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Algorithms of Numerical Linear Algebra
Assignment 4

Exercise 1 (*Uniqueness of Cholesky Factorization*) **7P.**

In this exercise we will prove Theorem 23.1. from [1], i.e., that the Cholesky factorization is unique. We will do this in three steps:

(a) Let $A, B \in \mathbb{C}^{m\times m}$ be upper triangular. Prove that $C = AB$ is also upper triangular with diagonal entries $c_{ii} = a_{ii}b_{ii}$ for $i = 1, \dots, m$.

(b) Let $A \in \mathbb{C}^{m\times m}$ be upper triangular and nonsingular with inverse $B = A^{-1}$. Prove that B is also upper triangular with diagonal entries $b_{ii} = 1/a_{ii}$ for $i = 1, \dots, m$. Hint: Use the Schur complement of the upper left submatrix and proceed recursively.

(c) Assume that for every Hermitian and positive definite matrix $A \in \mathbb{C}^{m\times m}$, there exists a Cholesky factorization $A = U^*U$ with upper triangular matrix U. Prove that this factorization is unique when demanding $u_{jj} > 0$ for all j.

Note: The fact that the algorithm presented in [1] yields a unique solution does not necessarily mean that the factorization is unique. There might be another algorithm to compute a different Cholesky factorization.

Exercise 2 (*Uniqueness of QR Factorization*) **3P.**

Let $A \in \mathbb{C}^{m\times m}$ be nonsingular. Assume that there exists a QR factorization $A = QR$ where Q is unitary and R is upper triangular. Prove that this factorization is unique when demanding $r_{jj} > 0$ for all j.

Note: This means that both R and Q are uniquely defined.

Hint: You may use the result from the above exercise.

Exercise 3 (*LU decomposition of Banded Matrices*) **6P.**

Prove the following proposition: *Let $A \in \mathbb{C}^{m\times m}$ be a non-singular banded matrix with bandwidth $2p+1$, i.e., $a_{ij} = 0$ for $|i-j| > p$. Furthermore, let $A = LU$ be a LU-factorization without pivoting. Then, in addition to being triangular, L and U also have bandwidth $2p + 1$.*

Hint: Show that the statement holds for the first row of U and use induction.

Exercise 4 (*Eigenvalue Properties*) **4P.**

For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m \times m}$ unless otherwise indicated, and "ew" stands for eigenvalue.

(a) If A is real and λ is an ew of A, then so is $-\lambda$.

(b) If λ is an ew of A, then $\bar{\lambda}$ is an ew of A^*.

(c) If λ is an ew of A and A is nonsingular, then λ^{-1} is an ew of A^{-1}.

(d) If A is Hermitian and λ is an ew of A, then $|\lambda|$ is a singular value of A.

(e) If all the ew's of A are zero, then $A = 0$.

(f) If λ is an ew of A, then $(\lambda - \mu)$ is an ew of $(A - \mu I)$.

(g) If A is diagonalizable and all its ew's are equal, then A is diagonal.

(h) The absolute value of the determinant of A is the product of it's singular values, i.e. $|\det A| = \sigma_1 \sigma_2 ... \sigma_m$

References

[1] L.N. Trefethen and D. Bau. *Numerical Linear Algebra*. Society for Industrial and Applied Mathematics, 1997.