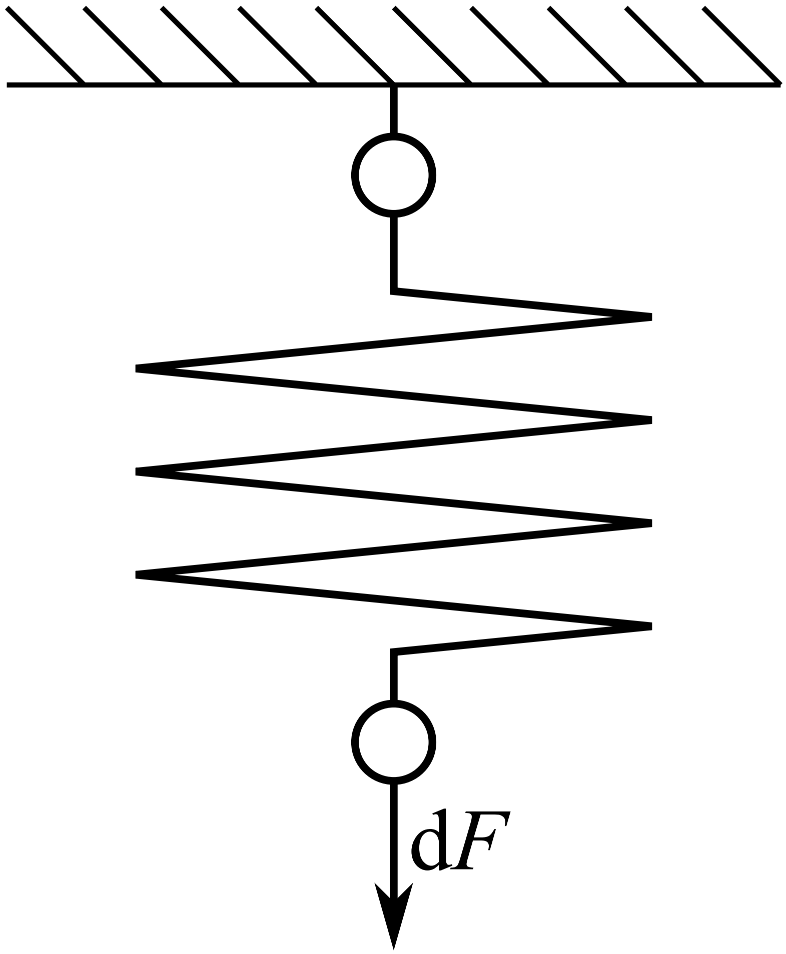
3 Finite Element Methods

3.1 Strain Energy

Strain Energy



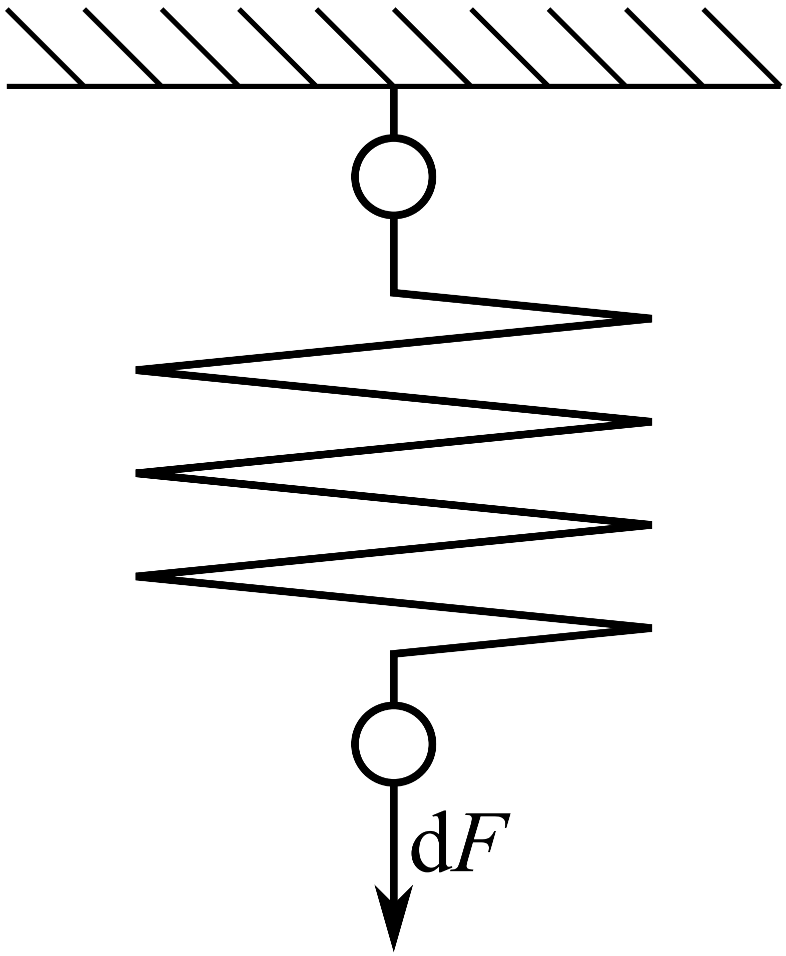
consider the linear one-dimensional case, with a spring which extends $\mathrm{d}x$ when force $\mathrm{d}F/k$ is applied

the work done by the applied force is:

$$\mathscr{W} = \int_a^b F \, \mathrm{d}x = \int_a^b kx \, \mathrm{d}x = \left.\frac{kx^2}{2}\right|_a^b$$

this work is stored as strain energy and is available to do further work

Strain Energy

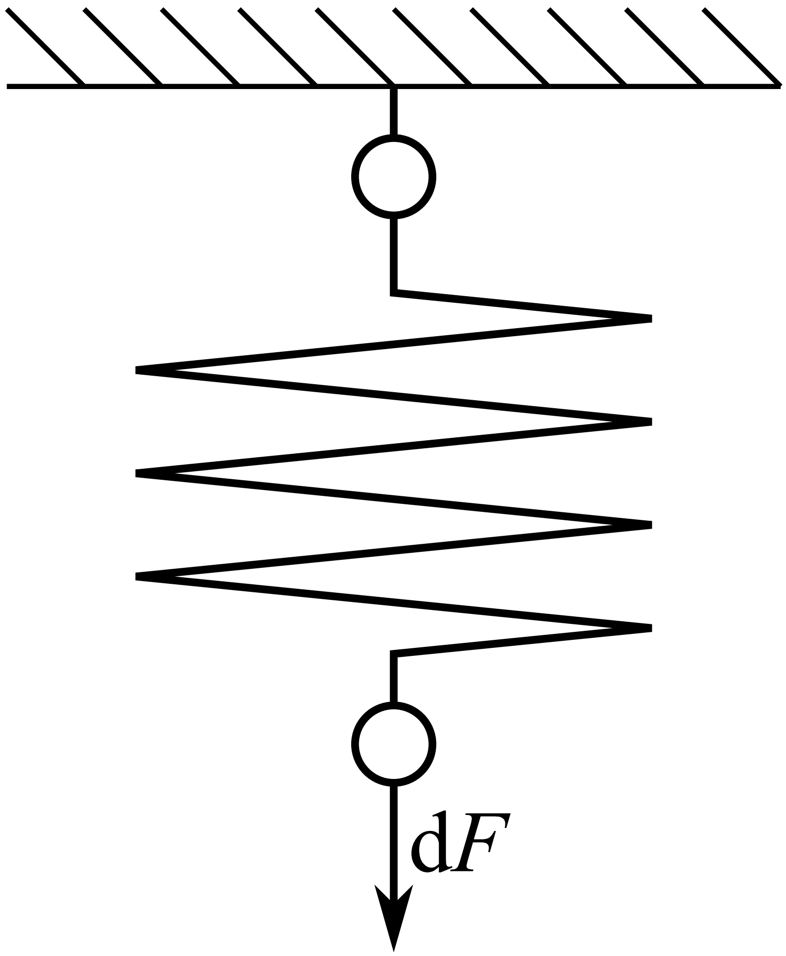


in the continuum approach, stress and strain are the analogues of force and displacement the strain energy density is determined through:

$$\mathscr{W} = \int \sigma_{ij}\, \mathrm{d}\epsilon_{ij}$$

W is the amount of energy per unit volume which is stored in a material due to elastic deformation

Strain Energy



strain energy is work that is *recoverable*

there are a great many deformation mechanisms which are dissipative: the energy involved in the deformation cannot be recovered examples are plastic deformation, fracture, viscous flow, friction

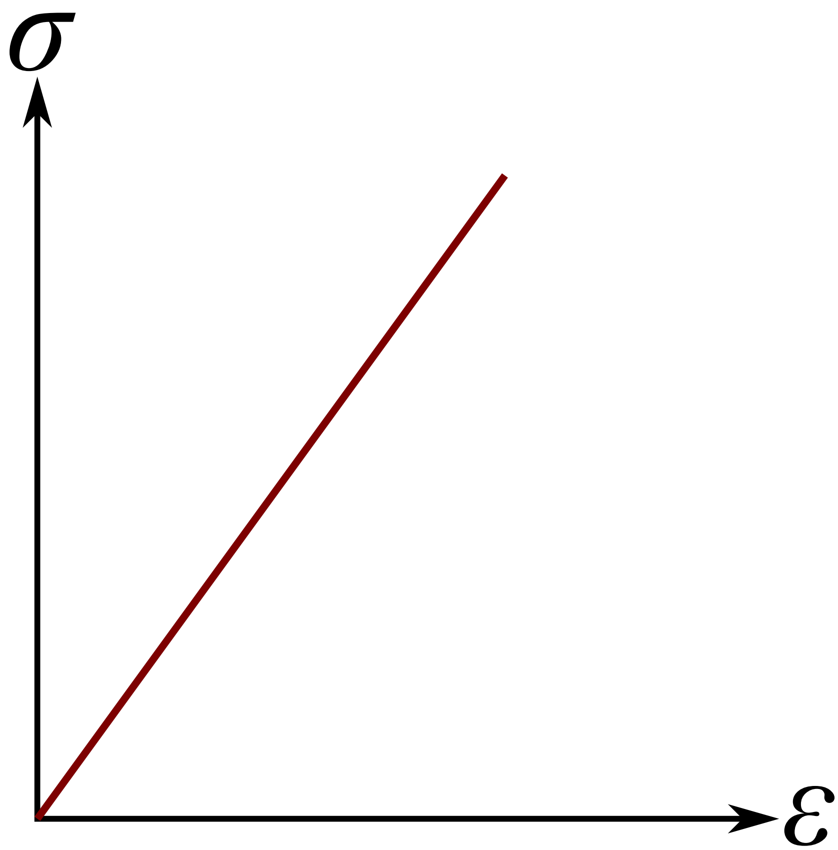
recoverability of the strain energy is essential to the definition of elastic deformation

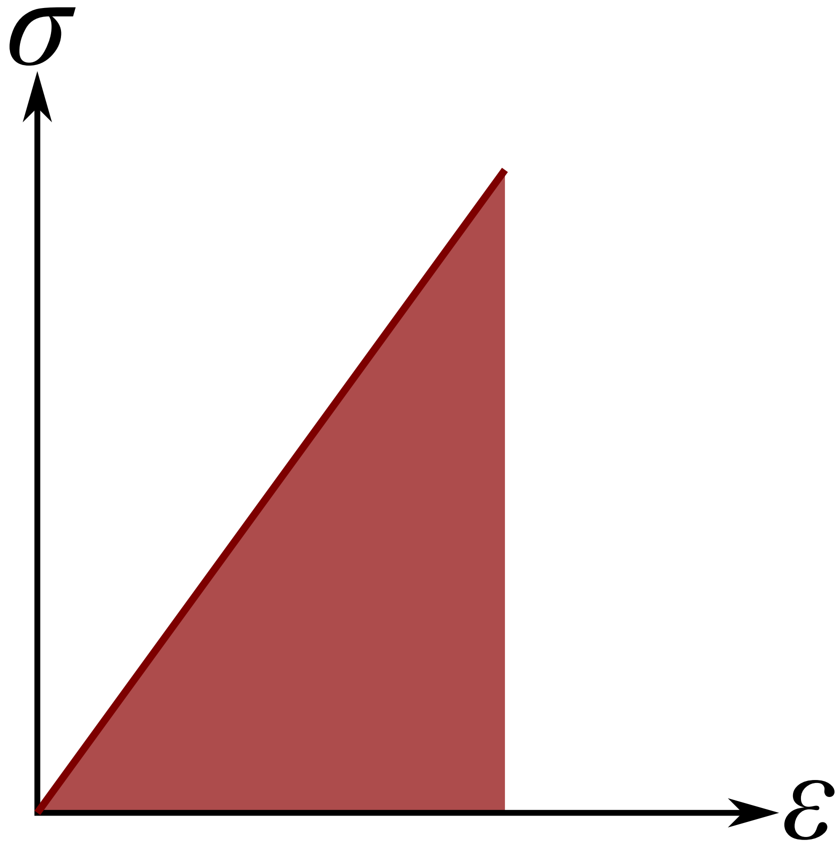
Strain Energy and Complementary Strain Energy

in the linear elastic case, stress is a linear function of strain

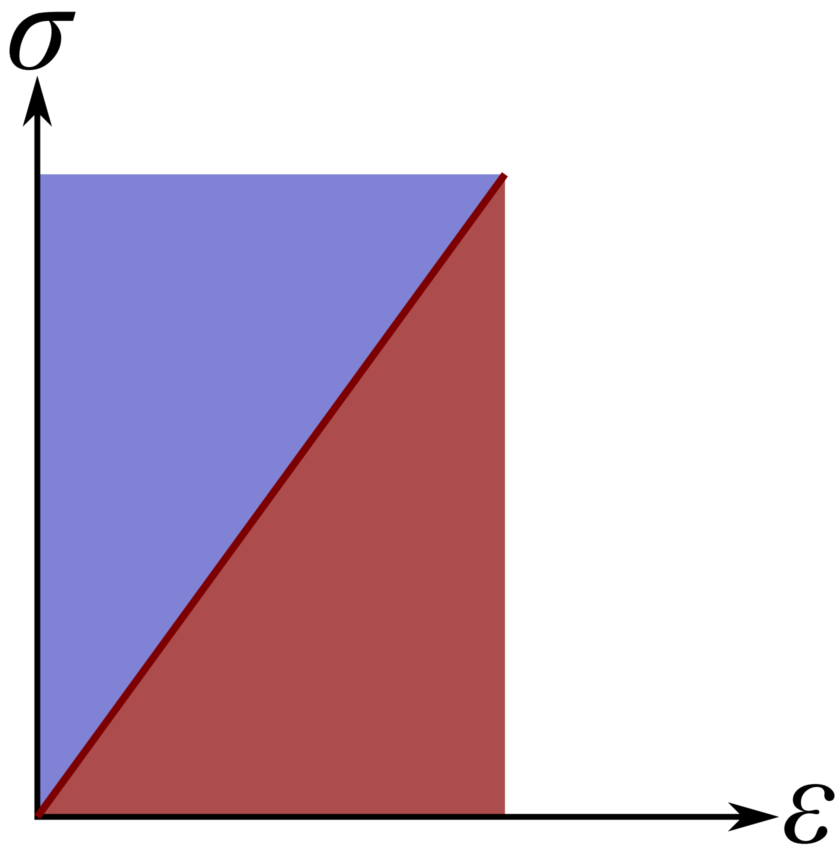
the strain energy density can be evaluated by integrating the stress with respect to strain; here it can be geometrically represented by the area under the stress-strain curve

we can also define the complementary strain energy density, which is the integral of strain with respect to stress; graphically it is the area between the stress-strain curve and the stress axis





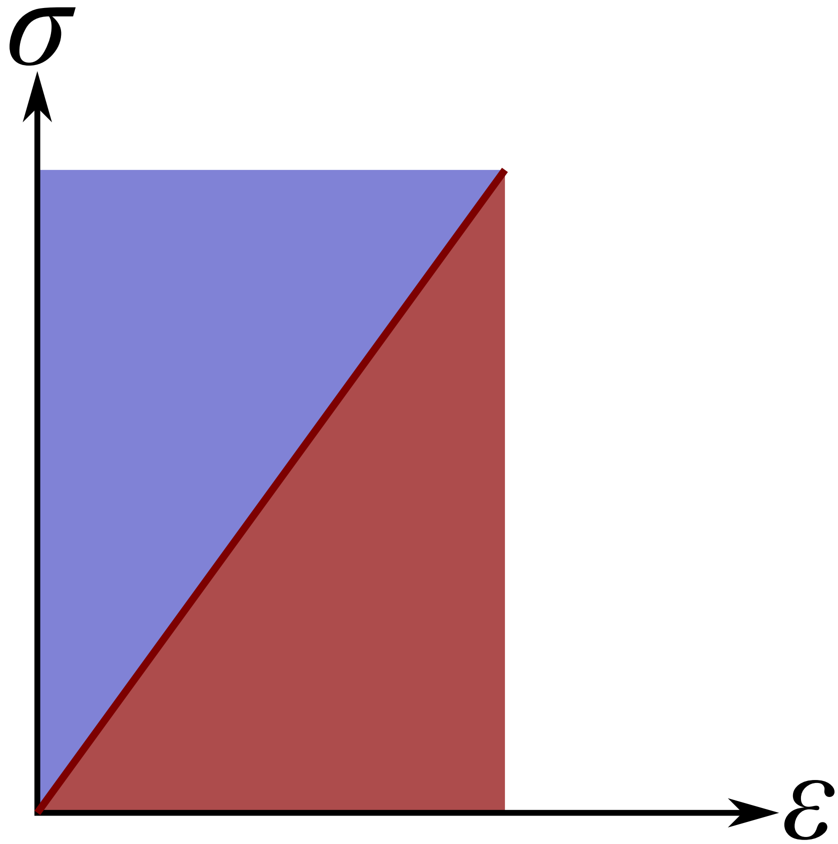
σ
ε



Complementary Strain Energy

if the stress-strain relation is linear, then the strain energy density is equal to the complementary strain energy density

however, if the stress-strain relation is non-linear, then strain energy density does not necessarily equal the complementary strain energy density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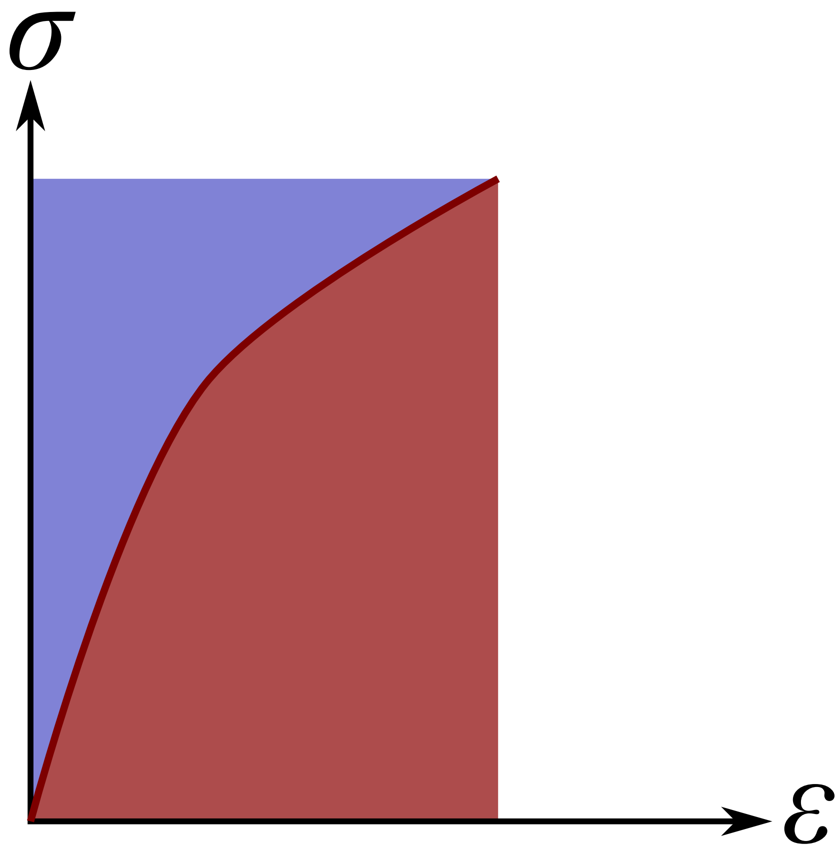


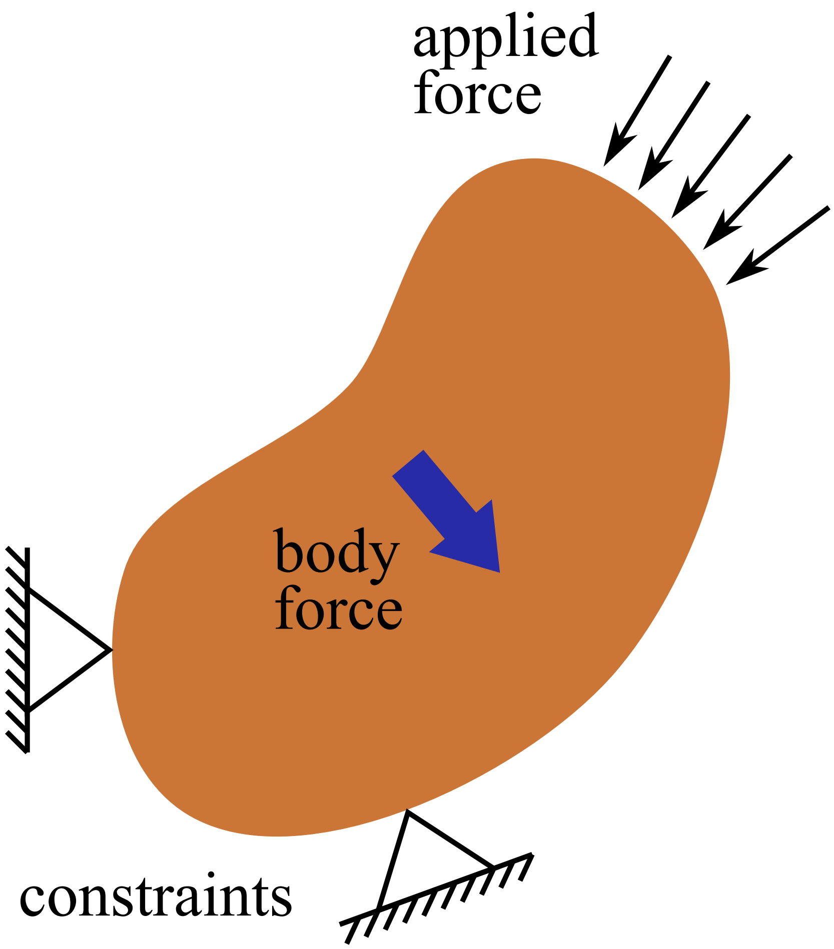
3.2 Virtual Work

The Principle of Virtual Work

take an arbitrary body with body force density B_i, subjected to tractions T_i over part of the surface labelled S_σ and constraints over the remainder of the surface labelled S_u

the displacement field over the body is given by u_i

within the body are fields of stress, strain and strain energy density



The Principle of Virtual Work

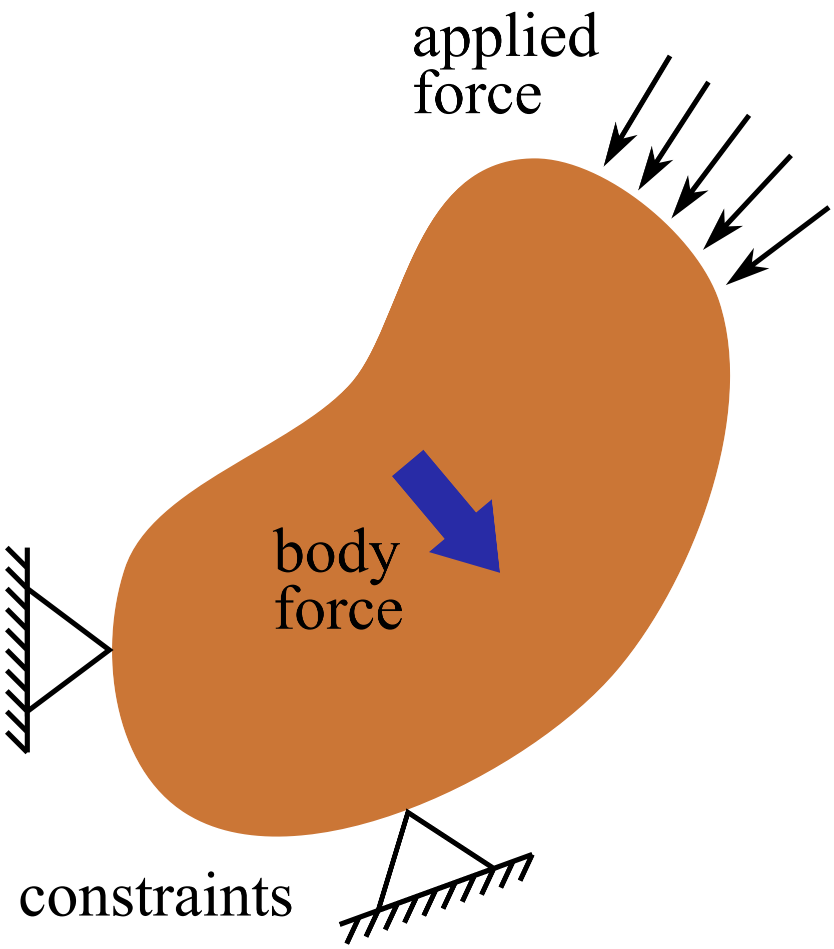
a *virtual displacement* δu_i is a kinematically admissible infinitesimal displacement field which is added to the real displacements u_i

this means that the virtual displacement must be zero along all of S_u

also, we require that the virtual displacement is smooth, thrice differentiable and is sufficiently small that all displacements remain elastic

the *virtual work* $\delta\mathscr{E}$ is the work done on a body by all the external forces as they travel through the virtual displacement δu_i

$$\delta\mathscr{E} = \int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S$$



The Principle of Virtual Work

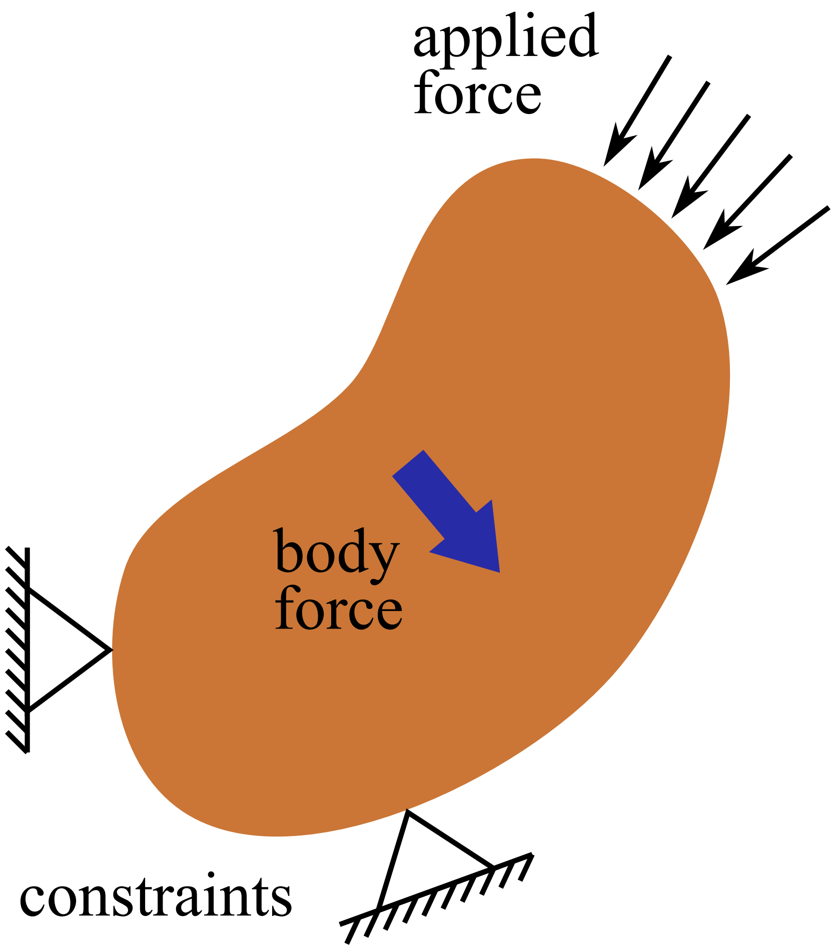
the actual forces and stresses acting on the body are *unaffected* by the virtual displacement

also define a *virtual strain* $\delta\epsilon_{ij}$ as a kinematically admissible infinitesimal strain field that is compatible with the virtual displacement field δu_i

the virtual strain energy of the body is:

$$\delta\mathscr{U} = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V$$

after integration this is a *linear* equation: in the real system, σ_{ij} is a function of ϵ_{ij}, but σ_{ij} is not a function of $\delta\epsilon_{ij}$



The Principle of Virtual Work

the principle of virtual work states that the external virtual work performed by the applied forces and body forces is equal to the internal virtual work, the virtual strain energy

$$\delta\mathscr{E} = \delta\mathscr{U}$$

in full:

$$\int_V B_i \delta u_i \,\mathrm{d}V + \int_S T_i \delta u_i \,\mathrm{d}S = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V$$

if we have linearised strains, this becomes:

$$\int_V B_i \delta u_i \,\mathrm{d}V + \int_S T_i \delta u_i \,\mathrm{d}S = \int_V \frac{1}{2}\sigma_{ij}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)\mathrm{d}V$$

the principle of virtual work may be viewed as combining a kinematically compatible deformation $(\delta u_i, \delta\epsilon_{ij})$ with a stress field σ_{ij} which for a given external force system

(B_i, T_i) satisfies equilibrium requirements

The Principle of Virtual Work

for a discrete system, like a truss, the external virtual work is:

$$\delta\mathscr{E} = \sum_{i=1}^{n} P_i \delta u_i$$

where the P_i are the applied forces and the δu_i are the displacements at the corresponding nodes (for n total nodes)

the internal virtual work (the virtual strain energy) is:

$$\delta\mathscr{U} = \sum_{i=1}^{m} F_i \delta e_i$$

where the F_i are the member forces and δe_i the member extensions (for m total members)

the real member forces F_i are in equilibrium with the real applied forces P_i, while the virtual displcements δu_i are compatible with the virtual member extensions δe_i

Virtual Work: Truss Example

consider the three-member pin-jointed truss shown to the right, loaded by forces P_1 and P_2

apply a system of virtual displacements and extensions

the external virtual work is:

$$\delta\mathscr{E} = P_1 \delta u_1 + P_2 \delta u_2$$

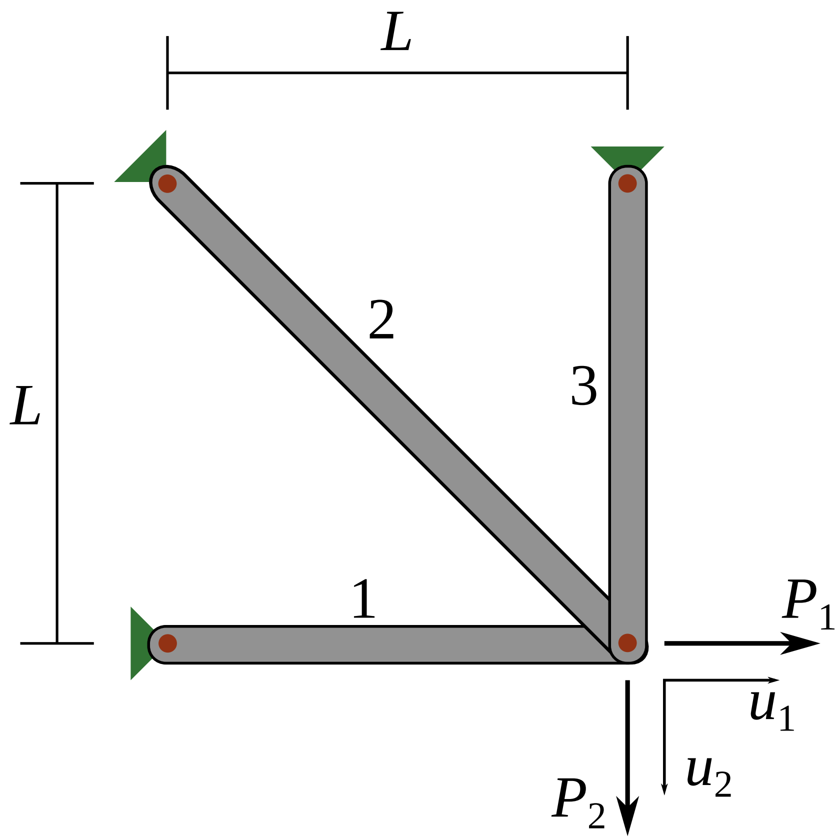
while the internal virtual work is:

$$\delta\mathscr{U} = F_1 \delta e_1 + F_2 \delta e_2 + F_3 \delta e_3$$

hence:

$$P_1 \delta u_1 + P_2 \delta u_2 = F_1 \delta e_1 + F_2 \delta e_2 + F_3 \delta e_3$$

provided (P_i, F_i) are an equilibrium system and $(\delta u_i, \delta e_i)$ are a compatible system



Virtual Work: Truss Example

we use displacement compatibility to substitute for the member extensions:

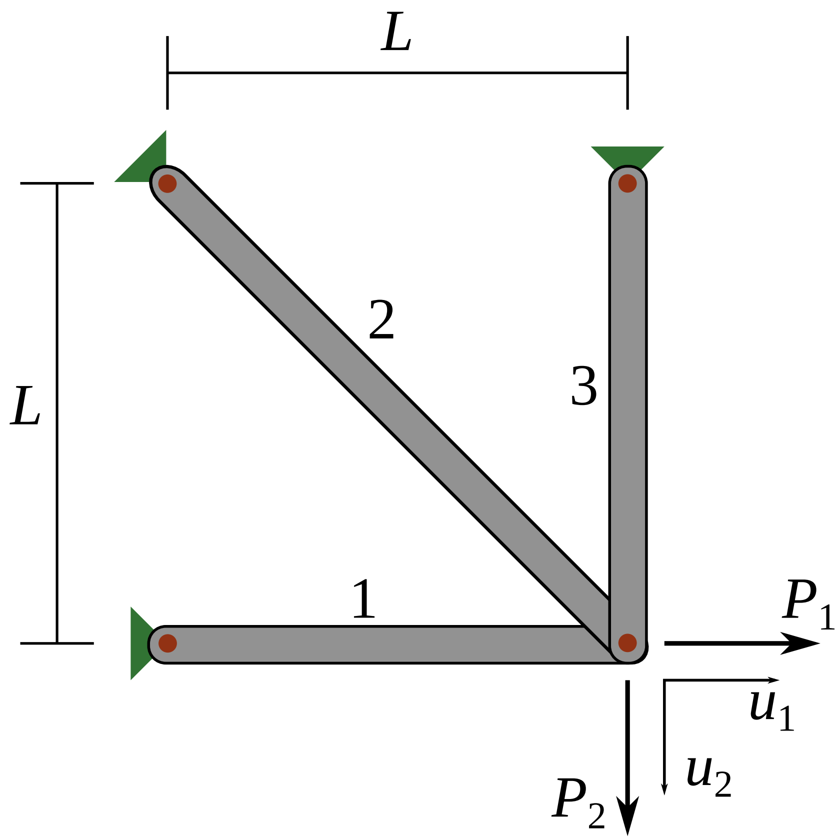
$$e_1 = u_1; \quad e_2 = \frac{1}{\sqrt{2}}(u_1 + u_2); \quad e_3 = u_2$$

in virtual terms:

$$\delta e_1 = \delta u_1; \quad \delta e_2 = \frac{1}{\sqrt{2}}(\delta u_1 + \delta u_2); \quad \delta e_3 = \delta u_2$$

eliminating the virtual extensions:

$$F_1 \delta u_1 + F_2 \frac{\delta u_1 + \delta u_2}{\sqrt{2}} + F_3 \delta u_2 = P_1 \delta u_1 + P_2 \delta u_2$$



Virtual Work: Truss Example

rearranging:

$$\left(F_1 + \frac{F_2}{\sqrt{2}} - P_1\right)\delta u_1 + \left(\frac{F_2}{\sqrt{2}} + F_3 - P_2\right)\delta u_2 = 0$$

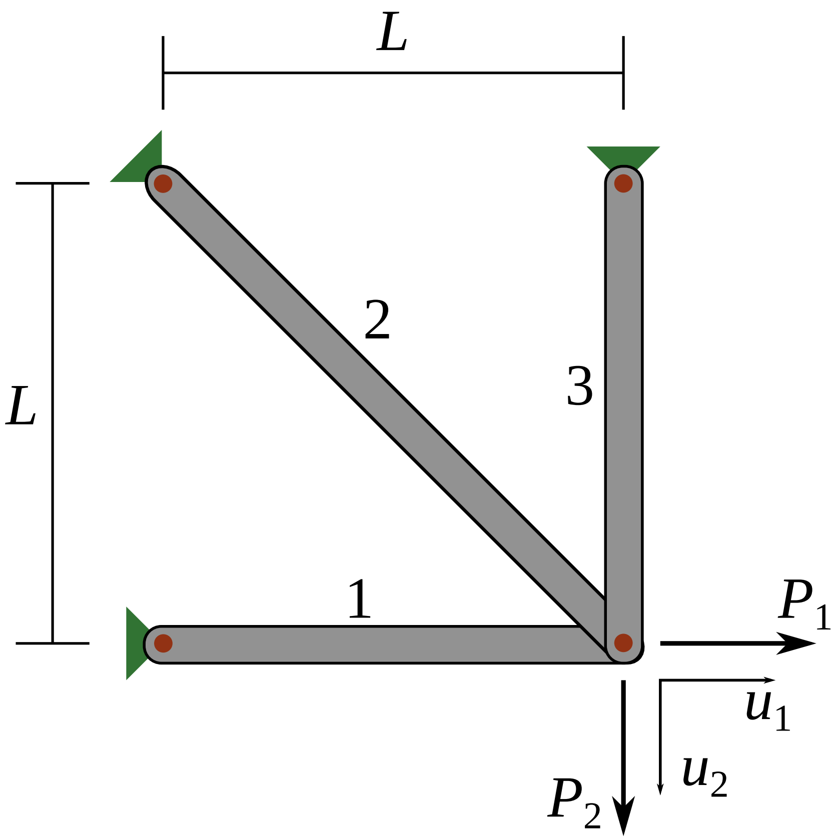
because the virtual displacements are arbitrary, we must have:

$$F_1 + \frac{F_2}{\sqrt{2}} - P_1 = 0$$

$$\frac{F_2}{\sqrt{2}} + F_3 - P_2 = 0$$

remarkably, this has recovered the equilibrium equations!

the equilibrium equations have been obtained simply through consideration of changes in energy in the system



Principle of Virtual Work

the previous result did not depend upon any particular constitutive law: it was not required that the system behave in a linear elastic manner, and hence can be used for any constitutive relation

to advance this calculation further, a constitutive law must be adopted

if we choose to use linear elasticity, some additional results can be obtained

consider an elastic system where the external forces can be derived from potential functions, so that:

$$B_i = -\frac{\partial \mathscr{G}}{\partial u_i}; \quad T_i = -\frac{\partial \mathscr{T}}{\partial u_i}$$

so $\mathscr{G}$ and $\mathscr{T}$ are the potentials of the body force and the surface tractions, respectively

Theorem of Minimum Potential Energy

this is the principle of virtual work for a continuum:

$$\int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S = \int_V \sigma_{ij} \delta \epsilon_{ij} \, \mathrm{d}V$$

from our previous definition of strain energy density,

$$\int_V \sigma_{ij}\delta\epsilon_{ij}\ \mathrm{d}V = \int_V \frac{\partial\mathscr{W}}{\partial\epsilon_{ij}}\delta\epsilon_{ij}\ \mathrm{d}V = \int_V \delta\mathscr{W}\ \mathrm{d}V = \delta\int_V \mathscr{W}\ \mathrm{d}V$$

consider this notation to be analogous to the notation used in the truss example, where a real force times a virtual displacement gave a virtual energy; here a real stress times a virtual strain gives a virtual energy density

if we also replace the body force and the surface tractions with their potentials, we get:

$$\delta\int_V \mathscr{W}\ \mathrm{d}V + \delta\int_V \mathscr{G}\ \mathrm{d}V + \delta\int_S \mathscr{T}\ \mathrm{d}S\ = 0$$

Theorem of Minimum Potential Energy

define the *potential energy* of the system as:

$$V = \int_V (\mathscr{W} + \mathscr{G})\ \mathrm{d}V + \int_S \mathscr{T}\ \mathrm{d}S$$

the result above states that the virtual potential energy must be zero:

$$\delta\mathscr{V}\ = 0$$

in more descriptive terms, the value of the potential energy at equilibrium must be stationary; unless it is a minimum the equilibrium is unstable: for stable equilibrium, the potential energy must be a minimum

change the terminology: *virtual* quantities are referred to as *variations* in the mathematical literature

Notation in Variational Calculus

typically the notation for the variation of a function J is:

$$\delta J$$

when $J[u]$ is a functional of u, involving, for example, u and u', then:

$$\delta J = \int_a^b \left(F(x, u + \delta u, u' + \delta u') - F(x, u, u')\right)\ \mathrm{d}x$$

for example, if $F = (\ \mathrm{d}u/\ \mathrm{d}x)^2$,

$$\begin{aligned}
\delta J &= \int_a^b \left(\left(\frac{\mathrm{d}(u+\delta u)}{\mathrm{d}x}\right)^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)\ \mathrm{d}x \\
&= \int_a^b \left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\delta u}{\mathrm{d}x} + \left(\frac{\mathrm{d}\delta u}{\mathrm{d}x}\right)^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\right)\ \mathrm{d}x \\
&= \int_a^b 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\delta u}{\mathrm{d}x}\ \mathrm{d}x = \int_a^b 2\frac{\mathrm{d}u}{\mathrm{d}x}\delta u'\ \mathrm{d}x
\end{aligned}$$

Minimum Potential Energy and Equilibrium

from the concepts of virtual work, the first variation of the potential energy is:

$$\delta\mathscr{V} = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V - \int_V B_i\delta u_i\,\mathrm{d}V - \int_S T_i\delta u_i\,\mathrm{d}S$$

note that all of these terms are energy-valued, ie *scalar*

recall that:

$$\begin{aligned}\int_S T_i\delta u_i\,\mathrm{d}S &= \int_S \sigma_{ij}\nu_j\delta u_i\,\mathrm{d}S = \int_V \left(\sigma_{ij}\delta u_i\right)_{,j}\,\mathrm{d}V \\ &= \int_V \sigma_{ij,j}\delta u_i\,\mathrm{d}V + \int_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V\end{aligned}$$

now:

$$\int_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V = \int_V \frac{1}{2}\sigma_{ij}\left(\delta u_{i,j} + \delta u_{j,i}\right)\,\mathrm{d}V = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V$$

because of the symmetry of σ_{ij}

Minimum Potential Energy and Equilibrium

this means that:

$$\int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V = \int_S \sigma_{ij}\nu_j\delta u_i\,\mathrm{d}S - \int_V \sigma_{ij,j}\delta u_i\,\mathrm{d}V$$

replacing this term in the principle of virtual work and collecting integrals gives:

$$\int_V \left(\sigma_{ij,j} + B_i\right)\delta u_i\,\mathrm{d}V + \int_S \left(\sigma_{ij}\nu_j - T_i\right)\delta u_i\,\mathrm{d}S = 0$$

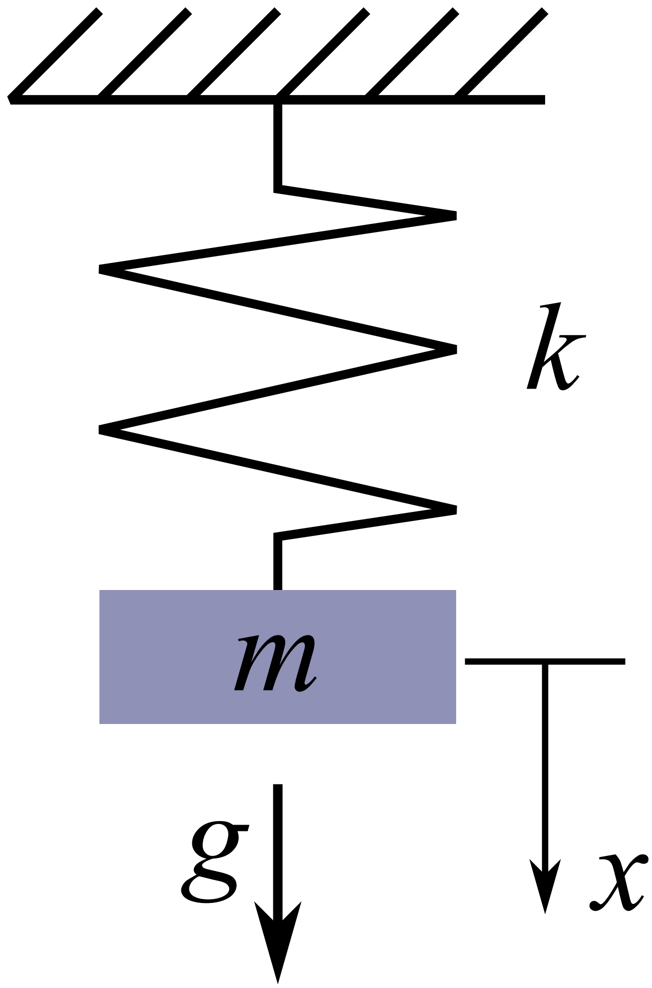
1. what conditions have to hold for this equation to be true?
2. what do these conditions represent?

Example: Direct Variations: Spring-Mass Problem

the mass hanging from a spring is a classic example of a problem that can be solved using the theorem of minimum potential energy (amongst other methods)

the unknown in the problem is the displacement x experienced by the object of mass m under the force of gravity g when hanging from a spring with constant k

take as datum the position of the object when the spring is unstretched, that is, when $x = 0$, and note that x is positive downward



Example: Direct Variations: Spring-Mass Problem

first, identify that the approach is to use the variational form of the theorem of minimum potential energy:

$$\delta\mathscr{V} = \delta\mathscr{U} + \delta\mathscr{E} = 0$$

next, calculate the internal potential energy; this is the strain energy that arises as a consequence of the combination of the spring extension and the force applied to cause this extension:

$$\mathscr{U} = \frac{1}{2}kx^2$$

calculate the first variation of U:

$$\begin{aligned}\delta\mathscr{U} &= \frac{1}{2}k\,(x+\delta x)^2 - \frac{1}{2}kx^2 \\ &= \frac{1}{2}k\left(x^2 + 2x\delta x + (\delta x)^2 - x^2\right) \\ &= kx\delta x\end{aligned}$$

Example: Direct Variations: Spring-Mass Problem

next, find the potential energy due to external forces; in this case it will be the motion of the mass acting under the force of gravity, and hence

$$\mathscr{E} = -mgx$$

and

$$\delta\mathscr{E} = -mg(x+\delta x) + mgx = -mg\delta x$$

combine the two expressions to complete the expression for the first variation of the total potential energy:

$$\delta\mathscr{V} = \delta\mathscr{U} + \delta\mathscr{E} = kx\delta x - mg\delta x = (kx - mg)\,\delta x = 0$$

because of the arbitrary value of δx, in order for this equation to be true, the term inside the brackets must be zero and hence:

$$x = \frac{mg}{k}$$

what is the physical meaning of the vanishing of the term inside the brackets?

Example: Direct Variations: Cantilever Under Self-Weight

the above cantilever beam of length L, height t and width b is composed of a material with constant density ρ and is unloaded except for its own mass due to gravity g

this beam will bend under its own self-weight

what is the deflected shape $u(x)$ that it will attain in equilibrium conditions?

Example: Direct Variations: Cantilever Under Self-Weight

because this is an equilibrium problem, use the theorem of minimum potential energy to solve it

for convenience use:

$$A = bt \qquad I = \frac{bt^3}{12}$$

the potential energy $\mathscr{V}$ is composed of the internal strain energy $\mathscr{U}$ due to bending and the external gravitational potential $\mathscr{E}$ associated with the displaced weight of the beam:

$$\mathscr{U} = \int_0^L \frac{1}{2} M\kappa \, \mathrm{d}x = \int_0^L \frac{1}{2} EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

where $M(x)$ is the bending moment and $\kappa(x)$ is the local curvature; $M = EI\kappa$

$$\mathscr{E} = -\int_0^L \rho g A u \, \mathrm{d}x$$

Example: Direct Variations: Cantilever Under Self-Weight

combine these into a single statement of the total potential energy in the system:

$$\mathscr{V} = \int_0^L \left(\frac{1}{2} EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 - \rho g A u \right) \mathrm{d}x$$

now, the goal is to find the function $u(x)$ that minimizes $\mathscr{V}$; what method can be used to do this? can this be solved by the methods of calculus you have already learned?

this type of problem that variational calculus is intended to solve

Example: Direct Variations: Cantilever Under Self-Weight

take the first variation of $\mathscr{V}$ with respect to the displacement u:

$$\delta\mathscr{V} = \mathscr{V}(u + \delta u) - \mathscr{V}(u) = \delta\mathscr{U} + \delta\mathscr{E} = \mathscr{U}(u + \delta u) - \mathscr{U}(u) + \mathscr{E}(u + \delta u) - \mathscr{E}(u)$$

start with the first variation of the internal strain energy:

$$\begin{aligned}
\delta\mathscr{U} = \mathscr{U}(u + \delta u) - \mathscr{U}(u) &= \frac{1}{2} EI \int_0^L \left(\left(\frac{\mathrm{d}^2 (u + \delta u)}{\mathrm{d}x^2} \right)^2 - \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \right) \mathrm{d}x \\
&= \frac{1}{2} EI \int_0^L \left(\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 + 2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right)^2 - \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \right) \mathrm{d}x \\
&= EI \int_0^L \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) \mathrm{d}x
\end{aligned}$$

Example: Direct Variations: Cantilever Under Self-Weight

next, find the first variation of the external potential energy:

$$\delta\mathscr{E} = \mathscr{E}(u+\delta u) - \mathscr{E}(u) = -\rho g A \int_0^L (u + \delta u - u)\ \mathrm{d}x$$

$$= -\rho g A \int_0^L \delta u \, \mathrm{d}x$$

hence, the first variation of the total potential energy is:

$$\delta\mathscr{V} = \int_0^L \left(EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) - \rho g A \delta u \right) \mathrm{d}x$$

the goal is to find the function $u(x)$ that makes the first variation of $\mathscr{V}$ equal to zero

Example: Direct Variations: Cantilever Under Self-Weight

integrate twice by parts the term containing the two second derivatives:

$$\begin{aligned}
\delta\mathscr{V} &= \int_0^L \left(EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L + \int_0^L \left(-EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(-EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \delta u \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \right) - \rho g A \right) \delta u \, \mathrm{d}x
\end{aligned}$$

Example: Direct Variations: Cantilever Under Self-Weight

we have the first variation of the total potential energy:

$$\delta\mathscr{V} = EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \right) - \rho g A \right) \delta u \, \mathrm{d}x$$

look at the two boundary terms generated by the integrations by parts; in each of the four cases that must be evaluated, one of the factors is zero because of the physical boundary conditions of the problem: at $x = 0$, δu and $\mathrm{d}\delta u/\,\mathrm{d}x$ are zero, while at $x = L$, $\mathrm{d}^2 u/\,\mathrm{d}x^2$ and $\mathrm{d}^3 u/\,\mathrm{d}x^3$ are zero hence all of the boundary terms are zero

what remains is:

$$\delta\mathscr{V} = \int_0^L \left(EI\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \rho g A \right) \delta u \, \mathrm{d}x = 0$$

in order for this always to be true, given the arbitrariness on δu, the bracketed part of the integrand must be zero:

$$EI\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \rho g A = 0$$

Example: Direct Variations: Cantilever Under Self-Weight
the equation we now have is an ordinary differential equation that we can solve by repeated integration; by replacing $\rho g A$ with w, this would become the Euler-Bernoulli beam equation with which we are familiar:

$$EI\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = \rho g A$$

by integrating four times and solving for the constants of integration using the physical boundary conditions, we have:

$$u(x) = \frac{\rho g A}{EI}\left(\frac{x^4}{24} - \frac{Lx^3}{6} + \frac{L^2 x^2}{4} \right)$$

where the maximum deflection, at $x = L$ is:

$$u(L) = \frac{\rho g A L^4}{8EI} = \frac{3\rho g L^4}{2Et^2}$$

3.3 Variational Methods

Variational Calculus
since we now have a problem of minimising a functional (a function of a function), the techniques of variational calculus are extremely important

the notation used in the description of virtual work (δu_i, δE etc) can be transferred directly to the notation of variational calculus: instead of referring to virtual quantities, we refer to variations of a function or a functional

stated more rigorously, if we have a function u defined between points a and b, and with known values $u(a)$ and $u(b)$, the variation of u is δu, where δu has the following properties:

- δu can be decomposed into the product $\epsilon\eta$
- ϵ is an infinitesimally small quantity

- η is an arbitrary function with $\eta(a) = \eta(b) = 0$

at this point, the function u may be unknown; u may be scalar-valued, vector-valued, or higher-order-valued

Variational Calculus

the archetypal problem in variational calculus is the brachistochrone problem: given two points $(a, u(a))$ and $(b, u(b))$ in a gravitational field, what path would a mass take to minimise the time to move from $(a, u(a))$ to $(b, u(b))$?

more generally, variational calculus looks to solve problems of minimisation of functionals; for example, consider the functional $J[u]$:

$$J[u] = \int_a^b F(x, u, u') \, \mathrm{d}x$$

where the function u satisfies the boundary conditions $u(a) = u_a$ and $u(b) = u_b$

make the assumption that F is continuous and differentiable with respect to x, u and u' up to second order partial derivatives, and that such derivatives are continuous

Variational Calculus

there is an infinitely large family of functions $u(x)$ which satisfy the boundary conditions and continuity requirements set out; however, only one of these functions – call it $y(x)$ – also minimises the functional $J[u]$

we would like to know which function this is

we have assumed that the problem of minimising $J[u]$ has the solution $y(x)$ which is a member of the family of functions $u(x)$ which satisfy the boundary conditions and continuity requirements; if so, then:

$$J[y] \leq J[u]$$

create the function $\eta(x)$ with the properties that $\eta(x)$, $\eta'(x)$ and $\eta''(x)$ are continuous on $a \leq x \leq b$ and that $\eta(a) = \eta(b) = 0$

then any function $u(x)$ can be expressed as:

$$u(x) = y(x) + \epsilon\eta(x)$$

for some $\eta(x)$ and ϵ

Variational Calculus

now introduce the function:

$$\phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F[x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)] \, \mathrm{d}x$$

since $y(x)$ is a particular function, assumed to be known, $\phi(\epsilon)$ is a function only of ϵ for a given $\eta(x)$

as a consequence of $J[y]$ minimising J, it must be that:

$$\phi(0) \leq \phi(\epsilon)$$

which implies:

$$\phi'(0) = 0$$

where the prime indicates differentiation with respect to ϵ

Variational Calculus

we can differentiate under the integral:

$$\begin{aligned} \phi'(\epsilon) &= \int_a^b \left[F_u \left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right) \eta(x) \right. \\ &+ \left. F_{u'} \left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right) \eta'(x) \right] \mathrm{d}x \end{aligned}$$

where:

$$F_u = \frac{\partial F}{\partial u}; \quad F_{u'} = \frac{\partial F}{\partial u'}$$

or, slightly more compactly:

$$\phi'(\epsilon) = \int_a^b \left(F_u \eta + F_{u'} \eta' \right) \mathrm{d}x$$

integrating the second term by parts produces:

$$\phi'(\epsilon) = \int_a^b \left[F_u - \frac{\mathrm{d}}{\mathrm{d}x} F_{u'} \right] \eta(x) \,\mathrm{d}x + F_{u'} \eta(x) \Big|_a^b$$

because $\eta(a) = \eta(b) = 0$, the last term vanishes

Variational Calculus

finally, we get the equation:

$$0 = \phi'(0) = \int_a^b \left[F_y \left(x, y, y'\right) - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} \left(x, y, y'\right) \right] \eta(x) \,\mathrm{d}x$$

where, because $\epsilon = 0$, the relevant member of the family u of functions is now y; the notation changes slightly to reflect this

the fundamental lemma of variational calculus states: *If $\psi(x)$ is a continuous function on $a \leq x \leq b$ and the relation*

$$\int_a^b \psi(x)\eta(x) \,\mathrm{d}x = 0$$

holds for all functions $\eta(x)$ which vanish at a and b and are continuous with continuous derivatives, then:

$$\psi(x) \equiv 0$$

the consequence of this lemma is that the term in the integrand multiplied by $\eta(x)$ must be uniformly zero

Variational Calculus

this leads directly to Euler's equation of variational calculus:

$$F_y\left(x, y, y'\right) - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\left(x, y, y'\right) = 0$$

writing this in expanded form gives us:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \frac{\partial^2 F}{\partial y' \partial y'} + \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\partial^2 F}{\partial y' \partial y} + \frac{\partial^2 F}{\partial y' \partial x} - \frac{\partial F}{\partial y} = 0$$

remember that we are looking for the function $y(x)$ which minimises $J[u]$; the satisfaction of Euler's equation is necessary for the function $y(x)$ to provide a local extremum of $J[u]$

Variational Calculus

to return to the more common notation, it is customary to call the function $\epsilon\eta(x)$ the variation of $u(x)$ by writing:

$$\epsilon\eta(x) = \delta u(x)$$

similarly, the first variation of the functional $J[u]$ is:

$$\delta J = \epsilon\phi'(\epsilon)$$

this notation is analogous to differential notation, where we say that the differential of f, $\mathrm{d}f = \epsilon f'(x)$

Example: Variational Calculus

say we have the functional:

$$J[u] = \int_0^1 \left(1 + (u')^2\right) \mathrm{d}x$$

with boundary conditions:

$$u(0) = 0 \qquad u(1) = 1$$

and we would like to find the function y which minimises J

we use Euler's equation where:

$$F = 1 + (u')^2$$

hence:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial u'} = 0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(2u'\right) = 0$$

Example: Variational Calculus

evaluating the previous expression:

$$-\frac{\mathrm{d}}{\mathrm{d}x}(2u') = -2u'' = 0$$

which we can integrate; after solving for the constants, it is the straight line:

$$y = x$$

what this result states is that the minimum distance between points $(0,0)$ and $(1,1)$ is a straight line (note that the initial integrand F is the square of the distance along a line)

Example: Variational Calculus

find the function y that minimises the functional:

$$J[u] = \int_0^{\pi/2} \left((u')^2 - u^2\right) \mathrm{d}x$$

with boundary conditions:

$$u(0) = 0 \qquad u\left(\frac{\pi}{2}\right) = 1$$

the integrand is:

$$F = (u')^2 - u^2$$

and applying Euler's equation gives us:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = -2u - 2u'' = 0$$

this is $y = \sin(x)$

Variations with Higher Derivatives

we may have functionals of the form:

$$J[u] = \int_a^b F\left(x, u, u', u''\right) \mathrm{d}x$$

once again we must have a family of functions u with continuous derivatives and known values of the function and its first derivative at the boundaries

as well, we again have the arbitrary function $\eta(x)$ which has continuous derivatives and it and its first derivative are zero at the boundaries

if we are looking for the function $y(x)$ which minimises $J[u]$, then we can form the function $u(x) = y(x) + \epsilon\eta(x)$

Variations with Higher Derivatives

hence we have the functional:

$$\phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F\left(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta''\right)\,\mathrm{d}x$$

again, differentiate under the integral, integrate by parts and remove terms equal to zero:

$$0 = \phi'(\epsilon) = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial u''}\right)\eta\,\mathrm{d}x$$

in general for an integrand F which is a function of the derivatives of y up to order n, the following holds:

$$\sum_{\nu=0}^{n} (-1)^{\nu}\,\frac{\mathrm{d}^{\nu}}{\mathrm{d}x^{\nu}}\left(\frac{\partial F}{\partial u^{(\nu)}}\right) = 0$$

with this it is possible to handle variational problems involving derivatives of any order; needless to say, solving the resulting differential equation is not necessarily easy

Example: Cantilever Under Self-Weight

recall the potential energy of a cantilever loaded only by its own self-weight:

$$\mathscr{V} = \int_0^L \left(\frac{1}{2}EI\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 - \rho g A u\right)\mathrm{d}x$$

here:

$$F = \frac{1}{2}EI\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 - \rho g A u$$

based on the last slide, the version of the Euler-Lagrange equation for functionals involving second derivatives can be employed:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial u''} = 0$$

Example: Cantilever Under Self-Weight

there are no terms involving u', but u and u'' appear:

$$\frac{\partial F}{\partial u} = -\rho g A \qquad\qquad \frac{\partial F}{\partial u''} = EIu''$$

evaluating the term involving u'' in the Euler-Lagrange equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}EIu'' = EIu''''$$

and hence the totality of the Euler-Lagrange equation applied to the cantilever loaded by its self-weight is:

$$EIu'''' - \rho g A = 0$$

3.4 Variational Equations of Motion

Variational Equations of Motion

for small displacements, the equation of motion in Eulerian coordinates is:

$$\sigma_{ji,j} + B_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

consider virtual displacements δu_i which are zero over the parts of the surface where the displacements are specified; if so, the external virtual work done by the surface tractions and the body forces is:

$$\delta \mathscr{E} = \int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S$$

transform the surface integral into a volume integral using Cauchy's Law and Gauss's theorem:

$$\begin{aligned} \int_S T_i \delta u_i \, \mathrm{d}S &= \int_S \sigma_{ji} \nu_j \delta u_i \, \mathrm{d}S = \int_V \left(\sigma_{ji} \delta u_i\right)_{,j} \, \mathrm{d}V \\ &= \int_V \sigma_{ji,j} \delta u_i \, \mathrm{d}V + \int_V \sigma_{ji} \delta u_{i,j} \, \mathrm{d}V \end{aligned}$$

Variational Equations of Motion

substituting in the equation of motion:

$$\int_V \sigma_{ji,j} \delta u_i \, \mathrm{d}V + \int_V \sigma_{ji} \delta u_{i,j} \, \mathrm{d}V = \int_V \left(\rho \frac{\partial^2 u_i}{\partial t^2} - B_i\right) \delta u_i \, \mathrm{d}V + \int_V \sigma_{ji} \delta \epsilon_{ij} \, \mathrm{d}V$$

combining all of this gives:

$$\int_V \sigma_{ij} \delta \epsilon_{ij} \, \mathrm{d}V = \int_V \left(B_i - \rho \frac{\partial^2 u_i}{\partial t^2}\right) \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S$$

which we call the variational equation of motion

if we have an elastic system with a strain energy density $\mathscr{W}$, then:

$$\delta \int_V \mathscr{W} \, \mathrm{d}V = \int_V \left(B_i - \rho \frac{\partial^2 u_i}{\partial t^2}\right) \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S$$

Variational Equations of Motion

now, we can integrate this over some arbitrary period of time, from t_0 to t_1:

$$\int_{t_0}^{t_1} \int_V \delta \mathscr{W} \, \mathrm{d}V \, \mathrm{d}t = \int_{t_0}^{t_1} \left[\int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S - \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \, \mathrm{d}V \right] \mathrm{d}t$$

take the acceleration term, reverse the order of integration and integrate by parts to get:

$$\int_{t_0}^{t_1} \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \, \mathrm{d}V \, \mathrm{d}t = \int_V \rho \frac{\partial u_i}{\partial t} \delta u_i \, \mathrm{d}V \bigg|_{t_0}^{t_1} - \int_V \int_{t_0}^{t_1} \frac{\partial u_i}{\partial t} \left(\rho \frac{\partial \delta u_i}{\partial t} + \frac{\partial \rho}{\partial t} \delta u_i \right) \mathrm{d}t \, \mathrm{d}V$$

we can require that $\delta u_i(t_0) = \delta u_i(t_1) = 0$ and take note of the density partial derivative, which by the equation of continuity becomes a partial derivative of velocity: this makes that term of higher order and hence negligible

Variational Equations of Motion

the result is that:

$$\int_{t_0}^{t_1} \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \, \mathrm{d}V \, \mathrm{d}t = - \int_{t_0}^{t_1} \int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial \delta u_i}{\partial t} \, \mathrm{d}V \, \mathrm{d}t = - \int_{t_0}^{t_1} \delta \mathscr{K} \, \mathrm{d}t$$

which is the variation of the kinetic energy of the moving body:

$$\mathscr{K} = \frac{1}{2} \int_V \rho \left(\frac{\partial u_i}{\partial t} \right)^2 \mathrm{d}V$$

recall how we defined the variation of potential energy:

$$\delta \mathscr{V} = \int_V \delta \mathscr{W} \, \mathrm{d}V - \int_V B_i \delta u_i \, \mathrm{d}V - \int_S T_i \delta u_i \, \mathrm{d}S$$

we can combine everything to get:

$$0 = \int_{t_0}^{t_1} \delta \mathscr{K} - \delta \mathscr{V} \, \mathrm{d}t$$

Lagrangian and Hamiltonian

our main result is:

$$0 = \int_{t_0}^{t_1} \delta \mathscr{K} - \delta \mathscr{V} \, \mathrm{d}t$$

where we say that the Lagrangian is:

$$\mathscr{L} = \mathscr{K} - \mathscr{V}$$

and the Hamiltonian is:

$$\mathscr{A} = \int_{t_0}^{t_1} \mathscr{L} \, \mathrm{d}t$$

and our manipulation of the variational equation of motion shows that

$$\delta\mathscr{A} = \int_{t_0}^{t_1} \delta\mathscr{K} - \delta\mathscr{V} \, \mathrm{d}t = 0$$

that is, the variation of the Hamiltonian is zero for an admissible motion

Hamilton's Principle

Hamilton's principle states that:

> the time integral of the Lagrangian function between two arbitrary times is an extremum for the "actual" motion with respect to all admissible virtual displacements which vanish at times t_0 and t_1 and on the parts of the boundary with displacements prescribed

3.5 Strong Forms and Weak Forms

Strong Form of the Equilibrium Equations

the standard ("strong") forms for the equilibrium equations in linearised elasticity are:

1. equilibrium:

$$\sigma_{ji,j} + B_i = 0$$

2. constitutive law:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

3. strain - displacement relation:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

4. traction boundary conditions:

$$\sigma_{ji}\nu_j = T_i \;\; \text{on} \;\; S_\sigma$$

5. displacement boundary conditions:

$$u_i = \bar{u} \;\; \text{on} \;\; S_u$$

we can combine 1, 2 and 3 to get:

$$\frac{1}{2}\left(C_{ijkl}\left(u_{k,l} + u_{l,k}\right)\right)_{,j} + B_i = 0$$

Weak Form of the Equilibrium Equations

take the equilibrium equation of the strong form as expressed above, multiply it by a "weight function" δw_i and integrate over the volume:

$$\int_V \sigma_{ji,j}\delta w_i \, \mathrm{d}V + \int_V B_i \delta w_i \, \mathrm{d}V = 0$$

using Gauss's theorem, transform the first integral into:

$$\int_V \sigma_{ji,j}\delta w_i \, \mathrm{d}V = \int_S \left(\delta w_i \sigma_{ji}\right) \nu_j \, \mathrm{d}S - \int_V \delta w_{i,j}\sigma_{ji} \, \mathrm{d}V$$

substituting, this generates:

$$\int_V \delta w_{i,j}\sigma_{ji} \, \mathrm{d}V = \int_V \delta w_i B_i \, \mathrm{d}V + \int_S \left(\delta w_i \sigma_{ji}\right) \nu_j \, \mathrm{d}S$$

which can be written as:

$$\int_V \delta w_{i,j} C_{ijkl}\epsilon_{kl} \, \mathrm{d}V = \int_V \delta w_i B_i \, \mathrm{d}V + \int_S \left(\delta w_i C_{ijkl}\epsilon_{kl}\right) \nu_j \, \mathrm{d}S$$

Why Have We Done This??

replacing the strains with the derivatives of displacement:

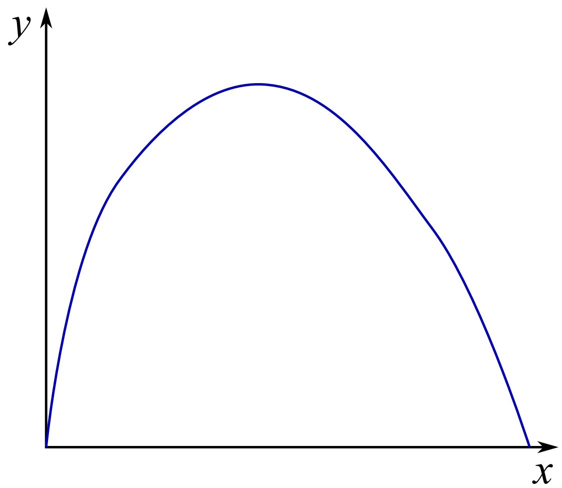
$$\frac{1}{2}\int_V \delta w_{i,j} C_{ijkl} \left(u_{k,l} + u_{l,k}\right) \, \mathrm{d}V = \int_V \delta w_i B_i \, \mathrm{d}V + \frac{1}{2}\int_S \left(\delta w_i C_{ijkl} \left(u_{k,l} + u_{l,k}\right)\right) \nu_j \, \mathrm{d}S$$

clearly this is a legitimate mathematical operation: multiplying by the variation of w_i, which is some arbitrary function, and then integrating over the domain should still produce zero if the original function is zero everywhere

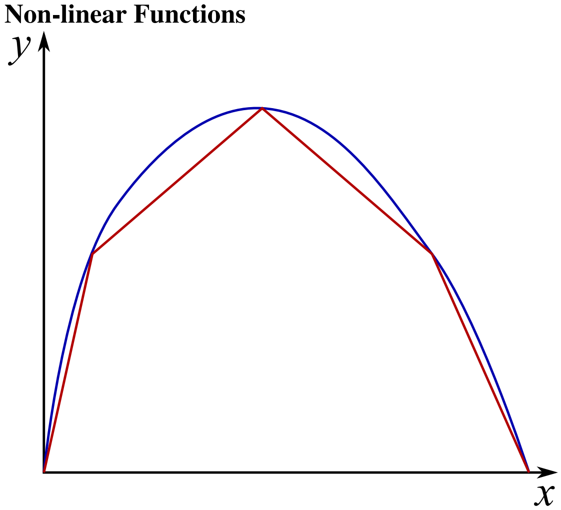
so why do we want to do this?

examine the order of the derivatives in the strong and weak forms, and recall that strain is a derivative of displacement: what benefit could this have?

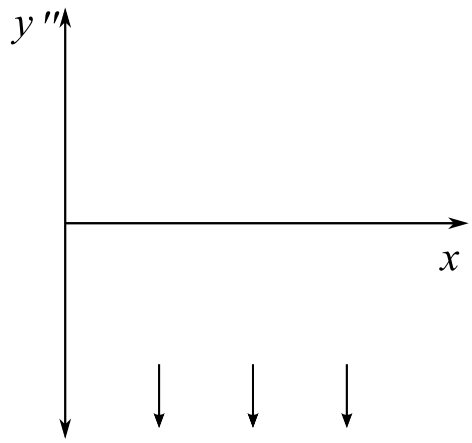
Non-linear Functions



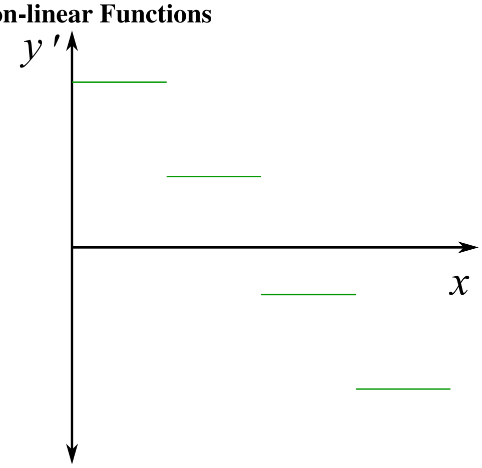
Approximating Non-linear Functions



Approximating Non-linear Functions



Approximating Non-linear Functions



The Value of the Weak Form

the key advantage to the weak form is that there are no derivatives higher than the first

if we would like to approximate a displacement function with a set of piecewise continuous functions, the first derivative can be integrated, but the second derivative is problematic

by using the weak form, we are able to have much looser requirements on the smoothness of the functions we are using to approximate the actual solution: this makes finite element analysis possible

3.6 Linking Energy Methods to Finite Elements

Background to Finite Element Methods

classical elasticity methods, based upon the principles discovered by Newton and Hooke and extended by Euler and Cauchy, provide elegant, exact and complete solutions to a very limited number of problems; a general method for solving solids problems is therefore required

by approaching problems in solids mechanics from the perspective of energy balances, it is possible to change the formulation of the problem to permit approximate solutions which satisfy (approximately) the equations of equilibrium

the finite element method is an example of a *Galerkin* method of solution; the fundamental concept of finite elements is that the problem domain is broken into discrete parts (finite elements) and the discretisation is performed by a conversion to the weak form and the use of linear interpolants both as the trial functions and the weight functions

Background to Finite Element Methods

in practice, the mechanics of the finite element method are generally hidden behind the interface of a commercial software package

Richard Courant published the "first" paper in finite element analysis 1943 wherein he used triangular elements to solve some vibrational problems

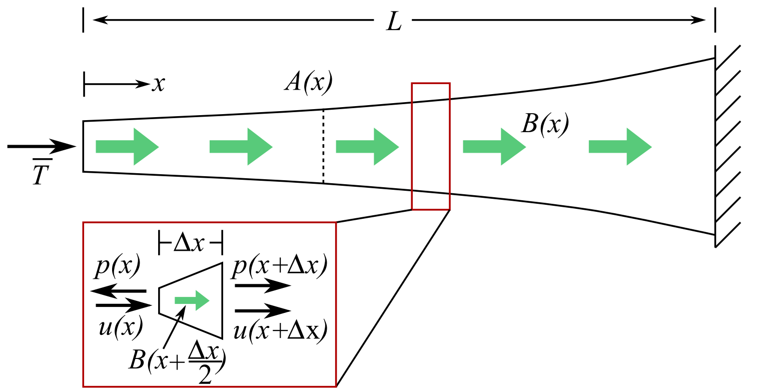
a stronger mathematical footing was found when it was shown that for benign conditions, the finite element method would always converge to the correct solution provided the number of elements went to infinity

the falling cost of computer power makes finite element analysis very widely available, and very widely used

today, billions of dollars are spent annually on finite element analysis, and many major commercial software packages are available, including ABAQUS, ANSYS, and LS-DYNA

3.7 1-D Element

Finite Elements in One Dimension



Finite Elements in One Dimension

take a one-dimensional beam which is oriented parallel to the x-axis, which has varying area $A(x)$, Young's modulus E, and has body force $B(x)$ acting in the x-direction

the beam is fixed at $x = L$, and has traction $\bar{T}$ acting at $x = 0$ which has units of force per area

all displacements are in the x-direction

equilibrium of a small element of the beam is shown in the inset

Finite Elements in One Dimension

if we assume that the material of which the bar is made is linear and elastic, the bar must satisfy several conditions:

1. it must be in equilibrium
2. it must satisfy Hooke's Law: $\sigma(x) = E\epsilon(x)$
3. the displacement field must be compatible
4. it must satisfy the one-dimensional strain - displacement relation: $\epsilon(x) = \mathrm{d}u/\,\mathrm{d}x$

Strong Form in One Dimension

the governing equation for equilibrium of the bar is determined from the local equilibrium of an element:

$$-p(x) + B\left(x + \frac{\Delta x}{2}\right)\Delta x + p(x + \Delta x) = 0$$

rearranging terms and dividing by Δx:

$$\frac{p(x + \Delta x) - p(x)}{\Delta x} + B\left(x + \frac{\Delta x}{2}\right) = 0$$

taking the limit as Δx goes to zero gives the one-dimensional analogue to the equation of equilibrium:

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} + B(x) = 0$$

Strong Form in One Dimension

in one dimension, define stress as force per area:

$$\sigma(x) = \frac{p(x)}{A(x)}$$

coupled with the constitutive law and the strain - displacement relation this gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B = 0$$

which is a second-order ordinary differential equation

to solve this, the boundary conditions at the two ends of the beam are needed; for this case they are:

$$\sigma(0) = -\bar{T}$$

$$u(L) = 0$$

these final three expressions make up the strong form in one dimension

Weak Form in One Dimension

to develop the weak form in one dimension, take the equilibrium equation and the traction boundary condition, multiply them by the "weight function" δw and integrate the equilibrium equation over the domain:

$$\int_0^L \delta w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right) \mathrm{d}x = 0$$

$$\left(\delta w A\left(E\frac{\mathrm{d}u}{\mathrm{d}x} + \bar{T}\right)\right)_{x=0} = 0$$

note that δw is restricted to be zero at $x = L$

integrate by parts:

$$\left[\delta w AE\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L - \int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x + \int_0^L \delta w B\,\mathrm{d}x = 0$$

and evaluate:

$$(\delta w A\sigma)_{x=L} - (\delta w A\sigma)_{x=0} - \int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x + \int_0^L \delta w B\,\mathrm{d}x = 0$$

Weak Form in One Dimension

the first term is zero because $\delta w(L) = 0$, and $\sigma(0) = -\bar{T}$:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x = \int_0^L \delta w B \, \mathrm{d}x + \left(\delta w A \bar{T}\right)_{x=0}$$

this is the weak form of the governing equations, and the function $u(x)$ with $u(L) = 0$ which satisfies the weak form for all δw with $\delta w(L) = 0$, is the solution to the equation

because the displacement boundary condition must be satisfied by any possible solution $u(x)$, it is called an *essential* boundary condition; the traction boundary condition is satisfied within the weak form and is called a *natural* boundary condition

Equivalence of the Weak and Strong Forms

we now have the weak form:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x = \int_0^L \delta w B \, \mathrm{d}x + \left(\delta w A \bar{T}\right)_{x=0}$$

it is necessary to show that the function $u(x)$ which satisfies the weak form is also the solution to the strong form by reversing the order of our analysis and showing that no information is lost or added

integration by parts shows us that:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\, \mathrm{d}x = \left[\delta w AE \frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L - \int_0^L \delta w \frac{\mathrm{d}}{\mathrm{d}x}\left(AE \frac{\mathrm{d}u}{\mathrm{d}x}\right) \mathrm{d}x$$

substituting into the weak form and collecting integrals generates:

$$\int_0^L \delta w \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE \frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right) \mathrm{d}x + \delta w A \left(\bar{T} + \sigma\right)_{x=0} = 0$$

Equivalence of the Weak and Strong Forms

because δw is an arbitrary function, it can be anything we need; set it to:

$$\delta w = \psi(x) \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE \frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right)$$

where $\psi(x)$ is smooth, positive, and vanishes on the boundaries

this yields:

$$\int_0^L \psi \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE \frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right)^2 \mathrm{d}x = 0$$

and the boundary term is zero because of how δw has just been defined because the integrand is the product of a positive function ψ and a quadratic term, the integral must always be positive, unless the integrand is zero, which requires:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B = 0$$

and the strong form of the equilibrium equation has been recovered

Equivalence of the Weak and Strong Forms

we can return a couple of steps and use an alternate δw and we have remaining:

$$\delta w A \left(\bar{T} + \sigma\right)_{x=0} = 0$$

the only way this can always be true is if

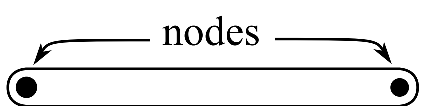
$$\sigma(0) = -\bar{T}$$

now all aspects of the strong form have been recovered, and hence a solution to the weak form must also be a solution to the strong form

Approximating the Solution and Weight Functions

a key aspect of finite element analysis is that the actual solution (displacements, in the case of solid mechanics) is approximated by dividing the structure into pieces (the "finite elements") and solving approximately in each element

a one-dimensional system can be divided into one-dimensional elements; at each end of an element it will be connected to other elements or a boundary at a location called a node



call the displacements at the nodes d_1 and d_2; we want to approximate the displacement at any position in the element if we know the displacements at the nodes

Approximating the Solution and Weight Functions

the easiest approximation is linear; use a system of local coordinates such that node 1 is at position $x_1 = 0$ and node 2 is at position $x_2 = \ell$ (the length of the element is $\ell = x_2 - x_1$)

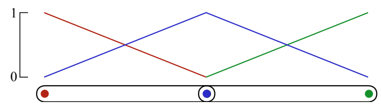
create a local position variable x which then ranges from zero to ℓ

a linear approximation of the displacements throughout the element, using only the nodal displacements d_1 and d_2, is:

$$u(x) = d_1\frac{\ell - x}{\ell} + d_2\frac{x}{\ell}$$

this linearisation has the property that the displacement at node 1 has no effect on the displacement at node 2 (and *vice versa*)

this figure shows the regions of *influence* of the displacements at the nodes; note that at any location the sum of the influence lines is 1



Approximating the Solution and Weight Functions

these linear functions are going to be used to approximate the actual solution within each element

note that it is very easy to take the derivatives of these functions, which are what we will need to use when we do the integrations:

$$u'(x) = \frac{1}{\ell}(d_2 - d_1)$$

the weight functions δw are also approximated by the same interpolating function as the solution functions: the values of the weight functions at the nodes are known and are used to approximate the value of the weight function anywhere in the element

$$\delta w = \delta w_1 \frac{\ell - x}{\ell} + \delta w_2 \frac{x}{\ell}$$

finite element methods which use the same approximation for the solution ('trial') and the weight ('test') function are called "Galerkin methods"

Discretisation of the Weak Form

at this point, it is easiest to write the equations in vector - matrix form

the solution function $u(x)$ is approximated by:

$$u(x) \approx \begin{bmatrix} \frac{\ell - x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \mathbf{Nd}$$

where **N** is called the element shape function and **d** is a vector of nodal displacements

select a value $0 \leq x \leq \ell$ and use the interpolation function to get an approximate displacement $u(x)$ at x

the derivative of the solution function, $u'(x)$ is:

$$u'(x) \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \mathbf{Hd}$$

for the weight function δw and its derivative, we have:

$$\delta w \approx \begin{bmatrix} \frac{\ell - x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{N}\mathbf{w} \qquad \delta w' \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{H}\mathbf{w}$$

Discretisation of the Weak Form

we started with the weak form:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta w B\,\mathrm{d}x + \left(\delta w A\bar{T}\right)_{x=0}$$

since we are now decomposing the system into elements, we want to replace the integral over the whole domain with a sum of the integrals over each element:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x - \int_{x_1}^{x_2} \delta w B\,\mathrm{d}x - \left(\delta w A\bar{T}\right)_{x=0} \right) = 0$$

we now also replace the approximated terms with their matrix representations:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \mathbf{w}^T \mathbf{H}^T AE\mathbf{H}\mathbf{d}\,\mathrm{d}x - \int_{x_1}^{x_2} \mathbf{w}^T \mathbf{N}^T B\,\mathrm{d}x - \left(\mathbf{w}^T \mathbf{N}^T A\bar{T}\right)_{x=0} \right) = 0$$

Discretisation of the Weak Form

since $\mathbf{w}$ is an arbitrary function, it can be removed from the integrations; similarly, $\mathbf{d}$ is not a function of x, so it can come outside the integral:

$$\sum^{\text{elems}} \mathbf{w}^T \left(\int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\,\mathrm{d}x\mathbf{d} - \int_{x_1}^{x_2} \mathbf{N}^T B\,\mathrm{d}x - \left(\mathbf{N}^T A\bar{T}\right)_{x=0} \right) = 0$$

the integral:

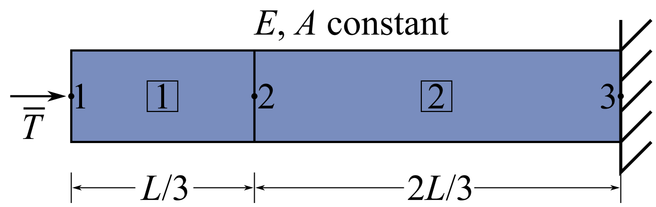
$$\mathbf{K} = \int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\,\mathrm{d}x$$

is the element stiffness matrix; assembling these for all of the elements generates the structural stiffness matrix

similarly, the external body and boundary forces are:

$$\mathbf{f}_\Omega = \int_{x_1}^{x_2} \mathbf{N}^T B\,\mathrm{d}x; \qquad \mathbf{f}_\Gamma = \left(\mathbf{N}^T A\bar{T}\right)_{x=0}$$

Example: One-Dimensional Finite Element



the bar shown above can be considered to be one-dimensional: all forces and displacements are axial to the bar

it is fixed at the right end and loaded with a traction $\bar{T}$ at the free end

it is divided into two one-dimensional elements, associated with three nodes

use the finite element method to find the nodal displacements for this bar

Example: One-Dimensional Finite Element

first, we need to evaluate the integral that provides the element stiffness matrix:

$$\mathbf{K}_{\text{elem}} = \int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\, \mathrm{d}x$$

in this case, E and A are constant and come outside the integral; expressing $\mathbf{H}^T$ and $\mathbf{H}$ in full and using $x_1 = 0$ and $x_2 = \ell$ gives:

$$\begin{aligned}
\mathbf{K}_{\text{elem}} &= AE \int_0^{\ell} \begin{bmatrix} -\frac{1}{\ell} \\ \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \mathrm{d}x \\
&= AE \int_0^{\ell} \begin{bmatrix} \frac{1}{\ell^2} & -\frac{1}{\ell^2} \\ -\frac{1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} \mathrm{d}x \\
&= \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

Example: One-Dimensional Finite Element

next, calculate the element stiffness matrices for each of the two elements:

$$\mathbf{K}_1 = \frac{3AE}{L} \cdot \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{K}_2 = \frac{3AE}{2L} \cdot \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

combine the two element matrices into a single structural stiffness matrix:

$$\mathbf{K} = \frac{3AE}{L} \cdot \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Example: One-Dimensional Finite Element

we also need to determine the force vector and the displacement vector; first the force vector:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \cancelto{0}{\mathbf{N}^T B}\, \mathrm{d}x - \left(\mathbf{N}^T A \bar{T} \right)_{x=0} \right)$$

the transpose of the element shape function, $\mathbf{N}^T$, is:

$$\mathbf{N}^T = \begin{bmatrix} \dfrac{\ell - x}{\ell} \\ \dfrac{x}{\ell} \end{bmatrix}$$

so for element 1

$$\mathbf{f}_1 = \left(\mathbf{N}^T A \bar{T} \right)_{x=0} = \begin{bmatrix} A\bar{T} \\ 0 \end{bmatrix}$$

and for the full force vector:

$$\mathbf{f} = \begin{bmatrix} A\bar{T} \\ 0 \\ f_3 \end{bmatrix}$$

Example: One-Dimensional Finite Element

the displacement vector is:

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix}$$

note that d_1, d_2 and f_3 are unknown and must be solved; for each pair d_i, f_i, one of the pair must be unknown

the problem to solve is:

$$\mathbf{f} = \mathbf{K}\mathbf{d}$$

Example: One-Dimensional Finite Element

because we know that part of the displacement vector, $\mathbf{D}$, is zero, we can partition the problem to solve the smallest linear system possible:

$$\begin{bmatrix} A\bar{T} \\ 0 \end{bmatrix} = \frac{3AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & \dfrac{3}{2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

solving:

$$d_1 = \frac{3}{2} d_2; \quad A\bar{T} = \frac{3AE}{L} \left(\frac{3}{2} d_2 - d_2 \right) \Longrightarrow d_2 = \frac{2\bar{T}L}{3E}; \quad d_1 = \frac{\bar{T}L}{E}$$

Example: One-Dimensional Finite Element

also, we can now solve for the reaction force, f_3, using the full displacement vector and the full stiffness matrix:

$$\begin{bmatrix} A\bar{T} \\ 0 \\ f_3 \end{bmatrix} = \frac{3AE}{L} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\bar{T}L}{E} \\ \frac{2\bar{T}L}{3E} \\ 0 \end{bmatrix}$$

$$f_3 = \frac{\bar{T}L}{E}\frac{3AE}{L} \cdot 0 + \frac{2\bar{T}L}{3E}\frac{3AE}{L} \cdot -\frac{1}{2} = -A\bar{T}$$