Chapter 6

Graph Theory

6.1 Basic Concepts and Definitions

Configurations of points and their connections occur in a great diversity of applications. For example, the points could be communication centres with the lines denoting communication links or they may represent other physical networks, such as electrical circuits, roadways, or organic molecules. They are also used in representing sociological relations, databases, or in a computer program. In fact, we see that many real-world situations can conveniently be described by means of such configurations, which we call **graphs**, modeled by combinatorial structures, consisting of two sets called points (vertices) and lines (edges) and an incidence relation between them. Thus we think of a **graph** as a set of points in a plane or in 3-space and a set of line segments, each of which either joins two points or joins a point to itself.

Graphs and Digraphs

A **graph** G is a mathematical structure consisting of two sets $V(G)$ and $E(G)$. V is a nonvoid set and the elements of V are called **points or vertices**, and the elements of E are called lines or **edges**, such that each edge e in E is assigned an unordered pair of vertices (u, v), called the end **vertices** of e. An edge is said to join its endpoints.

A **trivial graph** is a graph consisting of one vertex and no edges.

A **null graph** is a graph whose vertex set has n vertices and edge set is empty.

A **Self-loop** is an edge that joins a single endpoint to itself.

A **proper edge** is an edge that it not a self-loop.

A **multiedge** is a collection of two or more edges having identical endpoints. These edges are called **multiple edges** or **parallel edges.**

A graph is called **simple graph** if it has no self-loops and no multiple edges.

A graph is called a **multigraph** if it has no self-loop but between a pair of vertices there can be more than one edge, *i.e.*, it has multiple edges.

A graph is called a **pseudograph** if both self-loops and multiple edges are allowed.

Two vertices u and v of a graph G, which are joined by an edge e are said to be **adjacent vertices.** We say that the vertex u and the edge e are **incident** with each other, similarly, the vertex v and the edge e are incident with each other. If two distinct edges are incident with a common vertex, then they are called **adjacent edges.**

A vertex of G which is not the end of any edge is called **isolated vertex.**

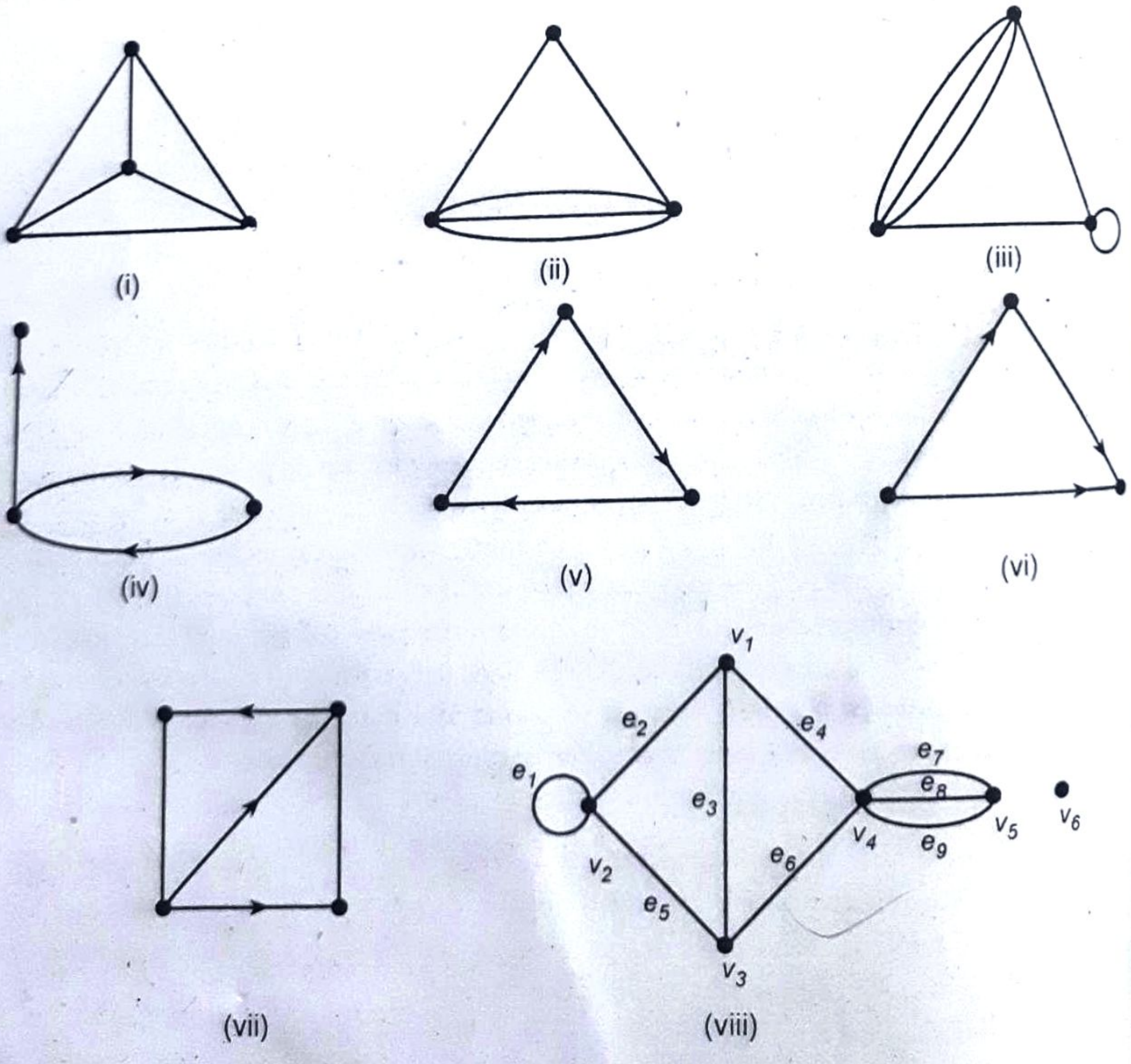

Fig. 6.1 : Some graphs

An edge between two vertices creates a connection in two opposite senses, forward and backward directions. In a line drawing of the graph the choice of forward direction is indicated by placing an arrow on an edge.

Fig. 6.2 : A simple diagraph

A **directed graph** (or **digraph**) is a graph each of whose edges is directed. A directed edge is represented by an ordered pair of vertices. In this case an edge (u, v) is said to join from u to v and (u, v) and (v, u) are two different lines.

The **underlying graph** of a directed graph is the graph that results by deleting all the edge-directions.

An **oriented graph** is a directed graph having no symmetric pair of directed edges.

A **partially directed graph** is a graph that has both undirected and directed edges. However, we can think of an undirected edge to be two edges between the same pair of vertices but in opposite directions.

Example 6.1 *The graph in Fig. 6.1(i) is a simple graph, in Fig. 6.1(ii) a multigraph and in Fig. 6.1(iii) a pseudograph. Fig. 6.1(iv), (v), (vi) are digraphs in which (v) and (vi) are oriented graphs. The graph in Fig. 6.1(viii) has vertex set*

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and edge set $$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}.$$

The edge e_1 is a self-loop and the edges e_7, e_8, e_9 are multiple edges. The vertex v_6 is a isolated vertex, and e_2, e_3, e_4 are adjacent edges. Fig. 6.1(vii) is a partially directed graph. □

Example 6.2 *The graph in Fig. 6.2 is a simple digraph. Recall that a graph or a directed graph is simple if it has neither self-loops nor multiedges. In this graph the edge (v_1, v_2) and (v_2, v_1) are two different edges. One is from v_1 to v_2 and other is from v_2 to v_1 so these are not parallel or multiedges. However its underlying graph is not simple since in the underlying graphs these edges becomes parallel edges.* □

Mathematical Modelling with Graphs

Graphs are essentially versatile tools for modelling the real-world problems. In situations where the graphs have no parallel edges, it is convenient to refer to a directed edge as an ordered pair of vertices, where the first component specifies the tail and the second component specifies the head. These formal specifications for a graph and a digraph can easily be implemented with a variety of programmer-defined data structures. However, from the perspective of object-oriented software design, the ordered-pair representation treats digraphs as a different class of objects from graphs, and therefore a large part of computer code have to be rewritten in order to adapt an algorithm that was original designed for a digraph to work on an undirected graph. For applying graph theory to real-world problems, one must model the problem mathematically first.

Example 6.3 *A model for a road map is represented by a partially directed graph in Fig. 6.3. Landmarks are represented by the vertices, and the directed and undirected edges represent the one-way and two-way streets, respectively.*

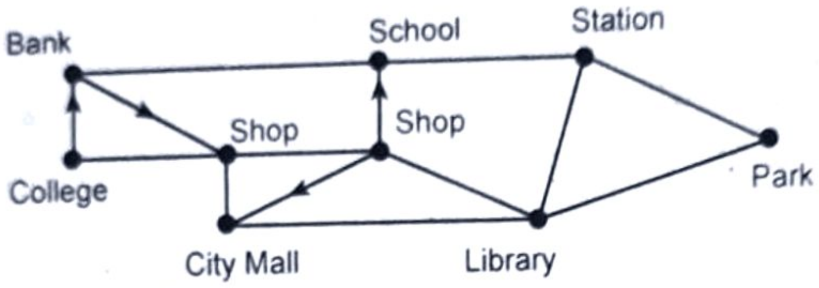

Fig. 6.3 : Graph for a road map

6.2 Degree of a Vertex in Graph

The **degree** of a vertex v in a graph G, denoted deg (v), is the number of proper edges incident on v plus twice the number of self-loops incident on v.

Thus a vertex v of degree zero is an **isolated vertex,** and an **endvertex** if deg $(v) = 1$.

The minimum of all the degrees of the vertices of a graph G is denoted by $\delta(G)$, and the maximum of all the degrees of the vertices of G is denoted by $\Delta(G)$. If $\delta(G) = \Delta(G) = m$, that is each vertex of G has the same degree m, then G is said to be a **regular graph** of degree m or **m-regular graph**. 3-regular graphs are called **cubic graphs.** Some interesting 3-regular graphs are shown in Fig. 6.5. In a simple graph G with p vertices, we have

$$0 \leq deg\ (v) \leq p - 1$$

for every vertex v in G.

The **degree sequence of** a graph is the sequence formed by arranging the vertex degrees in nondecreasing order. For example, the degree sequence of the graph in Fig. 6.1*(viii)* is (0, 3, 3, 3, 4, 5). Two structurally different graphs can have identical degree sequences. For example, two different graphs G and G′ in Fig. 6.4 have the same degree sequence (2, 2, 2, 2, 2, 2, 2, 2).

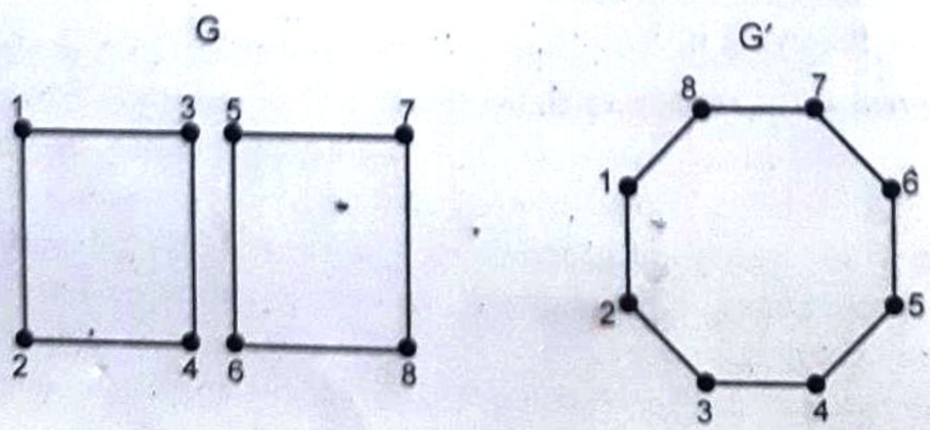

Fig. 6.4 : Two graphs with same degree sequence

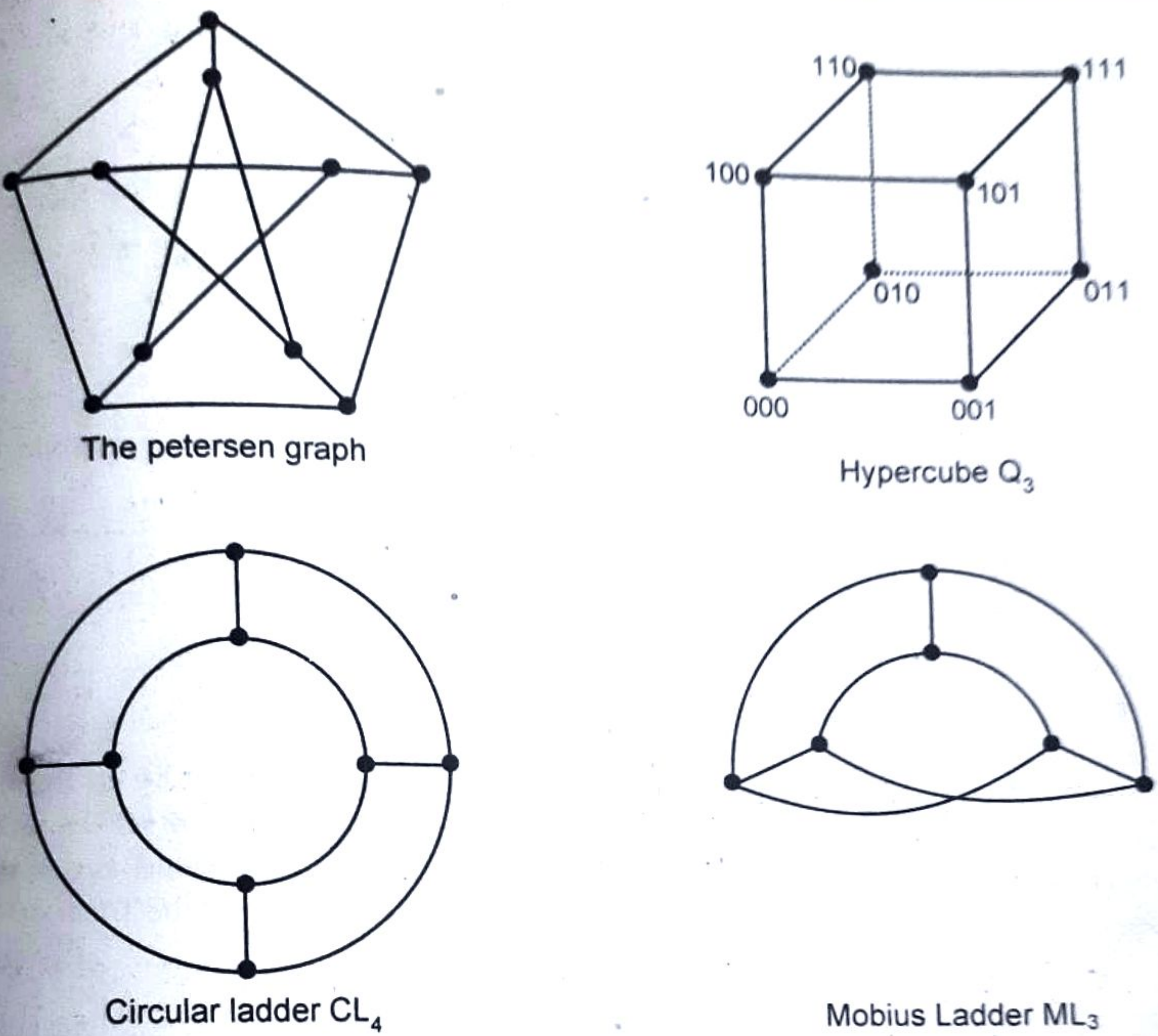

Fig. 6.5 : Some 3-regular graphs

A **vertex** v of a graph G is called **even** and **odd** depending on whether its degree, $deg\ (v)$ is even or odd.

The work of Euler is regarded as the starting of graph theory as a mathematical discipline. The following is the first theorem is Graph Theory.

Theorem 6.1 *For any graph G with e edges and n vertices $v_1, v_2, \ldots, v_n$*

$$\sum_{i=1}^{n} deg\ (v_i) = 2e$$

That is, the sum of the degrees of the vertices of a graph is twice the number of edges.

Proof. Each edge contributes two to the degree sum since every edge is incident to two end vertices. Even a self-loop contributes two to the sum although the two ends are identical. □

Corollary 1.

In a graph G there is an even number of odd vertices.

Proof. Let U be the set of even vertices u_i and W be the set of odd vertices w_j in G then by the above theorem,

$$\Sigma \, deg\,(u_i) + \Sigma \, deg\,(w_j) = 2e,$$

where e are the number of edges in G.

Thus $$\Sigma \, deg\,(w_j) = 2e - \Sigma \, deg\,(u_i).$$

It is an even number, being the difference of two even numbers. As all the terms in the L.H.S. sum are odd and the sum is even, there must be an even number of them, *i.e.*, odd vertices in a graph must be even in number. □

Corollary 2.

Every cubic graph G has an even number of vertices.

Proof. Since every vertex in a cubic graph(3-regular graph) G is of degree odd, G has an even number of vertices. □

Corollary 3.

The degree sequence of a graph G is a nondecreasing sequence of nonnegative integers whose sum is even.

Proof. If n are the number of vertices v_i of G, then

$$0 \le deg\,(v_i) \le n - 1$$

if G is simple. Thus degrees of vertices are nonnegative integers and can be arranged as non-decreasing sequence. Even if G is not simple and has finite edges, may be self-loops or multi-edges, degrees of vertices are nonnegative integers and hence degree sequence is nondecreasing sequence of nonnegative integers and their sum is even follows directly by the Theorem 6.1. □

Theorem 6.2 *A nontrivial simple graph G must have atleast two vertices with equal degree.*

Proof. Let G be a graph with n vertices, then in the degree sequence, n possible degree values, 0, 1, 2,..., $n - 1$, may appear. In a graph, there can not be a vertex of degree 0 and also a vertex of degree $n - 1$, both. Because a vertex of degree zero is isolated and each of the remaining $n - 1$ is adjacent to atmost $n - 2$ other vertices of the graph. Hence the n vertices of G can have atmost $n - 1$ possible degree values, so by the pigeonhole principle atleast two of the n vertices must have equal degree values. □

Theorem 6.3 *If $(d_1, d_2, \ldots, d_n)$ is a sequence of nonnegative integers whose sum is even, then there exists a graph with n vertices v_i, such that for $i = 1, 2, \ldots, n$, $deg\,(v_i) = d_i$.*

Proof. Draw n isolated vertices $v_1, v_2, v_3, v_4, v_5, \ldots, v_n$. Then for each i, if d_i is even, draw $d_i/2$ self-loops on vertex v_i, and if d_i is odd, then draw $(d_i - 1)/2$ self-loops. Since there is an even number of odd vertices (corollary 1), group the vertices associated with the odd degrees into pairs then join each pair by a single edge. Thus the construction of a graph is completed. □

Example 6.4 *Construct graphs whose degree sequences are*

(i) (0, 1,2, 2, 3, 4)

(ii) (1, 1, 2, 3, 3, 4)

Sol. (*i*) Start with six isolated vertices $v_1, v_2, v_2, v_4, v_5, v_6$.

For even-valued terms of the sequence, *i.e.*, for 2, 2, 4 draw one self-loop to each vertex v_3 and v_4 and two self-loops to the vertex v_6. Vertex v_1 remains isolated. For the odd-valued terms, *i.e.*, for 1 and 3, draw a self-loop to v_5 and then join the pair v_1, v_5 by a single-edge. The resulting graph is shown in Fig. 6.6(*a*).

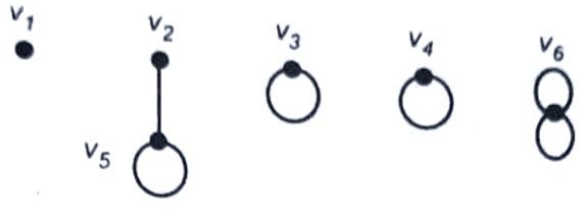

Fig. 6.6(*a*) : Graph with degree sequence (0, 1, 2, 2, 3, 4)

(*ii*) Similarly, we can construct the graph for the degree sequence (1, 1, 2, 3, 3, 4),and the resulting graph is shown in Fig. 6.6(*b*).

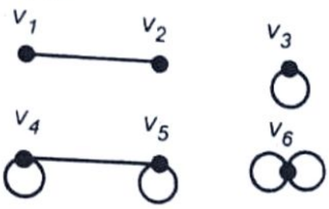

Fig. 6.6(*b*) : Graph with degree sequence (1, 1, 2, 3, 3, 4)

□

6.3 Degree of a Vertex in Digraph

The definition of vertex degree in a digraph is slightly refined and is given as follows :

Let G be a digraph and v_i be the vertices in G.

The indegree of a vertex v_i in G is the number of edges incident into v_i, and is written as $deg^-(v_i)$.

The outdegree of a vertex v_i in G is the number of edges incident out of v_i, and is written as $deg^+(v_i)$.

In Fig. 6.7, for example,

$deg^+(v_1) = 2, \quad deg^-(v_1) = 1,$
$deg^+(v_2) = 2, \quad deg^-(v_2) = 2,$
$deg^+(v_3) = 3, \quad deg^-(v_3) = 1,$
$deg^+(v_4) = 2, \quad deg^-(v_4) = 2,$
$deg^+(v_5) = 1, \quad deg^-(v_5) = 3,$
$deg^+(v_6) = 1, \quad deg^-(v_6) = 2.$

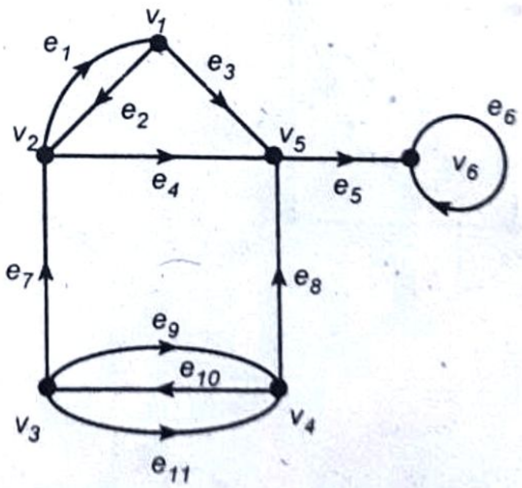

Fig. 6.7 : Digraph with six vertices and eleven edges

Theorem 6.4 *In a digraph G, with n vertices v_i, the sum of the indegrees and the some of the outdegrees both equal the number of edges. That is*

$$\sum_{i=1}^{n} deg^+(v_i) = \sum_{i=1}^{n} deg^-(v_i) = e, \quad \text{where } e \text{ is the number of edges in G.}$$