Markov Chains answer sheet

Problem 1

1) Because it is square and the rows sum to 1.

2) $p(z|x) = 0.1$

3) $p(y|x) + p(y|z) = 0.2 + 0.8 = 1$.

4) The invariant distribution is found from

$$\begin{aligned}
\pi(x) &= 0.7\pi(x) + 0.3\pi(y) \\
\pi(y) &= 0.2\pi(x) + 0.4\pi(y) + 0.8\pi(z) \\
\pi(z) &= 0.1\pi(x) + 0.3\pi(y) + 0.2\pi(z)
\end{aligned}$$

The first equation gives $\pi(x) = \pi(y)$, using this in the second gives $\pi(y) = 2\pi(z)$. Using both in the third gives $\pi(z) = \pi(z)$, as expected. The extra condition needed is $\pi(x) + \pi(y) + \pi(z) = 1$, leading eventually to $\pi(x) = \pi(y) = 0.4$ and $\pi(z) = 0.2$.

5) The Markov Chain is homogeneous. The transition distribution shows it is impossible to move from z to x directly, since $p(x|z) = 0$. Still ergodicity looks tricky, because, x can be reached from z via y, so by the two-step Markov Chain. So in principle we can redefine a new Markov Chain as P^2 that allows one to reach each state from all of the other states. The point is, however, that all states can be reached from all other states in a finite number of steps, so the chain will be ergodic.

Problem 2

1) It is not of interest because all samples will be the same.

2) Use the conditions for the invariant distribution, and that P has to be a stochastic matrix, so its rows have to add up to 1. The solution is

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{pmatrix}$$

3) Check that $\pi(x)P(y|x) = \pi(y)P(x|y)$ for each combination x and y.

4) This leads to

$$P = \begin{pmatrix} 0.91 & 0.09 \\ 0.01 & 0.99 \end{pmatrix}$$

5) The chain of 4) is quite close to the identity chain of 1), so its samples will look very similar. The chain of 4) mixes the states better, so it can be expected that it leads to more diverse samples, leading to better statistics of the invariant density.