Q: Let $x_1, x_2 \ldots x_n \overset{iid}{\sim} Bin(n, p)$ where n is known and p is unknown. Find the MLE for p?

Sol^n

If $X \sim Bin(n, p)$

$$f_X(x,p) = P_X(x) = \binom{n}{x} p^x q^{n-x} \; ; \quad x = 0, 1, 2 \ldots n$$
$$q = 1-p$$

Likelihood function

$$L(p) = f(x_1, p) \cdot f(x_2, p) \cdots f(x_n, p)$$
$$= \prod_{i=1}^{n} f(x_i, p).$$
$$= \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$
$$L(p) = p^{\sum_{i=1}^{n} x_i} (1-p)^{n^2 - \sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} \binom{n}{x_i}$$

$$(1-p)^{(n-x_1)+(n-x_2)+\cdots(n-x_n)} = (1-p)^{\sum_{i=1}^{n}(n-x_i)} = (1-p)^{n^2 - \sum_{i=1}^{n} x_i}$$

Taking log

$$l = \ln L(p) = \left(\sum_{i=1}^{n} x_i\right) \cdot \ln p + \left(n^2 - \sum_{i=1}^{n} x_i\right) \ln(1-p) + \ln\left(\prod_{i=1}^{n} \binom{n}{x_i}\right)$$

$$\frac{\partial l}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n^2 - \sum_{i=1}^{n} x_i\right)}{(1-p)} + 0$$

$$\frac{\partial l}{\partial p} = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{p} = \frac{n^2 - \sum_{i=1}^{n} x_i}{1-p} \Rightarrow \sum_{i=1}^{n} x_i - \cancel{p \cdot \sum_{i=1}^{n} x_i} = n^2 p - \cancel{p \sum_{i=1}^{n} x_i}$$

$$\Rightarrow p = \frac{\sum_{i=1}^{n} x_i}{n^2} = \frac{1}{n} \bar{x}$$

Now
$$\frac{\partial^2 l}{\partial p^2} = -\frac{\sum_{i=1}^{n} x_i}{p^2} - \frac{\left(n^2 - \sum_{i=1}^{n} x_i\right)}{(1-p)^2} < 0 \text{ at } p = \frac{\bar{x}}{n}$$

Thus $\hat{p} = \frac{\bar{x}}{n}$ is MLE for p. #

Q: Let $X_1, X_2 \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

(i) μ is unknown, σ^2 known, find MLE for μ

(ii) μ is known, σ^2 unknown, find MLE for σ^2

(iii) μ and σ^2 both unknown; find MLE for μ and σ^2

Solⁿ (i) σ^2 is known (constant)

Put $\sigma^2 = \sigma_0^2$ (constant)

so if $X \sim N(\mu, \sigma_0^2)$ then

$$f_X(\mu) = \frac{1}{\sigma_0\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma_0^2}}; \quad x \in \mathbb{R}, \ \mu \in \mathbb{R}$$

The likelihood function for μ

$$L(\mu) = f(x_1, \mu) \cdot f(x_2, \mu) \cdots f(x_n, \mu)$$

$$= \prod_{i=1}^{n} f(x_i; \mu)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma_0^2}}$$

$$= \left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n \cdot e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma_0^2}}$$

$$l = \ln L(\mu) = -n \ln(\sigma_0\sqrt{2\pi}) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma_0^2}$$

$$\frac{\partial l}{\partial \mu} = 0 - \sum_{i=1}^{n}\frac{\cancel{2}(x_i-\mu)}{\cancel{2}\sigma_0^2}\cdot(-1)$$

$$= \sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma_0^2}$$

For max; $\frac{\partial l}{\partial \mu} = 0 \Rightarrow \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma_0^2}\right) = 0$

$$\Rightarrow \sum_{i=1}^{n}(x_i-\mu) = 0 \Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\Rightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{n}{\sigma_0^2} < 0 \text{ at } \mu = \bar{x}$$

Thus $\hat{\mu} = \bar{x}$ is MLE for μ

(ii) ③

μ is Known ; σ^2 is unknown (We have to find estimate for σ^2 not for σ)

Put $\mu = \mu_0$ and $\sigma^2 = \vartheta$

Then if $X \sim N(\mu_0, \vartheta)$

$$f_X(\vartheta) = \frac{1}{\sqrt{2\pi \cdot \vartheta}} e^{-\frac{(x-\mu_0)^2}{2\vartheta}} \quad ; \quad \vartheta > 0, \; x \in \mathbb{R}$$

$$L(\vartheta) = f(x_1, \vartheta) \cdot f(x_2, \vartheta) \cdots f(x_n, \vartheta)$$

$$= \prod_{i=1}^{n} f(x_i, \vartheta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \cdot \vartheta}} e^{-\frac{(x_i-\mu_0)^2}{2\vartheta}}$$

$$= \left(\frac{1}{\sqrt{2\pi\vartheta}}\right)^n \cdot e^{-\sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\vartheta}}$$

$$l = \ln L(\vartheta) = -n \ln(\sqrt{2\pi\vartheta}) - \sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\vartheta}$$

$$l = -\frac{n}{2} \ln(2\pi\vartheta) - \sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\vartheta}$$

$$\frac{\partial l}{\partial \vartheta} = -\frac{n}{2} \cdot \frac{\cancel{2\pi}}{\cancel{2\pi} \cdot \vartheta} + \frac{\sum_{i=1}^{n} (x_i-\mu_0)^2}{2\vartheta^2}$$

$$= -\frac{n}{2\vartheta} + \sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{2\vartheta^2}$$

$$\frac{\partial l}{\partial \vartheta} = 0 \Rightarrow \quad \vartheta = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu_0)^2 \quad — (*)$$

Now; $$\frac{\partial^2 l}{\partial \vartheta^2} = \frac{n}{2\vartheta^2} - \sum_{i=1}^{n} \frac{(x_i-\mu_0)^2}{\vartheta^3}$$

$$\left.\frac{\partial^2 l}{\partial \vartheta^2}\right|_{\vartheta = (*)} = \frac{n^2}{2 \sum_{i=1}^{n} (x_i-\mu_0)^2} - \frac{n^3 \sum_{i=1}^{n} (x_i-\mu_0)^2}{\left(\sum_{i=1}^{n} (x_i-\mu_0)^2\right)^3}$$

$\because$ second term is more dominating term so

$\frac{\partial^2 l}{\partial \vartheta^2} < 0$ thus $\hat{\vartheta} = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu_0)^2$ is MLE for ϑ

Thus $\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu_0)^2$ is MLE for σ^2 #

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(4)

(iii) Both are unknown

If $X \sim N(\mu, \sigma^2)$

$$f_X(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

For our convenience we will take $\sigma^2 = v$

then

$$L(\mu, v) = f(x_1, \mu, v) \cdot f(x_2, \mu, v) \dots f(x_n, \mu, v)$$

$$= \prod_{i=1}^{n} f(x_i, \mu, v)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi v}} \cdot e^{-\frac{(x_i-\mu)^2}{2v}}$$

$$L(\mu, v) = \left(\frac{1}{\sqrt{2\pi v}}\right)^n \cdot e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2v}}$$

$$l = \ln(L(\mu, v)) = -\frac{n}{2} \ln(2\pi v) - \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2v}$$

$$\cancel{\frac{\partial l}{\partial v} = -\frac{n}{2} \cdot 1}$$

Now partially diff w.r.t μ and v respectively

$$\frac{\partial l}{\partial \mu} = 0 - \sum_{i=1}^{n} \frac{\cancel{2}(x_i-\mu) \cdot (-1)}{\cancel{2} v}$$

$$\frac{\partial l}{\partial \mu} = \sum_{i=1}^{n} \left(\frac{x_i-\mu}{v}\right) \quad \text{eq}^n (1)$$

Now;

$$\frac{\partial l}{\partial v} = -\frac{n}{2} \cdot \frac{1 \cdot \cancel{2\pi}}{\cancel{2\pi} v} + \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2v^2}$$

$$\frac{\partial l}{\partial v} = -\frac{n}{2v} + \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2v^2} \quad - \text{eq}^n (2)$$

Now from eq^n (1) $\frac{\partial l}{\partial \mu} = 0 \Rightarrow \mu = \bar{x}$

Now, putting the value of μ in eq^n (2)

$$\frac{\partial l}{\partial v} = -\frac{n}{2v} + \sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{2v^2}$$

Now $\frac{\partial l}{\partial v} = 0 \Rightarrow v = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{n}{v} = \frac{-n}{\sum_{i=1}^{n}(x_i-\bar{x})^2} < 0$$

Also $$\frac{\partial^2 l}{\partial v^2} = \frac{n}{2v^2} - \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2v^3}$$

Again second term will be more dominating at value of v. so $\frac{\partial^2 l}{\partial v^2} < 0$

Thus $\hat{\mu} = \bar{x}$ and $\hat{v} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$

is MLE for μ and v and so for μ and σ^2.

Thus

<u>Result</u>: Invariance Properties of MLE :-

Let MLE of θ be $\hat{\theta}$. Consider $g(\theta)$ be one - one function from Θ then MLE of $g(\theta)$ be $g(\hat{\theta})$.

Ex : If $\bar{X}$ is MLE for θ then $\bar{X}^3$ will be MLE for θ^3 #