Statistical Analysis in Fin Mkts

MSF 502

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4 Introduction to Probability

CHAPTER

Chapter 4 Learning Objectives (LOs)

LO 4.1: Describe fundamental probability concepts.

LO 4.2: Formulate and explain subjective, empirical, and a priori probabilities.

LO 4.3: Calculate and interpret the probability of the complement of an event, the probability that at least one of two events will occur, and a joint probability.

LO 4.4: Calculate and interpret a conditional probability.

Chapter 4 Learning Objectives (LOs)

LO 4.5: Distinguish between independent and dependent events.

LO 4.6: Calculate and interpret probabilities from a contingency table.

LO 4.7: Apply the total probability rule and Bayes' theorem.

LO 4.8: Use a counting rule to solve a particular counting problem.

Sportsware Brands

- Annabel Gonzalez, chief retail analyst at marketing firm Longmeadow Consultants is tracking the sales of compression-gear produced by Under Armour, Inc., Nike, Inc., and Adidas Group.
- After collecting data from 600 recent purchases, Annabel wants to determine weather age influences brand choice.

Age Group	Brand Name		
	Under Armour	Nike	Adidas
Under 35 years	174	132	90
35 years and older	54	72	78

4.1 Fundamental Probability Concepts

LO 4.1 Describe fundamental probability concepts.

- **A probability is a numerical value that measures the likelihood that an uncertain event occurs.**
- **The value of a probability is between zero (0) and one (1).**
  - **A probability of zero indicates *impossible* events.**
  - **A probability of one indicates *definite* events.**

LO 4.1

4.1 Fundamental Probability Concepts

- **An experiment is a trial that results in one of several uncertain outcomes.**
  - **Example: Trying to assess the probability of a snowboarder winning a medal in the ladies' halfpipe event while competing in the Winter Olympic Games.**
  - **Solution: The athlete's attempt to predict her chances of medaling is an experiment because the outcome is unknown.**
    - **The athlete's competition has four possible outcomes: gold medal, silver medal, bronze medal, and no medal. We formally write the sample space as**
      S = {gold, silver, bronze, no medal}.

LO 4.1

4.1 Fundamental Probability Concepts

- **A sample space, denoted *S*, of an experiment includes all possible outcomes of the experiment.**
  - **For example, a sample space containing letter grades is:**

$$S = \{A, B, C, D, F\}$$

$$A, B, C, D \qquad\qquad F$$

- **An event is a subset of the sample space.**

The event "passing grades" is a subset of S.

The *simple event* "failing grades" is a subset of S.

LO 4.1

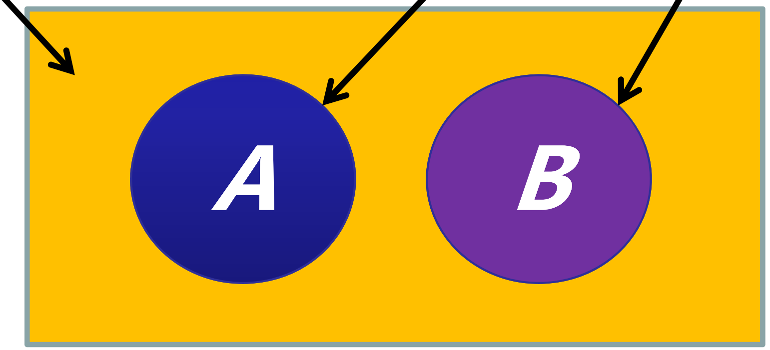
4.1 Fundamental Probability Concepts

- **Events are considered to be**
  - **Exhaustive**
    - **If all possible outcomes of a random experiment are included in the events. For example, the events "earning a medal" and "failing to earn a medal" in a single Olympic event are exhaustive since these are the only outcomes.**
  - **Mutually exclusive**
    - **If they do not share any common outcome of a random experiment. For example, the events "earning a medal" and "failing to earn a medal" in a single Olympic event are mutually exclusive.**

LO 4.1

4.1 Fundamental Probability Concepts

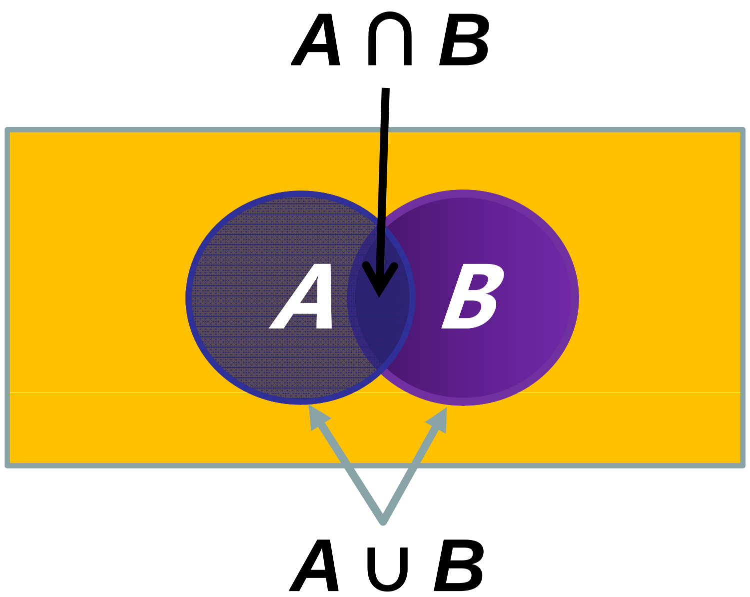
- A ***Venn Diagram*** **represents the** ***sample space*** **for the event(s).**
  - **For example, this Venn Diagram illustrates the sample space for events *A* and *B*.**



  - **The union of two events (*A* ∪ *B*) is the event consisting of all simple events in *A* or *B*.**

4.1 Fundamental Probability Concepts

- **The intersection of two events (*A* ∩ *B*) consists of all simple events in both *A* <u>and</u> *B*.**



- **The complement of event *A* (i.e., A^c) is the event consisting of all simple events in the sample space *S* that are not in *A*.**

LO 4.1

4.1 Fundamental Probability Concepts

- **Example: Recall the snowboarder's sample space defined as S = {gold, silver, bronze, no medal}. Given the following, find $A \cup B$, $A \cap B$, $A \cap C$, and B^c.**
  - **A = {gold, silver, bronze}.**
  - **B = {silver, bronze, no medal}.**
  - **C = {no medal}.**
- **Solution:**
  - **$A \cup B$ = {gold, silver, bronze, no medal}. Note that there is no double counting.**
  - **$A \cap B$ = {silver, bronze}. $A \cap C = \varnothing$ (null or empty set).**
  - **B^c = {gold}.**

4.1 Fundamental Probability Concepts

LO 4.2 Formulate and explain subjective, empirical, and a priori probabilities.

- **Assigning Probabilities**
  - **Subjective probabilities**
    - **Draws on personal and subjective judgment.**
  - **Objective probabilities**
    - **Empirical probability: a relative frequency of occurrence.**
    - **a priori probability: logical analysis.**

LO 4.2

4.1 Fundamental Probability Concepts

- **Two defining properties of a probability:**
  - **The probability of any event *A* is a value between 0 and 1.**
  - **The sum of the probabilities of any list of mutually exclusive and exhaustive events equals 1.**
- **Calculating an empirical probability**
  - **Use relative frequency*:***

$$P(A) = \frac{\text{the number of outcomes in } A}{\text{the number of outcomes in } S}$$

LO 4.1

4.1 Fundamental Probability Concepts

- **Example: Let event A be the probability of earning a medal:**

$$P(A) = P(\{\text{gold}\}) + P(\{\text{silver}\}) + P(\{\text{bronze}\}) = 0.10 + 0.15 + 0.20 = 0.45.$$

- $P(B \cup C) = P(\{\text{silver}\}) + P(\{\text{bronze}\}) + P(\{\text{no medal}\}) = 0.15 + 0.20 + 0.55 = 0.90.$

- $P(A \cap C) = 0$; **recall that there are no common outcomes in A and C.**

- $P(B^c) = P(\{\text{gold}\}) = 0.10.$

LO 4.2

4.1 Fundamental Probability Concepts

- **Probabilities expressed as odds.**
  - ***Percentages* and *odds* are an alternative approach to expressing probabilities include.**
- **Converting an odds ratio to a probability:**
  - **Given odds *for* event *A* occurring of "a to b," the probability of *A* is:**

$$\frac{a}{a+b}$$

  - **Given odds *against* event *A* occurring of "a to b," the probability of *A* is:**

$$\frac{b}{a+b}$$

LO 4.2

4.1 Fundamental Probability Concepts

- **Converting a probability to an odds ratio:**
  - **The odds *for* event *A* occurring is equal to**

$$\frac{P(A)}{1-P(A)}$$

  - **The odds *against* *A* occurring is equal to**

$$\frac{1-P(A)}{P(A)}$$

LO 4.2

4.1 Fundamental Probability Concepts

- **Example: Converting an odds ratio to a probability.**
  - **Given the odds of 2:1 for beating the Cardinals, what was the probability of the Steelers' winning just prior to the 2009 Super Bowl?**

$$\frac{a}{a+b} = \frac{2}{2+1} = \frac{2}{3} = 0.67$$

- **Example: Converting a probability to an odds ratio.**
  - **Given that the probability of an on-time arrival for New York's Kennedy Airport is 0.56, what are the odds for a plane arriving on-time at Kennedy Airport?**

$$\frac{P(A)}{1-P(A)} = \frac{0.56}{1-0.56} = \frac{0.56}{0.44} = 1.27 \text{ or } 1.27{:}1$$

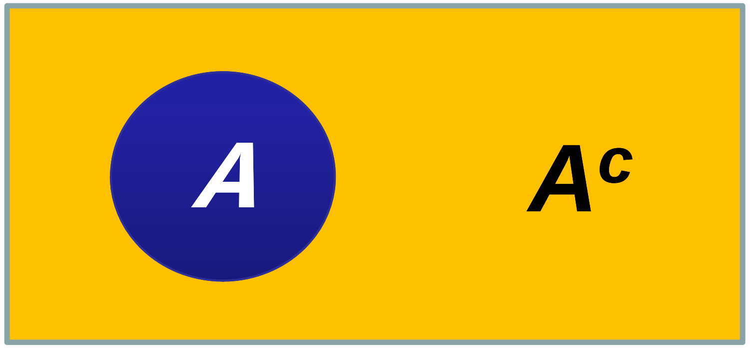
4.2 Rules of Probability

LO 4.3 Calculate and interpret the probability of the complement of an event, the probability that at least one of two events will occur, and a joint probability.

- **The Complement Rule**
  - **The probability of the complement of an event, $P(A^c)$, is equal to one minus the probability of the event.**

$$P\left(A^c\right) = 1 - P\left(A\right)$$

Sample Space *S*



LO 4.3

4.2 Rules of Probability

- **The Addition Rule**
  - **The probability that event *A* or *B* occurs, or that at least one of these events occurs, is:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

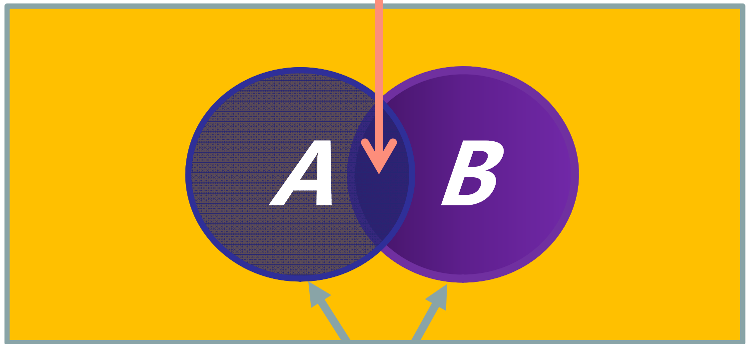
LO 4.3

4.2 Rules of Probability

- **Illustrating the Addition Rule with the Venn Diagram.**

$A \cap B$

Events *A and B* *both* occur.



$A \cup B$

***A* occurs *or* *B* occurs *or* both occur.**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

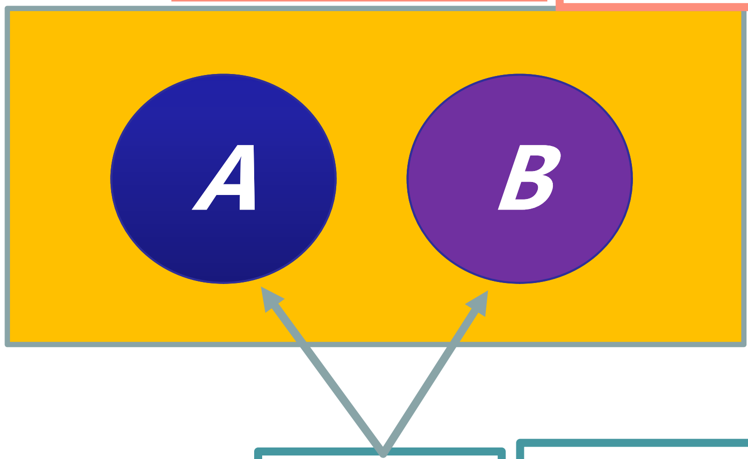
LO 4.3

4.2 Rules of Probability

- **The Addition Rule for Two Mutually Exclusive Events**

Events *A and B* *both* cannot occur.

$A \cap B = 0$



$A \cup B$

***A* occurs *or* *B* occurs**

$$P(A \cup B) = P(A) + P(B)$$

LO 4.3

4.2 Rules of Probability

- **Example: The addition rule.**
  - **Anthony feels that he has a 75% chance of getting an A in Statistics, a 55% chance of getting an A in Managerial Economics and a 40% chance of getting an A in both classes. What is the probability that he gets an A in at least one of these courses?**

$$P(A_S \cup A_M) = P(A_S) + P(A_M) - P(A_S \cap A_M)$$
$$= 0.75 + 0.55 - 0.40 = 0.90$$

  - **What is the probability that he does not get an A in either of these courses? Using the compliment rule, we find**

$$P\left((A_S \cup A_M)^C\right) = 1 - P(A_S \cup A_M) = 1 - 0.90 = 0.10$$

LO 4.3

4.2 Rules of Probability

- **Example: The addition rule for mutually exclusive events.**
  - **Samantha Greene, a college senior, contemplates her future immediately after graduation. She thinks there is a 25% chance that she will join the Peace Corps and a 35% chance that she will enroll in a full-time law school program in the United States.**

$$P(A \cup B) = P(A) + P(B) = 0.25 + 0.35 = 0.60$$

  - **What is the probability that she does not choose either of these options?**

$$P\left((A \cup B)^c\right) = 1 - P(A \cup B) = 1 - 0.60 = 0.40$$

4.2 Rules of Probability

LO 4.4 Calculate and interpret a conditional probability.

- **Unconditional (Marginal) Probability**
  - **The probability of an event without any restriction.**
  - **For example, $P(A)$ = probability of finding a job, and $P(B)$ = probability of prior work experience.**

LO 4.4

4.2 Rules of Probability

- **Conditional Probability**
  - **The probability of an event given that another event has already occurred.**
    - **In the conditional probability statement, the symbol " | " means "given."**
    - **Whatever follows " | " has already occurred.**
  - **For example, $P(A \mid B)$ = probability of finding a job *given* prior work experience.**

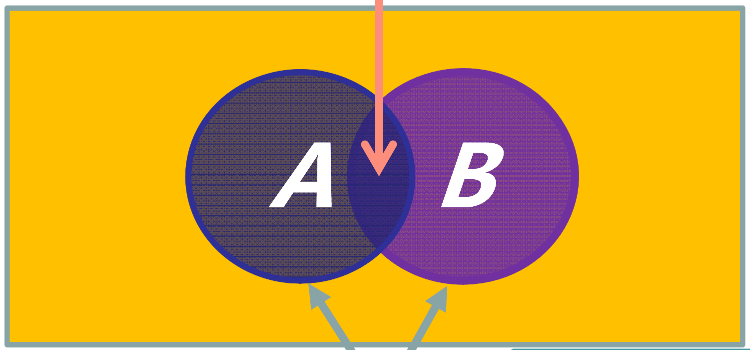
LO 4.4

4.2 Rules of Probability

- **Illustrating Conditional Probabilities with the Venn Diagram**

$A \cap B$

Events *A and B* *both* occur.



$A \cup B$

***A* occurs *or* *B* occurs *or* both occur.**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

LO 4.4

4.2 Rules of Probability

- **Calculating a Conditional Probability**
  - **Given two events *A* and *B*, each with a positive probability of occurring, the probability that *A* occurs given that *B* has occurred (*A* conditioned on *B*) is equal to**

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

  - **Similarly, the probability that *B* occurs given that *A* has occurred (*B* conditioned on *A*) is equal to**

$$P(B|A)=\frac{P(A\cap B)}{P(A)}$$

LO 4.4

4.2 Rules of Probability

- **Example: Conditional Probabilities**
  - **An economist predicts a 60% chance that country *A* will perform poorly economically and a 25% chance that country *B* will perform poorly economically. There is also a 16% chance that both countries will perform poorly. What is the probability that country *A* performs poorly given that country *B* performs poorly?**
  - **Let P(*A*) = 0.60, P(*B*) = 0.25, and P(*A* ∩ *B*) = 0.16**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.16}{0.25} = 0.64$$

  - **Since P(*A*|*B*) = 0.64 ≠ P(*A*) = 0.60, events *A and B* are not independent.**

4.2 Rules of Probability

LO 4.5 Distinguish between independent and dependent events.

- **Independent and Dependent Events**
  - **Two events are *independent* if the occurrence of one event does not affect the probability of the occurrence of the other event.**
  - **Events are considered *dependent* if the occurrence of one is related to the probability of the occurrence of the other.**

- **Two events are independent *if and only if***

$$P(A|B) = P(A)$$

or

LO 4.5

4.2 Rules of Probability

- **The Multiplication Rule**
  - **The probability that *A* and *B* both occur (referred to as a joint probability), is equal to**

$$P\left(A \cap B\right) = P\left(A \mid B\right) * P(B) = P\left(B \mid A\right) * P\left(A\right)$$

LO 4.5

4.2 Rules of Probability

- **The Multiplication Rule for Independent Events**
  - **The joint probability of *A* and *B* equals the product of the individual probabilities of *A* and *B.***

$$P(A \cap B) = P(A)P(B)$$

  - **The multiplication rule may also be used to determine independence. That is, two events are independent if the above equality holds.**

4.3 Contingency Tables and Probabilities

LO 4.6 Calculate and interpret probabilities from a contingency table.

- **Contingency Tables**
  - **A contingency table generally shows frequencies for two qualitative or categorical variables, *x* and *y*.**
  - **Each cell represents a mutually exclusive combination of the pair of *x* and *y* values.**
  - **Here, *x* is 'Age Group' with two outcomes while *y* is 'Brand Name' with three outcomes.**

	Brand Name		
Age Group	Under Armour	Nike	Adidas
Under 35 years	174	132	90
35 years and older	54	72	78

LO 4.6

4.3 Contingency Tables and Probabilities

Contingency Tables

- Note that each cell in the contingency table represents a frequency.

Age Group	Brand Name		
	Under Armour	Nike	Adidas
Under 35 years	174	132	90
35 years and older	54	72	78

- In the above table, 174 customers under the age of 35 purchased an Under Armour product.
- 54 customers at least 35 years old purchased an Under Armour product.

LO 4.6

4.3 Contingency Tables and Probabilities

- **The contingency table may be used to calculate probabilities using relative frequency.**
  - **Note: Abbreviated labels have been used in place of the class names in the table.**

Age Group	Brand Name			
	B_1	B_2	B_3	Total
A	174	132	90	396
A^c	54	72	78	204
Total	228	204	168	600

  - **First obtain the row and column totals.**
  - **Sample size is equal to the total of the row totals or column totals. In this case, n = 600.**

LO 4.6

4.3 Contingency Tables and Probabilities

Joint Probability Table

- The joint probability is determined by dividing each cell frequency by the grand total.

Age Group	Brand Name B_1	B_2	B_3	Total
A	0.29	0.22	0.15	0.66
A^c	0.09	0.12	0.13	0.34
Total	0.38	0.34	0.28	1.00

Joint Probabilities

Marginal Probabilities

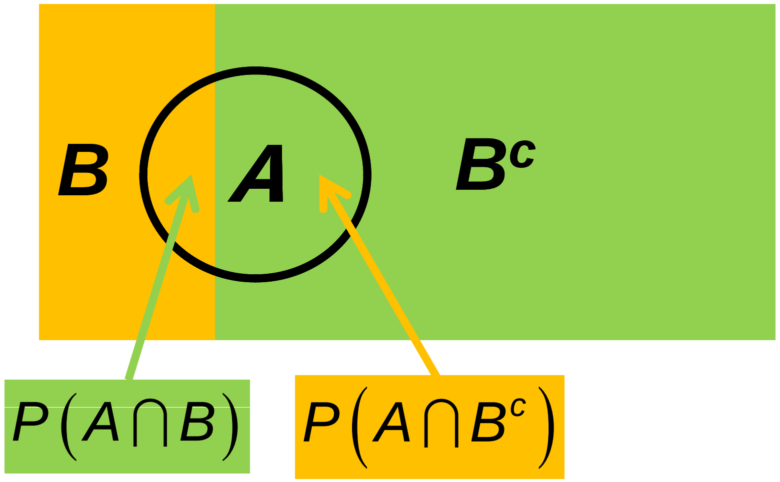
- For example, the probability that a randomly selected person is under 35 years of age and makes an Under Armour purchase is

$$P(A \cap B_1) = \frac{174}{600} = 0.29$$

LO 4.7 Apply the total probability rule and Bayes' theorem.

The Total Probability Rule

- $P(A)$ is the sum of its intersections with some mutually exclusive and exhaustive events corresponding to an experiment.
- Consider event B and its complement B^c. These two events are mutually exclusive and exhaustive.
- The circle, representing event A, consists entirely of its intersections with B and B^c.



LO 4.7

4.4 The Total Probability Rule and Bayes' Theorem

- **The Total Probability Rule conditional on two outcomes**
  - **The total probability rule conditional on two events, *B* and *B*^c^, is**

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

  - **or equivalently,**

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$

LO 4.7

4.4 The Total Probability Rule and Bayes' Theorem

- **Bayes' Theorem**
  - **A procedure for updating probabilities based on new information.**
    - **Prior probability is the original (unconditional) probability (e.g., $P(B)$).**
    - **Posterior probability is the updated (conditional) probability (e.g., $P(B \mid A)$).**

LO 4.7

4.4 The Total Probability Rule and Bayes' Theorem

- **Bayes' Theorem**
  - **Given a set of prior probabilities for an event and some new information, the rule for updating the probability of the event is called Bayes' theorem.**

$$P(B \mid A) = \frac{P(A \cap B)}{P(A \cap B) + P(A \cap B^c)}$$

or

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^c)P(B^c)}$$

LO 4.7

4.4 The Total Probability Rule and Bayes' Theorem

- **Example: Bayes' Theorem**
  - **Assume that 99% of the individuals taking a polygraph test tell the truth. These tests are considered to be 95% reliable (i.e., a 95% chance of actually detecting a lie). Let there also be a 0.5% chance that the test erroneously detects a lie even when the individual is telling the truth.**
  - **An individual has just taken a polygraph test and the test has detected a lie. What is the probability that the individual was actually telling the truth?**
  - **Let *D* denote the outcome that the polygraph detects a lie and *T* represent the outcome that an individual is telling the truth.**

LO 4.7

4.4 The Total Probability Rule and Bayes' Theorem

- **Example: Bayes' Theorem**
  - **Given the following probabilities,**

Prior Probability	Conditional Probability	Joint Probability	Posterior Probability
$P(T) = 0.99$	$P(D \mid T) = 0.005$	$P(D \cap T) = 0.00495$	$P(T \mid D) = 0.34256$
$P(T^c) = 0.01$	$P(D \mid T^c) = 0.95$	$P(D \cap T^c) = 0.00950$	$P(T^c \mid D) = 0.65744$
$P(T) + P(T^c) = 1$		$P(D) = 0.01445$	$P(T \mid D) + P(T \mid D^c) = 1$

We find

$$P(T \mid D) = \frac{(0.005)(0.99)}{(0.005)(0.99) + (0.95)(0.01)} = \frac{0.00495}{0.01445} = 0.34256$$

4.5 Counting Rules

LO 4.8 Use a counting rule to solve a particular counting problem.

- **The Factorial Formula**
  - **The number of ways to assign every member of a group of size *n* to *n* slots is calculated using the *factorial formula*:** $n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 1$
  - **By definition, 0! = 1.**
  - **For example, in how many ways can a little-league coach assign nine players to each of the nine team positions (pitcher, catcher, first base, etc.)?**
  - **Solution:** $9! = 9 \times 8 \times 7 \times \ldots \times 1 = 362{,}880$

LO 4.8

4.5 Counting Rules

- **The Combination Formula**
  - **The number of ways to choose *x* objects from a total of *n* objects, where the order in which the *x* objects are listed *does not matter*, is referred to as a combination and is calculated as:**

$$ {}_nC_x = \binom{n}{x} = \frac{n!}{(n-x)!\,x!} $$

LO 4.8

4.5 Counting Rules

- **The Permutation Formula**
  - **The number of ways to choose *x* objects from a total of *n* objects, where the order in which the *x* objects is listed *does matter*, is referred to as a permutation and is calculated as:**

$$_nP_x = \frac{n!}{(n-x)!}$$

LO 4.8

4.5 Counting Rules

- **Example: The Permutation Formula**
  - **The little-league coach recruits three more players so that his team has backups in case of injury. Now his team totals 12. In how many ways can the coach select nine players from the 12-player roster?**

  - **Combination: What if the order in which the players are selected is not important.**

$$_{12}C_9 = \frac{12!}{(12-9)!9!} = 220$$

  - **Permutation: What if the order in which the players are selected is important.**

$$_{12}P_9 = \frac{12!}{(12-9)!} = 79{,}833{,}600$$

End of Chapter