Aircraft Flight Dynamics

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Overview

1. **Equations of motion**
   - Full Nonlinear EOM
   - Decoupling of EOM
   - Simplified Models

2. **Aerodynamics**
   - Dimensionless coefficients
   - Stability & Control Derivatives

3. **Trim Analysis**
   - Level, climb and glide
   - Turning maneuver

4. **Linearized Dynamics Analysis**
   - Longitudinal
   - Lateral

Equations of Motion

- Dynamical system is defined by a transition function, mapping states & control inputs to future states

$$\dot{X} = f(X, \delta)$$

States and Control Inputs

$$X = \left[\begin{array}{c} x \\ y \\ z \\ u \\ v \\ w \\ \theta \\ \phi \\ \psi \\ p \\ q \\ r \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} x \\ y \\ z \end{array}}\right\} \text{position} \\ \left.\vphantom{\begin{array}{c} u \\ v \\ w \end{array}}\right\} \text{velocity} \\ \left.\vphantom{\begin{array}{c} \theta \\ \phi \\ \psi \end{array}}\right\} \text{attitude} \\ \left.\vphantom{\begin{array}{c} p \\ q \\ r \end{array}}\right\} \text{angular velocity} \end{array}$$

$$\delta = \left[\begin{array}{c} \delta_e \\ \delta_t \\ \delta_a \\ \delta_r \end{array}\right] \begin{array}{l} \text{elevator} \\ \text{throttle} \\ \text{aileron} \\ \text{rudder} \end{array}$$

There are alternative ways of defining states and control inputs

Full Nonlinear EOM

- System of 12 Nonlinear ODEs

Dynamics Eqs.*

Linear Acceleration = Aero + gravity + Gyro

$$m\dot{u} = X - mg\sin(\theta) + m(rv - qw)$$
$$m\dot{v} = Y + mg\sin(\phi)\cos(\theta) + m(pw - ru)$$
$$m\dot{w} = Z + mg\cos(\phi)\cos(\theta) + m(qu - pv)$$

Angular Acceleration = Aero + Gyro

$$I_{xx}\dot{p} = l + (I_{yy} - I_{zz})qr$$
$$I_{yy}\dot{q} = m + (I_{zz} - I_{xx})pr$$
$$I_{zz}\dot{r} = n + (I_{xx} - I_{yy})pq$$

*Assuming calm atmosphere and symmetric aircraft (Neglecting cross-products of inertia

Kinematic Eqs.:

Relation between position and velocity

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = {}^{N}R^{B}_{(\phi,\theta,\psi)} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Relation between attitude and angular velocity

$$\dot{\phi} = p + q\sin(\phi)\tan(\theta) + r\cos(\phi)\tan(\theta)$$
$$\dot{\theta} = q\cos(\phi) - r\sin(\phi)$$
$$\dot{\psi} = q\frac{\sin(\phi)}{\cos(\theta)} + r\frac{\cos(\phi)}{\cos(\theta)}$$

Nonlinearity and Model Uncertainty

- Sources of nonlinearity:
  - Trigonometric projections (dependent on attitude)
  - Gyroscopic effects
  - Aerodynamics
    - Dynamic pressure
    - Reynolds dependencies
    - Stall & partial separation

- Model uncertainties:
  - Gravity & Gyroscopic terms are straightforward, provided we can measure mass, inertias and attitude accurately
  - **Aerodynamics is harder**, especially viscous effects: lifting surface drag, propeller & fuselage aerodynamics

Flight Dynamics and Navigation Decoupling

The full nonlinear EOMs have a cascade structure

$$\xrightarrow{\delta_{dyn}} \boxed{\text{Flight Dyn.}} \xrightarrow{X_{dyn} = \delta_{nav}} \boxed{\text{Nav.}} \xrightarrow{X_{nav}}$$

<u>Flight Dynamics</u>

$$X_{dyn} = \begin{bmatrix} u \\ v \\ w \\ \theta \\ \phi \\ p \\ q \\ r \end{bmatrix} \qquad \delta_{dyn} = \begin{bmatrix} \delta_e \\ \delta_t \\ \delta_a \\ \delta_r \end{bmatrix}$$

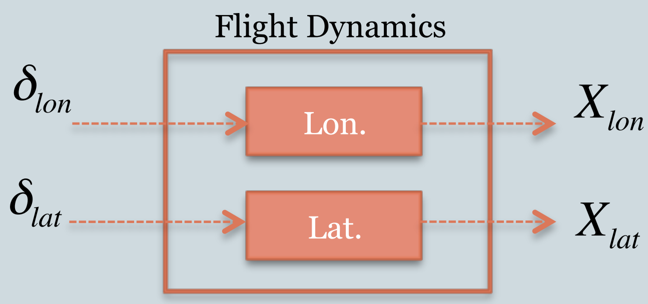
<u>Navigation</u>

$$X_{nav} = \begin{bmatrix} x \\ y \\ z \\ \psi \end{bmatrix} \qquad \delta_{nav} = \begin{bmatrix} u \\ v \\ w \\ \theta \\ \phi \\ p \\ q \\ r \end{bmatrix}$$

- Flight Dynamics is the “inner dynamics”
- Navigation is “outer dynamics”: usually what we care about

Longitudinal & Lateral Decoupling

For a symmetric aircraft near a symmetric flight condition, the Flight Dynamics can be further decoupled in two independent parts



Longitudinal Dynamics

$$X_{lon} = \begin{bmatrix} u \\ w \\ \theta \\ q \end{bmatrix} \qquad \delta_{lon} = \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

Lateral-Directional Dynamics

$$X_{lat} = \begin{bmatrix} v \\ \phi \\ p \\ r \end{bmatrix} \qquad \delta_{lat}{*} = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

* As we shall see, throttle also has effects on the lateral dynamics, but these can be eliminated with appropriate aileron and rudder

- Although usually used in perturbational (linear) models, many times this decoupling can also be used for nonlinear analysis (e.g. symmetric flight with large vertical motion)

Alternative State Descriptions

- Translational dynamics:
  1. {u, v, w}: most useful in 6 DOF flight simulation
  2. {V, alfa, beta}: easiest to describe aerodynamics

Transformations:

$$u = V\cos(\alpha)\cos(\beta)$$

$$v = V\sin(\beta)$$

$$w = V\sin(\alpha)\cos(\beta)$$

- Longitudinal dynamics:
  1. {V, alfa, theta, q}: conventional description
  2. {V, CL, gamma, q}: best for nonlinear trajectory optimization
  3. {V, CL, theta, q}: all states are measurable, more natural for controls

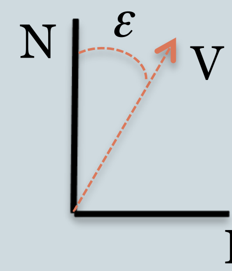
$$C_L = a_o(\alpha - \alpha_{L_o})$$

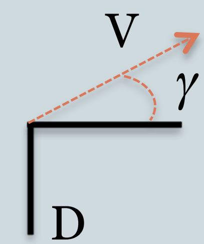
$$\gamma = \theta - \alpha$$

$$C_L \approx -\frac{m}{\frac{1}{2}\rho S}\frac{accel_z}{V^2}$$

Simplified Models

- Many times we can neglect or assume aspects of the system and look at the overall behavior
- Its important to know what you want to investigate





x,y,z: North, East, Down position coordinates
ε : course over ground

Model	States	Controls	EOM	Constraints
2 DOF navigation+ 1 DOF point mass	x, y, ε	$\dot{\varepsilon}$	$\dot{x} = V_o \sin(\varepsilon)$ $\dot{y} = V_o \cos(\varepsilon)$ $\varepsilon = \int \dot{\varepsilon}\, dt$	$\dot{\varepsilon} \le \frac{V_o}{R_{\min}}$
3 DOF navigation + 1 DOF point mass	x, y, z, ε	$\dot{\varepsilon}, \gamma$	$\dot{x} = V_o \sin(\varepsilon)\cos(\gamma)$ $\dot{y} = V_o \cos(\varepsilon)\cos(\gamma)$ $\dot{z} = -V_o \sin(\gamma)$	$\gamma \le \gamma_{\max}$
3 DOF navigation + 2 DOF point mass	$x, y, z, \gamma, \varepsilon$	V, C_L, ϕ	$\dot{\varepsilon} = \frac{1}{2}\rho \frac{m}{S}^{-1} V C_L \sin(\phi)$ $\dot{\gamma} = \frac{1}{2}\rho \frac{m}{S}^{-1} V C_L \cos(\phi) - \frac{g\cos(\gamma)}{V}$	$V \le V_{\max}$ $C_L < C_{L_{\max}}$ $\phi \le \phi_{\max}$
3 DOF navigation + 3 DOF point mass	$x, y, z, \gamma, \varepsilon, V$	T, C_L, ϕ	$\dot{V} = \frac{T}{m} - \frac{g}{m}\sin(\gamma) - \frac{1}{2}\rho \frac{m}{S}^{-1} V^2 C_D(C_L)$	$T \le T_{\max}$
Many more models!	....	....	....	....

+ dynamics
+ variables
+ details

"Everything should be made as simple as possible, but not more" (~A. Einstein)

Aerodynamics

- In the full nonlinear EOM aerodynamic forces and moments are:

$$X, Y, Z, l, m, n$$

- Given how experimental data is presented, and to separate different aerodynamic effects, its easier to use:

$$L, D, Y, T, l, m, n$$

- Dimensional analysis allows to factor different contributions:
  - Dynamic pressure
  - Aircraft size
  - Aircraft geometry
  - Relative flow angles
  - Reynolds number

$$L = \underbrace{\tfrac{1}{2}\rho V^2}_{\text{dyn. pressure}} \underbrace{S}_{\text{size}} \underbrace{C_L}_{\text{Aircraft and flow geometry, and Reyonolds}}$$

Aerodynamics II

- The dimensionless forces and moments $C_L, C_D, C_Y, C_T, C_l, C_m, C_n$ are a function of:

  i. Aircraft geometry (fixed): AR, taper, dihedral, etc.

  ii. Control surface deflections $\delta_e, \delta_a, \delta_r$

  iii. Relative flow angles: $\alpha \approx \frac{w}{V}, \quad \beta \approx \frac{v}{V}, \quad \lambda = \frac{V}{\Omega R}, \quad \hat{p} = \frac{pb}{2V}, \quad \hat{q} = \frac{qc}{2V}, \quad \hat{r} = \frac{rb}{2V}$

  iv. Reynolds number: $\mathrm{Re} = \frac{\rho c V}{\mu}$ if the variation of speeds is small, it can be assumed constant and factored out

- Alfa and lambda: dependence is nonlinear and should be preserved if possible

- The rest can be represented with linear terms (Stability and Control Derivatives)

- At low AoA some stability derivatives depend on alfa, and at high angles of attack all are affected by alf

Stability and Control Derivatives

	Stability Derivtaives					Control Derivatives			Nonlinear/Trim	
	α'	β	$\hat{p}$	$\hat{q}$	$\hat{r}$	δ_e	δ_a	δ_r	α	λ
CL	C_{L_α}			C_{L_q}		$C_{L_{\delta_e}}$			$\bar{C}_L(\alpha,\lambda)$	
CD	C_{D_α}			C_{D_q}		$C_{D_{\delta_e}}$			$\bar{C}_D(\alpha,\lambda)$	
CY		C_{Y_β}			C_{Y_r}			$C_{Y_{\delta_r}}$		
Cl		C_{l_β}	C_{l_p}		C_{l_r}		$C_{l_{\delta_a}}$	$C_{l_{\delta_r}}$		$\bar{C}_l(\lambda)$
Cm	C_{m_α}			C_{m_q}		$C_{m_{\delta_e}}$			$\bar{C}_m(\alpha,\lambda)$	
Cn		C_{n_β}	C_{n_p}		C_{n_r}		$C_{n_{\delta_a}}$	$C_{n_{\delta_r}}$		$\bar{C}_n(\lambda)$
CT										$C_T(\lambda)$

δ are small angular deflections w.r.t. a zero position, usually the trim deflection

α' is an angle of attack perturbation around α

- ~zero
- Minor importance
- Estimate via calculations
- Estimate via calculations or flight testing
- Estimate or trim out via flight testing
- Hard to estimate

Stability and Control Derivatives II

- Examples of force and moment expressions:

Lift: $$C_L = \bar{C}_L(\alpha_0, \lambda) + C_{L_\alpha}\alpha' + C_{L_q}\hat{q} + C_{L_{\delta_e}}(\delta_e - \delta_{e_0})$$

Pitching moment: $$C_m = \bar{C}_m(\alpha_0, \lambda) + C_{m_\alpha}\alpha' + C_{m_q}\hat{q} + C_{m_{\delta_e}}(\delta_e - \delta_{e_0})$$

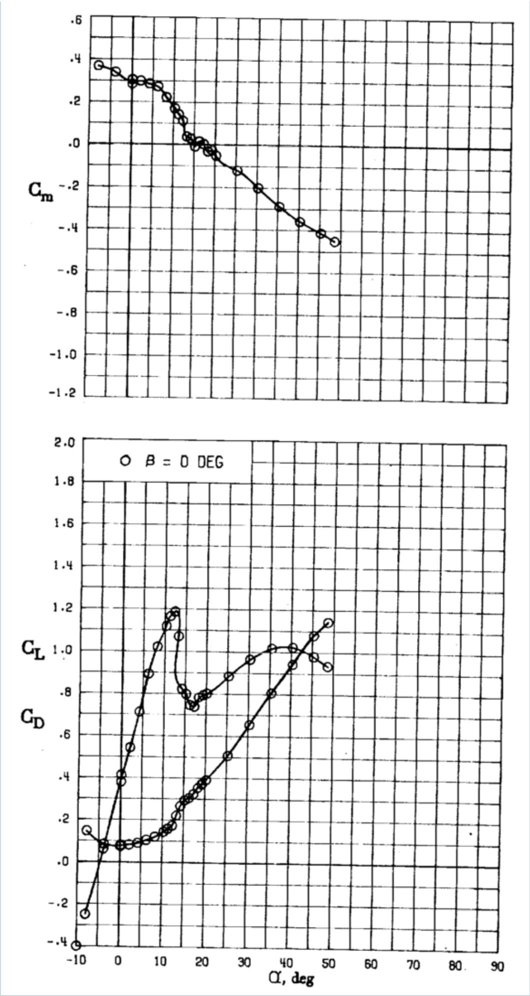
Rolling moment: $$C_l = C_{l_p}\hat{p} + C_{l_{\delta_a}}\delta_a + C_{l_\beta}\beta + C_{l_{\delta_r}}\delta_r$$

- Example dimensionless pitching equation*:

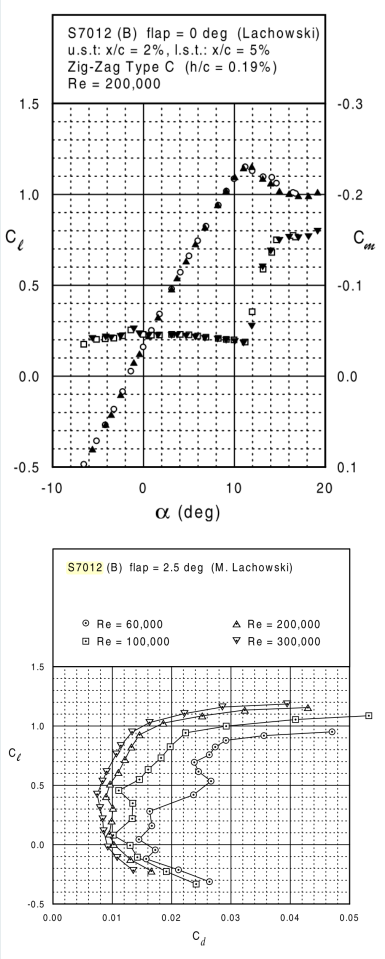
$$\hat{I}_{xx}\dot{\hat{q}} = C_m = \bar{C}_m(\alpha_0, \lambda) + C_{m_\alpha}\alpha' + C_{m_q}\hat{q} + C_{m_{\delta_e}}(\delta_e - \delta_{e_0})$$

* Neglecting gyroscopic terms

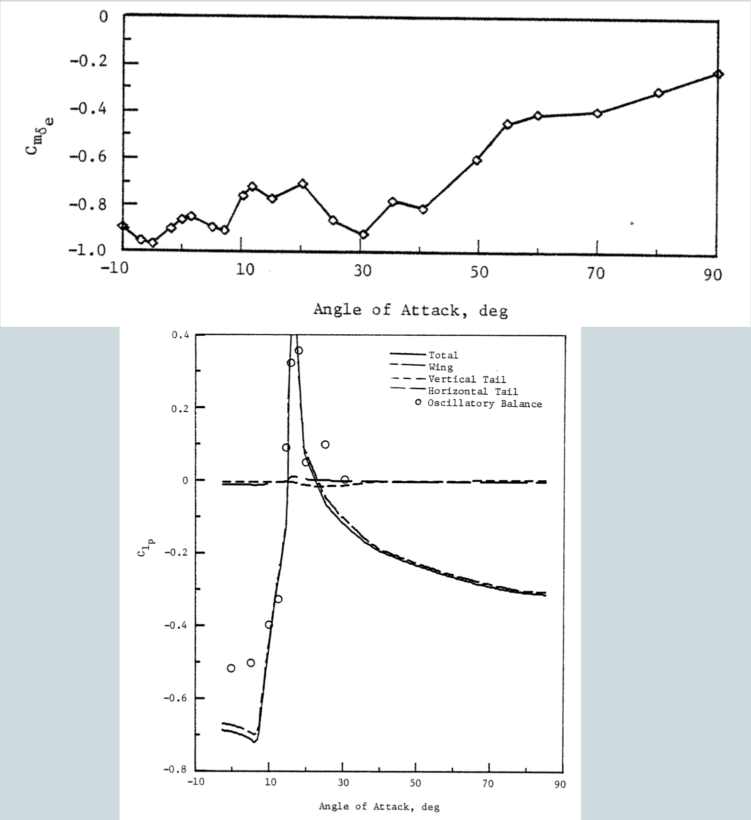
Aerodynamics IV



Bihrle, et. al. (1978)



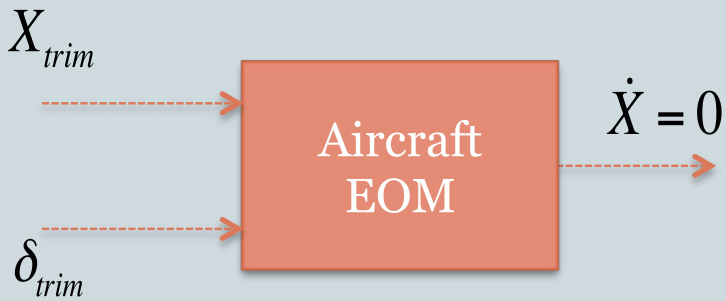
LSPAD, Selig, et. al. (1997)



Zilliac (1983)

Trim Analysis

- Flight conditions at which if we keep controls fixed, the aircraft will remain at that same state (provided no external disturbances)



- For each aircraft there is a mapping between trim states and trim control inputs
  - Analogy: car going at constant speed, requires a constant throttle position

$$\dot{X} = f(X_{trim}, \delta_{trim}) = 0 \quad \Rightarrow \quad X_{trim} = g_{trim}(\delta_{trim})$$

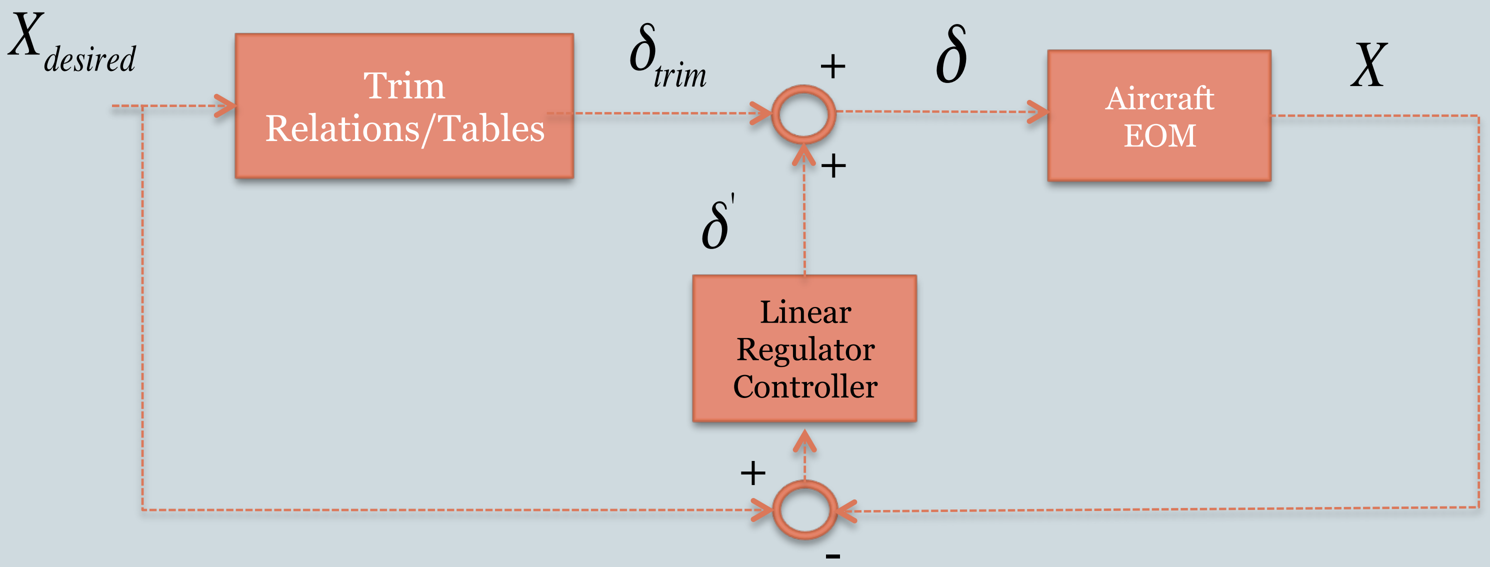
- The mapping g() is not always one-to-one, could be many-to-many!

- If internal dynamics are stable, then flight condition converges on trim condition

An idea: Trim + Regulator Controller

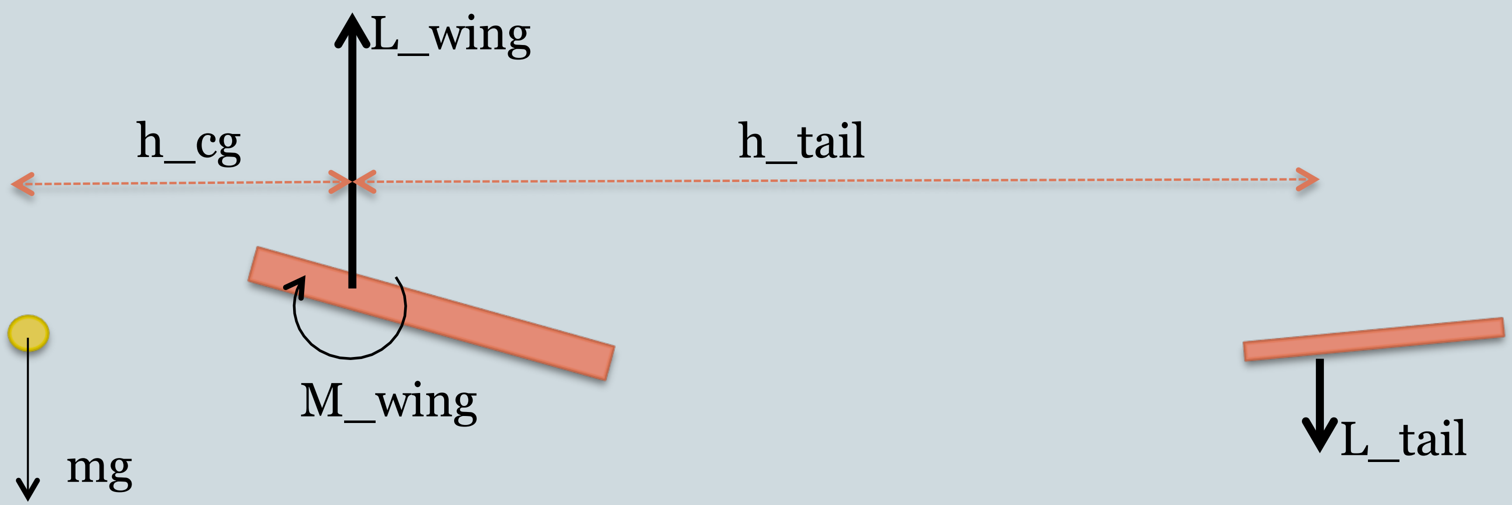
I. Inverse trim: set control inputs that will take us to the desired state

II. Regulator: to stabilize modes and bring us back to desired trim state in the presence of disturbances



Longitudinal Trim

- Simple wing-tail system



Longitudinal Trim (II)

- Moment balance:

$$0 = M_{wing} - h_{CG} L_{wing} + x_{tail} L_{tail}$$

$$\rightarrow 0 = 1/2 \rho V^2 \left[c_{wing} S_{wing} C_{m_{wing}} - h_{CG} S_{wing} C_{L_{wing}} + h_{tail} S_{tail} C_{L_{tail}} \right]$$

$$\Rightarrow \frac{h_{CG}}{c_{wing}} C_{L_{wing}}(\alpha_{trim}) = \frac{h_{tail} S_{tail}}{c_{wing} S_{wing}} C_{L_{tail}}(\alpha_{trim}, \delta e_{trim}) - C_{m_{wing}}$$

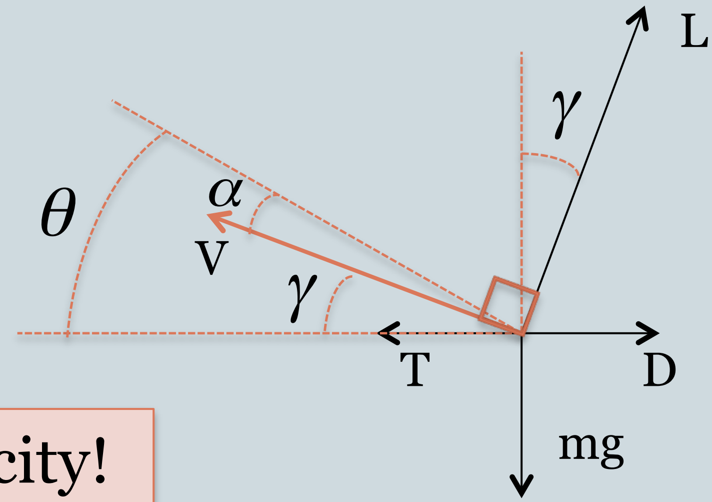
Elevator trim defines trim AoA, and consequently trim CL

Longitudinal Trim (III)

- Force balance*

$$mg = L\cos(\gamma) \approx L = \tfrac{1}{2}\rho V^2 S C_L$$

$$\Rightarrow V^2 = \frac{mg}{\tfrac{1}{2}\rho S C_L(\delta e_{trim})}$$



Trim Elevator defines trim Velocity!

$$T = D + L\sin(\gamma) \approx D + L\gamma$$

$$\Rightarrow \gamma = \frac{T_{(\delta t_{trim})}}{mg} - \frac{1}{(L/D)_{(\delta e_{trim})}}$$

Elevator & Thrust **both** define Gamma!

*Assuming small Gamma

Longitudinal Trim (IV)

- How do we get an aircraft to climb? (Gamma > 0)

- Two ways:
  1. Elevator up
     - Elevator up increases AoA, which increases CL
     - Increased CL, accelerates aircraft up
     - Up acceleration, increases Gamma
     - Increased Gamma rotates Lift backwards, slowing down the aircraft

  2. Increase Thrust
     - Increased thrust increases velocity, which increases overall Lift
     - Increased Lift, accelerates aircraft up
     - Up acceleration, increases Gamma
     - Increased Gamma rotates Lift backwards, slowing down the aircraft to original speed (set by Elevator, remember!)

- Elevator has its limitations
  - When L/D max is reached, we start going down
  - When CL max is reached, we go down even faster!

Experimental Trim Relations

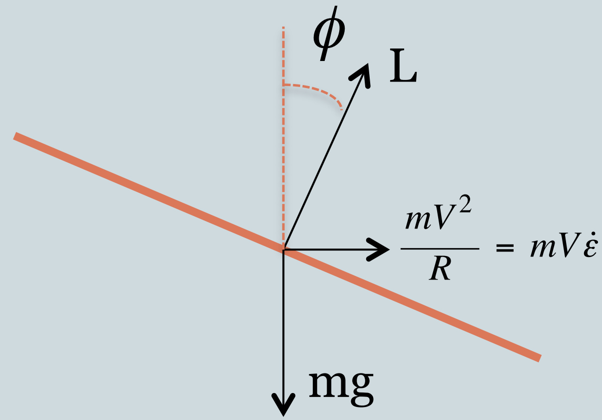
- Theoretical relations hold to some degree experimentally
- In reality:
  - Propeller downwash on horizontal tail has a significant distorting effect
  - Reynolds variations with speed, distort aerodynamics
- One can build trim tables experimentally
  - Trim flight at different throttle and elevator positions
  - Measure:
    - Average airspeed
    - Average flight path angle Gamma
  - Phugoid damper would be very helpful
- One could almost fly open loop with trim tables!

Turning Maneuver

- Centripetal force balance:

$$L\sin(\phi) = \frac{mV^2}{R}$$

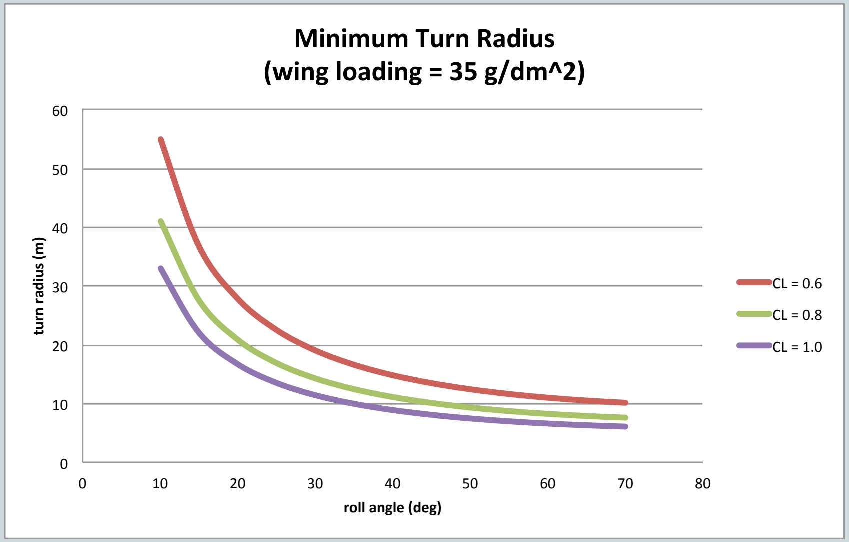
$$\Rightarrow R = \frac{mV^2}{\frac{1}{2}\rho V^2 S C_L \sin(\phi)} = \frac{m}{S}\frac{1}{\frac{1}{2}\rho C_L \sin(\phi)}$$



- Minimum turn radius:

$$R_{\min} = \frac{m}{S}\frac{1}{\frac{1}{2}\rho C_{L_{\max}} \sin(\phi_{\max})}$$

- Depends on: $\phi_{\max}, \frac{m}{S} \,\&\, C_{L_{\max}}$



Turning Maneuver II

- What are the constraints on maximum turn?
  1. Elevator deflection to achieve high CL in a turn
  2. Do we care about loosing altitude?
  3. Maximum speed and thrust
  4. Controls: maneuver can be short lived, so high bandwidth is require for tracking tracking
     1. Roll tracking, etc.
     2. Sensor bandwidth
  5. Maximum G-loading
  6. Maximum CL and stall
  7. Aerolasticity of controls at high loading

- Elevator to achieve CL:
  - The pitching moment balance equation in dimensionless form:

$$\hat{I}_{xx}\dot{\hat{q}} = \bar{C}_m(\alpha,\lambda_0) + C_{m_q}\hat{q} + C_{m_{\delta_e}}(\delta_e - \delta_{e_0})$$

  - Assume that before the turn we have trimmed the aircraft in level flight at the desired alfa (CL):

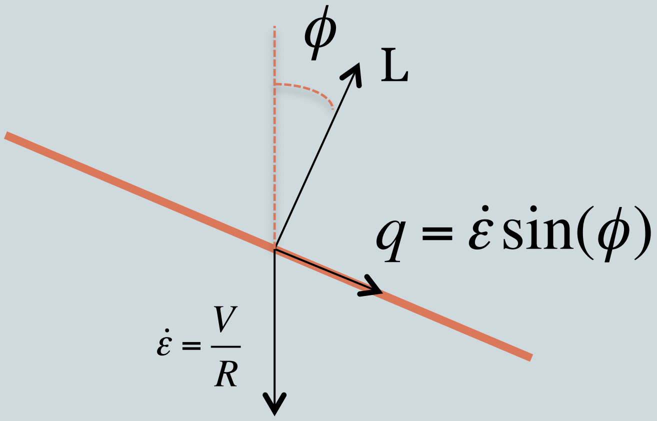
$$\Rightarrow C_{m_{\delta_e}}\delta_{e_0} = \bar{C}_m(\alpha,\lambda_0)$$

Turning Maneuver III

- The pitch rate is the projection of the turn rate onto the pitch axis:

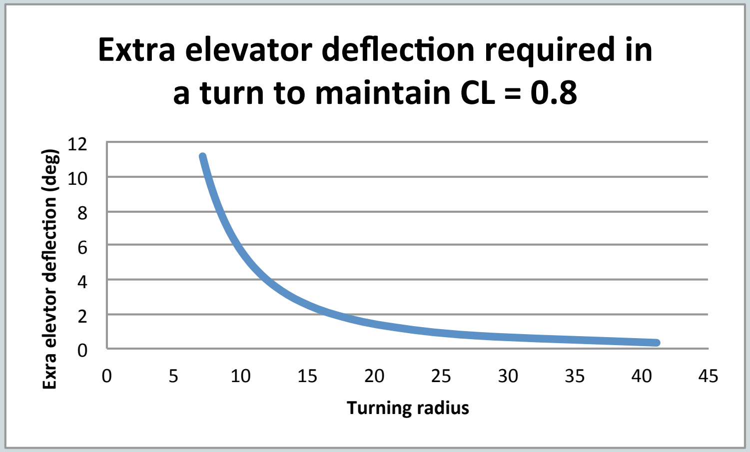
$$q = \dot{\varepsilon}\sin(\phi) = \frac{V}{R}\sin(\phi)$$

$$\Rightarrow \hat{q} = \frac{qc}{2V} = \frac{c\sin(\phi)}{2R}$$



- To maintain the same alfa (CL), extra elevator is required to counter the pitch rate

$$\delta_e = -C_{m_q}\frac{c\sin(\phi)}{2R}$$



- To take advantage of elevator throw, horizontal tail incidence has set appropriately, otherwise turning ability might be limited

Linearized Dynamics Analysis

- Many flight dynamic effects can be analyzed & explained with Linearized Dynamics
- Most of the times we linearize dynamics around Trim conditions

- Useful to synthesize linear regulator controllers
  - Provide stability in the face of uncertainty in different dynamic parameters
  - They help in rejecting disturbances
  - They can also help in going from one trim state to the another, provided they are not "too far away"

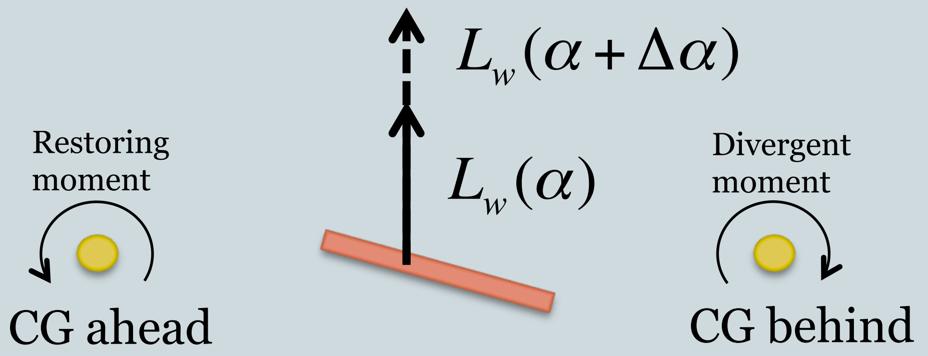
Linearized Dynamics

- Limited to a small region (what does “small” mean?)
  - Especially due to trigonometric projections and nonlinear alfa dependences
- In practice, nonlinear dynamics bear great resemblance
  - We can gain a lot of insight by studying dynamics in the vicinity a flight condition
- We can separate into longitudinal and lateral dynamics

  (If aircraft and flight condition are symmetric)
- Linearized models also provide some information about trim relations

Longitudinal Static Stability

- Static stability
  - Does pitching moment increase when AoA increases?
  - If so, then divergent pitch motion (a.k.a statically unstable)

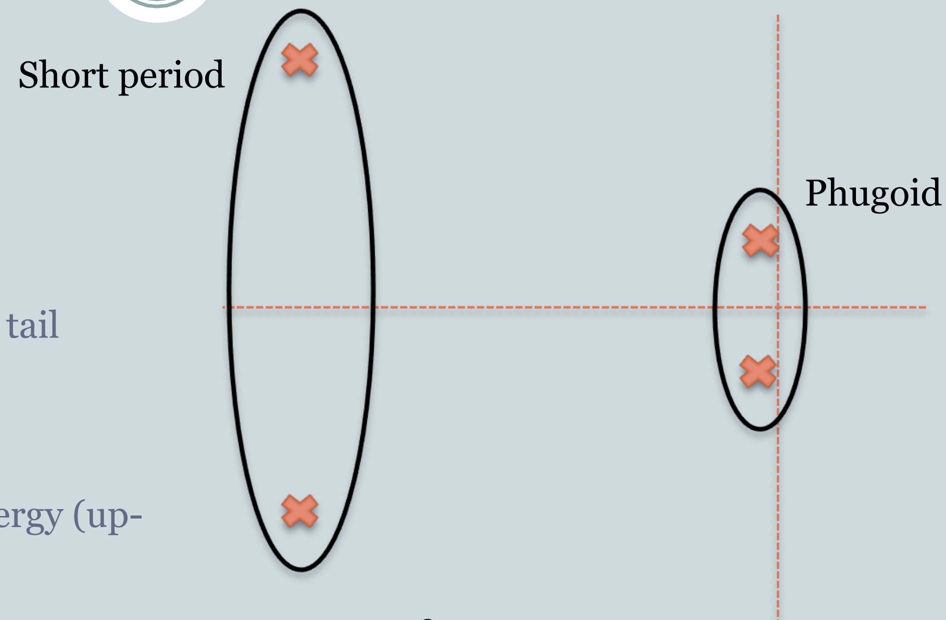


- CG needs to be ahead of quarter chord!

- As CG goes forward, static margin increases, but… more elevator deflection is required for trim and trim drag increases

Longitudinal Dynamics

- Longitudinal modes
  1. Short period
  2. Phugoid
- Short period:
  - Weather cock effect of horizontal tail
  - Usually highly damped, if you have a tail
  - Dynamics is on AoA
- Phugoid:
  - Exchange of potential and kinetic energy (up->speed down, down-> speed up)
  - Lightly damped, but slow
  - Causes "bouncing" around pitch trim conditions
  - Damping depends on drag: low drag, low damping!
  - How can we stabilize/damp it?
- Propeller dynamics: as a first order lag
- Idea for Phugoid damper design: reduced 2nd order longitudinal system



$$\omega_{ph} \approx \sqrt{2}\frac{g}{V_o}$$

$$\zeta_{ph} \approx \frac{1}{\sqrt{2}}\frac{C_{D_0}}{C_{L_0}}$$

Lateral Dynamics

- Lateral-Directional modes:
  1. Roll subsidence
  2. Dutch roll
  3. Spiral

- Roll subsidence:
  - Naturally highly damped
  - "Rolling in honey" effect

$$\sigma_{roll} \approx \frac{\frac{1}{2}\rho V_o S b^2}{I_{xx}} C_{l_p}$$

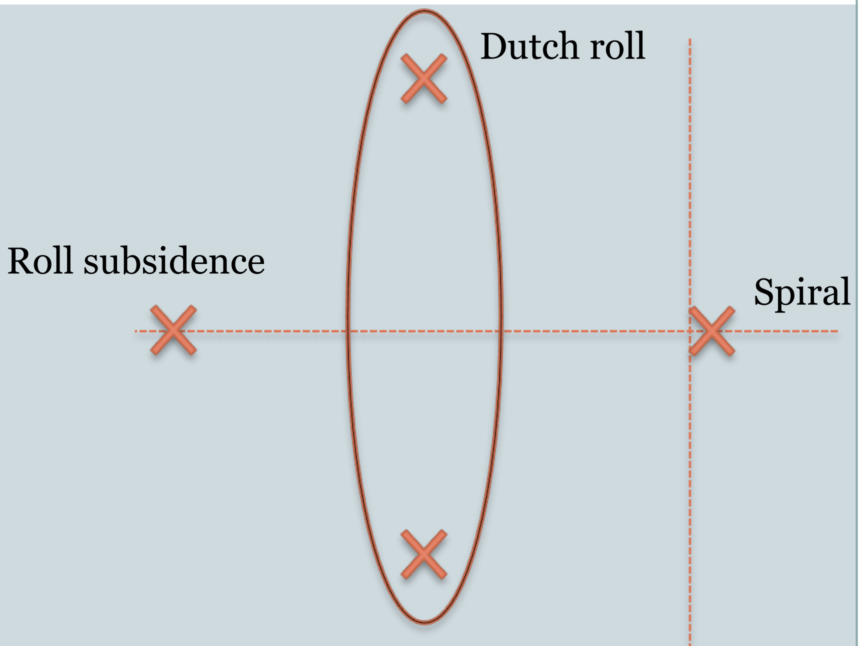
- Dutch roll:
  - Oscillatory motion
  - Usually stable, and sometimes lightly damped
  - Exchange between yaw rate, sideslip and roll rate

- Spiral:
  - Usually unstable, but slow enough to be easily stabilized

$$\sigma_{spiral} \approx \frac{\frac{1}{2}\rho V_o S b^2}{I_{zz}} \left(C_{n_r} - C_{n_\beta} \frac{C_{l_r}}{C_{l_\beta}}\right)$$

- Dutch Roll and Spiral stability are competing factors
  - Dihedral and vertical tail volume dominate these

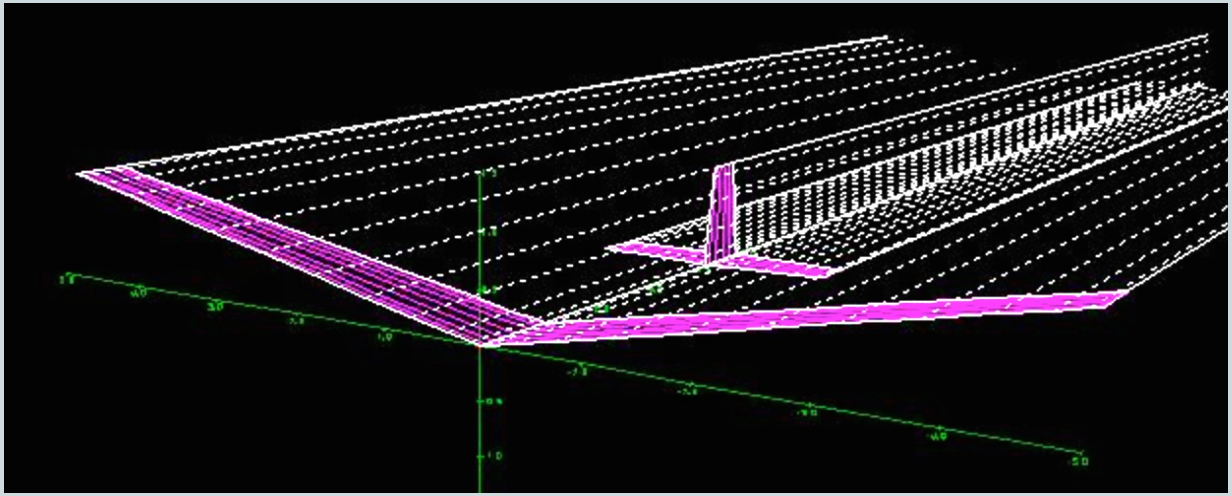
- Note: see "Flight Vehicle Aerodynamics", Ch. 9 for more details



Vortex Lattice Codes

- Good at predicting **inviscid** part of **attached** flow around **moderate aspect ratio** lifting surfaces

- Represents potential flow around a wing by a lattice of horseshoe vortices



VLM Codes (II)

- Viscous drag on a wing, can be added for with "strip theory"
  - Calculate local Cl with VLM
  - Calculate 2D Cd(Cl) either from a polar plot of airfoil
  - Add drag force in the direction of the local velocity

- Usually not included:
  - Fuselage
    - can be roughly accounted by adding a "+" lifting surface
  - Propeller downwash

- VLMs can roughly predict:
  - Aerodynamic performance (L/D vs CL)
  - Stall speed (CLmax)
  - Trim relations
  - Stability Derivatives
    - Linear control system design
    - Nonlinear Flight simulation (non-dimensional aerodynamics is linear, but dimensional aerodynamics are nonlinear and EOMs are nonlinear)

VLM Codes (III)

- AVL:
  - Reliable output
  - Viscous strip theory
  - No GUI & cumbersome to define geometry

- XFLR:
  - Reliable output
  - Viscous strip theory
  - GUI to define geometry
  - Good analysis and visualization tools

- Tornado
  - I've had some discrepancies when validating against AVL
  - Written in Matlab

- QuadAir
  - Good match with AVL
  - Written in Matlab
  - Easy to define geometries
  - Viscous strip theory soon
  - Originally intended for flight simulation, not aircraft design
    - Very little native visualization and performance analysis tools

Recommended Readings

1. Fundamentals of Flight, Shevell
   - Big picture of Aerodynamics, Flight Dynamics and Aircraft Design
2. Dynamics of Flight, Etkin
   - Very good development of trim and linearized flight dynamics and aerodynamics. Some ideas for control
3. Flight Vehicle Aerodynamics, Drela
   - Great mix between real world and mathematical aerodynamics and flight dynamics. No controls. Ch. 9 very clear and useful development of linearized models
4. Automatic Control of Aircraft and Missiles, Blakelock
   - In depth description of flight EOMs and many ideas for linear regulators
5. Low-speed Aerodynamics, Plotkin & Katz
   - Great book on panel methods (only if you want to write your own panel code)