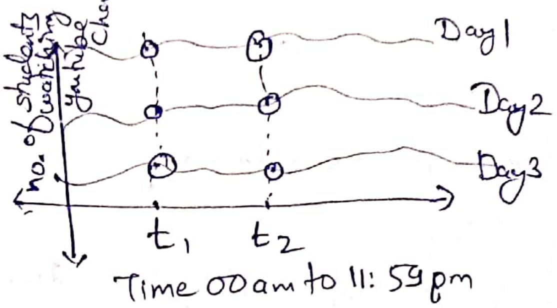
Random Processes

①

A random process is a collection (or ~~ensemp~~ ensemble) of RVs $\{X(s,t)\}$, that are functions of a real variable namely time t where $s \in S$ (sample space) and $t \in T$ (Parameter set or index set).

Sample Space : Collection of all sample pt.
Ensemble : Collection of all sample fn.



→ The set of possible values of any individual member of the random process is called state space.

→ Any individual member itself is called a sample fn or realisation of the process.

⊛ (i) If s & t are fixed, $\{X(s,t)\}$ is a number.

(ii) If s is fixed $\{X(s,t)\}$ is a single time function.

(iii) If t is fixed $\{X(s,t)\}$ is a R.V.

(iv) If s & t are variables $\{X(s,t)\}$ is collection of RVs that are time functions.

⊛ Dependence of RP on s is ~~obin~~ obvious. So, we denote RP by $\{X(n)\}$ or $\{X_n\}$ for T discrete.

OR $\{X(t)\}$ for T continuous.

⊛ Classification:

(i) T & S both discrete, RP is called discrete random sequence.

(ii) T: Discrete & S is continuous, RS is called continuous random sequence.

(iii) T : continuous & S is discrete, then RP is called discrete random process.

(iv) If T & S both are continuous then RP is called continuous R.P.

(4) <u>Methods of Description of R.P.</u>

Since a R.P. is an indexed set of RVs, we use the joint probability distribution funs to describe a RP.

$F(x,t) = P[X(t) \le x]$ is called the first-order distribution of the process $\{X(t)\}$, & $f(x,t) = \frac{\partial}{\partial x} F(x,t)$ is called the first order density of $\{X(t)\}$.

$F(x_1, x_2, t_1, t_2) = P[X(t_1) \le x_1;\ X(t_2) \le x_2]$ is joint distribution of the RVs $X(t_1)$ & $X(t_2)$ & is called the second-order distribution of the process $\{X(t)\}$ &

$f(x_1, x_2, t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2, t_1, t_2)$ is second-order density of $\{X(t)\}$.

Similarly, n^{th} order distribution $F(x_1, x_2, \ldots x_n; t_1, t_2 \ldots, t_n)$ of the RVs $X(t_1), X(t_2), \ldots X(t_n)$.

2.

(4) Average values of Random Processes

→ Mean of the process $\{X(t)\}$ is the expected value of a typical member $X(t)$ of the Process i.e. $\mu(t) = E[X(t)]$

→ Autocorrelation, denoted by $R_{x}(t_1, t_2)$ or $R(t_1, t_2)$

& $R(t_1, t_2) = E[X(t_1) \times X(t_2)]$

→ Auto covariance, denoted by $C_{x}(t_1, t_2)$ or $C(t_1, t_2)$

& $C(t_1, t_2) = E[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}]$

$= R(t_1, t_2) - \mu(t_1) \times \mu(t_2)$

→ Correlation co-efficients of the process $\{X(t)\}$, is

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1) \times C(t_2, t_2)}}$$

where $C(t_1, t_1)$ is variance of $X(t_1)$.

(5) For 2 or more random processes $\{X(t)\}$ & $\{Y(t)\}$

Cross-correlation $R_{xy}(t_1, t_2) = E[X(t_1) \times Y(t_2)]$

Cross-Covariance $C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1) \times \mu_y(t_2)$

~~Cor~~ Cross-correlation co-efficient of 2 process is

$$\rho_{xy}(t_1, t_2) = \frac{C_{xy}(t_1, t_2)}{\sqrt{C_{xx}(t_1, t_1) \times C_{yy}(t_2, t_2)}}$$

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Stationary:

If the joint distribution of $X(t_1), X(t_2), \dots X(t_n)$ is the same as that of $X(t_1+h), X(t_2+h), \dots X(t_n+h)$ for all $t_1, t_2, \dots, t_n$ & $(h>0)$ & for all $n \geq 1$, then the process $\{X(t)\}$ is called a strict sense stationary process (SSS).

If above definition holds only for $n = 1, 2, \dots, k$ and & not for $n > k$, then the process is called k^{th} order stationary.

(1) For R.P. to be SSS, the densities of $x(t)$ & $x(t+b)$ are the same. $\Rightarrow f(x,t) = f(x, t+b)$ this only possible for $f(x,t)$ independent of t.
Therefore, first order densities of a SSS process are independent of time.
As a consequence $E[X(t)] = \mu =$ constant is also independent of timet.

(2) The second order densities must be invariant under translation of time, i.e. the joint pdf of $\{X(t_1), X(t_2)\}$ is the same as that of $\{X(t_1+b), X(t_2+b)\}$.
i.e. $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+b, t_2+b)$
This is possible only if $f(x_1, x_2, t_1, t_2)$ is function of $t = t_1 - t_2$.
$\therefore$ 2^{nd} order densities of a SSS process are fun of $\tau = t_1 - t_2$

And hence, $R(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$ is also function of $\tau = t_1 - t_2$
But for $E(X(t))$ constant & $R(t_1, t_2)$ a fun of $(t_1 - t_2)$, R.P. ~~always not~~ the R.P. $\{X(t)\}$ need not to be a SSS process.

(3) for SSS rps $E(X(t))$ & var$[X(t)]$ must be independent of time.

* **Wide-sense stationarity**: A R.P. $\{X(t)\}$ with finite first & second order moments is called a weakly stationary process OR wide-sense stationary process (WSS), if its mean is a constant & the autocorrelation depends only on the time difference. i.e. if $E[X(t)] = \mu$ & $E[X(t) \times X(t-\tau)] = R(\tau)$

→ SSS process with finite first & second order moments is a WSS process, while a WSS process need not be a SSS process.

→ Two R.P. $\{X(t)\}$ & $\{Y(t)\}$ are said to be jointly stationary in the wide sense, if each process is individually a WSS process & $R_{xy}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only.

* A R.P. that is not stationary in any sense is called an evolutionary process.

Ex: In a fair coin experiment, the r.p $\{X(t)\}$ is defined as $X(t) = \sin \pi t$, if head shows, &
$= 2t$, if tail shows.

Find $E[X(t)]$, & $F(x,t)$ for $t = 0.25$.

Soln: $P[X(t) = \sin \pi t] = \frac{1}{2}$ $\quad P[X(t) = 2t] = \frac{1}{2}$

$\therefore E[X(t)] = \frac{1}{2} \sin \pi t + 2t \cdot \frac{1}{2} = \frac{1}{2} \sin \pi t + t.$

Now, for $t = 0.25$, $P[X(t) = \frac{1}{2}] = P[X(t) = \sin \pi \cdot 0.25] = \frac{1}{2}$ &
$P[X(t) = \frac{1}{\sqrt{2}}] = \frac{1}{2}.$

$\therefore F(x, 0.25) = 0$, if $x < \frac{1}{2}$
$= \frac{1}{2}$, if $\frac{1}{2} \le x \le \frac{1}{\sqrt{2}}$
$= 1$, if $\frac{1}{\sqrt{2}} \le x.$

Ex 2. Is the Poisson process $\{X(t)\}$, given by the probability law $P[X(t)=r] = \frac{e^{-\lambda t}(\lambda t)^r}{\lfloor r}$, $r = 0,1,2,\ldots$ WSS?

Soln: The probability distribution of $X(t)$ is Poisson distribution with parameter λt.

$\Rightarrow E[X(t)] = \lambda t$ not a constant.

$\therefore$ the Poisson process is not WSS.

Ex 3: Consider rp $\{X(t)\}$. probability distribution given by

$$P[X(t)=n] = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n = 1,2,\ldots$$

$$= \frac{at}{1+at}, \quad n=0.$$

Show that this rp is not stationary.

$P[X(t)=0] = at/1+at$, $P[X(t)=1] = \frac{1}{(1+at)^2}$, $P[X(t)=2] = \frac{at}{(1+at)^3}$

$\therefore E[X(t)] = \sum_{n=0}^{\infty} n \cdot P_n = \cancel{\frac{at}{(1+at)^2}} + \cancel{2at}$

$$= 0 \cdot \frac{at}{1+at} + \frac{1}{(1+at)^2} + \frac{2 \cdot at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\left\{1 + \frac{2\,at}{1+at} + \frac{3at^2}{(1+at)^2} + \cdots\right\}$$

$$= \frac{1}{(1+at)^2}(1 + 2\alpha + 3\alpha^2 + \cdots) \qquad \alpha = \frac{at}{1+at}$$

$$= \frac{1}{(1+at)^2}(1-\alpha)^{-2} = \frac{1}{(1+at)^2}\left(1 - \frac{at}{1+at}\right)^{-2} = \frac{1}{(1+at)^2} \cdot (1+at)^2 = 1.$$

$$E[X^2(t)] = \sum_{n=0}^{\infty} n^2 P_n = \sum_{n=1}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad \boxed{+\sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$= \frac{1}{(1+at)^2}\left[\frac{2}{\left(1-\frac{at}{1+at}\right)^3} - \frac{1}{\left(1-\frac{at}{1+at}\right)^2}\right] = 1+2at$$

$\therefore \operatorname{Var}[X(t)] = 2at$ not independent of time.

$\therefore X(t)$ is is not stationary.