A Problem on Approximating Mean of a Function of a Random Variable

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May 27, 2022

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Question

Q29 [12^{th} Papoulis Textbook Exercise 5]:

Using the equation,

$$\eta_y = E(g(x)) \approx g(\eta) + g^{``}(\eta)\frac{\sigma^2}{2} \tag{1}$$

find $E(x^3)$ if $\eta_x = 10$ and $\sigma_x = 2$.

Solution

Before finding the answer, let us prove equation (1)
Proof: If $\mathbf{y} = g(\mathbf{x})$ is new random variable, formed from original random variable $\mathbf{x}$, then

$$E(g(x)) = \int_{-\infty}^{\infty} g(x)f(x)dx \tag{2}$$

Generally, to determine this value, knowledge of f(x) is required. But if $\mathbf{x}$ is concentrated towards it's mean, then $E(g(\mathbf{x}))$ can be approximated. Suppose $f(x)$ is negligible outside the interval $(\eta - \epsilon, \eta + \epsilon)$ and ϵ is very small, such that in this interval $g(x) \approx g(\eta)$, then $g(x)$ can be approximated with

$$g(x) \approx g(\eta) + g^{'}(\eta)(x - \eta) + ...g^{n}(\eta)\frac{(x-\eta)^n}{n!} \tag{3}$$

Solution

Approximating $g(x)$ to two degree polynomial by neglecting higher terms and putting into equation (2) gives us

$$\begin{aligned} E(g(x)) = g(\eta)\int_{-\infty}^{\infty} f(x)dx + g^{'}(\eta)\int_{-\infty}^{\infty}(x-\eta)f(x)dx \\ + g^{''}(\eta)\int_{-\infty}^{\infty}\frac{(x-\eta)^2}{2}f(x)dx \end{aligned} \tag{4}$$

$$E(g(x)) = g(\eta) + g^{''}(\eta)\frac{\sigma^2}{2} \tag{5}$$

Given,

$$g(x) = x^3, \eta = 10, \sigma = 2 \tag{6}$$

After substituting, $E(g(x)) = 1120$