Assignment
AI1110: Probability and Random Variables

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Rutheeshreddy/AI1110/blob/main/Assignment_revision-1/codes/1.1.c

gcc 1.1.c -o 1.1 && ./1.1
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1

```
wget https://github.com/Rutheeshreddy/AI1110/blob/main/Assignment_revision-1/codes/uni_cdf_plot.py

python3 uni_cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$. **Solution:** For uniform distribution , the pdf is a constant. Let,

$$p_U(x) = a, 0 \leq x \leq 1 \tag{1.2}$$
$$\text{then, } F_U(x) = \int_0^x a dx \tag{1.3}$$
$$F_U(x) = ax, 0 \leq x \leq 1 \tag{1.4}$$
$$\text{As, } F_U(1) = 1 \implies a = 1 \tag{1.5}$$
$$\therefore F_U(x) = x, 0 \leq x \leq 1 \tag{1.6}$$

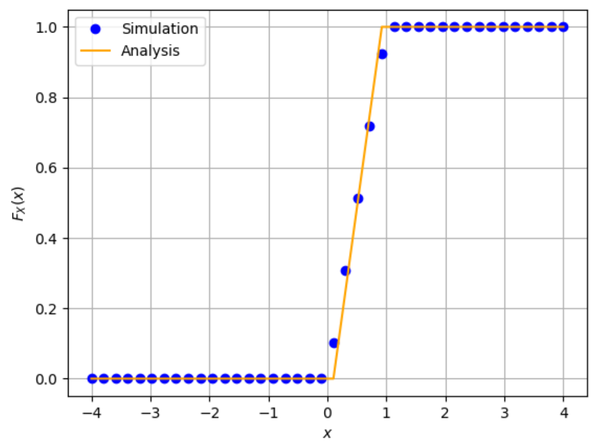

Fig. 1: The CDF of U

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.7}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

Write a C program to find the mean and variance of U.
Solution: The following code calculates mean and variance

```
wget https://github.com/Rutheeshreddy/AI1110/blob/main/Assignment_revision-1/codes/1.4.c

gcc 1.4.c -o 1.4 && ./1.4
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.9}$$

Solution:

$$\text{As, } E(X) = \int_{-\infty}^{\infty} x.p_X(x).dx \tag{1.10}$$

$$E(X) = \int_{0}^{1} x.dx \tag{1.11}$$

$$E(X) = \left[\frac{x^2}{2}\right]_0^1 \tag{1.12}$$

$$E(X) = \frac{1}{2} \tag{1.13}$$

$$\text{And, } E(X^2) = \int_{-\infty}^{\infty} x^2 p_X(x).dx \tag{1.14}$$

$$E(X^2) = \int_{0}^{1} x^2.dx \tag{1.15}$$

$$E(X^2) = \frac{1}{3} \tag{1.16}$$

$$Var(X) = E(X^2) - [E(X)]^2 \tag{1.17}$$

$$Var(X) = \frac{1}{12} \tag{1.18}$$

Theoretical values

$$E(X) = 0.5 \tag{1.19}$$

$$Var(X) = 0.08333 \tag{1.20}$$

Numerical values calculated in C program

$$E(X) = 0.500007 \tag{1.21}$$

$$Var(X) = 0.083301 \tag{1.22}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/2.1.c

gcc 2.1.c -o 2.1 && ./2.1
```

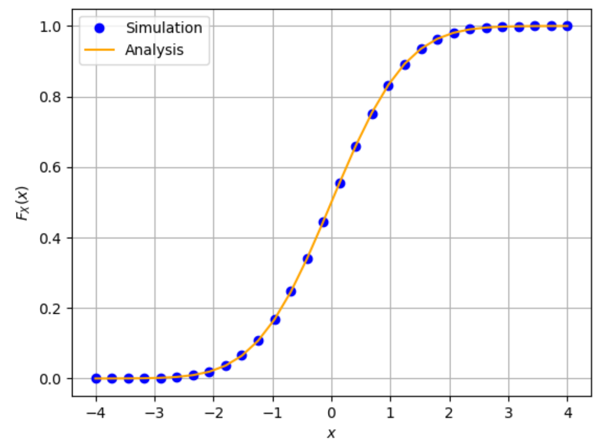

Fig. 2: The CDF of X

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2
The following code plots Fig. 2

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/gau_cdf_plot.py

python3 gau_cdf_plot.py
```

(i)The function is symmetric

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2 using the code below

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/gau_pdf_plot.py

python3 gau_pdf_plot.py
```

(i)The function is symmetric about y-axis.

2.4 Find the mean and variance of X by writing a C program.
Solution: The following code calculates mean and variance of X

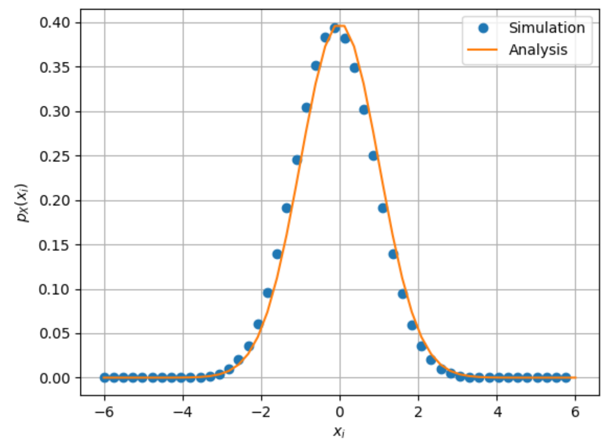

Fig. 2: The PDF of X

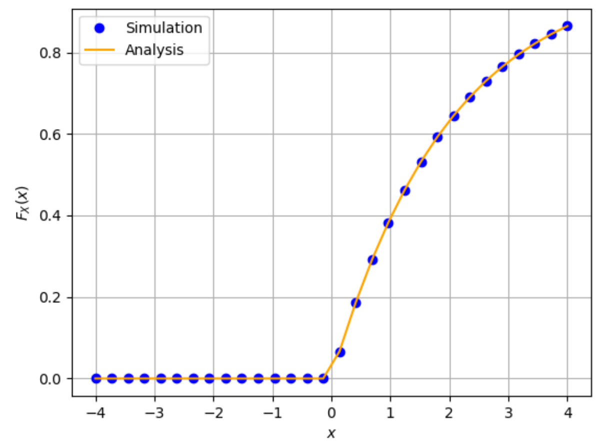

Fig. 3: The CDF of V

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/2.4.c

gcc 2.4.c -o 2.4 && ./2.4
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Solution: Mean, $U(X) = \int_{\infty}^{\infty} x p_X(x).dx$

$$\text{So, } U(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{(\frac{-x^2}{2})}.dx \tag{2.4}$$

$$U(X) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^t.dt, \text{where, } t = \frac{-x^2}{2} \tag{2.5}$$

$$U(X) = 0 \tag{2.6}$$

As $E(X) = 0$, $Var(X) = E(X^2)$, So

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{(\frac{-x^2}{2})}.dx \tag{2.7}$$

$$E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{(\frac{-x^2}{2})}.dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2\pi}}{1} \tag{2.8}$$

$$E(X^2) = 1 \tag{2.9}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.

Solution: The following codes generate samples and plot cdf in Fig.3

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/3.1.c

gcc 3.1.c -o 3.1 && ./3.1

wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/3.1_cdf.py

python3 3.1_cdf.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution: If Y = g(X), we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$X = -2ln(1 - Y) \tag{3.2}$$

$$ln(1 - Y) = e^{\frac{-X}{2}} \tag{3.3}$$

$$Y = 1 - e^{\frac{-X}{2}} \tag{3.4}$$

$$F_V(X) = F_U(1 - e^{\frac{-X}{2}}) \tag{3.5}$$

when , $0 \leq 1 - e^{\frac{-X}{2}} \leq 1$

$$0 \leq e^{\frac{-X}{2}} \leq 1 \tag{3.6}$$

$$X \geq 0, \text{So,} \tag{3.7}$$

$$F_V(X) = 1 - e^{\frac{-X}{2}}, X \geq 0 \tag{3.8}$$

$$F_V(X) = 0, X < 0 \tag{3.9}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program.

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/3.1_cdf.py
```

Download the above files and execute the following commands

```
$ gcc 4.1.c
$ ./a.out
```

4.2 Find the CDF of T.
Solution: The CDF of T is plotted in Fig. **??** using the code below

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/3.1_cdf.py
```

Download the above files and execute the following commands to produce Fig.4

```
$ python3 4.5cdf.py
```

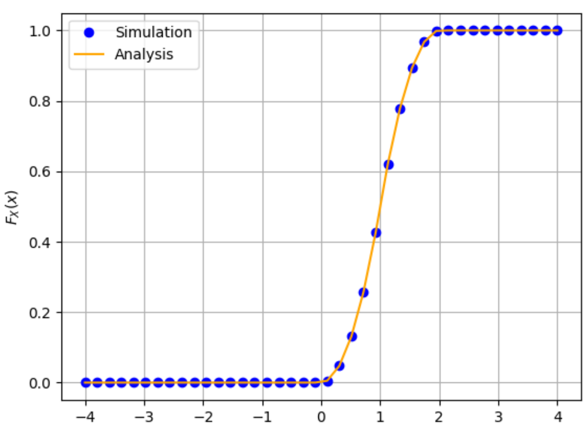

Fig. 4: The CDF of T

4.3 Find the PDF of T.
Solution: The PDF of T is plotted in Fig. 4 using the code below

```
wget https://github.com/Rutheeshreddy/
    AI1110/blob/main/Assignment_revision
    -1/codes/3.1_cdf.py
```

Download the above files and execute the following commands to produce Fig.4

```
$ python3 4.5pdf.py
```

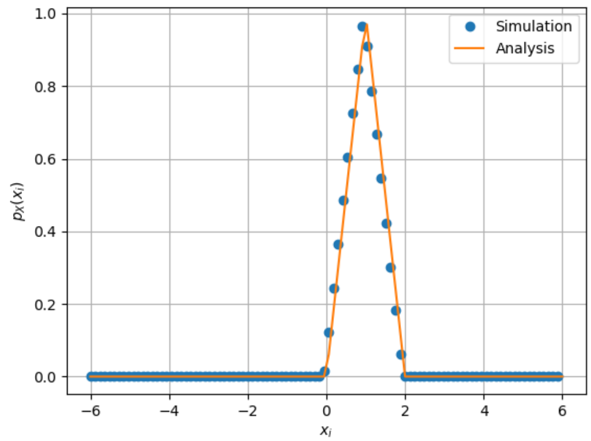

Fig. 4: The PDF of T

4.4 Find the Theoretical Expression for the PDF and CDF of T
Solution:

$$T = U_1 + U_2 \tag{4.2}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_{U1}(x)p_{U2}(y)dx \tag{4.3}$$

$$\text{As,} p_{U1}(x) = p_{U1}(y) = p_U(u) \tag{4.4}$$

$$\implies p_T(t) = \int_{-\infty}^{t} p_U(u)p_U(t-u)du \tag{4.5}$$

a) Theoretical PDF

i) $t \leq 1$

$$p_T(t) = \int_{0}^{t} p_U(t-u)du \tag{4.6}$$

$$\implies p_T(t) = \int_{0}^{t} du = t \tag{4.7}$$

ii) $t > 1$

$$p_T(t) = \int_{0}^{1} p_U(t-u)du \tag{4.8}$$

$$\implies p_T(t) = \int_{t-1}^{1} du = 2-t \tag{4.9}$$

$$\implies \boxed{P_T(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & t < 0 \text{ or } t > 2 \end{cases}}$$

b) Theoretical CDF

$$F_T(t) = \int_{-\infty}^{t} p_T(u)du \tag{4.10}$$

$$\implies \boxed{F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \le t \le 1 \\ 2t - 1 - \frac{t^2}{2} & 1 < t \le 2 \\ 1 & t > 2 \end{cases}}$$

4.5 Verify your results through a plot
Solution: The Results are verfied in the plots Fig 4 and Fig 4