Normal Distribution / Gaussian Dist^n :-

Even function: A function $f(x)$ is said to be even function if $f(-x) = f(x) \quad \forall x \in D$

Domain

e.g. $\cos x$, x^2, x^4 ...

Odd function: A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$

e.g. $\sin x$, x^3, x^5, ..

Integration of even

Note :- ① $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even.

② $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd function

Normal Distribution :- A r.v X is said to have Normal dist^n if its pdf is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \quad x \in \mathbb{R}, \ \mu \in \mathbb{R}, \ \sigma > 0$$

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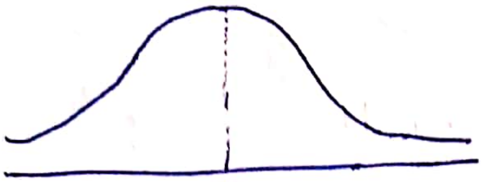
Properties :

① The curve of normal $dist^n$ is **bell shaped** .



μ

No.- 64

② This is symmetric about its mean value

③ The curve of normal $dist^n$ is fixed

④ Normal $dist^n$ is used to model negative values as well.

Q: Verify that it is proper pdf. Also find mean and variance for this $dist^n$.

Sol^n Claim: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

From LHS $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Put $\frac{x-\mu}{\sigma} = t \Rightarrow \frac{1}{\sigma}dx = dt \Rightarrow dx = \sigma dt$

When $x \to \infty$; $t \to \infty$

When $x \to -\infty$; $t \to -\infty$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} \cdot \cancel{\sigma}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2}\, dt \qquad \left(\because\ e^{-t^2/2} \text{ is even function}\right)$$

Now put $\frac{t^2}{2} = u \Rightarrow t = \sqrt{2u}$

$$\Rightarrow \frac{\cancel{2}\, t\, dt}{\cancel{2}} = du$$

$$\Rightarrow dt = \frac{du}{\sqrt{2u}}$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-u} \cdot (2u)^{-1/2}\, du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{2}} \int_{0}^{\infty} u^{-1/2} e^{-u}\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} u^{-\frac{1}{2}+1-1} e^{-u}\, du$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty u^{\frac{1}{2}-1} e^{-u} du$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\tfrac{1}{2}$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}}$$

$$= 1$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = t \quad \Rightarrow \quad x = \mu + \sigma t$

$$\Rightarrow \frac{dx}{\sigma} = dt$$

$$\Rightarrow dx = \sigma dt$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma t)\, e^{-\frac{t^2}{2}} \cdot \cancel{\sigma}\, dt$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t\, e^{-t^2/2}\, dt$$

even function

Odd function

$$= \frac{2\cdot\mu}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2}\, dt + \frac{\sigma}{\sqrt{2\pi}}\cdot 0$$

$$= \mu \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$\boxed{E(X) = \mu}$$

Now: $\boxed{\int_0^\infty e^{-t^2/2}\, dt = \sqrt{\frac{\pi}{2}}}$ (Claim)

Put $t^2/2 = u$

$\Rightarrow \frac{\cancel{2}t}{\cancel{2}}\, dt = du$

$\Rightarrow t\,dt = du$

$\Rightarrow \sqrt{2u}\, dt = du$

$\Rightarrow dt = \frac{du}{\sqrt{2u}}$

When $t = 0$; $u = 0$

$t \to \infty$; $u \to \infty$

$$\int_0^\infty e^{-u} \cdot \frac{du}{\sqrt{2u}} = \int_0^\infty (2u)^{-1/2} e^{-u}\, du$$

$$= 2^{-1/2} \int_0^\infty u^{-1/2} e^{-u}\, du$$

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$$= \frac{1}{\sqrt{2}} \int_0^\infty u^{1/2 - 1} e^{-u} du$$

$$= \frac{1}{\sqrt{2}} \cdot \Gamma\tfrac{1}{2}$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$= \sqrt{\frac{\pi}{2}}$$

Now ; $V(X) = EX^2 - (EX)^2$

$$EX^2 = \int_{-\infty}^{\infty} x^2 \cdot f_X(x) dx \quad = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = t \Rightarrow x = \mu + \sigma t$ | When $x \to \infty$; $t \to \infty$
$x \to -\infty$; $t \to -\infty$

$dx = \sigma dt$

$$EX^2 = \int_{-\infty}^{\infty} (\mu + \sigma t)^2 \cdot \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-t^2/2} \cdot \cancel{\sigma} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma t)^2 e^{-t^2/2} dt$$

$$= \frac{\mu^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2} dt}_{\text{even}} + \frac{\sigma^2}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} t^2 e^{-t^2/2} dt}_{\text{even}} + \overbrace{\cancel{\frac{2\mu\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} t e^{-t^2/2} dt}_{\text{Odd}}}}^{0}$$

$$= \frac{\mu^2 \cdot 2}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2}\, dt + \frac{\sigma^2 \cdot 2}{\sqrt{2\pi}} \int_0^\infty t^2 \cdot e^{-t^2/2}\, dt$$

$$= \mu^2 \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} + \sigma^2 \sqrt{\frac{2}{\pi}} \int_0^\infty t^2 e^{-t^2/2}\, dt$$

$$= \mu^2 + \sigma^2 \sqrt{\frac{2}{\pi}} \int_0^\infty t^2 e^{-t^2/2}\, dt$$

Now $\int_0^\infty t^2 e^{-t^2/2}\, dt$

Put $t^2/2 = u \Rightarrow t = \sqrt{2u}$ | When $t = 0;\ u = 0$; $t \to \infty;\ u \to \infty$

$\Rightarrow \frac{\cancel{2}t}{\cancel{2}}\, dt = du$

$\Rightarrow dt = \frac{du}{\sqrt{2u}}$

$$\int_0^\infty t^2 e^{-t^2/2}\, dt = \int_0^\infty (2u)\, e^{-u} \cdot \frac{du}{\sqrt{2u}}$$

$$= \sqrt{2} \int_0^\infty u^{1/2} e^{-u}\, du$$

$$= \sqrt{2} \int_0^\infty u^{(1/2+1)-1} e^{-u}\, du$$

$$= \sqrt{2} \int_0^\infty u^{3/2-1} e^{-u}\, du$$

$$= \sqrt{2} \cdot \Gamma 3/2$$

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$$= \sqrt{2}\cdot \Gamma\left(1+\tfrac{1}{2}\right)$$

$$= \sqrt{2}\cdot \frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}}\cdot\sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{2}}$$

Thus

$$Ex^2 = \mu^2 + \sigma^2\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty t^2 e^{-t^2/2}\,dt$$

$$= \mu^2 + \sigma^2\cdot\sqrt{\frac{2}{\pi}}\cdot\sqrt{\frac{\pi}{2}}$$

$$= \mu^2 + \sigma^2$$

So, $V(X) = Ex^2 - (EX)^2$

$$= \cancel{\mu^2} + \sigma^2 - \cancel{\mu^2}$$

$$\boxed{V(x) = \sigma^2}$$