Gibbs problem sheet

Consider the problem in which the prior density is Gaussian, and also the density of the observations, the likelihood in terms of the observations, is Gaussian. This is the standard problem considered in 3D- and 4DVar, and, when the likelihood as function of the state is also Gaussian, the (Ensemble) Kalman filter. It is given by:

$$p(x) \propto \exp\left[-\frac{1}{2}(x-x_0)B^{-1}(x-x_0) - \frac{1}{2}(y-H(x))R^{-1}(y-H(x))\right] \tag{1}$$

in which $H(x)$ is the measurement operator, which can include the model used to evaluate the states (in the case of 4Dvar).

1) First assume $H(x) = Hx$, i.e. the measurement operator is linear. Show (or argue) that the posterior can be written as:

$$p(x) \propto \exp\left[-\frac{1}{2}(x-x_n)P^{-1}(x-x_n)\right] \tag{2}$$

in which $x_n = x_0 + K(y - Hx_0)$ and $P = (1 - KH^T)B$, with $K = BH^T(HBH^T + R)^{-1}$.

2) Assume a 2D problem, so $x = (x_1, x_2)^T$, and $x_0 = (x_{0_1}, x_{0_2})^T$. For the Gibbs sampler we need to be able to sample from $p(x_1|x_2)$ and $p(x_2|x_1)$. Show that

$$p(x_1|x_2) \propto \exp\left[-\frac{1}{2}\frac{(x_1-m_1)}{\sigma_1^2}\right] \tag{3}$$

in which $m_1 = x_{0_1} + P_{12}(x_2 - x_{0_2})/P_{22}$ and $\sigma_1 = P_{11}(1-r)$, with $r = P_{12}^2/(P_{11}P_{22})$.

3) Argue, using symmetry, that

$$p(x_2|x_1) \propto \exp\left[-\frac{1}{2}\frac{(x_2-m_2)}{\sigma_2^2}\right] \tag{4}$$

in which $m_2 = x_{0_2} + P_{12}(x_1 - x_{0_1})/P_{11}$ and $\sigma_2 = P_{22}(1-r)$.

4) Describe how one can sample from $p(x_1|x_2)$ given above given samples from $N(0,1)$.

5) Describe how one could draw samples from $p(x)$ using samples from $N(0,1)$.

6) What happens if $P_{12} \approx 1$? What could one do to avoid this problem?

7) How would you use the Gibbs sampler when $H(x)$ is a nonlinear function of x?