Relation: If A and B are two sets, then relation R from A to B is the subset of the Cartesian product $A \times B$.

$\{(a,b):$ ———

If a is related to b under the relation R, then we write aRb.

Thus $\quad R = \{(x,y) : x \in A, y \in B \text{ and } xRy\}$

The set of first entries of the ordered pairs in a relation is called the **domain of the relation.** The set of second entries of the ordered pairs in a relation is called the **range of the relation.**

$(a,1)$

Example: If $A = \{2,3,4,5,6\}, B = \{2,4,6,8\}$ and $\boldsymbol{xRy}$: x divides y, then

$\subseteq A \times B$

$$R = \{(2,2),(2,4),(2,6),(2,8),(3,6),(4,4),(4,8),(6,6)\}$$

Domain of $R = \{2,3,4,6\}$,

Range of $R = \{2,4,6,8\}$

Note: For finite sets A and B with $|A|=m$ and $|B|=n$, there are 2^{mn} relations from A to B. There are also 2^{mn} relations from B to A.

PROPERTIES OF RELATIONS

Reflexive Relation

Let R be a relation defined in a set A; then R is reflexive if aRa holds for all $a \in A$, i.e., if $(a, a) \in R$ for all $a \in A$.

Example 1: Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c)\}$ then R is a reflexive relation in A.

Example 2: 'Equality' is a reflexive relation, since an element equals itself.

Symmetric Relation

A relation R defined in set A is said to be 'symmetric' if bRa holds whenever aRb holds for $b \in A$, i.e., R is symmetric in A if

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

Example: Let R be relation 'is perpendicular to' in the set of all straight lines, then R is a symmetric relation.

Transitive Relation

A relation R in set A is said to be transitive if

$$(a, b) \in R \;\; (b, c) \in R \Rightarrow \; (a, c) \in R$$

i.e., if aRb and $bRc \Rightarrow aRc$, $a, b, c \in A$.

Example 1: Let A denote the set of straight lines in a plane and R be a relation in A defined by 'is parallel to' then R is a transitive relation in A.

Example 2: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ then R is transitive.

(a, b) ; (b, a)

$\boxed{a=b}$

Anti-symmetric Relation

Let R be a relation in a set A, then R is called anti-symmetric.

$$\underline{(a, b) \in R}, \ \underline{(b, a) \in R} \Rightarrow \underline{a = b} \ \forall \ a, b \in R$$

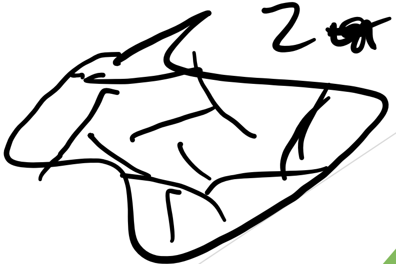
i.e., aRb and $bRb \Rightarrow a = b$

Example: Let N denote the set of Natural Numbers R be a relation in N, defined by 'a is a divisor' of b, i.e., aRb if a divides b then R is anti-symmetric since a divides b and b divides $a \Rightarrow a = b$.

Equivalence Relation

A relation R in a set A is said to be an equivalence relation in A, if R is reflexive, symmetric and transitive.

2u



Example: (*i*) Let A be the set of all triangle in a plane and let R be a relation in A defined by 'is congruent to', then R is reflexive, symmetric and transitive.

$\therefore$ R is an Equivalence relation in A.

(*ii*) Let $A = \{a, b, c\}$, and $R = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ then R is an equivalence relation in A.

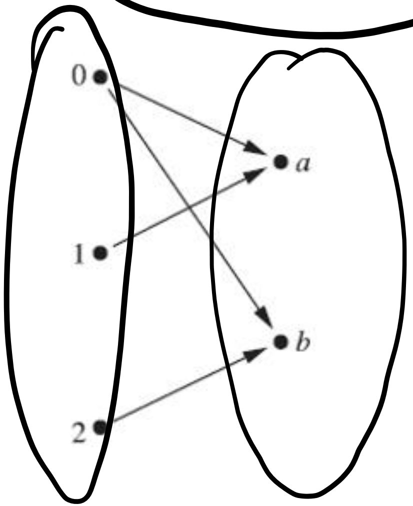
Example: Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. ◀

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0 \, R \, a$, but that $1 \not R \, b$. Relations can be represented graphically,



RELATIONS AND DIGRAPHS

A relation can be represented pictorially by drawing its graph. Let R be a relation on the set $A = \{a_1, a_2, \ldots a_n\}$. The element a_i of A are represented by points (or circles) called nodes (or vertices). If $(a_i, a_j) \in R_j$ then we connect the vertices a_i and a_j by means of an arc and put an arrow in the direction from a_i to a_j. If $(a_i, a_j) \in R$ and $(a_j, a_i) \in R$ then we draw two arcs between a_i and a_j (sometimes by one arc which starts from node a_i and relatives to node x_i (such an arc is called a loop). When all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, we get a graph of the relation R. If R is reflexive, then there must be a loop at each node in the graph of R. If R is symmetric, then $(a_i, a_j) \in R$ implies $(a_j, a_i) \in R$ and the nodes a_i and a_j will be connected by two arcs (edges) one from a_i to a_j and the other from a_j to a_i.

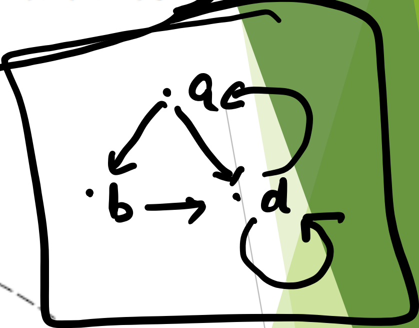
Example 1: Let $A = \{a, b, d\}$ and R be a relation on A given by

$$R = \{(a, b), (a, d), (b, d), (d, a), (d, d)\}$$

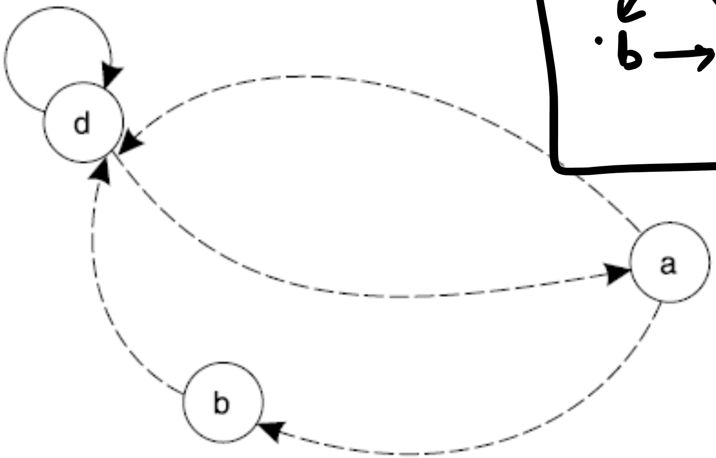
A x A

Construct the digraph of R.

Solution: The digraph of R is as shown in Fig. 3.7.



a < b

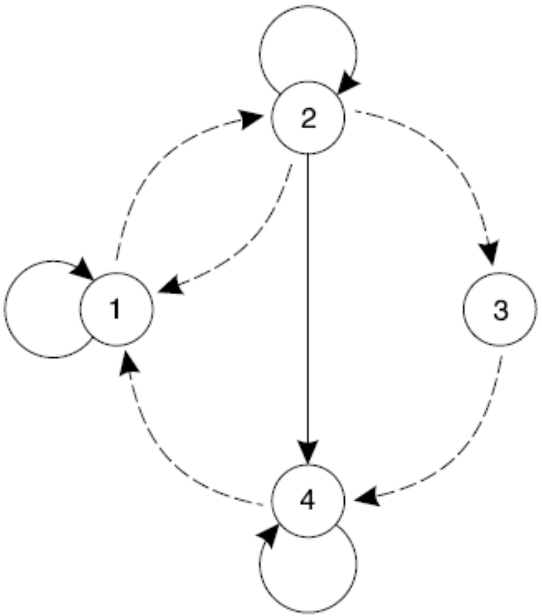


Example 2: Let $A = \{1, 2, 3, 4\}$

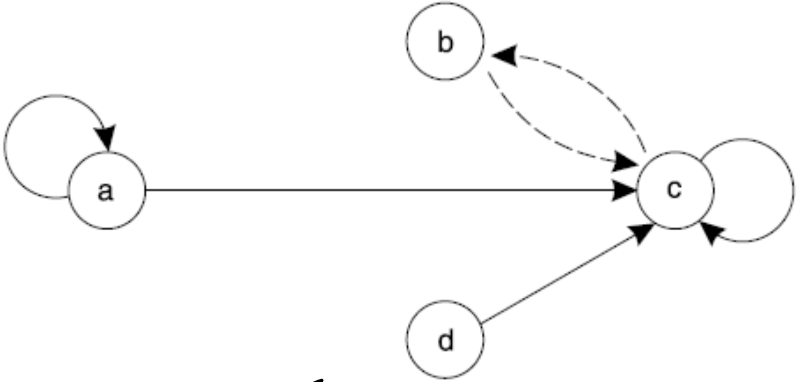
$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 4)\}$$

Construct the digraph of R.

Solution: The digraph of R is shown in Fig. 3.8.



Example 3: Find the relation determined by Fig. 3.9.



Solution: The relation R of the digraph is

$$R = \{(a, a), (a, c), (b, c), (c, b), (c, c), (d, c)\}$$

PARTIAL ORDERING

A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

$\leq$ $\leq$ $a \leq b \leq c$ a, b, c $a \leq b$

Example 1: If A is a non-empty set and $P(A)$ denotes the power set of A, then the relation set inclusion denoted by $\leq$ in $P(A)$ is a partial ordering.

Example 2: Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be a relation in A which is defined by "a divides b". Then R is a partial order in A.

ϕ $X, Y \in P(A)$

$(A, \subseteq)$ poset

$XRY \iff X \subseteq Y.$

$\leq$ $\leq$

aRb

$a | b.$

$(2, 6)$ $2 \mid 6$

Comparability

Let R be a partial order on A and $a,\ b \in A$ whenever $aR\,b$ or $bR\,a$, we say that a and b are comparable otherwise a and b are non-comparable.

TOTALLY ORDERED SET

$aRb.$

Let $(A, \leq)$ be a partially order set. If for every $a,\ b \in A,$ we have $a \leq b$ or $\boxed{b \leq a}$, then $\leq$ is called a simple ordering (or linear ordering) on A, and the set $(A, \leq)$ is called a totally ordered set or a chain.

Note: If A is a partially ordered set, then some of the elements of A are non-comparable. On the other land, if A is totally ordered then every pair of elements of A are comparable.

Example 1: Let N be the set of positive integers ordered by divisibility. The elements 5 and 15 are comparable. Since 5/15 on similarly the elements 7 and 21 are comparable since 7/21. The positive integers 3 and 5 are non-comparable since neither 3/5 nor 5/3. Similarly the integers 5 and 7 are non- comparable.

Example 2: The set N of positive integers with the usual order $\leq$ (less than equal) is a linear order on N. The set $(N, \leq)$ is a totally ordered set.

$2 \leq 3$

$5 \mid 15 \Leftrightarrow 5R15 \Rightarrow \boxed{5 \leq 15}$

$\leq$ $<$

$5, 15$

$(N, \leq)$

$2 \nmid 3$

Hasse Diagrams

- Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a).

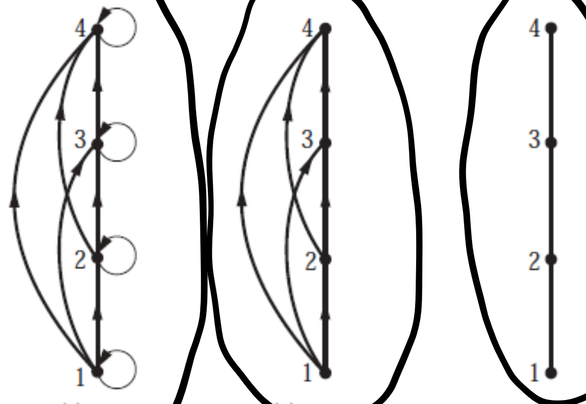
Because this relation is a partial ordering, it is reflexive and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown.

Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity.

$(A = \{1, 2, 3, 4\}, \leq)$

aRb

$a \leq b$



(a) (b) (c)

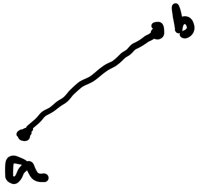
R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3) (3,4), (4,4)}

Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c).

$\boxed{\leq}$ $a \preceq b$

aRb.



$\boxed{aRb}$

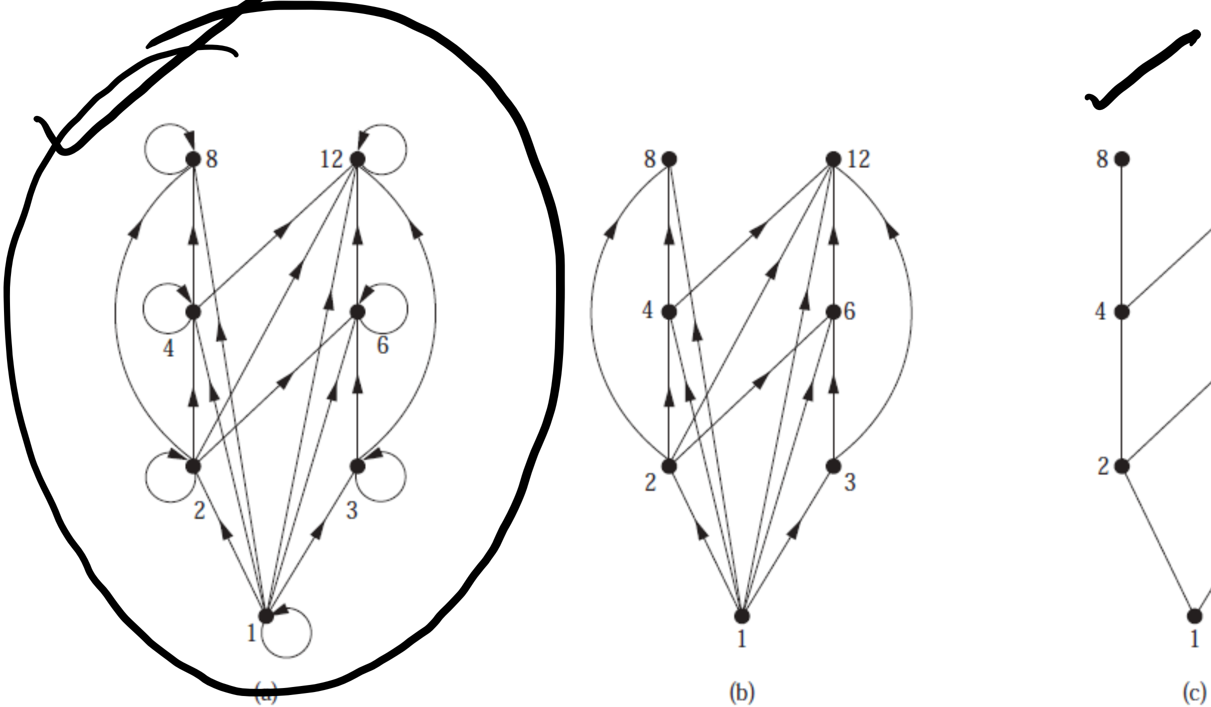


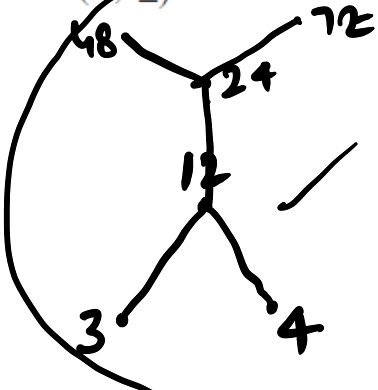
FIGURE 3 Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

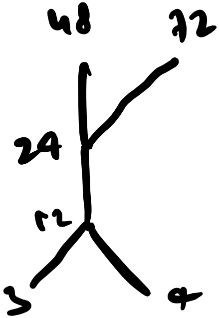
HASSE DIAGRAM

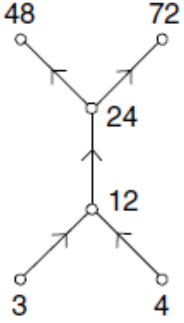
A Hasse diagram is a pictorial representation of a finite partial order on a set. In this representation, the objects i.e., the elements are shown as vertices (or dots).

Two related vertices in the Hasse diagram of a partial order are connected by a line if and only if they are related.

Example 1: Let $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\le$ be such that $a \le b$ if a divides b. The Hasse diagram of $(A, \le)$ is shown in Fig. 5.1.

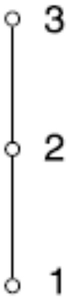




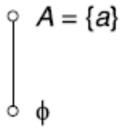


We avoid arrows in a Hasse diagram and draw lines to show that the elements are related. Hasse diagrams can be drawn for any relation which is anti-symmetric and transitive but not necessarily reflexive.

Example 2: Let $A = \{1, 2, 3\}$, and $\leq$ be the relation "less than of equal to" on A. Then the Hasse diagram of $(A, \leq)$ is as shown in Fig.



Example 3: Let $A = \{a\}$, and $\leq$ be the inclusion relation on the elements of $P(A)$. The Hasse diagram of $(P(A), \leq)$, can drawn as shown in Fig. 5.3.



Example 4: Draw the Hasse diagram representing the positive divisions of 36 (i.e. D_{36})

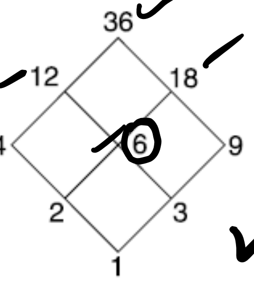
Solution: We have $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Let R denote the partial order relation on D_{36}, i.e., aRb if and only a divides b. The Hasse diagram for R is shown in Fig. 5.3 (a).

3

D_{36}

2



36

1

Note:

If $(A, \leq)$ is partially ordered set, the Hasse diagram of $(A, \leq)$ is not unique. For example, consider the set $A = \{a, b\}$. The relation of inclusion $(\leq)$ on $P(A)$ is a partial ordering. The Hasse diagrams of $(P(A), \leq)$ are given in Fig.