Q1 Suppose $R() : v$ is the inbuilt function that returns a random decimal value $w \epsilon (0, 1)$ from a uniform distribution. Let $f() : (u, v)$ be a function that returns a random pair $(u, v) \epsilon D$ from a uniform distribution over D.

(a)

$$D = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$$

(a=1 and b=2)
If the points (u, v) are from D and uniformly distributed, then
$p(x, y)dxdy = c \; \forall (x, y) \epsilon D$. and zero otherwise.
where $p(x, y)dxdy$ is the probability that $(u, v) \epsilon (x, x + dx) \times (y, y + dy)$.
c can be found by integrating $p(x, y)$ over D.
$\int_{(x,y) \, \epsilon \, D} p(x, y)dxdy = 1$
$c \int_{(x,y) \, \epsilon \, D} dxdy = 1 \implies cA = 1 \implies c = \frac{1}{A}$
where A is the area of D, which is πab for the ellipse.
Now,
$p(x, y)dxdy = p_x(x)p_y(\frac{y}{x})dxdy \implies p_y(\frac{y}{x}) = \frac{p(x,y)}{p_x(x)}$ where $p_x(x)$ is probability that $u \, \epsilon \, (x, x + dx)$ and $p_y(\frac{y}{x})$ is the probability that $v \, \epsilon \, (y, y + dy)$, given that $u \, \epsilon \, (x, x + dx)$.

$p_x(x)$ is not a uniform distribution, as the fraction of points in the ellipse with a given x coordinate varies. Precisely, $p_x(x) = \frac{l(x)}{A}$ where $l(x)$ is the difference between the y coordinates on the boundary of D at x. $l(x) = 2b\sqrt{1 - \frac{x^2}{a^2}}$ for the ellipse. This gives us a way to sample u from a uniform distribution. We find the $cdf_x(x)$ and a function to sample u points is $f_u() = cdf_x^{-1}(R())$ (This result is from Assigment 1.)
For reducing computation time, we have explicitly calculated the $cdf_x(x)$ and used it in the program.

$$\begin{aligned}
cdf_x(x) &= \textstyle\int_{-a}^{x} p_x(u)du / \text{ Put t = u/a} \\
&\implies \textstyle\int_{-1}^{x/a} p_x(at)adt \\
&= a \textstyle\int_{-1}^{x/a} \frac{2b\sqrt{1-\frac{a^2t^2}{a^2}}}{\pi ab} dt \\
&= \tfrac{2}{\pi} \textstyle\int_{-1}^{x/a} \sqrt{1 - t^2}dt \\
&= \tfrac{2}{\pi}(t\sqrt{1 - t^2} + \tfrac{1}{2}sin^{-1}(t)|_{-1}^{x/a}) \\
&= \tfrac{1}{2} + \tfrac{1}{\pi} \cdot \sin^{-1}(x/a) \; + \; \tfrac{x}{a\pi}\sqrt{\left(1 - (\tfrac{x}{a})^2\right)}
\end{aligned}$$

Having fixed a value of u, obtained from $f_u()$, we know v is picked from a uniform distribution over $(-\frac{l(x)}{2}, \frac{l(x)}{2})$. A function $f_v()$ can be: $f_v() = l(x)(R() - \frac{1}{2})$
Then, $f() = (f_u(), f_v()) = (cdf_x^{-1}(R()), l(x)(R() - \frac{1}{2}))$ is the algorithm to obtain a point (u,v) from a uniform distribution over D.

(b) Please check Q1ab.py for the documented code. Here is the histogram:

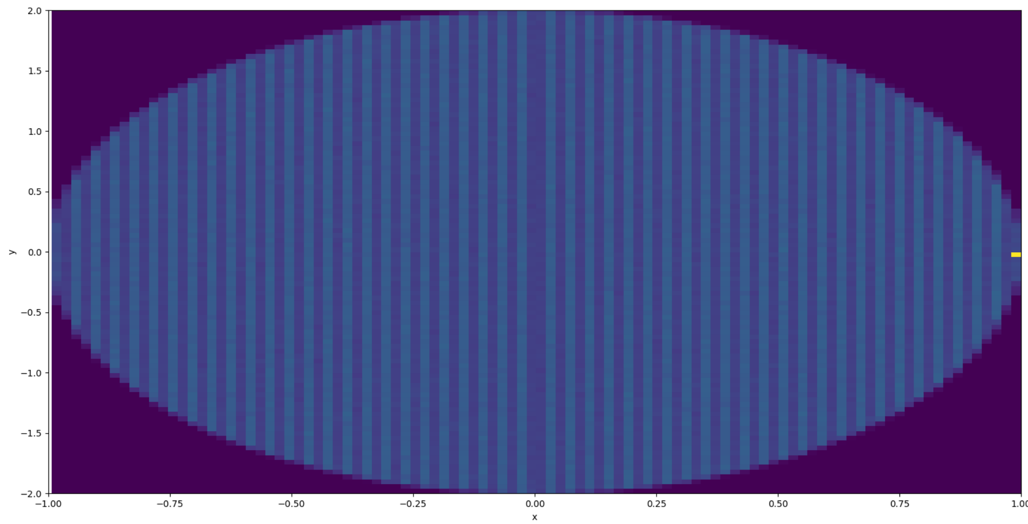

Figure 1: $N = 10^7$ points drawn from a uniform distribution over D.

(c)

$$D = \{(x, y) \mid \text{(x,y) lies in the triangle. }\}$$

A similar strategy can be applied here, but the pdf and cdf functions of x are a bit complicated. Note that the function $l(x)$ is replaced by $l'(x)$, where $l'(x)$ is the height of the point with (x, y) on the upper boundary of the triangle measured from the x-axis.

$$l'(x) = \left\{ \begin{array}{ll} \frac{3xe}{\pi}, & \text{for } 0 \le x \le \pi/3 \\ \\ \frac{3e}{2}\left(1 - \frac{x}{\pi}\right), & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

Also, $A = \pi e/2$ (the triangle's area, $A = bh/2$).

$$p_x(x) = \left\{ \begin{array}{ll} \frac{6x}{\pi^2}, & \text{for } 0 \le x \le \pi/3 \\ \\ \frac{3}{\pi}\left(1 - \frac{x}{\pi}\right), & \text{for } \pi/3 \le x \le \pi \end{array} \right\}, cdf_x(x) = \left\{ \begin{array}{ll} \frac{3x^2}{\pi^2}, & \text{for } 0 \le x \le \pi/3 \\ \\ \frac{3x}{\pi} - \frac{3x^2}{2\pi^2} - \frac{1}{2}, & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

Proof for $cdf_x(x)$:

$$cdf_x(x) = \int_0^x p_x(u)du = \left\{ \begin{array}{ll} \int_0^x \frac{6u}{\pi^2}du, & \text{for } 0 \le x \le \pi/3 \\ \\ \int_0^{\pi/3} \frac{6u}{\pi^2}du + \int_{\pi/3}^x \frac{3}{\pi}\left(1 - \frac{u}{\pi}\right)du, & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} \frac{3u^2}{\pi^2}\Big|_0^x, & \text{for } 0 \le x \le \pi/3 \\ \\ \frac{3u^2}{\pi^2}\Big|_0^{\pi/3} + \frac{3}{\pi}\left(u - \frac{u^2}{2\pi}\right)\Big|_{\pi/3}^x & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} \frac{3x^2}{\pi^2} - 0 & \text{for } 0 \le x \le \pi/3 \\ \frac{1}{3} + \frac{3}{\pi}\left(x - \frac{x^2}{2\pi}\right) - \frac{3}{\pi}\left(\frac{\pi}{3} - \frac{\pi^2}{18}\right) & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

$$cdf_x(x) = \left\{ \begin{array}{ll} \frac{3x^2}{\pi^2}, & \text{for } 0 \le x \le \pi/3 \\ \frac{3x}{\pi} - \frac{3x^2}{2\pi^2} - \frac{1}{2}, & \text{for } \pi/3 \le x \le \pi \end{array} \right\}$$

The algorithm to generate points within the triangle is now:

$$f() = (f_u(), f_v()) = (cdf_x^{-1}(R()), l'(x)R())$$

(d) Please see Q1cd.py for the documented code. Here is the histogram:

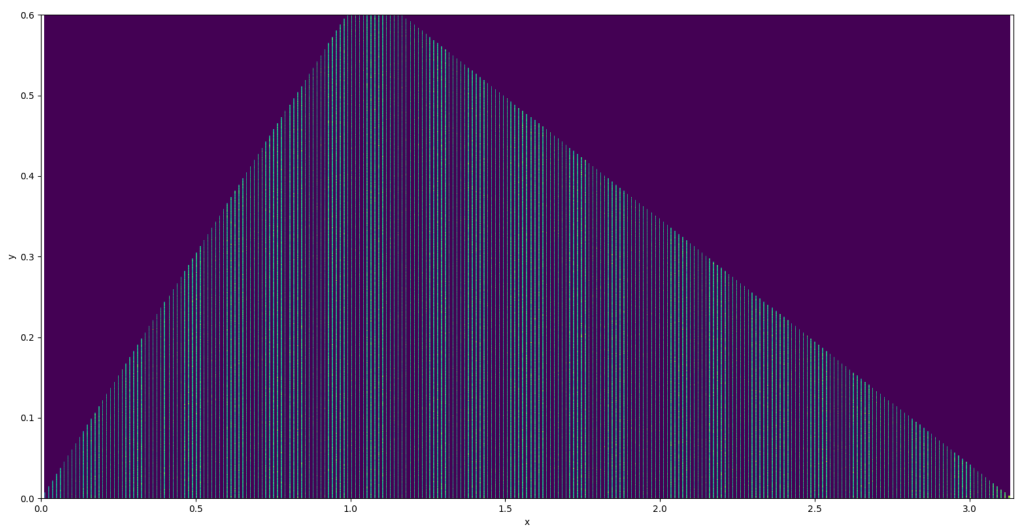

Figure 2: $N = 10^7$ points drawn from a uniform distribution over D.

Note: To reduce computation time, the precision of the inverse functions and the bins has been restricted, hence the apparent linear segregation in the images.