ANLA- Assignment 7

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Exercise 3

(a) Preconditioner applied in benchmark: Diagonal scaling / Jacobi.

(b) The `magic` function seems to be an attempt to solve a linear system of equations using the **CGN method** with a preconditioner.

This version applies the preconditioner (P* and A*) directly to A, making it inefficient. Matrix multiplication is *O(m^3)* whereas multiplying matrix with a vector is just *O(m^2)* that's why this kind of implementation should never be done.

$$P^T \underbrace{(A^* P^{-T} P^{-1} A)}_{O(\mathrm{m}^3)} P = ||\underbrace{P^{-1}(AP)}_{O(\mathrm{m}^2)}||_2^2$$

(c) The plot generated is shown below:

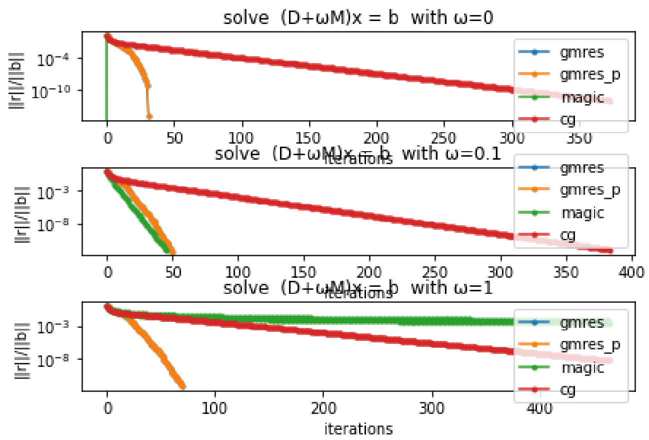

Figure 1 Convergence plots from `krylov_comparison.py`

- The convergence of the four methods depends on the value of omega (ω), which affects the condition number and stability of the system matrix (D+ωM)x = b.
- When ω = 0, the system matrix is diagonal and has a condition number of 1, which means it is well-conditioned and stable. All methods converge rapidly within 50 iterations, as expected for a simple and symmetric matrix.
- When ω = 0.1, the system matrix is less diagonal and has a higher condition number, which means it is less well-conditioned and less stable. The convergence is slower than in the first case, and there are noticeable differences between the methods. Magic and preconditioned GMRES perform better than non-preconditioned GMRES and CG, because they use a preconditioner that reduces the condition number and improves the stability of the system. This is consistent with the theory of preconditioning.
- When ω = 1, the system matrix is not diagonal and has a very high condition number, which means it is ill-conditioned and unstable. The convergence is even slower than in the second case, and CG fails to converge, indicating that the system matrix is no longer positive definite. This is consistent with the theory of CG, which requires a symmetric and positive definite matrix. Magic is slower than both GMRES methods, because the preconditioner increases the condition number and worsens the stability of the system. This is an example of a bad preconditioner. The GMRES methods show similar behavior, because they are more robust to the properties of the system matrix. They can find better approximations by using polynomials

that have zeros near the center of mass of the oval-like region where the eigenvalues of the system matrix are clustered. This is consistent with the theory of GMRES.

In the first scenario, the matrix A is diagonal, and as a result, the Jacobi preconditioner P^{-1} is exactly A^{-1}. This means magic() converges in a single step, followed closely by preconditioned GMRES. The CG and non-preconditioned GMRES iterations converge in the same number of steps, as expected for a symmetric positive definite (SPD) A.

In the second scenario, convergence of CG is slower than non-preconditioned GMRES, as A is less diagonally dominant. magic() and preconditioned GMRES converge quickly, as this is as expected since the condition number of the preconditioned system $P^{-1}A$ is still close to 1.

In the third scenario, CG fails to converge, indicating that A is no longer positive definite. Also, magic() takes longer to converge than both preconditioned and non-preconditioned GMRES. This can be explained by the condition number of $P^{-1}A$, leading to magic() requiring $\sim$ 100 iterations to converge.