Note: ① CDF of Gamma distⁿ is not in closed form

i.e. $F_X(x) = P(X \le x)$

$$= \int_0^x \frac{t^{\alpha+1} e^{-t/\beta}}{\Gamma\alpha \, \beta^\alpha} \, dt$$ which is

not integrable. This is drawback of Gamma distⁿ.

Note: ② CDF of Uniform distⁿ :-

$$F_X(x) = P(X \le x) = \int_a^x \frac{1}{b-a} \, dt$$ $\left(\because f_X(x) = \frac{1}{b-a} ; \right.$

$a \le x \le b$

$$\boxed{F_X(x) = \frac{1}{b-a}(x-a)}$$ #

Exponential Distribution :- Let X be a continuous rv, then X is said to have

~~its pdf~~ exponential distn if its pdf is given as

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 ; \lambda > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

<u>Note:</u> Exponential distn is special case of Gamma distn. with condition

$$\frac{1}{\beta} = \lambda , \alpha = 1$$

$$f_X(x) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma\alpha \, \beta^{\alpha}} ; x > 0 ; \alpha > 0 ; \beta > 0 \quad \text{(Gamma distn)}$$

$$= \frac{x^{1-1} \cdot e^{-x \cdot \lambda}}{\Gamma 1 \cdot \left(\frac{1}{\lambda}\right)^1}$$

$$= \frac{\lambda x^0 e^{-\lambda x}}{1.}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 & ; \text{other wise} \end{cases} \quad \text{(Exponential distn)}$$

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Q: Verify that it is proper pdf. Also find $E(X)$, $V(X)$ and $F_X(x)$?

No.- 61

Soln :: $\int_0^\infty f_X(x)\,dx = 1$

From L.H.S $\int_0^\infty \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-\lambda x} dx$

$= \lambda \dfrac{[e^{-\lambda x}]_0^\infty}{-\lambda}$

$= -[0 - e^{0}]$

$= 1$

$$E(X) = \int_0^\infty x \cdot f_X(x)\, dx$$

$$= \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x\, e^{-\lambda x}\, dx$$

Put $\lambda x = u$ | When $x = 0$; $u = 0$

$\Rightarrow \lambda\, dx = du$ | When $x \to \infty$; $u \to \infty$

$$= \cancel{\lambda} \int_0^\infty \left(\frac{u}{\lambda}\right) e^{-u}\, \frac{du}{\cancel{\lambda}}$$

$$= \frac{1}{\lambda} \int_0^\infty u\, e^{-u}\, du$$

$$= \frac{1}{\lambda} \int_0^\infty u^{2-1} e^{-u} du$$

$\left(\because \int_0^\infty x^{n-1} e^{-x} dx = \Gamma n \right.$

$\because \Gamma n+1 = n \Gamma n$

No.- 62

$(\because \Gamma 1 = 1)$

$$= \frac{1}{\lambda} \Gamma 2 = \frac{\Gamma 1+1}{\lambda} = \frac{1 \Gamma 1}{\lambda}$$

$$\boxed{E(X) = \frac{1}{\lambda}}$$

Now, $V(X) = EX^2 - (EX)^2$

So; $E(X^2) = \int_0^\infty x^2 . f_X(x) dx$

$$= \int_0^\infty x^2 . \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x} dx$$

Put $\lambda x = u$ | When $x = 0$; $u = 0$

$\Rightarrow \lambda dx = du$ | When $x \to \infty$; $u \to \infty$

$$= \cancel{\lambda} \int_0^\infty \left(\frac{u}{\lambda}\right)^2 e^{-u} \frac{du}{\cancel{\lambda}}$$

$$= \frac{1}{\lambda^2} \int_0^\infty u^2 e^{-u} du$$

$$= \frac{1}{\lambda^2} \int_0^\infty u^{3-1} e^{-u} du$$

$$= \frac{1}{\lambda^2} \Gamma 3$$

$$= \frac{1}{\lambda^2} \Gamma 2+1 = \frac{2 \Gamma 2}{\lambda^2}$$ $(\because n\Gamma n = \Gamma n+1)$

$$= \frac{2 \Gamma 1+1}{\lambda^2}$$

$$= \frac{2 \cdot 1 \cdot \Gamma 1}{\lambda^2}$$ $(\because \Gamma 1 = 1)$

$$= \frac{2}{\lambda^2}$$

$$\therefore V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{V(X) = \frac{1}{\lambda^2}}$$

$$F_X(x) = P(X \leq x) = \int_0^x f_X(t)\,dt$$

$$= \int_0^x \lambda e^{-\lambda t}\,dt$$

$$= \lambda \int_0^{\infty} e^{-\lambda t}\,dt$$

$$= \cancel{\lambda} \left[\frac{e^{-\lambda t}}{-\cancel{\lambda}}\right]_0^x$$

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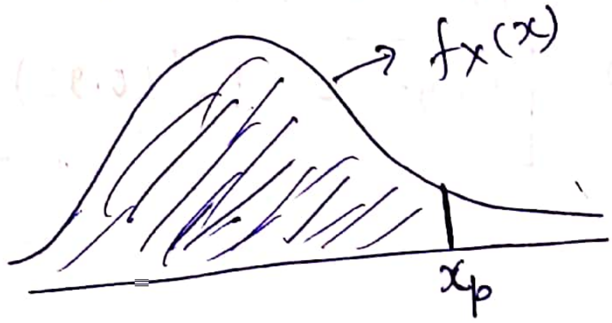
$$= -[e^{-\lambda x} - e^{0}]$$

$$= -[e^{-\lambda x} - 1]$$

$$\boxed{F_X(x) = 1 - e^{-\lambda x}}$$

Thus the CDF of exponential distn is in closed form but Gamma distn is more flexible as it contains more number of parameters.

Percentile :- Percentile gives the point below which required percentage of area lies.



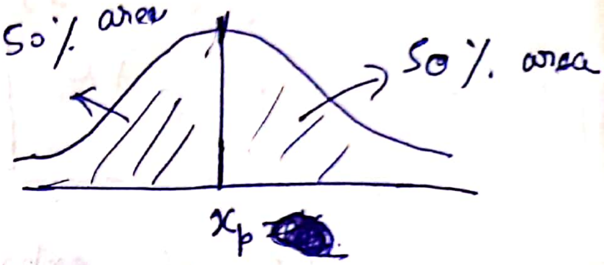
It is denoted as $F_X(x_p)$ and defined as

$$F_X(x_p) = P(X \le x_p)$$

$$= \int_{-\infty}^{x_p} f_X(x)\, dx$$

Note :

1. The point which divide whole are into two equal part is called Median



i.e $F_X(x_{0.5}) = 0.5$

i.e $\int_{-\infty}^{x_{0.5}} f_X(x)\,dx = 0.5$

i.e $\boxed{x_{0.5} = F_X^{-1}(0.5)}$

2. 95^{th} percentile is defined as

$$\int_{-\infty}^{x_{0.95}} f_X(x)\,dx = 0.95$$

i.e $F_X(x_{0.95}) = 0.95$

$\Rightarrow \boxed{x_{0.95} = F_X^{-1}(0.95)}$

EXERCISES Section 4.2 (11–27)

11. Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^2}{4} & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

a. Calculate $P(X \le 1)$.
b. Calculate $P(.5 \le X \le 1)$.
c. Calculate $P(X > 1.5)$.
d. What is the median checkout duration $\tilde{\mu}$? [solve $.5 = F(\tilde{\mu})$].
e. Obtain the density function $f(x)$.
f. Calculate $E(X)$.
g. Calculate $V(X)$ and σ_X.
h. If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E[h(X)]$.

12. The cdf for X (= measurement error) of Exercise 3 is

$$F(x) = \begin{cases} 0 & x < -2 \\ \dfrac{1}{2} + \dfrac{3}{32}\left(4x - \dfrac{x^3}{3}\right) & -2 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

a. Compute $P(X < 0)$.
b. Compute $P(-1 < X < 1)$.
c. Compute $P(.5 < X)$.

4.2

Q.11

$$F_X(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{x^2}{4} & ; \quad 0 \le x < 2 \\ 1 & ; \quad x \geq 2 \end{cases}$$

(a) $P(X \le 1) = F_X(1)$

$= \frac{(1)^2}{4} = \frac{1}{4}$

(b) $P(0.5 \leq X \leq 1) = F_X(1) - F_X(0.5)$

$$= \frac{1^2}{4} - \frac{(0.5)^2}{4}$$

$$= \frac{1}{4}(1 - 0.25)$$

$$= \frac{0.75}{4}$$

$$= 0.1875$$

c) $P(X > 1.5) = 1 - P(X \leq 1.5)$

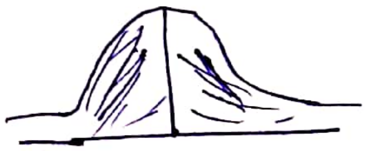
$$= 1 - F_X(1.5)$$

$$= 1 - \frac{(1.5)^2}{4}$$

$$= 1 - \frac{2.25}{4}$$

$$= 0.4375$$

(d) <u>Median</u> $F_X(x_{0.5}) = 0.5$



$$\frac{x_{0.5}^2}{4} = 0.5$$

$$\Rightarrow x_{0.5}^2 = 2$$

$$\Rightarrow \boxed{x_{0.5} = \sqrt{2}}$$

(e) $$f_X(x) = \begin{cases} 0 & ; \quad x < 0 \\ \frac{2x}{4} & ; \quad 0 \le x < 2 \\ 0 & ; \quad x > 2 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{x}{2} & ; \quad 0 \le x \le 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

(f) $$E(X) = \int_0^2 x \cdot f_X(x)\, dx$$

$$= \int_0^2 x \; \frac{x}{2}\, dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{1}{6} [8 - 0] = \frac{4}{3}$$

ie $\boxed{E(X) = \frac{4}{3}}$

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13. Example 4.5 introduced the concept of time headway in traffic flow and proposed a particular distribution for $X =$ the headway between two randomly selected consecutive cars (sec). Suppose that in a different traffic environment, the distribution of time headway has the form

$$f(x) = \begin{cases} \dfrac{k}{x^4} & x > 1 \\ 0 & x \le 1 \end{cases}$$

a. Determine the value of k for which $f(x)$ is a legitimate pdf.

b. Obtain the cumulative distribution function.

c. Use the cdf from (b) to determine the probability that headway exceeds 2 sec and also the probability that headway is between 2 and 3 sec.

d. Obtain the mean value of headway and the standard deviation of headway.

e. What is the probability that headway is within 1 standard deviation of the mean value?

13

13 $f_X(x) = \begin{cases} \frac{K}{x^4} & ; \quad x > 1 \\ 0 & ; \quad x \le 1 \end{cases}$

a) $\int_{1}^{\infty} f_X(x)\,dx = 1$

$\Rightarrow \int_{1}^{\infty} \frac{K}{x^4}\,dx = 1 \Rightarrow K \int_{1}^{\infty} x^{-4}\,dx = 1$

$\Rightarrow K \left[\frac{x^{-4+1}}{-4+1} \right]_{1}^{\infty} = 1$

$\Rightarrow \frac{K}{-3} \left(\frac{1}{x^3} \right)_{1}^{\infty} = 1$

$\Rightarrow \frac{K}{-3} \left(0 - \frac{1}{1} \right) = 1$

$\Rightarrow \boxed{K = 3}$

(b) $F_X(x) = P(X \le x) = \int_{1}^{x} f_X(t)\,dt$

$= \int_{1}^{x} \frac{3}{t^4}\,dt = \cancel{3} \left[\frac{t^{-4+1}}{\cancel{-4+1}} \right]_{1}^{x}$

$= -(x^{-3} - 1)$

$\boxed{F_X(x) = 1 - x^{-3}} \quad ; \quad x > 1$

(c) $P(X > 2) = 1 - P(X \leq 2)$

$= 1 - F_X(2)$

$= 1 - (1 - 2^{-3})$

$= 1 - 1 + \frac{1}{2^3}$

$\boxed{P(X > 2) = \frac{1}{8}}$

$P(2 \leq X \leq 3) = F_X(3) - F_X(2)$

$= (1 - 3^{-3}) - (1 - 2^{-3})$

$= \cancel{1} - \frac{1}{27} - \cancel{1} + \frac{1}{8}$

$= \frac{1}{8} - \frac{1}{27}$

$= 0.0879$

(d) $E(X) = \int_1^\infty x \cdot f_X(x)\, dx$

$= \int_1^\infty x \cdot \frac{3}{x^4}\, dx$

$= 3 \int_1^\infty \frac{1}{x^3}\, dx$

$= 3 \cdot \left[\frac{x^{-2}}{-2}\right]_1^\infty$

$= \frac{-3}{2}(0 - 1)$

$\mu = E(X) = \frac{3}{2}$

Now $\sigma_x = \sqrt{V(X)}$

where $V(X) = EX^2 - (EX)^2$

$$EX^2 = \int_1^{\infty} x^2 \cdot \frac{3}{x^4}\, dx$$

$$= 3 \int_1^{\infty} \frac{1}{x^2}\, dx = 3 \frac{[x^{-1}]_1^{\infty}}{-1}$$

$$= -3(0 - 1)$$

$$EX^2 = 3$$

$$V(X) = 3 - \left(\frac{3}{2}\right)^2$$

$$= 3 - \frac{9}{4}$$

$$V(X) = \frac{3}{4}$$

$$\boxed{\sigma_x = \frac{\sqrt{3}}{2}}$$

(e) $P(|X - \mu| < 1 \cdot \sigma_x) = P(\mu - \sigma_x < X < \mu + \sigma_x)$

$$= P\left(\frac{3}{2} - \frac{\sqrt{3}}{2} < X < \frac{3}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \int_{3/2 - \sqrt{3}/2}^{3/2 + \sqrt{3}/2} \frac{3}{x^4}\, dx$$