CS-204: Design and Analysis of Algorithms

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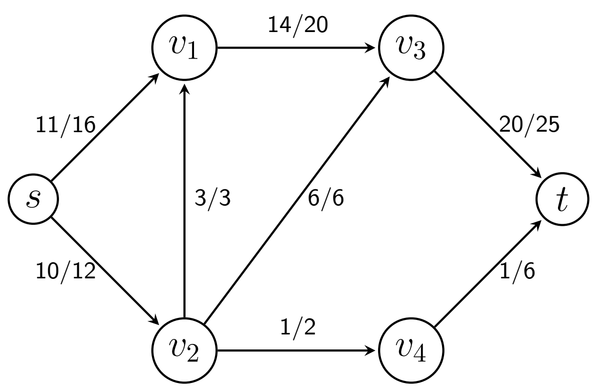
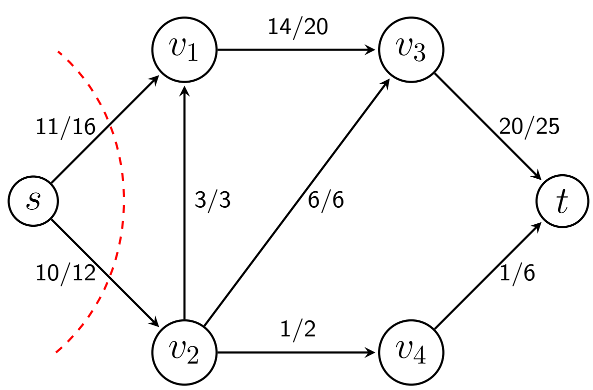
1 Cut

A cut (S, T) of a flow network $G = (V, E)$ is a partition of V into S and T, such that $S \cup T = V$ and $S \cap T = \phi$ where $s(source) \in S$ and $t(sink) \in T$.



The net flow of the above graph, $|f| = 11 + 10 = 21$ i.e. the total flow going out from the source.

Let us assume a cut as described in the following graph



Here, $S = \{s\}$ and $T = V \setminus \{s\}$.
We can also define a new term called as **Capacity** of cut, here it is 16 + 12 = 28.

1.1 Capacity of Cut

Capacity of the cut (S,T) is:

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

1.2 Minimum Cut

Minimum Cut of a network is a cut whose capacity is minimum over all cuts of the network.

Lemma: Let f be a flow in a flow network G with source s and sink t, and let (S,T) be any cut of G. Then the net flow across (S,T) is $f(S,T) = |f|$.

Proof: We can rewrite the flow-conservation condition for any node $u \in V \setminus \{s,t\}$ as

$$\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) = 0$$

Taking the definition of $|f|$ and adding the left-hand side of above equation, which equals 0, summed over all vertices in $S \setminus \{s\}$, gives

$$|f| = \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right)$$

Expanding the right-hand summation and regrouping terms yields

$$\begin{aligned}
|f| &= \sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u,v) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v,u) \\
&= \sum_{v \in V} \left(f(s,v) + \sum_{u \in S-\{s\}} f(u,v) \right) - \sum_{v \in V} \left(f(v,s) + \sum_{u \in S-\{s\}} f(v,u) \right) \\
&= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u)
\end{aligned}$$

Because $V = S \cup T$ and $S \cap T = \emptyset$ we can split each summation over V into summations over S and T to obtain

$$|f| = \sum_{v \in S} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u)$$

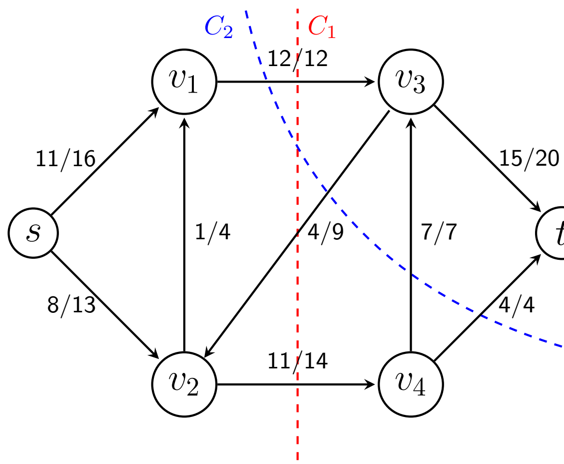
$$= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) + \left(\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) \right)$$

The two summations within the parentheses are actually the same, since for all vertices $u, v \in V$, the term $f(u,v)$ appears once in each summation. Hence, these summations cancel, and we have

$$|f| = \sum_{u \in S} \sum_{v \in T} f(u,v) + \sum_{u \in S} \sum_{v \in T} f(v,u)$$

$$= f(S,T)$$

1.3 Example



Note that $|f| = 11 + 8 = 19$.

Here,
Cut $(C_1) = \{(v_1, v_3), (v_3, v_2), (v_2, v_4)\}$
$S = \{s, v_1, v_2\}$
$T = \{t, v_3, v_4\}$
$|f| = f(v_1, v_3) + f(v_2, v_4) - f(v_3, v_2) = 12 + 11 - 4 = 19.$
$|c| = c(v_1, v_3) + c(v_2, v_4) = 12 + 14 = 26.$
Note that we have not included $c(u, v)$, s.t. $u \in T$ and $v \in S$.

Cut $(C_2) = \{(v_1, v_3), (v_3, v_2), (v_4, v_3), (v_4, t)\}$
$S = \{s, v_1, v_2, v_4\}$
$T = \{t, v_3\}$
$|f| = f(v_1, v_3) + f(v_4, v_3) + f(v_4, t) - f(v_3, v_2) = 12 + 7 + 4 - 4 = 19.$
$|c| = c(v_1, v_3) + c(v_4, v_3) + c(v_4, t) = 12 + 7 + 4 = 23.$

Note that the net flow of the above graph is equal to $|f|$ of above cuts! This is the result that we proved above.

Also notice that for any of the above cuts, $|f| \leq |c|$.
Actually we can prove that the value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

Proof: Let (S, T) be any cut of G.
From the definition of flow, we have:

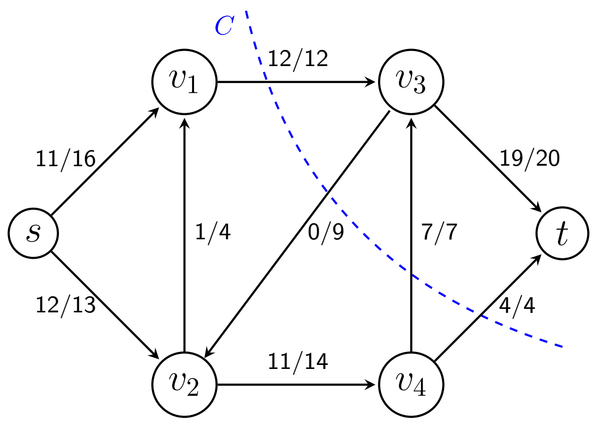
$$|f| = f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

Now, using the capacity constraint, we have:

$$|f| \leq \sum_{u \in S} \sum_{v \in T} f(u,v) \leq \sum_{u \in S} \sum_{v \in T} c(u,v) = c(S,T)$$

Therefore, the value of any flow f in the flow network G is bounded from above by the capacity of any cut (S,T) of G.

1.4 Increasing flow of the above graph

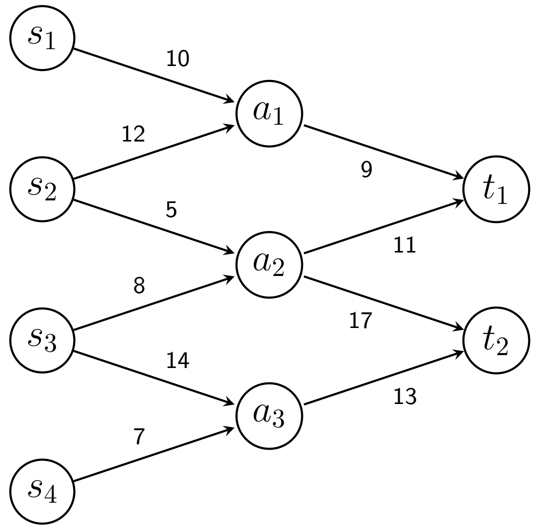


After increasing the flow in the above graph via the augmented path $s \to v_2 \to v_3 \to t$, the updated flow $|f|' = 19 + 4 = 23$.

Consider cut $(C) = \{(v_1, v_3), (v_3, v_2), (v_4, v_3), (v_4, t)\}$
$S = \{s, v_1, v_2, v_4\}$
$T = \{t, v_3\}$
$|f| = f(v_1, v_3) + f(v_4, v_3) + f(v_4, t) - f(v_3, v_2) = 12 + 7 + 4 - 0 = 23.$
$|c| = c(v_1, v_3) + c(v_4, v_3) + c(v_4, t) = 12 + 7 + 4 = 23.$

Since $\exists$ a cut for which $|f| = |c|$, we can no longer find an augmenting path in the augmented graph of the above flow graph from the source to the sink, and hence the flow of the above graph is the maximum flow.

2 Converting a multiple-source, multiple-sink maximum-flow problem into a problem with a single source and a single sink



We have a flow network with four sources $S = \{s_1, s_2, s_3, s_4\}$ and two sinks $T = \{t_1, t_2\}$
We can convert the above network to an equivalent single-source, single-sink flow network. We add a supersource s and an edge with infinite capacity from s to each of the multiple sources. We also add a supersink t and an edge with infinite capacity from each of the multiple sinks to t.