

Pushouts in topological spaces

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Practical training course: Formalizing mathematics in Lean

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Mathematical background

Definition of the adjunction space

Definition

Let X and Y be topological spaces, $X \sqcup Y$ be the disjoint union and $\varphi_1 : X \rightarrow X \sqcup Y$ and $\varphi_2 : Y \rightarrow X \sqcup Y$ be the canonical inclusion maps. The topology \mathcal{O} on $X \sqcup Y$ is given by

$$\mathcal{O} := \{U \subseteq X \sqcup Y \mid \varphi_1^{-1}(U) \text{ is open in } X \text{ or } \varphi_2^{-1}(U) \text{ is open in } Y\}.$$

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Definition

Let X be a topological space and \sim be an equivalence relation on X . Then, the **quotient space** X/\sim is the set $\{[x] : x \in X\}$ of equivalence classes together with the topology

$$\mathcal{O} := \{U \subseteq X/\sim \mid \exists V \subseteq X \text{ open} : x \in V \Leftrightarrow [x] \in U\}.$$

Mathematical background

Definition of the adjunction space

Definition

Let X and Y be topological spaces and let A be a subspace of Y . Moreover, let $f_1 : A \rightarrow X$ be continuous and $f_2 : A \rightarrow Y$ be the inclusion map.

Moreover, let \sim be the equivalence relation on $X \sqcup Y$ generated by $f_1(a) \sim f_2(a)$ for all $a \in A$. Then, the quotient

$$X \cup_{f_1} Y := (X \sqcup Y) / \sim$$

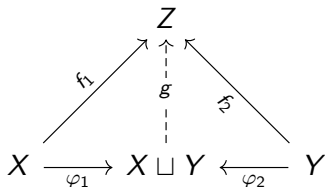
with the quotient topology is called the adjunction space (or pushout).

Mathematical background

Universal properties

Theorem (Universal property of the disjoint union)

Let X, Y, Z be topological spaces and $f_1 : X \rightarrow Z$ and $f_2 : Y \rightarrow Z$ be continuous maps. Then there exists exactly one continuous map $g : X \sqcup Y \rightarrow Z$ with $f_1 = g \circ \varphi_1$ and $f_2 = g \circ \varphi_2$.



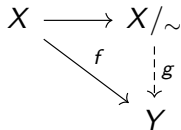
The disjoint union is the coproduct in the category of topological spaces.

Mathematical background

Universal properties

Theorem (Universal property of the quotient space)

Let X, Z be topological spaces, \sim be an equivalence relation on X and $f : X \rightarrow Z$ be continuous with $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$. Then there exists exactly one continuous map $g : (X/\sim) \rightarrow Y$ with $g([x]) = f(x)$ for all $x \in X$.



Mathematical background

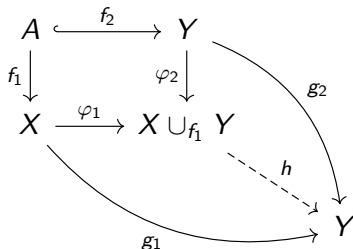
Universal properties

Theorem (Universal property of the pushout)

Let X, Y, Z be topological spaces, A a subspace of Y , $f_1 : A \rightarrow X$ be continuous and $f_2 : A \rightarrow Y$ be the inclusion map.

Moreover, let $g_1 : X \rightarrow Z$ and $g_2 : Y \rightarrow Z$ be continuous maps with $g_1 \circ f_1 = g_2 \circ f_2$.

Then there exists exactly one continuous map $h : X \cup_{f_1} Y \rightarrow Z$ with $g_1 = h \circ \varphi_1$ and $g_2 = h \circ \varphi_2$.



Mathematical background

Interesting lemmas

Lemma

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Lemma

If f_1 is a quotient map, φ_2 is a quotient map as well.

Mathematical background

Interesting lemmas

Definition (Connectedness)

A topological space X is called **connected** if and only if the only subsets of X that are both open and closed are \emptyset and X .

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A topological space X is called **path-connected** if and only if for all $x, y \in X$ there exists a continuous map $p : [0, 1] \rightarrow X$ with $p(0) = x$ and $p(1) = y$.

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Lemma

If A is nonempty and X and Y are connected, $X \cup_{f_1} Y$ is connected as well.

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Lemma

If A is nonempty and X and Y are path connected, $X \cup_{f_2} Y$ is path connected as well.

Mathematical background

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A nonempty topological space X is called a T_1 -**space** if and only if every set S with $|S| = 1$ is closed.

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Lemma

Let A be closed in Y and X and Y be T_1 -spaces. Then $X \cup_{f_1} Y$ is a T_1 -space as well.

Mathematical background

Interesting lemmas

Definition (Normal space)

A topological space X is called **normal** if and only if for all disjoint closed sets $C, D \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

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Definition (Normal space)

A topological space X is called **normal** if and only if for all disjoint closed sets $C, D \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

Definition (T_4 space)

A nonempty topological space X is called **T_4 -space** iff it is both T_1 and normal.

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Interesting lemmas

Definition (Normal space)

A topological space X is called **normal** if and only if for all disjoint closed sets $C, D \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

Definition (T_4 space)

A nonempty topological space X is called **T_4 -space** iff it is both T_1 and normal.

Theorem (Tietze's extension theorem)

Let X be a normal space, $C \subseteq X$ closed and $f : C \rightarrow \mathbb{R}$ be a continuous map. Then there exists a continuous map $f' : X \rightarrow \mathbb{R}$ with $f'|_C = f$.

Mathematical background

Interesting lemmas

Lemma

Let X, Y be T_4 -spaces, $A \subseteq Y$ a nonempty closed subspace of Y , $f_1 : A \rightarrow X$ be continuous and $f_2 : A \rightarrow Y$ be the inclusion map. Then $X \cup_{f_1} Y$ is a T_4 -space as well.

Formalization

Implementing the definition

Disjoint unions and quotient spaces are already implemented in mathlib as more general concepts:

```
--
`Sum α β`, or `α ⊔ β`, is the disjoint union of types `α` and `β`.
An element of `α ⊔ β` is either of the form `.inl a` where `a : α`,
or `.inr b` where `b : β`.
-/
inductive Sum (α : Type u) (β : Type v) where
  /- Left injection into the sum type `α ⊔ β`. If `a : α` then `.inl a : α ⊔ β`. -/
  | inl (val : α) : Sum α β
  /- Right injection into the sum type `α ⊔ β`. If `b : β` then `.inr b : α ⊔ β`. -/
  | inr (val : β) : Sum α β

@[inherit_doc] inductive Sum "⊔" => Sum
```

```
--
`Quotient α s` is the same as `Quot α r`, but it is specialized to a setoid `s`
(that is, an equivalence relation) instead of an arbitrary relation.
Prefer `Quotient` over `Quot` if your relation is actually an equivalence relation.
-/
def quotient {α : Sort u} (s : Setoid α) :=
  @Quot α Setoid.r
```

Formalization

Implementing the definition

The topologies on the disjoint union and the quotient space are given as instances of `TopologicalSpace`:

```
instance instTopologicalSpaceSum [t1 : TopologicalSpace X] [t2 : TopologicalSpace Y] :  
  TopologicalSpace (X ⋈ Y) :=  
    coinduced Sum.inl t1 ∪ coinduced Sum.inr t2  
  
instance instTopologicalSpaceQuotient {s : Setoid X} [t : TopologicalSpace X] :  
  TopologicalSpace (Quotient s) :=  
    coinduced Quotient.mk' t
```


Formalization

Implementing the definition

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- 2 Defining A and Y as separate types and defining the subspace relation by requiring f_2 to be an embedding.

Formalization

Implementing the definition

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- 1 Defining the equivalence relation on the disjoint union (`equivalence_of_images` $f_1 \ h f_2$)
- 2 Defining A and Y as separate types and defining the subspace relation by requiring f_2 to be an embedding.
- 3 Defining `AdjunctionSpace` $f_1 \ h f_2$ as the quotient of the disjoint union with respect to `equivalence_of_images`.

Formalization

Implementing the definition

Steps when implementing the definition of `AdjunctionSpace`:

- 1 Defining the equivalence relation on the disjoint union (`equivalence_of_images` $f_1 \ h f_2$)
- 2 Defining A and Y as separate types and defining the subspace relation by requiring f_2 to be an embedding.
- 3 Defining `AdjunctionSpace` $f_1 \ h f_2$ as the quotient of the disjoint union with respect to `equivalence_of_images`.
- 4 The topology on `AdjunctionSpace` $f_1 \ h f_2$ is now naturally given since the topology on the disjoint union and the quotient topology are already defined.

Formalization

Universal properties

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Universal properties

The universal properties of the disjoint union and the quotient space seem to be not implemented in Lean yet, so I proved them from scratch. The universal property of the pushout is an easy consequence of them (purely categorical diagram chasing proof).

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Formalization

Challenging aspects

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- Difficulties when trying to use the Tietze extension theorem.

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Challenging aspects

- Proving that φ_1 is an embedding.
- Missing framework when trying to prove connectedness
- Difficulties with coercions when trying to prove path-connectedness.
- Difficulties when trying to use the Tietze extension theorem.
- The script I was working with turned out contain some errors, so I discovered some missing prerequisites in some theorems in the working process.

What could one do in the future?

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- Implement the wedge product as a special case.
- Implement some examples (e.g. projective spaces, cylinder, Klein bottle) as instances of `AdjunctionSpace`.
- There is already a good category theory framework in Lean, in particular, the categorical pushout is already implemented (`CategoryTheory.IsPushout`). Maybe my work could somehow be connected with the category theory framework?