Lecture 10 - 12 ①

Central Limit Theorem ÷ Let $X_1, X_2, \dots X_n$ be i.i.d r.v with finite mean μ and finite variance σ^2. Let $\bar{X}$ be the sample mean of $X_1, X_2, \dots X_n$. Then

$$\frac{\bar{X} - E(\bar{X})}{\sqrt{V(\bar{X})}} = \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right) \sim N(0,1) \text{ as } n \to \infty.$$

Note ÷ We can use C.L.T, when $n > 30$.

Note ÷ Most of the distribution can be approximated into standard Normal distribution using CLT.

(EX) Let $X_1, X_2, \dots X_n$ be i.i.d and taken from Bernoulli distribution with parameter p i.e $Ber(p)$.

Define $S_n = \sum_{i=1}^{n} X_i$, then

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \sim N(0,1) \text{ as } n \to \infty$$

~~i.e S_n~~

Note ÷ Bernoulli distⁿ is special case of Binomial distⁿ when $n=1$. The sum of n Bernoullian trial is again turns into Binomial distⁿ.

So $S_n \sim Bin(n, p)$.

$$\Rightarrow \lim_{n \to \infty} \frac{S_n - np}{\sqrt{npq}} \longrightarrow N(0,1) \text{ (By using C.L.T)}$$

(2)

Ex Let $X_1, X_2 \ldots X_n \overset{iid}{\sim}$ Poission (λ). Then show that

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \sim N(0,1) \quad \text{when } n \to \infty$$

where $S_n = X_1 + X_2 \cdots + X_n$

Sol^n^

$\because S_n = X_1 + X_2 + \cdots + X_n$

$$E(S_n) = E(X_1 + X_2 \cdots + X_n)$$
$$= E(X_1) + E(X_2) + \cdots + E(X_n)$$
$$= \lambda + \lambda + \cdots + \lambda$$
$$= n\lambda$$

$$V(S_n) = V(X_1 + X_2 \cdots + X_n) \qquad X_i \sim iid$$
$$= V(X_1) + V(X_2) + \cdots + V(X_n) + 0 + 0 \cdots$$
$$= \lambda + \lambda + \cdots + \lambda$$
$$= n\lambda$$

Covariance term will be zero since X_i are independent

Thus
$$\lim_{n \to \infty} \frac{S_n - n\lambda}{\sqrt{n\lambda}} \to N(0,1) \qquad \#$$

(By C.L.T)

Q: A fair coin is tossed 720 times. Use CLT to find the probability of getting 100 to 140 sixes.

Sol^n^ $n = 720$; $p = \frac{1}{6}$; $q = \frac{5}{6}$

$X \sim Bin(720, \frac{1}{6})$

$P(100 \le X \le 140)$?

Sol^n^ $P(100 \le X \le 140)$? $= P\left(\frac{100 - E(X)}{\sqrt{V(X)}} \le \frac{X - E(X)}{\sqrt{V(X)}} \le \frac{140 - E(X)}{\sqrt{V(X)}}\right)$

$\because E(X) = np = 720 \times \frac{1}{6} = 120$

$V(X) = npq = 720 \times \frac{1}{6} \times \frac{5}{6} = 100$

$$P(100 \le X \le 140) = P\left(\frac{100-120}{\sqrt{100}} < \frac{X-120}{\sqrt{100}} < \frac{140-120}{\sqrt{100}}\right)$$

$$= P(-2 \le Z \le 2)$$

$$= \Phi(2) - \Phi(-2)$$

$$= 2\,\Phi(2) - 1$$

$$= 2 \times 0.9772 - 1 \quad \text{(from table)}$$

$$= 0.954 \quad \text{Ans}$$

Chapter - 6

Point Estimation

Estimator and Estimate :- Estimator is a function of random sample say $\underline{X} = (X_1, X_2 \cdots X_n)$. Then $T(\underline{X})$ is said to be estimator. and $T(\underline{x})$ when $\underline{x}$ is observed values of $\underline{X}$ say $(x_1, x_2 \cdot x_n)$ is known as estimate. Estimator is used to estimate the unknown parameters present in the population.

Parameter Space : This is the set of all possible values of the parameters. and it is donted by $\Theta \to$ capital theta.

(Ex) suppose $X \sim N(\mu, \sigma^2)$ where μ is unknown and σ^2 is known the

$$\Theta = \{\mu : -\infty < \mu < \infty\}$$

Desired Properties of Estimators :-

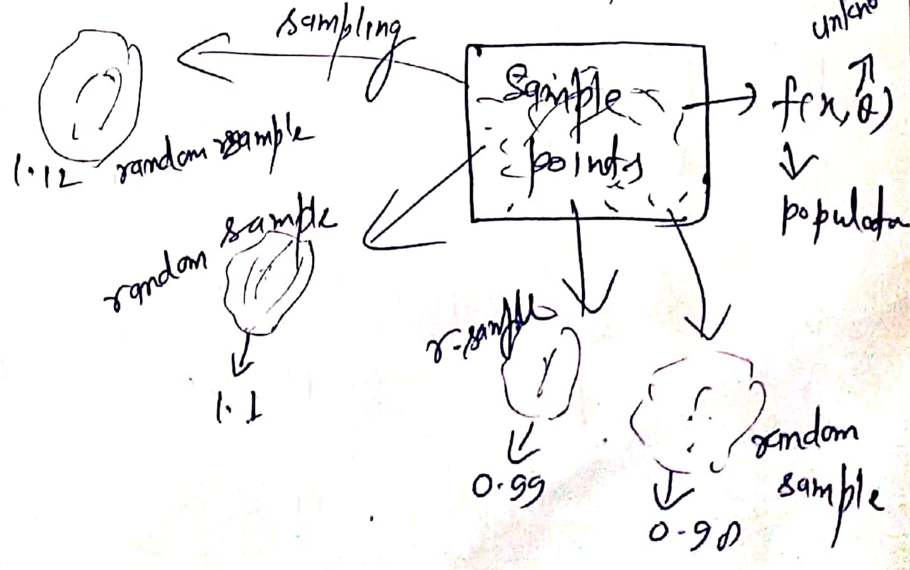
(i) Unbiased (ii) Consistent (iii) sufficient (iv) Efficient. (Not in syllabus)

Point Estimate :- It is defined as a particular value of statistic which is used to estimate a given parameter. Point estimation is a single valued estimation and is also called the estimation of the parameter.

(Ex) Suppose we want to estimate a true value of the parameter θ say 1 ie $\theta = 1$ by point estimation.

Then if we provide a single pt for example 1.1 or 0.99 or 0.98 or 1.12.

So these values are point estimate for $\theta = 1$.



Method of point Estimation.

imp (i) Maximum likelihood estimation

(ii) Method of Moments

(iii) Least square method

Likelihood Function: Let $X_1, X_2 . \, X_n$ be a random sample of size n from a population $f(x, \theta)$.

Then the likelihood function of the sample values $x_1, x_2 - x_n$ usually denoted by $L = L(\theta)$ is their joint density or joint p.m.f and given as

$$L(\underline{x}, \theta) = f(x_1, \theta) . f(x_2, \theta) -- f(x_n, \theta)$$

$$= \prod_{i=1}^{n} f(x_i; \theta) \quad ; \theta \in \Theta$$

The values of θ, say $\hat{\theta}(\underline{x})$ for which

$$L(\hat{\theta}, \underline{x}) \geq L(\theta, \underline{x}) \quad \forall \, \theta \in \Theta$$ is

called MLE of θ.

Steps for MLE :- Let $X_1, X_2 \dots X_n \overset{iid}{\sim} f(x, \theta)$.

(1) First write the likelihood function of θ.

$$L(\theta) = f(x_1, \theta) \cdot f(x_2, \theta) \dots f(x_n, \theta)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

(2) Take logarithm of likelihood function. (base e)

Since log is increasing function so resultant will be same either we maximize $L(\theta)$ or $\log L(\theta)$. But taking log will reduce the calculation difficulties.

$$l = \log L(\theta) = \ln f(x_1, \theta) + \ln f(x_2, \theta) + \dots + \ln f(x_n, \theta)$$

$$= \sum_{i=1}^{n} \ln f(x_i, \theta).$$

$\left(\because \ln(x_1, x_2 \dots x_n) = \ln x_1 + \ln x_2 \dots + \ln x_n \right)$

(3) Take partial derivative w.r.t. θ

$$\frac{\partial l}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=1}^{n} \ln f(x_i, \theta)$$

(4) For maxima put $\frac{\partial l}{\partial \theta} = 0$ and find the value of θ in terms of x

and check that $\frac{\partial^2 l}{\partial \theta^2} < 0$ then

$\hat{\theta} = \theta(x)$ is MLE for θ. #

Ex: Let $X_1, X_2, \ldots X_n \overset{iid}{\sim} P(\lambda)$. Find MLE for λ.

Soln $P(\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$L(\lambda) = \frac{e^{-\lambda} \lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \cdots \frac{e^{-\lambda} \lambda^{x_n}}{x_n!}$$

$$l = L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{1}{x_i!}$$

Next, take log both side

$$\ln l = -n\lambda + \left(\sum_{i=1}^{n} x_i\right) \ln \lambda + \ln\left(\prod_{i=1}^{n} \frac{1}{x_i!}\right)$$

Next take partial diff. w.r.t λ

$$\frac{\partial \ln l}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} + 0$$

For maxima

$$\frac{\partial \ln l}{\partial \lambda} = 0 \Rightarrow -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} = 0$$

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

Now; $\frac{\partial^2 \ln l}{\partial \lambda^2} = -\frac{\sum_{i=1}^{n} x_i}{\lambda^2} < 0$ for $\lambda = \bar{x}$

Thus $\hat{\lambda} = \bar{x}$ is MLE for λ.

Q: let $X_1, X_2 \ldots X_n \overset{iid}{\sim} \exp(\theta)$. Then find MLE for θ

Sol^n

$$f_X(x;\theta) = \theta e^{-\theta x} \quad ; \quad x > 0 ; \theta > 0$$

(exponential distribution)

$$L(\theta) = \theta . e^{-\theta x_1} . \theta e^{-\theta x_2} \ldots \theta e^{-\theta x_n}$$

$$= \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$= \theta^n . e^{-\theta \sum_{i=1}^{n} x_i}$$

$$l = \ln L(\theta) = n \ln \theta - \theta \sum_{i=1}^{n} x_i$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \sum_{i=1}^{n} x_i \Rightarrow \theta = \frac{\cancel{\sum_{i=1}^{n} x_i}}{n} =$$

$$\Rightarrow \theta = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

Now; $\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} = -\frac{n}{(1/\bar{x})^2} < 0 \quad \because \bar{x} > 0$

Thus $\hat{\theta} = \frac{1}{\bar{x}}$ is MLE for θ #