a point, i.e., material properties are direction-dependent. An *isotropic* body is one for which every material property in all directions at a point is the same. An isotropic or anisotropic material can be nonhomogeneous or homogeneous.

A material body is said to be *ideally elastic* when, under isothermal conditions, the body recovers its original form completely upon removal of the forces causing deformation, and there is a one-to-one relationship between the state of stress and the state of strain. The constitutive equations described here do not include creep at constant stress and stress relaxation at constant strain. Thus, the material coefficients that specify the constitutive relationship between the stress and strain components are assumed to be constant during the deformation. This does not automatically imply that we neglect temperature effects on deformation. We account for the thermal expansion of the material, which can produce strains or stresses as large as those produced by the applied mechanical forces. A more detailed discussion of the thermoelastic constitutive relations will be presented in the chapter on mechanical behavior of a lamina. Here we review the basic constitutive equations of linear elasticity (i.e., generalized Hooke's law) for small displacements.

The generalized Hooke's law relates the six components of stress to the six components of strain as

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{matrix}\right\} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix} \left\{\begin{matrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{matrix}\right\} \tag{1.3.28}$$

where C_{ij} are the elastic coefficients and

$$\begin{aligned} &\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23}\ ,\ \sigma_5 = \sigma_{13},\ \sigma_6 = \sigma_{12}\\ &\varepsilon_1 = \varepsilon_{11},\ \varepsilon_2 = \varepsilon_{22},\ \varepsilon_3 = \varepsilon_{33},\ \varepsilon_4 = 2\varepsilon_{23},\ \varepsilon_5 = 2\varepsilon_{13},\ \varepsilon_6 = 2\varepsilon_{12} \end{aligned} \tag{1.3.29}$$

The single subscript notation (1.3.29) used for stresses and strains is called the *contracted notation* or the Voigt–Kelvin notation. The two-subscript components C_{ij} are obtained from $C_{ijk\ell}$ by the following change of subscripts:

$$11 \rightarrow 1 \quad 22 \rightarrow 2 \quad 33 \rightarrow 3 \quad 23 \rightarrow 4 \quad 13 \rightarrow 5 \quad 12 \rightarrow 6$$

The resulting C_{ij} are also symmetric ($C_{ij} = C_{ji}$) by virtue of the assumption that there exists a potential function $U_0 = U_0(\varepsilon_{ij})$, called the *strain energy density function*, whose derivative with respect to a strain component determines the corresponding stress component

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}} \tag{1.3.30}$$

Such materials are termed *hyperelastic* materials. For hyperelastic materials, there are only 21 independent coefficients of the matrix $[C]$.

When three mutually orthogonal planes of material symmetry exist, the number of elastic coefficients is reduced to 9, and such materials are called *orthotropic*. The stress–strain relations for an orthotropic material take the form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \tag{1.3.31}$$

The inverse relations, strain–stress relations, are given by

$$\begin{aligned} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} &= \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \\ &= \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \end{aligned} \tag{1.3.32}$$

where S_{ij} denote the compliance coefficients, $[S] = [C]^{-1}$; E_1, E_2, E_3 are Young's moduli in 1, 2, and 3 material directions, respectively; ν_{ij} is Poisson's ratio, defined as the ratio of transverse strain in the jth direction to the axial strain in the ith direction when stressed in the i-direction; and G_{23}, G_{13}, G_{12} = shear moduli in the 2-3, 1-3, and 1-2 planes, respectively. Since the compliance matrix $[S]$ is the inverse of the stiffness matrix $[C]$ and the inverse of a symmetric matrix is symmetric, it follows that the compliance matrix $[S]$ is also a symmetric matrix. This in turn implies, in view of Eq. (1.3.31), that the following reciprocal relations hold:

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \quad \text{or} \quad \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (\text{no sum on } \; i, j) \tag{1.3.33}$$

for $i, j = 1, 2, 3$. Thus, there are only nine independent material coefficients

$$E_1,\ E_2,\ E_3,\ G_{23},\ G_{13},\ G_{12},\ \nu_{12},\ \nu_{13},\ \nu_{23}$$

for an orthotropic material.

When there exist no preferred directions in the material (i.e., the material has infinite number of planes of material symmetry), the number of independent elastic coefficients reduces to 2. Such materials are called *isotropic*. For isotropic materials, we have $E_1 = E_2 = E_3 = E$, $G_{12} = G_{13} = G_{23} \equiv G$, and $\nu_{12} = \nu_{23} = \nu_{13} \equiv \nu$. Of the three constants (E, ν, G), only two are independent and the third one is related to the other. For example, G can be determined from E and ν by the relation

$$G = \frac{E}{2(1+\nu)} \tag{1.3.34}$$

A *plane stress state* is defined to be one in which all transverse stresses are negligible. The strain–stress relations of an orthotropic body in plane stress of state can be written as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} \tag{1.3.35}$$

The strain–stress relations (1.3.35) are inverted to obtain the stress–strain relations

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} \tag{1.3.36}$$

where the Q_{ij}, called the *plane stress-reduced stiffnesses*, are given by

$$\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \\ Q_{12} &= \frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \\ Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\ Q_{66} &= \frac{1}{S_{66}} = G_{12} \end{aligned} \tag{1.3.37}$$

Note that the reduced stiffnesses involve four independent material constants, E_1, E_2, ν_{12}, and G_{12}.

The transverse shear stresses are related to the transverse shear strains in an orthotropic material by the relations

$$\begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{1.3.38}$$

where $Q_{44} = C_{44} = G_{23}$ and $Q_{55} = C_{55} = G_{13}$.

When temperature changes occur in the elastic body, we account for the thermal expansion of the material, even though the variation of elastic constants with temperature is neglected. When the strains, geometric changes, and temperature variations are sufficiently small, all governing equations are linear and superposition of mechanical and thermal effects is possible. The linear thermoelastic constitutive equations have the form

$$\sigma_j = C_{ji}[-\alpha_i(T - T_0) + \varepsilon_i] \tag{1.3.39}$$

$$\varepsilon_j = S_{ji}\sigma_i + \alpha_i(T - T_0) \tag{1.3.40}$$

where α_i $(i = 1, 2, 3)$ are the linear coefficients of thermal expansion, T denotes temperature, and T_0 is the reference temperature of the undeformed body. In writing Eqs. (1.3.39) and (1.3.40), it is assumed that α_i and C_{ij} are independent of strain and temperature. For an isotropic material, we have $\alpha_1 = \alpha_2 = \alpha_3 \equiv \alpha$. The plane stress strain-stress relations for a thermoelastic case are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 - \alpha_1 \Delta T \\ \varepsilon_2 - \alpha_2 \Delta T \\ \varepsilon_6 \end{Bmatrix} \tag{1.3.41}$$

where $\Delta T = T - T_0$ is the temperature change from the reference temperature T_0 and α_i $(i = 1, 2)$ are the coefficients of thermal expansion of an orthotropic material in the x_i-coordinate direction. In addition, Eq. (1.3.38) holds for the thermoelastic case.

Tables 1.3.1 and 1.3.2 contain the values of typical engineering constants for some commonly used metals and polymeric materials. The engineering properties of polymeric materials depend on the processing conditions, which can be different for different manufacturers.

Table 1.3.1 Values of the engineering constants* for several materials.

Material	E_1	E_2	G_{12}	G_{13}	G_{23}	ν_{12}
Aluminum	10.6	10.6	3.38	3.38	3.38	0.33
Copper	18.0	18.0	6.39	6.39	6.39	0.33
Steel	30.0	30.0	11.24	11.24	11.24	0.29
Gr-Ep (AS)	20.0	1.3	1.03	1.03	0.90	0.30
Gr-Ep (T)	19.0	1.5	1.00	0.90	0.90	0.22
Gl-Ep (1)	7.8	2.6	1.30	1.30	0.50	0.25
Gl-Ep (2)	5.6	1.2	0.60	0.60	0.50	0.26
Br-Ep	30.0	3.0	1.00	1.00	0.60	0.30

*Moduli are in msi = million psi; 1 psi = 6.895 kN/m^2.
Gr-Ep (AS) = graphite-epoxy (AS/3501); Gr-Ep (T) = graphite-epoxy (T300/934); Gl-Ep = glass-epoxy; Br-Ep = boron-epoxy.

Table 1.3.2 Values of additional engineering constants* for the materials listed in Table 1.3.1.

Material	E_3	ν_{13}	ν_{23}	α_1	α_2
Aluminum	10.6	0.33	0.33	13.1	13.1
Copper	18.0	0.33	0.33	18.0	18.0
Steel	30.0	0.29	0.29	10.0	10.0
Gr-Ep (AS)	1.3	0.30	0.49	1.0	30.0
Gr-Ep (T)	1.5	0.22	0.49	–0.167	15.6
Gl-Ep (1)	2.6	0.25	0.34	3.5	11.4
Gl-Ep (2)	1.3	0.26	0.34	4.8	12.3
Br-Ep	3.0	0.25	0.25	2.5	8.0

* Units of E_3 are msi, and the units of α_1 and α_2 are 10^{-6} in./in./°F.

1.4 Transformation of Stresses, Strains, and Stiffnesses

1.4.1 Introduction

In general, the coordinate system used in the solution of a problem does not coincide with the material coordinate system. Therefore, there is a need to establish transformation relations among stresses and strains in one coordinate system to the corresponding quantities in another coordinate system. These relations can be used to transform constitutive equations from the material coordinates to the coordinates used in the solution of a problem. Here we consider a special coordinate transformation. Let the principal material coordinates be denoted by (x_1, x_2, x_3) and the plate coordinates be (x, y, z). Further, assume that the x_1x_2-plane is parallel to the xy-plane, but rotated counterclockwise about the $z = x_3$-axis by an angle θ (Figure 1.4.1). Conversely, (x, y, z) is obtained from (x_1, x_2, x_3) by rotating the x_1x_2-plane clockwise by an angle θ. Then the coordinates of a material point in the two coordinate systems are related by Eq. (1.2.63). We wish to write the relationships between the components of stress and strain in the two coordinate systems.

1.4.2 Transformation of Stress Components

Let $\sigma_{11}, \sigma_{12}, \cdots, \sigma_{33}$ denote the stress components in the material coordinates (x_1, x_2, x_3) and $\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz}$ be the stress components referred to the plate coordinates (x, y, z). In the notation of Section 1.2, (x_1, x_2, x_3) is the system with bars, (x, y, z) is the system without bars, and θ is the angle of rotation about the $x_3 = z$ axis in the counter-clockwise direction. Since stress is a second-order tensor, we have from Eqs. (1.2.73) and (1.2.74) the following relations among the two sets of components:

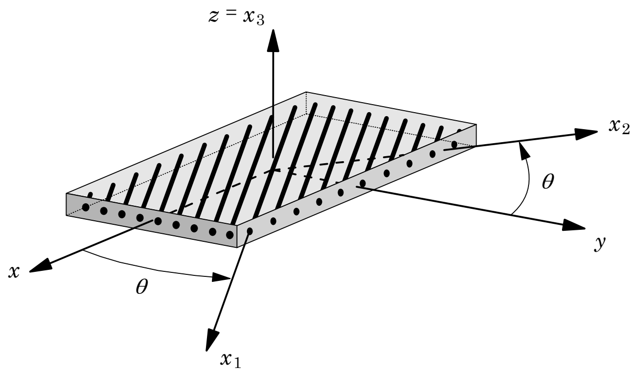


Figure 1.4.1 A plate with global and material coordinate systems.

$$
\begin{aligned}
\sigma_{11} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta \\
\sigma_{12} &= (\sigma_{yy} - \sigma_{xx})\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) \\
\sigma_{13} &= \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta \\
\sigma_{22} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\cos\theta\sin\theta \\
\sigma_{23} &= \sigma_{yz}\cos\theta - \sigma_{xz}\sin\theta \\
\sigma_{33} &= \sigma_{zz}
\end{aligned}
\tag{1.4.1}
$$

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta - 2\sigma_{12}\cos\theta\sin\theta \\
\sigma_{xy} &= (\sigma_{11} - \sigma_{22})\cos\theta\sin\theta + \sigma_{12}(\cos^2\theta - \sin^2\theta) \\
\sigma_{xz} &= \sigma_{13}\cos\theta - \sigma_{23}\sin\theta \\
\sigma_{yy} &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta + 2\sigma_{12}\cos\theta\sin\theta \\
\sigma_{yz} &= \sigma_{23}\cos\theta + \sigma_{13}\sin\theta \\
\sigma_{zz} &= \sigma_{33}
\end{aligned}
\tag{1.4.2}
$$

1.4.3 Transformation of Strain Components

Since strains are also tensor quantities, transformation equations derived for stresses are also valid for *tensor* components of strains. We have (see Problem 1.20)

$$
\begin{aligned}
\varepsilon_{11} &= \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + 2\varepsilon_{xy}\cos\theta\sin\theta \\
\varepsilon_{12} &= (\varepsilon_{yy} - \varepsilon_{xx})\cos\theta\sin\theta + \varepsilon_{xy}(\cos^2\theta - \sin^2\theta) \\
\varepsilon_{13} &= \varepsilon_{xz}\cos\theta + \varepsilon_{yz}\sin\theta \\
\varepsilon_{22} &= \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - 2\varepsilon_{xy}\cos\theta\sin\theta \\
\varepsilon_{23} &= \varepsilon_{yz}\cos\theta - \varepsilon_{xz}\sin\theta \\
\varepsilon_{33} &= \varepsilon_{zz}
\end{aligned}
\tag{1.4.3}
$$

$$
\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta - 2\varepsilon_{12}\cos\theta\sin\theta \\
\varepsilon_{xy} &= (\varepsilon_{11} - \varepsilon_{22})\cos\theta\sin\theta + \varepsilon_{12}(\cos^2\theta - \sin^2\theta) \\
\varepsilon_{xz} &= \varepsilon_{13}\cos\theta - \varepsilon_{23}\sin\theta \\
\varepsilon_{yy} &= \varepsilon_{11}\sin^2\theta + \varepsilon_{22}\cos^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta \\
\varepsilon_{yz} &= \varepsilon_{23}\cos\theta + \varepsilon_{13}\sin\theta \\
\varepsilon_{zz} &= \varepsilon_{33}
\end{aligned}
\tag{1.4.4}
$$

1.4.4 Transformation of Material Stiffnesses

The material stiffnesses $C_{ijk\ell}$ in their original form are the components of a fourth-order tensor. Hence, the tensor transformation law holds again. The fourth-order elasticity tensor components $\bar{C}_{ijk\ell}$ in the problem coordinates can be related to the components C_{mnpq} in the material coordinates by the tensor transformation law

$$\bar{C}_{ijk\ell} = a_{im}a_{jn}a_{kp}a_{\ell q}C_{mnpq} \tag{1.4.5}$$

where a_{ij} are the direction cosines defined in (1.2.57). However, the above equation involves five matrix multiplications with four-subscript material coefficients. Alternatively, the same result can be obtained by using the stress-strain relations in the principal material coordinates and the stress and strain transformation equations in Eqs. (1.4.1)–(1.4.4).

For the plane stress case, the elastic stiffnesses Q_{ij} in the principal material system are related to $\bar{Q}_{ij}$ in the reference coordinate system by the equations

$$
\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\ \cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\ \cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\ \cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \\
\bar{Q}_{45} &= (Q_{55} - Q_{44})\cos\theta\sin\theta \\
\bar{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta
\end{aligned}
\tag{1.4.6}
$$

The thermal coefficients of expansion α_{ij} are the components of a second-order tensor and, therefore, they transform like the stress and strain components. For an orthotropic material, the only nonzero components of thermal expansion tensor referred to the principal materials coordinates are α_1, α_2, and α_3. Hence, we can write the transformation relations among α_{ij} and $(\alpha_{xx}, \alpha_{yy}, \alpha_{xy})$ [see Eq. (1.4.4)]

$$
\begin{aligned}
\alpha_{xx} &= \alpha_1\cos^2\theta + \alpha_2\sin^2\theta \\
\alpha_{yy} &= \alpha_1\sin^2\theta + \alpha_2\cos^2\theta \\
\alpha_{xy} &= 2\,(\alpha_1 - \alpha_2)\sin\theta\cos\theta \\
\alpha_{xz} &= 0, \quad \alpha_{yz} = 0, \quad \alpha_{zz} = \alpha_3
\end{aligned}
\tag{1.4.7}
$$

The stress–strain relations in the problem coordinates (x, y, z) for the thermoelastic case can now be expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha_{xx}\Delta T \\ \varepsilon_{yy} - \alpha_{yy}\Delta T \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} - \alpha_{xy}\Delta T \end{Bmatrix} \tag{1.4.8}$$

1.5 Summary

In this chapter, a brief review of vectors, tensors, matrices, and equations of elasticity is presented. In particular, elements from vector and tensor calculus, summation convention, coordinate transformations, concepts of stress and strain, strain compatibility conditions, equations of motion and constitutive equations of linearized elasticity are presented. The concepts and equations presented in this chapter are useful in the forthcoming chapters of this book.

Problems

1.1 Prove the following properties of δ_{ij} and ε_{ijk} (assume $i, j = 1, 2, 3$ when they are dummy indices):

(a) $F_{ij}\delta_{jk} = F_{ik}$

(b) $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

(c) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

(d) $\varepsilon_{ijk}F_{ij} = 0$ whenever $F_{ij} = F_{ji}$ (symmetric)

1.2 Let $\mathbf{r}$ denote a position vector. Show that:

(a) grad $(r^n) = nr^{n-2}\mathbf{r}$

(b) $\nabla^2(r^n) = n(n+1)r^{n-2}$

(c) div $(\mathbf{r}) = 3$

(d) curl $(\mathbf{r}f(r)) = \mathbf{0}$, where $f(r)$ is an arbitrary continuous function of r with continuous first derivatives

1.3 If $[B]$ is a symmetric $n \times n$ matrix and $[C]$ is any $n \times n$ matrix, show that $[C]^{\mathrm{T}}[B][C]$ is symmetric.

1.4 Establish the ε-δ identity of Eq. (1.2.15).

1.5 Prove that the determinant of a 3×3 matrix $[C]$ can be expressed in the form

$$|C| = \varepsilon_{ijk}\, c_{1i}\, c_{2j}\, c_{3k} \tag{a}$$

and, thus, prove

$$|C| = \frac{1}{6}\varepsilon_{ijk}\, \varepsilon_{rst}\, c_{ir}\, c_{js}\, c_{kt} \tag{b}$$

where c_{ij} is the element occupying the ith row and the jth column of $[C]$.

1.6 Using Cramer's rule determine the solution to the following equations:

(a) (*Ans:* $x_1 = \frac{9}{4},\ x_2 = \frac{7}{2},\ x_3 = \frac{11}{4}$)

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 2 \\ -x_2 + 2x_3 &= 2 \end{aligned}$$

(b) [*Ans:* $x_1 = \frac{3}{2}\alpha,\ x_2 = \frac{1}{2h}\alpha,\ x_3 = -\frac{2}{h}\alpha,\ \alpha = (f_0 h^4/24b)$]

$$\frac{2b}{h^3}\begin{bmatrix} 12 & 0 & 3h \\ 0 & 4h^2 & h^2 \\ 3h & h^2 & 2h^2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{f_0 h}{12}\begin{Bmatrix} 12 \\ 0 \\ h \end{Bmatrix}$$

where b, f_0, and h are constants

1.7 Let [C] be a 3×3 matrix, [I] be a 3×3 identity matrix, and λ be a scalar. Show that

$$\det[C - \lambda I] = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3$$

where

$$I_1 = c_{ii},\ I_2 = \frac{1}{2}(c_{ii}c_{jj} - c_{ij}c_{ij}),\ I_3 = |C|$$

1.8 If we identify a second-order tensor $\mathbf{A}$ associated with the direction cosines $a_{ij} = \hat{\bar{\mathbf{e}}}_i \cdot \hat{\mathbf{e}}_j$ [see Eq. (1.2.57)]

$$\mathbf{A} = a_{ij}\,\hat{\bar{\mathbf{e}}}_i\hat{\mathbf{e}}_j$$

show that (a) $\mathbf{A}\cdot\mathbf{A} = \mathbf{A}$, (b) $\mathbf{A}\cdot\mathbf{A}^{\mathrm{T}} = \mathbf{I}$, and (c) $\mathbf{A}:\mathbf{A} = 3$.

1.9 Use the definition $\nabla^2 = \nabla\cdot\nabla$ to show that the Laplacian operator in the cylindrical coordinate system is given by

$$\nabla^2 = \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r}\frac{\partial^2}{\partial\theta^2} + r\frac{\partial^2}{\partial z^2}\Big]$$

1.10 Show that the gradient of a vector $\mathbf{u}$ in the cylindrical coordinate system is given by

$$\begin{aligned} \nabla\mathbf{u} = {} & \frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta + \frac{\partial u_z}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_z \\ & + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z + \frac{\partial u_\theta}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial u_r}{\partial z}\hat{\mathbf{e}}_z\hat{\mathbf{e}}_r + \frac{\partial u_z}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_z \\ & + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r \end{aligned}$$

1.11 Show that the curl of a vector $\mathbf{u}$ in the cylindrical coordinate system is given by

$$\nabla \times \mathbf{u} = \hat{\mathbf{e}}_r \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) + \hat{\mathbf{e}}_\theta \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) + \hat{\mathbf{e}}_z \left(\frac{\partial u_\theta}{\partial r} + \frac{u\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right)$$

1.12 For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \cdot \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \cdot \mathbf{S} = & \left[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial \theta} + \frac{\partial S_{zr}}{\partial z} + \frac{1}{r}\left(S_{rr} - S_{\theta\theta}\right) \right] \hat{\mathbf{e}}_r \\ & + \left[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{\partial S_{z\theta}}{\partial z} + \frac{1}{r}\left(S_{r\theta} + S_{\theta r}\right) \right] \hat{\mathbf{e}}_\theta \\ & + \left[\frac{\partial S_{rz}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r}S_{rz} \right] \hat{\mathbf{e}}_z\end{aligned}$$

1.13 For an arbitrary second-order tensor $\mathbf{S}$, show that $\nabla \times \mathbf{S}$ in the cylindrical coordinate system is given by

$$\begin{aligned}\nabla \times \mathbf{S} = \; & \hat{\mathbf{e}}_r\hat{\mathbf{e}}_r \left(\frac{1}{r}\frac{\partial S_{zr}}{\partial \theta} - \frac{\partial S_{\theta r}}{\partial z} - \frac{1}{r}S_{z\theta} \right) + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta \left(\frac{\partial S_{r\theta}}{\partial z} - \frac{\partial S_{z\theta}}{\partial r} \right) + \\ & \hat{\mathbf{e}}_z\hat{\mathbf{e}}_z \left(\frac{1}{r}S_{\theta z} - \frac{1}{r}\frac{\partial S_{rz}}{\partial \theta} + \frac{\partial S_{\theta z}}{\partial r} \right) + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta \left(\frac{1}{r}\frac{\partial S_{z\theta}}{\partial \theta} - \frac{\partial S_{\theta\theta}}{\partial z} + \frac{1}{r}S_{zr} \right) + \\ & \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r \left(\frac{\partial S_{rr}}{\partial z} - \frac{\partial S_{zr}}{\partial r} \right) + \hat{\mathbf{e}}_r\hat{\mathbf{e}}_z \left(\frac{1}{r}\frac{\partial S_{zz}}{\partial \theta} - \frac{\partial S_{\theta z}}{\partial z} \right) + \\ & \hat{\mathbf{e}}_z\hat{\mathbf{e}}_r \left(\frac{\partial S_{\theta r}}{\partial r} - \frac{1}{r}\frac{\partial S_{rr}}{\partial \theta} + \frac{1}{r}S_{r\theta} + \frac{1}{r}S_{\theta r} \right) + \hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_z \left(\frac{\partial S_{rz}}{\partial z} - \frac{\partial S_{zz}}{\partial r} \right) + \\ & \hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta \left(\frac{\partial S_{\theta\theta}}{\partial r} + \frac{1}{r}S_{\theta\theta} - \frac{1}{r}S_{rr} - \frac{1}{r}\frac{\partial S_{r\theta}}{\partial \theta} \right)\end{aligned}$$

1.14 Show that the curl of a tensor $\mathbf{E} = E_{rr}(r,z)\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + E_{\theta\theta}(r,z)\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta$ in the cylindrical coordinate system is given by

$$\nabla \times \mathbf{E} = \left(\frac{\partial E_{\theta\theta}}{\partial r} + \frac{E_{\theta\theta} - E_{rr}}{r} \right) \hat{\mathbf{e}}_z\hat{\mathbf{e}}_\theta + \frac{\partial E_{rr}}{\partial z}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r - \frac{\partial E_{\theta\theta}}{\partial z}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta$$

1.15 Show that $\nabla \mathbf{u}$ in the spherical coordinate system is given by

$$\begin{aligned}\nabla \mathbf{u} = & \frac{\partial u_r}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_r + \frac{\partial u_\phi}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_\phi + \frac{\partial u_\theta}{\partial r}\hat{\mathbf{e}}_r\,\hat{\mathbf{e}}_\theta \\ & + \frac{1}{r}\left(\frac{\partial u_r}{\partial \phi} - u_\phi \right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_r + \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \phi} + u_r \right)\hat{\mathbf{e}}_\phi\,\hat{\mathbf{e}}_\phi + \frac{1}{r}\frac{\partial u_\theta}{\partial \phi}\hat{\mathbf{e}}_\phi\hat{\mathbf{e}}_\theta \\ & + \frac{1}{r\sin\phi}\left[\left(\frac{\partial u_r}{\partial \theta} - u_\theta\sin\phi \right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_r + \left(\frac{\partial u_\phi}{\partial \theta} - u_\theta\,\cos\phi \right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\phi \right. \\ & \left. + \left(\frac{\partial u_\theta}{\partial \theta} + u_r\,\sin\phi + u_\phi\cos\phi \right)\hat{\mathbf{e}}_\theta\,\hat{\mathbf{e}}_\theta \right]\end{aligned}$$

1.16 Find the linear strains associated with the 2-D displacement field

$$
\begin{aligned}
u_1 &= -c_0 x_1 x_2 + c_1 x_2 + c_2 x_3 + c_4 \\
u_2 &= c_0 \left[\nu \left(x_2^2 - x_1^2 \right) + x_1^2 \right] + c_4 x_3 + c_5 x_1 + c_6 \\
u_3 &= 0
\end{aligned}
$$

where c_i and ν are constants.

1.17 Consider the displacement vector in polar axisymmetric system

$$\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_z \hat{\mathbf{e}}_z$$

The strain tensor in the cylindrical coordinate system can be written in the dyadic form as

$$
\begin{aligned}
\varepsilon = \varepsilon_{rr} \hat{\mathbf{e}}_{rr} + \varepsilon_{\theta\theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \varepsilon_{zz} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z + \varepsilon_{r\theta} \left(\hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r \right) \\
+ \varepsilon_{rz} \left(\hat{\mathbf{e}}_r \hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r \right) + \varepsilon_{\theta z} \left(\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta \right)
\end{aligned}
$$

Show that the only nonzero linear strains are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

1.18 Using the definitions of ∇ and $\mathbf{u}$ in the cylindrical coordinate system, show that the linear strains (1.3.7) in the cylindrical coordinate system are given by

$$
\begin{aligned}
\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \quad \varepsilon_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\
\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta}, \quad \varepsilon_{z\theta} = \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right), \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}
\end{aligned}
$$

1.19 Show that the equations of motion in the cylindrical coordinate system are given by

$$
\begin{aligned}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} \left(\sigma_{rr} - \sigma_{\theta\theta} \right) + \rho f_r &= \rho \frac{\partial^2 u_r}{\partial t^2} \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho f_\theta &= \rho \frac{\partial^2 u_\theta}{\partial t^2} \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho f_z &= \rho \frac{\partial^2 u_z}{\partial t^2}
\end{aligned}
$$

1.20 Derive the transformation relations (1.4.3) between the rectangular Cartesian components of strain $(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy})$ and the components of strain $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{r\theta})$ in the cylindrical coordinate system.

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