3.4. Field.

A commutative skew field is a field.

In other words, a non-trivial ring R with unity is a field if it be commutative and each non-zero element of R is a unit.

Therefore a non-empty set F forms a field with respect to two binary compositions $+$ and ., if

(i) $a + b \in F$ for all a, b in F;

(ii) $a + (b + c) = (a + b) + c$ for all a, b, c in F;

(iii) there exists an element, called the zero element and denoted by 0, in F such that $a + 0 = a$ for all a in F;

(iv) for each element a in F there exists an element, denoted by $-a$, in F such that $a + (-a) = 0$;

(v) $a + b = b + a$ for all a, b in F;

(vi) $a.b \in F$ for all a, b in F;

(vii) $a.(b.c) = (a.b).c$ for all a, b, c in F;

(viii) there exists an element, called the identity element and denoted by I, in F such that $a.I = a$ for all a in F;

(ix) for each *non-zero* element a in F there exists an element, denoted by a^{-1}, in F such that $a.(a^{-1}) = I$;

(x) $a.b = b.a$ for all a, b in F;

(xi) $a.(b + c) = a.b + a.c$ for all a, b, c in F.

The field is denoted by $(F, +, .)$, or by F.

Examples.

1. The rings $(\mathbb{Q},+,.)$, $(\mathbb{R},+,.)$, $(\mathbb{C},+,.)$ are familiar examples of a field. They are respectively called the field of all rational numbers, often denoted by $\mathbb{Q}$; the field of all real numbers, often denoted by $\mathbb{R}$; the field of all complex numbers, often denoted by $\mathbb{C}$.

2. The set $\{a+b\sqrt{2} : a, b \in \mathbb{Q}\}$ forms a commutative ring with unity under addition and multiplication. The multiplicative inverse of $a+b\sqrt{2}$ where $(a,b) \neq (0,0)$ is $\frac{a}{a^2-2b^2} + \frac{-b}{a^2-2b^2}\sqrt{2}$ and this belongs to the set because $a^2-2b^2 \neq 0$ and $\frac{a}{a^2-2b^2} \in \mathbb{Q}$, $\frac{-b}{a^2-2b^2} \in \mathbb{Q}$. Thus each non-zero element is a unit. Therefore the set forms a field. This is denoted by $\mathbb{Q}[\sqrt{2}]$.

Similarly, $\mathbb{Q}[\sqrt{3}]$, $\mathbb{Q}[\sqrt{5}]$, $\mathbb{Q}[\sqrt{7}]$, ... are fields.

3. The ring $(\mathbb{Z}_5,+,.)$ is a commutative ring with unity and each non-zero element of the ring is a unit. Therefore the ring $(\mathbb{Z}_5,+,.)$ is a field. As it contains a finite number of elements, it is a *finite* field.

Similarly, $(\mathbb{Z}_3,+,.)$, $(\mathbb{Z}_7,+,.)$, ... are finite fields.

Theorem