**Solutions for Problem Sheet 1**

NB There is more than one way to complete most of these problems. I give here only an indication of one way to do the problems and some simple steps may be missed out for brevity - these should not be regarded as model answers. Please however feel free to ask questions about any aspect of the problems. Let me know if you find any errors.

1. By definition,

$$\begin{aligned} E\left[I_N[f]\right] &= E\left[\frac{1}{N}\sum_{n=1}^{N} f(X_n)\right] \\ &= \frac{1}{N}\sum_{n=1}^{N}\int f(x_n)p(x_n)d\mu(x_n) \\ &= \int f(x)p(x)d\mu(x) \\ &= I[f], \end{aligned}$$

as required.

2. The CLT result states that for large $N$,

$$\lim_{N\to\infty} Pr\left\{a < \frac{\sqrt{N}}{\sigma[g]}\varepsilon_N < b\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}dx$$

The RHS of this expression is equal to 95% when $b = -a \simeq 1.96$ (from tables or computing the value of the inverse erf function). Hence we can be 95% confident that

$$\left|\frac{\sqrt{N}}{\sigma[g]}\varepsilon_N\right| < 1.96$$

for sufficiently large $N$. So,

$$|\varepsilon| < 0.1$$

with probability 0.95 if

$$1.96 \times \frac{\sigma[g]}{\sqrt{N}} \leq 0.1.$$

Now,

$$\begin{aligned} V[g] &= \int_0^1 \sin^2 2\pi y dy - \left(\int_0^1 \sin 2\pi y dy\right)^2 \\ &= \frac{1}{2}\int_0^1 (1-\cos 4\pi y)dy \\ &= \frac{1}{2}. \end{aligned}$$

So, we must have

$$N \geq \left(\frac{1.96}{0.1\times\sqrt{2}}\right)^2 = 192.$$

In general the exact value of the variance is unknown, (it may be as hard to compute as the integral itself) so we must determine the error and variance empirically (see Caflisch (1998), section 2.1).