Pushouts in topological spaces

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Practical training course: Formalizing mathematics in Lean

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Outline

1. Mathematical background
   - Definition of the adjunction space
   - Universal properties
   - Interesting lemmas
2. Formalization
   - Implementing the definition
   - Challenging aspects
3. Future work?

Definition

Let X and Y be topological spaces, $X \sqcup Y$ be the disjoint union and $\varphi_1 : X \to X \sqcup Y$ and $\varphi_2 : Y \to X \sqcup Y$ be the canonical inclusion maps. The topology $\mathcal{O}$ on $X \sqcup Y$ is given by

$$\mathcal{O} := \{U \subseteq X \sqcup Y \mid \varphi_1^{-1}(U) \text{ is open in } X \text{ or } \varphi_2^{-1}(U) \text{ is open in } Y\}.$$

Definition

Let X be a topological space and $\sim$ be an equivalence relation on X. Then, the **quotient space** $X/_\sim$ is the set $\{[x] : x \in X\}$ of equivalence classes together with the topology

$$\mathcal{O} := \{U \subseteq X/_\sim \mid \exists V \subseteq X \text{ open} : x \in V \Leftrightarrow [x] \in U\}.$$

Definition

Let X and Y be topological spaces and let A be a subspace of Y.
Moreover, let $f_1 : A \to X$ be continuous and $f_2 : A \to Y$ be the inclusion map.
Moreover, let $\sim$ be the equivalence relation on $X \sqcup Y$ generated by $f_1(a) \sim f_2(a)$ for all $a \in A$. Then, the quotient

$$X \cup_{f_1} Y := (X \sqcup Y)/_{\sim}$$

with the quotient topology is called the adjunction space (or pushout).

Mathematical background

Universal properties

Theorem (Universal property of the disjoint union)

Let X, Y, Z be topological spaces and $f_1 : X \to Z$ and $f_2 : Y \to Z$ be continuous maps. Then there exists exactly one continuous map $g : X \sqcup Y \to Z$ with $f_1 = g \circ \varphi_1$ and $f_2 = g \circ \varphi_2$.

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f_1"] \arrow[r, "\varphi_1"'] & X \sqcup Y \arrow[u, dashed, "g"] & Y \arrow[ul, "f_2"'] \arrow[l, "\varphi_2"]
\end{tikzcd}$$

The disjoint union is the coproduct in the category of topological spaces.

Theorem (Universal property of the quotient space)

Let X, Z be topological spaces, $\sim$ be an equivalence relation on X and $f : X \to Z$ be continuous with $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$. Then there exists exactly one continuous map $g : (X/_\sim) \to Y$ with $g([x]) = f(x)$ for all $x \in X$.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X/_\sim \arrow[d, dashed, "g"] \\ & Y \end{tikzcd}$$

Mathematical background

Universal properties

Theorem (Universal property of the pushout)

Let X, Y, Z be topological spaces, A a subspace of Y, $f_1 : A \to X$ be continuous and $f_2 : A \to Y$ be the inclusion map.
Moreover, let $g_1 : X \to Z$ and $g_2 : Y \to Z$ be continuous maps with $g_1 \circ f_1 = g_2 \circ f_2$.
Then there exists exactly one continuous map $h : X \cup_{f_1} Y \to Z$ with $g_1 = h \circ \varphi_1$ and $g_2 = h \circ \varphi_2$.

$$
\begin{tikzcd}
A \arrow[r, hook, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "\varphi_2"'] \arrow[ddr, bend left, "g_2"] & \\
X \arrow[r, "\varphi_1"] \arrow[drr, bend right, "g_1"'] & X \cup_{f_1} Y \arrow[dr, dashed, "h"] & \\
 & & Y
\end{tikzcd}
$$

Lemma

φ_1 *is an embedding.*

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If A is closed in X, φ_1 is a closed embedding.

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If A is closed in X, $\varphi_2|_{Y \setminus A}$ is an open embedding.

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Lemma

If f_1 is a quotient map, φ_2 is a quotient map as well.

Definition (Connectedness)

A topological space X is called **connected** if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.

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Definition (Path-connectedness)

A topological space X is called **path-connected** if and only if for all $x, y \in X$ there exists a continuous map $p : [0,1] \to X$ with $p(0) = x$ and $p(1) = y$.

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Lemma

If A is nonempty and X and Y are connected, $X \cup_{f_1} Y$ is connected as well.

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If A is nonempty and X and Y are path connected, $X \cup_{f_2} Y$ is path connected as well.

Definition

A nonempty topological space X is called a T_1-**space** if and only if every set S with $|S| = 1$ is closed.

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Lemma

Let A be closed in Y and X and Y be T_1-spaces. Then $X \cup_{f_1} Y$ is a T_1-space as well.

Definition (Normal space)

A topological space X is called **normal** if and only if for all disjoint closed sets $C, D \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

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Definition (T4 space)

A nonempty topological space X is called T_4**-space** iff it is both T_1 and normal.

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Definition (T4 space)

A nonempty topological space X is called T_4**-space** iff it is both T_1 and normal.

Theorem (Tietze's extension theorem)

Let X be a normal space, $C \subseteq X$ closed and $f : C \to \mathbb{R}$ be a continuous map. Then there exists a continuous map $f' : X \to \mathbb{R}$ with $f'|_C = f$.

Lemma

Let X, Y be T_4-spaces, $A \subseteq Y$ a nonempty closed subspace of Y, $f_1 : A \to X$ be continuous and $f_2 : A \to Y$ be the inclusion map. Then $X \cup_{f_1} Y$ is a T_4-space as well.

Disjoint unions and quotient spaces are already implemented in mathlib as more general concepts:

```
/--
`Sum α β`, or `α ⊕ β`, is the disjoint union of types `α` and `β`.
An element of `α ⊕ β` is either of the form `.inl a` where `a : α`,
or `.inr b` where `b : β`.
-/
inductive Sum (α : Type u) (β : Type v) where
  /-- Left injection into the sum type `α ⊕ β`. If `a : α` then `.inl a : α ⊕ β`. -/
  | inl (val : α) : Sum α β
  /-- Right injection into the sum type `α ⊕ β`. If `b : β` then `.inr b : α ⊕ β`. -/
  | inr (val : β) : Sum α β

@[inherit_doc] infixr:30 " ⊕ " => Sum
```

```
/--
`Quotient α s` is the same as `Quot α r`, but it is specialized to a setoid `s`
(that is, an equivalence relation) instead of an arbitrary relation.
Prefer `Quotient` over `Quot` if your relation is actually an equivalence relation.
-/
def Quotient {α : Sort u} (s : Setoid α) :=
  @Quot α Setoid.r
```

The topologies on the disjoint union and the quotient space are given as instances of TopologicalSpace:

```
instance instTopologicalSpaceSum [t₁ : TopologicalSpace X] [t₂ : TopologicalSpace Y] :
    TopologicalSpace (X ⊕ Y) :=
  coinduced Sum.inl t₁ ⊔ coinduced Sum.inr t₂
```

```
instance instTopologicalSpaceQuotient {s : Setoid X} [t : TopologicalSpace X] :
    TopologicalSpace (Quotient s) :=
  coinduced Quotient.mk' t
```

Steps when implementing the definition of AdjunctionSpace:

1. Defining the equivalence relation on the disjoint union (equivalence_of_images f_1 hf_2)
2. Defining A and Y as seperate types and defining the subspace relation by requiring f_2 to be an embedding.