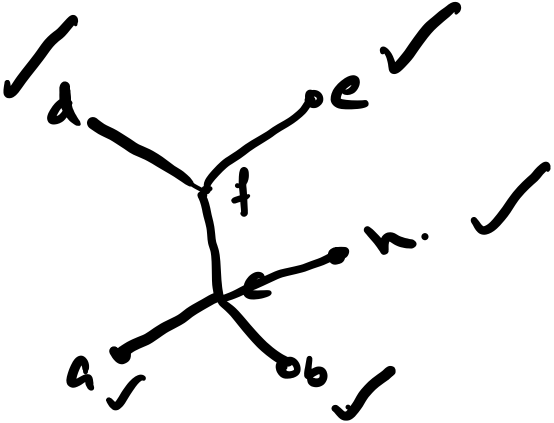
Maximum
Minimum.

MINIMAL AND MAXIMAL ELEMENTS (MEMBERS)

Maximal elements: Let $(S, \leq)$ be a poset an element $a \in S$, is called a maximal element of S if there is no element $b \in S$ such that $a < b$.

Minimal elements: Let $(S, \leq)$ be a poset an element $a \in S$, is called a minimal element of S if there is no element $b \in S$ such that $b < a$.



$a < b$

aRb ✗
aRc ✓
cRf ✓
dRe ✗

Example 1: Consider the poset shown in Fig. 5.6

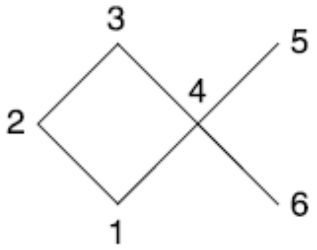


Fig. 5.6

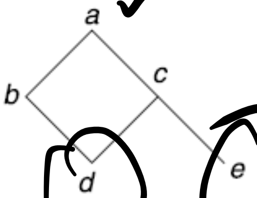
There are two maximal elements and two minimal elements.

The elements 3, 5 are maximal and the elements 1 and 6 are minimal.

b max of S.

$\forall a \in S$ $\boxed{a < b}$

Example 2: Let $A = \{a, b, c, d, e\}$ and let Fig. 5.7 represent the partial order on A in the natural way. The element a is maximal. The elements d and e are minimal.

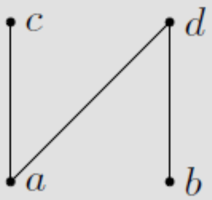


a is maximal

also a is maximum

Distinct minimal members of a partially ordered set are incomparable and distinct maximal members of a poset are also incomparable.

Example



What are the minimal, maximal, minimum, maximum elements?

- Minimal: $\{a, b\}$
- Maximal: $\{c, d\}$
- There are no unique minimal or maximal elements.

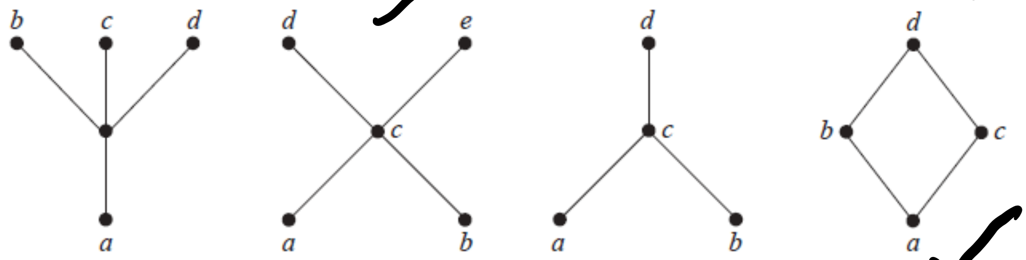
$\leq$

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists

Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists

Example: Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

Max - greatest
Min → least.



(a) (b) (c) (d)

Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d.

$S = \{1,2,3,4,5\}$

$A = \{1,2\}$.

$1,2 \le x$

$A \subseteq S$

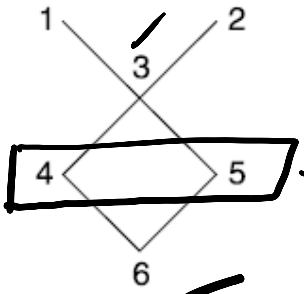
$\boxed{a \le 4}$

UPPER AND LOWER BOUNDS

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

Example 1: $A = \{1, 2, 3, \ldots, 6\}$ be ordered as pictured in

B =



B l

$l \leq b$
$\forall b \in B.$

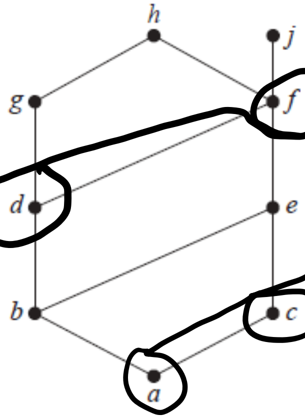
If $B = \{4, 5\}$ then

The upper bounds of B are 1, 2, 3

The lower bound of B is 6.

$C = \{3, 4\}$
ubs → 1, 2, 3
→ 4, 6

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.



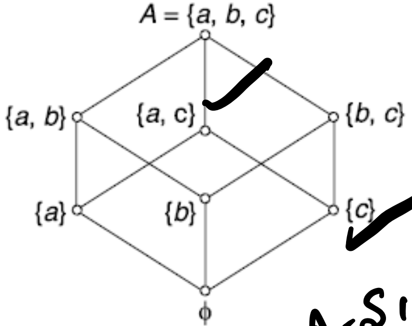
$\boxed{f, j, h}$

UBs of $\{a, b, c\}$ $= e, f, j, h$

LBs $= a$.

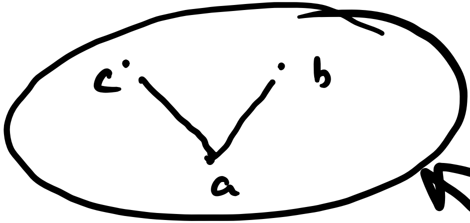
Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a.

Example 2: Let $A = \{a, b, c\}$ and $(P(A), \leq)$ be the partially ordered set. The Hasse diagram of the Poset be as pictured in Fig. 5.9.



A = {1, 2 -- 10}

≤ {0, 11, 2, --- ∞}



If B ~~is the subset~~ $\{\{a, c\}, \{c\}\}$. Then the upper bounds of B are $\{a, c\}$ and A, while the lower bounds of B are $\{c\}$ and $\varnothing$.

a, b, c

a ≤ b; a ≤ c

Least $\boxed{\text{Upper Bound}}$ (Supremum)

Set A be a partially ordered set and B a subset of A. An element $M \in A$ is called the least upper bound of B if M is an upper bound of B and $M \le M'$ whenever M' is an upper bound of B.

A least upper bound of a partially ordered set if it exist is unique.

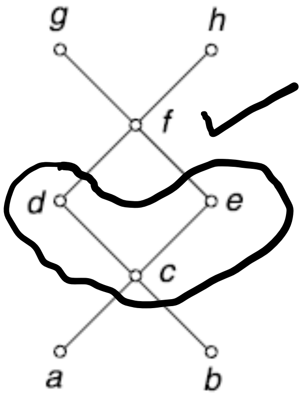
a, b

$a \le b$

$b \le a$.

$a = b$

Example: Let $A = \{a, b, c, d, e, f, g, h\}$ denote a partially ordered set. Whose Hasse diagram is shown



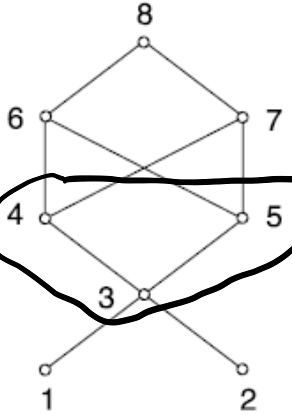
If $B = \{c, d, e\}$ then f, g, h are upper bounds of B. The elements f is least upper bound.

The Greatest Lower Bound (Infimum)

Let A be a partially ordered set and B denote a subset of A. An element L is called a greatest lower bound of B if l is a lower of B and $L' \leq L$ whenever L' in a lower bound of B.

The greatest lower bound of a poset if it exists is unique.

Example: Consider the poset $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ whose Hasse diagram is shown in Fig. 5.11 and let $B = \{3, 4, 5\}$

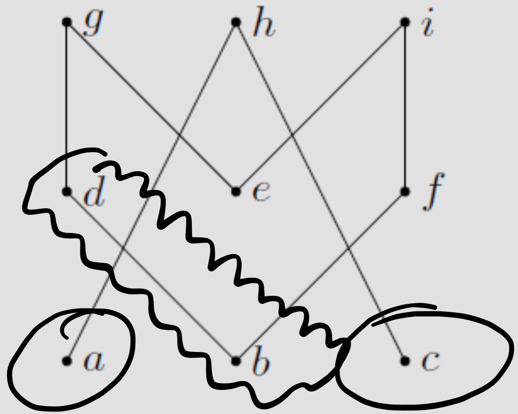


· LUB → does not exist

The elements 1, 2, 3 are lower bounds of B. 3 is greatest lower bound

The least upper bound (LUB) and the greatest lower bound (GLB) of subset B are also called the supremum and infimum of the subset B.

Example



What are the lower/upper bounds and glb/lub of the sets $\{d, e, f\}$, $\{a, c\}$ and $\{b, d\}$

↳ $\{ \times \} \to \{ \times \}$

$\{d, e, f\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\emptyset$, thus no lub either.

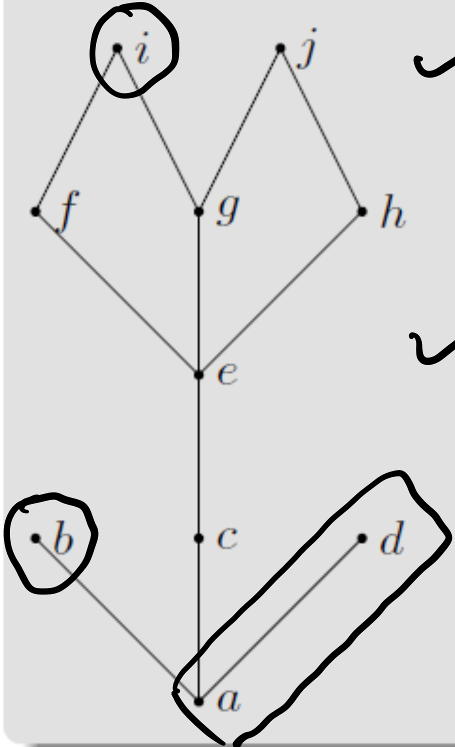
$\{a, c\}$

- Lower Bounds: $\emptyset$, thus no glb either.
- Upper Bounds: $\{h\}$, since its unique, lub is also h.

$\{b, d\}$

- Lower Bounds: $\{b\}$ and so also glb.
- Upper Bounds: $\{d, g\}$ and since $d \prec g$, the lub is d.

Example



Minimal/Maximal elements?

- Minimal & Minimum Element: a.
- Maximal Elements: b, d, i, j.

Bounds, glb, lub of $\{c, e\}$?

- Lower Bounds: $\{a, c\}$, thus glb is c.
- Upper Bounds: $\{e, f, g, h, i.j\}$ thus lub is e

Bounds, glb, lub of $\{b, i\}$?

- Lower Bounds: $\{a\}$, thus glb is a.
- Upper Bounds: $\emptyset$, thus lub DNE.