Linear systems and complexity

Question 1 **10 marks**

Consider the following matrices and vector:

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 4 & 1 \\ -2 & -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & -4 \\ 2 & -4 & 1 \\ 2 & 10 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 50 & 1 \end{pmatrix} \quad R = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

(**a**) Compute the decomposition $PA = LU$. Show the intermediary steps like we did in lecture 6.

Solution:

$$\begin{pmatrix} 1 & 2 & -4 \\ 2 & 4 & 1 \\ -2 & -1 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 1 \\ 1 & 2 & -4 \\ -2 & -1 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & -9/2 \\ 0 & 3 & 4 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & -9/2 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(**b**) Use the decomposition you found at (**a**) to solve $Ax = R$.

Solution:

$$Ly = PR \;\Rightarrow\; y = \begin{pmatrix} -3 \\ -2 \\ 7/2 \end{pmatrix} \quad \text{(by forward substitution)}$$

$$Ux = y \;\Rightarrow\; x = \begin{pmatrix} -50/27 \\ 10/27 \\ -7/9 \end{pmatrix} \quad \text{(by backward substitution)}$$

(**c**) Use Gaussian elimination to solve $Bx = R$. Show intermediary steps like we did in lecture 5.

Solution:

$$\left(\begin{array}{ccc|c} 2 & 0 & -4 & 2 \\ 2 & -4 & 1 & -3 \\ 2 & 10 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 0 & -4 & 2 \\ 0 & -4 & 5 & -5 \\ 0 & 10 & 7 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 0 & -4 & 2 \\ 0 & -4 & 5 & -5 \\ 0 & 0 & 39/2 & -27/2 \end{array}\right) \rightarrow x = \begin{pmatrix} -5/13 \\ 5/13 \\ -9/13 \end{pmatrix}$$

using backward substitution in the last step.

(**d**) Compute the condition number of C. If we solve $Cx = Q$, where Q is a 2-vector of which we know the entries up to 5 digits of precision, then how accurately can we compute x?

Solution: the singular values of C are the square roots of the eigenvalues of C^tC:

$$C^tCv = \lambda v \;\text{ gives } \lambda_1 = 2.5 \times 10^3,\; \lambda_2 = 1.6 \times 10^{-3},\; \text{so that } K(C) = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} = \frac{\sigma_1}{\sigma_2} \approx 1.25 \times 10^3$$

This means that we can compute x up to two digits of precision. You can compute this simply using numpy.linalg.cond:

```
> A = numpy.matrix([[2,0],[50,1]])
> numpy.linalg.cond(A,2)
1252.4992015962975
```

Question 2 **10 marks**

The banded, upper triangular matrix $A \in \mathbb{R}^{n\times n}$ can be written as

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & & & & \\ & a_2 & b_2 & c_2 & & & \\ & & a_3 & b_3 & c_3 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & a_{n-2} & b_{n-2} & c_{n-2} \\ & & & & & a_{n-1} & b_{n-1} \\ & & & & & & a_n \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} \vec{a} &= (a_1, a_2, \dots, a_{n-2}, a_{n-1}, a_n) \in \mathbb{R}^n, \\ \vec{b} &= (b_1, b_2, \dots, b_{n-2}, b_{n-1}) \in \mathbb{R}^{n-1}, \\ \vec{c} &= (c_1, c_2, \dots, c_{n-2}) \in \mathbb{R}^{n-2}. \end{aligned}$$

The entries of $\vec{a}$, $\vec{b}$, and $\vec{c}$ are all assumed to be nonzero.

(**a**) Write a pseudocode for computing the matrix-vector product of A with a vector x. That is, given vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ that define A as in definition (1), and given a vector $x \in \mathbb{R}^n$, your algorithm should compute the matrix-vector product $\vec{y} = A\vec{x}$.
Your pseudocode should have the following form:

Input: vector $\vec{x} \in \mathbb{R}^n$ and vectors $\vec{a} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^{n-1}$ and $\vec{c} \in \mathbb{R}^{n-2}$.

1. $n \leftarrow \text{length}(x)$
2. $y = 0$
2. for $i = 1 : n-2$
   a. $y_i = a_i x_i + b_i x_{i+1} + c_i x_{i+2}$

   end
3. $y_{n-1} \leftarrow a_{n-1}x_{n-1} + b_{i-1}x_n$
4. $y_n \leftarrow a_n x_n$

Output: vector $\vec{y} \in \mathbb{R}^n$ such that $\vec{y} = A\vec{x}$

(**b**) Analyse the complexity of the algorithm from part (**a**). That is, determine how many flops are required to compute the product $\vec{y} = A\vec{x}$ with your algorithm.

There are 5 FLOPs in line 2a, which is inside a loop for i from 1 to $n-2$, there are 3 FLOPs in line 3 and finally there is 1 in line 4. We get:

$$\#\text{flops} = 3 + 1 + \sum_{i=1}^{n-2} 5 = 5n - 6$$

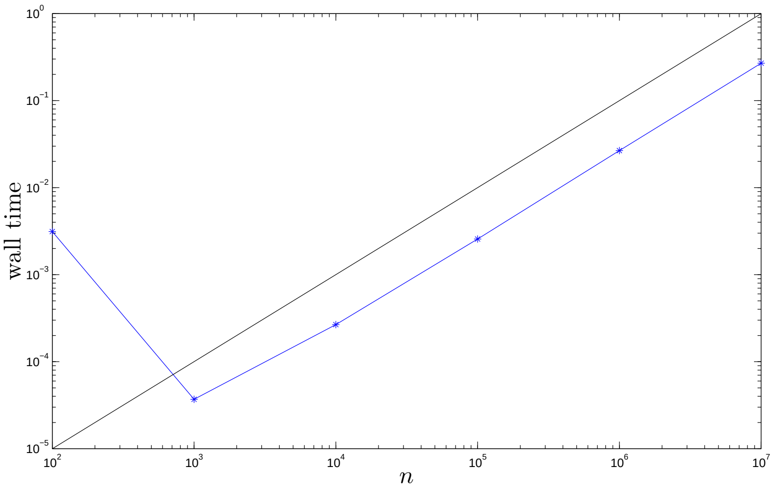
so this matrix-vector product is $O(n)$.

(**c**) Implement your pseudo-code in as a function called `tridag_matvec.py`. Also, write a script called `compare.py` to generate a tridiagonal test matrix and a test vector for a $n = 10^k$, $k = 2, \dots, 7$, compute the product and measure the time your code takes to complete. Produce a plot of the time taken versus n on a logarithmic scale, along with your prediction.

Example code:

```
import scipy
def tridiag(a,b,c,x):
    n = scipy.size(x)
    y = scipy.zeros((n,1))
    for i in range(0,n-2):
        y[i,0]=a[i]*x[i]+b[i]*x[i+1]+c[i]*x[i+2]
    y[n-2,0]=a[n-2]*x[n-2]+b[n-2]*x[n-1]
    y[n-1,0]=a[n-1]*x[n-1]
    return y
```

Measuring the wall time (in seconds) gives the following picture (black is $O(n)$ scaling):



(d) Repeat the test, but now using the built-in matrix-vector product (`scipy.dot`/`scipy.matmul`) and the matrix A defined as an $n \times n$ matrix with mostly zeros. Plot the time taken in the same plot as for (**c**). Which algorithm is faster? Is the difference as great as you expected?

Of course, the matrix-vector multiplication with a full matrix is slower, namely $O(n^2)$. Moreover, for some $n > 10^4$, your machine runs out of memory. In the figure below you see the wall time in seconds for the function we programmed in red, and for the full matrix-vector product in green.