Prof. Dr. U. Rüde
Benjamin Mann

Winter Term
2023/2024

Algorithms of Numerical Linear Algebra
Assignment 1

Exercise 1 (*Induced Matrix Norms*) **1P.**

Let $\|\cdot\|$ denote both a norm on $\mathbb{C}^m$ and the corresponding induced matrix norm (operator norm) on $\mathbb{C}^{m\times m}$. Show that for all $A \in \mathbb{C}^{m\times m}$ the spectral radius $\varrho(A)$ is bounded from above by $\|A\|$.

Note: The spectral radius and induced matrix norm are defined by

$$\varrho(A) = \max_{i=1}^{m} |\lambda_i| \quad \text{and} \quad \|A\| = \sup_{u\in\mathbb{C}^m\setminus\{0\}} \frac{\|Au\|}{\|u\|}, \quad \text{respectively.}$$

Exercise 2 (*Unitary Matrices*) **3P.**

Let $Q \in \mathbb{C}^{m\times m}$ be unitary, i.e., $Q^* = Q^{-1}$. Give an answer together with a short proof for each of the following questions.

(a) Let $x \in \mathbb{C}^m$. What can be said about the Euclidian norm of Qx?

(b) What can be said about the eigenvalues of Q?

(b) How about the determinant of Q?

Exercise 3 (*Hermitian Matrices*) **2P.**

Give a formal proof for the following statement: A matrix $A \in \mathbb{C}^{m\times m}$ is Hermitian if an only if it is both unitary diagonalizable and all its eigenvalues are real.

Hint: Recall that every $m \times m$ matrix has a Schur decomposition $A = QUQ^*$ with unitary matrix Q and upper triangular matrix U.

Exercise 4 (*Rank-one Perturbations*) **8P.**

For $u, v \in \mathbb{C}^m\setminus\{0\}$, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity*.

(a) Under which conditions on u and v is A singular? Give an expression of $\mathrm{null}(A)$. Prove your answers!

(b) Prove the following statement: If A is nonsingular, there exists $\alpha \in \mathbb{C}$, such that $A^{-1} = I + \alpha uv^*$. Give an expression for α.

It can be shown that, if this matrix A is also unitary, u and v are linearly dependent, i.e., for given $u \in \mathbb{C}^m\setminus\{0\}$ there exists $\omega \in \mathbb{C}$ s.th. $A = I + \omega uu^*$.

(c) Find $\Omega_u \subset \mathbb{C}$ s.th. A is unitary if and only if $\omega \in \Omega_u$ (proof!). Sketch Ω_u in the complex plane.

(d) What can be said about α in this case? Your statement should describe α as accurate as possible but independent of ω and u.

Exercise 5 (*Singular Value Decompositions*) **2P.**

Determine SVDs of the following matrices by determining the matrix Σ and guessing the matrices U and V. Think about matrices which switch signs, swap entries and scale entries when multiplied from the left or right side. Do *not* use a computer package! This exercise should make you familiar with elementary transformations such as rotations and reflections.

$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$

Exercise 6 (*More SVD*) **4P.**

Consider the matrix

$$\begin{pmatrix} -2 & 11 \\ -10 & 5 \end{pmatrix}$$

(a) Determine a real SVD of A in the form $A = U\Sigma V^T$.

(b) Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A. Also add the image of the right singular vectors, i.e., AV and mark the coordinates of their vertices.

(c) What are the 1-, 2-, ∞-, and Frobenius norms of A?

(d) Find A^{-1} not directly, but via the SVD.