# AER1216: Fundamentals of UAVs: Multi-Rotor Performance

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# Introduction

Multi-rotor performance analysis/prediction less developed than fixed wing, number of factors contribute:

- Multi-rotors more recent development
  - fixed wing aircraft been around for over 110 years
  - helicopters been around for over 80 years
- Most multi-rotors are small, low-cost $\rightarrow$ experimentation reasonable method for optimizing performance
- Aerodynamics of multiple rotating propellers close to each other, combined with unstreamlined frame is extremely complex
- While helicopter aerodynamics can be applied there are significant differences:
  - helicopter blades flap, lag and twist (i.e. variable pitch, changes throughout rotation!)
  - helicopters tend to have one main rotor
  - Disc-loading unexpectedly similar : Robinson 22 T/A $\approx$ 88 $N/m^2$, Phantom 4 T/A $\approx$ 90 $N/m^2$
    - Scaling law helps quad-rotors

# Hover

Multi-rotors spend significant amount of time hovering

- Hovering endurance an important performance metric
- For steady-state hover thrust/rotor is:

$$T_h = W/4$$

- Doing this with minimum power will give maximum endurance
- From actuator disc theory we know:

$$P_{ind} = \sqrt{\frac{W^3}{2\rho A}} \quad \text{and induced velocity } v_h = \sqrt{\frac{W}{2\rho A}} \text{ where A is total disc area}$$

- Lower disc loading leads to lower power requirements and slower induced velocity (downwash)
- For given weight want large propellers for endurance

# Hover

- For fixed size propeller what propeller is most efficient?
  - Aerodynamic efficiency of blades important and pitch, airfoil section, ...
- Measured static thrust and power propellers can be used to compare props:
  - Thrust constant and considering $D$ constant yields,

$$T_1 = C_{T_1} \rho n_1^2 D^4 = T_2 = C_{T_2} \rho n_2^2 D^4 \rightarrow \frac{C_{T_1}}{C_{T_2}} = \frac{n_2^2}{n_1^2}$$

  - Ratio of rotational speeds dependent on ratio of known thrust coefficients
  - From known $C_{P_1}$ and $C_{P_2}$, can find $P_1/P_2$,

$$\frac{P_1}{P_2} = \frac{C_{P_1} \rho \ n_1^3 \ D^5}{C_{P_2} \rho \ n_2^3 \ D^5} = \frac{C_{P_1} n_1^3}{C_{P_2} n_2^3}$$

  - but from previous expression $n_1/n_2 = \sqrt{C_{T_2}/C_{T_1}}$ so,

$$\frac{P_1}{P_2} = \frac{C_{P_1}}{C_{P_2}} \left( \frac{C_{T_2}}{C_{T_1}} \right)^{3/2}$$

  - Thus can find ratio hover of power from measured $C_P$'s and $C_T$'s

# Hover

- Apply to to APC 9xX SF (Slow-flyer) propellers, from UIUC database:

| Propeller | $C_T$ | $C_P$ | RPM* |
|---|---|---|---|
| 1. 9x3.8 | 0.1025 | 0.0401 | 5597 |
| 2. 9x4.7 | 0.1190 | 0.0484 | 5736 |
| 3. 9x6 | 0.1557 | 0.0809 | 5696 |
| 4. 9x7.5 | 0.1797 | 0.1249 | 5607 |

- Plugging in $C_T$'s and $C_P$'s into power ratio expressions yields $P_1/P_2$:

| | 9x3.8 | 9x4.7 | 9x6 | 9x7.5 |
|---|---|---|---|---|
| 9x3.8 | - | 0.93 | 0.92 | 0.75 |
| 9x4.7 | - | - | 0.99 | 0.80 |
| 9x6 | - | - | - | 0.80 |

- Lower pitch better for hover efficiency
- Lower pitch operating at higher aerodynamic efficiency on average (will vary along blades)
- Large pitch$\rightarrow$ large $C_L$ $\rightarrow$ increase in induced drag that is larger than reduction in parasite drag (due to lower speed i.e. dynamic pressure)
    - Trend could reverse with very very small pitch?

# Hover

- Momentum theory indicates larger prop is more efficient
  - assume additional losses, not considered by momentum theory, don't make smaller propeller more efficient than larger propeller
- Smaller propeller needs either larger pitch, higher $C_{\ell_\alpha}$ airfoil, or faster rotation to generate same thrust
- Do induced power effects dominate aerodynamic efficiency?
  - difficult to answer absolutely as many factors (like Re), but we'll do a quick check on one propeller
  - using same procedure as earlier, but now allowing D to vary between two propellers being compared

$$\frac{P_1}{P_2} = \frac{C_{P_1}}{C_{P_2}} \left( \frac{C_{T_2}}{C_{T_1}} \right)^{3/2} \left( \frac{D_2}{D_1} \right)$$

# Hover

- Comparing two different APC slow flyers we have:

| Propeller | $C_T$ | $C_P$ | D | RPM |
|---|---|---|---|---|
| 1. 9x6 | 0.1557 | 0.0809 | 9 | 5696 |
| 2. 8x3.8 | 0.1087 | 0.0464 | 8 | 6212 |

- $P_1/P_2 = 0.904$ in this case induced power effects are more important, which for most situations should hold as long as a reasonable pitch was chosen for the two props
- As final check calculate power to generate thrust of 0.33kg at sea-level we get (disc loading 78N/m$^2$);

| Propeller | RPM | Power, W | Act. Disc Power, W |
|---|---|---|---|
| 9x3.8 | 5829 | 28.1 | 18.4 |
| 9x4.7 | 5604 | 30.1 | 18.4 |
| 9x6 | 4729 | 30.3 | 18.4 |
| 9x7.5 | 4402 | 37.7 | 18.4 |
| 8x3.8 | 7163 | 33.5 | 20.7 |

- Analysis Assumes motor/ESC/battery combination can achieve same efficiency for each case

# Hover

- Once we have selected the propeller we can easily find the endurance using either the 0th order or 1st order model of the batteries we covered
- **0th order model:**
  - Note: don't need detailed info on motor, ESC or battery, just efficiencies!
  - Find the total Energy in the battery,
    $E_b = (\text{No. Cells}) \times 3.7 \times \text{mA-hr}/1000 \times 3600$
  - Then the endurance in seconds is,

$$t_e = \frac{E_b \eta_m \eta_e}{P_{prop}}$$

  - if motor and ESC sized correctly and quality motor used, motor and ESC efficiencies are approximately $\eta_m \approx 0.85$, and $\eta_e \approx 0.95$.

# Hover

- **1st order model:**
  - Detailed motor, ESC, batter info required
  - Using propeller charts (at J=0) or static thrust figures determine n required for hover thrust, $T_h$ (static thrust)
  - Using propeller charts (at J=0) and $n$, determine required torque $Q$
  - Using motor $K_t$ and no-load current $i_0$ find motor current $i_m$ using Equation (28) in Propulsion notes ($Q = K_t(i_m - i_0)$ )
  - Find motor voltage, $\mathrm{v}_{m_i}$ from Equation (27) in Propulsion notes ($\Omega = K_v(v_{m_i} - i_m r_m)$ )
  - Find ESC voltage $\mathrm{v}_{e_t}$ (using $\mathrm{v}_{e_t} = \mathrm{v}_{m_i} + i_m r_e$)
  - Now you will need to time march a numerical solution to find the endurance as follows:
    1. Initialize the battery discharge $D = 0$ and $t = 0$
    2. Find $\mathrm{v}_b i_b^{0.05}(D)$ value from battery discharge curve (page 55 propulsion notes, call it $g$)
    3. Solve 3 equations, 3 unknowns $(v_b, i_b, k_e)$, $\mathrm{v}_b = \mathrm{v}_{e_t}/k_e$, $i_b = i_m k_e$, $\mathrm{v}_b i_b^{0.05} = g$
    4. $D = D + i_b \Delta t$, and $t = t + \Delta t$ (careful with units, discharge curve may be in mAhr)
    5. Stop if $D >$ mA-hr or $k_e > 1$, otherwise go to step 2
    6. $t_e = t$

# Vertical Climb

- Will use propeller charts to estimate max climb rate
  - assume load equally carried by propellers so just analyze one propeller
  - assume motor/battery has sufficient power to achieve static thrust = 2W
  - assume drag of frame is negligble
- Need more available thrust than required for hover to maneuver
- Many rules of thumb out there but assume need static available thrust of $T_A = 2T_h$, max. vertical acceleration is then

$$a_z = (2mg - mg)/m = 1g$$

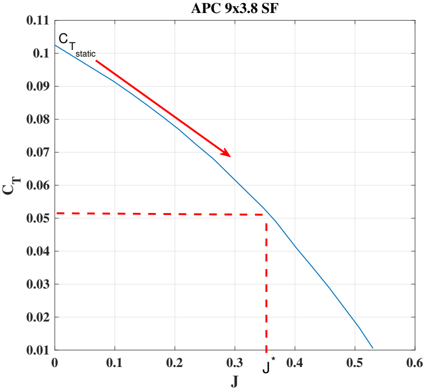
- Achieved by increasing $n$ to $\sqrt{2}n_h$ (need to double thrust at $C_{T_{static}}$)
- As vertical speed builds, thrust drops as blade unloads (see $C_T$ versus $J$ curve)
  - assume $n$ fixed at $\sqrt{2}n_h$ (motor can spin prop faster as it unloads but not much so ignore)
  - Maximum climb rate can be estimated from $C_T$ versus $J$ chart

- Assume at $n = \sqrt{2}n_h$
- Find $C_T$ where $C_T = C_{T_{static}}/2$
  - find $J$, call it $J^*$
- Max climb speed is then

$$V_{c_{max}} = J^* \sqrt{2}n_h D = v_h J^* \sqrt{\frac{\pi}{C_{T_{static}}}}$$

- So for this 9x3.8 propeller lifting 0.33kg $v_h$ =5.67m/s so

$$V_{c_{max}} = 0.34(5.67)\sqrt{\frac{\pi}{0.1025}} = 10.7m/s$$

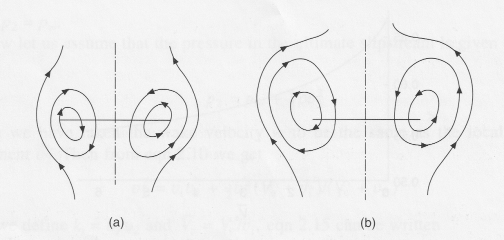


# Descent

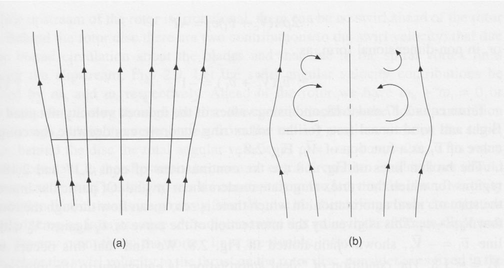
As mentioned in Propulsion lecture descent needs special attention

- If $V_c/v_h > -0.2$ (note the $V_c$ is negative for descent) then can use actuator disc theory as is
- If $-1.5 < V_c/v_h < -0.2$ no flow through disc, stream-tube breaks down, actuator disc theory doesn't work, rotor in own wake, vortex ring state
- if $-2 < V_c/v_h < -1.5$ actuator disc theory does work, rotor in turbulent wake state
- if $-V_c/v_h < -2.5$ rotor in windmill break state, actuator disc theory works again but,
  - Need to use $|V_c + v|$ in expression so thrust is still positive
  - Motor being forced to turn by airflow over blades
  - Auto-rotate mode for helicopter, used to land when engine failure
- **Note:** in forward flight at low speeds can enter vortex ring or turbulent state at small decent rates –increase descent rate to get out

# Descent




(a)Vortex Ring State Slow (b) Vortex Ring State Fast

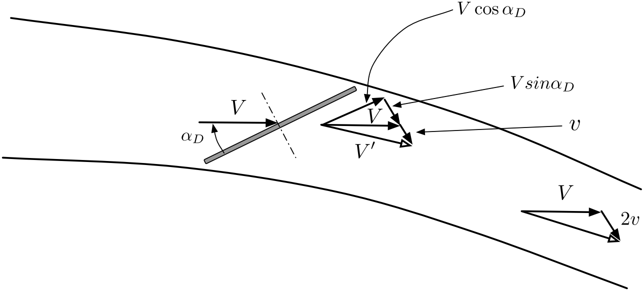



(a) Windmill Wake State (b) Turbulence Wake State

# Forward Flight-Momentum Theory

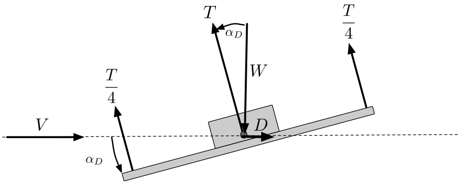
- We will start by using momentum theory since this leads to relatively simple expressions that give insight into the important factors for performance
- Recall from propulsion lecture that we have the following expression for forward flight:

$$v = \frac{T}{2\rho A\sqrt{V^2cos^2\alpha_D + (v + V sin\alpha_D)^2}} \tag{1}$$



# Forward Flight-Momentum Theory

- Now for horizontal steady-level flight of a quadrotor (this generalizes to multi-rotors) we have




- from the vertical and longitudinal force balance we have

$$T = \sqrt{W^2 + D^2}$$

- where we will assume the drag is given by $D = 1/2\rho S C_D V^2$
- and then we can find $\alpha_D$ from,

$$\alpha_D = \tan^{-1}(D/W)$$

# Forward Flight-Momentum Theory

- At a given forward speed, $V$ we can then solve for $\alpha_D,\ v,\ T,\ P_{ind},\ P_{tot}$ as follows:

1. Find quadrotor drag $D = 1/2\rho S C_D V^2$ at $\alpha_D$=0
   - assume $\alpha_D$ does not effect the drag, otherwise need to iterate to find solution
2. Solve $\alpha_D = \tan^{-1}(D/W)$
3. Square both sides of Eqn 1 replace $T^2$ by $W^2 + D^2$ and re-arrange to get,

$$v^4 + (2V \sin \alpha_D)v^3 + V^2 v^2 - (W^2 + D^2)/(2\rho A)^2 = 0$$

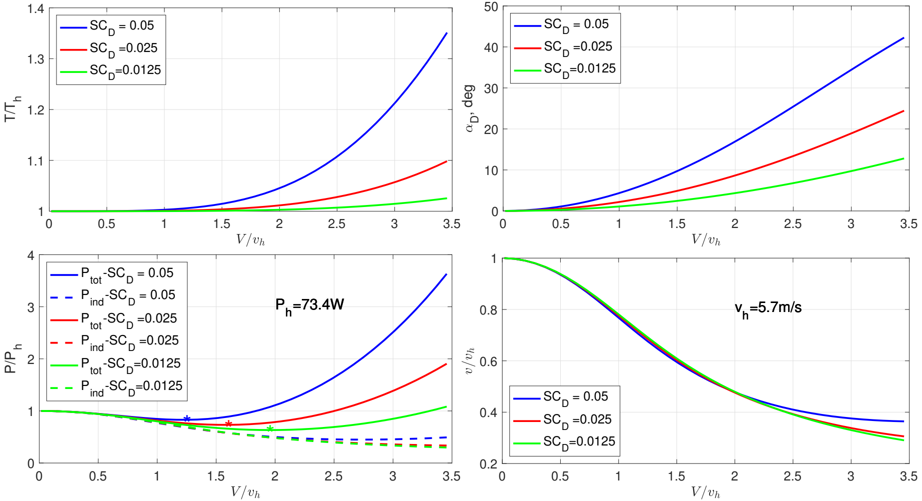
4. Positive real root of equation gives $v$, then

$$P_{ind} = Tv \quad \text{and} \quad P_{tot} = T(v + V \sin \alpha_D)$$

   - Notice that this *total* power does not include profile drag, swirl, or additional losses due to non-uniform induced velocity!
5. Solve for a range of speeds and plot results versus $V$

# Forward Flight-Momentum Theory

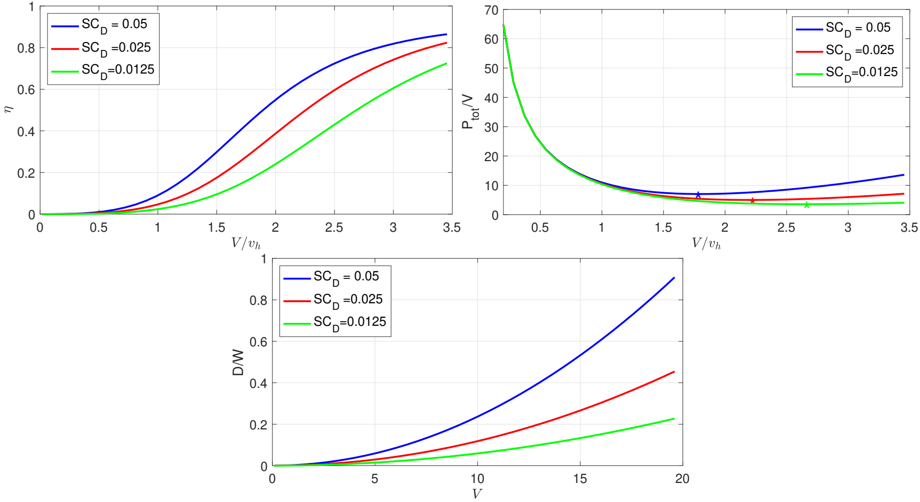
Ex: $C_D = 0.5$, $S=[0.1, 0.05, 0.025]$ $W=4(0.33)$kg, $A = 4(0.041)m^2$ (four 9 in props)




Max Endurance (min $P_{tot}$) speed: $V/v_h$=[1.96, 1.60, 1.25] or $V$=[11.1, 9.1, 7.1]m/s

# Forward Flight-Momentum Theory

Ex: $C_D = 0.5$, $S{=}[0.1, 0.05, 0.025]$ $W{=}4(0.33)$kg, $A = 4(0.041)m^2$ (four 9 in props)




Max Range (min $P_{tot}/V$) speed: $V/v_h$=[2.66, 2.22 , 1.78] or $V$=[15.1, 12.6, 10.1]m/s

# Forward Flight-Momentum Theory

- Due to the limitations of momentum theory these results will not be particularly accurate, trends reasonable
- Power neglects profile drag, swirl and non-uniform induced velocity, all potentially a function of $V$, $\alpha_D$
- Thus even speeds for max Range and Endurance should be considered rough estimates
- In addition fixed propeller pitch has a HUGE effect on efficiency, thrust, and power so need propeller for optimal efficiency at given flight condition (equivalent J found using $V sin(-\alpha_D)/nD$)
- Motor/battery/ESC efficiency all need to be considered but these can be considered close to constant if sized properly

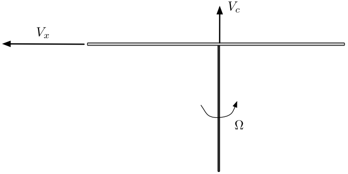
# General Comments

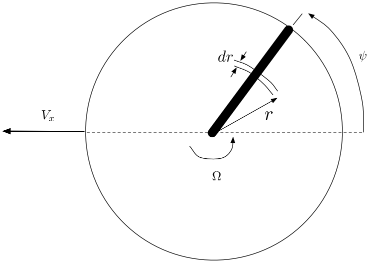
- Other factors need to be considered other than what was covered:
  - Inertia of propellers (which will increase quickly with D) must be considered, motor will not be able to spin up the bigger propellers as quickly may effect maneuverability and stability
  - Very large propellers may lead to safety issues
  - Very large propellers may lead to interference (aerodynamic or mechanical) between the the 4 propellers
  - Motors/propellers need to selected together speed/torque requirements lead to high motor efficiency

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

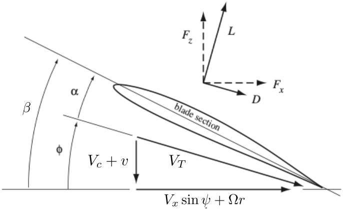
- The details of this method will not be on the exam or problem set!
- This is the next level of detail for a blade element model than can handle forward flight with changing conditions
- We'll adjust our nomenclature a little bit to make things a bit simpler

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model



# General Blade Element Theory with Pitt-Peters Dynamic Inflow model




- Non-dimesionalize everything!

$$\begin{aligned}
\mu &= \frac{V_x}{\Omega R} - \text{advance ratio} \\
\bar{V}_c &= \frac{V_c}{\Omega R} \\
C_T &= \frac{T}{\rho \pi R^2 \Omega^2 R^2} \\
C_L &= \frac{L}{\rho \pi R^2 \Omega^2 R^3} - \text{non-dimensional rotor aerodynamic roll moment( L is roll moment)}
\end{aligned}$$

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

- Non-dimensionalize everything!

$$
\begin{aligned}
C_M &= \frac{M}{\rho \pi R^2 \Omega^2 R^3} - \text{non-dimensional rotor aerodynamic pitch moment} \\
\lambda_0 &= \frac{v_0}{\Omega R} - \text{non-dimensional mean induced inflow} \\
\lambda_1 &= \frac{v_1}{\Omega R} - \text{non-dimensional longitudinal inflow gradient across rotor disc} \\
\lambda_2 &= \frac{v_2}{\Omega R} - \text{non-dimensional lateral inflow gradient across rotor disc} \\
\chi &= tan^{-1}\left(\frac{\mu}{\lambda_0 + \bar{V}_c}\right) - \text{wake skew angle} \\
X &= tan\left(\chi/2\right) \\
x &= r/R
\end{aligned}
$$

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

- First let's assume we know the rotor forces, then we can solve the inflow problem using the following:

$$[\mathfrak{M}] \begin{bmatrix} \acute{\lambda}_0 \\ \acute{\lambda}_1 \\ \acute{\lambda}_2 \end{bmatrix} + [V]\,[\mathfrak{L}]^{-1} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} C_T \\ -C_L \\ -C_M \end{bmatrix}$$

- where $\acute{()}$ is derivative with respect to non-dimensional time (note $C_L$ is the roll moment here)

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

- also where

$$[\mathfrak{M}] = \begin{bmatrix} \frac{128}{75\pi} & 0 & 0 \\ 0 & \frac{16}{45\pi} & 0 \\ 0 & 0 & \frac{16}{45\pi} \end{bmatrix}$$

$$[\mathfrak{L}] = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{15\pi}{64}X \\ 0 & 2(1+X^2) & 0 \\ \frac{15\pi}{64}X & 0 & 2(1-X^2) \end{bmatrix}$$

$$[V] = \begin{bmatrix} V_m & 0 & 0 \\ 0 & \bar{V} & 0 \\ 0 & 0 & \bar{V} \end{bmatrix}$$

- where

$$V_m = \sqrt{\mu^2 + \left(\lambda_0 + \bar{V}_c\right)^2}$$

$$\bar{V} = \frac{\mu^2 + \left(\lambda_0 + \bar{V}_c\right)\left(2\lambda_0 + \bar{V}_c\right)}{V_m}$$

- once the induced velocities are known can find the inflow at any point on the actuator disc using:

$$\lambda(x,\psi) = \lambda_0 + x\left(\lambda_1 \sin\psi + \lambda_2 \cos\psi\right)$$

- and the induced inflow velocity at any location is then $v(x,\psi) = \lambda(x,\psi)\Omega R$

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

- Now using geometry and knowing the blade rotation $\psi$ we can find $\phi$ at a given non-dimensional radius x using

$$\phi(x,\psi) = \tan^{-1}\left(\frac{V_c + v(x,\psi)}{V_x \sin\psi + \Omega x R}\right)$$

- and then $\alpha$ can be found using $\alpha = \beta(x) - \phi(x,\psi)$ and therefore the lift and drag per span on a blade are:

$$\begin{aligned} \ell &= 1/2\rho V_T^2 c a \alpha \\ d &= 1/2\rho V_T^2 c c_d \end{aligned}$$

- Note there is no induced drag term since the induced velocity effects have already been included

# General Blade Element Theory with Pitt-Peters Dynamic Inflow model

- The thrust, torque, roll moment, and pitch moment contributions for a small segment of blade of width $\Delta r$ are,

$$\begin{aligned} \Delta T &= (\ell\cos(\phi) - d\sin(\phi))\,\Delta r \\ \Delta Q &= (\ell\sin(\phi) + d\cos(\phi))\,r\Delta r \\ \Delta L &= \Delta T r\sin(\psi) \\ \Delta M &= \Delta T r\cos(\psi) \end{aligned}$$

- Contributions summed up over all segments of blade and all blades

$$T = \sum_{\text{No. blades}} \sum_{\text{segments}} \Delta T \quad \text{similarly for Q, L, and M}$$

- Note inflow depends on thrust and moments and thrust and moments depend on inflow