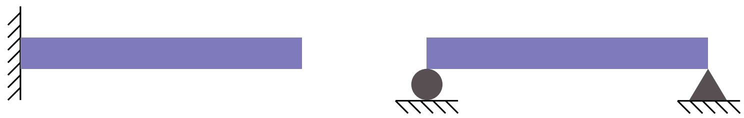
2 Plates and Shells

2.1 Euler - Bernoulli Beam Theory

Euler - Bernoulli Beam Theory

a standard structural element is the beam, which is a long, slender solid that carries transverse forces by bending



concentrate on statically determinate beams: cantilevers and simply supported beams

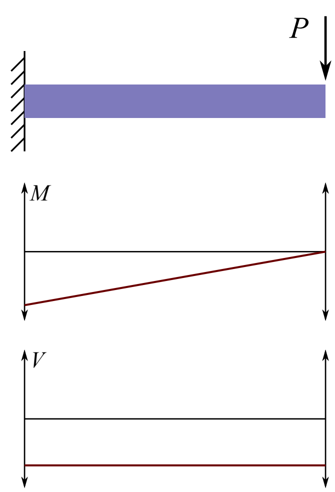
fundamental modelling simplifications are:

1. the beam is a one-dimensional object;
2. plane sections remain plane; and
3. there is no axial extension or contraction of the beam.

note that the principal variable is u, the vertical deflection of the midplane of the beam

Euler - Bernoulli Beam Theory

we use several graphical tools to analyse beams



Euler - Bernoulli Beam Theory

what is the relationship between the shear diagram and the moment diagram?

all of the derivatives of the displacement u have a physical meaning

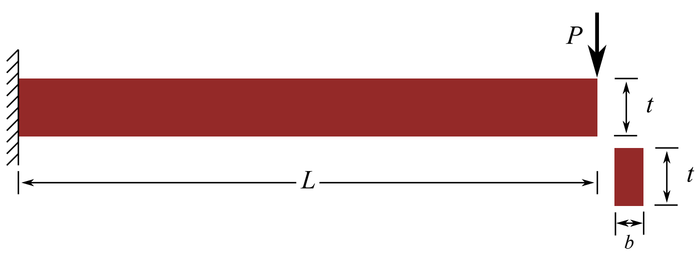
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \theta \quad \text{slope of the beam}$$
$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}\theta}{\mathrm{d}x} = \kappa \quad \text{curvature of the beam}$$
$$EI\kappa = M \quad \text{moment in the beam}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{\mathrm{d}M}{\mathrm{d}x} = V \quad \text{shear force in the beam}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{\mathrm{d}V}{\mathrm{d}x} = w \quad \text{distributed force on the beam}$$

Euler - Bernoulli Beam Theory

there are several key assumptions associated with Euler-Bernoulli beam theory:

1. the beam is slender: the bending deformations are much larger than the shear deformations
2. there is no through-thickness deformation: the thickness of the beam remains constant
3. cross-sections of the beam remain normal to the axis of bending

Euler - Bernoulli Beam Theory



Euler-Bernoulli beam theory enables us to calculate the deflected shape of beams of various sorts: the cantilever shown above is an example

a complete set of boundary conditions is necessary to use Euler-Bernoulli beam theory

in general, start with the distributed load and integrate it to get the shear: in this case, the distributed load is zero

$$V = -\int 0 \, \mathrm{d}x = C_1$$

because the shear force at any cross-section is $-P$, we find that $C_1 = -P$

Euler - Bernoulli Beam Theory

second, integrate the shear force to find the moment distribution:

$$M = -\int V\,\mathrm{d}x = -\int (-P)\,\mathrm{d}x = Px + C_2$$

we know that the moment at the free end of the beam (where $x = L$) is zero, so:

$$M(L) = 0 = PL + C_2$$

and therefore:

$$C_2 = -PL$$

third, convert the moment to a curvature:

$$\kappa = \frac{M}{EI}$$

Euler - Bernoulli Beam Theory

fourth, integrate the curvature to get the slope:

$$\theta = \int \kappa\,\mathrm{d}x = \frac{1}{EI}\int (Px - PL)\ \mathrm{d}x = \frac{1}{EI}\left(\frac{Px^2}{2} - PLx + C_3\right)$$

the slope at the wall is zero, so evaluating $\theta(0)$ gives:

$$\theta(0) = C_3 = 0$$

and hence $C_3 = 0$

Euler - Bernoulli Beam Theory

finally, integrate the slope to get the deflected shape:

$$u = \int \theta\,\mathrm{d}x = \frac{1}{EI}\int\left(\frac{Px^2}{2} - PLx\right)\mathrm{d}x = \frac{1}{EI}\left(\frac{Px^3}{6} - \frac{PLx^2}{2} + C_4\right)$$

we know that $u(0) = 0$, so we can solve for the final constant of integration:

$$C_4 = 0$$

thus, the complete deflected shape of the end-loaded cantilever beam is:

$$u = \frac{1}{EI}\left(\frac{Px^3}{6} - \frac{PLx^2}{2}\right)$$

the maximum deflection, at $x = L$, is:

$$u(L) = -\frac{PL^3}{3EI}$$

Algorithm: Euler-Bernoulli Beam Theory

1. ensure that the beam you are analyzing is statically determinate: if it is not, you will have to use a *superposition* method
2. identify four boundary conditions on the displacement, slope, moment or shear force to solve for the constants of integration
3. starting with the distributed load w (often this is zero), integrate four times to get the deformed shape u
4. in general solve for the constants of integration as early as possible rather than carrying multiple constants

2.2 Plate Theory

Plates

beams have one dimension that is much larger than the other two: beams can be treated as effectively one-dimensional

plates, shells and membranes have two dimensions that are much larger than the third dimension, and hence can be treated as two-dimensional objects

the key difference between plates and membranes is that plates can support loads both in-plane and in bending, while membranes can only support in-plane loads

Plate Midplane

much like in Euler - Bernoulli beam theory, we deal primarily with the midplane of a plate; we make assumptions that the forces applied to a plate are acting on the midplane

provided the plate material is isotropic and homogeneous, and the plate has a constant thickness, identification of the midplane is easy: it is at the middle of the plate thickness

much like our development of laminate plate theory, once there is material and geometric complexity, such as non-symmetric layers in a plate, the midplane must be found from additional considerations

Modulus Weighted Plate Midplane

if the stiffness of the plate changes through its thickness (as is the case for composite laminates) the modulus-weighted plate thickness, h^*, is:

$$h^* = \int_t \frac{E}{E_0}\, \mathrm{d}z_0$$

where t is the thickness of the plate and z_0 is the transverse coordinate with a conveniently chosen origin

E is the modulus at location z, while E_0 is a conveniently chosen reference modulus

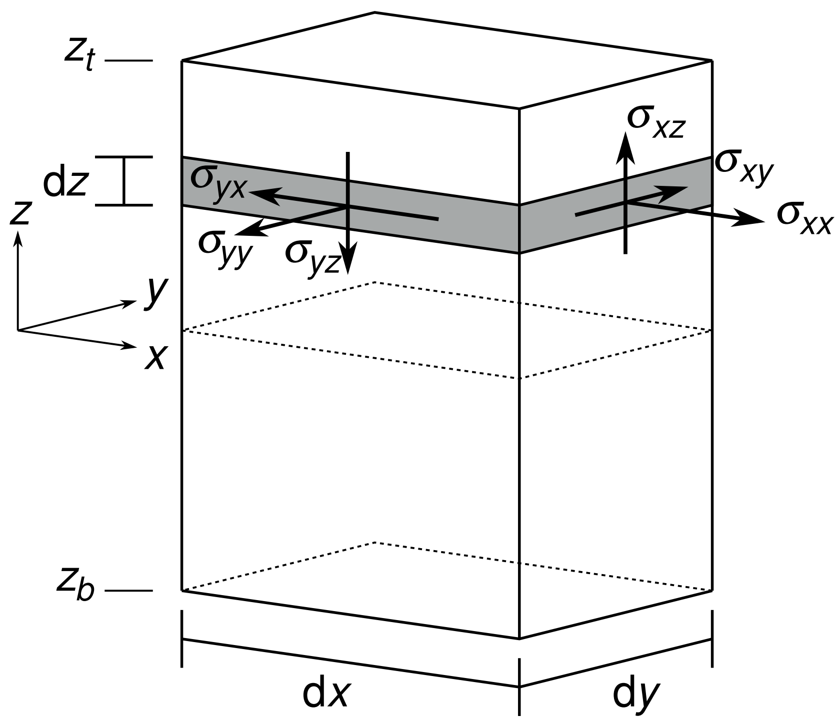
we want the modulus-weighted midplane to be the origin of the z coordinate; this will be true provided:

$$0 = \int_t z \frac{E}{E_0}\,\mathrm{d}z$$

the two coordinates, z and z_0 are related by: $z = z_0 - z_0^*$; z_0^* is the distance from the arbitrary system origin to the modulus weighted midplane, and can be found by:

$$z_0^* h^* = \int_t z_0 \frac{E}{E_0}\,\mathrm{d}z_0 = \sum_i (\bar{z}_0)_i \left(\frac{Eh}{E_0}\right)_i$$

Plate Stresses

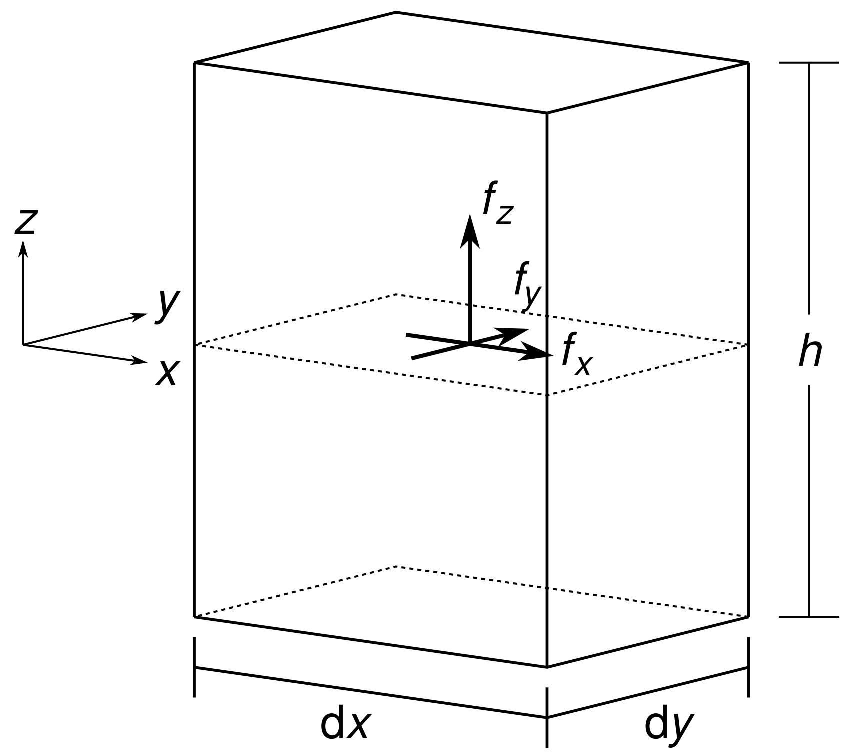


we use the notation at left to identify the stresses acting on the faces of a representative element of the plate

this is consistent with standard notation for stresses

note that the stresses may vary in x, y and z; for beams, stresses varied only in x and z

Applied Forces

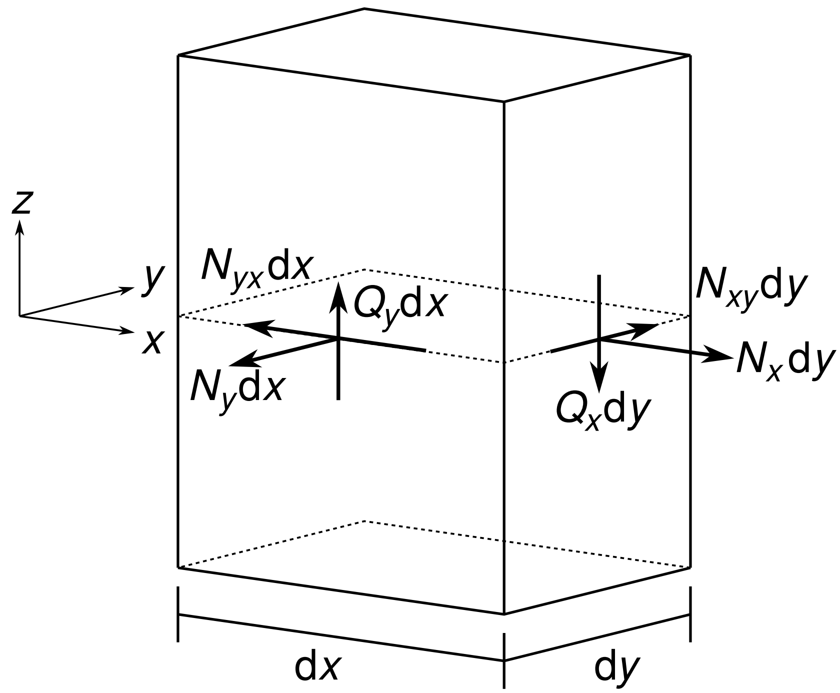


the applied forces are decomposed into three components, aligned with the coordinate axes

the applied forces are assumed to act at the midplane of the plate

in all cases, the applied forces are tractions and have units of force per unit area

Stress Resultants - Normal and Shear



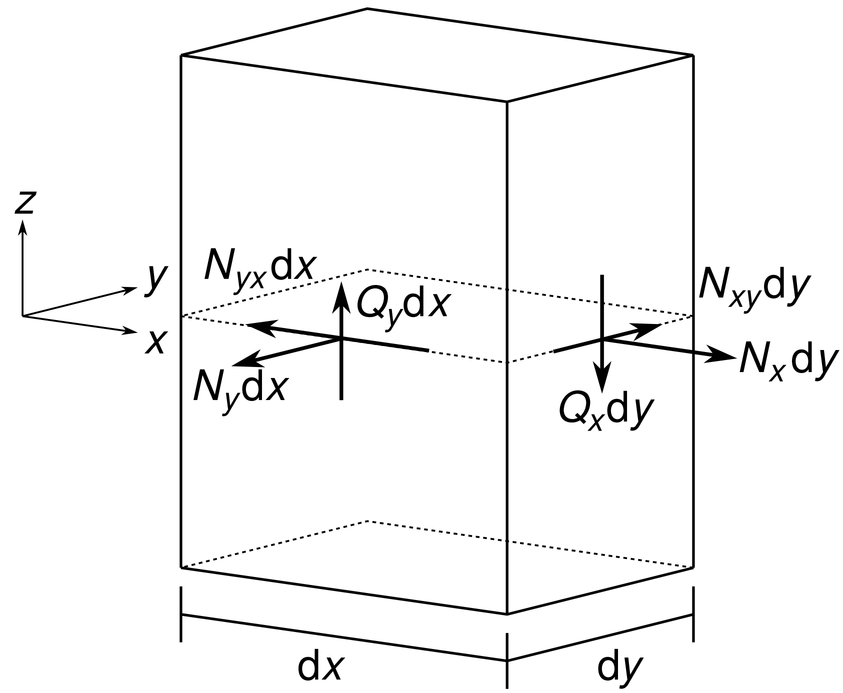
integrating the stresses gives us the resultants (per unit width):

$$N_x = \int_{z_b}^{z_t} \sigma_{xx} \, \mathrm{d}z$$

$$N_y = \int_{z_b}^{z_t} \sigma_{yy} \, \mathrm{d}z$$

$$N_{xy} = N_{yx} = \int_{z_b}^{z_t} \sigma_{xy} \, \mathrm{d}z$$

Stress Resultants - Normal and Shear

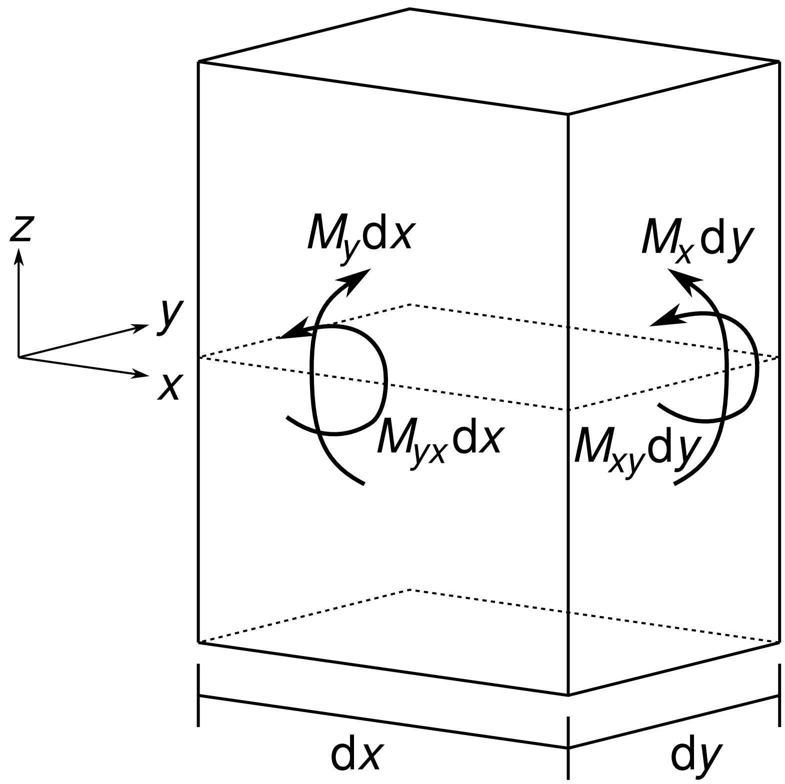


the transverse shear stress resultants are:

$$Q_x = -\int_{z_b}^{z_t} \sigma_{xz}\,\mathrm{d}z$$

$$Q_y = -\int_{z_b}^{z_t} \sigma_{yz}\,\mathrm{d}z$$

Stress Resultants - Moments



finally, the resultant moments are:

$$M_x = -\int_{z_b}^{z_t} z\sigma_{xx}\,\mathrm{d}z$$

$$M_y = -\int_{z_b}^{z_t} z\sigma_{yy}\,\mathrm{d}z$$

$$M_{xy} = -M_{yx} = -\int_{z_b}^{z_t} z\sigma_{xy}\,\mathrm{d}z$$

Plate Displacements

we make many of the same approximations for plate theory as for Euler - Bernoulli beam theory and laminate theory:

the most important are:

1. straight lines perpendicular to the modulus weighted midplane before bending remain straight and perpendicular after bending
2. the perpendiculars to the modulus weighted midplane remain unchanged in length after deflection

the consequence is that we need only consider the modulus weighted midplane deflections; deflections at other locations through the thickness can be derived:

$$u(x, y, z) = u^\circ(x, y, 0) - z\theta_y(x, y)$$

$$v(x, y, z) = v^\circ(x, y, 0) - z\theta_x(x, y)$$

$$w(x, y, z) = w^\circ(x, y, 0)$$

Strains

there are only three non-zero strains, which we can determine by differentiating the modulus weighted midplane displacements and adding terms to account for the curvature of the plate:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u^\circ}{\partial x} - z\frac{\partial^2 w^\circ}{\partial x^2}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v^\circ}{\partial y} - z\frac{\partial^2 w^\circ}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} - 2z\frac{\partial^2 w^\circ}{\partial x \partial y}$$

Stress - Strain Relations

we assume a linear elastic material model and hence apply Hooke's Law to relate the stresses and the strains, in this case in a plane strain manner:

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1 - \nu^2}\left(\epsilon_{yy} + \nu\epsilon_{xx}\right)$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E\gamma_{xy}}{2(1 + \nu)}$$

Stress - Displacement Relations

we express the stresses in terms of displacements; to do so, substitute the definitions of strains into the expressions of Hooke's Law, using commas in the subscripts to indicate differentiation in order to shorten the expressions:

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\left(u^\circ_{,x} + \nu v^\circ_{,y} - z\left(w^\circ_{,xx} + \nu w^\circ_{,yy}\right)\right)$$

$$\sigma_{yy} = \frac{E}{1 - \nu^2}\left(v^\circ_{,y} + \nu u^\circ_{,x} - z\left(w^\circ_{,yy} + \nu w^\circ_{,xx}\right)\right)$$

$$\sigma_{xy} = \frac{E}{2(1 + \nu)}\left(u^\circ_{,y} + v^\circ_{,x} - 2zw^\circ_{,xy}\right)$$

Stress Resultant - Displacement Relations

finally this produces the relationship between the stress resultants and the displacements:

$$N_x = K^* \left(u^\circ_{,x} + \nu v^\circ_{,y} \right)$$

$$N_y = K^* \left(v^\circ_{,y} + \nu u^\circ_{,x} \right)$$

$$N_{xy} = \frac{1}{2} (1 - \nu) K^* \left(u^\circ_{,y} + v^\circ_{,x} \right)$$

where:

$$K^* = \frac{E_0 h^*}{1 - \nu^2}$$

Stress Resultant - Displacement Relations

similarly:

$$M_x = D^* \left(w^\circ_{,xx} + \nu w^\circ_{,yy} \right)$$

$$M_y = D^* \left(w^\circ_{,yy} + \nu w^\circ_{,xx} \right)$$

$$M_{xy} = \left(1 - \nu^2 \right) D^* w^\circ_{,xy}$$

where:

$$D^* = E_0 I^*$$

and

$$I^* = \int_t \frac{z^2}{1 - \nu^2} \frac{E}{E_0} \, \mathrm{d}z$$

Homogeneous Plates

note that in the above we use K^*, D^*, I^* and h^*

this is to provide for the case where we have a layered material and the layers are not homogeneous, as we will find in composites

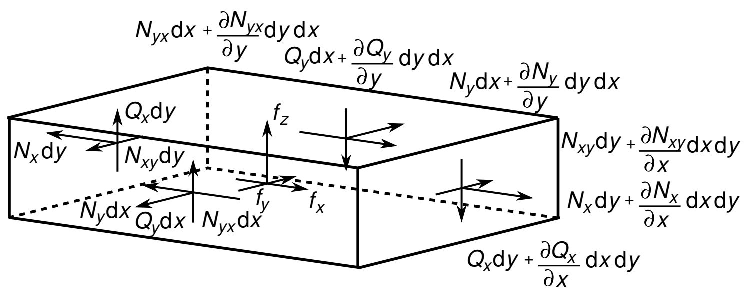
if we have a homogeneous plate, then:

$$K = \frac{Eh}{1 - \nu^2}$$

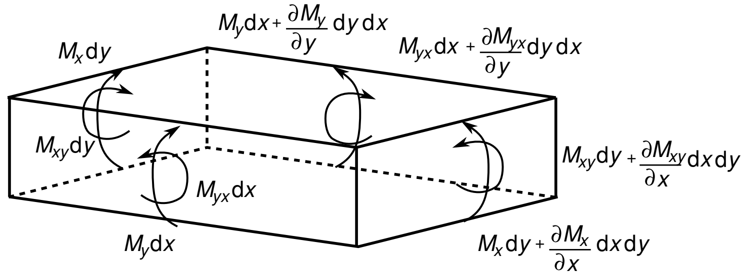
$$I = \frac{h^3}{12 \left(1 - \nu^2 \right)}$$

$$D = \frac{Eh^3}{12 \left(1 - \nu^2 \right)}$$

Equilibrium of a Plate Element



Equilibrium of a Plate Element



Equilibrium of a Plate Element

apply equilibrium equations to the free body diagrams on the previous two slides; the key results are, for forces:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = -f_x$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = -f_y$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -f_z$$

and for moments:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

Equilibrium of a Plate Element

substitute the stress resultant - displacement equations into the equilibrium equations (again using comma differentiation notation for compactness):

$$K^* \left(u^\circ_{,xx} + \nu v^\circ_{,yx}\right) + \frac{1}{2}(1-\nu) K^* \left(v^\circ_{,xy} + u^\circ_{,yy}\right) = -f_x$$

$$K^* \left(v^\circ_{,yy} + \nu u^\circ_{,xy}\right) + \frac{1}{2}(1-\nu) K^* \left(v^\circ_{,xx} + u^\circ_{,yx}\right) = -f_y$$

and moment equilibrium gives us:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = f_z$$

using this and the resultant moment - displacement relation produces, after some simplification:

$$D^* \left(w^\circ_{,xxxx} + 2w^\circ_{,xxyy} + w^\circ_{,yyyy}\right) = f_z$$

this is the constant-thickness, small deflection, isotropic thin plate bending equation; it incorporates equilibrium, continuity of deflection and a linear elastic material model