Conditional PMF :- The conditional pmf of X given $Y = y_j$ is defined as

$$P(X = x_i \mid Y = y_j) = p_{X|Y=y_j}(x_i|y_j) = \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)}; \quad x_i \in R_X \qquad -(1)$$

provided $p_Y(y_j) > 0$.

* $X|Y$ → means X given Y ($Y = y_j$ is fixed)

Note :- ~~Condi~~ Verify that it is proper pmf

(i) $p_{X|Y=y_j}(x_i|y_j) \geq 0$

(ii) Claim: $\sum_{x_i \in R_X} p_{X|Y=y_j}(x_i|y_j) = 1$

Proof:

$$\sum_{x_i \in R_X} p_{X|Y=y_j}(x_i|y_j)$$

$$= \sum_{x_i \in R_X} \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)} \quad (\text{By def}^n)$$

$$= \frac{1}{p_Y(y_j)} \underbrace{\sum_{x_i \in R_X} p_{X,Y}(x_i, y_j)}_{\text{Marginal of } Y}$$

$$= \frac{1}{p_Y(y_j)} \cdot p_Y(y_j)$$

$$= 1$$

Similarly, conditional of Y given X is defined as

$$P(Y = y_j \mid X = x_i) = p_{Y|X=x_i}(y_j \mid x_i) = \frac{p_{X,Y}(x_i, y_j)}{p_X(x_i)}; \ y_j \in R_Y$$

$$; \ p_X(x_i) > 0$$

Note :- Again we can verify that it is proper p.m.f

(i) $p_{Y|X=x_i}(y_j \mid x_i) \geq 0$

(ii) $\sum\limits_{y_j \in R_Y} p_{Y|X=x_i}(y_j \mid x_i) = \sum\limits_{y_j \in R_Y} \frac{p_{X,Y}(x_i, y_j)}{p_X(x_i)}$

$= \frac{1}{p_X(x_i)} \cdot p_X(x_i) \longrightarrow$ Marginal of X

$= 1.$

Q: In previous ~~question~~ question, find (i) $P(X \mid Y = 0)$

(ii) $P(X \mid Y = 1)$ (iii) $P(Y \mid X = 1)$

Soln (i) $X \to 0, 1, 2$ (possible values of X)

Claim: $P(X = x_0 \mid Y = 0)$?

$$P_{X|Y}(X = 0 \mid Y = 0) = \frac{p_{X,Y}(0, 0)}{p_Y(0)} = \frac{\frac{1}{21}}{\frac{5}{21}} = \frac{1}{5}$$

Next,

$$p_{X|Y}(X=1|Y=0) = \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{\frac{1}{7}}{\frac{5}{21}} = \frac{1}{7}\times\frac{21}{5} = \frac{3}{5}$$

$$p_{X|Y}(X=2|Y=0) = \frac{p_{X,Y}(2,0)}{p_Y(0)} = \frac{\frac{1}{21}}{\frac{5}{21}} = \frac{1}{5}$$

So; Conditional p.m.f of $X|Y=0$ is given as

$X\|Y=0$	0	1	2
$p_{X\|Y}(x_i\|y_j)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Here, again we can verify that it is proper p.m.f.

Note:- Whatever given in conditional pmf, that is fixed values, thats why we have written $x_i \in R_X$ in eq (1) because $p_Y(y_j)$ is fixed value.

Similarly we can calculate other probabilities.

Independent Random Variable :- Two random variable X and Y are said to be independent

if $\underbrace{p_{X,Y}(x_i, y_j)}_{\text{Joint p.m.f}} = \underbrace{p_X(x_i)}_{\text{Marginal of } X} \cdot \underbrace{p_Y(y_j)}_{\text{Marginal of } Y}$; $\forall (x_i, y_j) \in R_{X \times Y}$

Note: If for any pair, the above defn fails, we say that X and Y are dependent.

e.g. $p_{X,Y}(0,0) = \frac{1}{21}$; $p_X(0) = \frac{5}{12}$; $p_Y(0) = \frac{5}{21}$ | $p_{X,Y}(0,0) \neq p_X(0).p_Y(0)$ so Dependent

Proceeding in similar fashion, all the defination of joint pmf can be extended to Joint pdf (continuous r-v)

Joint Pdf: If (X, Y) are continuous bivariate r-v then joint pdf of (X,Y) is denoted as $f_{X,Y}(x,y)$

and satisfies the following properties,

(i) $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y)$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$

Marginal Pdf: Marginal density of X is given as

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

→ function of x

→ Integrate joint density w.r.t y

Similarly, marginal of Y is defined as

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

→ function of y.

Note: (1) To find marginal of X, we need to integrate the joint pdf w.r.t Y means limit of integration is the range of Y and vice-versa

(2) Again, we can verify that these marginals are proper pdf.

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$ (joint density)

$= 1$

Conditional Pdf: Conditional pdf of $X|Y$ is defined as

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \; ; \; f_Y(y) > 0$$

$f_{X|Y=y}(x|y) \downarrow$ fixed; $f_Y(y) \rightarrow$ fixed

Similarly, conditional of $Y|X$ is defined as

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \; ; \; f_X(x) > 0$$

$f_X(x) \rightarrow$ fixed

Note:- Verify that it is proper pdf.

(i) $f_{X|Y=y}(x|y) \geq 0$

(ii) Claim $\int_{-\infty}^{\infty} f_{X|Y=y}(x|y)\,dx = 1$

$$\int_{-\infty}^{\infty} f_{X|Y=y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{1}{f_Y(y)} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}_{\text{marginal of } Y}$$

$$= \frac{1}{\cancel{f_Y(y)}} \cdot \cancel{f_Y(y)} = 1$$

Lecture - 5 ①

Change of ~~Integ~~ Order of Integration :-

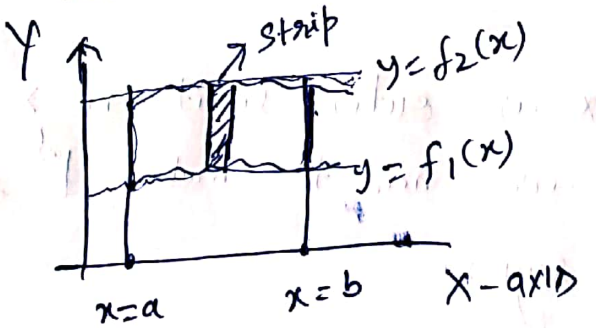
$$\int_{x=a}^{b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx = \int_{y=c}^{d} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

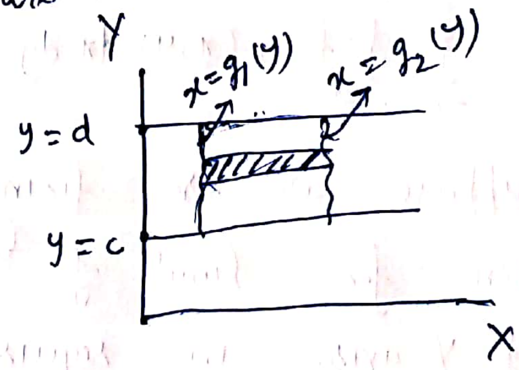
Note: ① If the double integral is given then first integrate w.r.t internal limit variable then w.r.t external limit variable.

② Outer limit of double integral should be fixed.

③ Change of order of integration is important because sometimes by changing the order of integration of variable, double integral can be solved easily.

④ If the limit of x is fixed on X-axis then we need to draw a strip for limit of y, which would be parallel to Y-axis. Similarly, if the limit of y is fixed on Y-axis, then for limit of x, we need to draw a strip parallel to X-axis.





Q: ②

$$f_{X,Y}(x,y) = \begin{cases} 2 & ; \ 0 \le x \le y \le 1 \\ 0 & ; \ \text{otherwise.} \end{cases}$$

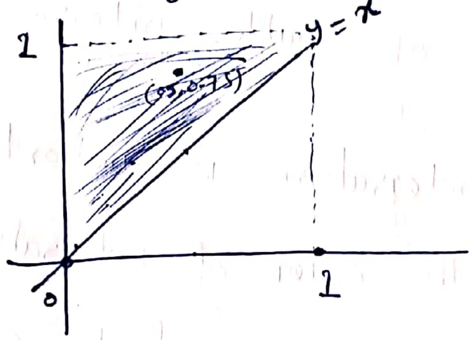
(i) Verify that it is joint pdf.

(ii) Marginal of x and y

(iii) Conditional of x given y and y given x.

(iv) $E(X)$; $V(X)$.

Solⁿ First draw the range of integration.



Now, you need to verify the region $x \le y$.
So take any random pt say (0.5, 0.75) and draw it on above graph so you can easily find the required region.

Thus shaded region is required region for integration.

Now,

(i) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1.$

Suppose we are fixing x as outer limit on X-axis. then for limit of y, we need to make $\parallel^{el}$ strip to Y-axis in required region.

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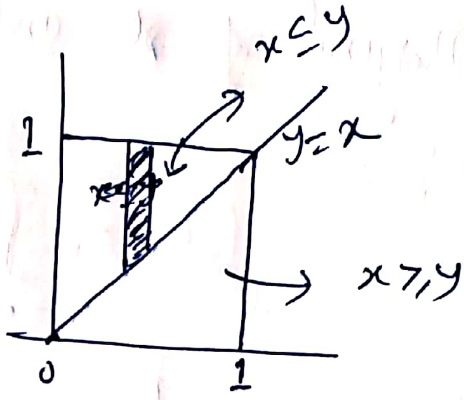
Claim:

$$\int_{x=0}^{1}\int_{y=x}^{1} f_{X,Y}(x,y)\,dy\,dx = 1$$



Note: y is varying from the curve $y = x$ to $y = 1$ and first integrate w.r.t internal limit y then w.r.t external (fix) limit x.

$$\int_{x=0}^{1}\int_{y=x}^{1} 2\,dy\,dx = \int_{x=0}^{1} \left[2y\right]_{y=x}^{1} dx$$

$$= 2\int_{x=0}^{1} (1-x)\,dx$$

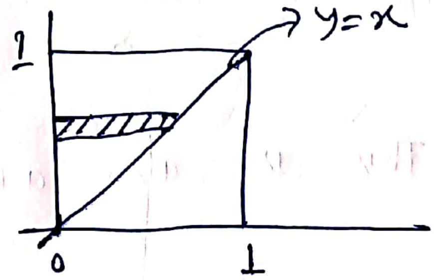
$$= 2\left[x - \frac{x^2}{2}\right]_0^1$$

$$= 2\left[\left(1-\frac{1}{2}\right) - 0\right] = 2\cdot\frac{1}{2} = 1$$

#

Note: We can also verify it by fixing outer limit as y then make $\parallel^{el}$ strip to x-axis for internal limit x.

$$\int_{y=0}^{1}\int_{x=0}^{y} f_{X,Y}(x,y)\,dx\,dy = 1$$



(verify)

$$\int_{y=0}^{1}\int_{x=0}^{y} 2\,dx\,dy = 1$$

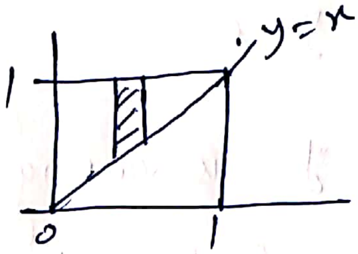
(9)

(ii) <u>Marginal of x and y.</u> $f_{X,Y}(x,y) = \begin{cases} 2 ; & 0 \le x \le y \le 1 \\ 0 ; & \text{elsewhere} \end{cases}$

<u>By defn:</u> $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \; ; \; -\infty < x < \infty$

Here

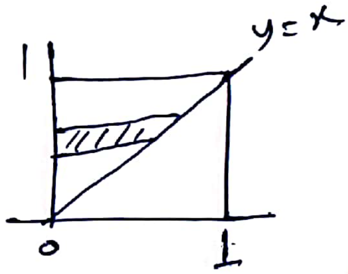
$$f_X(x) = \int_{y=x}^{1} 2\, dy$$



$$f_X(x) = \begin{cases} 2(1-x) ; & 0 \le x \le 1 \\ 0 ; & \text{otherwise} \end{cases}$$

We can verify that this is also a pdf.

Now; $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$



$$= \int_{x=0}^{y} f_{X,Y}(x,y)\, dx$$

$$= \int_{x=0}^{y} 2\, dx$$

$$f_Y(y) = 2 \cdot y \; ; \; 0 \le y \le 1$$

$$= 0 \; ; \; \text{elsewhere}$$

This is also a pdf.

5

Conditional pdf of $X|Y=y$ and $Y|X=x$

By defn

$$f_{X|Y=y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} ; \quad -\infty \leq x \leq \infty$$

$$f_{X|Y=y}(x|y) = \frac{2}{2y} ; \quad 0 \leq x \leq y$$

is constant because it is given

$$f_{X|Y=y}(x|y) = \frac{1}{y} ; \quad 0 \leq x \leq y$$

constant

function of x (fortunately in this quest it is constant)

Thus

$$f_{X|Y=y}(x|y) = \begin{cases} \frac{1}{y} ; & 0 \leq x \leq y \\ 0 ; & \text{otherwise} \end{cases}$$

This is also a pdf.

We can show it.

$$\int_{x=0}^{y} f_{X|Y=y}(x|y)\, dx = 1$$

$$\int_{x=0}^{y} \frac{1}{y} \cdot dx = \frac{1}{y} \int_{x=0}^{y} dx = \frac{1}{\cancel{y}} \cancel{y} = 1$$

constant

Similarly,

conditional of $Y|X=x$

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad ; -\infty \le y < \infty$$

$$= \frac{\cancel{2}}{\cancel{2}(1-x)} \quad ; \underbrace{x}_{\text{fixed (constant)}} \le y \le 1$$

$$f_{Y|X=x}(y|x) = \begin{cases} \frac{1}{1-x} & ; x \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

You can verify that it is pdf for that

$$\int_{y \in R_Y} f_{Y|X=x}(y|x)\, dy = 1$$

$$\Rightarrow \int_{y=x}^{1} \frac{1}{1-x} \cdot dy = \frac{1}{1-x} \int_{y=x}^{1} dy$$

$$= \frac{1}{\cancel{1-x}} \cancel{(1-x)} = 1$$

#