Sampling Distributions: ①

Chi-Square Distribution: A r.v X is said to have χ^2 distⁿ of n degree of freedom if its pdf is given as.

$$f_X(x) = \frac{x^{n/2 - 1} \, e^{-x/2}}{2^{n/2} \, \Gamma\frac{n}{2}} \; ; \quad x > 0$$

$$E(X) = n$$

$$V(X) = 2n$$

$$M_X(t) = \frac{1}{(1-2t)^{n/2}}$$

Degree of freedom: No of free components

or

no. of pieces of information required to estimate population values

Note: It is special case of Gamma distⁿ.

Take $\alpha = \frac{n}{2}$, $\beta = 2$

Note: If $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(0,1)$

then $S_n = \sum_{i=1}^{n} X_i^2$ has $\chi^2_{(n)}$.

Note: (i) If $X \sim N(0,1)$ then $X^2 \sim \chi^2_{(1)}$

(ii) If $X_1, X_2 \overset{iid}{\sim} N(0,1)$ then $X_1^2 + X_2^2 \sim \chi^2_{(2)}$

and $E(X_1^2 + X_2^2) = 2$

$V(X_1^2 + X_2^2) = 2 \cdot 2 = 4$

(iii) If $X \sim N(\mu, \sigma^2)$ then

$$Z = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}.$$

Note! If $X_1, X_2 \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. then

$$\sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}.$$

③

Prove that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$ where

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ and s^2 is sample variance

Proof: $\because X_i \sim N(\mu, \sigma^2)$

$$\frac{X_i - \mu}{\sigma} \sim N(0,1)$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1)$$

Now consider

$$W = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$$

$$= \sum_{i=1}^{n}\left(\frac{(X_i - \bar{X}) + (\bar{X} - \mu)}{\sigma}\right)^2$$

$$= \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2 + \sum_{i=1}^{n}\left(\frac{\bar{X} - \mu}{\sigma}\right)^2 + \frac{2(\bar{X} - \mu)}{\sigma}\sum_{i=1}^{n}(X_i - \bar{X})$$

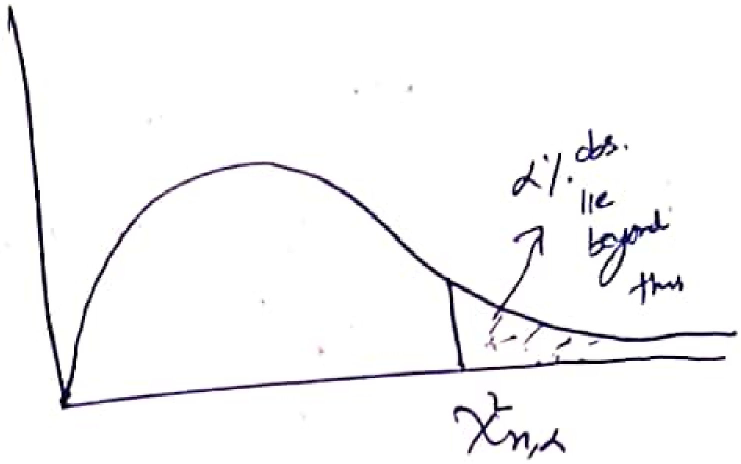
$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 + \frac{n}{\sigma^2}(\bar{X} - \mu)^2 + \frac{2}{\sigma}(\bar{X} - \mu)(n\bar{X} - n\bar{X})$$

$$W = \frac{1}{\sigma^2}\cdot\frac{(n-1)}{(n-1)}\cdot\sum_{i=1}^{n}(X_i - \bar{X})^2 + \underbrace{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2_{(1)}}$$

$W \downarrow \chi^2_{(n)}$

$$\Rightarrow \frac{(n-1)\cdot s^2}{\sigma^2} \sim \chi^2_{(n-1)}.$$

Curve of $\chi^2_{(n)}$:

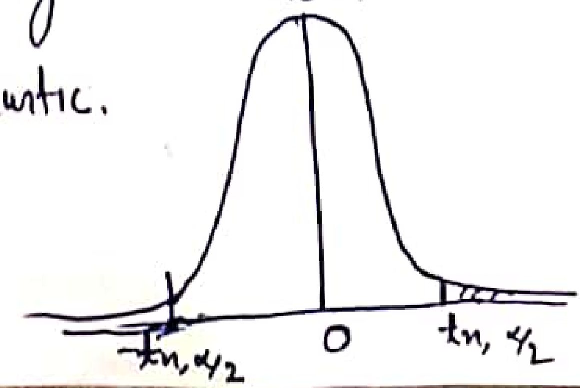


Student - t distribution: Let $X \sim N(0,1)$ and $Y \sim \chi^2_{(n)}$ and X and Y are independent then $T = \dfrac{X}{\sqrt{Y/n}}$ is said to have Student - t distⁿ with p.d.f

$$f_T(t) = \frac{\Gamma\frac{n+1}{2}}{\sqrt{n\pi}\,\Gamma\frac{n}{2}\left(\frac{t^2}{n}+1\right)^{\frac{n+1}{2}}} \; ; -\infty < t < \infty$$

<u>Properties:</u> (i). $f_T(t)$ is symmetric about 0.

(ii) It is leptokurtic.



<u>Note:</u> If $X \sim$ t-distⁿ then $E(X) = 0$

F- distribution:

Let X and Y be independent χ^2 r.vs with m and n degree of freedom respectively then r.v

$$F = \frac{X/m}{Y/n}$$

is said to have a F-distribution with (m, n) degree of freedom. We write $F \sim F(m,n)$ and pdf is given as

$$g_F(f) = \frac{\Gamma\frac{m+n}{2} \cdot \left(\frac{m}{n}\right) \left(\frac{m}{n} \cdot f\right)^{\frac{m}{2}-1} \left(1+\frac{m}{n} f\right)^{-\left(\frac{m+n}{2}\right)}}{\Gamma\frac{m}{2} \cdot \Gamma\frac{n}{2}} ; f > 0$$

Note! If $X \sim F(m,n)$ then $\frac{1}{X} \sim F(n,m)$

Note! If $m = 1$ then $F(1,n) \sim \chi^2_{(n)}$

(Ex): $\bar{X}$ and S^2 are independent, where $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$ for $i = 1, 2 \cdots n$.

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \sim N(0, 1)$$

Also $$\frac{(n-1) \cdot S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

then $$\frac{\sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right)}{\sqrt{\frac{(n-1) \cdot S^2}{\sigma^2 \cdot (n-1)}}} \sim t_{(n-1)}$$

$$\Rightarrow \sqrt{n}\left(\frac{\bar{X}-\mu}{S}\right) \sim t_{(n-1)}.$$

(Ex): indep.

$X_1, X_2, \cdots, X_m \sim N(\mu_1, \sigma_1^2)$, $\bar{X}$, S_1^2

$Y_1, Y_2, \cdots, Y_n \sim N(\mu_2, \sigma_2^2)$, $\bar{Y}$, S_2^2

$$S_1^2 = \frac{1}{m-1} \sum_{i=1}^{m} (X_i - \bar{X})^2, \quad S_2^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

$$\frac{(m-1) \cdot S_1^2}{\sigma_1^2} \sim \chi^2_{(m-1)} \text{ and } \frac{(n-1) \cdot S_2^2}{\sigma_2^2} \sim \chi^2_{(n-1)}$$

$$\Rightarrow \frac{(m-1) \cdot S_1^2}{\sigma_1^2 (m-1)} \Big/ \frac{(n-1) \cdot S_2^2}{\sigma_2^2 (n-1)} \sim F_{(m-1, n-1)}$$

$$\Rightarrow \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F_{(m-1, n-1)}.$$

Note: If $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$\Rightarrow \frac{S_1^2}{S_2^2} \sim F_{(m-1, n-1)}.$$

Result : $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$$E\left(\frac{(n-1)s^2}{\sigma^2}\right) = n-1$$

$$\frac{(n-1)}{\sigma^2} E(s^2) = (n-1)$$

$$\Rightarrow \boxed{E(s^2) = \sigma^2}$$

$$V\left(\frac{(n-1)s^2}{\sigma^2}\right) = 2(n-1)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} \cdot V(s^2) = 2(n-1)$$

$$\Rightarrow V(s^2) = \frac{2(n-1)\cdot\sigma^4}{(n-1)^2}$$

$$\Rightarrow \boxed{V(s^2) = \frac{2\cdot\sigma^4}{(n-1)}}$$

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