Note. Let $B = \{b_1, b_2, \ldots\ldots, b_r\}$. If $a = LUB(B)$, then a is the first vertex that can be reached from $b_1, b_2, \ldots\ldots, b_r$ by upward paths. Similarly, if $a = GLB(B)$, then a is the first vertex that can be reached from $b_1, b_2, \ldots\ldots, b_r$ by downward paths.

LATTICES

Lattice

A lattice L is a poset in which every pair of elements has a least upper bound (LUB) or supremum and a greatest lower bound (GLB) or infimum.

Join

Consider a poset L under the ordering $\leq$. Let $a, b \in L$. Then LUB (a, b) or SUP (a, b) is denoted by $a \vee b$ or $a \cup b$ and is called the join of a and b *i.e.*, $a \vee b = \text{SUP}(a, b)$.

Meet

Consider a poset L under the ordering $\leq$. Let $a, b \in L$. Then GLB (a, b) or inf (a, b) is denoted by $a \wedge b$ or $a \cap b$ and is called the meet of a and b *i.e.*, $a \wedge b = \inf(a, b)$.

From the above, it follows that a lattice L is a mathematical structure with two binary operations $\vee$(Join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$. The lattice L for any elements a, b and c satisfies the following properties :

(*a*) **Commutative Property**

(*i*) $a \wedge b = b \wedge a$

(*ii*) $a \vee b = b \vee a$.

(*b*) **Associative Property**

(*i*) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(*ii*) $(a \vee b) \vee c = a \vee (b \vee c)$.

(*c*) **Absorption Property**

(*i*) $a \wedge (a \vee b) = a$

(*ii*) $a \vee (a \wedge b) = a$.

Theorem III. *Prove that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b = b$.*

Proof. Let us first assume that $a \wedge b = a$.

Using absorption property, we have

$$b = b \vee (b \wedge a) = b \vee (a \wedge b) = b \vee a = a \vee b \quad \ldots(i)$$

Conversely, let us assume $a \vee b = b$.

Again using the absorption property, we have

$$a = a \wedge (a \vee b) = a \vee b \quad \ldots(ii)$$

From eqn. (*i*) and (*ii*), we have

$$a \wedge b = a \text{ if and only if } a \vee b = b.$$

Theorem IV. *Prove that for elements of lattice*

(*i*) $a \wedge a = a$ (*ii*) $a \vee a = a$ Idempotent property

Proof. (*i*) $a \wedge a = a \wedge (a \vee (a \wedge b))$ {Using absorption property c(*ii*)}

$= a$ {Using absorption property c(*i*)}

(*ii*) $a \vee a = a \vee (a \wedge (a \vee b))$ {Using absorption property c(*i*)}

$= a$ {Using absorption property c(*ii*)}

Theorem V. *Consider a lattice L. Prove that the relation $a \leq b$ defined by either $a \wedge b = a$ or $a \vee b = b$ is a partial ordering on lattice L.*

Proof. For any element $a \in L$, we have $a \wedge a = a$

i.e., $a \leq a$. Therefore, the relation $\leq$ is reflexive. Now assume $a \leq b$ and $b \leq a$. Then we have

$$a \wedge b = a \quad \text{and} \quad b \wedge a = b$$

Thus, $a = a \wedge b = b \wedge a = b$, therefore the relation $\leq$ is antisymmetric.

At last, we assume $a \leq b$ and $b \leq c$. So, we have

$$a \wedge b = a \text{ and } b \wedge c = b$$

$$a \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

i.e.,

$$= a \wedge b = a$$

Therefore, $a \leq c$. So, the relation $\leq$ is transitive. From above, we can say, $\leq$ is a partial order on L.

Example 22. *Let P(S) be the power set of the set S = {1, 2, 3}. Construct the Hasse diagram of the partial order induced on P(S) by the lattice (P(S), $\wedge$, $\vee$).*

Sol. The Hasse diagram obtained by a lattice is same as obtained under the partial ordering of set inclusion. In the lattice, $a \leq b$ whenever $a \wedge b = a$. Thus, in the above case $a \leq b$, whenever $a \wedge b = a$

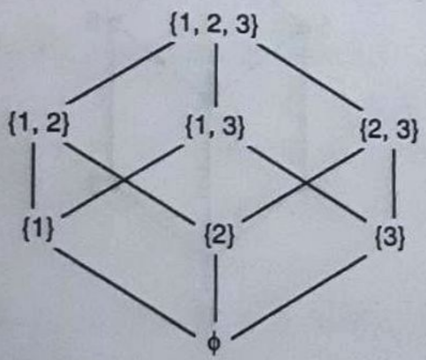

Fig. 22.

Example 23. *Determine which of the posets shown in Fig. 23 are lattices.*

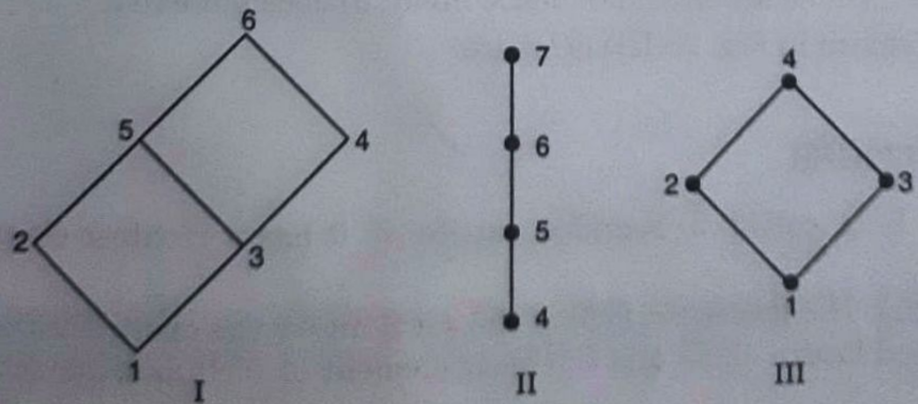

Fig. 23.

Sol. All the posets shown in Fig. 23 are lattices.

Example 24. *Determine whether the posets shown in Fig. 24 are lattices or not.*

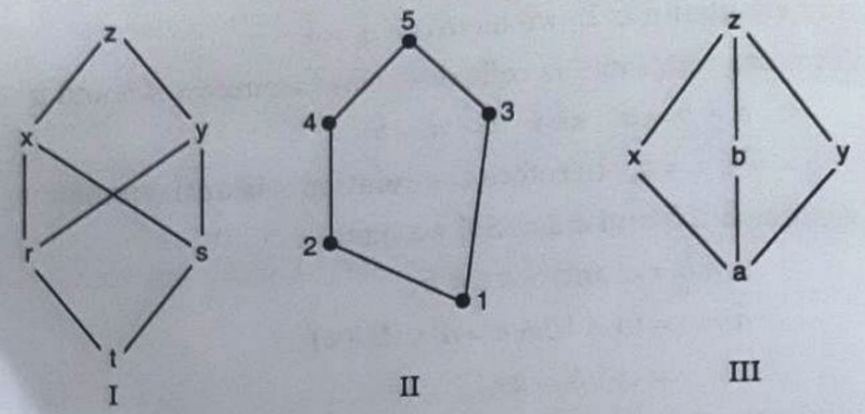

Fig. 24.

Sol. The posets shown in Figs. 24(II) and 24(III) are lattices. The posets shown in Fig. 24(I) is not lattice as (x, y) has three lower bounds *i.e.*, r, s and t but the inf (x, y) does not exist. Also, sup (r, s) does not exist.

Example 25. *Determine whether the posets shown in Fig. 25 are lattices or not.*

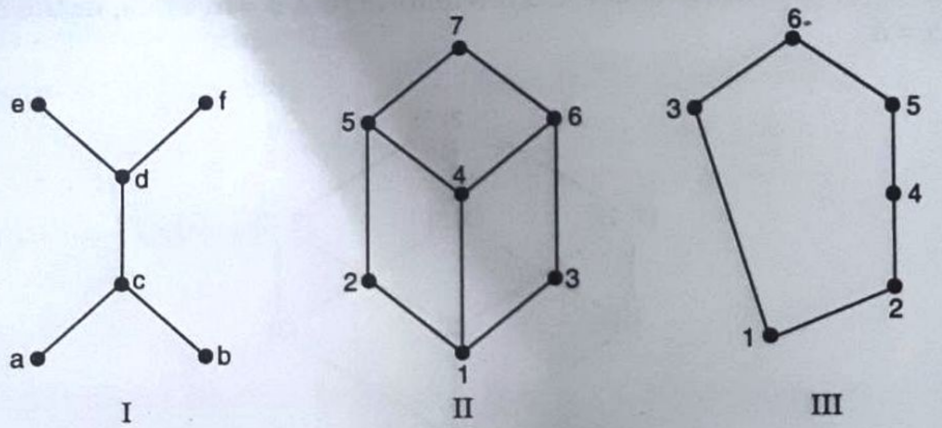

Fig. 25.

Sol. The poset shown in Fig. 25(II) is a lattice. The poset shown in Fig. 25(I) is not lattice since the elements e and f have no upper bound, hence sup (e, f) does not exist. Similarly, the elements a and b have no lower bound, hence inf (a, b) does not exist.

The poset shown in Fig. 25(III) is lattice.

BOUNDED LATTICES

A lattice L is called a bounded lattice if it has a greatest element 1 and a least element 0.

Example (*i*). The power set P(S) of the set S under the operations of intersection and union is a bounded lattice since ϕ is the least element of P(S) and the set S is the greatest element of P(S).

(*ii*) The set of +ve integers I_+ under the usual order of $\leq$ is not a bounded lattice since it has a least element 1 but the greatest element does not exist.

Properties of Bounded Lattices

It L is a bounded lattice, then for any element $a \in L$, we have the following identities :

(*i*) $a \vee 1 = 1$ (*ii*) $a \wedge 1 = a$ (*iii*) $a \vee 0 = a$ (*iv*) $a \wedge 0 = 0$.

Theorem VI. *Prove that every finite lattice* $L = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$ *is bounded.*

Proof. We have given the finite lattice :

$$L = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$$

Thus, the greatest element of lattice L is $a_1 \vee a_2 \vee a_3 \vee \ldots\ldots \vee a_n$.

Also, the least element of lattice L is $a_1 \wedge a_2 \wedge a_3 \wedge \ldots\ldots \wedge a_n$.

Since, the greatest and least elements exist for every finite lattice. Hence L is bounded.

SUBLATTICES

Consider a non-empty subset L_1 of a lattice L. Then L_1 is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L *i.e.*, if $a \vee b \in L_1$ and $a \wedge b \in L_1$ whenever $a \in L_1$ and $b \in L_1$.

Example 26. *Consider the lattice of all +ve integers* I_+ *under the operation of divisibility. The lattice* D_n *of all divisors of* $n > 1$ *is a sublattice of* I_+.

Determine all the sublattices of D_{30} *that contain at least four elements,* $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

Sol. The sublattices of D_{30} that contain at least four elements are as follows :

(*i*) {1, 2, 6, 30} (*ii*) {1, 2, 3, 30}

(*iii*) {1, 5, 15, 30} (*iv*) {1, 3, 6, 30}

(*v*) {1, 5, 10, 30} (*vi*) {1, 3, 15, 30}

(*vii*) {2, 6, 10, 30}.

Example 27. *Consider the lattice* $L = \{1, 2, 3, 4, 5\}$ *as shown in Fig. 26. Determine all the sublattices with three or more elements.*

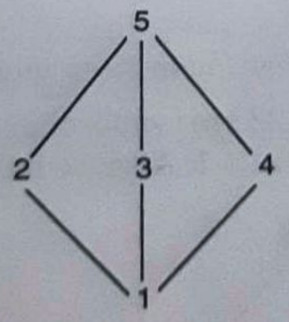

Fig. 26.

Sol. All the sublattices with three or more elements are those whose supremum and infimum exists for every pair of elements which are as follows :

(*i*) {1, 2, 5} (*ii*) {1, 3, 5}

(*iii*) {1, 4, 5} (*iv*) {1, 2, 3, 5}

(*v*) {1, 3, 4, 5} (*vi*) {1, 2, 3, 4, 5}

(*vii*) {1, 2, 4, 5}.

Example 28. *Consider the lattice L as shown in Fig. 27. Determine whether or not each of the following is a sublattice of L.*

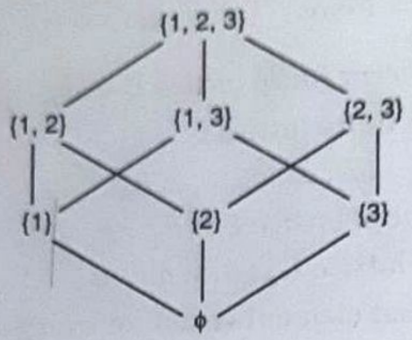

Fig. 27.

$A = \{\phi, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$
$B = \{\phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$
$C = \{\phi, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$
$D = \{\{1\}, \{3\}, \{1, 3\}, \{1, 2, 3\}\}$
$E = \{\phi, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$.

Sol. A is not a sublattice since $\{1, 2\} \wedge \{2, 3\} = \{2\}$ which does not exist in A.
B is a sublattice since sup and inf of every pair of elements exist.
C is a sublattice since sup and inf of every pair of elements exist.
D is not a sublattice since $\{1\} \wedge \{3\} = \phi$ which does not exist in D.
E is a sublattice since sup and inf of every pair of elements exist.

ISOMORPHIC LATTICES

Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from L_1 to L_2 *i.e.*, $f : L_1 \to L_2$, such that

$$f(a \wedge b) = f(a) \wedge f(b) \text{ and } f(a \vee b) = f(a) \vee f(b)$$

for every element a, b belongs to L_1.

Example 29. *Determine whether the lattices shown in Fig. 28 are isomorphic.*

Sol. The lattices shown in Fig. 28 are isomorphic. Consider the mapping $f = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$. For example $f(b \wedge c) = f(a) = 1$. Also we have $f(b) \wedge f(c) = 2 \wedge 3 = 1$.

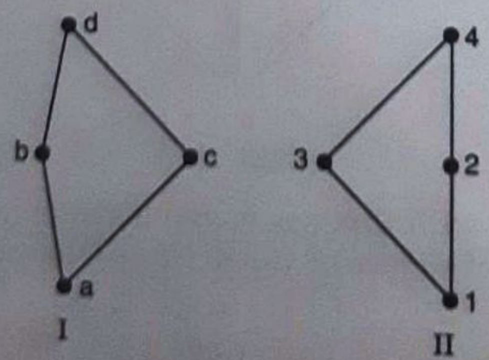

Fig. 28.

Example 30. *Determine whether the lattices shown in Fig. 29 are isomorphic.*

Sol. The lattices shown in Fig. 29 are not isomorphic since bijection is not possible as the element of the two lattices are not same.

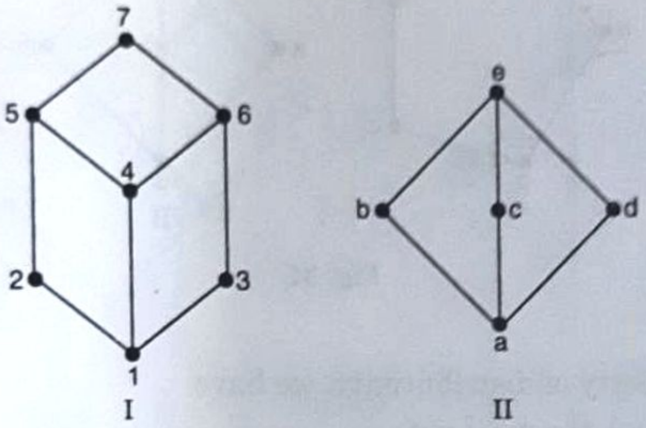

Fig. 29.

DISTRIBUTIVE LATTICE

A lattice L is called distributive lattice if for any elements a, b and c of L, it satisfies following distributive properties :

(*i*) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(*ii*) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

For example :

1. The power set P(S) of the set S under the operations of intersection and union is a distributive function. Since,

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

and also $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ for any sets a, b and c of P(S).

2. The lattice shown in Fig. 30 is distributive. Since it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3 and 4.

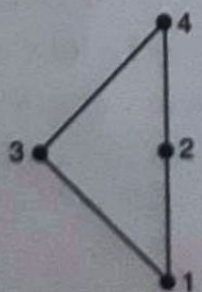

Fig. 30

Example 31. *Show that the lattices shown in Fig. 31 are non-distributive.*

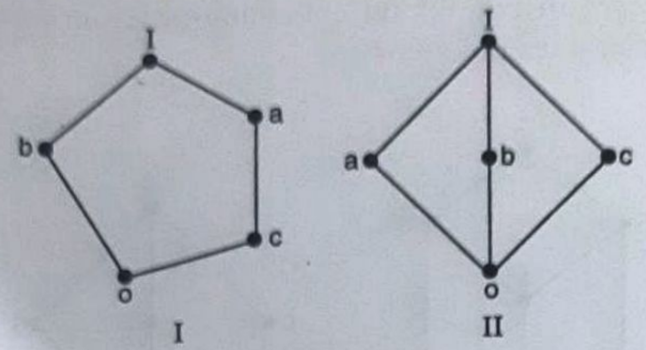

Fig. 31.

Sol.

I. From the (*i*) property of distributivity, we have

$$a \wedge (b \vee c) = a \wedge I = a$$

But, $(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

II. Again, from the (*i*) property of distributivity, we have

$$a \wedge (b \vee c) = a \wedge I = a$$

But $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

Note. A lattice L is non-distributive if and only if it contains a sublattice that is isomorphic to one of the two lattices shown in Figs. 31(I) and (II).

Theorem VII. *Prove that in a distributive lattice* $(L, \wedge, \vee)$, $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ *holds for all* $a, b, c \in L$.

Proof. We have given that L is distributive, so using distributive property, we have

$$\begin{aligned}(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) &= \{[(a \vee b) \vee b] \wedge [(a \wedge b) \vee c)]\} \vee (c \wedge a)\\ &= [b \wedge \{(a \vee c) \wedge (b \vee c)\}] \vee (c \wedge a)\\ &= [(a \vee c) \wedge \{b \wedge (b \vee c)\}] \vee (c \wedge a)\\ &= [(a \vee c) \wedge b] \vee (c \wedge a)\\ &= [(a \vee c) \vee (c \wedge a)] \wedge [b \vee (c \wedge a)]\\ &= [\{(a \vee c) \vee c\} \wedge \{(a \vee c) \vee a\}] \wedge [(b \vee c) \wedge (b \vee a)]\\ &= (a \vee c) \wedge (a \vee c) \wedge (b \vee c) \wedge (a \vee b)\\ &= (a \vee c) \wedge (b \vee c) \wedge (a \vee b) = \text{R.H.S.}\end{aligned}$$

JOIN-IRREDUCIBLE

Consider a lattice $(L, \wedge, \vee)$. An element $a \in L$ is called join-irreducible if it can be expressed as the join of two distinct elements of L *i.e.*, $a \in L$ is join-irreducible if

$$a = x \vee y \Rightarrow a = x \text{ or } a = y, \text{ where } x, y \in L.$$

Example 32. *Determine the join-irreducible elements of the lattices as shown in Fig. 32.*

Sol. 1. The join-irreducible elements of Fig. 32(I) are a, b and d.

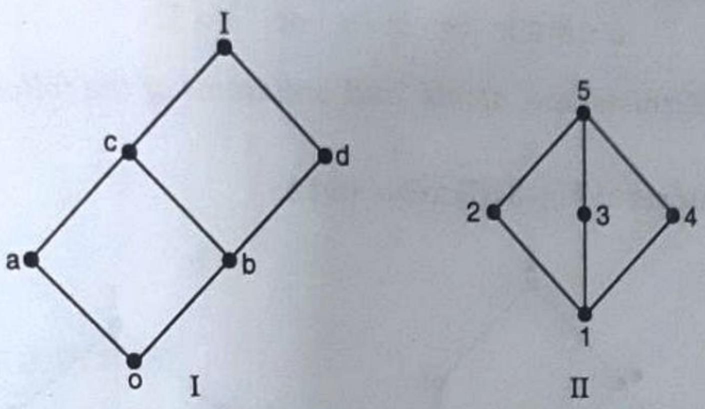

Fig. 32.

2. The join-irreducible elements of Fig. 32(II) are 2, 3 and 4.

MEET-IRREDUCIBLE

Consider a lattice (L, $\wedge$, $\vee$). An element $a \in$ L is called meet-irreducible if it can be expressed as the meet of two distinct elements of L *i.e.*, $a \in$ L is meet-irreducible if

$$a = x \wedge y \;\Rightarrow\; a = x \text{ or } a = y,$$

where $x, y \in$ L.

Example 33. *Determine the meet-irreducible elements of the lattices as shown in Fig. 33.*

Sol. 1. The meet-irreducible elements of Fig. 33(I) are a, b and c.

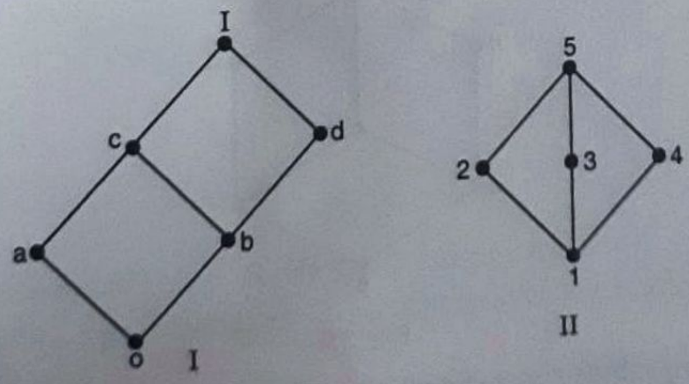

Fig. 33.

2. The meet-irreducible elements of Fig. 33(II) are 2, 3 and 4.

ATOM

Consider a lattice (L, $\wedge$, $\vee$) with a lower bound (1). An element a is called an atom if it is an immediate successor of 0 *i.e.*, $a \neq 0$ is an atom if $0 \le b \le a \Rightarrow b = 0$ or $b = a$.

ANTIATOM

Consider a lattice (L, $\wedge$, $\vee$) with an upper bound 1. An element a is called an antiatom if it is an immediate predecessor of 1 *i.e.*, $a \neq 1$ is an antiatom if

$$a \leq b \leq 1 \;\Rightarrow\; b = a \quad \text{or} \quad b = 1.$$

Example 34. *Determine the atoms and antiatoms of the following lattices shown in Fig. 34.*

Sol. Atoms. 1. Atoms of Fig. 34(I) are a and b.

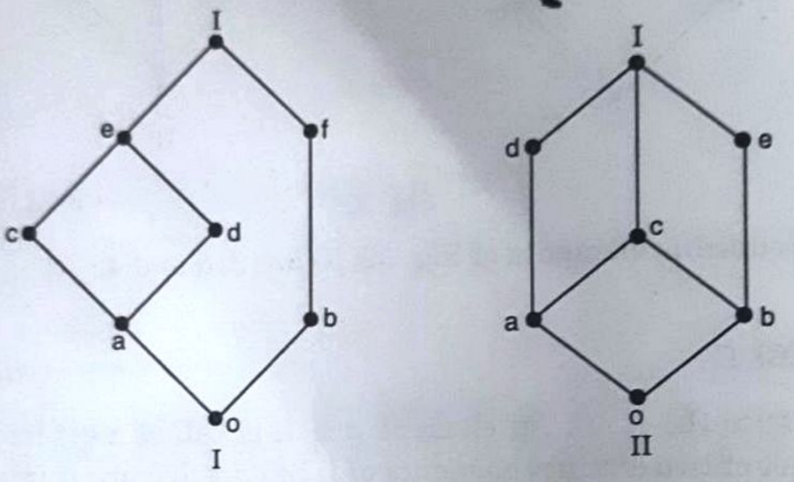

Fig. 34.

2. Atoms of Fig. 34(II) are a and b.

Antiatoms. 1. Antiatoms of Fig. 34(I) are e and f.

2. Antiatoms of Fig. 34(II) are d and e.

Example 35. *Construct the meet and join table of the lattice* (L, $\vee$, $\wedge$) *as shown in Fig. 35.*

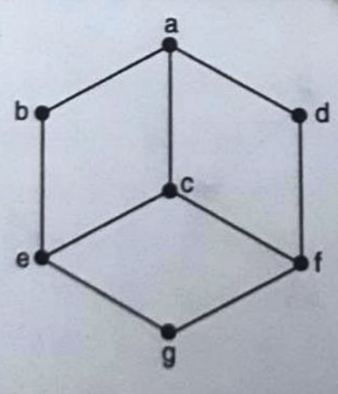

Fig. 35.

Sol. The following tables show the meet and join table of the lattice (L, $\wedge$, $\vee$).

$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

COMPLEMENTED LATTICES

Consider a bounded lattice L with greatest element I and the least element 0. An element $x \in$ L is called a complement of x if $x \vee x' = 1$ and $x \wedge x' = 0$.

From the definition of complement, if x' is a complement of x, then x is a complement of x'. It is not necessary that an element x has a complement. Also the complements need not be unique *i.e.*, an element have more than one complement.

Note. That $1' = 0$ and $0' = 1$.

Definition. A lattice L is called a complemented lattice if L is bounded and every element in L has a complement.

Example 36. *Determine the complement of a and c in Fig. 36.*

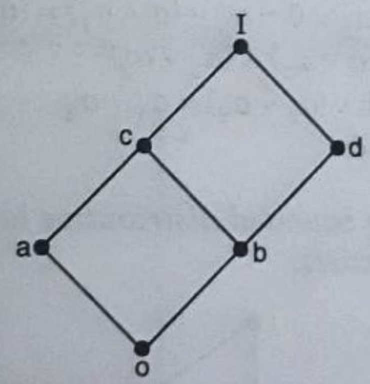

Fig. 36.

Sol. The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$.

The complement of c does not exist. Since, there does not exist any element c such that $c \vee c' = 1$ and $c \wedge c' = 0$.

Theorem VIII. *Prove that 0 and 1 are complement of each other.*

Proof. To show that 1 is the only complement of 0, consider that $c \neq 1$ is a complement of 0 and $c \in$ L.

Then, $\quad 0 \wedge c = 0 \quad$ and $\quad 0 \vee c = 1 \quad$ [Property of bounded lattice]

But $\quad 0 \vee c = c$

and $c \neq 1$ leads to a contradiction.

Similarly, we can show that 0 is the only complement of 1.

Example. The power set P(S) of the set S under the operations of intersection and union is a complemented lattice L since each element of L has a unique complement.

Example. The lattices shown below are complemented lattices Fig. 37.

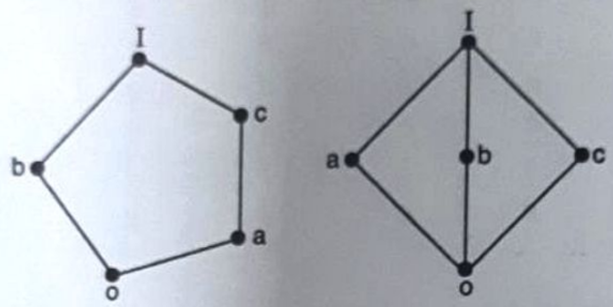

Fig. 37.

But the complements of some of the elements are not unique *e.g.*, b has two complements a and c in both the cases.

Theorem IX. *Prove that for a bounded distributive lattice L, the complements are unique if they exist.*

Proof. Consider a_1 and a_2 be complements of some elements $a \in$ L. Then, we have

$$a \vee a_1 = I \quad \text{and} \quad a \vee a_2 = I$$

Also $$a \wedge a_1 = 0 \quad \text{and} \quad a \wedge a_2 = 0$$

Now using the distributive property, we have

$$\begin{aligned} a_1 &= a_1 \vee 0 = a_1 \vee (a \wedge a_2) \\ &= (a_1 \vee a) \wedge (a_1 \vee a_2) = (a \vee a_1) \wedge (a_1 \vee a_2) = I \wedge (a_1 \vee a_2) = a_1 \vee a_2 \end{aligned}$$

Similarly, $$\begin{aligned} a_2 &= a_2 \vee 0 = a_2 \vee (a \wedge a_1) = (a_2 \vee a) \wedge (a_2 \vee a_1) \\ &= (a \vee a_2) \wedge (a_1 \vee a_2) \\ &= I \wedge (a_1 \vee a_2) = a_1 \vee a_2 \end{aligned}$$

Thus, $a_1 = a_2$. Hence proved.

Example 37. *Consider the bounded distributive lattice as shown in Fig. 38. Show that every complement is unique if it exists.*

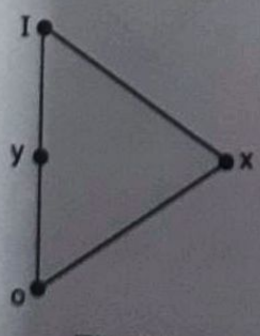

Fig. 38.

Sol. The complement of x is y and vice-versa. Similarly, the complement of 0 is I and vice-versa. Hence, all the complements are unique.

MODULAR LATTICE

A lattice (L, $\wedge$, $\vee$) is called a modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$.

Theorem X. *Show that* $a \vee (b \wedge c) = (a \vee b) \wedge c$ *whenever* $a \leq c$.

Proof. If $a \le c$, then $a \vee c = c$

If L is distributive, then, we have

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c \qquad | \because a \vee c = c$$

Note. (*i*) Every distributive lattice is modular.

(*ii*) Every modular lattice is not distributive.

Example. The lattice shown in Fig. 39 is a non-distributive modular lattice.

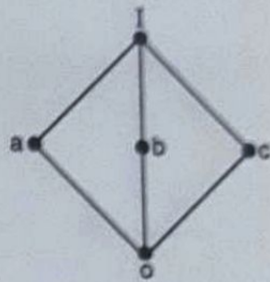

Fig. 39.

We can check that the lattice shown in Fig. 39 is not distributive.

$$a \wedge (b \vee c) = a \wedge 1 = a$$

but $$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

To, check the modularity holds for $a = c$.

DIRECT PRODUCT OF LATTICES

Let $(L_1, \vee_1, \wedge_1)$ and $(L_2, \vee_2, \wedge_2)$ be two lattices. Then $(L, \vee, \wedge)$ is the direct product of lattices, where $L = L_1 \times L_2$ in which the binary operations $\vee$(join) and $\wedge$(meet) on L are such that for any (a_1, b_1) and (a_2, b_2) in L.

$$(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee_1 a_2, b_1 \vee_2 b_2)$$

and $$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge_1 a_2, b_1 \wedge_2 b_2).$$

Example 38. *Consider a lattice $(L, \le)$ as shown in Fig. 40, where $L = \{1, 2\}$. Determine the lattice $(L^2, \le)$, where $L^2 = L \times L$.*

Fig. 40.

Sol. The lattice $(L^2, \le)$ is as shown in Fig. 41.

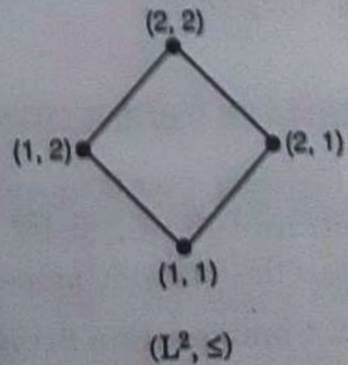

Fig. 41.