Lecture -13 ①

Interval Estimation :- Let $X_1, X_2 \ldots X_n$ be a random sample from a population with pdf $f(x, \theta)$ $\theta \in \Theta$.

i) A random interval is an interval whose end points are random variables.

ii) A confidence interval for θ with confidence coefficient $(1-\alpha)$; $0<\alpha<1$ is a random interval whose end pts are statistics, say $L(X_1, X_2 \ldots X_n)$ & $U(X_1, X_2 \ldots X_n)$ s.t $L(\underline{X}) \leq U(\underline{X})$ where $\underline{X} = (X_1, X_2 \ldots X_n)$ and

$$P(L(\underline{X}) \leq \theta \leq U(\underline{X})) = 1-\alpha$$

Then $[L(\underline{X}), U(\underline{X})]$ is called $100(1-\alpha)\%$ confidence interval for θ.

Chi - Square Distribution :-

If $X \sim N(0, 1)$ then $X^2 \sim \chi^2_{(1)}$ (chi-square with 1-degree of freedom)

Degree of Freedom :- The degree of freedom of distⁿ is sum of square of standard normal distⁿ.

(Ex) ① If $X \sim N(0,1)$ then $X^2 \sim \chi^2_{(1)\,d.f}$

② If $X_1, X_2 \overset{iid}{\sim} N(0,1)$ then $X_1^2 + X_2^2 \sim \chi^2_{(2)\,d.f}$

⋮

③ If $X_1 \ldots X_n \overset{iid}{\sim} N(0,1)$ then $\sum_{i=1}^{n} X_i^2 \sim \chi^2_{(n)}$

The p.d.f of Chi-square distⁿ is given as.

$$f_X(x) = \frac{x^{\frac{n}{2}-1} \, e^{-\frac{x}{2}}}{\Gamma\frac{n}{2} \cdot 2^{\frac{n}{2}}} \; ; \quad x > 0. \quad \text{This is } \chi^2_{(n)}$$

Student's t-distribution : If $X \sim N(0,1)$

and $Y \sim \chi^2_{(n)}$

and X and Y are independent then

$$\frac{X}{\sqrt{Y/n}} \sim t_{(n)}. \qquad \#$$

Confidence Interval for μ of the Normal distribution with known σ^2 :

Step-1 Choose a confidence level γ (95%, 99% or the like).

Step-2 Determine the corresponding c ;

γ	0.90	0.95	0.99	0.999
c	1.645	1.960	2.576	3.291

Step-3 Compute the mean $\bar{x}$ of the sample $x_1, x_2 \cdots x_n$.

Step-4 Compute $k = \frac{c\sigma}{\sqrt{n}}$.

So the confidence interval for μ is
$$\{ \bar{x} - k \le \mu \le \bar{x} + k \}.$$

Q) Find a 95% confidence interval for the mean of a normal distn with variance $\sigma^2 = 9$, using a sample of $n = 100$ values with mean $\bar{x} = 5$

Soln

Step-I $\gamma = 0.95$

Step-2 $c = 1.960$

Step-3 $\bar{x} = 5$ → given

Step 4 $K = \frac{c\sigma}{\sqrt{n}} = \frac{1.960 \times 3}{\sqrt{100}} = 0.588$

Hence $\bar{x} - K = 5 - 0.588 = 4.412$

$\bar{x} + K = 5 + 0.588 = 5.588$

Thus 95% Confidence interval for μ is

$(4.412, 5.588)$ Ans

Scanned with CamScanner

No.- 68

Q/ Find a 95% CI for μ of a normal population with standard deviation 4 from the sample

30, 42, 40, 34, 48, 50.

Sol

Step I : $\gamma = 0.95$

Step 2 : $c = 1.960$

Step3 : $\bar{x} = \dfrac{30 + 42 + 40 + 34 + 48 + 50}{6}$

$= 40.66$

Step 4 : $K = \dfrac{c\sigma}{\sqrt{n}} = \dfrac{1.96 \times 4}{\sqrt{6}} = 3.2006$

Now $(\bar{x} - K, \bar{x} + K)$ will be 95% C.I for μ

i.e $(37.459, 43.8607)$ " " #

Determine of a Confidence Interval for the mean μ of a Normal Distribution with Unknown Variance σ^2 :

Step 1: Choose a confidence level γ (95%, 99% or the like)

Step 2: Determine the sol^n c of the eq^n

$$F(c) = \frac{1}{2}(1+\gamma)$$ from the table of

t - distribution with n-1 degree of freedom (Table A9 in Appendix ; where n = sample size)

Step 3: Compute the mean $\bar{x}$ and the sample variance s^2 of the sample $x_1, x_2, \ldots x_n$.

Step 4: Compute $k = \frac{c \cdot s}{\sqrt{n}}$.

The confidence interval for μ is

$(\bar{x} - k, \quad \bar{x} + k)$.

Q! Five independent measurements of the point of inflammation (flash point) of Diesel oil gave the values
No.- 68
(In °F) 144, 147, 146, 142, 144.
Assuming normality, determine a 99% CI for the mean?

Sol^n Step 1 : $\gamma = 0.99$

Step 2 : $F(c) = \frac{1}{2}(1+\gamma) = \frac{1}{2}(1+0.99)$

$= 0.995$

Here $n = 5$

So the value of c for which $F(c) = 0.995$ with degree of freedom $(5-1) = 4$ is

4.60

i.e $\boxed{c = 4.60}$

Step 3: $\bar{x} = \frac{144+147+146+142+144}{5} = 144.6$

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$

$= \frac{1}{5-1}\left[(144-144.6)^2 + (147-144.6)^2 + \cdots + (144-144.6)^2\right]$

$= 3.8$

$$K = \frac{4.60 \times \sqrt{3.8}}{\sqrt{5}} = 4.01$$

Thus $(144.6 - 4.01,\ 144 + 4.01)$ is 99% C.I for μ

(3)

Determine the Confidence Interval for σ^2 of a Normal Distribution whose Mean need not be Known :-

1: Choose a confidence level γ

2: Find c_1 and c_2 s.t.

$$F(c_1) = \frac{1}{2}(1-\gamma) \quad \text{and} \quad F(c_2) = \frac{1}{2}(1+\gamma)$$

From the table of chi-square distribution with $(n-1)$ degree of freedom.

3: Compute $k_1 = \frac{(n-1)s^2}{c_1}$ and $k_2 = \frac{(n-1)s^2}{c_2}$.

so that $P(k_1 \le \sigma^2 \le k_2) = \gamma$

Q: Determine a 95% C.I for the variance with sample 89, 84, 87, 81, 89, 86, 91, 90, 78, 89, 87, 99, 83, 89.

Sol^n

1: $\gamma = 95\% = 0.95$

2: $n = 14 \Rightarrow n-1 = 13$

$F(c_1) = \frac{1}{2}(1-\gamma) = \frac{1}{2}(1-0.95) = 0.025$

$\Rightarrow c_1 = F^{-1}(0.025) = 5.01$ (From χ^2 table with 13 degree of freedom)

$F(c_2) = \frac{1}{2}(1+\gamma) = \frac{1}{2}(1+0.95) = 0.975$

$\Rightarrow c_2 = F^{-1}(0.975) = 24.74$ (From χ^2 table with 13 degree of freedom)

3: $K_1 = \frac{13 \cdot s^2}{5.01}$; $K_2 = \frac{13 \cdot s^2}{24.74}$

where $s^2 = \frac{1}{14-1} \sum_{i=1}^{14} (x_i - \bar{x})^2$; $\bar{x} = \frac{1}{14}(89 + 84 + \cdots 83 + 89)$