Statistical Analysis in Fin Mkts

MSF 502

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6 Continuous Probability Distributions

CHAPTER

Chapter 6 Learning Objectives (LOs)

LO 6.1: Describe a continuous random variable.

LO 6.2: Describe a continuous uniform distribution and calculate associated probabilities.

LO 6.3: Explain the characteristics of the normal distribution.

LO 6.4: Use the standard normal table or the z table.

LO 6.5: **Calculate and interpret probabilities for a random variable that follows the normal distribution.**

LO 6.6: **Calculate and interpret probabilities for a random variable that follows the exponential distribution.**

LO 6.7: **Calculate and interpret probabilities for a random variable that follows the lognormal distribution.**

6.1 Continuous Random Variables and the Uniform Probability Distribution

LO 6.1 Describe a continuous random variable.

- **Remember that random variables may be classified as**
- **Discrete**
  - **The random variable assumes a countable number of distinct values.**
- **Continuous**
  - **The random variable is characterized by (infinitely) uncountable values within any interval.**

LO 6.1

6.1 Continuous Random Variables and the Uniform Probability Distribution

- **When computing probabilities for a continuous random variable, keep in mind that $P(X = x) = 0$.**
  - **We cannot assign a nonzero probability to each infinitely uncountable value and still have the probabilities sum to one.**
  - **Thus, since $P(X = a)$ and $P(X = b)$ both equal zero, the following holds for continuous random variables:**

$$P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b)$$

LO 6.1

6.1 Continuous Random Variables and the Uniform Probability Distribution

- **Probability Density Function $f(x)$ of a continuous random variable X**
  - **Describes the relative likelihood that X assumes a value within a given interval (e.g., $P(a \leq X \leq b)$), where**
    - **$f(x) > 0$ for all possible values of X.**
    - **The area under $f(x)$ over all values of x equals one.**

LO 6.1 6.1 Continuous Random Variables and the Uniform Probability Distribution

- **Cumulative Density Function $F(x)$ of a continuous random variable X**
  - **For any value x of the random variable X, the cumulative distribution function $F(x)$ is computed as**

    $$F(x) = P(X \leq x)$$
  - **As a result, $P(a \leq X \leq b) = F(b) - F(a)$**

6.1 Continuous Random Variables and the Uniform Probability Distribution

LO 6.2 Describe a continuous uniform distribution and calculate associated probabilities.

- **The Continuous Uniform Distribution**
  - **Describes a random variable that has an equally likely chance of assuming a value within a specified range.**
  - **Probability density function:**

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \text{, and} \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$

  - **where *a* and *b* are the lower and upper limits, respectively.**

LO 6.2

6.1 Continuous Random Variables and the Uniform Probability Distribution

- **The Continuous Uniform Distribution**
  - **The expected value and standard deviation of *X* are:**

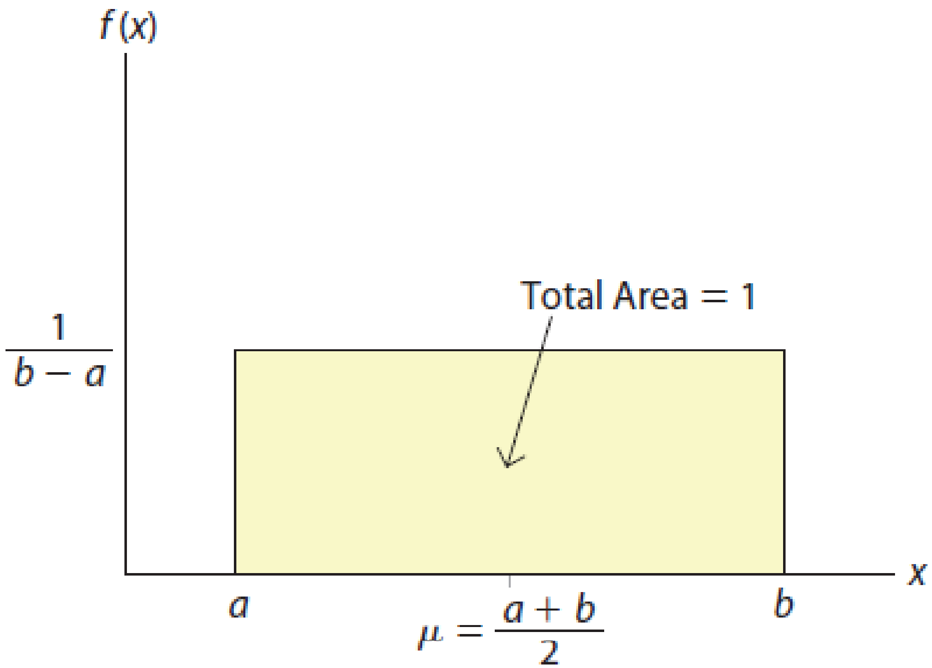
$$E(X) = \mu = \frac{a+b}{2}$$

$$SD(X) = \sigma = \sqrt{(b-a)^2/12}$$

LO 6.2

6.1 Continuous Random Variables and the Uniform Probability Distribution

- **Graph of the continuous uniform distribution:**
  - **The values *a* and *b* on the horizontal axis represent the lower and upper limits, respectively.**
  - **The height of the distribution does not directly represent a probability.**
  - **It is the area under *f*(*x*) that corresponds to probability.**



LO 6.2 6.1 Continuous Random Variables and the Uniform Probability Distribution

- **Example: Based on historical data, sales for a particular cosmetic line follow a continuous uniform distribution with a lower limit of $2,500 and an upper limit of $5,000.**
  - **What are the mean and standard deviation of this uniform distribution?**
    - **Let the lower limit *a* = $2,500 and the upper limit *b* = $5,000, then**

$$\mu = \frac{a+b}{2} = \frac{\$2{,}500 + \$5{,}000}{2} = \$3{,}750, \text{ and}$$

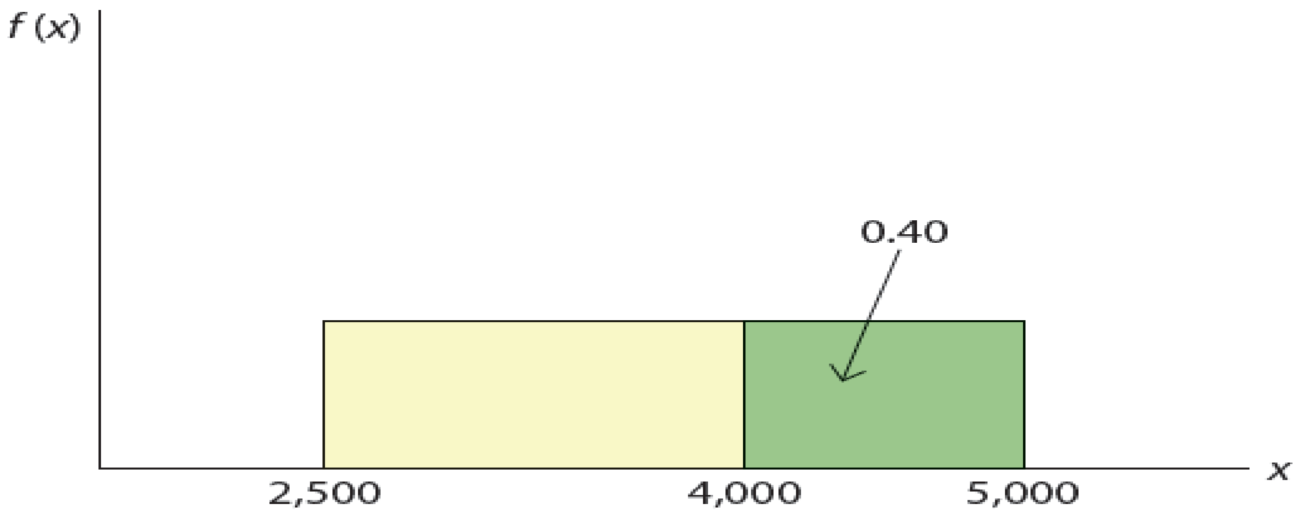
$$\sigma = \sqrt{(b-a)^2/12} = \sqrt{(5{,}000 - 2{,}500)^2/12} = \$721.69$$

LO 6.2

6.1 Continuous Random Variables and the Uniform Probability Distribution

- **What is the probability that sales exceed $4,000?**
  - **P(*X* > 4,000) = base × height =**

$$(5{,}000-4{,}000)\times(1/(5{,}000-2{,}500)=1{,}000\times 0.0004=0.4$$



6.2 The Normal Distribution

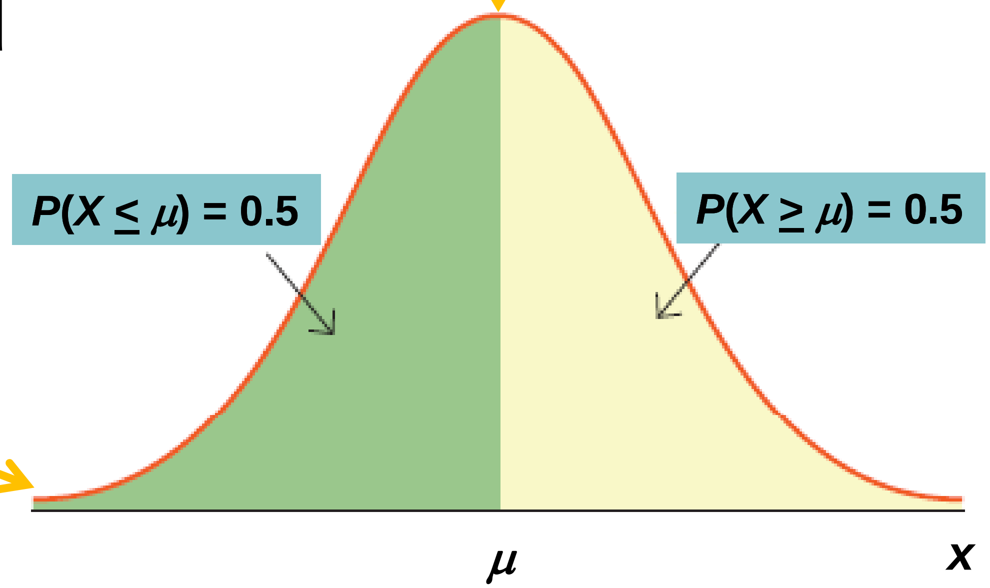
LO 6.3 Explain the characteristics of the normal distribution.

- **The Normal Distribution**
  - **Symmetric**
  - **Bell-shaped**
  - **Closely approximates the probability distribution of a wide range of random variables, such as the**
    - **Heights and weights of newborn babies**
    - **Scores on SAT**
    - **Cumulative debt of college graduates**
  - **Serves as the cornerstone of statistical inference.**

LO 6.3

6.2 The Normal Distribution

- **Characteristics of the Normal Distribution**
  - **Symmetric about its mean**
    - **Mean = Median = Mode**
  - **Asymptotic—that is, the tails get closer and closer to the horizontal axis, but never touch it.**



LO 6.3

6.2 The Normal Distribution

- **Characteristics of the Normal Distribution**
  - **The normal distribution is completely described by two parameters: μ and σ^2.**
    - **μ is the population mean which describes the central location of the distribution.**
    - **σ^2 is the population variance which describes the dispersion of the distribution.**

LO 6.3

6.2 The Normal Distribution

- **Probability Density Function of the Normal Distribution**
  - **For a random variable *X* with mean μ and variance σ^2**

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where $\pi = 3.14159$ and $\exp(x) = e^x$

$e \approx 2.718$ is the base of the natural logarithm

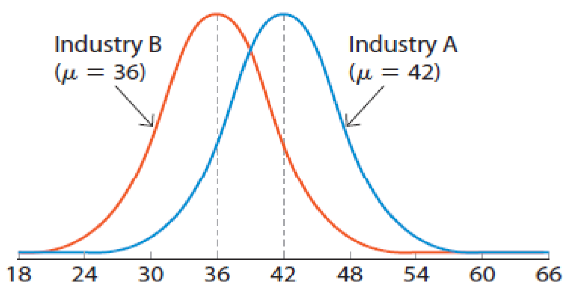
LO 6.3

6.2 The Normal Distribution

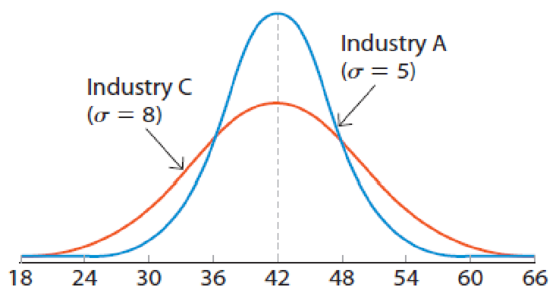
- **Example: Suppose the ages of employees in Industries A, B, and C are normally distributed.**
  - **Here are the relevant parameters:**

Industry A	Industry B	Industry C
$\mu = 42$ years	$\mu = 36$ years	$\mu = 42$ years
$\sigma = 5$ years	$\sigma = 5$ years	$\sigma = 8$ years

  - **Let’s compare industries using the Normal curves.**



σ is the same, μ is different.

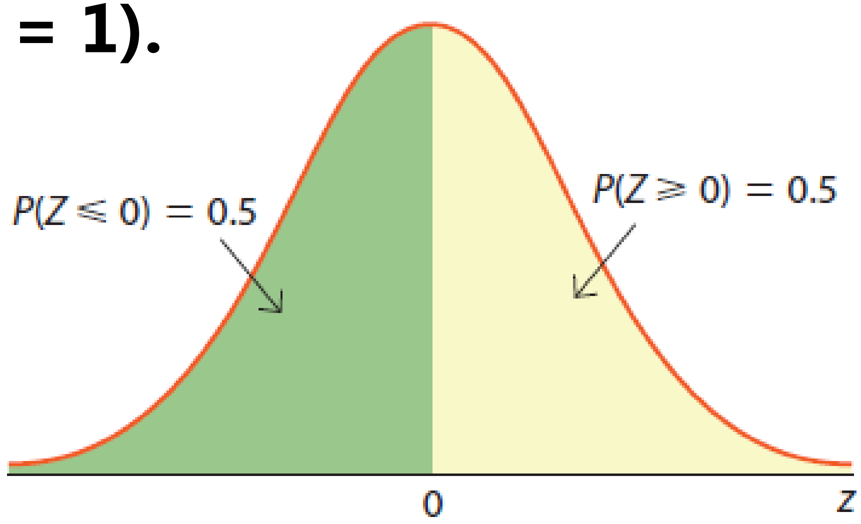


μ is the same, σ is different.

6.2 The Normal Distribution

LO 6.4 Use the standard normal table or the *z* table.

- **The Standard Normal (*Z*) Distribution.**
  - **A special case of the normal distribution:**
    - **Mean (μ) is equal to zero ($E(Z) = 0$).**
    - **Standard deviation (σ) is equal to one ($SD(Z) = 1$).**



LO 6.4

6.2 The Standard Normal Distribution

- **Standard Normal Table (Z Table).**
  - **Gives the cumulative probabilities $P(Z \leq z)$ for positive and negative values of z.**
  - **Since the random variable Z is symmetric about its mean of 0,**

$$P(Z < 0) = P(Z > 0) = 0.5.$$

  - **To obtain the $P(Z < z)$, read down the z column first, then across the top.**

LO 6.4

6.2 The Standard Normal Distribution

- **Standard Normal Table (*Z* Table).**

Table for positive *z* values.

z	0.00	0.01	0.02	0.03	0.04
0.0	0.5000	0.5040	0.5080	0.5120	0.5160
0.1	0.5398	0.5438	0.5478	0.5517	0.5557
0.2	0.5793	0.5832	0.5871	0.5910	0.5948
0.3	0.6179	0.6217	0.6255	0.6293	0.6331
0.4	0.6554	0.6591	0.6628	0.6664	0.6700

Table for negative *z* values.

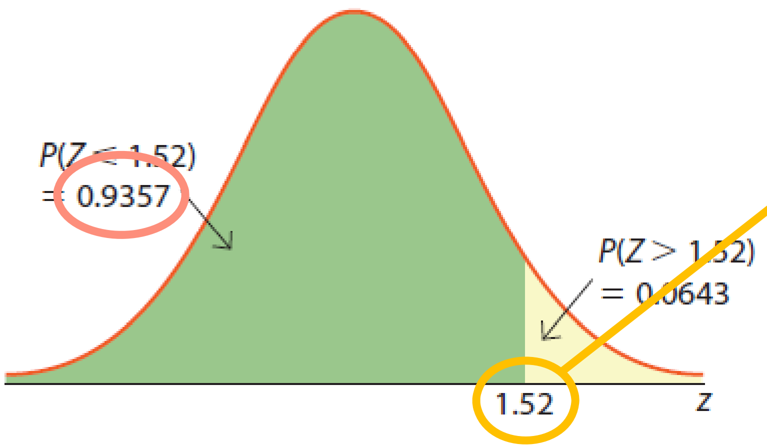
z	0.00	0.01	0.02	0.03	0.04
-3.9	0.0000	0.0000	0.0000	0.0000	0.0000
-3.8	0.0001	0.0001	0.0001	0.0001	0.0001
-3.7	0.0001	0.0001	0.0001	0.0001	0.0001
-3.6	0.0002	0.0002	0.0001	0.0001	0.0001
-3.5	0.0002	0.0002	0.0002	0.0002	0.0002

LO 6.4

6.2 The Standard Normal Distribution

- **Finding the Probability for a Given *z* Value.**
  - **Transform normally distributed random variables into standard normal random variables and use the *z* table to compute the relevant probabilities.**
  - **The *z* table provides cumulative probabilities $P(Z \leq z)$ for a given *z*.**

Portion of right-hand page of *z* table.
If *z* = 1.52, then look up



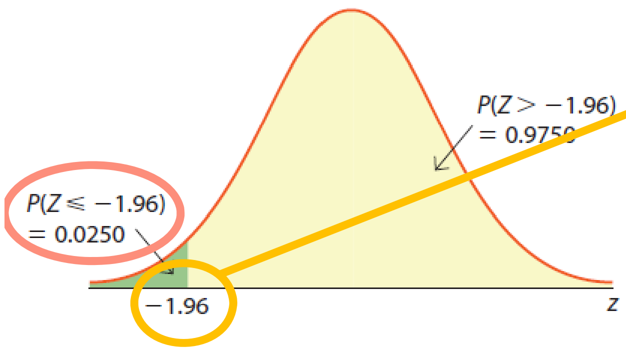
z	0.00	0.01	0.02
0.0	0.5000	0.5040	↓
0.1	0.5398	0.5438	↓
	⋮	⋮	↓
1.5	→	→	0.9357

LO 6.4

6.2 The Standard Normal Distribution

- **Finding the Probability for a Given *z* Value.**
  - **Remember that the *z* table provides cumulative probabilities $P(Z \leq z)$ for a given *z*.**
  - **Since z is negative, we can look up this probability from the left-hand page of the z table.**

Portion of left-hand page of *Z* Table.
If *z* = –1.96, then look up



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
−3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	↓
−3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	↓
⋮	⋮	⋮	⋮	⋮	⋮	⋮	↓
−1.9	→	→	→	→	→	→	0.0250

LO 6.4

6.2 The Standard Normal Distribution

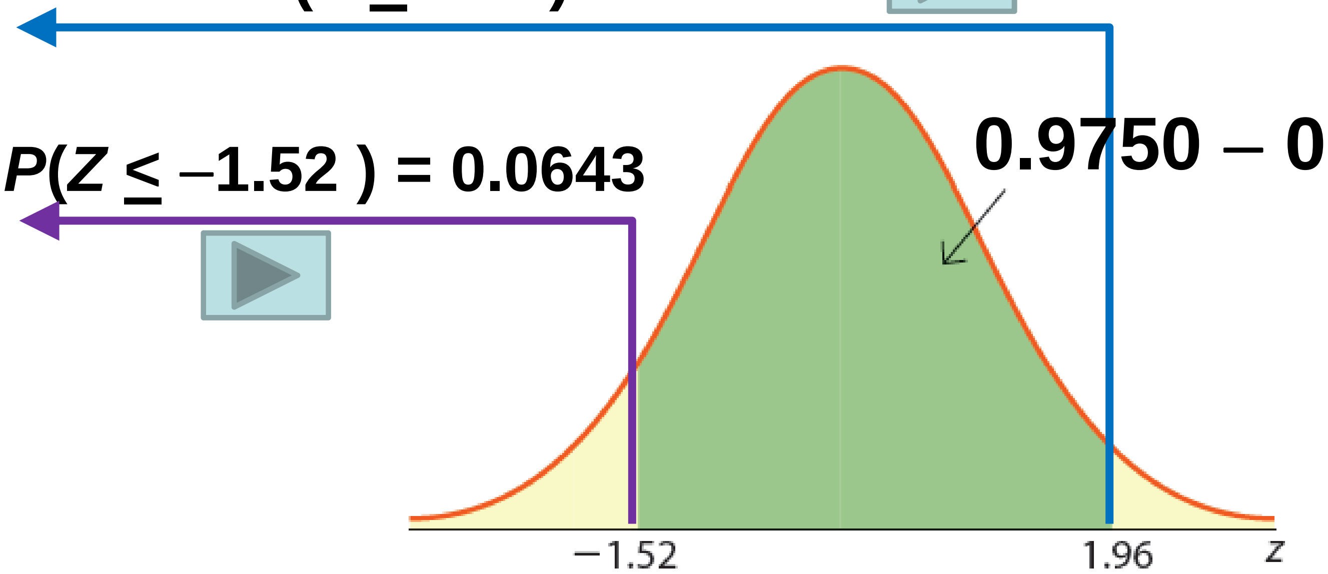
- **Example: Finding Probabilities for a Standard Normal Random Variable *Z*.**
  - **Find $P(-1.52 \leq Z \leq 1.96)$ =**

$P(Z \leq 1.96) - P(Z \leq -1.52) =$

$P(Z \leq 1.96) = 0.9750$

$P(Z \leq -1.52) = 0.0643$

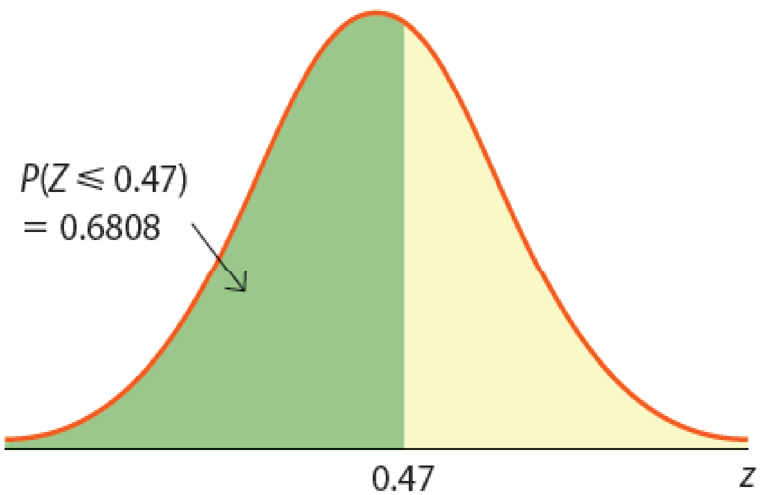
$0.9750 - 0.0643 = 0.9107$



LO 6.4

6.2 The Standard Normal Distribution

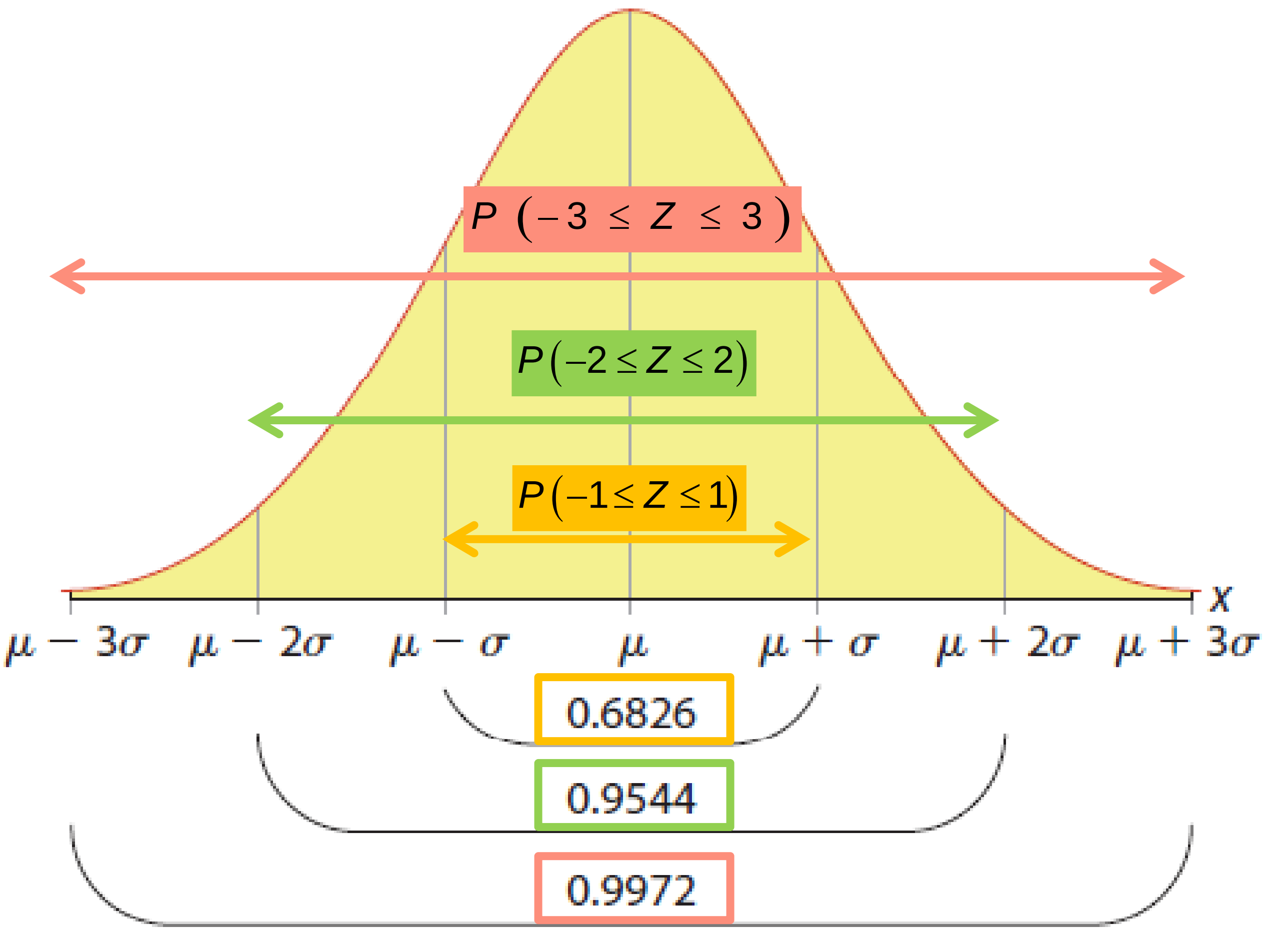
- **Example: Finding a z value for a given probability.**
  - **For a standard normal variable Z, find the z values that satisfy $P(Z \leq z) = 0.6808$.**
  - **Go to the standard normal table and find 0.6808 in the body of the table.**
  - **Find the corresponding z value from the row/column of z.**
  - **$z = 0.47$.**



LO 6.4

6.2 The Standard Normal Distribution

- **Revisiting the Empirical Rule.**



LO 6.4

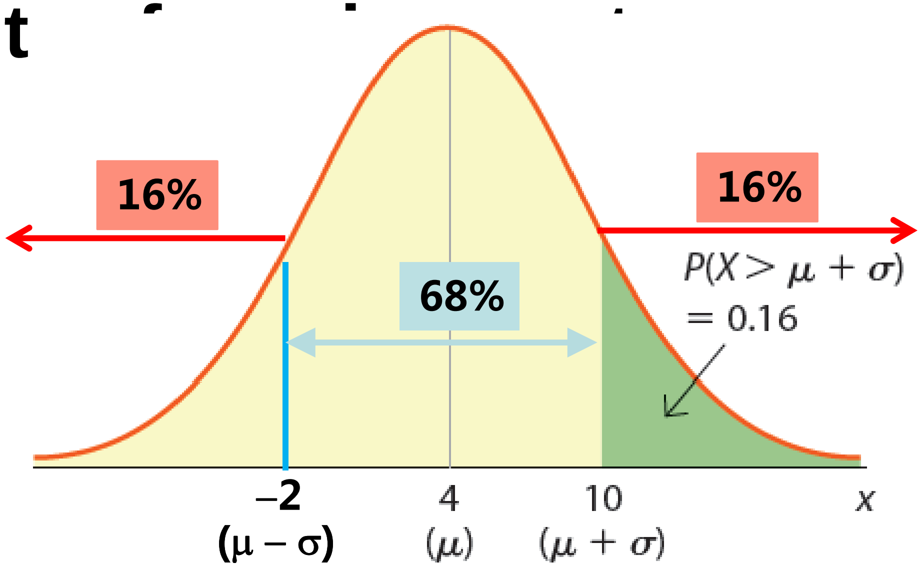
6.2 The Standard Normal Distribution

- **Example: The Empirical Rule**
  - **An investment strategy has an expected return of 4% and a standard deviation of 6%. Assume that investment returns are normally distributed.**
  - **What is the probability of earning a return greater than 10%?**
    - **A return of 10% is one standard deviation above the mean, or 10 = μ + 1σ = 4 + 6.**
    - **Since about 68% of observations fall within one standard deviation of the mean, 32% (100% – 68%) are outside the range.**

LO 6.4

6.2 The Standard Normal Distribution

- **Example: The Empirical Rule**
  - **An investment strategy has an expected return of 4% and a standard deviation of 6%. Assume that investment returns are normally distributed.**
  - **What is the probabilit[y] … greater than 10%?**
    - **Using symmetry, we conclude that 16% (half of 32%) of the observations are greater than 10%.**



6.3 Solving Problems with the Normal Distribution

LO 6.5 Calculate and interpret probabilities for a random variable that follows the normal distribution.

- **The Normal Transformation**
  - **Any normally distributed random variable *X* with mean μ and standard deviation σ can be transformed into the standard normal random variable *Z* as:**

$$Z = \frac{X - \mu}{\sigma}$$ **with corresponding values** $$z = \frac{x - \mu}{\sigma}$$

As constructed: $E(Z) = 0$ and $SD(Z) = 1$.

LO 6.5

6.3 Solving Problems with the Normal Distribution

- **A *z* value specifies by how many standard deviations the corresponding *x* value falls above ($z > 0$) or below ($z < 0$) the mean.**
  - **A positive *z* indicates by how many standard deviations the corresponding *x* lies above μ.**
  - **A zero *z* indicates that the corresponding *x* equals μ.**
  - **A negative *z* indicates by how many standard deviations the corresponding *x* lies below μ.**

LO 6.5

6.3 Solving Problems with the Normal Distribution

- **Use the Inverse Transformation to compute probabilities for given *x* values.**
  - **A standard normal variable *Z* can be transformed to the normally distributed random variable *X* with mean μ and standard deviation σ as**

$X = \mu + Z\sigma$ **with corresponding values** $X = \mu + Z\sigma$

LO 6.5

6.3 Solving Problems with the Normal Distribution

- **Example: Scores on a management aptitude exam are normally distributed with a mean of 72 (μ) and a standard deviation of 8 (σ).**
  - **What is the probability that a randomly selected manager will score above 60?**
    - **First transform the random variable *X* to Z using the transformation formula:** $$z = \frac{x - \mu}{\sigma} = \frac{60 - 72}{8} = -1.5$$
    - **Using the standard normal table, find**
    - $P(Z > -1.5) = 1 - P(Z < -1.5) = 1 - 0.0668 = 0.9332$

6.4 Other Continuous Probability Distributions

LO 6.6 Calculate and interpret probabilities for a random variable that follows the exponential distribution.

- **The Exponential Distribution**
  - **A random variable *X* follows the exponential distribution if its probability density function is:**

$$f(x) = \lambda e^{-\lambda x}$$

for $x \geq 0$

where λ is the rate parameter

$e \approx 2.718$

and

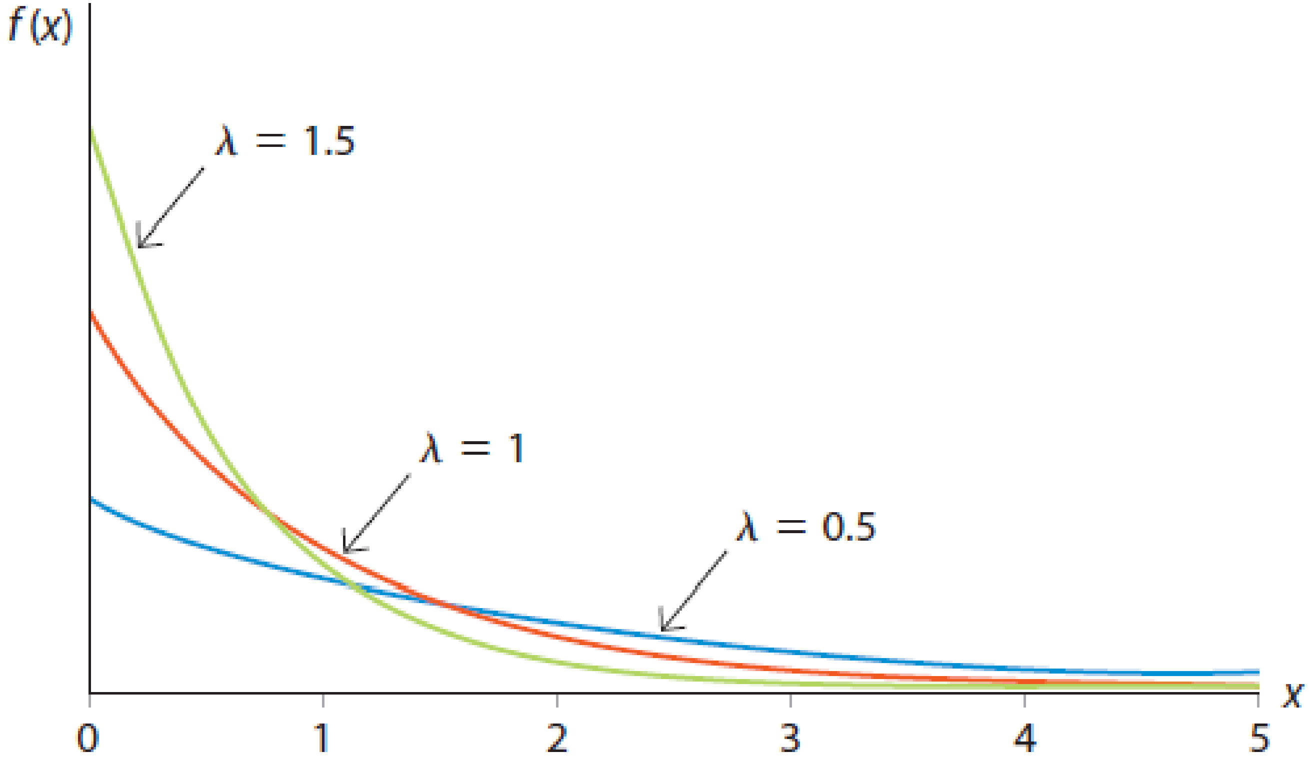
$$E(X) = SD(X) = \frac{1}{\lambda}$$

  - **The cumulative distribution function is:** $P(X \leq x) = 1 - e^{-\lambda x}$

LO 6.6

6.4 Other Continuous Probability Distributions

- **The exponential distribution is based entirely on one parameter, λ > 0, as illustrated below.**



LO 6.6

6.4 Other Continuous Probability Distributions

- Let the time between e-mail messages during work hours be exponentially distributed with a mean of 25 minutes.
  - a. Calculate the rate parameter λ.
  - b. What is the probability that you do not get an e-mail for more than one hour?
  - c. What is the probability that you get an e-mail within 10 minutes?

SOLUTION:

a. Since the mean $E(X)$ equals $\frac{1}{\lambda}$, we compute $\lambda = \frac{1}{E(X)} = \frac{1}{25} = 0.04$.

b. The probability that you do not get an e-mail for more than an hour is $P(X > 60)$. Since $P(X \leq x) = 1 - e^{-\lambda x}$, we have $P(X > x) = 1 - P(X \leq x) = e^{-\lambda x}$. Therefore, $P(X > 60) = e^{-0.04(60)} = e^{-2.40} = 0.0907$. The probability of not getting an e-mail for more than one hour is 0.0907.

c. Here, $P(X \leq 10) = 1 - e^{-0.04(10)} = 1 - e^{-0.40} = 1 - 0.6703 = 0.3297$. The probability of getting an e-mail within 10 minutes is 0.3297.

6.4 Other Continuous Probability Distributions

LO 6.7 Calculate and interpret probabilities for a random variable that follows the lognormal distribution.

- **The Lognormal Distribution**
  - **Defined for a positive random variable, the lognormal distribution is positively skewed.**
  - **Useful for describing variables such as**
    - **Income**
    - **Real estate values**
    - **Asset prices**
  - **Failure rate may increase or decrease over time.**

LO 6.7

6.4 Other Continuous Probability Distributions

- **Let *X* be a normally distributed random variable with mean μ and standard deviation σ. The random variable *Y* = *e*^*X* follows the lognormal distribution with a probability density function as**

$$f(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{\left(\ln(y) - \mu\right)^2}{2\sigma^2} \right) \text{ for } y > 0,$$

where π equals approximately 3.14159

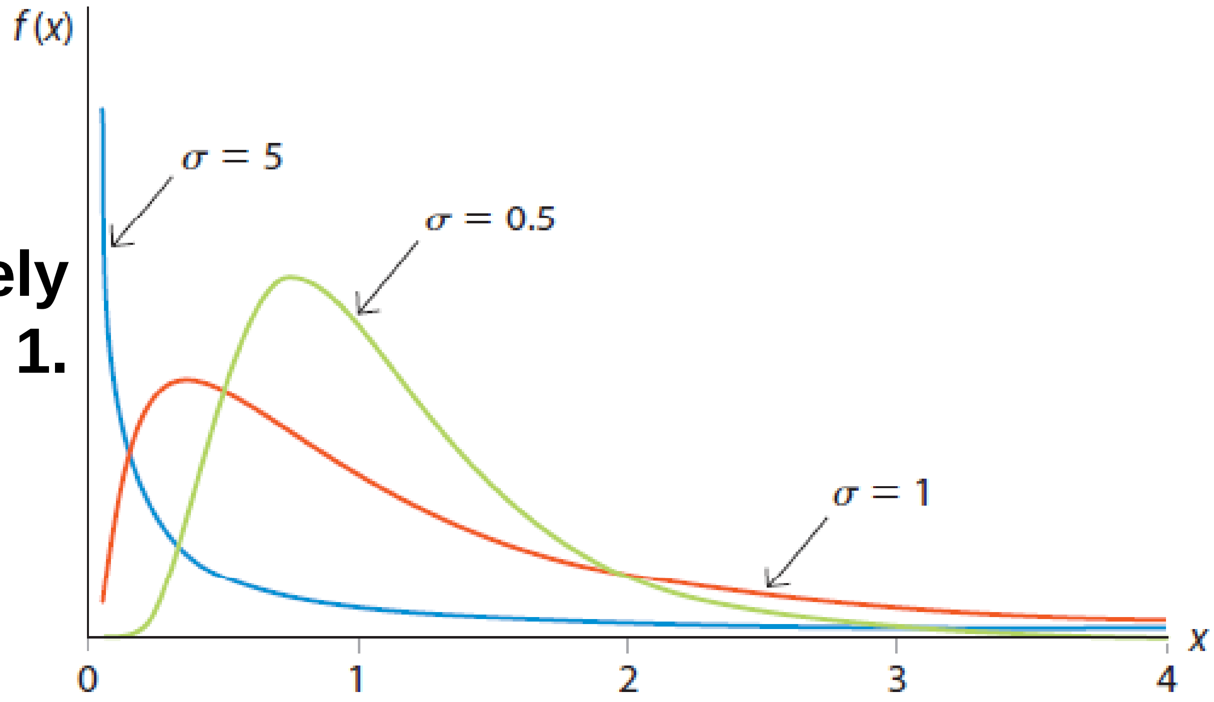
$\exp(x) = e^x$ is the exponential function

$e \approx 2.718$

LO 6.7

6.4 Other Continuous Probability Distributions

- **The graphs below show the shapes of the lognormal density function based on various values of σ.**
  - **The lognormal distribution is clearly positively skewed for σ > 1. For σ < 1, the lognormal distribution somewhat resembles the normal distribution.**



LO 6.7

6.4 Other Continuous Probability Distributions

- Compute the mean and standard deviation of a lognormal random variable if the mean and the standard deviation of the underlying normal random variable are as follows:
  a. $\mu = 0, \sigma = 1$
  b. $\mu = 2, \sigma = 1$
  c. $\mu = 2, \sigma = 1.5$

SOLUTION: Since X is normal, $Y = e^X$ is lognormal with mean $\mu_Y = \exp\left(\frac{2\mu + \sigma^2}{2}\right)$ and standard deviation $\sigma_Y = \sqrt{(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)}$.

a. We compute $\mu_Y = \exp\left(\frac{0 + 1^2}{2}\right) = 1.65$ and $\sigma_Y = \sqrt{(\exp(1^2) - 1)\exp(0 + 1^2)} = 2.16$.

b. Here $\mu_Y = \exp\left(\frac{4 + 1^2}{2}\right) = 12.18$ and $\sigma_Y = \sqrt{(\exp(1^2) - 1)\exp(4 + 1^2)} = 15.97$.

c. Here $\mu_Y = \exp\left(\frac{4 + 1.5^2}{2}\right) = 22.76$ and $\sigma_Y = \sqrt{(\exp(1.5^2) - 1)\exp(4 + 1.5^2)} = 66.31$.

LO 6.7

6.4 Other Continuous Probability Distributions

- **Expected values and standard deviations of the lognormal and normal distributions.**
  - **Let X be a normal random variable with mean μ and standard deviation σ and let $Y = e^X$ be the corresponding lognormal variable. The mean μ_Y and standard deviation σ_Y of Y are derived as**

$$\mu_Y = \exp\left(\frac{2\mu + \sigma^2}{2}\right)$$

$$\sigma_Y = \sqrt{\left(\exp\left(\sigma^2\right) - 1\right)\exp\left(2\mu + \sigma^2\right)}$$

LO 6.7

6.4 Other Continuous Probability Distributions

- **Expected values and standard deviations of the lognormal and normal distributions.**
  - **Equivalently, the mean and standard deviation of the normal variable *X* = ln(*Y*) are derived as**

$$\mu = \ln\left(\frac{\mu_Y^2}{\sqrt{\mu_Y^2 + \sigma_Y^2}}\right)$$

$$\sigma = \sqrt{\ln\left(1 + \frac{\sigma_Y^2}{\mu_Y^2}\right)}$$

End of Chapter