imp
Q1

Lecture 7 - 8 ①

Show that

$$\text{corr}(aX+b,\ cY+d) = \text{corr}(X,Y)$$

Proof: $\because$ $\text{corr}(X,Y) = \dfrac{\text{cov}(X,Y)}{\sqrt{V(X)}\cdot\sqrt{V(Y)}}$ $\left(\sigma_x = \sqrt{V(X)},\ \sigma_y = \sqrt{V(Y)}\right)$

So; $\text{corr}(aX+b,\ cY+d) = \dfrac{\text{cov}(aX+b,\ cY+d)}{\sqrt{V(aX+b)}\ \sqrt{V(cY+d)}}$

$$= \frac{E(aX+b)(cY+d) - E(aX+b).E(cY+d)}{\sqrt{V(aX+b)}\ \sqrt{V(cY+d)}} \quad \because \text{cov}(X,Y) = EXY - EXEY$$

$$= \frac{E(acXY + adX + bcY + bd) - E(aX+b)\,E(cY+d)}{\sqrt{a^2V(X)}\ \sqrt{c^2V(Y)}} \quad \because V(aX+b) = a^2V(X)$$

$$= \frac{ac\,E(XY) + ad\,E(X) + bc\,E(Y) + bd - (a\,E(X)+b)(c\,E(Y)+d)}{a\sqrt{V(X)}\cdot c\sqrt{V(Y)}}$$

$$= \frac{ac\,E(XY) + \cancel{ad\,E(X)} + \cancel{bc\,E(Y)} + \cancel{bd} - ac\,EX.EY - \cancel{ad\,EX} - \cancel{bc\,E(Y)} - \cancel{bd}}{ac.\sigma_x.\sigma_y}$$

$$= \frac{\cancel{ac}\,(E(XY) - E(X).E(Y))}{\cancel{ac}\,\sigma_x.\sigma_y}$$

$$= \frac{E(XY) - E(X).E(Y)}{\sigma_x.\sigma_y}$$

$$= \text{corr}(X,Y) \quad \#$$

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Result: ②

Show that

$$V(aX+bY) = a^2V(X) + b^2V(Y) + 2ab\ \mathrm{cov}(X,Y)$$

Proof:

$\because V(X) = EX^2 - (EX)^2$

$$V(aX+bY) = E(aX+bY)^2 - \left(E(aX+bY)\right)^2$$

$$= E(a^2X^2 + b^2Y^2 + 2abXY) - (aEX + bEY)^2$$

$$= a^2EX^2 + b^2EY^2 + 2ab\,EXY - a^2(EX)^2 - b^2(EY)^2 - 2ab\,EX\cdot EY$$

$$= a^2\left(EX^2 - (EX)^2\right) + b^2\left(EY^2 - (EY)^2\right) + 2ab\left(EXY - EX\cdot EY\right)$$

$$= a^2V(X) + b^2V(Y) + 2ab.\ \mathrm{cov}(X,Y) \quad \#$$

Do it

Result! $V(aX-bY) = a^2V(X) + b^2V(Y) - 2ab\ \mathrm{cov}(X,Y)$

Result: $\mathrm{cov}(aX, bY) = ab.\ \mathrm{cov}(X,Y)$

Q!
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For any two r.v X, Y ② show that

$-1 \leq corr(X,Y) \leq 1$; i.e range of correlation coefficient is between 1 and -1.

Proof: We know that $V(X) \geq 0$

Consider,

$$V\left(\frac{X}{\sigma_x} + \frac{Y}{\sigma_y}\right) \geq 0$$; where σ_x and σ_y are standard deviation of X and Y respec.

$\because V(aX+bY) = a^2V(X)+b^2V(Y)+2ab\,cov(X,Y)$

$$\Rightarrow V\left(\frac{X}{\sigma_x}\right) + V\left(\frac{Y}{\sigma_y}\right) + 2.cov\left(\frac{X}{\sigma_x}, \frac{Y}{\sigma_y}\right) \geq 0$$

$$\Rightarrow \frac{V(X)}{\sigma_x^2} + \frac{V(Y)}{\sigma_y^2} + 2.\frac{1}{\sigma_x \sigma_y} cov(X,Y) \geq 0$$

$\because V(aX)=a^2V(X)$
$Cov(aX,bY)$
$= ab\,cov(X,Y)$
You can easily prove it

$$\Rightarrow \frac{V(X)}{V(X)} + \frac{V(Y)}{V(Y)} + 2.\frac{1}{\sigma_x.\sigma_y} cov(X,Y) \geq 0$$

$$\Rightarrow 1+1+2\,corr(X,Y) \geq 0$$

$$\Rightarrow corr(X,Y) \geq -1 \quad \text{— (i)}$$

Similarly, consider

$$V\left(\frac{X}{\sigma_x} - \frac{Y}{\sigma_y}\right) \geq 0$$

$\because V(aX-bY) = a^2V(X)+b^2V(Y)-2ab\,cov(X,Y)$

$$\Rightarrow \frac{1}{\sigma_x^2}.V(X) + \frac{1}{\sigma_y^2}V(Y) - 2.\frac{1}{\sigma_x}.\frac{1}{\sigma_y} cov(X,Y) \geq 0$$

$$\Rightarrow 1 + 1 - 2\,corr(X,Y) \geq 0$$

$$\Rightarrow corr(X,Y) \leq 1 \quad \text{— (ii)}$$

From (i) and (ii) $\boxed{-1 \leq corr(X,Y) \leq 1}$

EXERCISES Section 5.1 (1–21)

1. A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint pmf of X and Y appears in the accompanying tabulation.

$p(x, y)$		y = 0	1	2
x	0	.10	.04	.02
	1	.08	.20	.06
	2	.06	.14	.30

a. What is $P(X = 1 \text{ and } Y = 1)$?
b. Compute $P(X \leq 1 \text{ and } Y \leq 1)$.
c. Give a word description of the event $\{X \neq 0 \text{ and } Y \neq 0\}$, and compute the probability of this event.
d. Compute the marginal pmf of X and of Y. Using $p_X(x)$, what is $P(X \leq 1)$?
e. Are X and Y independent rv's? Explain.

2. When an automobile is stopped by a roving safety patrol, each tire is checked for tire wear, and each headlight is checked to see whether it is properly aimed. Let X denote the number of headlights that need adjustment, and let Y denote the number of defective tires.
a. If X and Y are independent with $p_X(0) = .5$, $p_X(1) = .3$, $p_X(2) = .2$, and $p_Y(0) = .6$, $p_Y(1) = .1$, $p_Y(2) = p_Y(3) = .05$, and $p_Y(4) = .2$, display the joint pmf of (X, Y) in a joint probability table.
b. Compute $P(X \leq 1 \text{ and } Y \leq 1)$ from the joint probability table, and verify that it equals the product $P(X \leq 1) \cdot P(Y \leq 1)$.
c. What is $P(X + Y = 0)$ (the probability of no violations)?
d. Compute $P(X + Y \leq 1)$.

3. A certain market has both an express checkout line and a superexpress checkout line. Let X_1 denote the number of customers in line at the express checkout at a particular time of day, and let X_2 denote the number of customers in line at the superexpress checkout at the same time. Suppose the joint pmf of X_1 and X_2 is as given in the accompanying table.

		x_2 = 0	1	2	3
x_1	0	.08	.07	.04	.00
	1	.06	.15	.05	.04
	2	.05	.04	.10	.06
	3	.00	.03	.04	.07
	4	.00	.01	.05	.06

a. What is $P(X_1 = 1, X_2 = 1)$, that is, the probability that there is exactly one customer in each line?
b. What is $P(X_1 = X_2)$, that is, the probability that the numbers of customers in the two lines are identical?
c. Let A denote the event that there are at least two more customers in one line than in the other line. Express A in terms of X_1 and X_2, and calculate the probability of this event.

1 5.1

Sol^n

(a) $P(X=1 \text{ and } Y=1) = 0.20$

(b) $P(X \leq 1, Y \leq 1) =$

X	Y
0	0
1	1

$= P(0,0) + P(0,1) + P(1,0) + P(1,1)$

$= 0.10 + 0.04 + 0.08 + 0.20$

(c) $P(X \neq 0, Y \neq 0) = 1 - P(X=0, Y=0)$

$= 1 - 0.10 = 0.90$

(d) Marginal of X

$P_X(x) = \sum_{y \in R_Y} P_{X,Y}(x,y) \quad ; \quad x \in R_X$

$x \to 0, 1, 2 \quad ; y \to 0, 1, 2$

$P_X(0) = \sum_{y=0}^{2} P_{X,Y}(0,y)$

$= P(0,0) + P(0,1) + P(0,2)$

$= 0.10 + 0.04 + 0.02$

$P_X(1) = \sum_{y=0}^{2} P_{X,Y}(1,y) = P(1,0) + P(1,1) + P(1,2)$

$= 0.08 + 0.20 + 0.06$

$P_X(2) = \sum_{y=0}^{2} P_{X,Y}(2,y) = P(2,0) + P(2,1) + P(2,2)$

$= 0.06 + 0.14 + 0.30$

X	0	1	2
$P_X(x)$	$P_X(0)$	$P_X(1)$	$P_X(2)$

; Similarly Marginal of Y

(e) $\because P_{X,Y}(x,y) \neq P_X(x) \cdot P_Y(y) \quad \forall x,y \in R_{X \times Y}$

$\Rightarrow X$ and Y are not independent

2 (5.1)

Sol^n

$\because$ X and Y are independent

$\Rightarrow P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$

Here

$P_X(0) = 0.5$; $P_X(1) = 0.3$; $P_X(2) = 0.2$

$P_Y(0) = 0.6$; $P_Y(1) = 0.1$; $P_Y(2) = P_Y(3) = 0.05$

X \ Y	0	1	2	3
0				
1				
2				

$P_{X,Y}(0,0) = P_X(0) \cdot P_Y(0) = 0.5 \times 0.6 = 0.30$

$P_{X,Y}(0,1) = P_X(0) \cdot P_Y(1) = 0.5 \times 0.1 = 0.05$

$P(0,2) = P(0) \cdot P(2) = 0.5 \times 0.05 = .025 = P(0,3)$

$P(1,0) = P(1) \cdot P(0) = 0.3 \times 0.6 = 0.18$

$P(1,1) = P(1)\, P(1) = 0.3 \times 0.1 = 0.03$

$P(1,2) = 0.3 \times 0.05 = 0.015$

$P(1,3) = - - -$

(b) $P(X \leq 1, Y \leq 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$

X	Y
0	0
1	1

$P(X \leq 1) = P(0) + P(1)$

$P(Y \leq 1) = P(0) + P(1)$

(c) $P(X+Y=0) = P(0,0)$

X	Y	X+Y
0	0	0

(d) $P(X+Y \leq 1)$

$= P(0,0) + P(1,0) + P(0,1)$

X	Y	X+Y
0	1	1
1	0	1
0	0	0

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~~End sem~~

Suppose that the r.v X and Y have a joint density function given by

$$f_{X,Y}(x,y) = \begin{cases} c(2x+y) & ; 2<x<6 ;\ 0<y<5 \\ 0 & ; \text{otherwise} \end{cases}$$

find 'c', the marginal of X and Y and $P(X+Y>4)$?

Soln

$\because \displaystyle\int_{x=2}^{6}\int_{y=0}^{5} f_{X,Y}(x,y)\,dy\,dx = 1$ ($\because$ it is density)

$$\Rightarrow c\int_{x=2}^{6}\int_{y=0}^{5}(2x+y)\,dy\,dx = 1 \Rightarrow c\int_{x=2}^{6}\left[2xy+\frac{y^2}{2}\right]_0^5 dx = 1$$

$$\Rightarrow c\int_{x=2}^{6}\left(10x+\frac{25}{2}\right)dx = 1$$

$$\Rightarrow c\left[10\cdot\frac{x^2}{2}+\frac{25}{2}\cdot x\right]_2^6 = 1 \Rightarrow c\left[(5\times 36+75)-(20+25)\right] = 1$$

$$\Rightarrow c\,[210] = 1$$

$$\Rightarrow \boxed{c = \frac{1}{210}}$$

Now, for Marginal of X

$$f_X(x) = \int_{y=0}^{5}\frac{1}{210}(2x+y)\,dy$$

$$= \frac{1}{210}\left(2xy+\frac{y^2}{2}\right)_0^5$$

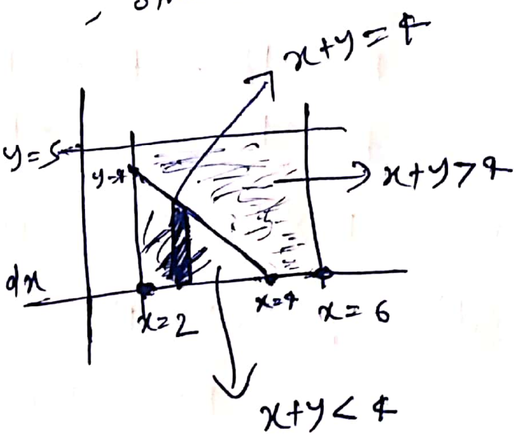
$$f_X(x) = \begin{cases} \frac{1}{210}\left(10x+\frac{25}{2}\right) & ; 2<x<6 \\ 0 & ; \text{otherwise} \end{cases}$$

Similarly, we can find $f_Y(y)$

$$P(X+Y>4) = 1 - P(X+Y\le 4)$$

$$= 1 - \int_{x=2}^{4}\int_{y=0}^{4-x}\frac{1}{210}(2x+y)\,dy\,dx$$

$$= \frac{11}{15}$$



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