AER1216 Lecture (Fall 2021)
Lecture 06: Fixed-Wing UAS Performance

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Outline

1. Force Equations of Motion for Performance Analysis

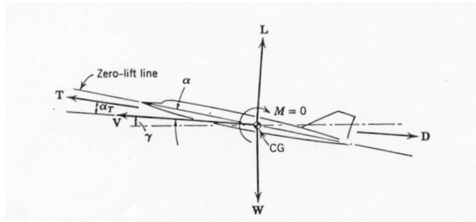


Figure 1: Force Diagram

There are four physical forces acting on the airplane:

1. Lift, which is perpendicular to the flight path direction
2. Drag, which is parallel to the flight path direction
3. Weight, which acts vertically toward the center of the earth
4. Thrust, which in general is inclined at the angle α_T with respect to the flight path direction

When the earth is considered flat, any earth-fixed frame of reference is an inertial system, in which Newton's second laws are valid. We will derive the desired differential equations. It is most convenient to express the forces in components $\perp$ and $\|$ to $\underset{\rightarrow}{\boldsymbol{V}}$.

$$T \cos \alpha_T - D - W \sin \gamma = m\dot{V} \quad (1)$$
$$T \sin \alpha_T + L - W \cos \gamma = mV\dot{\gamma} \quad (2)$$

Note that α_T is very small, implies that $\sin \alpha_T \approx 0, \cos \alpha_T \approx 1$. That leads to

$$T - D - W \sin \gamma = m\dot{V} \quad (3)$$
$$L - W \cos \gamma = mV\dot{\gamma} \quad (4)$$

2. Aerodynamics Forces: Lift and Drag

The actual magnitude of lift L depends not only on α, but also on (free-stream) velocity V_∞, altitude ρ_∞, size of the aerodynamic surface represented by the wing area S, shape of the airfoil, viscosity coefficient μ (Reynolds number), and compressibility of the airflow (Mach number). Hence, we can write that, for a given shape airfoil at a given angle of attack,

$$L = L(V_\infty, \rho_\infty, S, \mu_\infty, a_\infty) \tag{5}$$

It is simplified by the following definition: lift is defined to be the aerodynamic force perpenticular to $\underset{\rightarrow}{\boldsymbol{V}}$ and its values are proportional to V^2:

$$L = \frac{1}{2}\rho_\infty V_\infty^2 S C_L = qSC_L \tag{6}$$

where $C_L = C_L(\alpha, M, Re)$, $q = \frac{1}{2}\rho_\infty V_\infty^2$ is called the *dynamic pressure*.

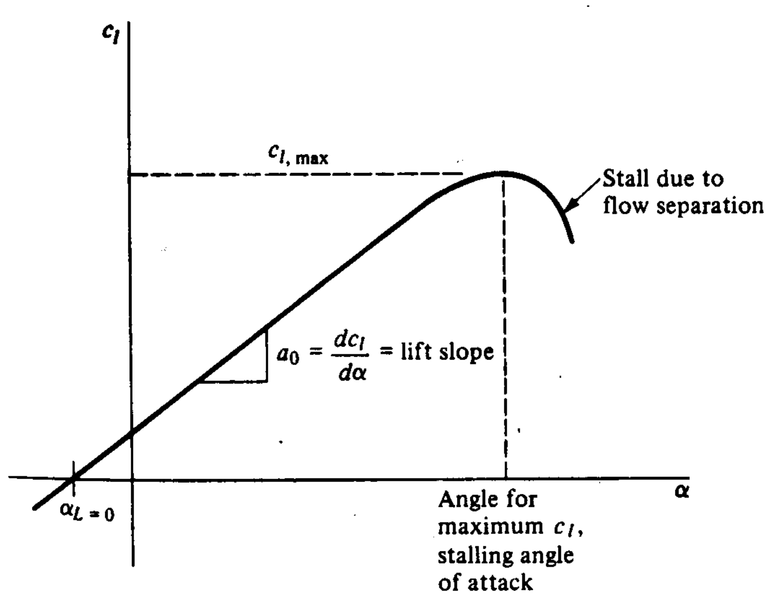


Figure 2: Lift Curve of an Airfoil

Lift of Wing

The fundamental difference between flows over finite wings as opposed to infinite wings is due to the *wing-tip vortices*.

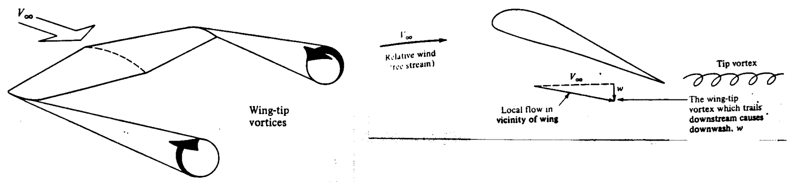


Figure 3: Wing-tip Vortices

for 2D airfoil,

$$c_l = c_{l_\alpha}\alpha$$

for 3D wing,

$$C_L = C_{L_\alpha}\alpha = a\alpha \tag{7}$$

Drag Polar

When we can neglect Mach number and Reynolds number effects, it is found that one can represent the C_D for a complete aircraft (or a wing alone) by the parabolic drag polar

$$C_D = C_{D_0} + KC_L^2 \tag{8}$$

where for a given aircraft C_{D_0} and K are constants. Note that $\frac{1}{2}\rho V^2 SC_{D_0}$ is the parasite drag, and $\frac{1}{2}\rho V^2 SKC_L^2$ is the induced drag. It can be shown (through wing theory) that

$$K = \frac{1}{\pi\epsilon AR} \tag{9}$$

where AR is the wing aspect ratio, and $0 \le \epsilon \le 1$ depends on the wing design (Oswald's efficiency factor).

for wing design only,

$$K = \frac{1}{\pi e AR}$$

where e represents the wing-span efficiency.
For the entire aircraft (e.g. wing-tail configuration),

$$K = \frac{1}{\pi \epsilon AR}$$

replaced by the Oswald's efficiency factor.

- normally the span efficiency is high $e > 0.9$,
- the Oswald's efficiency factor is lower $\epsilon \approx 0.8$

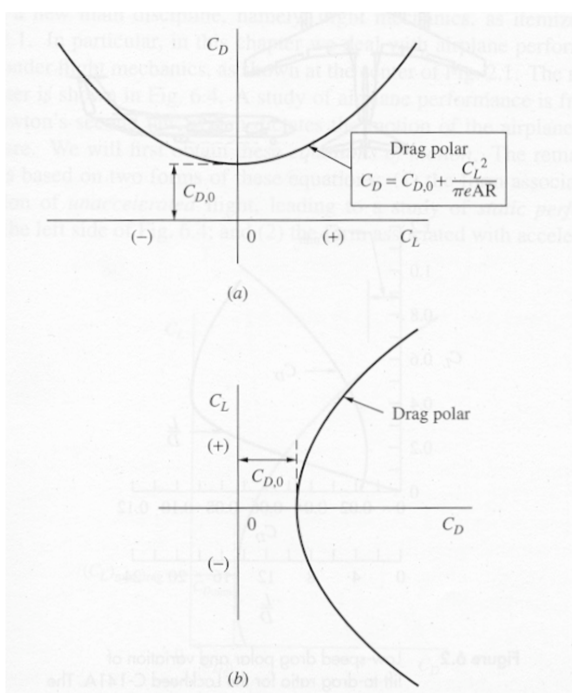


Figure 4: Drag Polar

3. Steady Level Flight Performance

For level ($\gamma = 0$), unaccelerated flight (right side of above equations are zero),

$$T = D \tag{10}$$
$$L = W \tag{11}$$

The Kinematic Equations w.r.t. F_E are

$$\dot{x} = V \cos\gamma \tag{12}$$
$$\dot{h} = V \sin\gamma \tag{13}$$

where x, h are the forward and altitude coordinates respectively.

3.1 Thrust Required

The thrust required for an aircraft to fly at a given velocity in level, unaccelerated flight is

$$T_R = \frac{W}{C_L/C_D} = \frac{W}{L/D} \tag{14}$$

since

$$\frac{T}{W} = \frac{D}{L} = \frac{qSC_D}{qSC_L} = \frac{C_D}{C_L}$$

Lift-to-Drag Ratio

The *lift-to-drag ratio* L/D is a measure of the aerodynamic efficiency of an aircraft. The maximum aerodynamic efficiency leads to the minimum thrust required.

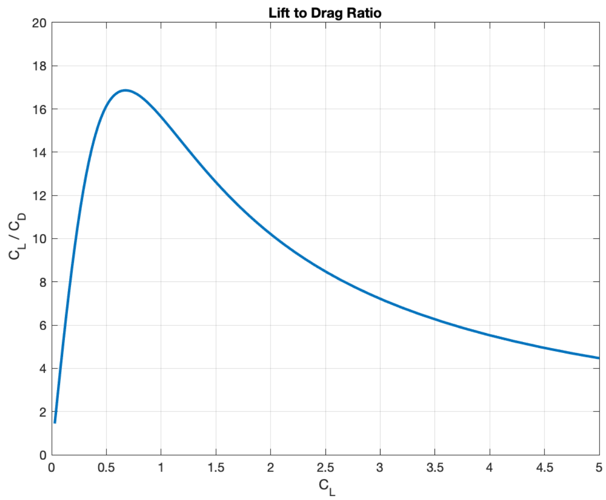


Figure 5: Lift to Drag Ratio

$$L = W = qSC_L \tag{15}$$

$$D_0 = qSC_{D_0} = \frac{1}{2}\rho S C_{D_0} V^2 \tag{16}$$

$$D_i = qSC_{D_i} = qSKC_L^2 = qSK(\frac{W}{qs})^2 = KS\frac{(W/S)^2}{q} \tag{17}$$

Table 1: Thrust required or Drag vs V and α

V	q	$C_L \propto \alpha$	D_0	D_i	$D = D_0 + D_i$	α
low	low	high	low	high	D_i dominant	high
$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$		
high	high	low	high	low	D_0 dominant	low

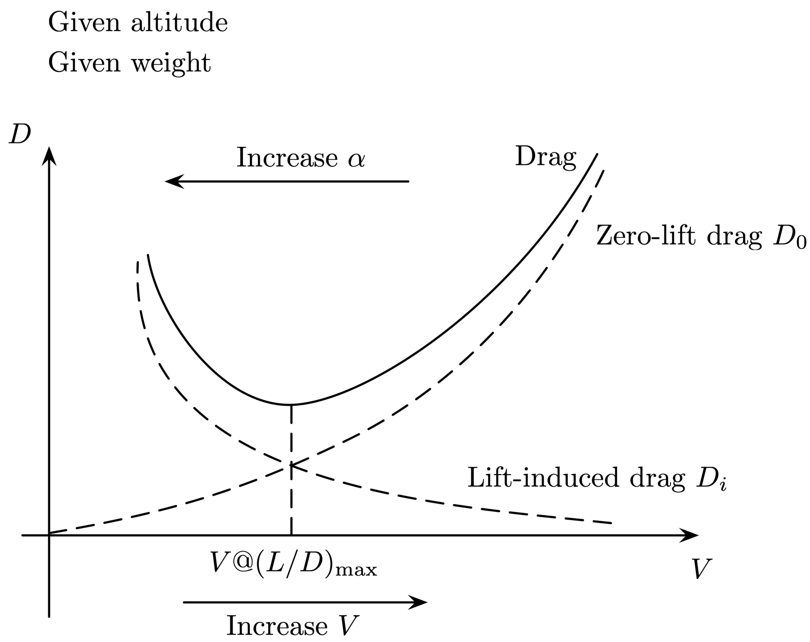


Figure 6: Thrust-required Curve

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + KC_L^2} \tag{18}$$

The maximum at $\frac{dC_L/C_D}{dC_L} = 0$:

$$\frac{dC_L/C_D}{dC_L} = \frac{(C_{D_0} + KC_L^2) - C_L(2KC_L)}{(C_{D_0} + KC_L^2)^2} \tag{19}$$

$$= \frac{C_{D_0} - KC_L^2}{(C_{D_0} + KC_L^2)^2} = 0 \tag{20}$$

leading to the maximum condition

$$C_{D_0} = C_{D_i} \tag{21}$$

and

$$\left(\frac{C_L}{C_D}\right)_{\max} = \frac{1}{2\sqrt{KC_{D_0}}} \tag{22}$$

In summary,

$$(T_R)_{\min} = 2\sqrt{KC_{D_0}}W \tag{23}$$

$$(C_L)_{\mathrm{TR,min}} = (C_L)_{\mathrm{LD,max}} = \sqrt{\frac{C_{D_0}}{K}} \tag{24}$$

$$V_{\mathrm{TR,min}} = \sqrt{\frac{2W/S}{\rho\,(C_L)_{\mathrm{TR,min}}}} \tag{25}$$

3.2 Thrust Available

While thrust required T_R is an airframe-associated phenomenon, the thrust available T_A is strictly associated with the engine of the aircraft. Piston engines with propellers exhibit different patterns from the turbojet engines.

- for piston reciprocating engines, power is relatively constant
- for turbojet engines, thrust is relatively constant w.r.t. velocity

For a given thrust available $T_A = T$, assume it is generated to balance the drag to maintain the steady level flight,

1. $T_A = T_R = D$, leading to

$$T - qSC_{D_0} - KS\frac{(W/S)^2}{q} = 0 \tag{26}$$

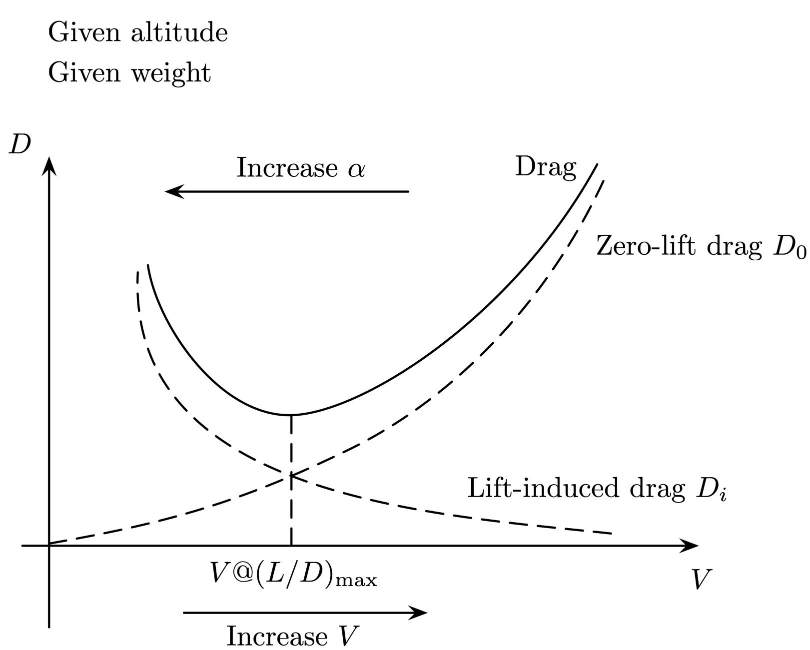
2. solve q for

$$q^2 - \left(\frac{T/S}{C_{D_0}}\right) q + K\frac{(W/S)^2}{C_{D_0}} = 0 \tag{27}$$

where $T \geq 2\sqrt{KC_{D_0}}W = T_{R_{\min}}$ guarantees the feasible solution

3. to identify low and high speed of steady level flight,

$$q = \frac{T/S \pm \sqrt{(T/S)^2 - 4KC_{D_0}(W/S)^2}}{2C_{D_0}} \tag{28}$$



The term W/S is called the *wing loading*, that is the total weight of an aircraft divided by the reference wing area; while the term T/S is called the *thrust loading*, the thrust divided by the wing area. Graphically, the corresponding flight speed under a given thrust available is the intersection of the T_R curve and the constant T_A line. Obviously, in order to make intersection valid, the thrust available needs to give at least the minimum thrust required value, that is consistent with the mathematical validation solving for Eq. (27).

Maximum Velocity

The intersection of the T_R curve (dependent on the airframe) and the maximum T_A curve (dependent on the engine) dfines the maximum velocity V_{max} of the airplane at the *given altitude*.

$$T_{R_{max}} = T_A = q_{max} S C_{D_0} + \frac{W^2}{q_{max} S \pi \epsilon AR} \tag{29}$$

Altitude Effects

turbojet engine is relatively constant with velocity. On the other hand, the thrust of a turbojet for a given throttle setting is directly proportional to the mass flow rate of the air through the engine. Consequently, as the density of the atmosphere decreases with an increase in altitude, so does the available thrust.
The **thrust** T (not temperature) at any given h can be expressed in terms of its sea-level value by:

$$\frac{T}{T_s} = \frac{\rho}{\rho_s} \tag{30}$$

3.3 Power Required

Power is defined as energy per unit time. Therefore, the *power required* for a level, unaccelerated flight at a given altitude and a given velocity is

$$P_R = T_R V = \frac{W}{C_L/C_D}\sqrt{\frac{2W}{\rho S C_L}} \propto \frac{1}{C_L^{3/2}/C_D} \tag{31}$$

The minimum power required happens when

$$\frac{dP_R}{dV} = 0 \Rightarrow C_{D_0} = \frac{1}{3}C_{D_i} \tag{32}$$

In summary,

$$(P_R)_{\min} = \frac{W}{(C_L)_{\text{PR,min}}}(4C_{D_0})\sqrt{\frac{2W/S}{\rho(C_L)_{\text{PR,min}}}} \quad (33)$$

$$(C_L)_{\text{PR,min}} = \sqrt{\frac{3C_{D_0}}{K}} \quad (34)$$

$$V_{\text{PR,min}} = \sqrt{\frac{2W/S}{\rho(C_L)_{\text{PR,min}}}} \quad (35)$$

3.4 Power Available

The *power available* P_A is a characteristic of the power plant.

- For piston engine with propeller, the power is constant

$$P_A = \eta P \quad (36)$$

  where $\eta < 1$ is the propeller efficiency.
- For jet engine, when the thrust is constant

$$P_A = T_A V \quad (37)$$

- For electrical system, we can also treat the power is constant (if the propeller is picked for optimal condition)

The intersection of the P_R curve (dependent on the airframe) and the maximum P_A curve (dependent on the engine) dfines the maximum velocity V_{max} of the airplane at the *given altitude*.

- η is close to constant for a variable pitch propeller
- the fixed pitch propeller is picked for max efficiency at min thrust or min power, this is the case for small UAVs often equipped with fixed pitch propellers
- the optimizations hold for fixed η, i.e., the propeller needs to be picked for max η at a given optimized condition

Altitude Effects

Both jet engines and propellers are air-breathing power plants. The power plant output varies with the air density, or there is a strong altitude effect on thrust. For propellers, the impact of altitude change (air density ρ) on the power available is governed by the square-root law:

$$\frac{P_A}{P_{A_s}} = \sqrt{\frac{\rho}{\rho_s}} \tag{38}$$

Thrust and Power Curve

```
% thrut/power required and available
for i = 1:400
    % at sea level
    Vs(i) = 10 + 1 * i;
    qsS = 0.5 * rho_s * Vs(i)^2 * S;
    C_Ls = W / qsS;
    C_Ds = C_d_0 + K * C_Ls^2;
    Ds(i) = qsS * C_Ds;
    Fs(i) = F_s;
    PR_s(i) = Ds(i) * Vs(i);
    PA_s(i) = Fs(i) * Vs(i);
```



```
    % at altitude (fix C_L)
    Vh(i) = Vs(i) * sqrt(rho_s / rho_h);
    qS = 0.5 * rho_h * Vh(i)^2 * S;
    C_L = C_Ls;
    C_D = C_Ds;
    Dh(i) = qS * C_D;
    Fh(i) = (rho_h / rho_s) * Fs(i);
    PR_h(i) = PR_s(i) * sqrt(rho_s / rho_h);
    PA_h(i) = PA_s(i) * (rho_h / rho_s);
```

with fixed C_L for calculation purpose, and from turbojet thrust property with altitude, one obtains

$$\text{velocity} \qquad V = \sqrt{\frac{\rho_s}{\rho}} V_s \tag{39}$$

$$\text{thrust required} \qquad D = D_S \tag{40}$$

$$\text{thrust available} \qquad T = \frac{\rho}{\rho_s} T_s \tag{41}$$

$$\text{power required} \qquad P_R = \sqrt{\frac{\rho_s}{\rho}} P_{R_s} \tag{42}$$

$$\text{power available} \qquad P_A = \frac{\rho}{\rho_s} P_{A_s} \tag{43}$$

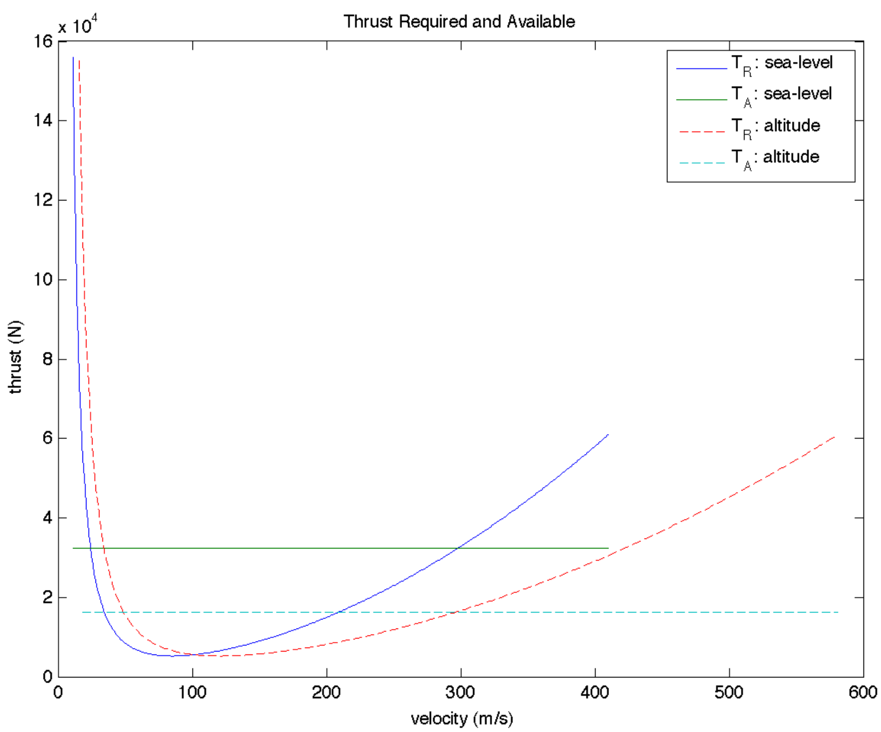


Figure 7: Thrust Required and Thrust Available Curve: Altitude 6705.6m

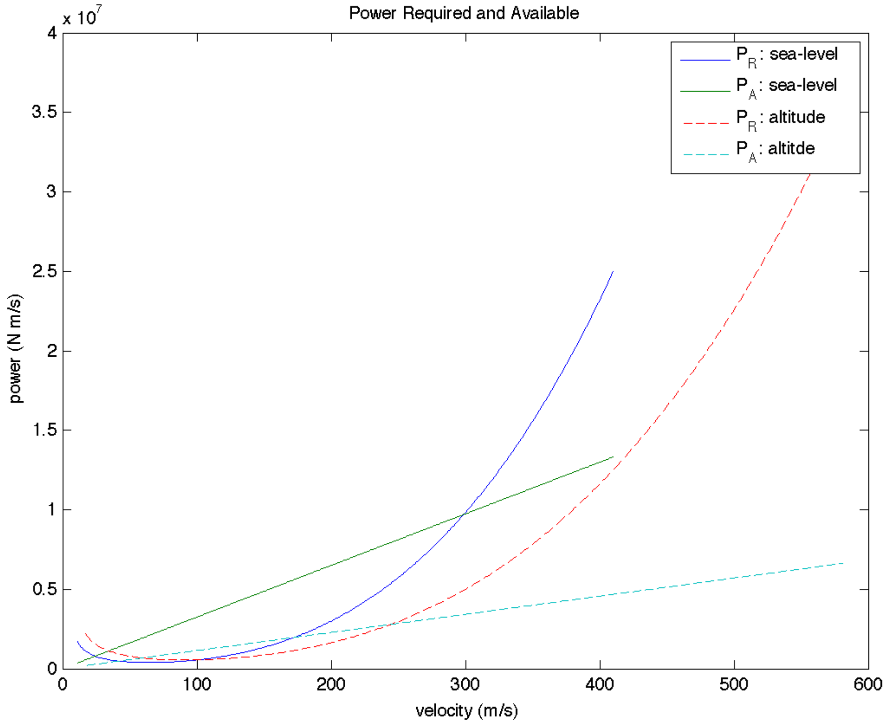


Figure 8: Power Required and Power-Available Curve: Altitude 6705.6m

4. Steady Climbing Performance

Consider a steady, unaccelerated, climbing flight, $\gamma \neq 0$, $\dot{V} = \dot{\gamma} = 0$ and the equation of motion become

$$\begin{aligned} T - D - W \sin\gamma &= m\dot{V} = 0 && (44) \\ L - W\cos\gamma &= mV\dot{\gamma} = 0 && (45) \\ \dot{x} &= V\cos\gamma && (46) \\ \dot{h} &= V\sin\gamma && (47) \end{aligned}$$

4.1 Rate of Climb and Climbing Angle

$$\dot{h} = V \sin\gamma \tag{48}$$
$$= \frac{TV - DV}{W} \tag{49}$$
$$= \frac{P_A - P_R}{W} \tag{50}$$

where $P_A = TV$ is the power available, $P_R = DV$ is the power required. $\dot{h} = V \sin\gamma$ is also called the *rate of climb R/C*. $(P_A - P_R)$ is called the *excess power*.

Graphical Approach

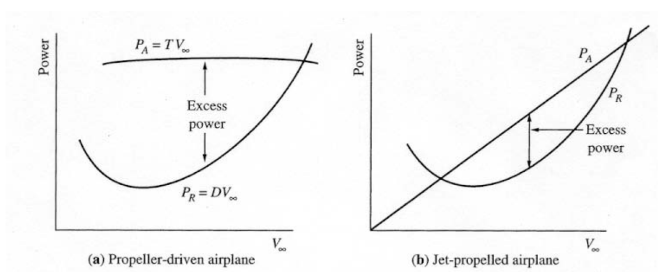


Figure 9: R/C Determined by $P_A - P_R$ Curve

- Note that Equation $(P_A - P_R)/W$ is only an approximation to the rate of climb, because $P_R = DV$ is based on level flight, and the drag for climbing is smaller than that in level flight.

Graphical Solution: Process

1. For a certain velocity V at a given altitude with fixed throttle setting
2. $L \approx W \Rightarrow C_L = \frac{W}{qS}$
3. $C_D = C_{D_0} + KC_L^2$
4. $D = qSC_D$
5. $R/C = \frac{TV - DV}{W}$, and
6. Climbing Angle $\gamma = \sin^{-1}\left(\frac{T-D}{W}\right)$

Observation: the maximum rate of climb $(R/C)_{max}$ achieves at the maximum excess power.

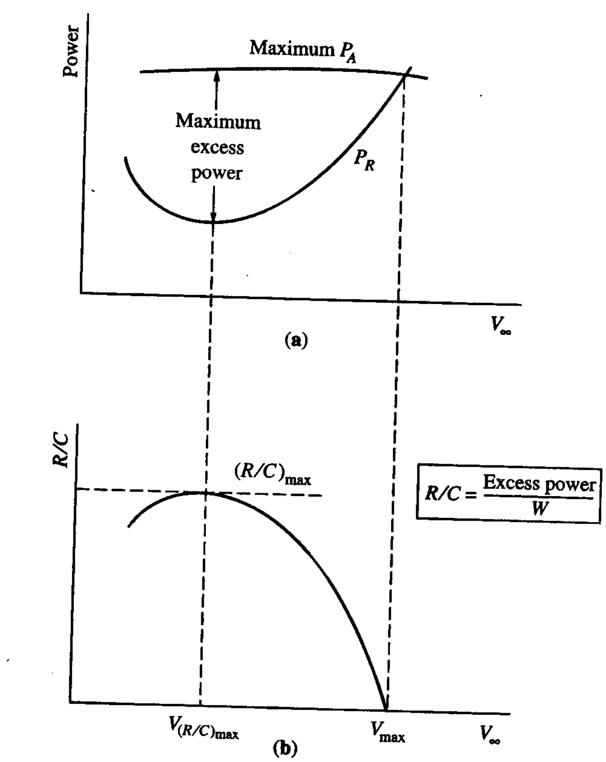


Figure 10: Maximum R/C

4.2 Climbing Performance for Jet

- climbing angle

$$\sin\gamma = \frac{T}{W} - q\frac{C_{D_0}}{W/S} - \frac{K(W/S)}{q} \tag{51}$$

- rate of climb

$$\begin{aligned} R/C &= V\left(\frac{T}{W} - q\frac{C_{D_0}}{W/S} - \frac{K(W/S)}{q}\right) \\ &= V\sin\gamma = V\left[\frac{T}{W} - \frac{1}{2}\rho V^2\frac{C_{D_0}}{(W/S)} - \frac{2K(W/S)}{\rho V^2}\right] \end{aligned} \tag{52}$$

Maximum Climbing Angle (Jet)

$$(\sin\gamma)_{max} = \left(\frac{T_A - T_R}{W}\right)_{max} = \frac{T_A - (T_R)_{min}}{W} \tag{53}$$

where the minimum thrust required happens at the maximum lift-to-drag ratio $(C_L/C_D)_{max}$, when $C_{D_0} = C_{D_i}$. Therefore

$$(T_R)_{min} = 2W\sqrt{KC_{D_0}} \tag{54}$$

$$\gamma_{max} = \sin^{-1}\left(\frac{T_A}{W} - 2\sqrt{KC_{D_0}}\right) \tag{55}$$

Maximum Climb Rate (Jet)

$$
\begin{aligned}
R/C &= V\left[\frac{T}{W} - \frac{1}{2}\rho V^2 \frac{C_{D_0}}{(W/S)} - \frac{2K(W/S)}{\rho V^2}\right] \\
V_{RC,max} &= \sqrt{\frac{(T/W)(W/S)}{3\rho C_{D_0}}\left[1 + \sqrt{1 + \frac{12 C_{D_0} K}{(T/W)^2}}\right]}
\end{aligned}
$$

4.3 Climbing Performance for Propeller

- rate of climb

$$
R/C = \frac{\eta P}{W} - \frac{1}{2}\rho V^3 \frac{C_{D_0}}{(W/S)} - \frac{2K(W/S)}{\rho V} \qquad (56)
$$

- climbing angle

$$
\sin\gamma = \frac{\eta P}{VW} - \frac{1}{2}\rho V^2 \frac{C_{D_0}}{(W/S)} - \frac{2K(W/S)}{\rho V^2} \qquad (57)
$$

Maximum Climbing Rate (Propeller)

$$(R/C)_{max} = \left(\frac{P_A - P_R}{W}\right)_{max} = \frac{P_A - (P_R)_{min}}{W} \tag{58}$$

where the minimum power required happens when

$$\frac{dP_R}{dV} = 0 \Rightarrow C_{D_0} = \frac{1}{3}C_{D_i} \tag{59}$$

therefore

$$(C_L)_{PR,min} = \sqrt{\frac{3C_{D_0}}{K}} \tag{60}$$

$$(V)_{PR,min} = \sqrt{\frac{2W}{\rho S (C_L)_{PR,min}}} \tag{61}$$

$$\gamma_{RC,max} = \sin^{-1}\frac{(R/C)_{max}}{V_{PR,min}} \tag{62}$$

Maximum Climb Angle for Propeller

$$\begin{aligned}
\sin\gamma &= \frac{\eta P}{VW} - \frac{1}{2}\rho V_\infty^2 \frac{C_{D_0}}{(W/S)} - \frac{2K(W/S)}{\rho V^2} \\
\frac{d\sin\gamma}{dV} &= 0 \\
V_{\gamma,max} &= \frac{4K(W/S)}{\rho(\eta P/W)} \Rightarrow (\sin\gamma)_{max}
\end{aligned}$$

Hodograph

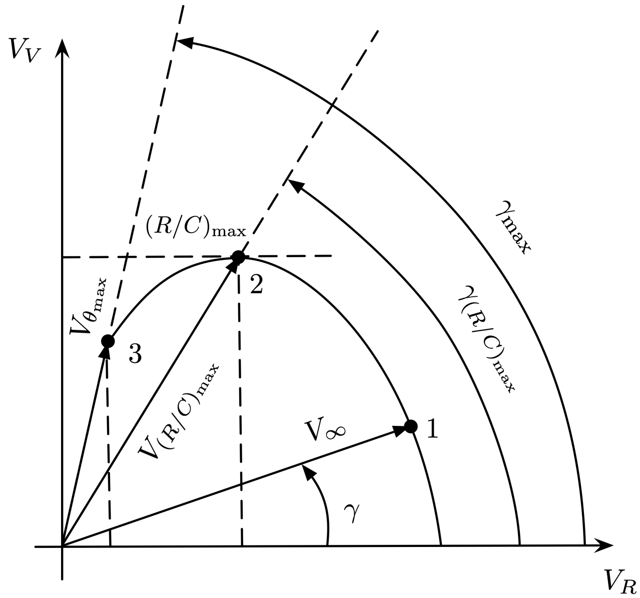


Figure 11: Hodograph diagram for climb performance at a given altitude

4.4 Absolute and Service Ceilings

In general, the thrust generated from the powerplant is a function of velocity, altitude, and throttle settings. Look at the thrust availabel and required curve again.

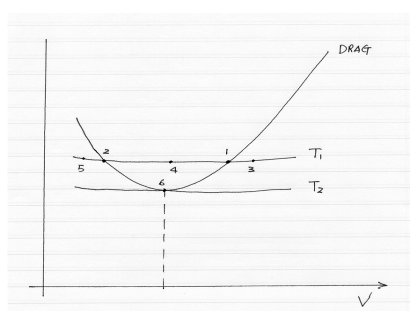


Figure 12: Level-flight equlibrium condition

One can reduce the rottle keeping the altitude, or one can climb to higher altitude with the fixed throttle setting, until a certain level of altitude that is the maximum height for this aircraft to maintain level flight (point 6).

Now we look at this condition from power point of view.

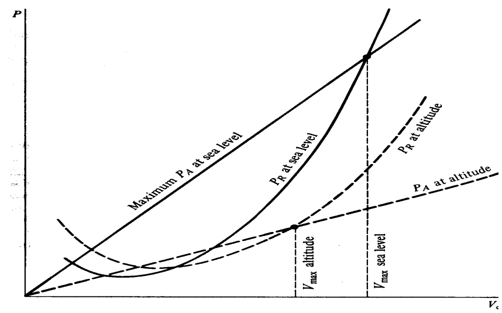


Figure 13: Power required and available at altitude

With excess power, the airplane can keep steady, unaccelerated climbing with the rate of climb R/C, and the altitude increases, resulting in decrease of excess power. In turn, rate of climb decreases ... until excess power disappears, ie. $R/C = 0$.

- The *absolute ceiling* is defined as the altitude where the maximum rate of climb is zero
- The *service ceiling*, from practice point of view, is defined as the altitude where the maximum rate of climb is 100*fpm* or approximately $0.5 m/s$

Graphical Approach

1. Calculate $(R/C)_{max}$ at a number of different altitudes.
2. Plot the results in h vs $(R/C)_{max}$ chart
3. Extrapolate the curve

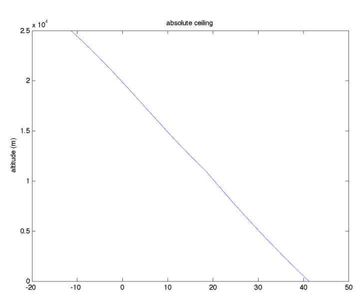


Figure 14: Absolute Ceiling, CJ1

5. Range

Range is technically defined as the total distance (measured w.r.t. the ground) traversed by the aircraft on one tank of fuel. It is closely connected with engine performance in terms fuel consumption. The following specifications are adopted:

- *Specific Fuel Consumption (SFC)* (for propeller)

$$SFC = \text{fuel weight loss (consumption) rate per power}$$

- *thrust specific fuel consumption (TSFC)* (for jet engine)

$$TSFC = \text{weight of fuel per hr per weight of thrust}$$

To cover the longest distance, common sense says that we must use the minimum fuel consumption per unit distance (e.g. km or mile).

$$c_p = -\frac{\dot{W}_f}{P} \tag{63}$$

$$c_t = -\frac{\dot{W}_f}{T} \tag{64}$$

$$c_t = \frac{c_p V}{\eta} \tag{65}$$

For steady, level flight, we have

$$\frac{-dW}{dt} = c_t T \tag{66}$$

In other words, the elemental change in weight of the airplane (loss of weight $-dW$) is due to fuel consumption over a time increment dt. Then, the inrement in distance traversed by the jet becomes:

$$ds = Vdt = \frac{-V}{c_t T} dW \tag{67}$$

Integrating from $s = 0$ where $W = W_0$ to $s = R$ where $W = W_1$, we have

$$R = \int_0^R ds = -\int_{W_0}^{W_1} \frac{V}{c_t T} dW \tag{68}$$

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Applied by $T = D$, $L = W$

$$R = \int_{W_1}^{W_0} \frac{V}{c_t} \frac{C_L}{C_D} \frac{dW}{W} \tag{69}$$

$$R = \int_{W_1}^{W_0} \frac{\eta}{c_p} \frac{C_L}{C_D} \frac{dW}{W} \tag{70}$$

Range Performance For Jet Engine

$$R = \int_{W_1}^{W_0} \frac{V}{c_t}\frac{C_L}{C_D}\frac{dW}{W} \quad (71)$$

$$= \int_{W_1}^{W_0} \sqrt{\frac{2}{\rho S}}\frac{C_L^{1/2}/C_D}{c_t}\frac{dW}{W^{1/2}} \quad (72)$$

$$R = 2\sqrt{\frac{2}{\rho S}}\frac{1}{c_t}\frac{C_L^{1/2}}{C_D}\left(W_0^{1/2} - W_1^{1/2}\right) \quad (73)$$

In order to achieve maximum range for a jet airplane, we want the following:

- maximum weight loss $W_0 - W_1 = W_f$, where W_f is the fuel weight.
- maximum $C_L^{1/2}/C_D$.

For a given altitude, the maximum range achieves when $C_L^{1/2}/C_D$ achieves the maximum.
Setting

$$\frac{d}{dC_L}\left(\frac{C_L^{1/2}}{C_D}\right) = 0 \quad (74)$$

yields

$$C_{D_0} = 3C_{D_i} \quad \text{for} \left(\frac{C_L^{1/2}}{C_D}\right)_{max} \quad (75)$$

therefore,

$$\left(\frac{C_L^{1/2}}{C_D}\right)_{max} = \frac{3}{4}\left(\frac{1}{3KC_{D_0}^3}\right)^{1/4} \quad (76)$$

$$R_{max} = 2\sqrt{\frac{2}{\rho S}}\frac{1}{c_t}\left(\frac{C_L^{1/2}}{C_D}\right)_{max}\left(W_0^{1/2} - W_1^{1/2}\right) \qquad (77)$$

Range Performance For Propeller

$$R = \int_{W_1}^{W_0} \frac{\eta}{c_p}\frac{C_L}{C_D}\frac{dW}{W} = \frac{\eta}{c_p}\frac{C_L}{C_D}\ln\frac{W_0}{W_1} \qquad (78)$$

$$R_{max} = \frac{\eta}{c_p}\left(\frac{C_L}{C_D}\right)_{max}\ln\frac{W_0}{W_1}$$

$$= \frac{\eta/c_p}{2\sqrt{KC_{D_0}}}\ln\frac{W_0}{W_1} \qquad (79)$$

Range for Electric Aircraft

Weight is constant as the battery is discharged. If we consider the total energy in the battery as E and the power required as $P_R = \eta T_R V$, then

$$\frac{dE}{dt} = \eta T_R V \tag{80}$$

and the distance is given as,

$$\frac{ds}{dt} = V \tag{81}$$

so

$$\frac{dE}{ds} = \frac{T_R}{\eta} \tag{82}$$

if we assume η is constant, this is minimized by always flying at the minimum thrust condition, $T_{R_{\min}}$ or maximum C_L/C_D, and the range is given as,

$$R = \frac{\eta E}{T_{R_{\min}}} \tag{83}$$

This is achieved at the same speed as for the Gas Powered Aircraft ($V_{TR,\min}$) but now the weight is constant so speed is constant for a given altitude

6. Endurance

The *endurance* is defined as the total time that an aircraft stays in the air on a tank of fuel.

- Again, it is common sense to use the minumum thrust per unit time (say hour) for maximum endurance.
- For steady, level flight where $L = W$ and $T = D$, the range (also called cruise range) and endurance have quantitative formulation.

$$\frac{-dW}{dt} = c_t T \tag{84}$$

we have

$$dt = \frac{-dW}{c_t T} \tag{85}$$

Integraing between $t = 0$ where $W = W_0$ and $t = E$ where $W = W_1$, we obtain:

$$E = -\int_{W_0}^{W_1} \frac{dW}{c_t T} \tag{86}$$

$$= \int_{W_1}^{W_0} \frac{1}{c_t} \frac{C_L}{C_D} \frac{dW}{W} \tag{87}$$

Endurance Performance for Jet Engine

$$E = \frac{1}{c_t}\frac{C_L}{C_D}\ln\frac{W_0}{W_1} \qquad (88)$$

$$E_{max} = \frac{1}{c_t}\left(\frac{C_L}{C_D}\right)_{max}\ln\frac{W_0}{W_1} = \frac{1}{2c_t\sqrt{KC_{D_0}}}\ln\frac{W_0}{W_1} \qquad (89)$$

In order to achieve maximum endurance for a jet airplane, we want the following:

- maximum fuel weight W_f
- maximum C_L/C_D.

Endurance Performance for Propeller

$$E = \int_{W_1}^{W_0}\frac{\eta}{c_p V}\frac{C_L}{C_D}\frac{dW}{W} \qquad (90)$$

$$= \int_{W_1}^{W_0}\frac{\eta}{c_p}\sqrt{\frac{\rho S}{2}}\frac{C_L^{3/2}}{C_D}\frac{dW}{W^{3/2}} \qquad (91)$$

$$E = \frac{\eta}{c_p}\sqrt{2\rho S}\left(\frac{C_L^{3/2}}{C_D}\right)\left(\frac{1}{\sqrt{W_1}}-\frac{1}{\sqrt{W_0}}\right) \qquad (92)$$

In order to achieve maximum endurance for a propeller driven airplane, we want the following:

- maximum weight loss $W_0 - W_1 = W_f$, where W_f is the fuel weight.
- maximum $C_L^{3/2}/C_D$.

For a given altitude, the maximum range achieves when $C_L^{3/2}/C_D$ achieves the maximum.
Setting

$$\frac{d}{dC_L}\Big(\frac{C_L^{3/2}}{C_D}\Big) = 0 \tag{93}$$

yields

$$C_{D_0} = \frac{1}{3}C_{D_i} \quad \text{for } \Big(\frac{C_L^{3/2}}{C_D}\Big)_{max} \tag{94}$$

therefore,

$$\left(\frac{C_L^{3/2}}{C_D}\right)_{max} = \frac{1}{4C_{D_0}}\left(3\frac{C_{D_0}}{K}\right)^{3/4} \tag{95}$$

$$E_{max} = \frac{\eta}{c}\sqrt{2\rho S}\left(\frac{C_L^{3/2}}{C_D}\right)_{max}\left(\frac{1}{\sqrt{W_1}} - \frac{1}{\sqrt{W_0}}\right) \tag{96}$$

Endurance for Electric Aircraft

Since the rate of energy use is given by the power the maximum endurance for an electric aircraft with a fixed available battery power E is achieved when the aircraft is flown at the minimum power condition, PRmin . The endurance is simply,

$$T_E = \frac{E}{\eta P} \tag{97}$$

and the speed is such that we are flying at maximum $C_L^{3/2}/C_D$. Note that this speed is less than the speed for maximum range! Also note that we considered η fixed or maximized at the same speed as $P_{R_{\min}}$. So we select our motor and propeller for optimal efficiency at this speed for this analysis to be meaningful.

7. Steady Level Turn

Even though an aircraft may spend most of a mission in straight flight in the vertical plane, there are times when it must change direction, i.e., turn. We shall continue our assumption of steady flight by neglecting any tangential accelerations, but must consider the radical acceleration what is normal to the coordinated flight.

The wings of the airplane are banked through angle ϕ; hence the lift vector is inclined at angle ϕ to the vertical. Therefore the airplane maintains a constant altitude, moving in the same horizontal plane. The resultant force is perpendiculat to the flight path, causing the airplane to turn in a circular path with a radius of curvature R. We wish to study the drag in turn, the radius R and the turn rate $d\theta/dt = \omega$.

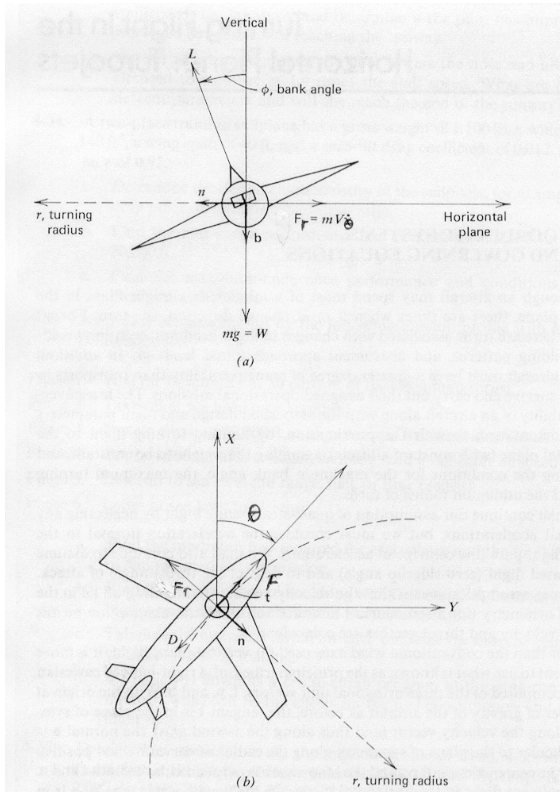

Figure 15: Steady Horizontal Turn

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The equations of motion are:

$$T = D \tag{98}$$

$$L\cos\phi = W \tag{99}$$

$$L\sin\phi = mV\frac{d\theta}{dt} = m\frac{V^2}{R} \tag{100}$$

Define *load factor*:

$$n = \frac{L}{W} = \sec\phi \tag{101}$$

Bank angle:

$$\tan\phi = \frac{V^2}{gR} \tag{102}$$

$$= \sqrt{\sec^2\phi - 1} = \sqrt{n^2 - 1} \tag{103}$$

7.1 Turning Performance

For better turning performance, i.e., smaller turning radius and higher turning rate, one can conduct analysis through the following basic formula. From the steady turn equation

$$\tan\phi = \frac{V^2}{gR} = \sqrt{\sec^2\phi - 1} = \sqrt{n^2 - 1} \tag{104}$$

one obtains:

$$R = \frac{V^2}{g\sqrt{n^2-1}} \tag{105}$$

$$\tag{106}$$

$$\omega_c = \frac{g\sqrt{n^2-1}}{V} \tag{107}$$

Generally speaking, we expect high load factor and low speed for better turning performance.

7.2 V-n Diagram

When we use $L = nW$ instead of $L = W$ the drag in steady horizontal turn becomes

$$T = D = qSC_{D_0} + K\frac{n^2W^2}{qS} \tag{108}$$

Now, let's go back to the drag in turn equation. Since $T = D$,

$$n = \left[\frac{q}{K(W/S)}\left(\frac{T}{W} - q\frac{C_{D_0}}{W/S}\right)\right]^{1/2} \tag{109}$$

illustrated by V-n plot.

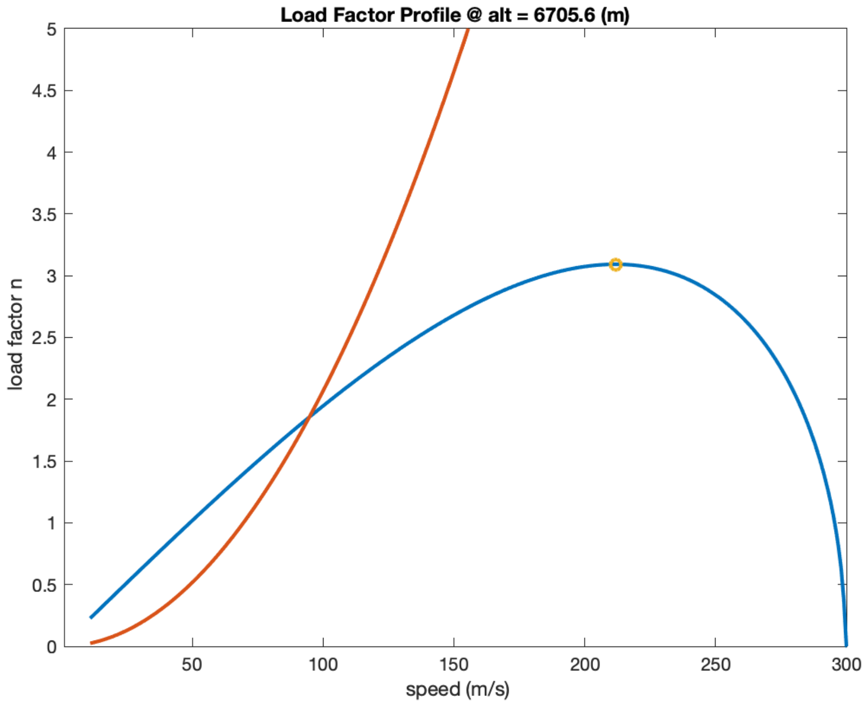

Figure 16: V-n Diagram of [CJ1]

From the performance analysis perspective, we care about the *extreme value* as we have done before. Here, the $n_{\max}$ can be obtained by taking $\frac{dn}{dV} = \frac{dn}{dq} = 0$,

$$\frac{dn^2}{dq} = \frac{1}{K(W/S)}\left(\frac{T}{W} - 2q\frac{C_{D_0}}{W/S}\right) \tag{110}$$

The result becomes: $C_{D_0} = KC_L^2$, or

$$\frac{C_L}{C_D} = \left(\frac{C_L}{C_D}\right)_{\max} = \frac{1}{2\sqrt{KC_{D_0}}} \tag{111}$$

Therefore,

$$n_{\max} = \frac{1}{2\sqrt{KC_{D_0}}}\frac{T}{W} \tag{112}$$

$$V_{n,\max} = \sqrt{\frac{2n_{\max}W/S}{\rho(C_L)_{LD,\max}}} = \sqrt{\frac{(T/W)(W/S)}{\rho C_{D_0}}} \tag{113}$$

In fact, a typical V-n diagram presents more information for performance analysis. In addition to the main V-n curve, if we consider the minimum speed (stall speed), that is

$$V_{stall} = \sqrt{\frac{2nW}{\rho S C_{L_{\max}}}}$$

then the load factor is constrained by the curve,
$n = \frac{1}{2} C_{L_{\max}} \rho/(W/S) V_{stall}^2$.
One shall also make sure the sustainable level turn must have $n \geq 1$, in other words, $n = 1$ becomes another constraint factor. Of course, the load factor is an important term from aircraft structural perspective. The structural limit to n can be plotted in the V-n diagram to complete the story.

7.3 Minimum Turning Radius

The minimum radius is obtained by setting $dR/dV = 0$:

$$n^2 - 1 - qn\frac{dn}{dq} = 0$$

leading to

$$(V)_{R,min} = \sqrt{\frac{4K(W/S)}{\rho(T/W)}} \quad (114)$$

$$n_{R,min} = \sqrt{2 - \frac{4KC_{D_0}}{(T/W)^2}} \quad (115)$$

$$R_{min} = \frac{4K(W/S)}{g\rho(T/W)\sqrt{1 - 4K\frac{C_{D_0}}{(T/W)^2}}} \quad (116)$$

7.4 Maximum Turning Rate

The maximum turning rate ω is obtained by setting $d\omega/dV = 0$:

$$nV\frac{dn}{dV} - (n^2 - 1) = 0$$

leading to

$$(V)_{\omega,max} = \sqrt{\frac{2(W/S)}{\rho}\sqrt{\frac{K}{C_{D_0}}}} \tag{117}$$

$$n_{\omega,max} = \sqrt{\frac{T/W}{\sqrt{KC_{D_0}}} - 1} \tag{118}$$

$$\omega_{max} = g\sqrt{\frac{\rho}{W/S}\left(\frac{T/W}{2K} - \sqrt{\frac{C_{D_0}}{K}}\right)} \tag{119}$$

8. The Pull-up and Pull-down Maneuvers

Consider an airplane initially in straight and level flight. The pilot suddenly pitches the airplane to a higher angle of attack such that the lift suddenly increases. The flight path becomes curved in the vertical plane, with a turn radius R and turn rate $\dot{\gamma}$

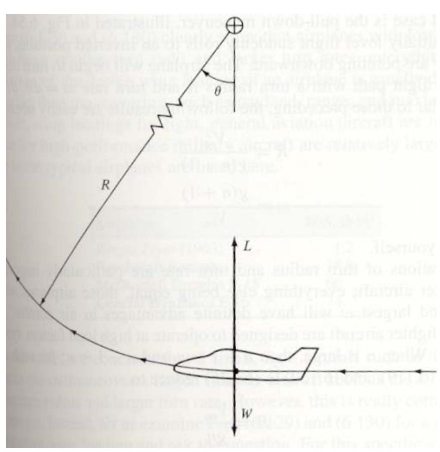

Figure 17: Pull-up Maneuver

Back to the general force equations again,

$$\begin{aligned} T - D - W \sin\gamma &= m\dot{V} \\ L\cos\mu - W\cos\gamma + Y\sin\mu &= mV\dot{\gamma} \\ L\sin\mu - Y\cos\mu &= mV\dot{\sigma}\cos\gamma \end{aligned}$$

This time, we will consider the vertical turn. Assume no sideslip $\beta = 0$ therefore no side force Y, $\sigma = \dot{\sigma} = 0$, $\dot{V} = 0, \mu = 0$. Consider an airplane initially in straight and level flight. The pilot suddenly pitches the airplane to a higher angle of attack such that the lift suddenly increases. The flight path becomes curved in the vertical plane, with a turn radius R_h and turn rate $\dot{\gamma} = \omega_h$.

$$T - D - W\sin\gamma = 0 \tag{120}$$

$$L - W\cos\gamma = mV\dot{\gamma} = mV\omega_h = m\frac{V^2}{R_h} \tag{121}$$

At the bottom, $\gamma = 0$, or *pull up*

$$L - W = m\frac{V^2}{R_h} \tag{122}$$

leading to

$$R_h = \frac{mV^2}{L - W} = \frac{V^2}{g(n-1)} \tag{123}$$

$$\omega_h = \dot{\gamma} = \frac{V}{R_h} = \frac{g(n-1)}{V} \tag{124}$$

At the top, $\gamma = 180^\circ$, or pull down, as illustrated by

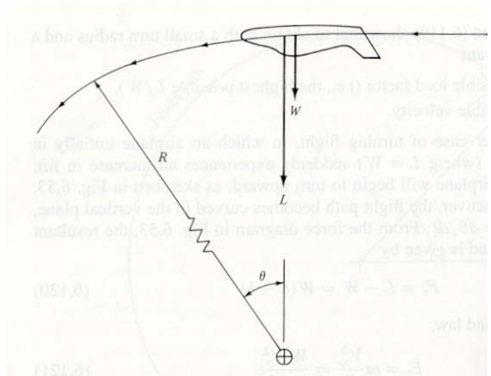

Figure 18: Pull-down Maneuver

$$L + W = m\frac{V^2}{R_h} \tag{125}$$

leading to

$$R_h = \frac{V^2}{g(n+1)} \tag{126}$$

$$\omega_h = \frac{V}{R_h} = \frac{g(n+1)}{V} \tag{127}$$

Please observe, pull up and pull down maneuvers represent

- instantaneous maneuver, the thrust limitations are not relevant (vs steady climbing).
- when $n >> 1$,

$$R_h = \frac{V^2}{gn} \tag{128}$$

$$\omega_h = \frac{gn}{V} \tag{129}$$

leading to

$$R_{h,\min} = \frac{2(W/S)}{\rho g C_{L_{max}}} \tag{130}$$

$$\omega_{h,\max} = g\sqrt{\frac{\rho C_{L_{max}} n_{max}}{2(W/S)}} \tag{131}$$

9. Accelerated Climb

The previous steady climb study shows that $V_{RC,max}$ varies as one climbs and thus if we wish to climb at $V_{RC,max}$ then our original assumption that $\dot{V} = 0$ cannot be correct. When $\dot{V}$ is very small this may not matter and it is not very important for civil aircraft. It may, however be critical for the analysis of the performance of military fighters. The corrected analysis can proceed on the basis of *energy method*.

Let's bring back the longitudinal force equations:

$$\begin{aligned} T - D - W\sin\gamma &= 0 \\ L - W\cos\gamma &= mV\dot{\gamma} = mV\omega_h = m\frac{V^2}{R_h} \end{aligned}$$

where the excess power $P_A - P_R = TV - DV = WV\sin\gamma + mV\dot{V}$ when the acceleration is also taken into account, which leads to

$$\frac{P_A - P_R}{W} = \frac{TV - DV}{W} = \overbrace{V\sin\gamma}^{dh/dt} + \frac{V}{g}\frac{dV}{dt} \qquad (132)$$

The term of $\frac{TV-DV}{W}$ is defined as *specific excess power*:

$$P_s = \frac{TV - DV}{W} \qquad (133)$$

Consider an airplane of mass m in flight at some altitude h and with some velocity V. Its total energy is the sum of potential energy due to altitude and kinetic energy due to speed:

$$E = Wh + \frac{1}{2}\frac{W}{g}V^2 \qquad (134)$$

where E is the total mechanical enery of aircraft with respect to the earth frame $\mathcal{F}_E$, Wh is the potential energy, $\frac{1}{2}\frac{W}{g}V^2$ is the kinetic energy.
Define *energy height* h_e (also called *specific energy*) as the total energy per unit weight:

$$h_e = h + \frac{V^2}{2g} \Longrightarrow E = Wh_e \qquad (135)$$

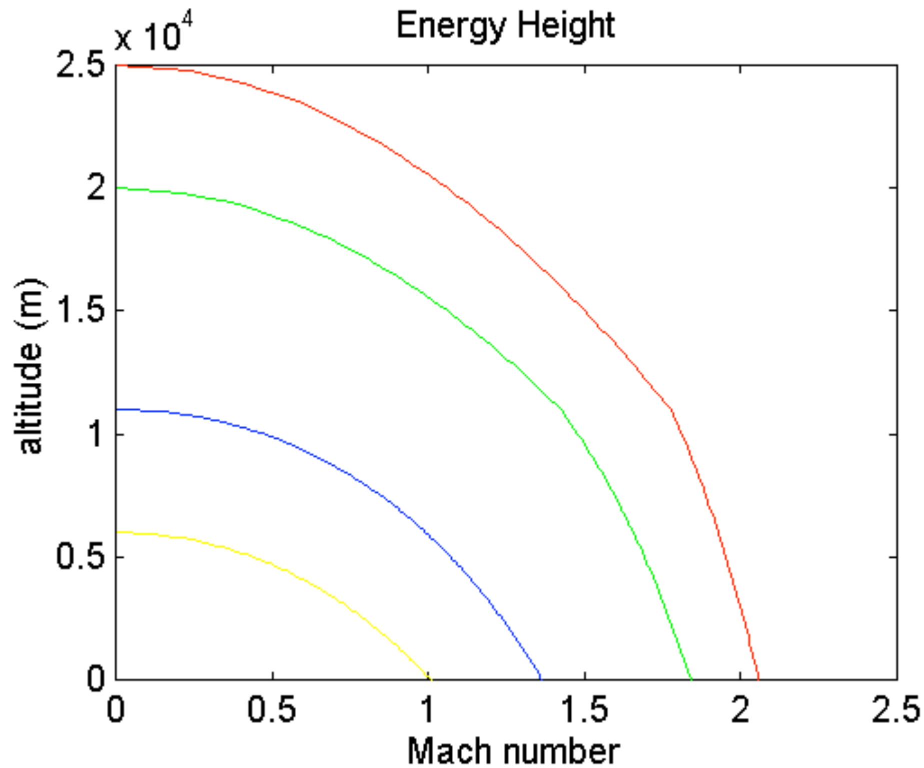

Figure 19: Energy Height

The state of increase of energy is $(T - D)V$ where engine adds power and drag removes power, thus $\dot{E} = P_A - P_R$. Therefore

$$P_s = \frac{\dot{E}}{W} = \frac{P_A - P_R}{W} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \dot{h}_e \qquad (136)$$

That is, the time rate of change of energy height is equal to the specific excess power. This answers the question of how airplane changing its energy state.

On the other hand, how can we ascertain whether a given airplane has enough P_s to reach a certain energy height? This question is answered by plotting altitude-Mach curve with energy heights (h_e contour) and specific excess power (P_s contour).

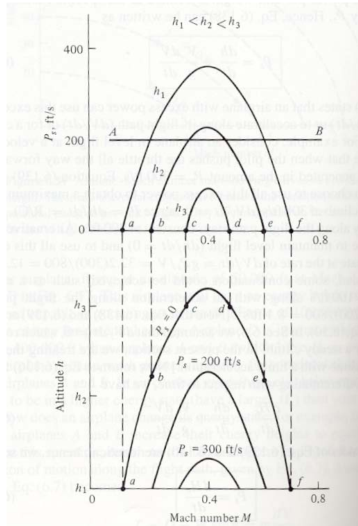

Figure 20: P_s Contour

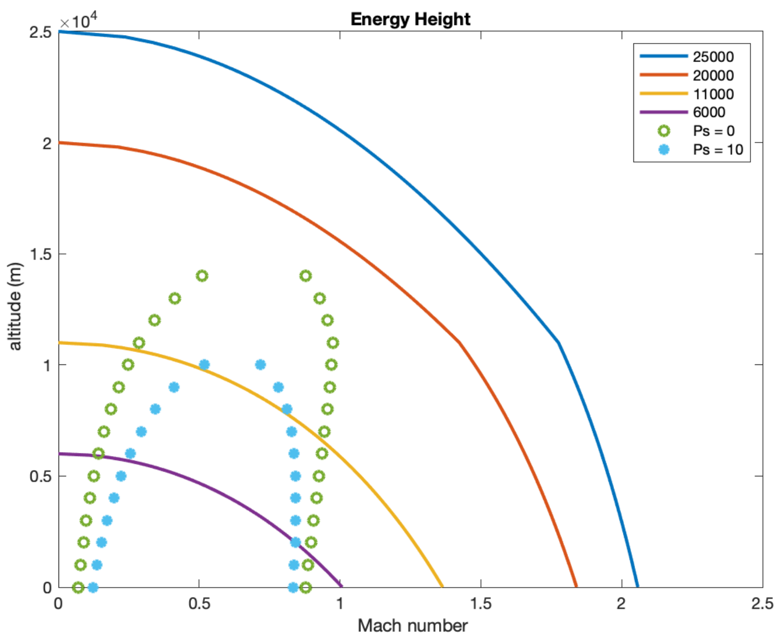

Figure 21: CJ1 P_s Profile

Summary

1. Force Equations of Motion for Performance Analysis
2. Aerodynamics Forces: Lift and Drag
3. Steady Level Flight Performance
4. Steady Climbing Performance
5. Range
6. Endurance
7. Steady Level Turn
8. The Pull-up and Pull-down Maneuvers
9. Accelerated Climb