Standard Uniform Distribution :- If $a=0$ and $b=1$ then the p.d.f of X in uniform distn turns as

$$f_X(x) = \begin{cases} \frac{1}{1-0} ; & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$E(X) = \frac{0+1}{2} = 1/2$$

$$V(X) = \frac{(1-0)^2}{12} = 1/12$$

Properties of CDF in Continuous Random Variable :-

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt$$

(i) $F_X(\infty) = P(X \le \infty)$

$$= \int_{-\infty}^{\infty} f_X(t)\,dt = 1 \qquad \left(\because \int_{-\infty}^{\infty} f_X(x)\,dx = 1\right)$$

(ii) $F_X(-\infty) = \int_{-\infty}^{-\infty} f_X(t)\,dt = 0$

(iii) $0 \le F_X(x) \le 1$

(iv) $P(a < X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dt$

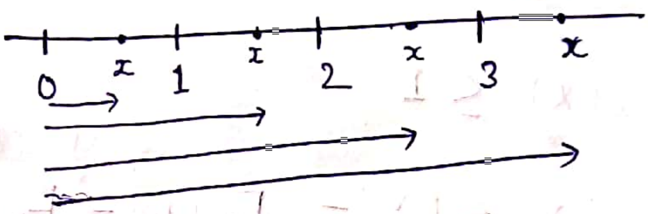
(v) $P(a < X < b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx$

(vi) $P(a \le X \le b) = \int_a^b f_X(x)\,dx = F_X(b) - F_X(a)$

Q: $f_X(x) = \begin{cases} \frac{x}{2} & ; \quad 0 \le x \le 1 \\ \frac{1}{2} & ; \quad 1 \le x \le 2 \\ \frac{-x}{2} + \frac{3}{2} & ; \quad 2 \le x \le 3 \\ 0 & ; \quad \text{otherwise} \end{cases}$

Compute CDF say $F_X(x)$? Also compute $P(1 \le X \le 1.5)$, $P(X = 1.5)$?

Sol^n



$F_X(x) = P(X \le x)$

$$= \begin{cases} \int_0^x f_X(t)\,dt = \int_0^x \frac{t}{2}\,dt = \frac{x^2}{4} & ; \quad 0 \le x \le 1 \\ \int_0^x f_X(t)\,dt = \int_0^1 f_X(t)\,dt + \int_1^x f_X(t)\,dt = \int_0^1 \frac{t}{2}\,dt + \int_1^x \frac{1}{2}\,dt = \frac{1}{4} + \frac{(x-1)}{2} & ; \quad 1 \le x \le 2 \\ \int_0^x f_X(t)\,dt = \int_0^1 f_X(t)\,dt + \int_1^2 f_X(t)\,dt + \int_2^x f_X(t)\,dt = \frac{1}{4} + \int_1^2 \frac{1}{2}\,dt + \int_2^x \left(-\frac{t}{2} + \frac{3}{2}\right)dt & ; \quad 2 \le x \le 3 \\ \quad = \frac{1}{4} + \frac{1}{2} + \left(-\frac{x^2}{4} + \frac{3x}{2} - 2\right) & \\ \quad = \left(\frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}\right) & ; \quad 2 \le x \le 3 \\ \int_0^x f_X(t)\,dt = \int_0^1 f_X(t)\,dt + \int_1^2 f_X(t)\,dt + \int_2^3 f_X(t)\,dt + \int_3^x f_X(t)\,dt & ; \quad x \ge 3 \\ \quad = 1 & \end{cases}$$

No.- 59

$$F_X(x) = \begin{cases} \frac{x^2}{4} & ; \quad 0 \le x \le 1 \\ \frac{2x-1}{4} & ; \quad 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} & ; \quad 2 \le x \le 3 \\ 1 & ; \quad x \geq 3 \end{cases}$$

Now, $P(1 \le X \le 1.5) = F_X(1.5) - F_X(1) = \int_1^{1.5} f_X(x)\,dx$

$$= \left(\frac{2 \times 1.5 - 1}{4}\right) - \left(\frac{2 \times 1 - 1}{4}\right)$$

$$= \frac{2}{4} - \frac{1}{4} = 1/4$$

$P(X = 1.5) = 0$

Gamma Distribution:

Gamma Function: (Γ)

(i) $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$; ($n > 0$, Convergent)

(ii) $\Gamma n+1 = n \Gamma n$

(iii) $\Gamma n+1 = n!$ if n is positive integer

(iv) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(v) $\Gamma 1 = 1$

Q: find $\Gamma \frac{7}{2}$, $\Gamma 9$?

Sol^n $\Gamma \frac{7}{2} = \Gamma 1+\frac{5}{2} = \frac{5}{2} \Gamma \frac{5}{2}$ ($\because \Gamma n+1 = n \Gamma n$

$= \frac{5}{2} \Gamma 1+\frac{3}{2} = \frac{5}{2} \cdot \frac{3}{2} \Gamma \frac{3}{2}$

$= \frac{5}{2} \cdot \frac{3}{2} \Gamma 1+\frac{1}{2}$

$= \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}$

$= \frac{15}{8} \times \sqrt{\pi}$ Ans

$\Gamma 9 = \Gamma 8+1 = 8!$ Ans

Scanned with CamScanner

Gamma Distribution :-

A r.v X is said to have Gamma distribution if

No.- 60

its pdf is given as

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma\alpha \ \beta^{\alpha}} & ; \ x > 0 \ ; \ \alpha > 0 ; \ \beta > 0 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$\alpha \rightarrow$ Shape parameter

$\beta \rightarrow$ Scale parameter.

Q: Verify that it is proper pdf and then compute $E(X)$ and $V(X)$?

Solⁿ

Claim $\int_0^{\infty} f_X(x)\, dx = 1$

$$\int_0^{\infty} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma\alpha \ \beta^{\alpha}} = \frac{1}{\Gamma\alpha \ \beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Put $\frac{x}{\beta} = u \Rightarrow x = \beta u$ | when $x = 0$; $u = 0$

$\Rightarrow dx = \beta du$ | when $x \to \infty$; $u \to \infty$

Scanned with CamScanner

$$= \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty (\beta u)^{\alpha-1} e^{-u} \beta \, du$$

$$= \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty \beta^{\alpha-1} \cdot \beta \; u^{\alpha-1} \cdot e^{-u} \, du$$

$$= \frac{\cancel{\beta^\alpha}}{\Gamma\alpha\, \cancel{\beta^\alpha}} \int_0^\infty u^{\alpha-1} e^{-u} \, du$$

$$= \frac{1}{\Gamma\alpha} \Gamma\alpha \qquad \left(\because \int_0^\infty x^{\alpha-1} \cdot e^{-x} dx = \Gamma\alpha \right.$$

$$= 1$$

Next,

$$E(X) = \int_0^\infty x \cdot f_X(x)\, dx = \int_0^\infty x \cdot \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma\alpha \;\beta^\alpha} dx$$

$$= \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty x^\alpha e^{-\frac{x}{\beta}} dx$$

Put ; $\frac{x}{\beta} = u \Rightarrow x = \beta u$

$\Rightarrow dx = \beta du$

when $x = 0$; $u = 0$

when $x \to \infty$; $u \to \infty$

$$E(X) = \frac{1}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty (\beta u)^\alpha e^{-u} \cdot \beta\, du$$

$$= \frac{1 \cdot \beta^{\alpha+1}}{\Gamma\alpha\, \beta^\alpha} \int_0^\infty u^\alpha e^{-u}\, du$$

$$= \frac{\beta}{\Gamma\alpha} \int_0^\infty u^{(\alpha+1)-1} e^{-u}\, du \qquad \left(\because \int_0^\infty x^{\alpha-1} e^{-x}\, dx = \Gamma\alpha\right.$$

$$= \frac{\beta}{\Gamma\alpha} \Gamma\alpha+1$$

$$= \frac{\beta \cdot \alpha \cancel{\Gamma\alpha}}{\cancel{\Gamma\alpha}} \qquad \left(\because \Gamma n+1 = n\Gamma n\right.$$

$$\boxed{E(X) = \alpha\beta} \quad \#$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^\infty x^2 \cdot f_X(x)\, dx$$

$$= \int_0^\infty x^2 \cdot \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma\alpha\, \beta^\alpha}\, dx$$

$$= \int_0^\infty \frac{x^{\alpha+1} \, e^{-\frac{x}{\beta}}}{\Gamma\alpha \, \beta^\alpha} \, dx$$

$$= \frac{1}{\Gamma\alpha \, \beta^\alpha} \int_0^\infty x^{\alpha+1} \cdot e^{-\frac{x}{\beta}} \, dx$$

Put $\frac{x}{\beta} = u \Rightarrow x = \beta u$

$dx = \beta \, du$

When $x = 0$; $u = 0$

When $x \to \infty$; $u \to \infty$

$$\therefore E(X^2) = \frac{1}{\Gamma\alpha \, \beta^\alpha} \int_0^\infty (\beta u)^{\alpha+1} \, e^{-u} \, \beta \, du$$

$$= \frac{\beta^{\alpha+2}}{\Gamma\alpha \, \beta^\alpha} \int_0^\infty u^{\alpha+1} \, e^{-u} \, du$$

$$= \frac{\beta^2}{\Gamma\alpha} \int_0^\infty u^{(\alpha+1+1)-1} \, e^{-u} \, du$$

$$\left(\because \int_0^\infty x^{\alpha-1} e^{-x} dx = \Gamma\alpha \right.$$

$$= \frac{\beta^2}{\Gamma\alpha} \Gamma\alpha+2$$

$$= \frac{\beta^2}{\Gamma\alpha} \Gamma(\alpha+1)+1 = \frac{\beta^2}{\Gamma\alpha} (\alpha+1) \, \Gamma\alpha+1$$

$$= \frac{\beta^2}{\cancel{\Gamma\alpha}} (\alpha+1) \cdot \alpha \, \cancel{\Gamma\alpha}$$

$$= \alpha\beta^2(\alpha+1)$$

Scanned with CamScanner

$$\therefore V(X) = EX^2 - (E(X))^2$$

$$= \alpha\beta^2(\alpha+1) - \alpha^2\beta^2$$

$$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2}$$

$$\boxed{V(X) = \alpha\beta^2}$$

#