Moment Generating Function:- Let X be a r.v, then moment generating function (MGF) of X is defined as

$$M_X(t) = E(e^{tX}) = \sum_{x \in R_X} e^{tx} \quad ; \quad |t| < h$$

where $h > 0$ (small) and $t \in \mathbb{R}$

Note! Any order non-central moment can be generated by the help of MGF.

$$\boxed{\left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0} = E(X^n)} \quad - (1)$$

Q: Find the MGF of Binomial distn ?

Soln:

$$E(e^{tX}) = \sum_{x \in R_X} e^{tx} \cdot p_X(x)$$

$M_X(t)$

$$= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x q^n$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^n$$

$$\boxed{M_X(t) = (pe^t + q)^n}$$

Scanned with CamScanner

Q. Find the mean for Binomial distⁿ from its MGF ?

No.- 40

Solⁿ

$M_X(t) = (pe^t + q)^n$; where $q = 1-p$

We know that first non-central moment is its mean.

Put $n=1$ in ①

$$\left.\frac{d\,M_X(t)}{dt}\right|_{t=0} = E(X)$$

$$\Rightarrow E(X) = \left.\frac{d}{dt}(pe^t+q)^n\right|_{t=0}$$

$$= \left. n(pe^t+q)^{n-1}\cdot pe^t \right|_{t=0}$$

$$= n(pe^0+q)^{n-1}\cdot pe^0$$

$$= n(p\cdot 1+q)^{n-1}\cdot p\cdot 1$$

$$= n(p+q)^{n-1}\cdot p$$

$$= n\cdot 1\cdot p$$

$$\boxed{E(X) = np}$$ #

Scanned with CamScanner

Note:- pmf of $Bin(n, p)$ is

$$p_X(x) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0, 1, 2 \ldots n$$

where $n \to$ no of trials.

If take $n = 1$ then $Bin(n, p)$ tends to Bernoulli distribution. It is denoted as $Ber(p)$ ($\because n = 1$)

so pmf is given as

$$p_X(x) = p^x q^{1-x} \quad ; \quad x = 0, 1$$

$$E(X) = p \qquad (\because np = 1 \cdot p)$$

$$V(X) = pq \qquad (\because npq = 1 \cdot p \cdot q)$$

$$M_X(t) = (pe^t + q)$$

Poission Distribution:

If n is very large i-e $n \to \infty$ and p is very small i-e $p \to 0$ s.t $np \to \lambda$ (constant)

then under these conditions, it is suggested to use Poission distn instead of $Bin(n, p)$.

Scanned with CamScanner

No.- 41

Proof: $P_X(x) = \binom{n}{x} p^x q^{n-x} \; ; x = 0,1,2,3 \cdots n$

$$= \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x} \; ; \; x = 0,1,2,3 \cdots n$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))\,\cancel{(n-x)!}\, p^x (1-p)^{n-x}}{x!\,\cancel{(n-x)!}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \qquad \because np \to \lambda \Rightarrow p \to \frac{\lambda}{n}$$

$$= \frac{n \cdot n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{(x-1)}{n}\right)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \underbrace{n \times n \times n \cdots \times n}_{x \text{ times}} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$P_X(x) = \cancel{n^x}\,\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!}\frac{\lambda^x}{\cancel{n^x}}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\lim_{n\to\infty} P_X(x) = \lim_{n\to\infty}\left[(1-0)(1-0)\cdots(1-0)\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{n}\right)^{n-x}\right] \quad (\because n \to \infty)$$

$$= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

Let $y = \left(1 - \frac{\lambda}{n}\right)^{n-x}$

Take log both sides,

$\because \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots$

$$\ln y = (n-x) \ln\left(1 - \frac{\lambda}{n}\right)$$

$$= (n-x)\left[-\left(\frac{\lambda}{n}\right) - \left(\frac{\lambda}{n}\right)^2 \cdot \frac{1}{2} - \left(\frac{\lambda}{n}\right)^3 \cdot \frac{1}{3} \cdots\right]$$

$$\ln y = -(n-x)\frac{\lambda}{n} - \frac{\lambda^2 (n-x)}{n^2 \cdot 2} - \frac{\lambda^3 (n-x)}{n^3 \cdot 3} - \cdots$$

$$\Rightarrow \lim_{n\to\infty} \ln y = \lim_{n\to\infty}\left(\frac{-\lambda \cdot \cancel{n}}{\cancel{n}} + \frac{x\lambda}{n}\right) - \frac{\lambda^2 (n-x)}{n^2 \cdot 2} - \frac{\lambda^3 (n-x)}{n^3 \cdot 3} - \cdots$$

$$\Rightarrow \lim_{n\to\infty} \ln y = -\lambda + 0 - 0 - 0$$

$$\Rightarrow y = e^{-\lambda}$$

Thus

$$p_X(x) = \frac{e^{-\lambda} \lambda^x}{x!} \; ; \; x = 0, 1, 2, 3 \cdots$$

This is p.m.f of Poission $dist^n$ and denoted by $P(\lambda)$.

Q: ① Verify that it is proper p.m.f

② find mean and Variance for poission distn.

No.- 41

③ Find MGF of poission distn.

Soln (1) $\because$ $p_X(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \dots$

Claim: $\sum_{x=0}^{\infty} p_X(x) = 1$

so, $\sum_{x=0}^{\infty} \dfrac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left[\sum_{x=0}^{\infty} \dfrac{\lambda^x}{x!}\right]$

$= e^{-\lambda}\left[\dfrac{\lambda^0}{0!} + \dfrac{\lambda^1}{1!} + \dfrac{\lambda^2}{2!} + \dfrac{\lambda^3}{3!} + \dots\right]$

$= e^{-\lambda}\left[1 + \dfrac{\lambda}{1!} + \dfrac{\lambda^2}{2!} + \dfrac{\lambda^3}{3!} + \dots\right]$

$= e^{-\lambda} \cdot e^{\lambda}$ $\quad \left(\because 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dots = e^x\right)$

$= e^{-\lambda+\lambda}$

$= e^0$

$= 1$ #