$X \sim \frac{1}{2} e^{-|x-\theta|}$ Q1 MLE AS12

Likelihood :

$$L(X_1 - - X_n) = \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^{n} |X_i - \theta|}$$

log-likelihood:

$$n \ln(1/2) - \sum_{i=1}^{n} |X_i - \theta|$$

$\Rightarrow$ The optimization becomes

$$\arg\max_{\theta} \left[n \ln(1/2) - \sum_{i=1}^{n} |X_i - \theta| \right]$$

$$= \arg\min_{\theta} \left(\sum_{i=1}^{n} |X_i - \theta| \right)$$

We know that,

$$\frac{\partial |x|}{\partial x} = \frac{\partial \sqrt{x^2}}{\partial x} = x (x^2)^{-1/2}$$

$$= \frac{x}{|x|} = \operatorname{sgn}(x)$$

$\Rightarrow$ Optimizing with respect to θ by taking derivative with respect to θ and equating with zero, we get

$$\boxed{\sum_{i=1}^{n} \operatorname{sgn}(X_i - \theta) = 0}$$

condition

Case 1 : n is odd

If we choose $\hat{\theta} = \text{median}(x_1 -- x_n)$, then we get $\frac{n-1}{2}$ cases where $x_i < \theta$ & other $\frac{n-1}{2}$ cases where $x_i > \theta$.

$\Rightarrow$ $\hat{\theta}$ satisfies the condition for MLE estimator of θ.

Case 2 : n is even

Now we cannot choose one x_i to satisfy the condition for MLE estimator

We can still perform the minimization by assuming that $x_1 —— x_n$ are ordered ($x_1 \leq x_2 \leq —— x_n$) and then choosing either $x_{n/2}$ or $x_{\frac{(n+1)}{2}}$.

$\Rightarrow$ Median is the MLE estimator for the given distribution.

AS 12 Q2 For L1 loss, $L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|$

We choose $\hat{\theta}$ to minimize

$$r(\hat{\theta}|x) = \int_{\theta} |\theta - \hat{\theta}| f(\theta|x)\, d\theta$$

$$= \int_{-\infty}^{\hat{\theta}} (\hat{\theta} - \theta) f(\theta|x)\, d\theta + \int_{\hat{\theta}}^{\infty} (\theta - \hat{\theta}) f(\theta|x)\, d\theta \quad (1)$$

Taking derivative of $r(\hat{\theta}|x)$ with respect to $\hat{\theta}$ & setting it equal to zero:

$$\int_{-\infty}^{\hat{\theta}} f(\theta|x)\, d\theta = \int_{\hat{\theta}}^{\infty} f(\theta|x)\, d\theta$$

$$\Rightarrow 2 \int_{-\infty}^{\hat{\theta}} f(\theta|x)\, d\theta = \int_{-\infty}^{\infty} f(\theta|x)\, d\theta = 1$$

$$\Rightarrow \int_{-\infty}^{\hat{\theta}} f(\theta|x)\, d\theta = \frac{1}{2}$$

& $\hat{\theta}$ is the median of $f(\theta|x)$.

The second derivative of (1) with respect to $\hat{\theta}$ gives $2 f(\hat{\theta}|x) \geq 0$.

$\Rightarrow$ The original function is convex & thus the median definitely corresponds to the minimum.

Hence, proved.