LATTICES

Definition:

A lattice is a partially ordered set $\langle L, \leq \rangle$ in which every pair of elements $a, b \in L$ has a greatest lower bound and least upper bound.

The greatest lower bound of a subset $\{a, b\} \in L$ will be denoted by $a * b$ and the least upper bound by $a \oplus b$.

Join or sum: The LUB of a subset $\{a, b\} \subseteq L$ is denoted by $a \oplus b$ (or $a \vee b$ or $a + b$) and is called the join or sum of a and b.

Meet or product: The GLB of a subset $\{a, b\} \subseteq L$ is denoted by $a * b$ (or $a \cdot b$ or $a \wedge b$) is called the meet or product of a and b.

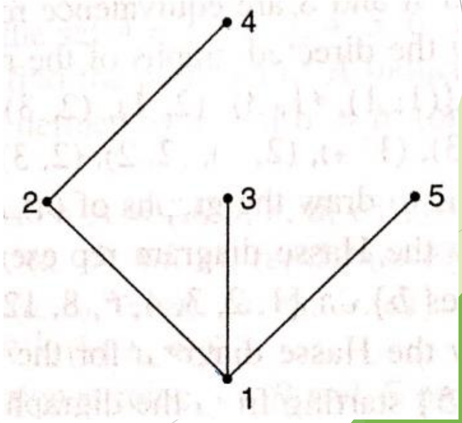
Example: $(\{1,2,4,8\},|)$, where | means 'divisor of '. The hasse diagram

LUB = 8,   GLB = 1

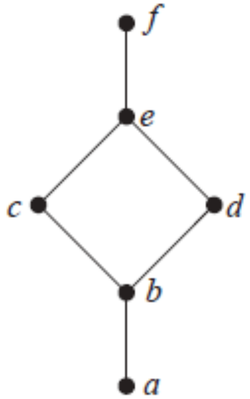
So, it is a lattice.

Example: $(\{1,2,3,4,5\},|)$

It is not a lattice, since LUB of the pair (2, 3) and (3, 5) do Not exist.

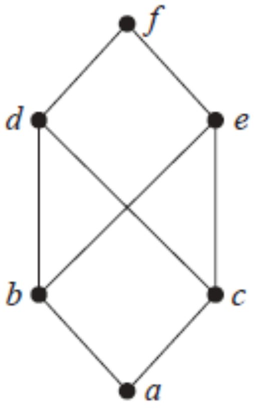


Example:



It is a lattice since every pair of elements has LUB and GLB.

Example:

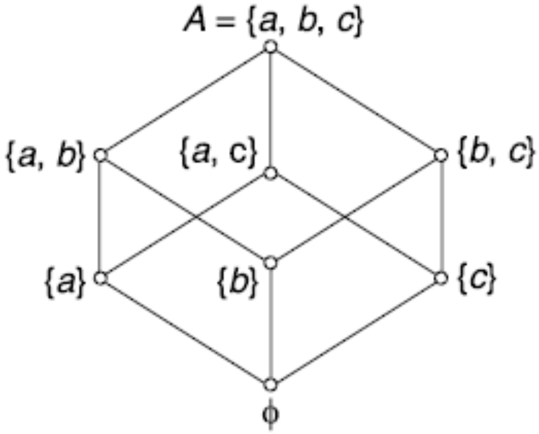


(b)

It is not a lattice since the pair (b, c) have no least upper bound.

Example: In the case of power set $P(S)$ of any set $S, (P(S), \subseteq)$ is a lattice

Here $LUB = A \cup B$ and $GLB = A \cap B$, where A and B are any subsets of $P(S)$.



Example: Is the poset $(Z^+, |)$ a lattice?

Solution: Let $a, b \in Z^+$, now, LUB of these two integers is the LCM (Least Common multiple) and GLB is the GCD (Greatest Common Divisor).

Principle of duality: If $\leq$ is a partial ordering relation on a set **S**, the converse $\geq$ is also a partial ordering relation on **S**.

Example: $\leq$ denotes the 'divisor of' then $\geq$ denotes 'multiple of'.

- The hasse diagram of $(S, \geq)$ can be obtained from that of $(S, \leq)$ by simply turning it upside down.

Example: $(\{1, 2, 4, 8, 16\}, multiple\ of)$

The lattice $\{L, \leq\}$ and $\{L, \geq\}$ are called the duals of each other.

Axioms of Algebraic lattice:

If $(L, \leq)$ is ordered lattice and let $a, b, c \in L$, then L will satisfy the following axioms:

1. $a \leq b \Leftrightarrow a * b = a$
2. $a \leq b \Leftrightarrow a \oplus b = b$
3. $a * b \leq a$ and $a * b \leq b$
4. $a \leq a \oplus b$ and $b \leq a \oplus b$
5. If $a \leq c,\ b \leq c$ then $a \oplus b \leq c$
6. If $c \leq a,\ c \leq b$ then $c \leq a * b$

Properties of Lattice:

We shall first list some of the properties of the two binary operations of meet and join denoted by $*$ and $\oplus$ on a lattice $(L, \leq)$.

For any $a, b, c \in L$, we have

Idempotent law

$(L-1)$ $\quad a * a = a,$ $\qquad$ $(L-1)'$ $\quad a \oplus a = a$

Commutative Law

$(L-2)$ $\quad a * b = b * a,$ $\qquad$ $(L-2)'$ $\quad a \oplus b = b \oplus a$

Associativity

$(L-3)$ $\quad (a * b) * c = a * (b * c),$ $\qquad$ $(L-3)'$ $\quad (a \oplus b) \oplus c = a \oplus (b \oplus c),$

Absorption Law

$(L-4)$ $\quad a * (a \oplus b) = a$ $\qquad$ $(L-4)'$ $\quad a \oplus (a * b) = a$

Theorem 1.: Let $(L, \leq)$ be a lattice in which $*$ and $\oplus$ denote the operations of meet and join respectively. For any $a, b \in L$

$$a \leq b \iff a * b = a \iff a \oplus b = b.$$

Proof:

First we will prove that $a \leq b \iff a * b = a$.

Let us assume that $a \leq b$,

since we know that $a \leq a$

$\Rightarrow a \leq a * b$,

From the definition of $a * b$, $a * b \leq a$

Hence, $a \leq b \Rightarrow a * b = a$ ...(1)

Next assume that $a * b = a$, but it is only possible if $a \leq b, i.e.,$

$$a * b = a \ \Rightarrow a \leq b \qquad \ldots(2)$$

Combining (1) and (2), we have

$$a \leq b \iff a * b = a.$$

Now, we will prove $a * b = a \iff a \oplus b = b,$

Let

$$a * b = a$$

We have

$$\begin{aligned} b \oplus (a * b) &= b \oplus a \\ &= a \oplus \mathrm{b} \end{aligned}$$

But, $b \oplus (a * b) = b$

Thus, we have $a \oplus b = b$,

$$\text{i.e, } a * b = a \Rightarrow a \oplus b = b,$$

Now to show that

$$\boldsymbol{a \oplus b = b \Rightarrow a * b = a}$$

Take

$$a \oplus b = b,$$

$$a * (a \oplus b) = a * b,$$

We know that

$$a * (a \oplus b) = a$$

$$\Rightarrow a * b = a$$

Hence

$$a \oplus b = b \Rightarrow a * b = a$$

Combining these two, we have

$$a * b = a \iff a \oplus b = b,$$

Isotonic Property:

Let $(L, \leq)$ be a lattice. For any $a\,, b, c, \in L$, the following properties hold

$(i) \quad b \leq c \;\Rightarrow\; a * b \leq a * c$

$(ii)\; b \leq c \;\Rightarrow\; a \bigoplus b \leq a \bigoplus c$

Proof: From Theorem 1

$$b \leq c \;\Leftrightarrow b * c = b$$

To show, $\quad b \leq c \;\Rightarrow a * b \leq a * c$

We will show that

$$(a * b) * (a * c) = a * b$$

L.H.S

$$
\begin{aligned}
&(a * b) * (a * c) \\
&\quad = a * b * a * c \\
&\quad = (a * a) * b * c \\
&\quad = a * (b * c) \\
&\quad = a * b
\end{aligned}
$$

Hence, $b \leq c \;\Rightarrow\; a * b \leq a * c$

Proof (ii): To show, $b \leq c \;\Rightarrow a \oplus b \leq a \oplus c$

We will show that

$$(a \oplus b) \oplus (a \oplus c) = a \oplus c$$

L.H.S

$$\begin{aligned}(a \oplus b) \oplus (a \oplus c) \\ &= a \oplus b \oplus a \oplus c \\ &= (a \oplus a) \oplus b \oplus c \\ &= a \oplus (b \oplus c) \\ &= a \oplus c\end{aligned}$$

Hence, $b \leq c \;\Rightarrow\; a \oplus b \leq a \oplus c$

Theorem 3. Let $(L, \leq)$ be a lattice, then for any $a, b, c, \in L$, the following properties hold

(i) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

(ii) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

Proof: Now $a \leq a \oplus b$ and $a \leq a \oplus c$ $\Longrightarrow a \leq (a \oplus b) * (a \oplus c)$

Again

$$b * c \leq b \leq a \oplus b$$
$$b * c \leq c \leq a \oplus c$$

$\Rightarrow$ $b * c \leq (a \oplus b) * (a \oplus c)$

$$\Rightarrow \quad a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

Similarly, we can prove the second property.

Modular inequality

Theorem 4. Let $(L, \leq)$ be a lattice, then for any $a\ , b, c, \in L$, the following properties hold

$$a \leq c \quad \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

Proof:

We have $\quad a \leq c \quad \Leftrightarrow (a \oplus c) = c$

Now,

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

$$a \oplus (b * c) \leq (a \oplus b) * \mathrm{c} \qquad \ldots (1)$$

Now let, $\quad a \oplus (b * c) \leq (a \oplus b) * \mathrm{c}$

$$a \ \leq a \oplus (b * c) \leq (a \oplus b) * \mathrm{c} \leq c$$

$$\Rightarrow a \leq c \qquad\qquad \ldots(2)$$

From (1) and (2),

$$a \leq c \iff a \oplus (b * c) \leq (a \oplus b) * c$$

Theorem 5: Let $(L, \leq)$ is a lattice, then for any $a, b, c \in L$,. If $a \leq b, c \leq d$

Then $a \oplus c \leq b \oplus d$ and $a * c \leq b * d$.

Proof: Given $a \leq b, c \leq d$

$a \leq b \Rightarrow c \oplus a \leq c \oplus b$ (Using Isotonic Property)

$a \oplus c \leq b \oplus c$ (Using commutative Prop) .....(1)

$c \leq d \Rightarrow b \oplus c \leq b \oplus d$ (Using Isotonic Property) ......(2)

From (1) and (2)

$$a \oplus c \leq b \oplus d$$

Similarly, we can prove

$$a * c \leq b * d$$