Sol. *(i)* Start with six isolated vertices $v_1, v_2, v_2, v_4, v_5, v_6$.

For even-valued terms of the sequence, *i.e.*, for 2, 2, 4 draw one self-loop to each vertex v_3 and v_4 and two self-loops to the vertex v_6. Vertex v_1 remains isolated. For the odd-valued terms, *i.e.*, for 1 and 3, draw a self-loop to v_5 and then join the pair v_1, v_5 by a single-edge. The resulting graph is shown in Fig. 6.6*(a)*.

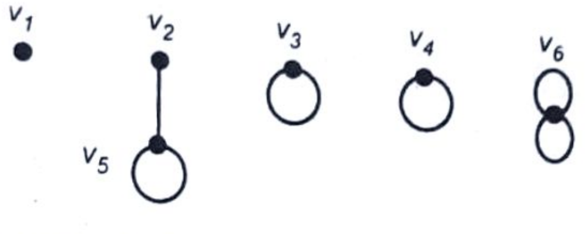


Fig. 6.6*(a)* : Graph with degree sequence (0, 1, 2, 2, 3, 4)

(ii) Similarly, we can construct the graph for the degree sequence (1, 1, 2, 3, 3, 4),and the resulting graph is shown in Fig. 6.6*(b)*.

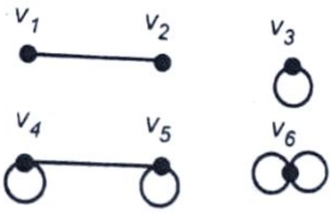


Fig. 6.6*(b)* : Graph with degree sequence (1, 1, 2, 3, 3, 4) □

6.3 Degree of a Vertex in Digraph

The definition of vertex degree in a digraph is slightly refined and is given as follows :

Let G be a digraph and v_i be the vertices in G.

The indegree of a vertex v_i in G is the number of edges incident into v_i, and is written as $deg^-(v_i)$.

The outdegree of a vertex v_i in G is the number of edges incident out of v_i, and is written as $deg^+(v_i)$.

In Fig. 6.7, for example,

$deg^+(v_1) = 2,\quad deg^-(v_1) = 1,$
$deg^+(v_2) = 2,\quad deg^-(v_2) = 2,$
$deg^+(v_3) = 3,\quad deg^-(v_3) = 1,$
$deg^+(v_4) = 2,\quad deg^-(v_4) = 2,$
$deg^+(v_5) = 1,\quad deg^-(v_5) = 3,$
$deg^+(v_6) = 1,\quad deg^-(v_6) = 2.$

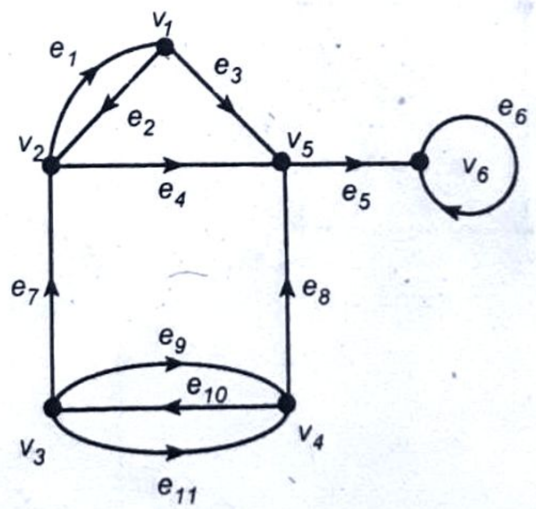


Fig. 6.7 : Digraph with six vertices and eleven edges

Theorem 6.4 *In a digraph G, with n vertices v_i, the sum of the indegrees and the some of the outdegrees both equal the number of edges. That is*

$$\sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v_i) = e, \quad \text{where } e \text{ is the number of edges in G.}$$

Proof. The result follows immediately by the fact that each directed edge contributes one to the indegree sum and one to the outdegree sum. □

Note that the above theorem is the digraph version of Euler's theorem on degree sum (Theorem 6.1).

In a digraph G, an **isolated vertex** is a vertex in which the indegree and the outdegree are both equal to zero.

In a digraph G, a vertex v_i is said to be **endvertex (Pendant)** if it is of degree one, *i.e.,* if

$$deg^{+}(v_i) + deg^{-}(v_i) = 1.$$

In a digraph G , two directed edges are said to be **multiple** or **parallel** if they are between the same ordered pair of vertices, that is, they are also directed in the same direction. For example, in Fig. 6.7, e_9 and e_{11} are parallel edges but e_9 and e_{10} are not parallel edges. Similarly e_1 and e_2 are also not parallel.

A digraph is called **symmetric digraph** if for every directed edge (u, v), there is also a directed edge (v, u).

Asymmetric or **antisymmetric digraphs** have atmost one directed edge between a pair of vertices, however they are allowed to have self-loops.

Recall that a digraph that has no self-loop or parallel edges is called a **simple digraph.**

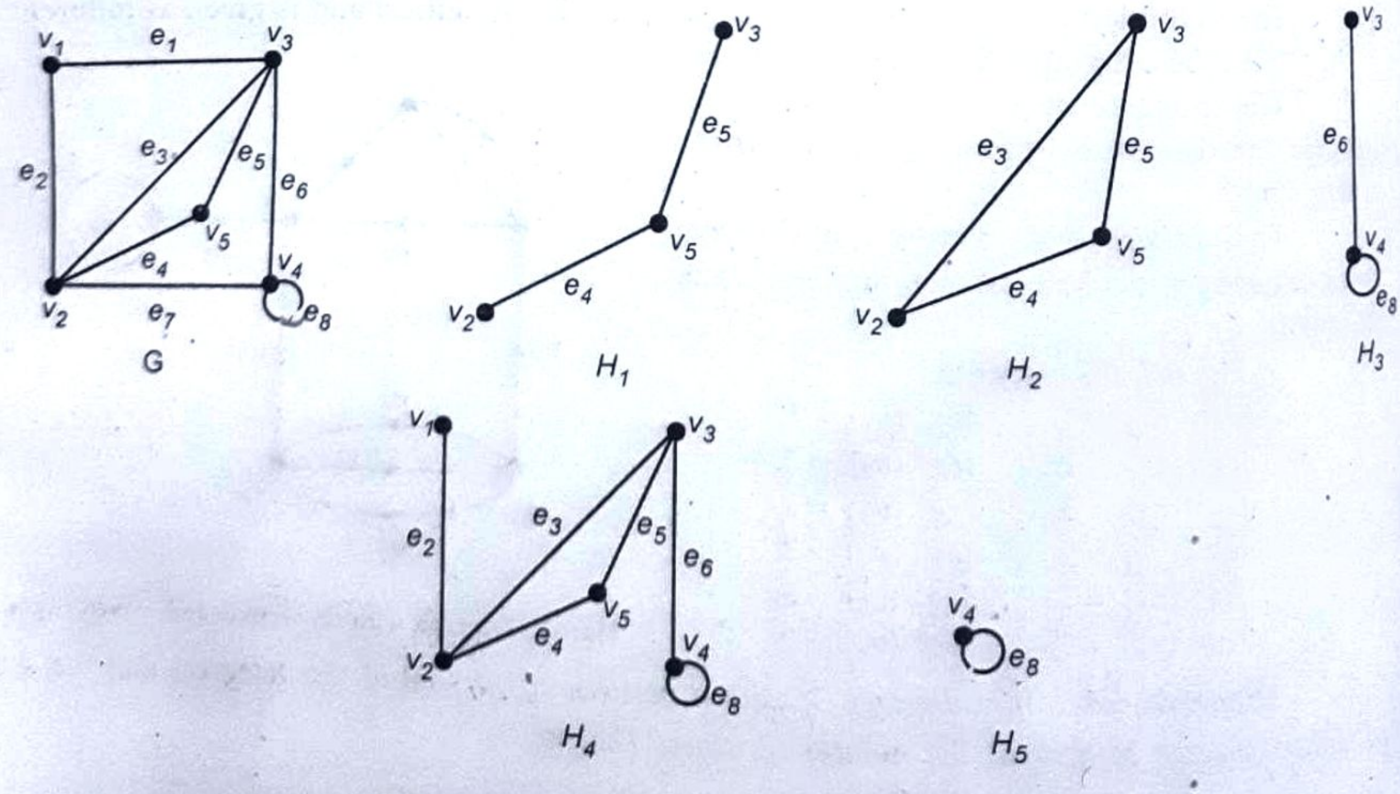


Fig. 6.8 : A graph G and its subgraphs H_1, H_2, H_3, H_4 and H_5

6.4 Subgraphs

A subgraph H of a graph G is a graph whose vertices and edges are all in G. If H is a subgraph of G, we say G is a **super graph** of H.

A subdigraph H of a digraph G is a digraph whose vertices and directed edges are all in G.

A **proper subgraph** H of G is a subgraph such that $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, *i.e.*, $V(H) \neq V(G)$ or $E(H) \neq E(G)$.

A subgraph H is said to **span** a graph G if $V(H) = V(G)$. The subgraph H is then called **spanning subgraph** of G.

For example, In fig. 6.8, the subgraphs H_1, H_2, H_3, H_4 and H_5 all are proper subgraphs of G, and subgraph H_4 spans G, *i.e.*, H_4 is a spanning subgraph of G.

For any subset U of $V(G)$,the **induced subgraph** $< U >$ is the maximal subgraph of G with vertex set U, *i.e.*, two vertices are adjacent in $< U >$ if and only if they are adjacent in G. Thus, for a given graph G, the **subgraph induced on a vertex set** U of $V(G)$is the subgraph of G whose vertex set is U and whose edge set consists of all edges in G that have both endvertices in U. For example, in Fig. 6.8, H_2 is the subgraph induced on the vertex subset $\{v_2, v_3, v_5\}$, and H_3 is the subgraph induced on the vertex subset $\{v_3, v_4\}$.

Two or more subgraphs of a graph G are said to be **edge-disjoint subgraphs** if they do not have any edges in common. For example, the subgraphs H_1 and H_3 are edge-disjoint subgraphs of the graph G shown in Fig. 6.8.

Subgraphs that do not have vertices in common are said to be **vertex-disjoint subgraphs.** For example, in Fig. 6.8, H_1 and H_5 are vertex-disjoint subgraphs of G.

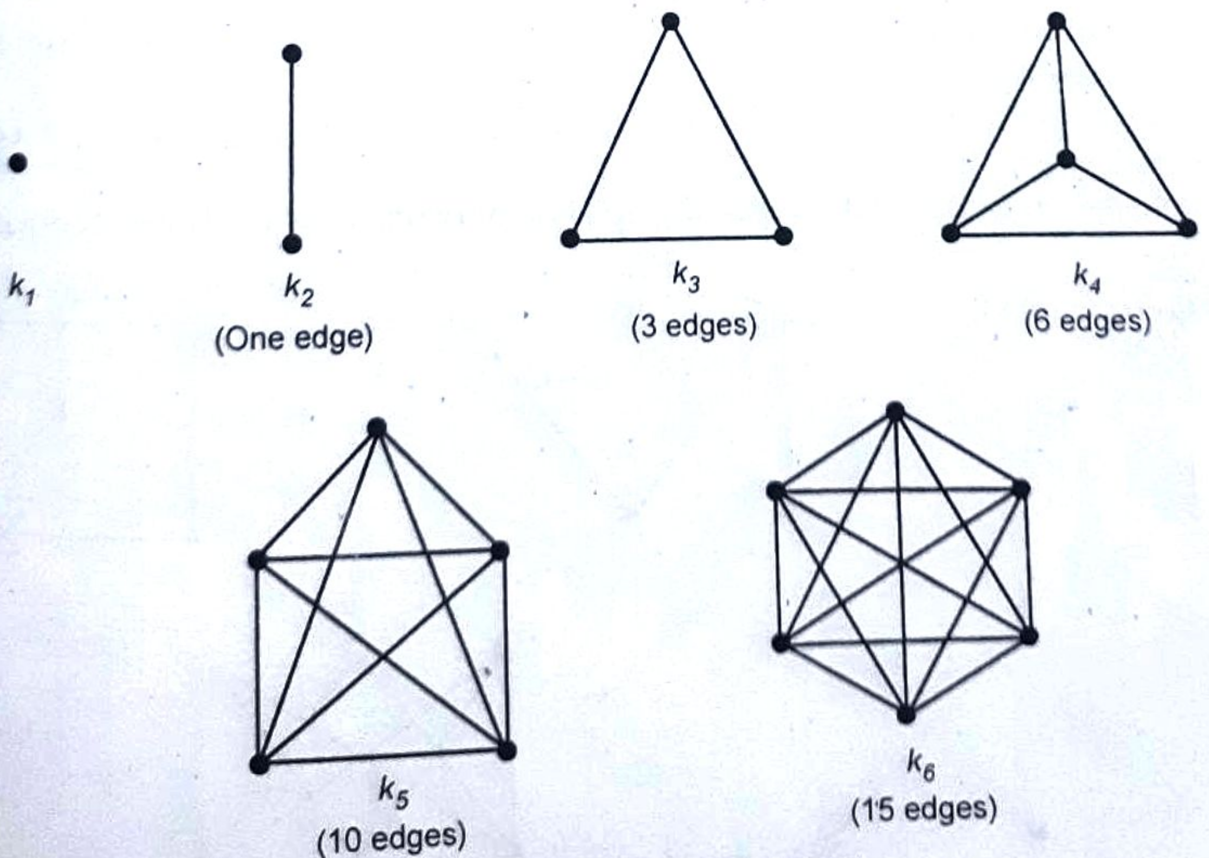


Fig. 6.9 : The first six complete graphs

Fig. 6.10 : Two complete digraphs

6.5 Complete Graphs, Digraphs, and Bigraphs

Complete Graphs

A **complete graph** is a simple graph in which each pair of distinct vertices is joined by an edge. Any complete graph on n vertices is denoted by K_n. It has every pair of its n vertices adjacent. Hence K_n has ${}^nC_2$ edges and is regular of degree $n - 1$.

For example, complete graphs on one, two, three, four, five, and six vertices are shown in Fig. 6.9. K_3 is usually called a triangle.

It follows easily from definitions that any simple graph on n vertices is a subgraph of the complete graph K_n.

A **tournament** is a digraph whose underlying graph is a complete graph.

Complete Digraphs

We have two types of complete digraphs.

A **complete symmetric digraph** is a simple digraph is which there is exactly one edge directed from every vertex to every other vertex.

A **complete asymmetric digraph** is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

For example, in Fig. 6.10, graphs G_1 is complete symmetric digraph and G_2 is a complete asymmetric digraph.

Note that a complete asymmetric digraph is a **tournament** or a complete tournament.

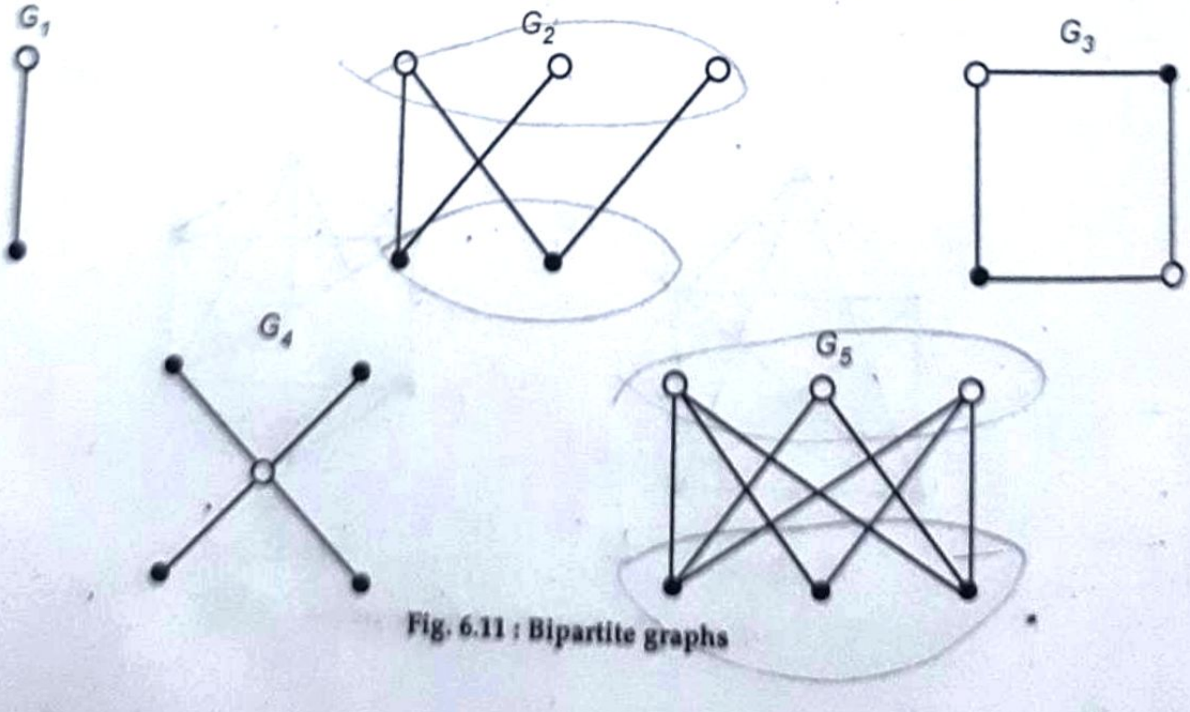


Fig. 6.11 : Bipartite graphs

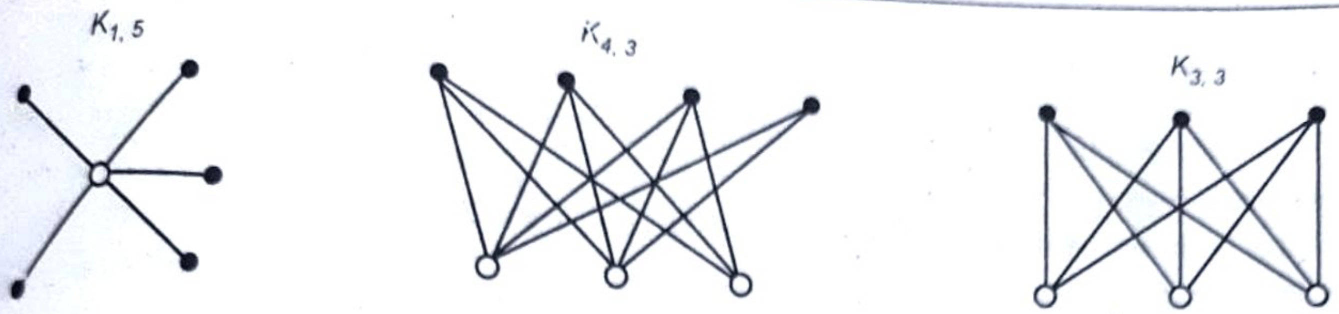


Fig. 6.12 : The complete bipartite graphs

Bipartite Graphs (Bigraphs)

A **bipartite graph** G is a graph whose vertex set $V(G)$ can be partioned into two subsets V_1 and V_2 such that every edge of G has one endvertex in V_1 and one endvertex in V_2.

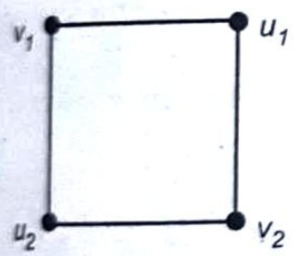
For example, graphs in Fig. 6.11 are bipartite graphs.

Note that by the definition a bipartite graph can not have any self-loop. And the smallest possible simple nontrivial graph that is not bipartite is the complete graph triangle, *i.e.,* K_3 (Fig. 6.9).

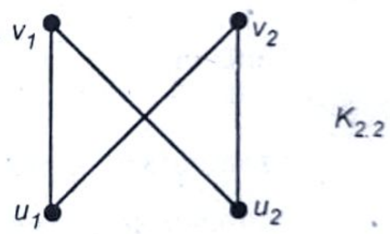
A bipartite graph G is said to be complete bipartite, if G contains every edge joining V_1 and V_2, *i.e.,* every vertex in V_1 is joined to every vertex in V_2. If V_1 and V_2 have m and n vertices respectively then the complete bipartite graph G is denoted by $K_{m,n}$.

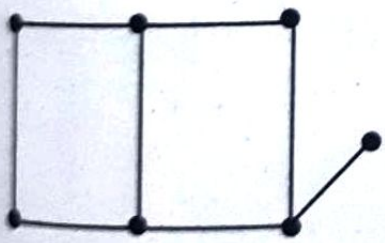
The complete bipartite graphs $K_{1,5}$, $K_{4,3}$, $K_{3,3}$ are shown in Fig. 6.12.

Example 6.5 *Redraw the following graphs to show that these are bipartite graphs :*

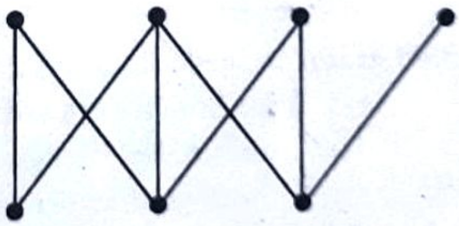


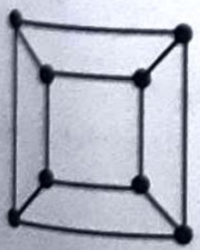
Redrawn as



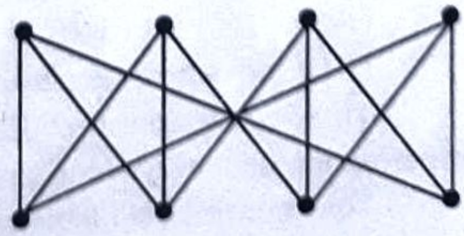


Redrawn as





Redrawn as

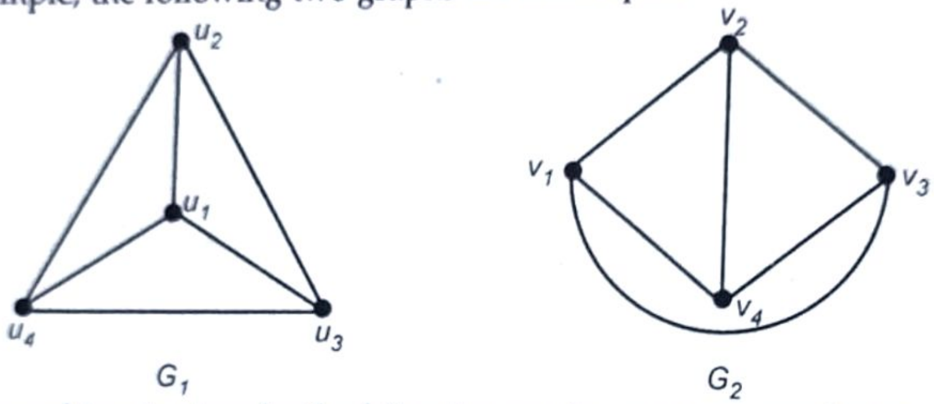


□

6.6 Graph Isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be **isomorphic**, if there exists a one-to-one correspondence between their vertex sets which preserves adjacency, that is, if e_1 is an edge with end vertices u_1 and v_1 in G_1 then the corresponding edge e_2 in G_2 has its endvertices u_2 and v_2 in G_2 which correspond to u_1 and v_1 respectively. Such a correspondence is called **graph isomorphism**. If G_1 and G_2 are isomorphic, we write $G_1 \equiv G_2$.

For example, the following two graphs are isomorphic :



The isomorphism is given by the following one-to-one correspondence of vertices :

$$u_i \leftrightarrow v_i \;, \quad i = 1, 2, 3, 4.$$

It is clear that the correspondence preserve adjacency. Hence

$$G_1 \equiv G_2.$$

Note that the relation "isomorphic to" is an equivalence relation on the set of all graphs.

Elementary Numerical Invariants

A **numerical graph invariant** is a numerical property of all graphs, such that any two isomorphic graphs have the same value.

We observe that some graph properties are preserved under isomorphism. These are called **graph invariants**. Some graph invariants are

(i) the number of vertices,

(ii) the number of edges,

(iii) the degree sequence, etc.

For example, if G and H are two isomorphic graphs, then

(i) $|V(G)| = |V(H)|$

(ii) $|E(G)| = |E(H)|$

(iii) the degree sequence of G and of H is same, *i.e.*, it is a graph invariant.

Thus, two graphs that differ in these properties can not be isomorphic. However, these properties are not sufficient to say that any two graphs are isomorphic.

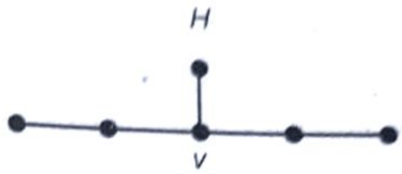
For example, the following two graphs are nonisomorphic graphs, although,

$$|V(G)| = |V(H)| = 6,$$

$$|E(G)| = |E(H)| = 5,$$

and degree sequence of G = degree sequence of H

$$= (1, 1, 1, 2, 2, 3)$$

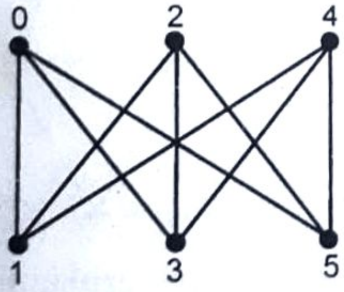


Here G and H are not isomorphic because, by definition vertex correspondence must be such that it preserves adjacency, which is not true in this case. We see that a graph isomorphism would have to map the vertex u in the graph G to the vertex v in the graph H since they are the only vertices of degree 3 in their respective graphs. However, the three adjacent vertices of u in G have degrees 1, 1 and 2, and the three adjacent vertices of v in H have degrees 1, 2, and 2.

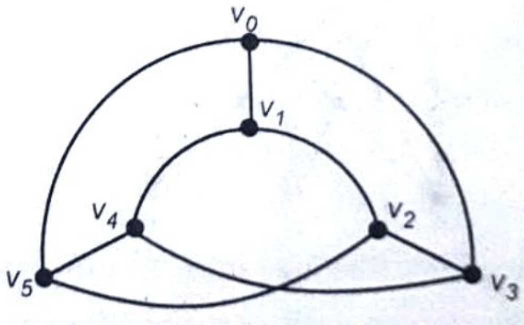
Hence, having the same values for arbitrarily many invariants is not assurance of isomorphism. However, if any two graphs that differ in even one such property are not isomorphic.

Example 6.6 *Show that the complete bipartite graph $K_{3,3}$ and the Möbius ladder ML_3 are isomorphic.*

Sol. The following two drawings represents the graphs $K_{3,3}$ and ML_3.



$K_{3,3}$



ML_3

The vertex labelings for the graph $K_{3,3}$ and the graph ML_3 specify a bijection that preserves adjacency. That is

$$0 \leftrightarrow v_0$$
$$1 \leftrightarrow v_1$$
$$2 \leftrightarrow v_2$$
$$3 \leftrightarrow v_3$$
$$4 \leftrightarrow v_4$$
$$5 \leftrightarrow v_5$$

Also

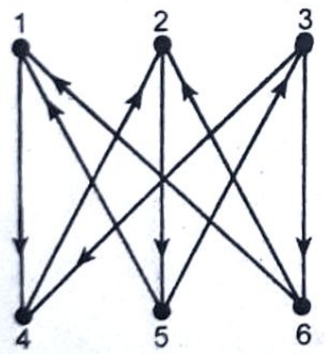
$$(0, 1) \leftrightarrow (v_0, v_1)$$
$$(0, 3) \leftrightarrow (v_0, v_3)$$
$$(0, 5) \leftrightarrow (v_0, v_5)$$
$$(2, 1) \leftrightarrow (v_2, v_1)$$
$$(2, 3) \leftrightarrow (v_2, v_3)$$
$$(2, 5) \leftrightarrow (v_2, v_5)$$
$$(4, 1) \leftrightarrow (v_4, v_1)$$
$$(4, 3) \leftrightarrow (v_4, v_3)$$
$$(4, 5) \leftrightarrow (v_4, v_5)$$

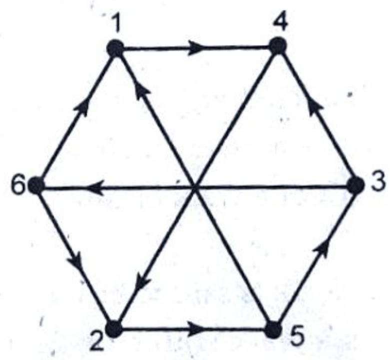
The $K_{3,3}$ and ML_3 are isomorphic. □

Isomorphism of Digraphs

Two digraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is an isomorphism between their underlying graphs that preserves the direction of each edge. That is, if e_1 is an edge in G_1 directed from u_1 to v_1 in G_1 then the corresponding edge e_2 in G_2 must be directed from u_2 to v_2 in G_2 which correspond to u_1 and v_1 respectively.

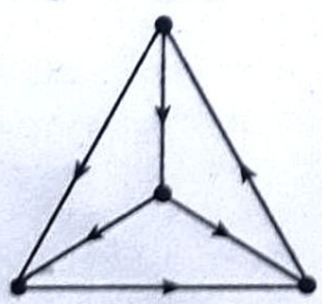
For example, the following two digraphs are isomorphic :

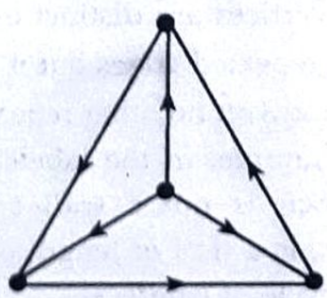




The vertex labelings show that the underlying graphs are isomorphic and arrows show that the direction of each corresponding edge is preserved. Hence these two digraphs are isomorphic.

The following two digraphs are nonisomorphic :





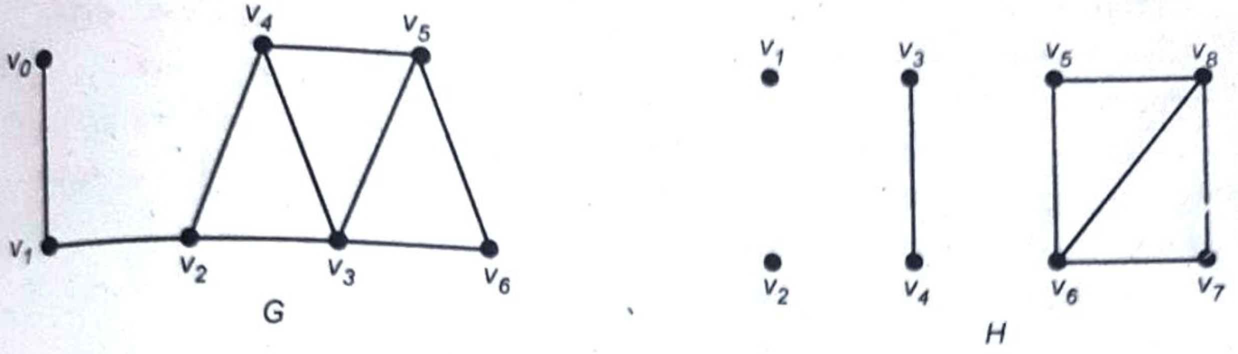


Fig. 6.13 : Connected graph G and disconnected graph H with four components

6.7 Walks and Connectedness

In a graph G, a **walk** from vertex v_0 to vertex v_n is an alternating sequence.

$$(v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n)$$

of vertices and edges such that endvertices of $e_i = \{v_{i-1}, v_i\}$, $i = 1, 2, ..., n$, that is each edge in the sequence is incident with two vertices immediately preceding and following it. Thus, a walk models a continuous traversal along some edges and vertices of a graph G. Above walk which joins v_0 and v_n is written as $v_0 - v_n$ walk and may also be denoted by $v_0v_1v_2...v_{n-1}v_n$. Note that in this walk v_0 and v_1 are adjacent, v_1 and v_2 are adjacent, and so on.

In a digraph G, a **directed walk** from v_0 to v_n is an alternating sequence

$$(v_0, e_1, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n)$$

of vertices and directed edges, such that tail of the edge $e_i = v_{i-1}$ and head of the edge $e_i = v_i$, $i = 1, 2, ..., n$, *i.e.*, e_i is a directed edge from the vertex v_{i-1} to the vertex v_i.

The **length** of a walk or directed walk is the number of occurrences of edges in the walk sequence.

A $v_0 - v_n$ walk is said to be **closed** if $v_0 = v_n$ and it is **open** otherwise.

A **trail** is a walk with no repeated edges, *i.e.*, all the edges in a trail are distinct.

A **path** is a trail with no repeated vertices, except possibly the initial and final. So path is a walk if all the verities are distinct and thus necessarily all graphs.

A walk, trail, or path is **trivial** if it has only one vertex and no edges.

A nontrivial closed path is called a **cycle**. Thus a walk which is closed is a cycle provided its vertices are distinct except the initial and the final vertex. A **circuit** is a closed walk with no repeated edges but it may have repeated vertices other than the end points. So a cycle is a circuit with no other repeated vertices other than the end points.

For example, in the labeled graph G of Fig. 6.13, $v_0v_1v_2v_4v_3v_2v_4$ is an open walk of length 6 which is not a trail and $v_2v_3v_4v_5v_3v_6v_5v_4v_2$ is a closed walk of length 8. $v_1v_2v_3v_5v_4v_3v_6$ is a trail of length 6 which is not a path. $v_1v_2v_4v_3v_6$ is a path of length 4 and $v_2v_3v_4v_2$ is a cycle of length 3.

The **girth** of a graph is the length of a shortest cycle, if any, in G.

The **circumference** of a graph G is the length of any longest cycle, if any, in G.

For example, the girth of G in Fig. 6.13 is 3 and the circumference of G is 5.

Directed trails, directed paths, and directed cycles are directed walks that satisfy conditions analogous to those of their undirected counterparts. For example, a directed trail is a directed walk in which no directed edge appears more than once.

The connectedness of a graph can be defined in terms of paths.

A graph G is **connected** if every pair of vertices are joined by a path.

A maximal connected subgraph of G is called a **connected component** or a **component** of G. Thus a **disconnected** graph has atleast two components. For example, the graph G in Fig. 6.13 is a connected graph and graph H is a disconnected graph and has 4 components.

A digraph is **connected** if its underlying graph is connected.

The **distance** between two vertices u and v in a graph, denoted $d(u, v)$, is the length of a shortest path joining them, if any. If there is no such path, we write $(u, v) = \infty$, *i.e.*, u and v are in different components of a disconnected graph.

The shortest $u - v$ path is usually called a **geodesic** and the length of any longest geodesic in a connected graph G is the **diameter** of G. For example, the diameter of G in Fig. 6.13 is 4.

Theorem 6.5 *A graph g is bipartite if and only if it has no cycles of odd length.*

Proof. Let G be bipartite and its vertex set has a bipartition such that each edge of G has one endvertex in one partition and other endvertex in another partition. So it requires an even number of steps (edge traversals) for a walk to return to the side from which it started. Thus, a cycle must have even length.

For the converse, let G be a graph with no cycles of odd length, and suppose that G is connected, for otherwise we can consider the components of G separately. Take any vertex u of G, and define a partition (V_1, V_2) of the vertex set $V(G)$ as follows,

$$V_1 = \{x \in G \mid d(u, x) \text{ is even}\}$$

$$V_2 = \{y \in G \mid d(u, y) \text{ is odd}\}.$$

Now (V_1, V_2) is a bipartition of $V(G)$ such that every edge of G joins a vertex of V_1 with a vertex of V_2, since all cycles of G are even. If (V_1, V_2) is not such a bipartition then suppose there is an edge wv joining two vertices w and v of V_1(or V_2). Then the geodesics from u to w and from u to v together with edge wv contains an odd cycle, contradicting the hypothesis. Hence (V_1, V_2) is a bipartition and G is a bipartite graph. □

Theorem 6.6 *Let G be a simple graph with n vertices and k components. Then it can have atmost* $\frac{1}{2}(n-k)(n-k+1)$ *edges*

Proof. Let n_i, $i = 1, 2, \ldots, k$, be number of vertices in each component of G, so

$$n_1 + n_2 + ... + n_k = n,$$

and $$n_i \geq 1, \quad i = 1, 2, ..., n. \qquad ....(1)$$

Now the maximum number of edges in the ith component of G is

$$^{n_i}C_2 = \frac{1}{2} n_i (n_i - 1)$$

Hence, the maximum number of edges in G is

$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) = \frac{1}{2} \Sigma n_i^2 - \frac{1}{2} \Sigma n_i$$

$$= \frac{1}{2} \Sigma n_i^2 - \frac{n}{2} \qquad ....(2)$$

Now, we have $$\sum_{i=1}^{k} (n_i - 1) = n - k$$

$$\Rightarrow \quad \left(\sum_{i=1}^{k} (n_i - 1) \right)^2 = (n - k)^2$$

$$\Rightarrow \quad \sum_{i=1}^{k} (n_i^2 - 2n_i) + k + \text{nonnegative terms} = (n - k)^2, \text{ since } (n_i - 1) \geq 0, \text{ for all } i$$

So, $$\sum_{i=1}^{k} n_i^2 + \text{nonnegative terms} = (n - k)^2 + 2n - k$$

i.e., $$\sum_{i=1}^{k} n_i^2 \leq (n - k)^2 + 2n - k \qquad ....(3)$$

Hence from (2) and (3),

$$\frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1) \leq \frac{1}{2} [(n - k)^2 + 2n - k] - \frac{n}{2}$$

$$= \frac{1}{2} (n - k)(n - k + 1).$$

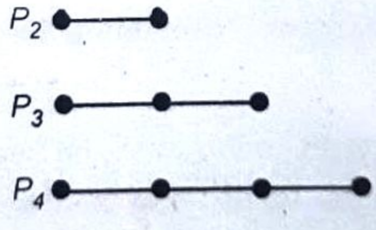
□

More Graphs

Path Graphs

A path graph P is a simple connected graph with $| V(P) | = | E(P) | + 1$.

An n-vertex path graph is denoted by P_n. For example,



Cycle Graphs

A cycle graph is a single vertex with a self-loop or a simple connected graph C with $| V(C) | = | E(C) |$ that can be drawn so that all of its vertices and edges lie on a circle.

An n-vertex cycle graph is denoted by C_n. For example,