Outline

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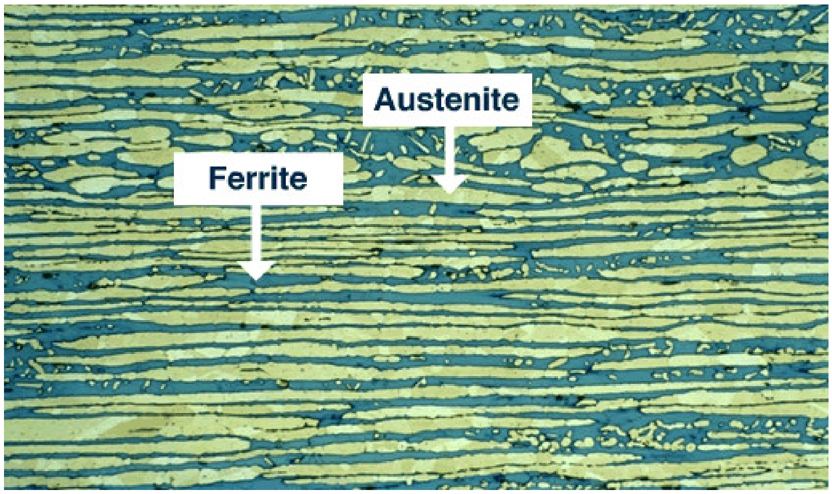
1.1 Introduction to Composites

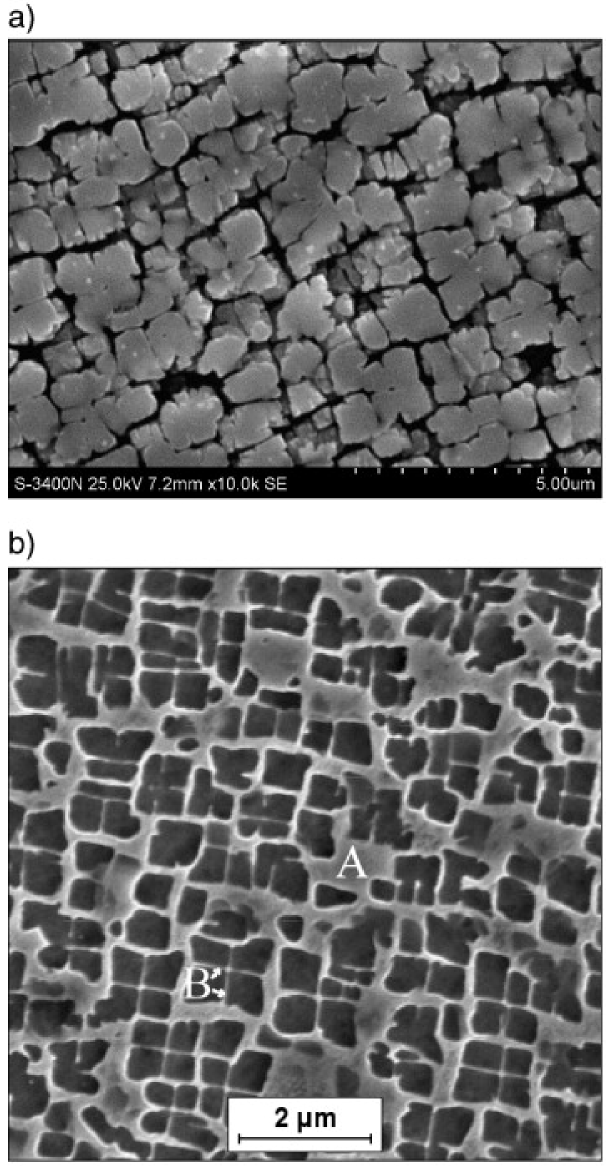
What Are Composites?

materials consisting of two or more phases

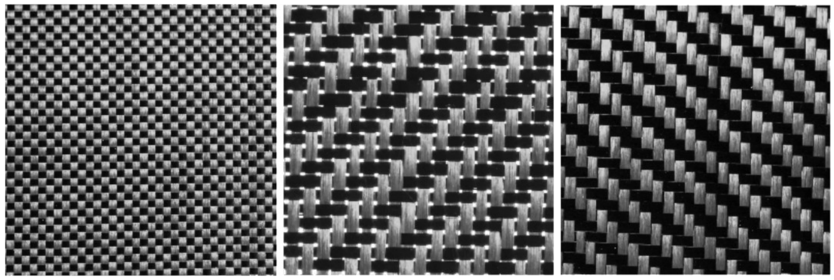


What Are Composites?

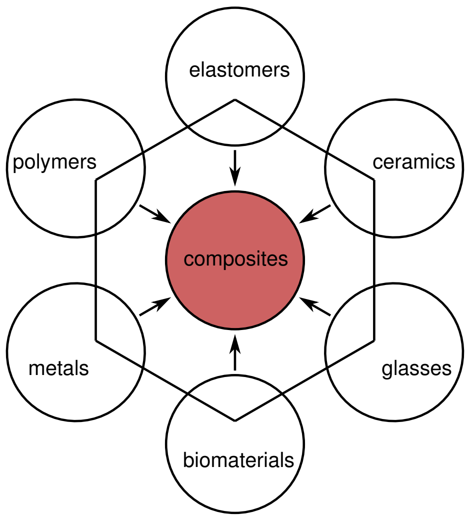
materials with two or more phases that are engineered so that the phases are geometrically related in a beneficial manner



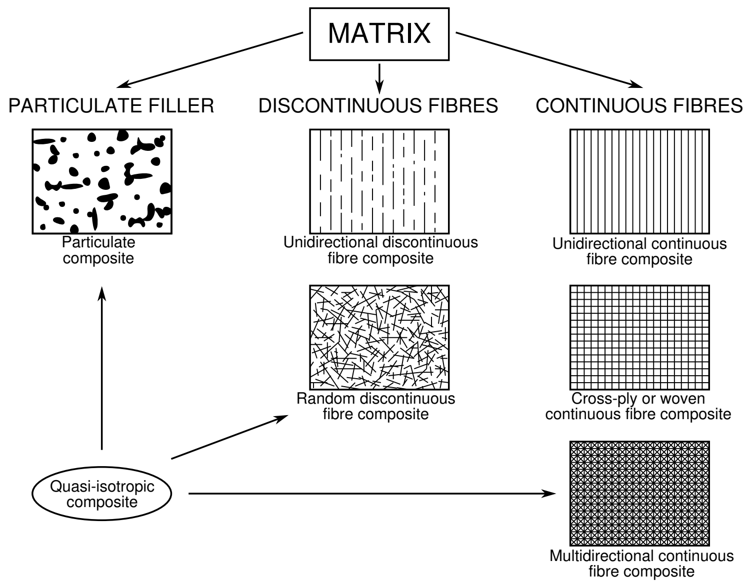
What Are Composites?



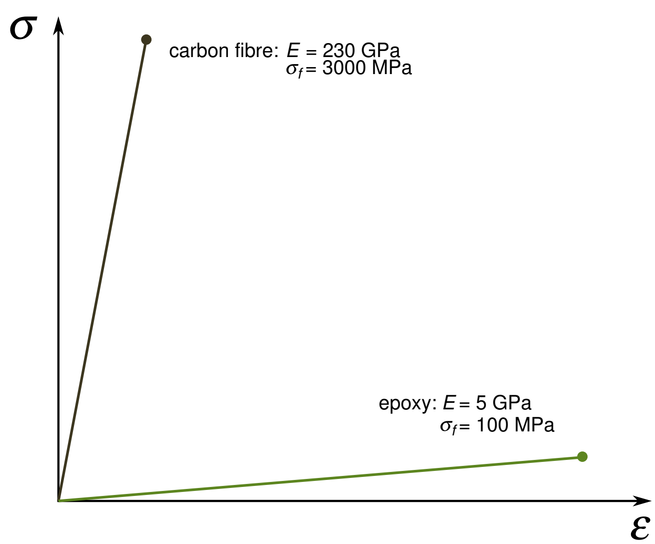
What Are Composites?

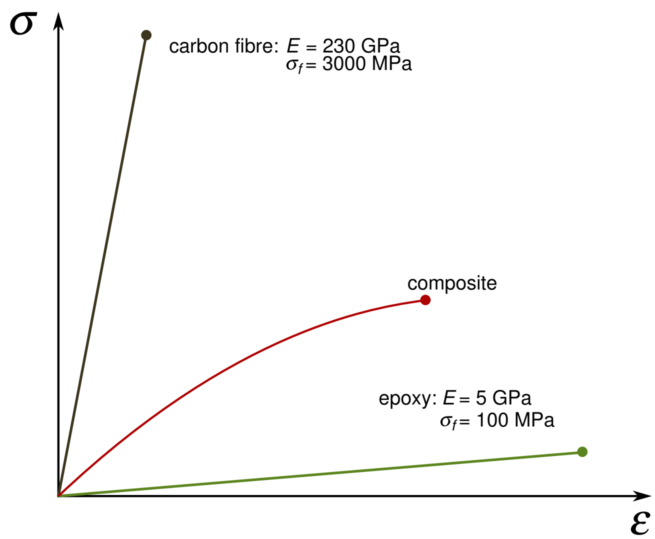


Classification of Composites



Why Composites?





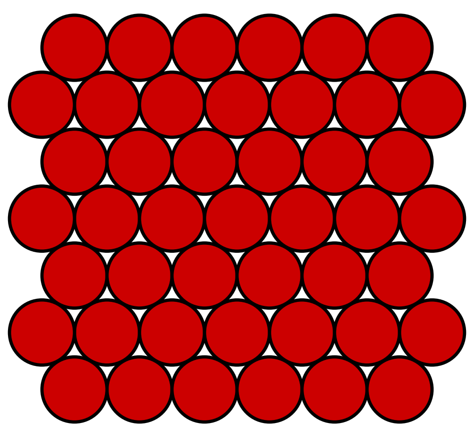
Composite Properties

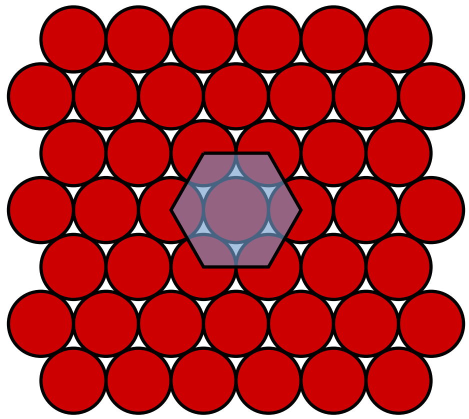
define:

v_f	volume fraction of the reinforcing material
ρ_m	density of the matrix material
ρ_f	density of the reinforcing material
E_m	Young's modulus of the matrix material
E_f	Young's modulus of the reinforcing material

Volume Fraction

looking at a unidirectional composite in cross-section, the cylindrical fibres look like closely packed circles





Composite Properties: Density

if we know the densities of the two components of the composites (ρ_f and ρ_m), as well as the volume fraction of the filler (v_f), we can directly calculate the density of the composite:

$$\rho = v_f \rho_f + \left(1 - v_f\right) \rho_m$$

this is the basic rule of mixtures, and is exact for the density of the composite material

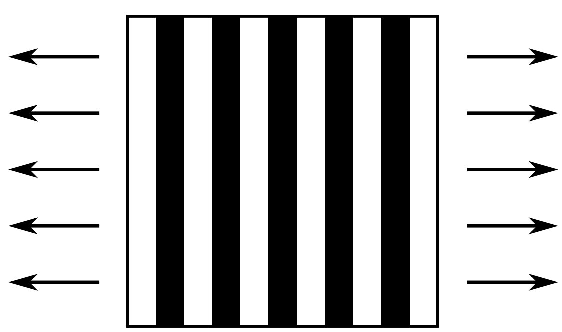
what other composite properties are described exactly by the rule of mixtures?

if we cannot calculate the property exactly, it is generally the case that we can find *bounds* on the possible values of the property

the bounds are the maximum and minimum values a property can have, independent of the arrangement of the reinforcement or the orientation of the composite

most importantly, we can calculate bounds on the elastic modulus of a composite

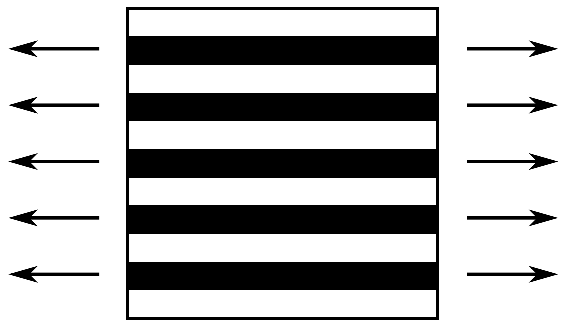
Bounds on Composite Properties: Modulus Lower (Reuss) Bound



the Reuss bound assumes that the *stress* is constant through all the layers of the composite and the strain is calculated:

$$\frac{1}{E} = \frac{v_f}{E_f} + \frac{\left(1 - v_f\right)}{E_m}$$

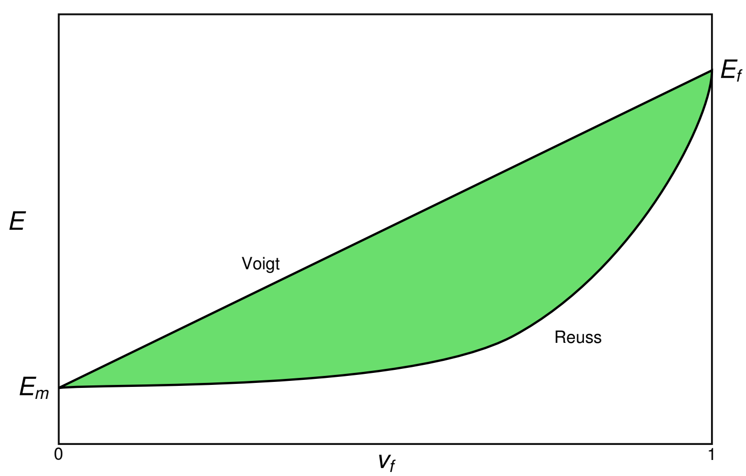
Bounds on Composite Properties: Modulus Upper (Voigt) Bound



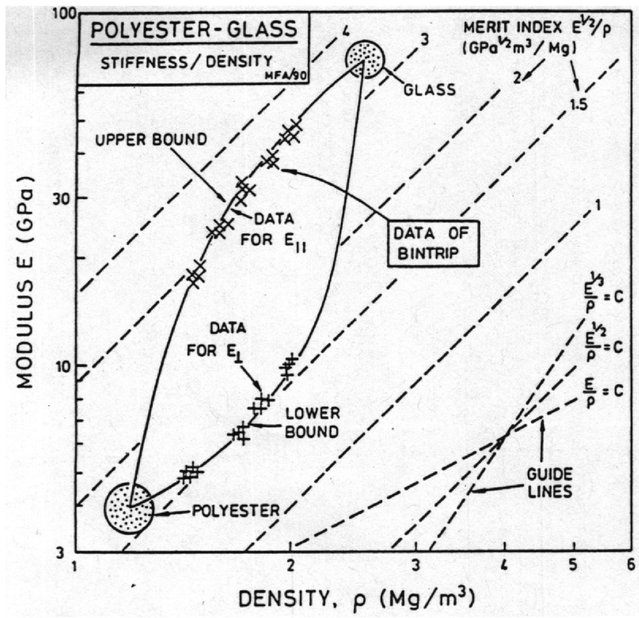
the Voigt bound assumes that the *strain* is constant through all the layers of the composite and the stress is calculated:

$$E = v_f E_f + \left(1 - v_f\right) E_m$$

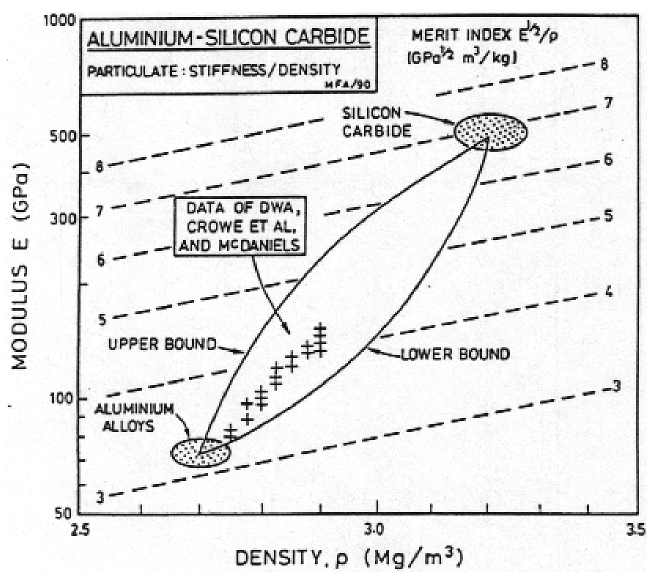
Modulus Bounds



Modulus Bounds: Polyester / Glass



Modulus Bounds: Aluminum / Silicon Carbide



Algorithm: Modulus Bounds

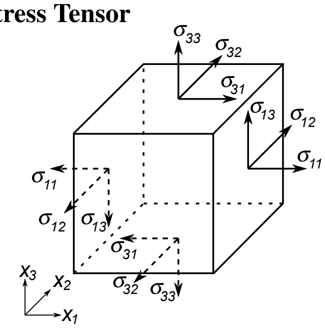
1. identify the materials of interest
2. determine the Young's modulus and density of the materials

3. for a given volume fraction, calculate the density and the Reuss and Voigt bounds for the composite
4. repeat for as many volume fractions are needed
5. plot the two bounds of composite modulus against volume fraction or density, as required

1.2 Elastic Properties of Composites

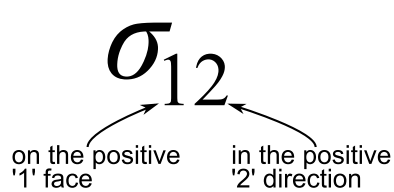
1.2.1 Linear Elasticity

Stress Tensor



this is a very small cube that surrounds a material point in a stressed medium

what is the stress at this material point?
we identify the various stresses on the cube using the following notation:



note that the stresses on the ‘2’ faces have been omitted for clarity

when expressed in a particular coordinate system, it has the components:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Strain Tensor; Elastic Relations

similarly, the strain tensor is given by:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

if the material behaves in a linear elastic manner, the stress and strain are related by a fourth order tensor:

$$\sigma_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} C_{ijkl}\epsilon_{kl} \equiv C_{ijkl}\epsilon_{kl}$$

where C_{ijkl} is the tensor of elastic constants

for a general material, this requires 81 constants; for an isotropic material, two constants are sufficient

composites are somewhere in between these two extremes

Vector Notation for Stresses and Strains

in the composites world, it is conventional to contract the second-order stress and strain tensors into six-component vectors, and the tensor of elastic constants into a two-dimensional matrix:

$$\begin{aligned} \sigma &= [\sigma_{11}\ \ \sigma_{22}\ \ \sigma_{33}\ \ \sigma_{23}\ \ \sigma_{13}\ \ \sigma_{12}]^T \\ &= [\sigma_1\ \ \sigma_2\ \ \sigma_3\ \ \sigma_4\ \ \sigma_5\ \ \sigma_6]^T \end{aligned}$$

and

$$\begin{aligned} \epsilon &= [\epsilon_{11}\ \ \epsilon_{22}\ \ \epsilon_{33}\ \ 2\epsilon_{23}\ \ 2\epsilon_{13}\ \ 2\epsilon_{12}]^T \\ &= [\epsilon_{11}\ \ \epsilon_{22}\ \ \epsilon_{33}\ \ \gamma_{23}\ \ \gamma_{13}\ \ \gamma_{12}]^T \\ &= [\epsilon_1\ \ \epsilon_2\ \ \epsilon_3\ \ \epsilon_4\ \ \epsilon_5\ \ \epsilon_6]^T \end{aligned}$$

stress is related to strain by:

$$\sigma_i = C_{ij}\epsilon_j$$

this should never be done because it discards all of the useful mathematics associated with tensors; unfortunately, in the modern composites literature it is conventional to make this modification

Why Multiply the Shear Strains by Two?

we do this to maintain an accurate work increment per unit volume:

$$\begin{aligned} \mathrm{d}W &= \sigma_{11}\,\mathrm{d}\epsilon_{11} + \sigma_{22}\,\mathrm{d}\epsilon_{22} + \sigma_{33}\,\mathrm{d}\epsilon_{33} + \sigma_{12}\,\mathrm{d}\epsilon_{12} + \sigma_{21}\,\mathrm{d}\epsilon_{21} \\ &+ \sigma_{13}\,\mathrm{d}\epsilon_{13} + \sigma_{31}\,\mathrm{d}\epsilon_{31} + \sigma_{23}\,\mathrm{d}\epsilon_{23} + \sigma_{32}\,\mathrm{d}\epsilon_{32} \end{aligned}$$

the corresponding expression for $\mathrm{d}W$ in the composites notation is:

$$\mathrm{d}W = \sigma_1\,\mathrm{d}\epsilon_1 + \sigma_2\,\mathrm{d}\epsilon_2 + \sigma_3\,\mathrm{d}\epsilon_3 + \sigma_4\,\mathrm{d}\epsilon_4 + \sigma_5\,\mathrm{d}\epsilon_5 + \sigma_6\,\mathrm{d}\epsilon_6$$

and hence $\epsilon_4 = 2\epsilon_{23}$

we have to be very careful how we handle the shear stresses, shear strains and the elastic constants relating them

Strain Energy

for a linearly elastic solid, the strain energy density is given by:

$$W = \frac{1}{2}\sigma_i\epsilon_i = \frac{1}{2}C_{ij}\epsilon_i\epsilon_j = \frac{1}{2}S_{ij}\sigma_i\sigma_j$$

where S_{ij} is the compliance matrix, and $S_{ij} = C_{ij}^{-1}$; note that W is scalar

to get the total strain energy, the strain energy density must be integrated over the volume of interest

now:

$$\frac{\partial W}{\partial \epsilon_i} = C_{ij}\epsilon_j$$

and

$$\frac{\partial^2 W}{\partial \epsilon_i \partial \epsilon_j} = C_{ij}$$

we can reverse the order of the differentiation and find that the stiffness matrix must be symmetric; the same can be said for the compliance matrix

hence, if strain energy exists, the stiffness and compliance matrices must be symmetric

Compliance for an Isotropic Solid

we are now looking for the **elastic compliance matrix** S that connects the stresses and strains:

$$\epsilon = S\sigma$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

recall, from the theory of elasticity:

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}$$

or, in the composites notation:

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E}$$

because

$$\epsilon_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6$$

this means that we can identify some of the components of the elastic compliance matrix:

$$S_{11} = \frac{1}{E}; \quad S_{12} = -\frac{\nu}{E}; \quad S_{13} = -\frac{\nu}{E} \quad S_{14} = S_{15} = S_{16} = 0$$

Compliance for an Isotropic Solid

also:

$$\epsilon_4 = \frac{\sigma_4}{G}$$

this means that another component of the compliance matrix is:

$$S_{44} = \frac{1}{G}$$

while

$$S_{41} = S_{42} = S_{43} = S_{45} = S_{46} = 0$$

recall that, for an isotropic solid:

$$G = \frac{E}{2(1+\nu)}$$

so there are two independent elastic constants

Compliance for an Isotropic Solid

the overall form of the compliance matrix for an **isotropic solid** in this notation is:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

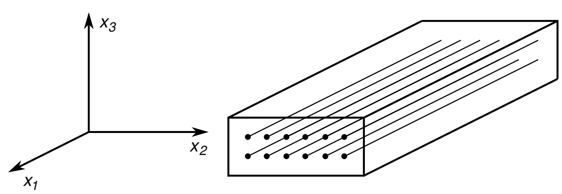
filling in the material constants:

$$
S_{ij} = \begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
$$

1.2.2 Orthotropic Solids

Compliance of Orthotropic Solids

however, the previous expressions were for **isotropic solids**; composites are **not** isotropic



typically, a composite lamina is a solid with three orthogonal axes of material symmetry, here shown oriented with the coordinate axes

this is what is called an **orthotropic** solid

the **forms** of the compliance and stiffness matrices are the same as for the isotropic case; how many elastic constants are needed?

Compliance Matrix for Orthogonal Laminae

the compliance matrix for a general orthotropic solid has the same non-zero components as the compliance matrix for an isotropic solid; however, more material constants are needed

$$
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}
$$

Unidirectional Laminae

composites are typically configured as stacks of laminae and each lamina is unidirectional, which means it has a constant fibre direction

for a unidirectional lamina, the fibre spacing is uniform in both transverse directions (approximately close-packed), so the response associated with the x_2 direction is the same as the response corresponding to the x_3 direction

the consequences of that are:

$$
\begin{aligned}
E_2 &= E_3 \\
G_{12} &= G_{13} \\
\nu_{12} &= \nu_{13} \\
\nu_{21} &= \nu_{31} \\
\nu_{23} &= \nu_{32} \\
G_{23} &= \frac{E_2}{2(1+\nu_{32})}
\end{aligned}
$$

Laminae in Plane Stress

in laminate plate theory, the through-thickness stresses are neglected and each lamina is considered to exist in a state of plane stress; we assume:

$$
\sigma_{33} = \sigma_{13} = \sigma_{23} = 0
$$

by neglecting the corresponding strain components ϵ_{33}, γ_{13} and γ_{23}, this simplifies the compliance relation to:

$$
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}
$$

where:

$$
S_{11} = \frac{1}{E_1}
$$

$$S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2}$$

$$S_{66} = \frac{1}{G_{12}}$$

Stiffness Matrix in Plane Stress

invert the compliance matrix to get the stiffness matrix:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

where:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

Example

consider T300/934 carbon-epoxy lamina with the properties: thickness $t = 0.125$ mm; fibre diameter $d = 6$ µm; fibre volume fraction $v_f = 0.65$

measured properties: $E_1 = 131$ GPa; $E_2 = 10.3$ GPa; $G_{12} = 6.0$ GPa; $\nu_{12} = 0.22$

the stiffness matrix is:

$$Q = \begin{bmatrix} 132 & 2.3 & 0 \\ 2.3 & 10.3 & 0 \\ 0 & 0 & 6.0 \end{bmatrix} \text{GPa}$$

what is the stress state if $\epsilon_{11} = 0.001$ and all other strains are zero?

what is the strain state if $\sigma_{11} = 100$ MPa and all other stresses are zero?

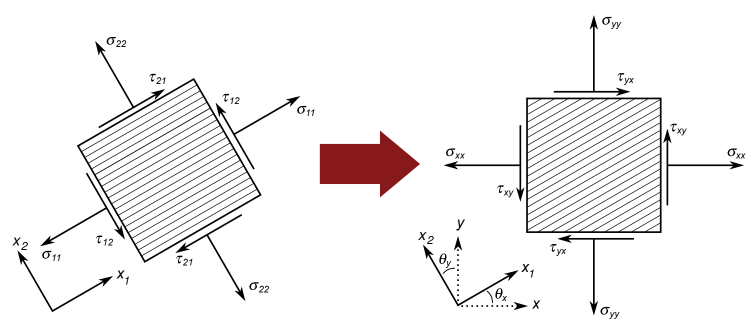
Algorithm: Calculating the Stiffness Matrix for a Unidirectional Lamina

1. collect all necessary material properties: E_1, E_2, ν_{12}, ν_{21}, G_{12}
2. calculate or estimate any properties that are missing
3. determine the compliance terms S_{11}, S_{22}, S_{12}, S_{66} using the equations on page 29
4. invert the resulting matrix to get the stiffness terms Q_{11}, Q_{22}, Q_{12}, Q_{66} using the equations on page 30

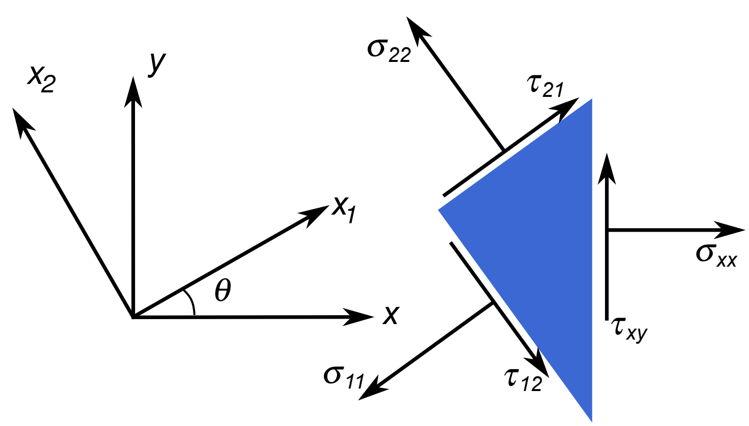
1.2.3 Rotation of Axes

Rotation of Axes

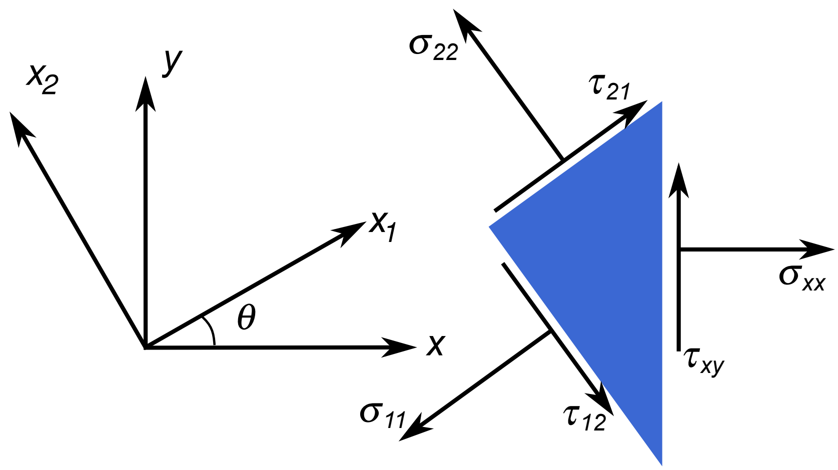
at present we can form the expressions for the elastic properties of a lamina oriented with the principal material axes (x_1, x_2); how do we transform these into the expressions for the elastic properties in global axes (x, y)?



Rotation of Axes



Rotation of Axes



determine the equilibrium of the infinitesimal triangle:

$$\begin{aligned}
\sigma_{xx}\,\mathrm{d}A &= \sigma_{11}\cos^2\theta\,\mathrm{d}A - \tau_{12}\sin\theta\cos\theta\,\mathrm{d}A \\
&\quad +\sigma_{22}\sin^2\theta\,\mathrm{d}A - \tau_{21}\cos\theta\sin\theta\,\mathrm{d}A
\end{aligned}$$

$$\begin{aligned}
\tau_{xy}\,\mathrm{d}A &= \sigma_{11}\sin\theta\cos\theta\,\mathrm{d}A + \tau_{12}\cos^2\theta\,\mathrm{d}A \\
&\quad -\sigma_{22}\cos\theta\sin\theta\,\mathrm{d}A - \tau_{21}\sin^2\theta\,\mathrm{d}A
\end{aligned}$$

simlarly for σ_{yy} and τ_{yx}

note that $\mathrm{d}A_1 = \mathrm{d}A\cos\theta$ and $\mathrm{d}A_2 = \mathrm{d}A\sin\theta$

Rotation of Axes

we can collect all of the terms into a matrix equation:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 2\cos\theta\sin\theta \\ \cos\theta\sin\theta & -\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix}$$

if we invert this relation, we get:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

we can write this as:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

Rotation of Axes

strains transform in the same manner:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12}/2 \end{pmatrix} = [T] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy}/2 \end{pmatrix}$$

transform this:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & 2\left(\cos^2\theta - \sin^2\theta\right) \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy}/2 \end{pmatrix}$$

and another transformation:

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Rotation of Axes

it turns out that:

$$\begin{bmatrix} \cos^2\theta & \sin^2\theta & \cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \left[T^T\right]^{-1}$$

hence we can invert the previous relation and get:

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \left[T^T\right] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

in full:

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & -\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & \cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

Rotation of Axes

what we are looking for is the relation between $\left(\sigma_{xx} \;\; \sigma_{yy} \;\; \tau_{xy}\right)^T$ and $\left(\epsilon_{xx} \;\; \epsilon_{yy} \;\; \gamma_{xy}\right)^T$; that is, we want to find $\left[\overline{Q}\right]$ where $(\sigma) = \left[\overline{Q}\right](\epsilon)$

recall that:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = [Q] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

then:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = [T]^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T]^{-1}[Q] \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = [T]^{-1}[Q]\left[T^T\right]^{-1} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

Rotation of Axes

that is the relation we want; we write it as:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \left[\overline{Q}\right] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix}$$

where

$$\left[\overline{Q}\right] = [T]^{-1}[Q]\left[T^T\right]^{-1}$$

$$\left[\overline{Q}\right] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}$$

note that $\overline{Q}_{21} = \overline{Q}_{12}$, $\overline{Q}_{62} = \overline{Q}_{26}$ and $\overline{Q}_{61} = \overline{Q}_{16}$: the matrix is symmetric

Rotation of Axes

the components of $\left[\overline{Q}\right]$ are given explicitly by:

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta \\
\overline{Q}_{12} &= \left(Q_{11} + Q_{22} - 4Q_{66}\right)\cos^2\theta\sin^2\theta + Q_{12}\left(\cos^4\theta + \sin^4\theta\right) \\
\overline{Q}_{22} &= Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2\left(Q_{12} + 2Q_{66}\right)\sin^2\theta\cos^2\theta \\
\overline{Q}_{16} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos^3\theta\sin\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta \\
\overline{Q}_{26} &= \left(Q_{11} - Q_{12} - 2Q_{66}\right)\cos\theta\sin^3\theta - \left(Q_{22} - Q_{12} - 2Q_{66}\right)\cos^3\theta\sin\theta \\
\overline{Q}_{66} &= \left(Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}\right)\cos^2\theta\sin^2\theta + Q_{66}\left(\cos^4\theta + \sin^4\theta\right)
\end{aligned}$$

note that this is a full matrix: this means that there is coupling between normal strains and shear stresses (and shear strains and normal stresses); uncoupling of direct and shear straining occurs only when the loading is aligned with the principal material directions

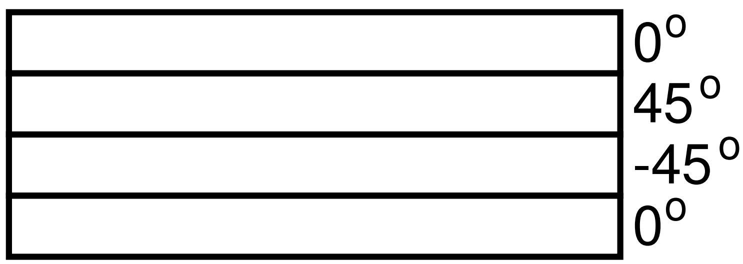
Algorithm: Calculating the Rotated Stiffness Matrix for a Unidirectional Lamina

1. find the stiffness matrix for the lamina in the material coordinate system
2. determine the angle θ by which the global coordinate system is rotated from the material system
3. use the equations on page 41 to determine the $\bar{Q}_{ij}$ components of the stiffness matrix

1.2.4 Laminate Plate Theory

Laminates

at present, composites are usually produced by stacking layers of fabric / prepreg to produce laminates



the laminate to the left is produced by stacking four layers of unidirectional prepreg in the order 0°, -45°, 45°, 0°

we use the conventional notation [0/-45/45/0] to identify this composite layup

note that this is from bottom to top: the first ply is the one closest to the tooling

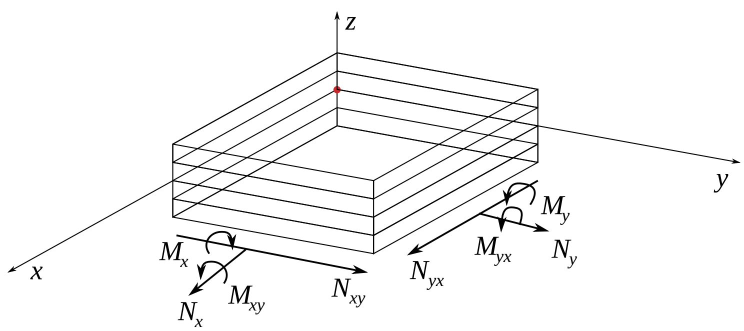
Laminate Notation

there is a standard convention for the identification of composite laminate configurations

1. a laminate description is enclosed by square brackets
2. the angles of the plies are separated by forward slashes
3. symmetry is indicated by a subscripted 's'
4. a bilayer consisting of a +45° ply and a -45° ply is often identified as [±45°]
5. the central ply of a symmetric laminate with an odd number of plies is indicated by an overbar
6. angles are always between +90 and -90

two more common terms: an 'angle-ply' laminate has $\pm\theta$ plies ([30 / -30] or [45 / -45]) and a 'cross-ply' laminate has 0° and 90° plies ([0 / 90]$_s$ or [0 / 90 / 0 / 90])

Laminate Plate Theory



the laminate is assumed to have its middle surface on the (x, y) plane

the displacements at a point (x, y, z) are (u, v, w)

Laminate Plate Theory

there are several important assumptions:

1. the plate consists of orthotropic laminae bonded together, with the principal material directions of the laminae oriented arbitrarily with respect to the (x, y) axes
2. each ply is linearly elastic
3. the thickness of the plate, t, is small compared to the other dimensions of the plate
4. the plate thickness t is constant
5. the displacements u and v are small compared to the plate thickness
6. tangential displacements u and v are are linear functions of the through-thickness z coordinate
7. the in-plane strains ϵ_x, ϵ_y and γ_{xy} are small
8. the transverse normal strain ϵ_z is neglected
9. transverse shear strains γ_{xz} and γ_{yz} are zero
10. the transverse shear stresses τ_{xz} and τ_{yz} vanish on the plate surfaces defined by $z = \pm\frac{t}{2}$

Laminate Plate Theory

identify the displacements and strains on the mid-plane of the laminate with superscripts: $u^\circ, v^\circ, w^\circ, \epsilon_x^\circ, \epsilon_y^\circ, \gamma_{xy}^\circ$

as a consequence, the displacements at points away from the mid-plane can be expressed as:

$$u(x, y, z) = u^\circ(x, y) + zF_1(x, y)$$

$$v(x, y, z) = v^\circ(x, y) + zF_2(x, y)$$

$$w(x, y, z) = w^\circ(x, y)$$

these are consequences of assumptions 4, 6 and 8

Laminate Plate Theory

differentiate the expressions for u and v with respect to z to get $\partial u/\partial z = F_1$ and $\partial v/\partial z = F_2$

substitute these into the definitions of transverse shear strain, recalling that they have been defined by assumption to be zero:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = F_1(x, y) + \frac{\partial w}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = F_2(x, y) + \frac{\partial w}{\partial y} = 0$$

consequently it must be the case that:

$$F_1(x, y) = -\frac{\partial w}{\partial x} \qquad \text{and} \qquad F_2(x, y) = -\frac{\partial w}{\partial y}$$

and

$$u = u^\circ - z\frac{\partial w}{\partial x} \qquad \text{and} \qquad v = v^\circ - z\frac{\partial w}{\partial y}$$

Laminate Plate Theory

substituting these results into the definitions of in-plane strains and differentiating gives:

$$\epsilon_x = \frac{\partial u}{\partial x} = \epsilon_x^\circ + z\kappa_x$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \epsilon_y^\circ + z\kappa_y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy}^\circ + z\kappa_{xy}$$

the strains on the mid-plane are:

$$\epsilon_x^\circ = \frac{\partial u^\circ}{\partial x} \quad \epsilon_y^\circ = \frac{\partial v^\circ}{\partial y} \quad \gamma_{xy}^\circ = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x}$$

and the curvatures of the mid-plane are:

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2} \quad \kappa_{xy} = -2\frac{\partial^2 w}{\partial x \partial y}$$

Laminate Plate Theory

these give the strains for any arbitrary distance z from the mid-plane; we can find the stresses in the kth lamina by substituting into the lamina stress-strain relations:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{pmatrix} \epsilon_x^\circ + z\kappa_x \\ \epsilon_y^\circ + z\kappa_y \\ \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}$$

it is convenient to use forces and moments per unit width instead of stresses, particularly when the component has relatively simple geometry

force per unit width is:

$$N_i = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_i \, \mathrm{d}z = \sum_{k=1}^{N} \left(\int_{z_{k-1}}^{z_k} (\sigma_i)_k \, \mathrm{d}z \right)$$

and moment per unit width is:

$$M_i = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_i z \, \mathrm{d}z = \sum_{k=1}^{N} \left(\int_{z_{k-1}}^{z_k} (\sigma_i)_k z \, \mathrm{d}z \right)$$

with z_k the z-coordinate of the top surface of lamina k and z_{k-1} the z-coordinate of the bottom surface of lamina k

Laminate Plate Theory

when we substitute the lamina stress-strain relations into the integrals, we get:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

the extensional stiffnesses are:

$$A_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k \, \mathrm{d}z = \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k (z_k - z_{k-1}) \right)$$

and the coupling stiffnesses are:

$$B_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z \, \mathrm{d}z = \frac{1}{2} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right) \right)$$

and the bending stiffnesses are:

$$D_{ij} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \left(\overline{Q}_{ij}\right)_k z^2 \, \mathrm{d}z = \frac{1}{3} \sum_{k=1}^{N} \left(\left(\overline{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right) \right)$$

Laminate Plate Theory

we often write this equation in partitioned form:

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{pmatrix} \epsilon^{\circ} \\ \kappa \end{pmatrix}$$

the extensional terms A relate the in-plane forces to the in-plane strains; the coupling terms B relate the the in-plane forces to bending and twisting, and the moments to in-plane strains; the bending terms D relate the moments and torques to the curvatures

NB: if a laminate is symmetric around the mid-plane, the B terms are all zero - no coupling

Algorithm: Calculating the Stiffness Matrix for a Laminate

1. calculate the $\bar{Q}_{ij}$ matrix for all N lamina in the laminate in a single global coordinate system
2. determine the locations z of the top and bottom of each lamina; z_k is at the top of the lamina and z_{k-1} is at the bottom (distances measured from the mid-plane)
3. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the layer thickness for all layers; this provides A_{ij} as shown on page 51
4. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the difference between the square of the distances to the top and bottom surfaces of the lamina; this provides B_{ij} (if the laminate is symmetric, the B_{ij} terms are all zero)
5. for N layers, calculate the sum of the $\bar{Q}_{ij}$ terms times the difference between the cube of the distances to the top and bottom surfaces of the lamina; this provides D_{ij}

Laminate Plate Theory - Calculating Stresses

the equation that we want to solve for the laminate is:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_x^\circ \\ \epsilon_y^\circ \\ \gamma_{xy}^\circ \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

typically we know the vector of applied forces:

$$\begin{pmatrix} N_x & N_y & N_{xy} & M_x & M_y & M_{xy} \end{pmatrix}^T$$

and solve for the strains:

$$\begin{pmatrix} \epsilon_x^\circ & \epsilon_y^\circ & \gamma_{xy}^\circ & \kappa_x & \kappa_y & \kappa_{xy} \end{pmatrix}^T$$

Laminate Plate Theory - Calculating Stresses

because we know the vector of laminate strains, we can assemble the vector of strains in each lamina, knowing the distance z of the individual laminae from the plate mid-plane:

$$\begin{pmatrix} \epsilon_x^\circ + z\kappa_x & \epsilon_y^\circ + z\kappa_y & \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}^T$$

here, z is usually taken to be the position of the mid-plane of the individual lamina

once we have this vector, we can use it with the stress-strain relation for lamina k:

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{61} & \overline{Q}_{62} & \overline{Q}_{66} \end{bmatrix}_k \begin{pmatrix} \epsilon_x^\circ + z\kappa_x \\ \epsilon_y^\circ + z\kappa_y \\ \gamma_{xy}^\circ + z\kappa_{xy} \end{pmatrix}$$

this gives us the vector of stresses in each lamina, which we can use to get the lamina stresses in the material coordinate system:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = [T] \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

failure analysis compares this stress state to some failure criterion

Algorithm: Calculating Stresses

1. using the laminate stiffness matrix and the vector of applied forces and moments, determine the vector of mid-plane strains and curvatures

2. for each lamina in the laminate, calculate the vector of strains for that lamina given by the equation on page 55

3. determine the vector of stresses in each lamina using the $\bar{Q}_{ij}$ matrix for that lamina and the vector of lamina strains; this generates stresses in the global coordinate system

4. for each lamina, convert the stresses in the global coordinate system to stresses in the material coordinate system using the vector of global stresses and the $[T]$ matrix for the appropriate lamina

1.3 Strength of Composites

Models of Composite Strength

we will look at two types of models for composite strength:

Engineering models: robust and simple enough to be used in everyday engineering calculations, but can be inaccurate if used in the wrong situation and therefore have to be supported by careful testing, both numerically and experimentally

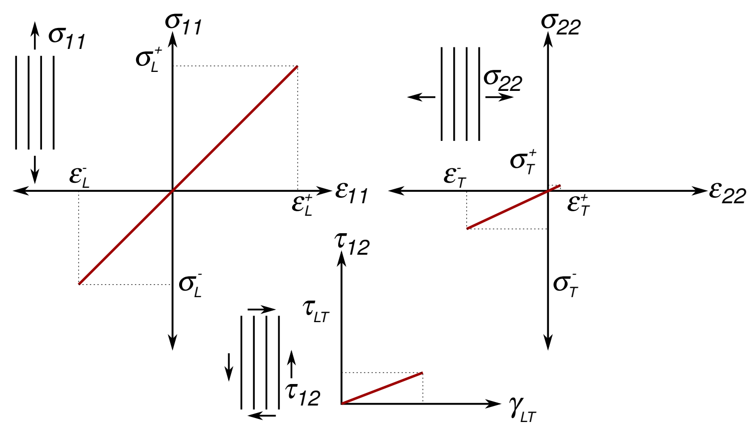
Micromechanical models: capture the essential physical mechanism for a particular failure mode and should be accurate for a range of materials and loading conditions, but are too complex for everyday use

for the most part, we are going to concentrate on engineering models in this course

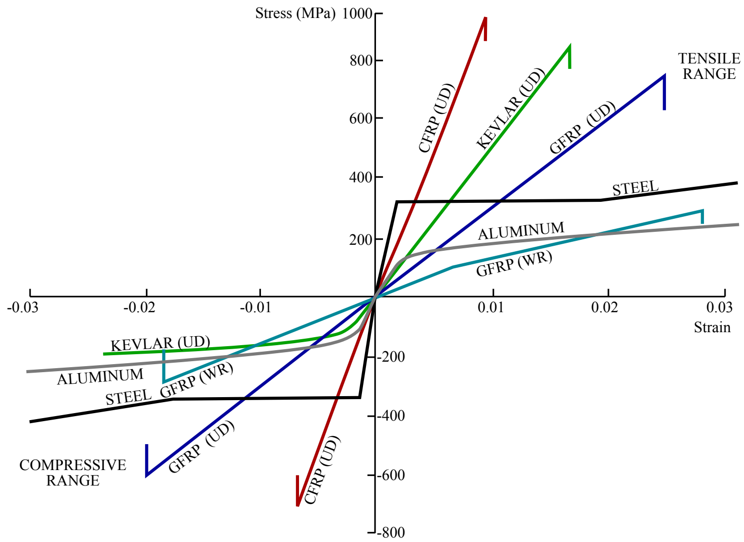
Stress-based Lamina Failure

consider the in-plane loading of a unidirectional composite lamina; we assume that we know the resolved stresses and strains in terms of the material axes of the lamina

this may be the result of simple calculations or of a large finite element simulation



Typical Stress-Strain Curves



Typical Failure Stress Data

Material	Longitudinal tension σ_L^+ (MPa)	Longitudinal compression σ_L^- (MPa)	Transverse tension σ_T^+ (MPa)	Transverse compression σ_T^- (MPa)	In-plane shear τ_{LT} (MPa)
Al 7075-T6	450	450	450	450	225
Glass - polyester	650-950	600-900	20-25	90-120	45-60
Carbon - epoxy	850-1500	700-1200	35-40	130-190	60-75
Kevlar - epoxy	1100-1250	240-290	20-30	110-140	40-60

note that while strengths are normally given as positive quantities, compressive strengths refers to negative stresses

for example, for Al 7075-T6, provided -450 MPa $\leq \sigma_{11} \leq 450$ MPa, the material will not yield in tension or compression

Typical Failure Strain Data

Material	Longitudinal tension ϵ_L^+	Longitudinal compression ϵ_L^-	Transverse tension ϵ_T^+	Transverse compression ϵ_T^-	In-plane shear γ_{LT}
Aluminum alloy	15%		15%		100%
Glass - polyester	2.5%	2%	0.5%	3%	10%
Carbon - epoxy	1-2%	1-2%	0.5%	2-3%	10%
Kevlar - epoxy	1.5%	2.5%	0.5%	1%	10%

1.3.1 Engineering Models

Standard Criteria for Composite Failure

the most common approach is to use laminate plate theory or finite element calculations to determine the stress state in a laminate, and from that analysis determine the stress state in all of the constituent laminae

empirical failure criteria are used to decide when laminate failure occurs; typically this is after failure of the first ply

this means that the stresses or strains in each ply are compared to **measured** material properties to determine whether failure has occurred

the are three main types of failure criteria:

1. maximum stress
2. maximum strain
3. quadratic / interaction

Maximum Stress Failure Criterion

the stresses in the lamina are compared to maximum allowable stresses

the stresses are resolved into the material coordinate system so that σ_{11} aligns with the fibre direction

failure will not occur provided the following conditions are met:

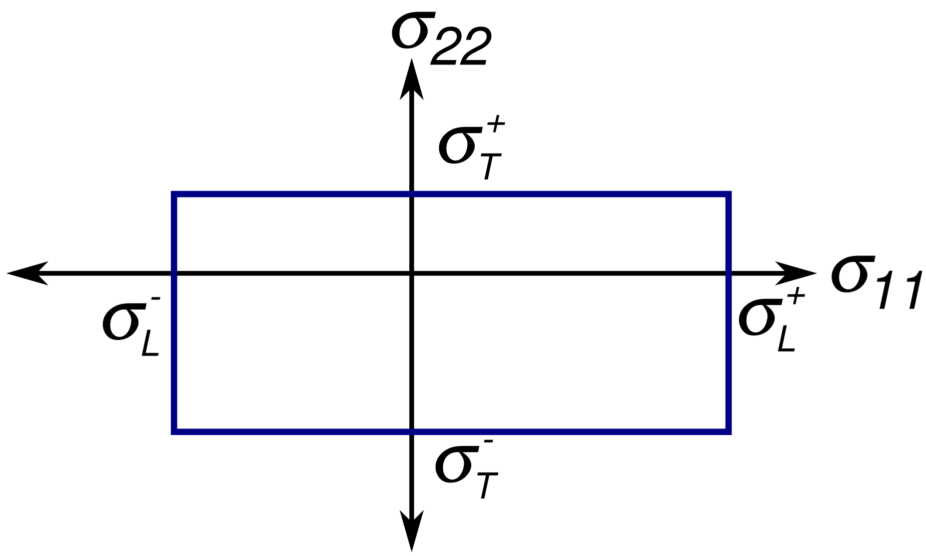
$$\sigma_L^- < \sigma_{11} < \sigma_L^+$$

$$\sigma_T^- < \sigma_{22} < \sigma_T^+$$

$$|\tau_{12}| < \tau_{LT}$$

this requires that all of the stresses are between the limits set by the strengths of the material

graphically, consider a **stress space** with axes σ_{11} and σ_{22}



this is a section of the failure surface for a maximum stress failure criterion; τ_{12} would lie in a third dimension

the stress point $(\sigma_{11}, \sigma_{22})$ must lie within the region set by σ_L and σ_T for the material to survive

Maximum Strain Failure Criterion

next consider the maximum allowable strain in the lamina

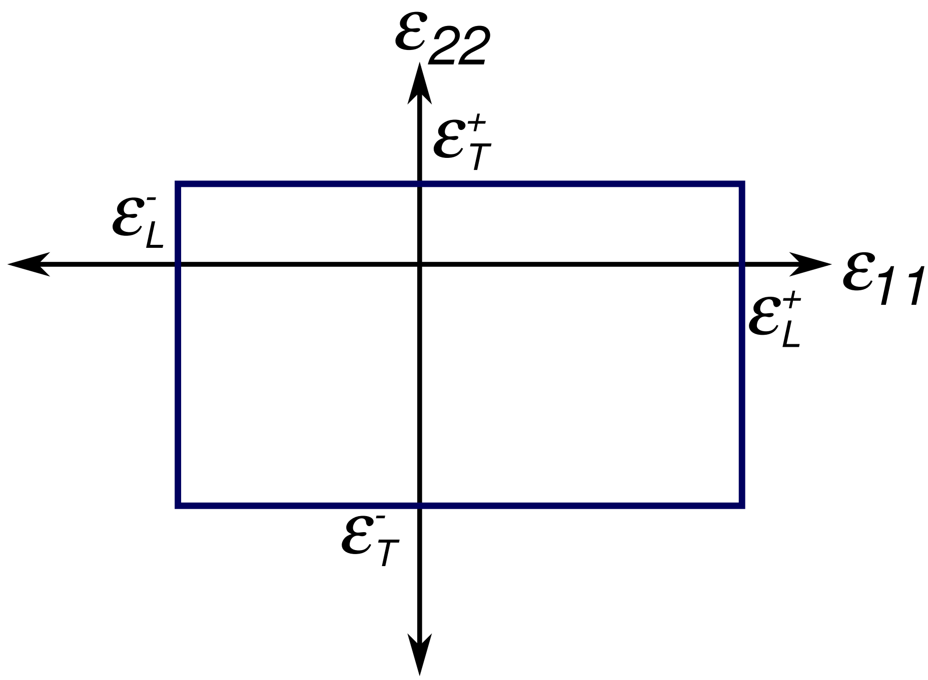
the strains are resolved such that ϵ_{11} aligns with the fibre direction

failure will not occur provided the following conditions are met:

$$\epsilon_L^- < \epsilon_{11} < \epsilon_L^+$$

$$\epsilon_T^- < \epsilon_{22} < \epsilon_T^+$$

$$|\gamma_{12}| < \gamma_{LT}$$



this is a section of the failure surface for a maximum strain failure criterion

Quadratic Failure Criterion - Tsai-Hill

failure criteria that are quadratic in the stresses allow for interaction between the stress components

consider plane stress with $\sigma_{33} = \tau_{13} = \tau_{23} = 0$

for homogeneous isotropic ductile solids, the von Mises criterion predicts yielding when:

$$\sigma_{11}^2 + \sigma_{22}^2 + (\sigma_{22} - \sigma_{11})^2 + 6\tau_{12}^2 \geq 2Y^2$$

where Y is the uniaxial yield strength

for **anisotropic** solids, the equivalent is the Tsai-Hill criterion:

$$\frac{\sigma_{11}^2}{\sigma_L^2} - \frac{\sigma_{11}\sigma_{22}}{\sigma_L^2} + \frac{\sigma_{22}^2}{\sigma_T^2} + \frac{\tau_{12}^2}{\tau_{LT}^2} \geq 1$$

where σ_L and σ_T are the strengths in either tension or compression, as appropriate for the respective signs of σ_{11} and σ_{22}

note that composite materials **do not yield**

Quadratic Failure Criterion - Tsai-Wu

the Tsai-Wu failure criterion uses a more general formulation for interaction between the various components of stresses

the most important terms of the relevant expression are:

$$F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{66}\tau_{12}^2 + F_1\sigma_{11} + F_2\sigma_{22} + 2F_{12}\sigma_{11}\sigma_{22} \geq 1$$

where:

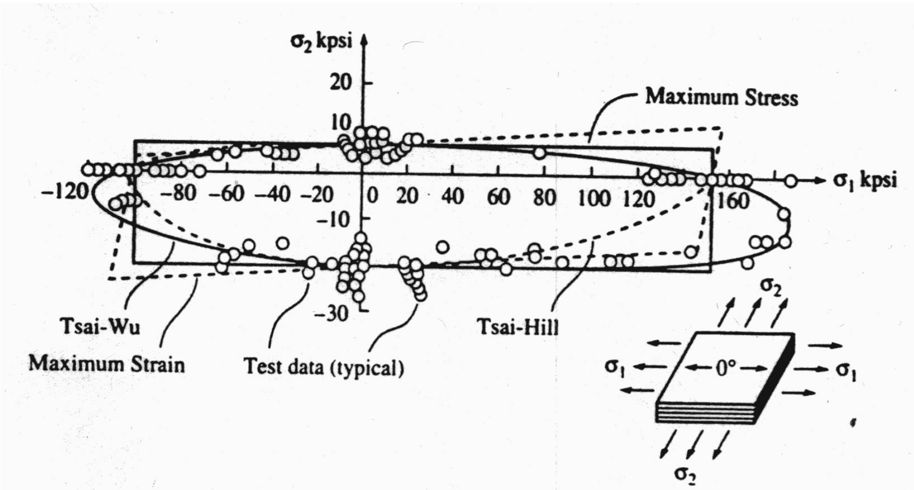
$$F_{11} = \frac{1}{|\sigma_L^+\sigma_L^-|} \qquad F_{22} = \frac{1}{|\sigma_T^+\sigma_T^-|}$$

$$F_1 = \frac{1}{\sigma_L^+} - \frac{1}{|\sigma_L^-|} \qquad F_2 = \frac{1}{\sigma_T^+} - \frac{1}{|\sigma_T^-|} \qquad F_{66} = \frac{1}{\tau_{LT}^2}$$

the interaction term should be optimised using experimental data, but is usually approximated as:

$$F_{12} \approx -\frac{(F_{11}F_{22})^{\frac{1}{2}}}{2}$$

Comparison of Failure Surfaces



Algorithm: Applying Failure Criteria

1. calculate the vector of stresses (or strains if needed) in the material coordinate system for the ply or plies of interest
2. select an appropriate failure criterion; for most composite design, a quadratic criterion is most likely to be used
3. identify the relevant parameters associated with the failure criterion
4. evaluate the equations associated with the chosen criterion

Failure as a Function of Ply Angle

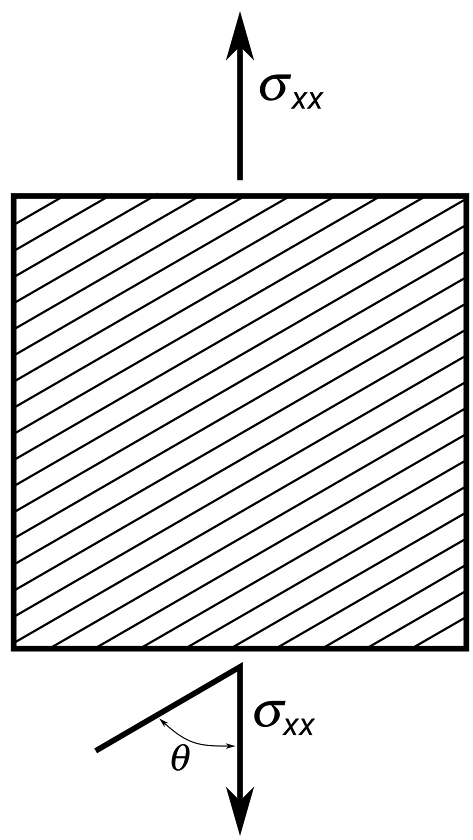
the maximum axial stress that can be applied to a ply is a strong function of the angle of the ply

here, the plies make angle θ with the loading direction

how does the strength of the ply vary with the angle of the ply?

resolve the stresses for each angle from 0° to 90° (after which the result is periodic)

apply a maximum stress or quadratic failure criterion to find the maximum σ_{xx} that can be applied before failure occurs



1.3.2 Micromechanical Models

Micromechanical Models

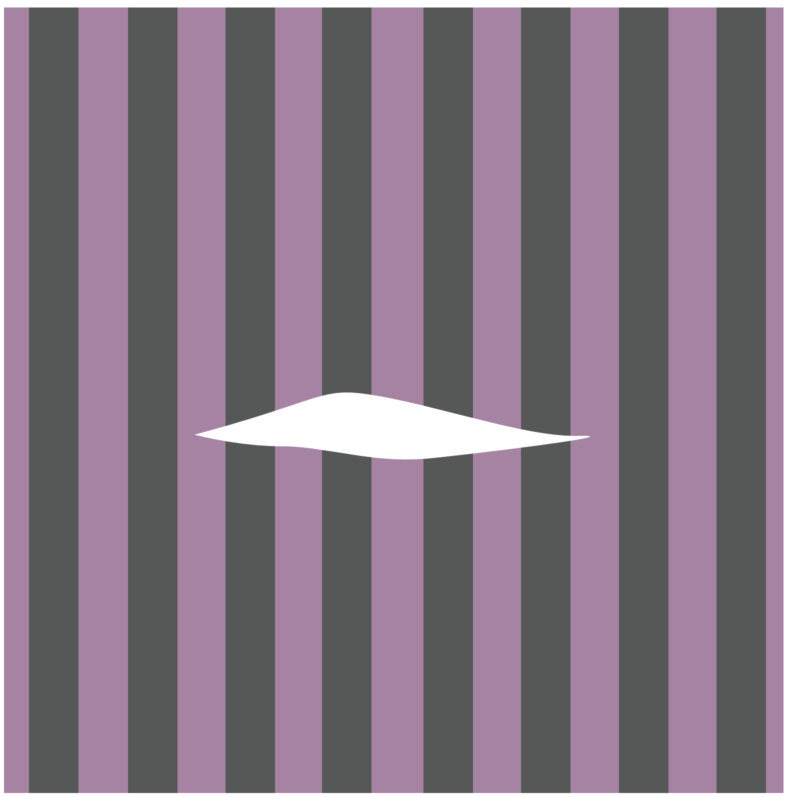
micromechanical models depend upon the accurate description of the material and geometry of a composite structure or specimen typically they involve far more effort than the application of empirical failure models (which tend to be highly phenomenological)

because micromechanical models accurately capture the physics of the behaviour of composites, they provide predictions of the strength of composite when strength is governed by a particular failure mechanism, even outside the range of experimental data

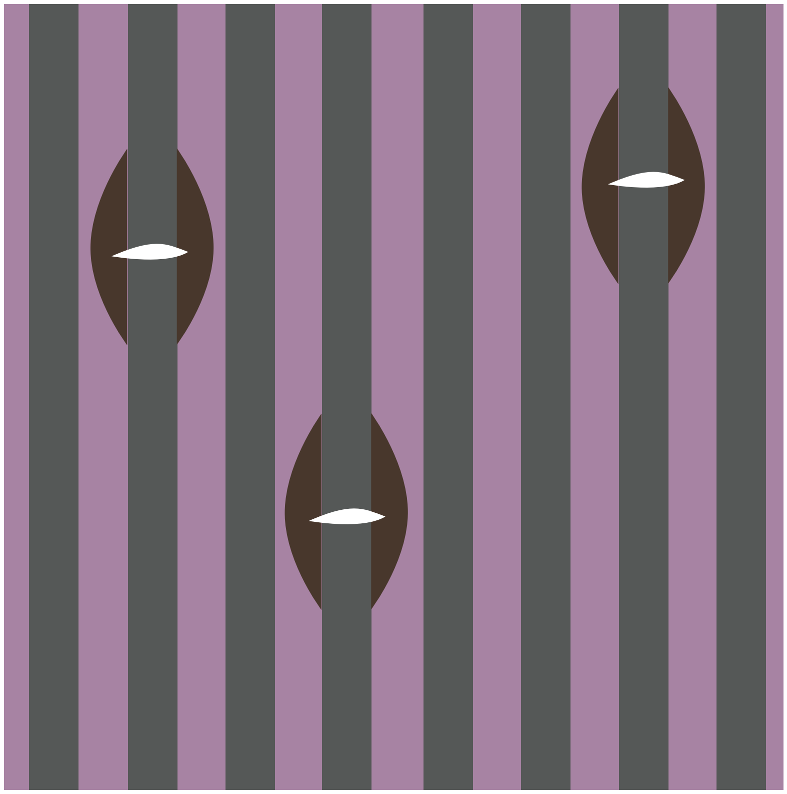
Micromechanisms of Failure - Fibre Failure

composite failure is governed by either fibre or matrix failure

stress concentrations due to fibre failure may lead to a cascading failure, where nearby fibres also break



alternately, shear lag zones enable fibres to pick up axial stress again some distance from the fibre break; the fibre breaks remain isolated

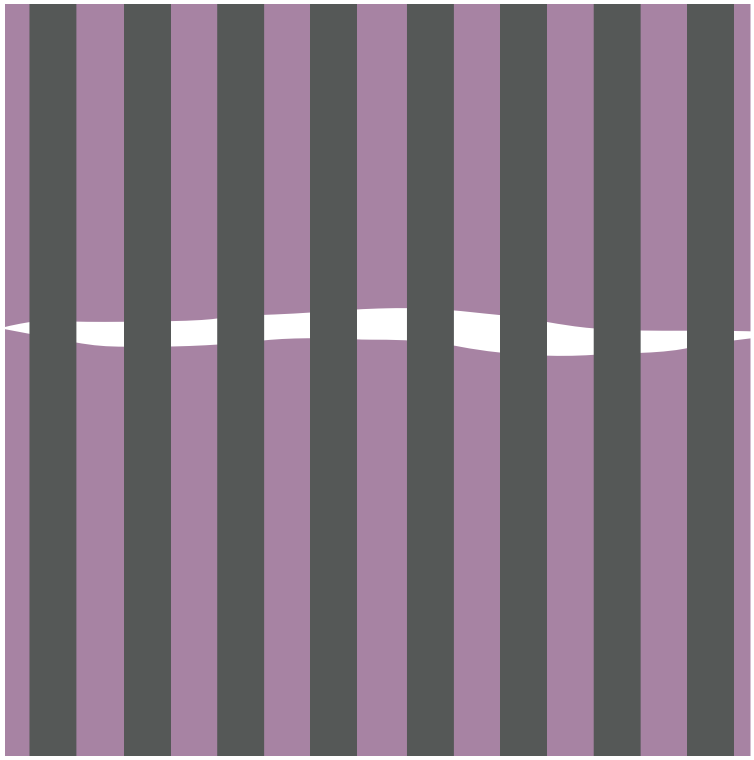


Micromechanisms of Failure - Matrix Failure

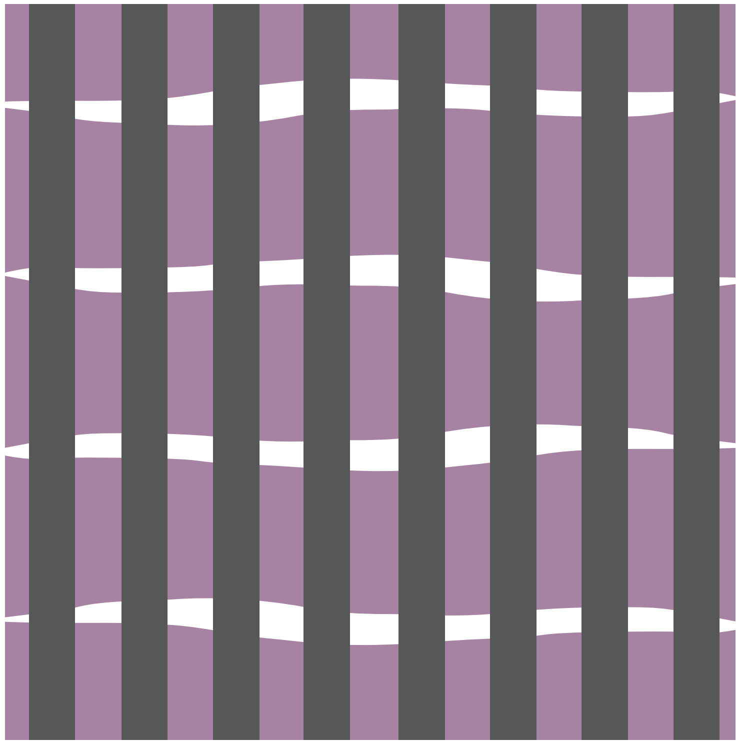
matrix failure occurs when the strain to failure of the matrix is less than half that of the fibres

fibres may bridge the matrix cracks

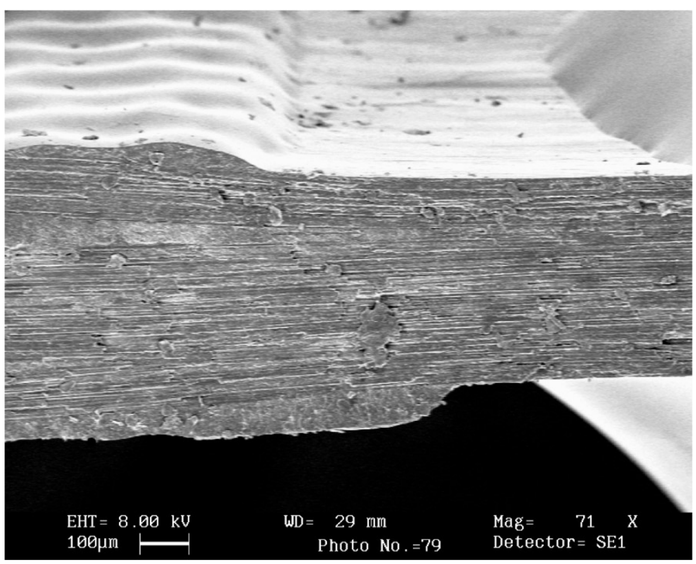
this is usually more common in ceramic matrix composites



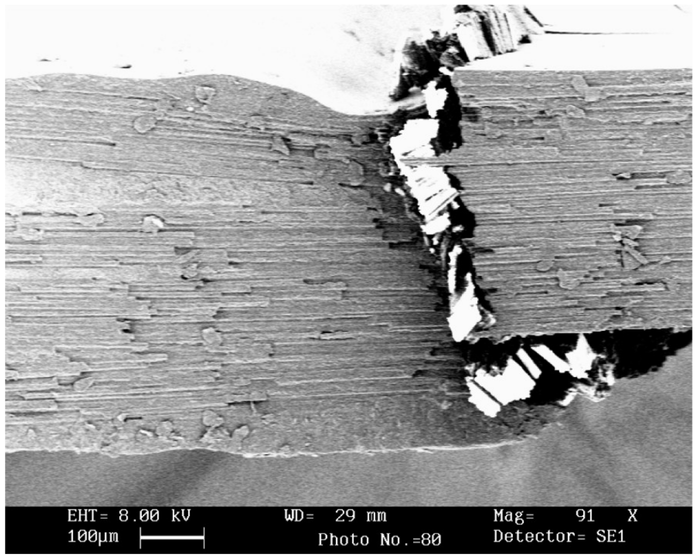
it is possible that multiple matrix fractures will occur; the distance between the fractures has a characteristic length scale depending upon the material properties of the fibre and matrix



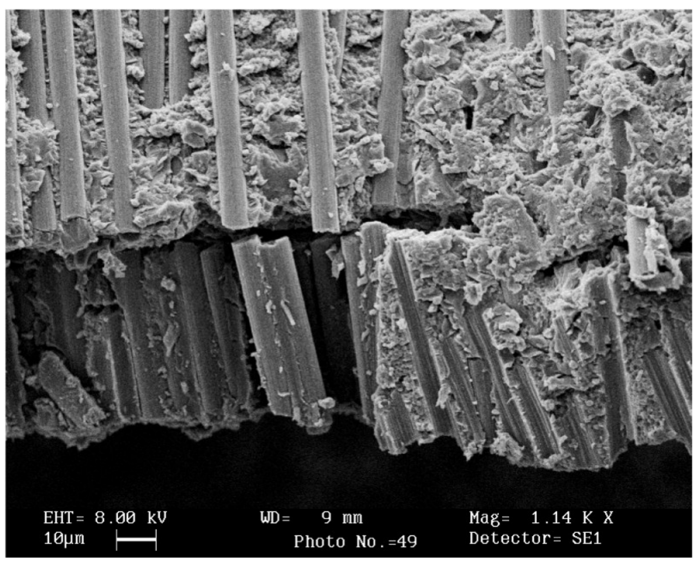
Microbuckling



Microbuckling



Microbuckling



Microbuckling

microbuckling is the standard mechanism of failure for long-fibre composites in compression

microbuckling is driven by an interaction between the misalignment of the primary load-bearing fibres and the shear failure of the matrix material when the matrix material shears plastically, kink bands of a characteristic length form in the fibres

the microbuckling strength of a composite can be estimated by:

$$\sigma_M = \frac{\tau_Y}{\phi_0}$$

see Fleck, *Advances in Applied Mechanics*, 1997, for comprehensive detail

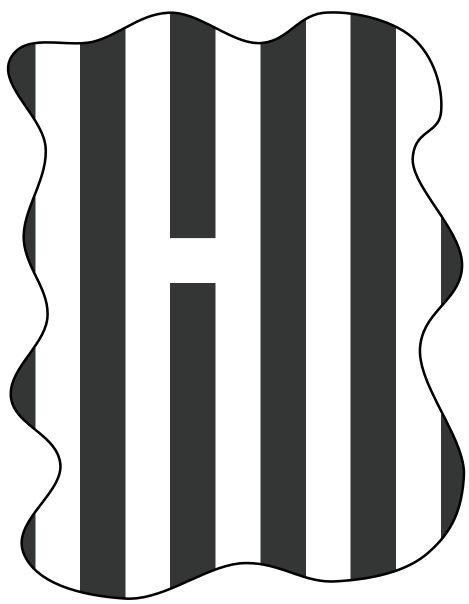
Shear Lag

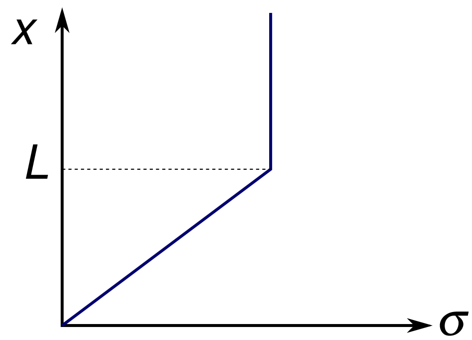
tensile failure of a composite can be associated with sudden, catastrophic large-scale damage, or with progressive fibre failure

in the second case, a micromechanical model of shear lag in fibres is useful

if there is a fibre break within the composite, the tensile stress on the end of the fibre must be zero

the stress in the fibre builds up as shear is transferred from the matrix to the fibre



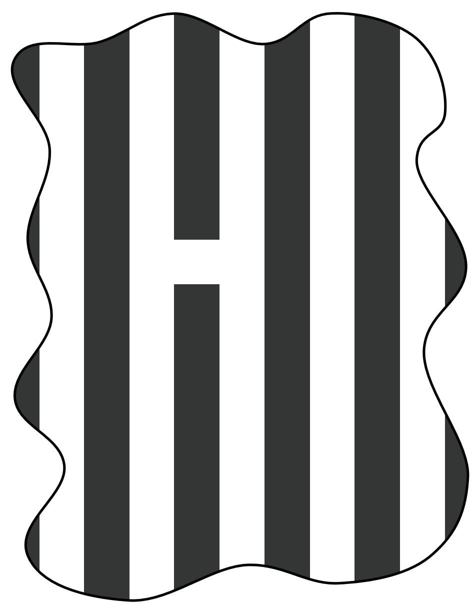


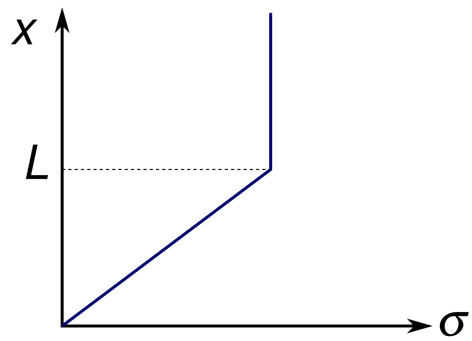
Shear Lag

Aveston, Cooper and Kelly developed a micromechanical model for shear lag: see Aveston and Kelly, *Journal of Material Science*, 1973

they found that the length of the shear lag zone is given by:

$$L = \frac{d\sigma_1 E_f}{4\tau_f E_1}$$





1.4 Sandwich Panels

Sandwich Panels and Beams

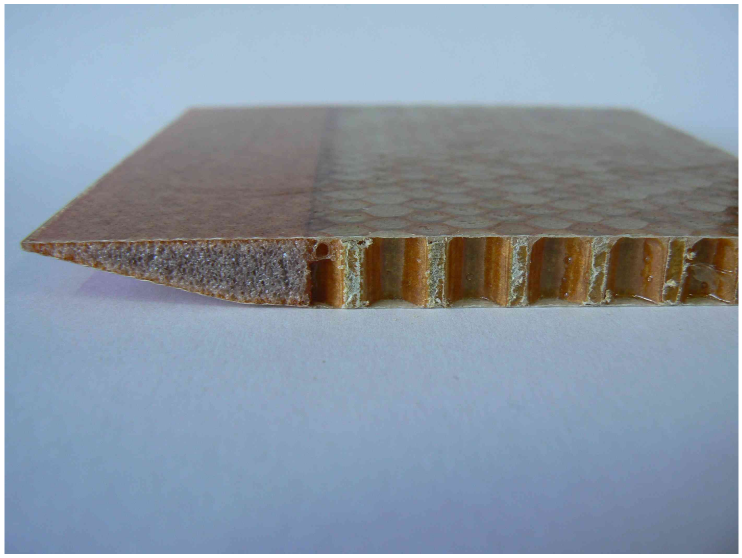
in aerospace engineering (and in many other areas) it is common to use composites in the form of sandwich structures, panels and beams

sandwich structures are composed of two strong, stiff face sheets separated by a lightweight core, which comes in a variety of configurations

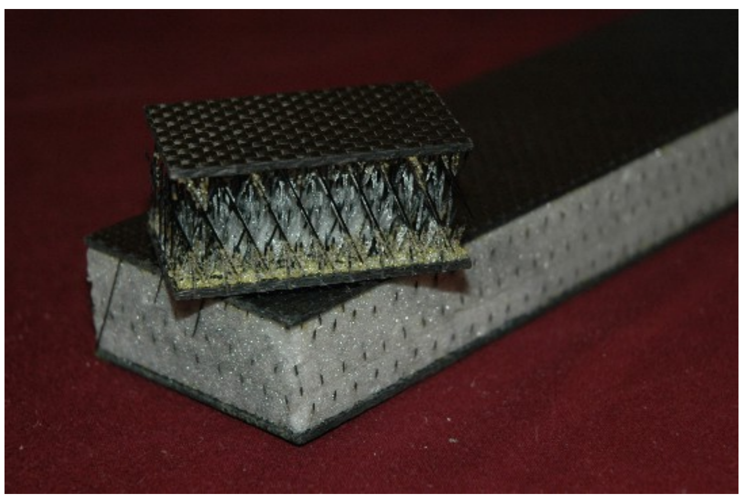
the purpose of this geometry is that the sandwich structure has a much higher ratio of bending stiffness to mass than do monolithic plates and beams; a sandwich is analogous to an I-beam

Polymer Foam Core Sandwiches

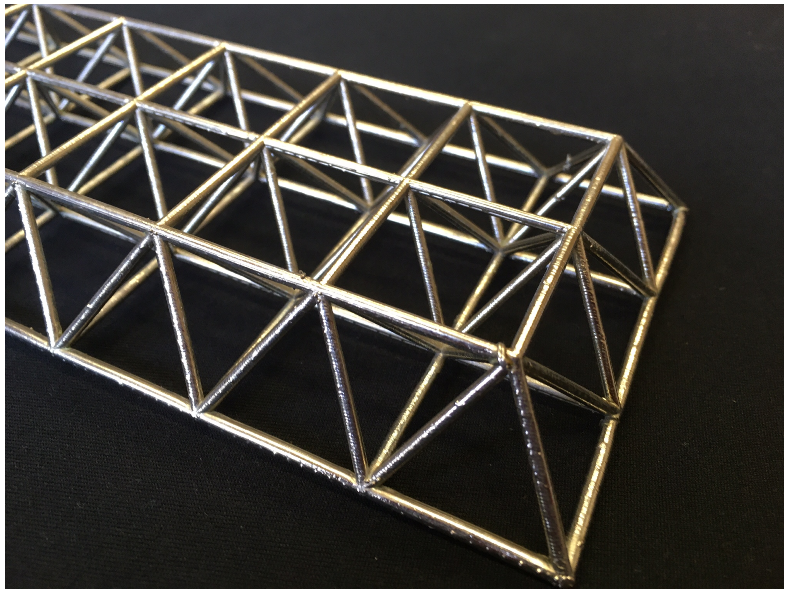
Honeycomb Core Sandwiches



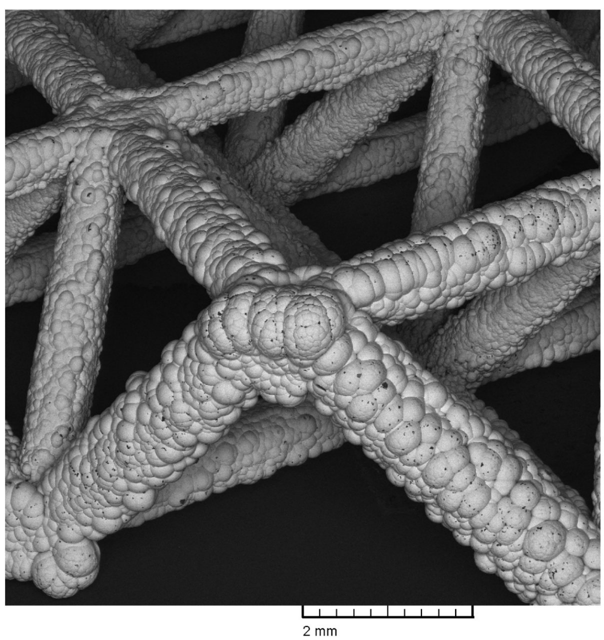
Truss Core Sandwiches



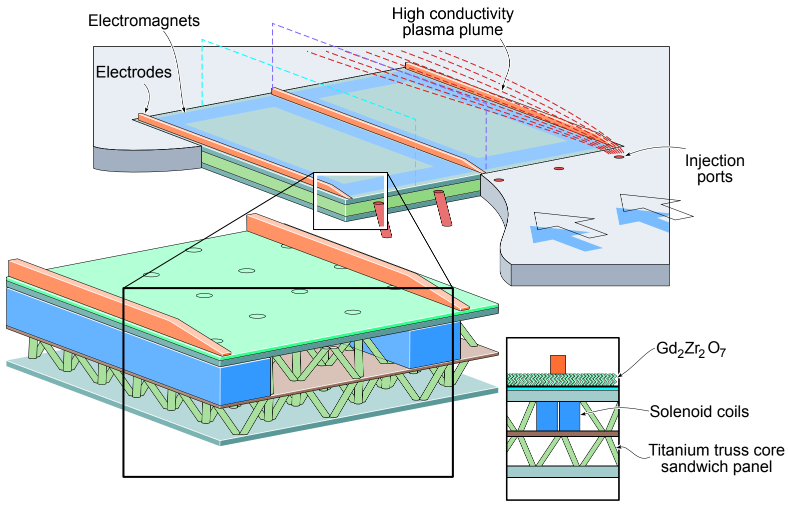
Hybrid Nanocrystalline Sandwiches



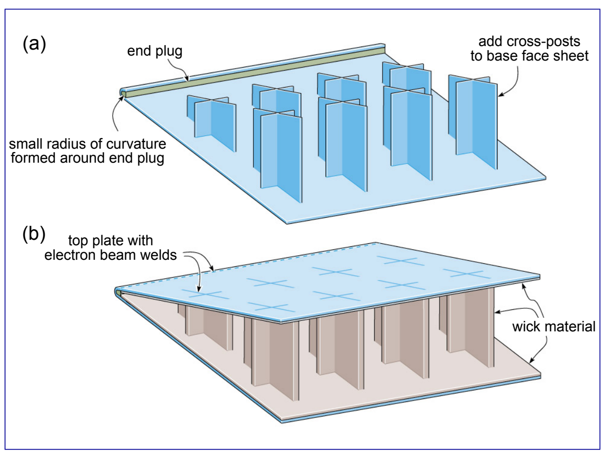
Hybrid Nanocrystalline Sandwiches



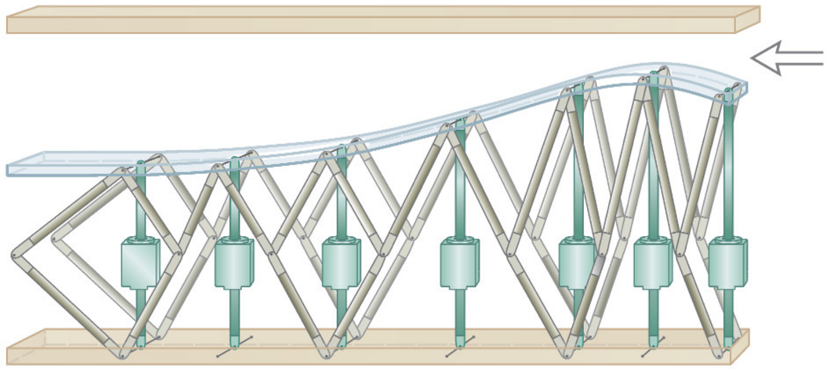
Multifunctionality - MHD Reentry Vehicle



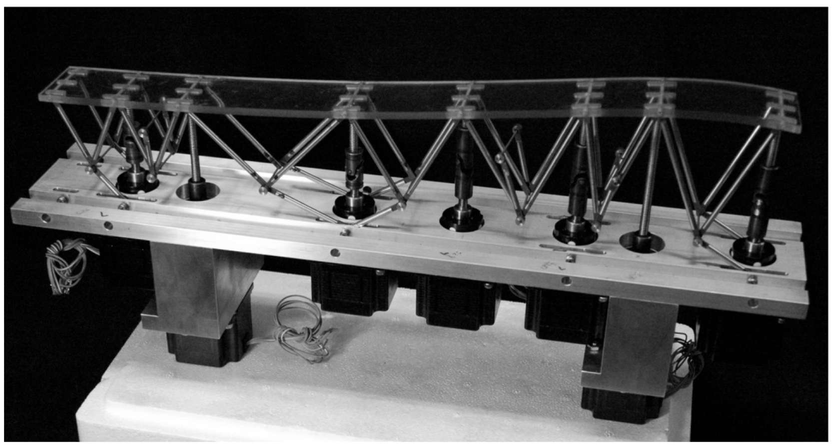
Multifunctionality - Leading Edge Heat Pipe



Multifunctionality - Morphing Wind Tunnel

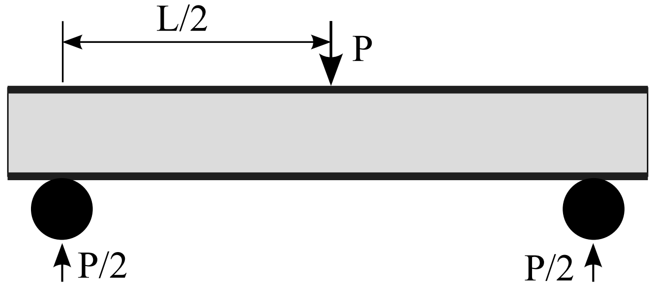


Multifunctionality - Morphing Wind Tunnel



Classical Sandwich Analysis

the prototypical sandwich loading configuration is three point bending:

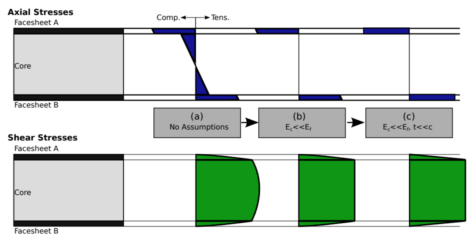


at all locations on the beam there is a bending moment and a transverse shear force

Classical Sandwich Analysis

classical sandwich analysis makes two crucial assumptions:

1. the face sheets carry all of the axial load associated with the bending moment; in the example geometry here, the top face sheet carries compressive load while the bottom face sheet is in tension
2. the core carries only shear load

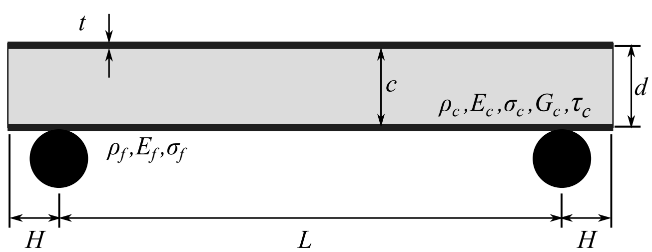


because of the large differences in the moduli and thicknesses of the face sheets and core, we make some additional assumptions:

1. the axial stress in the face sheets is constant through the thickness
2. the shear stress in the core is constant through the thickness

Classical Sandwich Analysis

to analyse sandwich beams, we need some nomenclature:



Classical Sandwich Analysis - Stiffness

the total deflection δ at the mid-point of a sandwich beam loaded in three-point bending is the sum of the deflections due to bending and shear:

$$\delta = \frac{PL^3}{48(EI)_{eq}} + \frac{PL}{4(AG)_{eq}}$$

$(EI)_{eq}$ is the equivalent flexural rigidity:

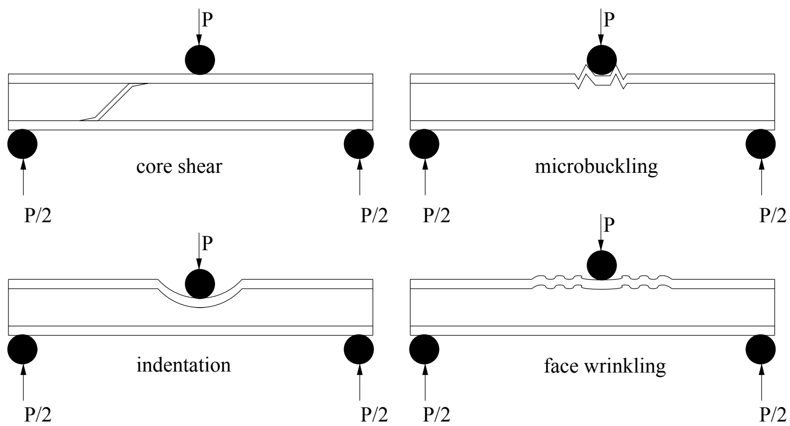
$$(EI)_{eq} = \frac{E_f btd^2}{2} + \frac{E_f bt^3}{6} + \frac{E_c bc^3}{12} \approx \frac{E_f btd^2}{2}$$

and $(AG)_{eq}$ is the equivalent shear rigidity:

$$(AG)_{eq} = \frac{bd^2 G_c}{c} \approx bdG_c$$

generally, we neglect the shear compliance of a sandwich beam

Classical Sandwich Analysis - Failure Mechanisms



Classical Sandwich Analysis - Strength

when analysing sandwich beam strength, the key thing to recognise is that, because they are complex structures, they are subject to complex failure mechanisms

for every combination of materials and geometry, we can calculate the maximum load that a sandwich beam can carry (in, for example, three point bending) for each mechanism

the mechanism that can carry the *least* load is the active failure mechanism for that geometry

later we will divide the design space into regions associated with each mechanism of failure

Classical Sandwich Analysis - Failure Mechanisms

there are well-developed analytical models for each of these mechanisms of failure (and several others as well):

microbuckling:

$$P = \frac{4bdt\sigma_f}{L}$$

core shear:

$$P = 2bd\tau_c$$

wrinkling:

$$P = \frac{2btd}{L}\sqrt[3]{E_f E_c G_c}$$

indentation:

$$P = bt\left(\frac{\pi^2 d E_f \sigma_c^2}{3L}\right)^{\frac{1}{3}} \qquad P = 2bt\left(\sigma_c \sigma_f\right)^{\frac{1}{2}}$$

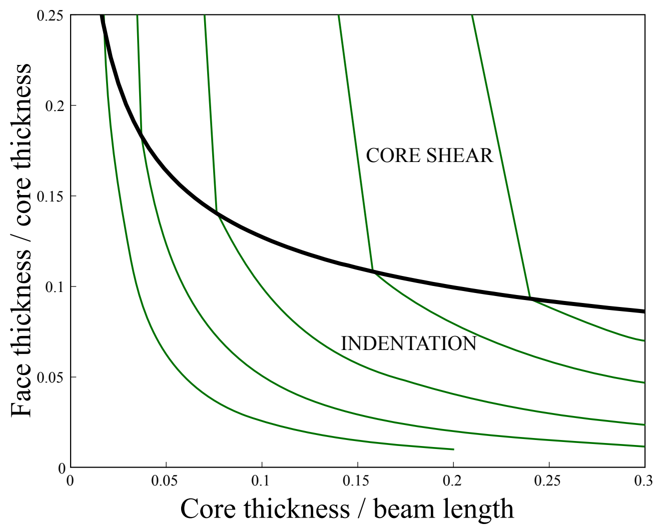
Classical Sandwich Analysis - Failure Mechanism Maps

once the material choice for a sandwich structure has been selected, and the length L and width b have been determined by the functional requirements, the only remaining variables are the thickness of the core c and the thickness of the face t

these two variables define a design space

as mentioned earlier, the design space can be divided into regions in which one mechanism of failure is dominant

Classical Sandwich Analysis - Failure Mechanism Maps



Classical Sandwich Analysis - Optimisation

at this point we can determine the strength of any sandwich beam loaded in three point bending, provided we know the geometry and several material properties; we can design beams that fulfil some functional requirement

a better question is: What is the lightest beam that we can design to fulfil the requirements?

we can calculate the mass of the sandwich beam:

$$M = 2bLt\rho_f + bLc\rho_c$$

Classical Sandwich Analysis - Optimisation

once we have this, we can also plot contours of mass on the failure mechanism map

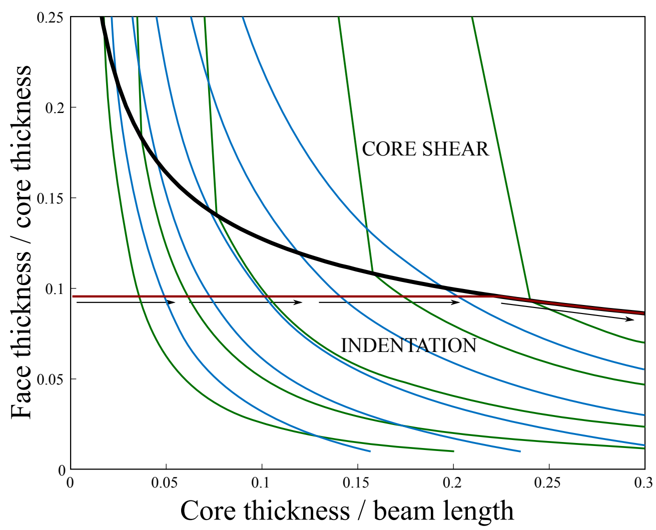
standard analytical techniques show us that mass is minimised for a required strength when the gradient of the mass is parallel to the gradient of the strength; mathematically speaking:

$$\nabla M = \lambda \nabla P$$

where λ is a Lagrange multiplier

we can find these locations analytically or on a failure mechanism map; the locus of all such points determines an optimal trajectory for minimum mass design

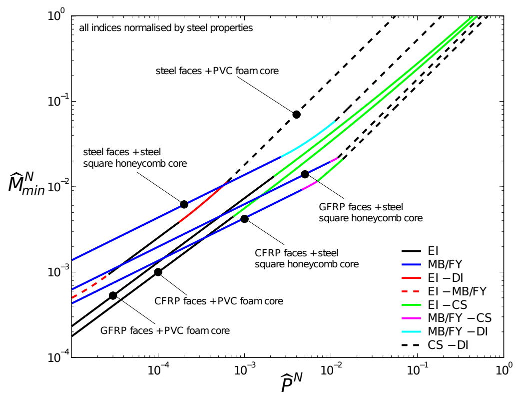
Classical Sandwich Analysis - Optimisation



Classical Sandwich Analysis - Optimisation

each trajectory in design space corresponds to a trajectory in load - mass space; we can plot the minimum mass as a function of the desired applied load

moreover, we can do this for a selection of different material choices; this reveals which material choice is lightest for a given load



Classical Sandwich Analysis - Optimisation

note that finding the optimal design for minimum mass for a given strength often results in sandwich beams which are exceptionally compliant; in general we will have to impose a stiffness constraint as well

in addition, we would like to know what is the ideal density for the core, rather than simply choosing the material *a priori*

if so, for a general sandwich beam, it is probable that the optimally designed panel will have just enough indentation strength and just enough stiffness to satisfy the requirements; the optimal values of the design variables are (in non-dimensional form):

$$\bar{\rho} = \left(\frac{\hat{P}_{req}}{2\gamma}\right)^{\frac{9}{14}} \left(\frac{3}{\tilde{\rho}}\right)^{\frac{5}{14}} \left(\frac{1}{\hat{E}I_{req}}\right)^{\frac{2}{7}}$$

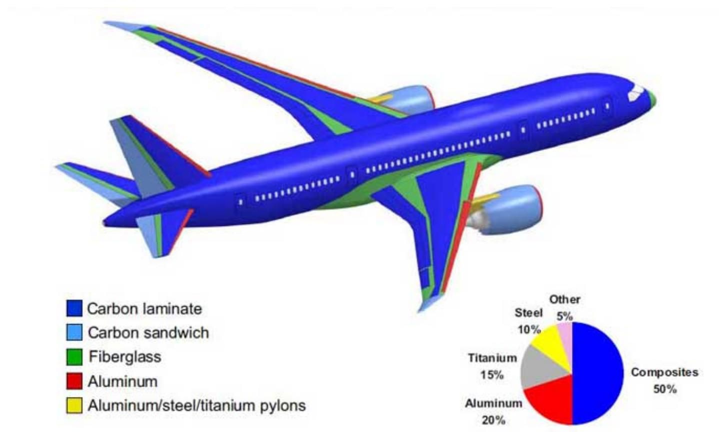
$$\bar{c} = \left(\frac{6\gamma \hat{E}I_{req}^2}{\hat{P}_{req}\tilde{\rho}}\right)^{\frac{3}{14}}$$

and

$$\bar{t} = \frac{2}{\hat{E}I_{req}^{\frac{2}{7}}} \left(\frac{\hat{P}_{req}\tilde{\rho}}{6\gamma}\right)^{\frac{9}{14}}$$

1.5 Fabrication of Composites

Composites in the Boeing 787



Breguet Range Equation

$$R = \frac{V}{C}\frac{C_L}{C_D}\ln\frac{w_i}{w_f}$$

V = cruise velocity C = specific fuel consumption of engines C_L = lift coefficient C_D = drag coefficient w_i = initial weight w_f = final weight

we are interested in how the weights change when we convert from metals to composites

Airbus A320

Aluminum	76.5%
Titanium	4.5%
Steel	13.5%
Composites	5.5%

Operating empty weight	42600 kg
Maximum zero-fuel weight	62500 kg
Maximum take-off weight	78000 kg
Maximum landing weight	64500 kg
Structural weight	22000 kg

what happens to the vehicle range when we convert all of the structural aluminum to composite?

Converting to Composites

assuming that the proper conversion from aluminum to composite represents the replacement of an aluminum tensile part with an equivalently stiff composite tensile part, the masses convert as:

$$\frac{m_c}{m_a} = \frac{\rho_c E_a}{\rho_a E_c}$$

if we replace a standard aerospace aluminum with a carbon fibre epoxy system, the A320 would be reduced in mass by over 11500 kg: this is over 50% of the total mass of the vehicle structure

consequently, the range of the A320 would increase by nearly 20% (or fuel burn would decrease commensurately): this justifies the effort needed to replace metal parts with composites

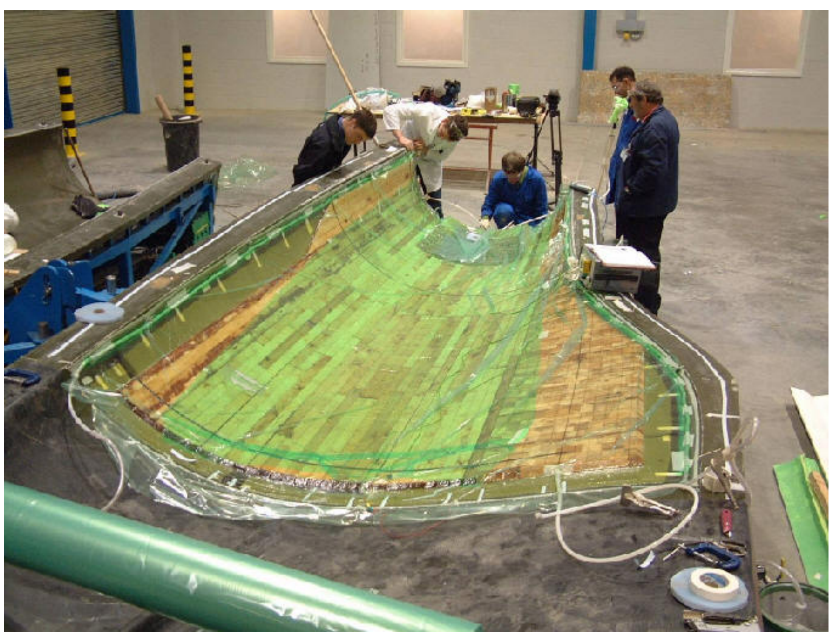
Fabrication Techniques (Polymer Matrix Composites)

OPEN MOULD TECHNIQUES:

1. hand lay-up
2. autoclaving
3. filament winding

CLOSED MOULD TECHNIQUES

1. resin transfer moulding
2. hot press moulding
3. pultrusion

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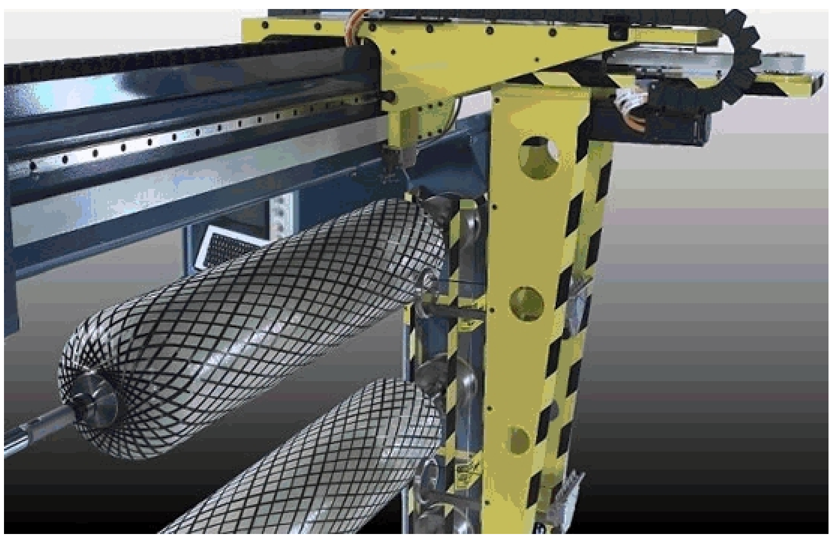
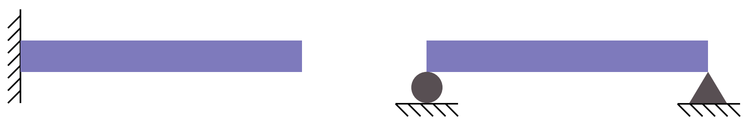


2 Plates and Shells

2.1 Euler - Bernoulli Beam Theory

Euler - Bernoulli Beam Theory

a standard structural element is the beam, which is a long, slender solid that carries transverse forces by bending



concentrate on statically determinate beams: cantilevers and simply supported beams

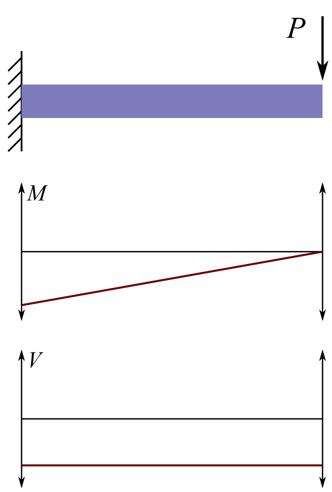
fundamental modelling simplifications are:

1. the beam is a one-dimensional object;
2. plane sections remain plane; and
3. there is no axial extension or contraction of the beam.

note that the principal variable is u, the vertical deflection of the midplane of the beam

Euler - Bernoulli Beam Theory

there are several graphical tools that are useful in the analysis of beams



Euler - Bernoulli Beam Theory

what is the relationship between the shear diagram and the moment diagram?

all of the derivatives of the displacement u have a physical meaning

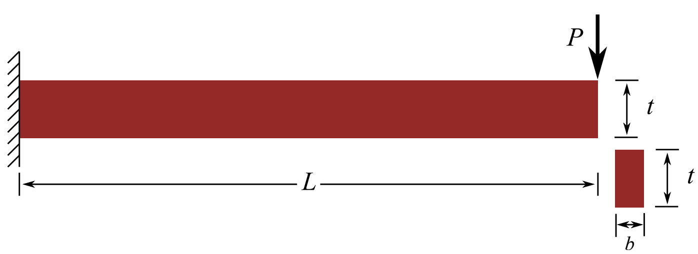
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \theta \;\; \text{slope of the beam}$$
$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}\theta}{\mathrm{d}x} = \kappa \;\; \text{curvature of the beam}$$
$$EI\kappa = M \;\; \text{moment in the beam}$$
$$-\frac{\mathrm{d}}{\mathrm{d}x}EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{\mathrm{d}M}{\mathrm{d}x} = V \;\; \text{shear force in the beam}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}EI\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -\frac{\mathrm{d}V}{\mathrm{d}x} = w \;\; \text{distributed force on the beam}$$

Euler - Bernoulli Beam Theory

there are several key assumptions associated with Euler-Bernoulli beam theory:

1. the beam is slender: the bending deformations are much larger than the shear deformations
2. there is no through-thickness deformation: the thickness of the beam remains constant
3. cross-sections of the beam initially normal to the neutral axis remain normal to the axis of bending after deformation

Euler - Bernoulli Beam Theory



Euler-Bernoulli beam theory enables the calculation of deflected shapes of beams of various sorts: the cantilever shown above is an example

a complete set of boundary conditions is necessary to use Euler-Bernoulli beam theory: these must be four conditions on the deflection u or its derivatives

in general, start with the distributed load $w(x)$ and integrate it to get the shear $V(x)$: in this case, the distributed load is zero

$$V(x) = -\int w \,\mathrm{d}x = -\int 0 \,\mathrm{d}x = C_1$$

because the shear force at any cross-section is $-P$, we find that $C_1 = -P$

Euler - Bernoulli Beam Theory

second, integrate the shear force to find the moment distribution $M(x)$:

$$M(x) = -\int V \, \mathrm{d}x = -\int (-P) \, \mathrm{d}x = Px + C_2$$

the moment at the free end of the beam (where $x = L$) is zero, so:

$$M(L) = 0 = PL + C_2$$

and therefore:

$$C_2 = -PL$$

third, convert the moment to a curvature, $\kappa(x)$:

$$\kappa(x) = \frac{M(x)}{EI}$$

Euler - Bernoulli Beam Theory

fourth, integrate the curvature to get the slope, $\theta(x)$:

$$\theta(x) = \int \kappa \, \mathrm{d}x = \frac{1}{EI} \int (Px - PL) \; \mathrm{d}x = \frac{1}{EI} \left(\frac{Px^2}{2} - PLx + C_3 \right)$$

the slope at the wall is zero because it is rigidly fixed and cannot rotate, so evaluating $\theta(0)$ gives:

$$\theta(0) = C_3 = 0$$

and hence $C_3 = 0$

Euler - Bernoulli Beam Theory

finally, integrate the slope to get the deflected shape, $u(x)$:

$$u(x) = \int \theta \, \mathrm{d}x = \frac{1}{EI} \int \left(\frac{Px^2}{2} - PLx \right) \mathrm{d}x = \frac{1}{EI} \left(\frac{Px^3}{6} - \frac{PLx^2}{2} + C_4 \right)$$

the displacement at the wall, $u(0) = 0$, so solve for the final constant of integration:

$$C_4 = 0$$

thus, the complete deflected shape of the end-loaded cantilever beam is:

$$u = \frac{1}{EI} \left(\frac{Px^3}{6} - \frac{PLx^2}{2} \right)$$

the maximum deflection, at $x = L$, is:

$$u(L) = -\frac{PL^3}{3EI}$$

Algorithm: Euler-Bernoulli Beam Theory

1. ensure that the beam you are analyzing is statically determinate: if it is not, you will have to use a *superposition* method
2. identify four boundary conditions on the displacement, slope, moment or shear force to solve for the constants of integration
3. starting with the distributed load w (often this is zero), integrate four times to get the deformed shape u
4. in general solve for the constants of integration as early as possible rather than carrying multiple constants

2.2 Plate Theory

Plates

beams have one dimension that is much larger than the other two: beams can be treated as effectively one-dimensional

plates, shells and membranes have two dimensions that are much larger than the third dimension, and hence can be treated as two-dimensional objects

the key difference between plates and membranes is that plates can support loads both in-plane and in bending, while membranes can only support in-plane loads

Plate Midplane

much like in Euler - Bernoulli beam theory, plate theory deals primarily with the midplane of a plate; it is assumed that the forces applied to a plate are acting on the midplane

provided the plate material is isotropic and homogeneous, and the plate has a constant thickness, identification of the midplane is easy: it is at the middle of the plate thickness

much like the development of laminate plate theory, once there is material and geometric complexity, such as non-symmetric layers in a plate, the midplane must be found from additional considerations

Modulus Weighted Plate Midplane

if the stiffness of the plate changes through its thickness (as is the case for composite laminates) the modulus-weighted plate thickness, h^*, is:

$$h^* = \int_t \frac{E}{E_0}\, \mathrm{d}z_0$$

where t is the thickness of the plate and z_0 is the transverse coordinate with a conveniently chosen origin

E is the modulus at location z, while E_0 is a conveniently chosen reference modulus

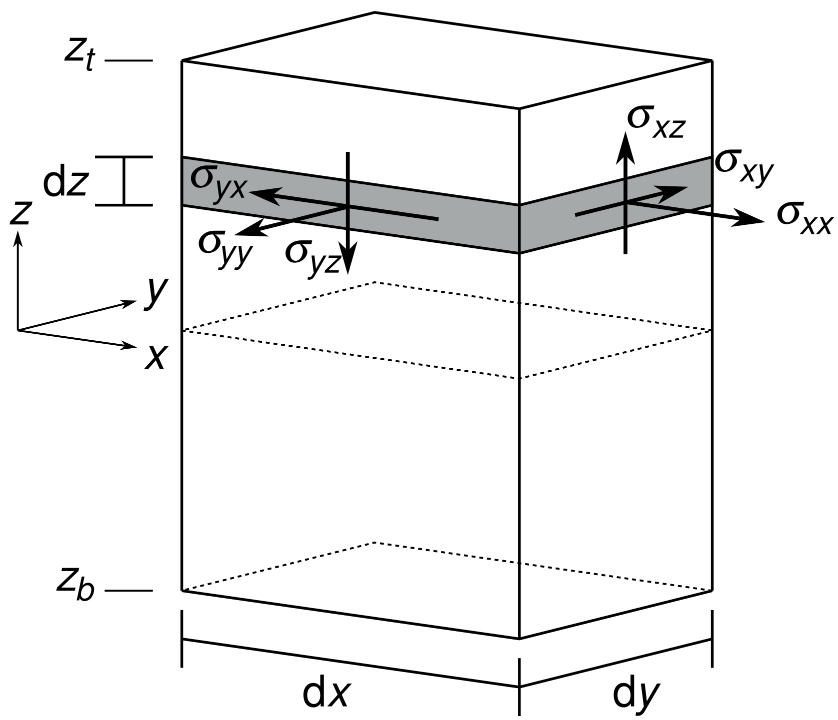
the modulus-weighted midplane is the origin of the z coordinate; this will be true provided:

$$0 = \int_t z \frac{E}{E_0} \, \mathrm{d}z$$

the two coordinates, z and z_0 are related by: $z = z_0 - z_0^*$; z_0^* is the distance from the arbitrary system origin to the modulus weighted midplane, and can be found by:

$$z_0^* h^* = \int_t z_0 \frac{E}{E_0} \, \mathrm{d}z_0 = \sum_i (\bar{z}_0)_i \left(\frac{Eh}{E_0}\right)_i$$

Plate Stresses

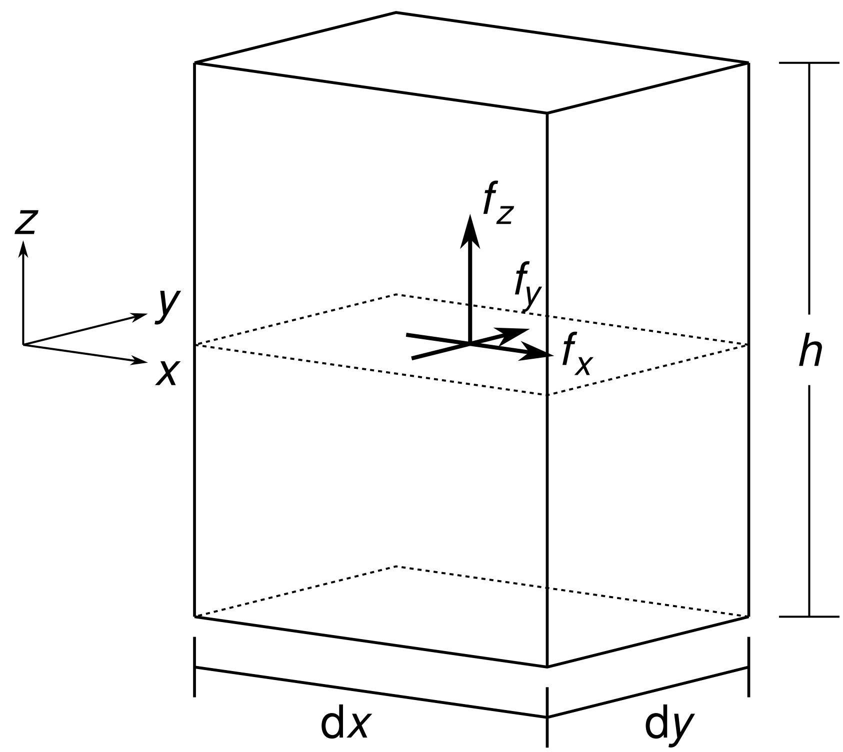


the notation at left to identifies the stresses acting on the faces of a representative element of the plate

this is consistent with standard notation for stresses

note that the stresses may vary in x, y and z; for beams, stresses varied only in x and z

Applied Forces

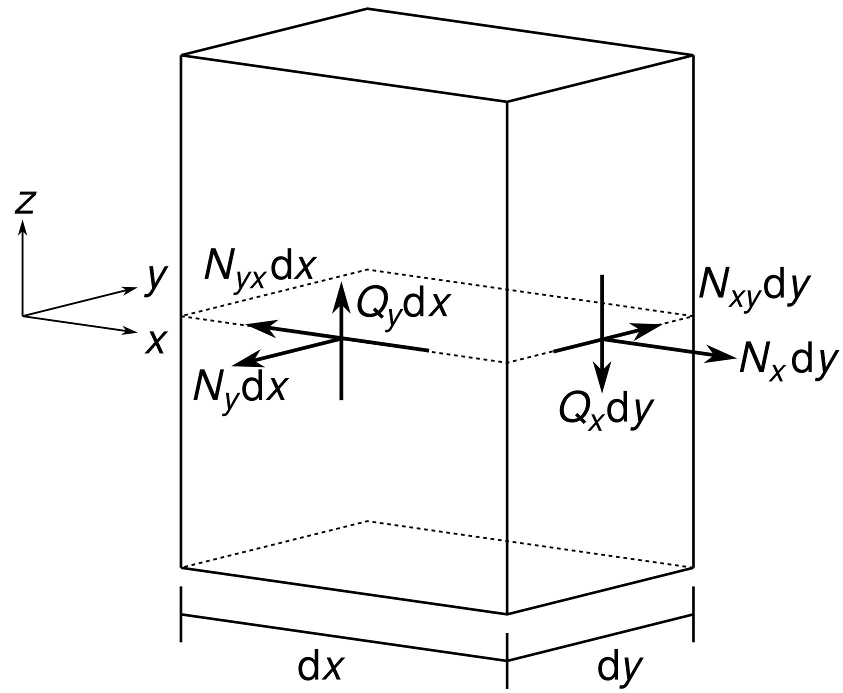


the applied forces are decomposed into three components, aligned with the coordinate axes

the applied forces are assumed to act at the midplane of the plate

in all cases, the applied forces are tractions and have units of force per unit area

Stress Resultants - Normal and Shear



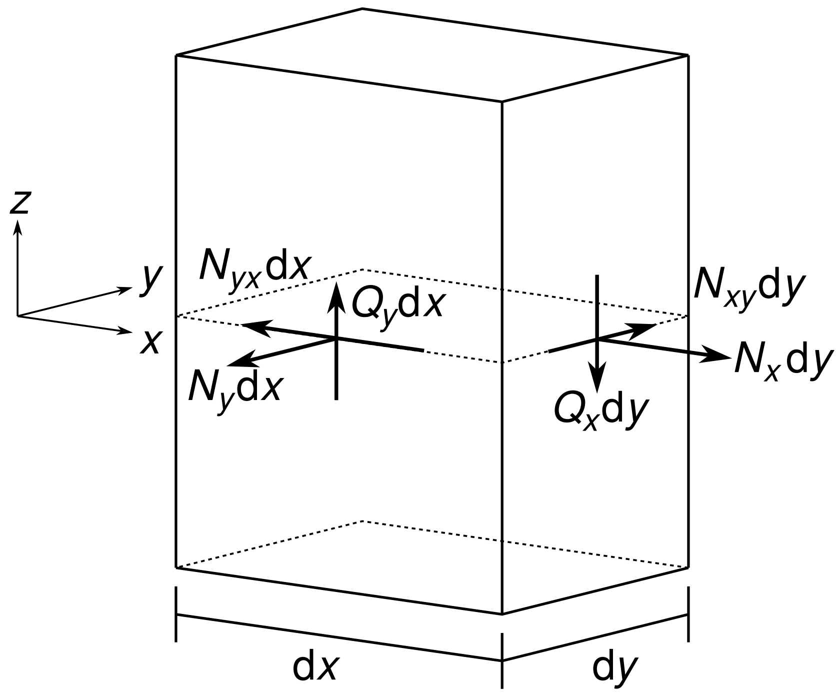
integrating the stresses gives the resultants (per unit width):

$$N_x = \int_{z_b}^{z_t} \sigma_{xx} \, \mathrm{d}z$$

$$N_y = \int_{z_b}^{z_t} \sigma_{yy} \, \mathrm{d}z$$

$$N_{xy} = N_{yx} = \int_{z_b}^{z_t} \sigma_{xy} \, \mathrm{d}z$$

Stress Resultants - Normal and Shear

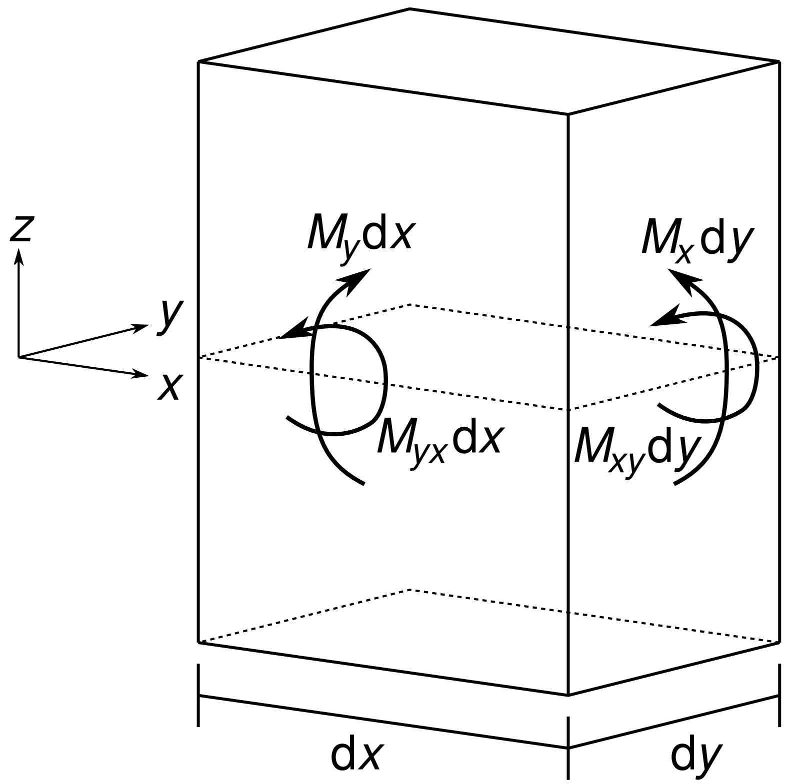


the transverse shear stress resultants are:

$$Q_x = -\int_{z_b}^{z_t} \sigma_{xz}\, \mathrm{d}z$$

$$Q_y = -\int_{z_b}^{z_t} \sigma_{yz}\, \mathrm{d}z$$

Stress Resultants - Moments



finally, the resultant moments are:

$$M_x = -\int_{z_b}^{z_t} z\sigma_{xx}\,\mathrm{d}z$$

$$M_y = -\int_{z_b}^{z_t} z\sigma_{yy}\,\mathrm{d}z$$

$$M_{xy} = -M_{yx} = -\int_{z_b}^{z_t} z\sigma_{xy}\,\mathrm{d}z$$

Plate Displacements

the approximations for plate theory are very similar to those for Euler - Bernoulli beam theory and composite laminate theory:

the most important are:

1. straight lines perpendicular to the modulus weighted midplane before bending remain straight and perpendicular after bending

2. the perpendiculars to the modulus weighted midplane remain unchanged in length after deflection

the consequence is that it is necessary only to consider the modulus weighted midplane deflections; deflections at other locations through the thickness can be derived:

$$u(x, y, z) = u^\circ(x, y, 0) - z\theta_y(x, y)$$

$$v(x, y, z) = v^\circ(x, y, 0) - z\theta_x(x, y)$$

$$w(x, y, z) = w^\circ(x, y, 0)$$

like in laminate plate theory, the superscripted 'o' indicates a midplane quantity

Strains

there are only three non-zero strains, which are found by differentiating the modulus weighted midplane displacements and adding terms to account for the curvature of the plate:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u^\circ}{\partial x} - z\frac{\partial^2 w^\circ}{\partial x^2}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v^\circ}{\partial y} - z\frac{\partial^2 w^\circ}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} - 2z\frac{\partial^2 w^\circ}{\partial x \partial y}$$

Stress - Strain Relations

assume a linear elastic material model and hence apply Hooke's Law to relate the stresses and the strains, in this case in a plane strain manner:

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\left(\epsilon_{xx} + \nu\epsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{1 - \nu^2}\left(\epsilon_{yy} + \nu\epsilon_{xx}\right)$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E\gamma_{xy}}{2(1 + \nu)}$$

Stress - Displacement Relations

express the stresses in terms of displacements; to do so, substitute the definitions of strains into the expressions of Hooke's Law, using commas in the subscripts to indicate differentiation in order to shorten the expressions:

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\left(u^\circ_{,x} + \nu v^\circ_{,y} - z\left(w^\circ_{,xx} + \nu w^\circ_{,yy}\right)\right)$$

$$\sigma_{yy} = \frac{E}{1 - \nu^2}\left(v^\circ_{,y} + \nu u^\circ_{,x} - z\left(w^\circ_{,yy} + \nu w^\circ_{,xx}\right)\right)$$

$$\sigma_{xy} = \frac{E}{2(1 + \nu)}\left(u^\circ_{,y} + v^\circ_{,x} - 2zw^\circ_{,xy}\right)$$

Stress Resultant - Displacement Relations

finally this produces the relationship between the stress resultants and the displacements:

$$N_x = K^* \left(u^\circ_{,x} + \nu v^\circ_{,y}\right)$$

$$N_y = K^* \left(v^\circ_{,y} + \nu u^\circ_{,x}\right)$$

$$N_{xy} = \frac{1}{2}\left(1-\nu\right) K^* \left(u^\circ_{,y} + v^\circ_{,x}\right)$$

where:

$$K^* = \frac{E_0 h^*}{1-\nu^2}$$

Stress Resultant - Displacement Relations

similarly:

$$M_x = D^* \left(w^\circ_{,xx} + \nu w^\circ_{,yy}\right)$$

$$M_y = D^* \left(w^\circ_{,yy} + \nu w^\circ_{,xx}\right)$$

$$M_{xy} = \left(1-\nu^2\right) D^* w^\circ_{,xy}$$

where:

$$D^* = E_0 I^*$$

and

$$I^* = \int_t \frac{z^2}{1-\nu^2} \frac{E}{E_0}\, \mathrm{d}z$$

Homogeneous Plates

note that in the equations above used K^*, D^*, I^* and h^*

this is to provide for the case with a layered material where the layers are not homogeneous, as found in composites

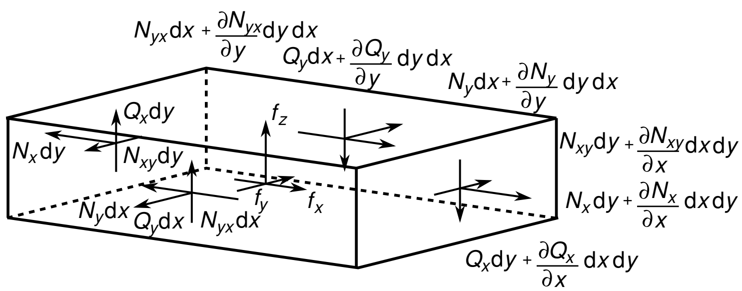
if the plate is homogeneous, then:

$$K = \frac{Eh}{1-\nu^2}$$

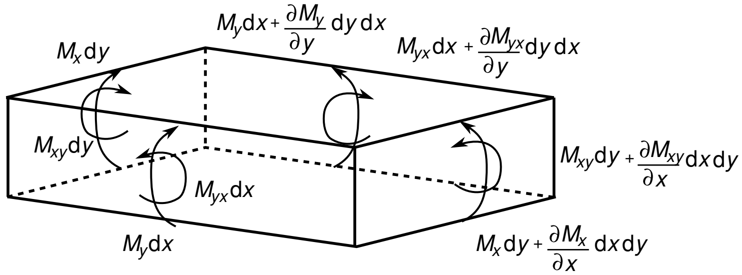
$$I = \frac{h^3}{12\left(1-\nu^2\right)}$$

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}$$

Equilibrium of a Plate Element



Equilibrium of a Plate Element



Equilibrium of a Plate Element

apply equilibrium equations to the free body diagrams on the previous two slides; the key results are, for forces:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = -f_x$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = -f_y$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -f_z$$

and for moments:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

Equilibrium of a Plate Element

substitute the stress resultant - displacement equations into the equilibrium equations (again using comma differentiation notation for compactness):

$$K^{*}\left(u^{\circ}_{,xx}+\nu v^{\circ}_{,yx}\right)+\frac{1}{2}\left(1-\nu\right)K^{*}\left(v^{\circ}_{,xy}+u^{\circ}_{,yy}\right)=-f_{x}$$

$$K^{*}\left(v^{\circ}_{,yy}+\nu u^{\circ}_{,xy}\right)+\frac{1}{2}\left(1-\nu\right)K^{*}\left(v^{\circ}_{,xx}+u^{\circ}_{,yx}\right)=-f_{y}$$

and moment equilibrium gives:

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}=f_z$$

using this and the resultant moment - displacement relation produces, after some simplification:

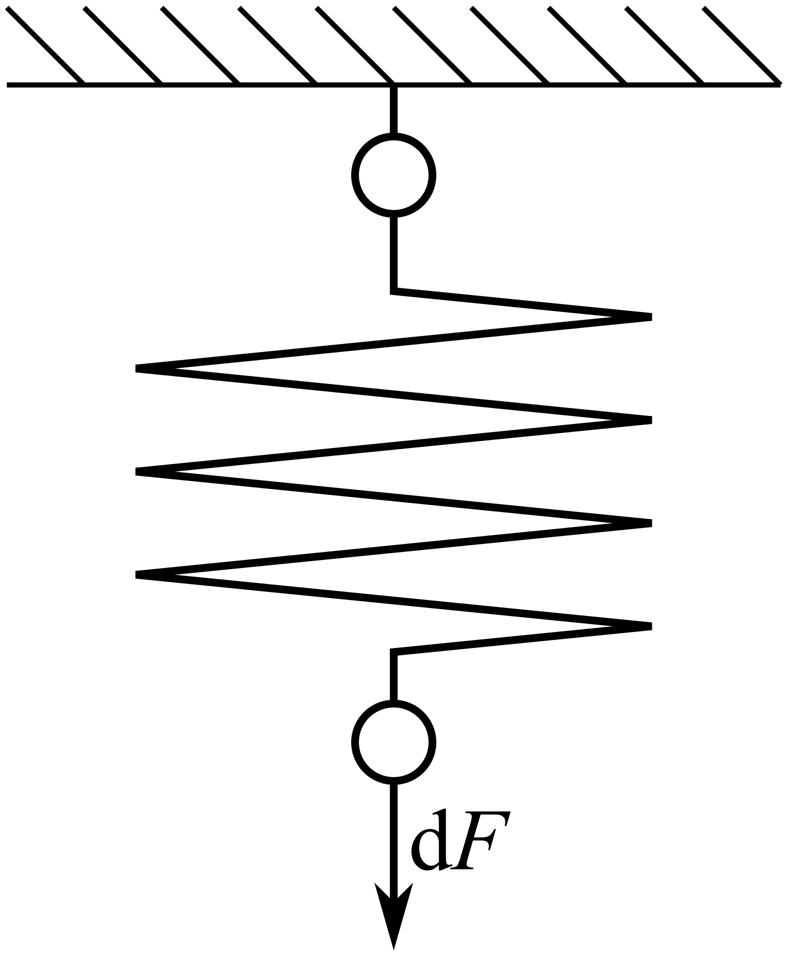
$$D^{*}\left(w^{\circ}_{,xxxx}+2w^{\circ}_{,xxyy}+w^{\circ}_{,yyyy}\right)=f_z$$

this is the constant-thickness, small deflection, isotropic thin plate bending equation; it incorporates equilibrium, continuity of deflection and a linear elastic material model

3 Finite Element Methods

3.1 Strain Energy

Strain Energy



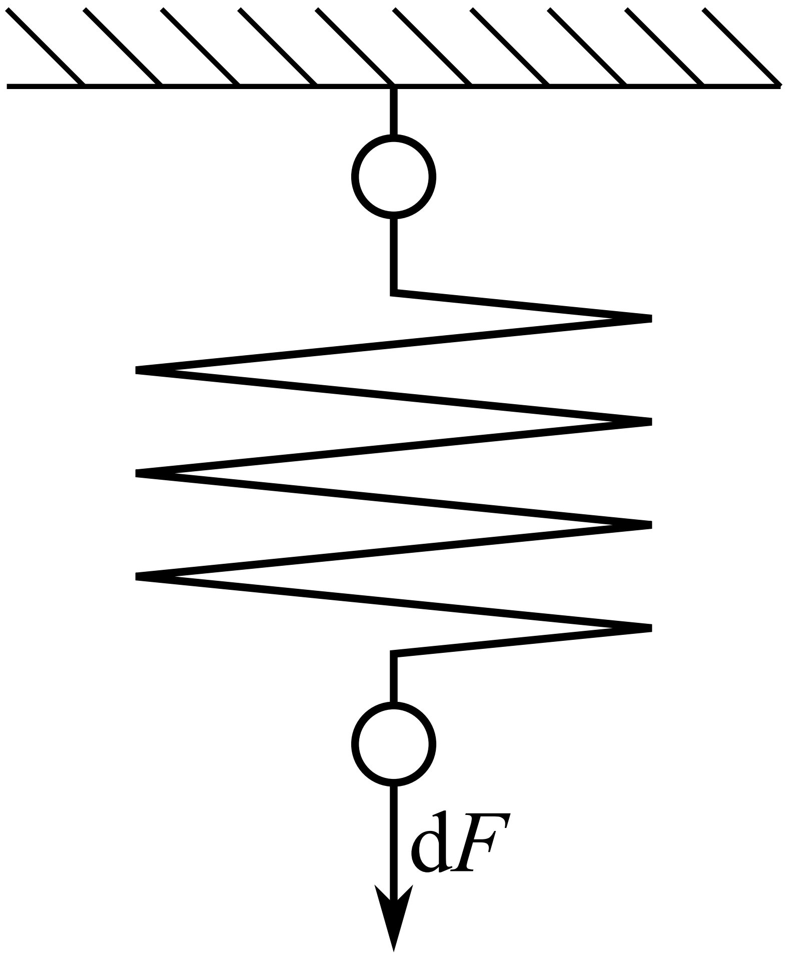
consider the linear one-dimensional case, with a spring which extends dx when force $\mathrm{d}F/k$ is applied

the work done by the applied force is:

$$W = \int_a^b F\,\mathrm{d}x = \int_a^b kx\,\mathrm{d}x = \left.\frac{kx^2}{2}\right|_a^b$$

this work is stored as strain energy and is available to do further work

Strain Energy



in the continuum approach, stress and strain are the analogues of force and displacement the strain energy density is determined through:

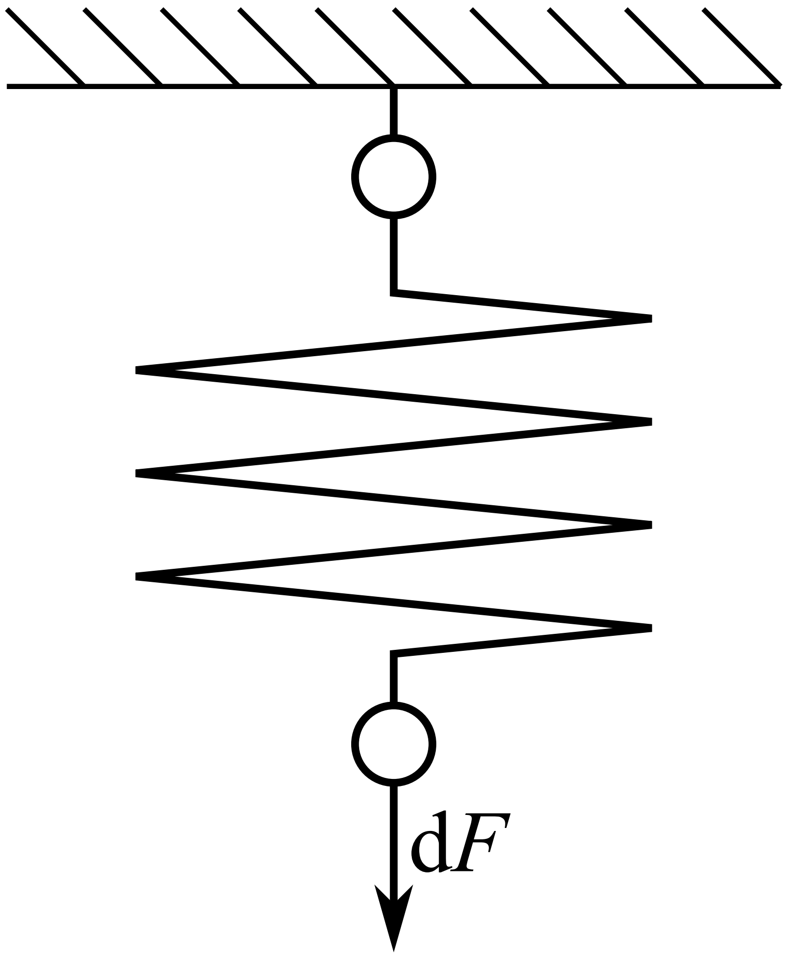
$$\mathscr{W} = \int \sigma_{ij} \, \mathrm{d}\epsilon_{ij}$$

$\mathscr{W}$ is the amount of energy per unit volume which is stored in a material due to elastic deformation

as a consequence:

$$\sigma_{ij} = \frac{\partial \mathscr{W}}{\partial \epsilon_{ij}}$$

Strain Energy



strain energy is work that is *recoverable*

there are a great many deformation mechanisms which are dissipative: the energy involved in the deformation cannot be recovered examples are plastic deformation, fracture, viscous flow, friction

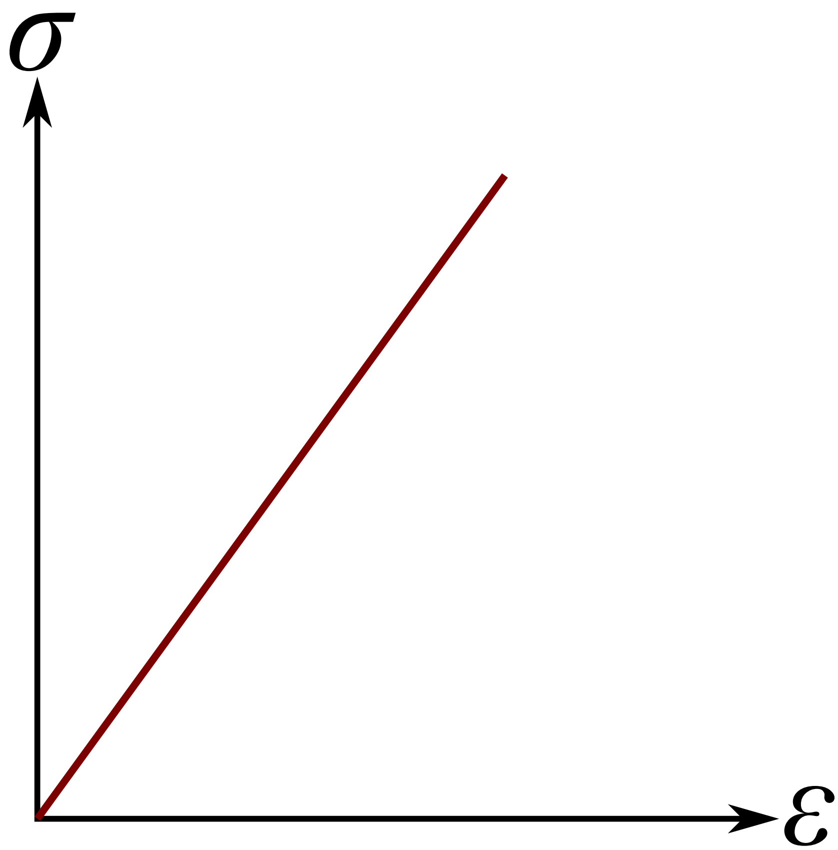
recoverability of the strain energy is essential to the definition of elastic deformation

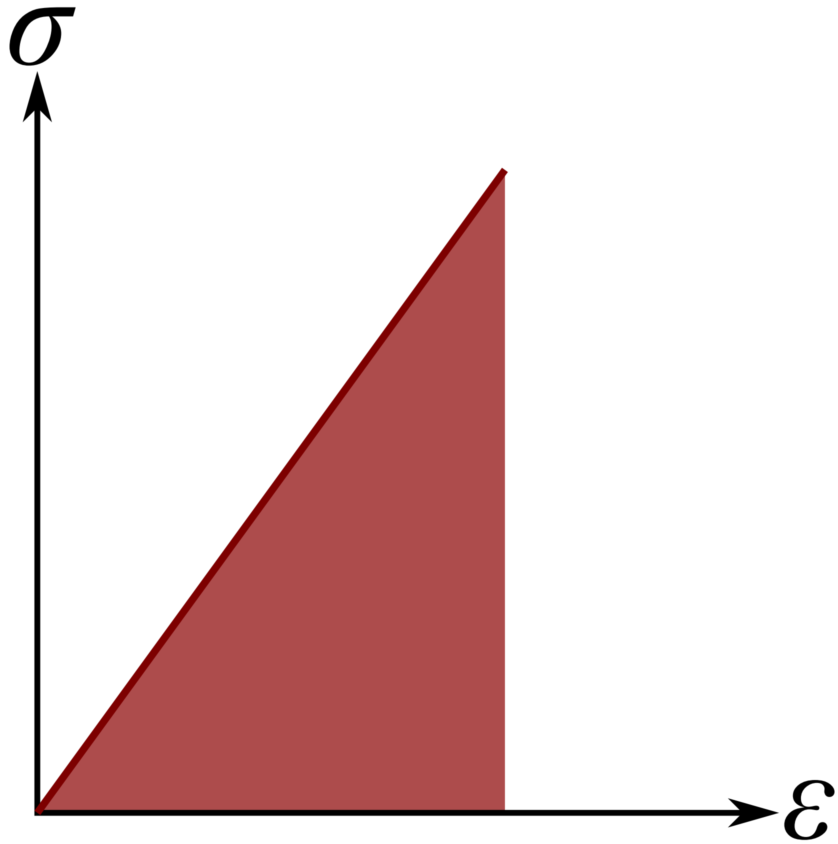
Strain Energy and Complementary Strain Energy

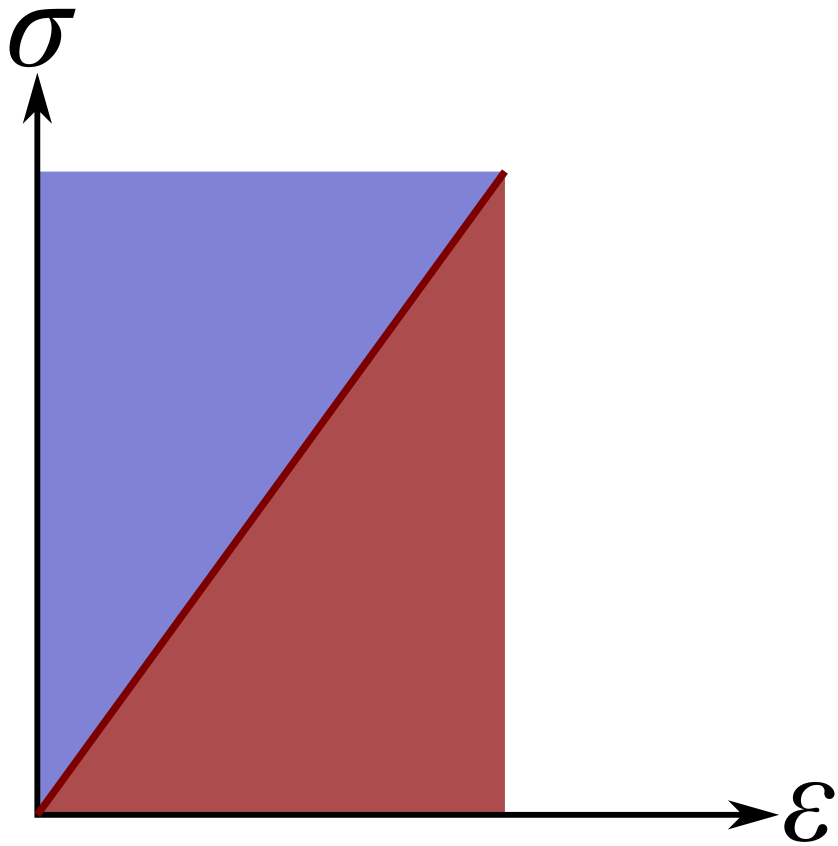
in the linear elastic case, stress is a linear function of strain

the strain energy density can be evaluated by integrating the stress with respect to strain; here it can be geometrically represented by the area under the stress-strain curve

we can also define the complementary strain energy density, which is the integral of strain with respect to stress; graphically it is the area between the stress-strain curve and the stress axis



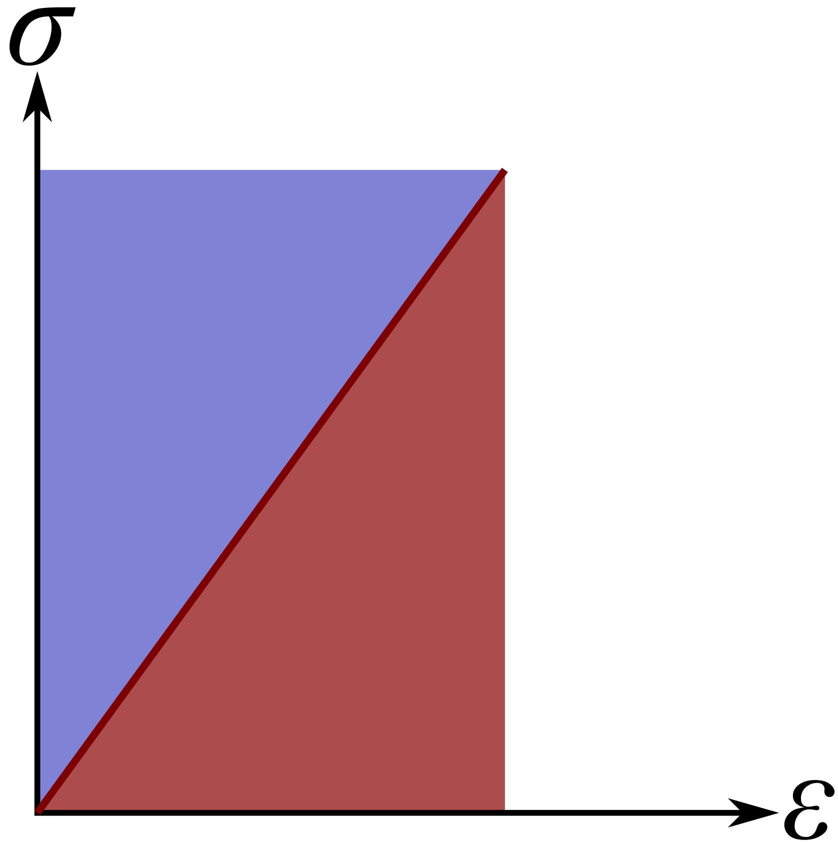

σ
ε



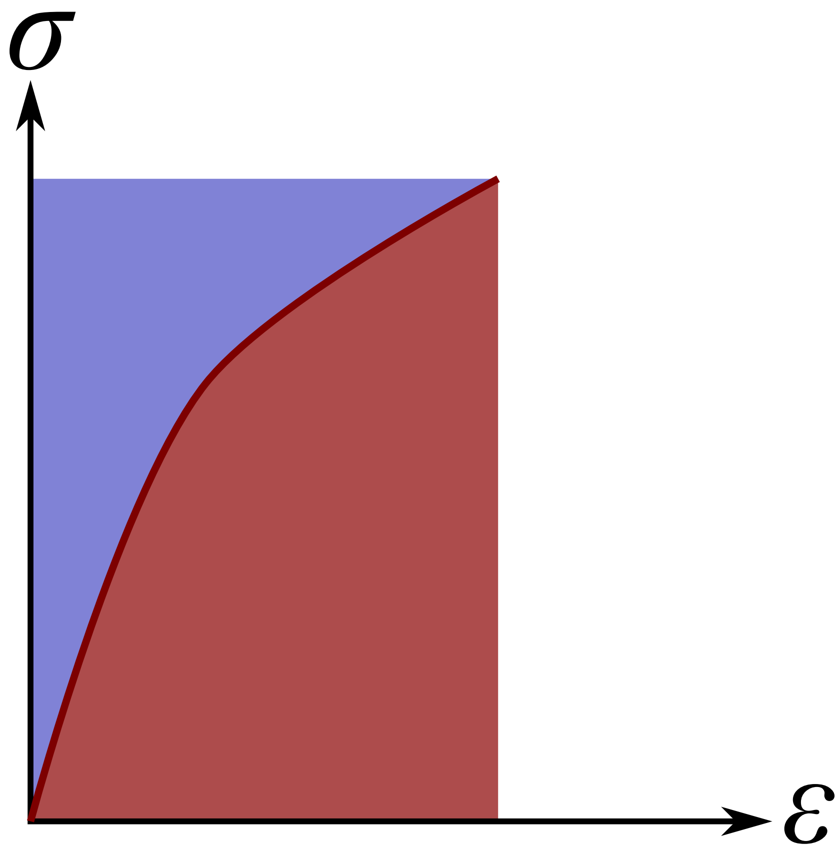
Complementary Strain Energy

if the stress-strain relation is linear, then the strain energy density is equal to the complementary strain energy density

however, if the stress-strain relation is non-linear, then strain energy density does not necessarily equal the complementary strain energy density



σ
ε



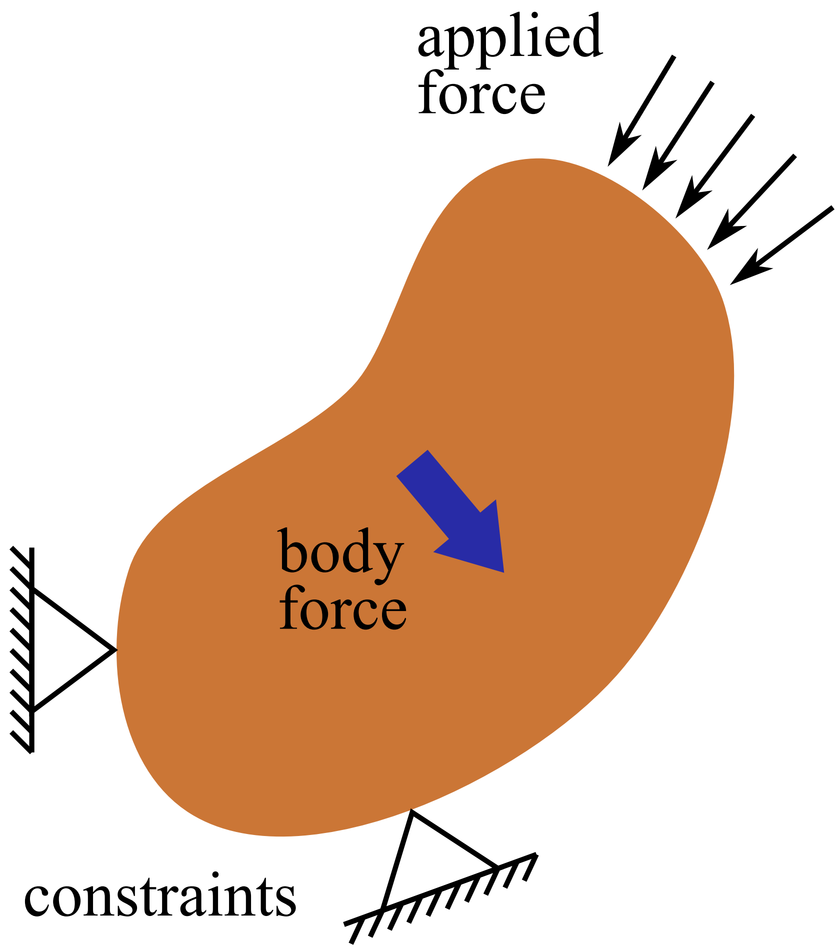
3.2 Virtual Work

The Principle of Virtual Work

take an arbitrary body with body force density B_i, subjected to tractions T_i over part of the surface labelled S_σ and constraints over the remainder of the surface labelled S_u

the displacement field over the body is given by u_i

within the body are fields of stress, strain and strain energy density



The Principle of Virtual Work

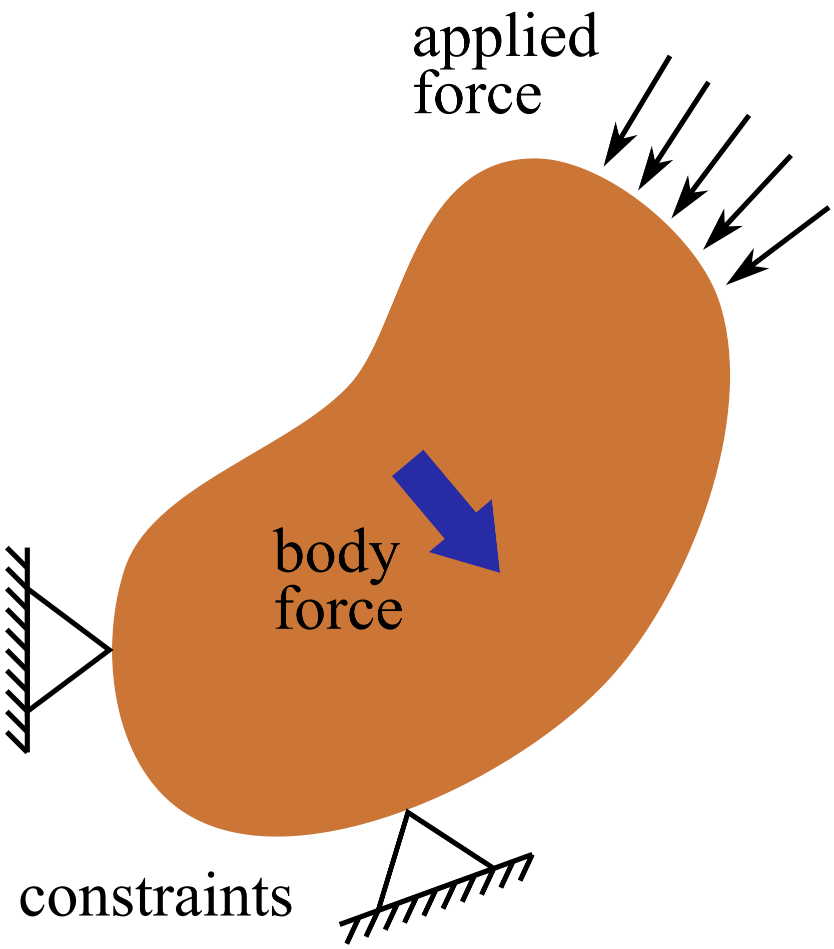
a *virtual displacement* δu_i is a kinematically admissible infinitesimal displacement field which is added to the real displacements u_i

at boundaries where the displacement is known the virtual displacement must be zero, specifically on S_u

also, the virtual displacement is smooth, thrice differentiable and sufficiently small that all displacements remain elastic

the *virtual work* $\delta\mathscr{E}$ is the work done on a body by all the external forces as they travel through the virtual displacement δu_i

$$\delta\mathscr{E} = \int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S$$



The Principle of Virtual Work

the actual forces and stresses acting on the body are *unaffected* by the virtual displacement

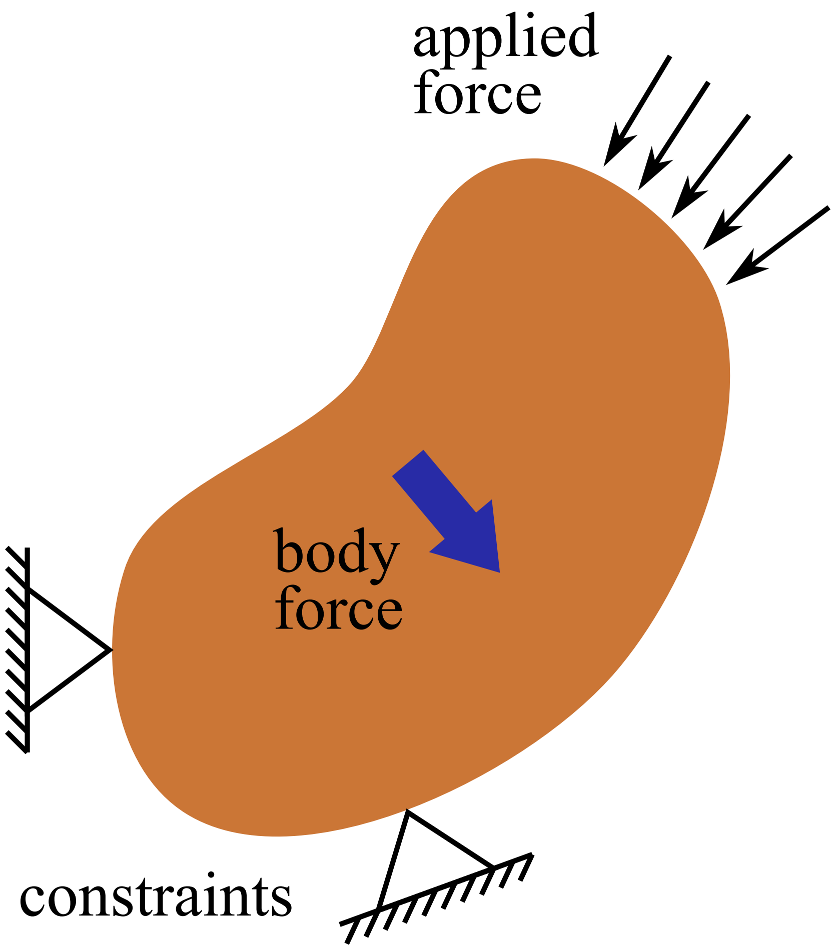
define a *virtual strain* $\delta\epsilon_{ij}$ as a kinematically admissible infinitesimal (linear) strain field that is compatible with the virtual displacement field δu_i:

$$\delta\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right)$$

the virtual strain energy of the body is:

$$\delta\mathscr{U} = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V$$

after integration this is a *linear* equation: in the real system, σ_{ij} is a function of ϵ_{ij}, but σ_{ij} is not a function of $\delta\epsilon_{ij}$



The Principle of Virtual Work

the principle of virtual work states that the external virtual work performed by the applied forces and body forces is equal to the internal virtual work, the virtual strain energy

$$\delta\mathscr{E} = \delta\mathscr{U}$$

in full:

$$\int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S = \int_V \sigma_{ij} \delta\epsilon_{ij} \, \mathrm{d}V$$

if the strains are linear, this becomes:

$$\int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S = \int_V \frac{1}{2}\sigma_{ij} \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right) \mathrm{d}V$$

the principle of virtual work combines a kinematically compatible virtual deformation $(\delta u_i, \delta\epsilon_{ij})$ with a stress field σ_{ij} which for a given external force system (B_i, T_i)

satisfies equilibrium requirements

The Principle of Virtual Work

for a discrete system, like a truss, the external virtual work is:

$$\delta\mathscr{E} = \sum_{i=1}^{n} P_i \delta u_i$$

where the P_i are the applied forces and the δu_i are the displacements at the corresponding nodes (for n total nodes)

the internal virtual work (the virtual strain energy) is:

$$\delta\mathscr{U} = \sum_{i=1}^{m} F_i \delta e_i$$

where the F_i are the member forces and δe_i the member extensions (for m total members)

the real member forces F_i are in equilibrium with the real applied forces P_i, while the virtual displcements δu_i are compatible with the virtual member extensions δe_i

Virtual Work: Truss Example

consider the three-member pin-jointed truss shown to the right, loaded by forces P_1 and P_2

apply a system of virtual displacements and extensions

the external virtual work is:

$$\delta\mathscr{E} = P_1 \delta u_1 + P_2 \delta u_2$$

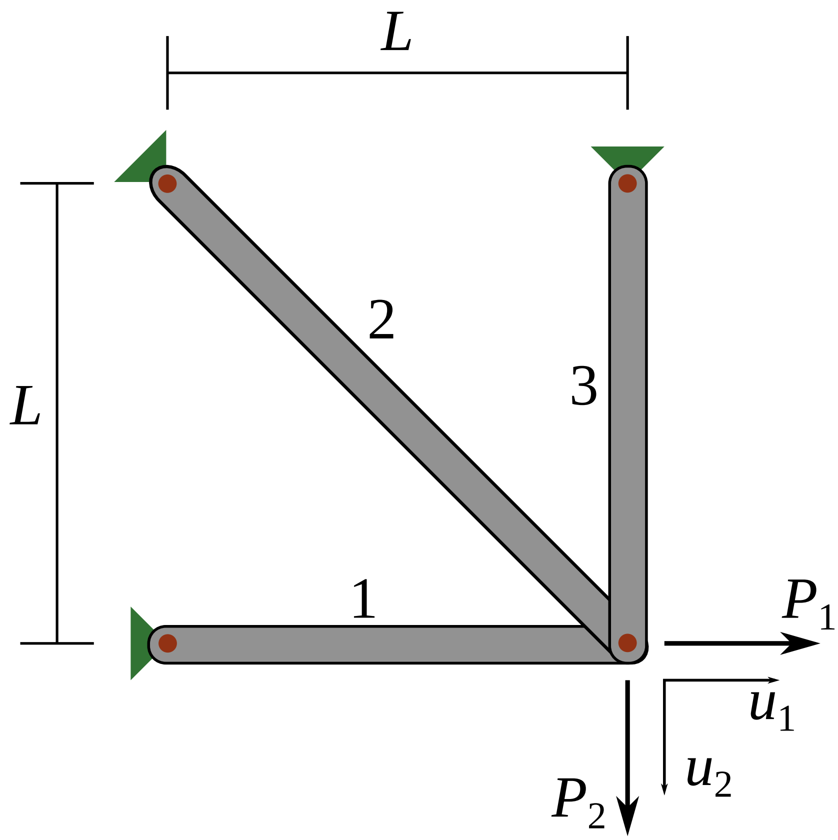
while the internal virtual work is:

$$\delta\mathscr{U} = F_1 \delta e_1 + F_2 \delta e_2 + F_3 \delta e_3$$

hence:

$$P_1 \delta u_1 + P_2 \delta u_2 = F_1 \delta e_1 + F_2 \delta e_2 + F_3 \delta e_3$$

provided (P_i, F_i) are an equilibrium system and $(\delta u_i, \delta e_i)$ are a compatible system



Virtual Work: Truss Example

use displacement compatibility to eliminate the member extensions:

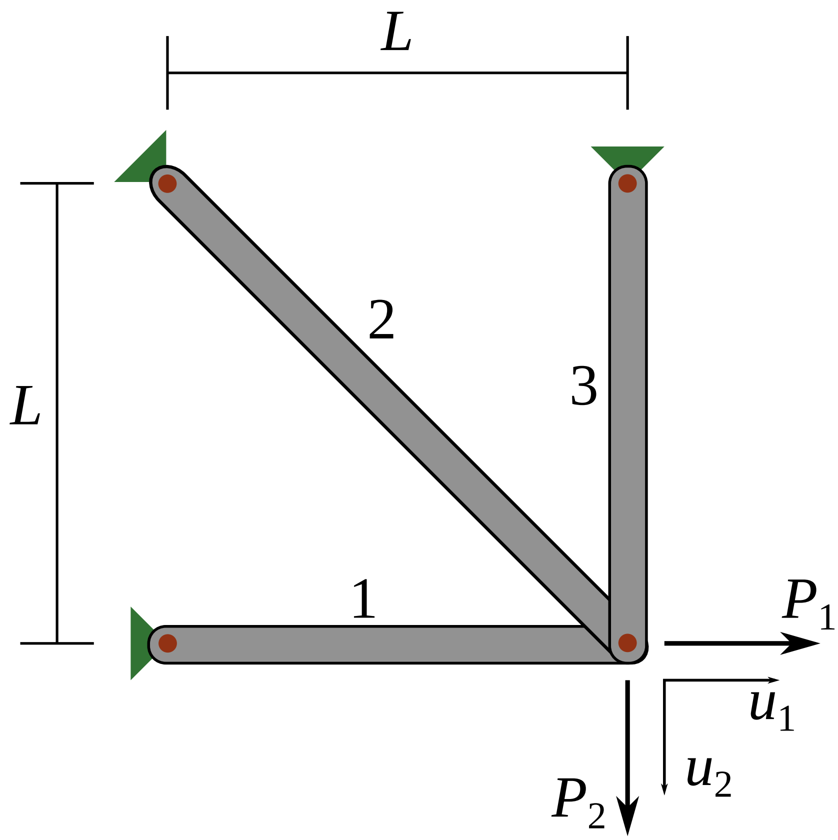
$$e_1 = u_1; \quad e_2 = \frac{1}{\sqrt{2}}(u_1 + u_2); \quad e_3 = u_2$$

in virtual terms:

$$\delta e_1 = \delta u_1; \quad \delta e_2 = \frac{1}{\sqrt{2}}(\delta u_1 + \delta u_2); \quad \delta e_3 = \delta u_2$$

eliminating the virtual extensions:

$$F_1 \delta u_1 + F_2 \frac{\delta u_1 + \delta u_2}{\sqrt{2}} + F_3 \delta u_2 = P_1 \delta u_1 + P_2 \delta u_2$$



Virtual Work: Truss Example

rearranging:

$$\left(F_1 + \frac{F_2}{\sqrt{2}} - P_1\right)\delta u_1 + \left(\frac{F_2}{\sqrt{2}} + F_3 - P_2\right)\delta u_2 = 0$$

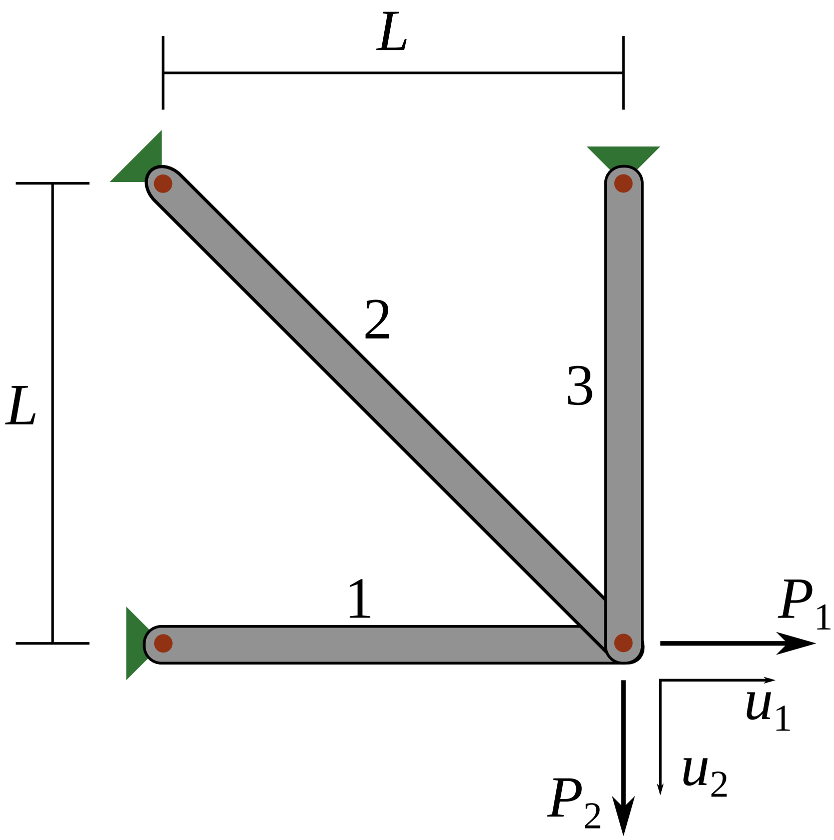
because the virtual displacements are arbitrary:

$$F_1 + \frac{F_2}{\sqrt{2}} - P_1 = 0$$

$$\frac{F_2}{\sqrt{2}} + F_3 - P_2 = 0$$

remarkably, this has recovered the equilibrium equations!

the equilibrium equations have been obtained simply through consideration of changes in energy in the system



Principle of Virtual Work

the previous result did not depend upon any particular constitutive law: it was not required that the system behave in a linear elastic manner, and hence can be used for any constitutive relation

if the system is assumed to behave in a linear elastic manner, additional results can be obtained

consider an elastic system where the external forces can be derived from potential functions, so that:

$$B_i = -\frac{\partial \mathscr{G}}{\partial u_i}; \quad T_i = -\frac{\partial \mathscr{T}}{\partial u_i}$$

so $\mathscr{G}$ and $\mathscr{T}$ are the potentials of the body force and the surface tractions, respectively

while this appears restrictive, most forces can be expressed in this manner (aeroelastic forces are an exception)

Theorem of Minimum Potential Energy

this is the principle of virtual work for a continuum:

$$\int_V B_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S = \int_V \sigma_{ij} \delta \epsilon_{ij} \, \mathrm{d}V$$

from the definition of strain energy density,

$$\int_V \sigma_{ij} \delta \epsilon_{ij} \, \mathrm{d}V = \int_V \frac{\partial \mathscr{W}}{\partial \epsilon_{ij}} \delta \epsilon_{ij} \, \mathrm{d}V = \int_V \delta \mathscr{W} \, \mathrm{d}V = \delta \int_V \mathscr{W} \, \mathrm{d}V$$

consider this notation to be analogous to the notation used in the truss example, where a real force times a virtual displacement gave a virtual energy; here a real stress times a virtual strain gives a virtual energy density

replace the body force and the surface tractions with their potentials:

$$\delta \int_V \mathscr{W} \, \mathrm{d}V + \delta \int_V \mathscr{G} \, \mathrm{d}V + \delta \int_S \mathscr{T} \, \mathrm{d}S = 0$$

Theorem of Minimum Potential Energy

define the *potential energy* of the system as:

$$\mathscr{V} = \int_V (\mathscr{W} + \mathscr{G}) \, \mathrm{d}V + \int_S \mathscr{T} \, \mathrm{d}S$$

the result above states that the virtual potential energy must be zero:

$$\delta \mathscr{V} = 0$$

in more descriptive terms, the value of the potential energy at equilibrium must be stationary; unless it is a minimum the equilibrium is unstable: for stable equilibrium, the potential energy must be a minimum

change the terminology: *virtual* quantities are referred to as *variations* in the mathematical literature

Notation in Variational Calculus

typically the notation for the variation of a function J is:

$$\delta J$$

when $J[u]$ is a functional of u, involving, for example, u and u', then:

$$\delta J = \int_a^b \left(F(x, u + \delta u, u' + \delta u') - F(x, u, u') \right) \mathrm{d}x$$

for example, if $F = (\mathrm{d}u/\mathrm{d}x)^2$,

$$\begin{aligned}
\delta J &= \int_a^b \left(\left(\frac{\mathrm{d}(u+\delta u)}{\mathrm{d}x} \right)^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \right) \mathrm{d}x \\
&= \int_a^b \left(\left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 + 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\delta u}{\mathrm{d}x} + \left(\frac{\mathrm{d}\delta u}{\mathrm{d}x} \right)^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}x} \right)^2 \right) \mathrm{d}x \\
&= \int_a^b 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\delta u}{\mathrm{d}x}\,\mathrm{d}x = \int_a^b 2\frac{\mathrm{d}u}{\mathrm{d}x}\delta u'\,\mathrm{d}x
\end{aligned}$$

Minimum Potential Energy and Equilibrium

from the concepts of virtual work, the first variation of the potential energy is:

$$\delta\mathscr{V} = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V - \int_V B_i\delta u_i\,\mathrm{d}V - \int_S T_i\delta u_i\,\mathrm{d}S$$

note that all of these terms are energy-valued, ie *scalar*

using Cauchy's Law, $T_i = \sigma_{ij}\nu_j$, and Gauss's divergence theorem:

$$\begin{aligned}
\int_S T_i\delta u_i\,\mathrm{d}S &= \int_S \sigma_{ij}\nu_j\delta u_i\,\mathrm{d}S = \int_V \left(\sigma_{ij}\delta u_i\right)_{,j}\,\mathrm{d}V \\
&= \int_V \sigma_{ij,j}\delta u_i\,\mathrm{d}V + \int_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V
\end{aligned}$$

because of the symmetry of σ_{ij}:

$$\int_V \sigma_{ij}\delta u_{i,j}\,\mathrm{d}V = \int_V \frac{1}{2}\sigma_{ij}\left(\delta u_{i,j} + \delta u_{j,i}\right)\,\mathrm{d}V = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V$$

Minimum Potential Energy and Equilibrium

this means that:

$$\int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V = \int_S \sigma_{ij}\nu_j\delta u_i\,\mathrm{d}S - \int_V \sigma_{ij,j}\delta u_i\,\mathrm{d}V$$

replacing this term in the expression for the first variation of potential energy and collecting integrals gives:

$$\int_V \left(\sigma_{ij,j} + B_i\right)\delta u_i\,\mathrm{d}V + \int_S \left(\sigma_{ij}\nu_j - T_i\right)\delta u_i\,\mathrm{d}S = 0$$

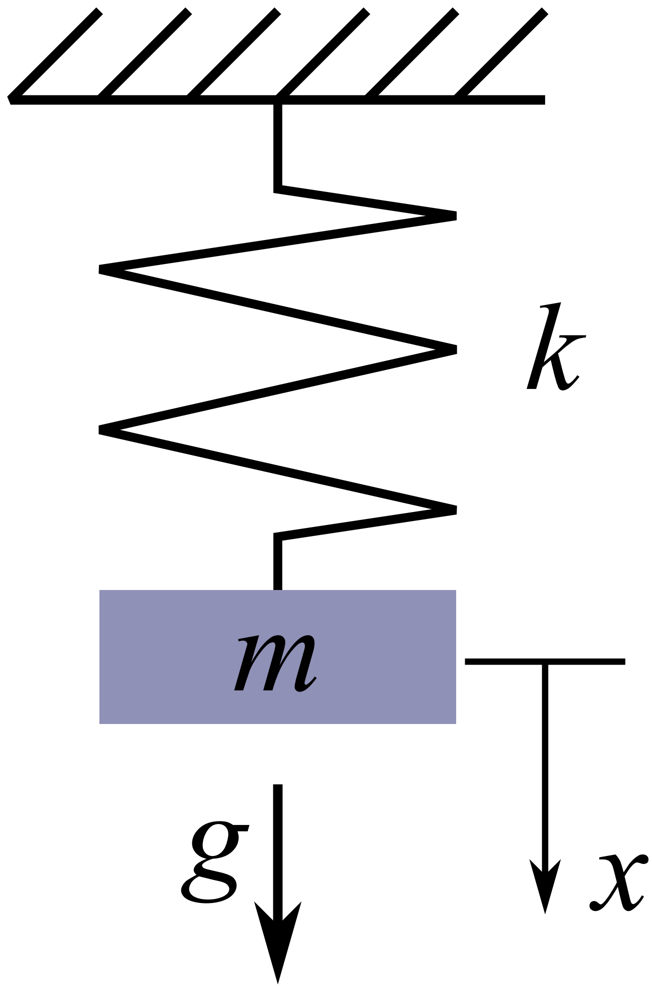
1. what conditions have to hold for this equation to be true?
2. what do these conditions represent?

Example: Direct Variations: Spring-Mass Problem

the mass hanging from a spring is a classic example of a problem that can be solved using the theorem of minimum potential energy (amongst other methods)

the unknown in the problem is the displacement x experienced by the object of mass m under the force of gravity g when hanging from a linear spring with constant k

take as datum the position of the object when the spring is unstretched, that is, when $x = 0$, and note that x is positive downward



Example: Direct Variations: Spring-Mass Problem

first, identify that the approach is to use the variational form of the theorem of minimum potential energy:

$$\delta\mathscr{V} = \delta\mathscr{U} + \delta\mathscr{E} = 0$$

second, calculate the internal potential energy; this is the strain energy that arises as a consequence of the combination of the spring extension and the force applied to cause this extension:

$$\mathscr{U} = \frac{1}{2}kx^2$$

calculate the first variation of U:

$$\begin{aligned}\delta\mathscr{U} &= \frac{1}{2}k\,(x+\delta x)^2 - \frac{1}{2}kx^2 \\ &= \frac{1}{2}k\left(x^2 + 2x\delta x + (\delta x)^2 - x^2\right) \\ &= kx\delta x\end{aligned}$$

Example: Direct Variations: Spring-Mass Problem

third, find the potential energy due to external forces; in this case it will be the motion of the mass acting under the force of gravity, and hence

$$\mathscr{E} = -mgx$$

and

$$\delta\mathscr{E} = -mg(x+\delta x) + mgx = -mg\delta x$$

fourth, combine the two expressions to complete the expression for the first variation of the total potential energy:

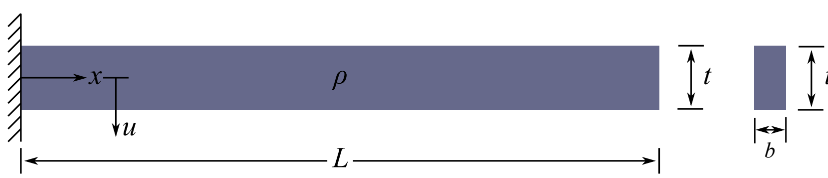
$$\delta\mathscr{V} = \delta\mathscr{U} + \delta\mathscr{E} = kx\delta x - mg\delta x = (kx - mg)\,\delta x = 0$$

because of the arbitrary value of δx, in order for this equation to be true, the term inside the brackets must be zero and hence:

$$x = \frac{mg}{k}$$

what is the physical meaning of the vanishing of the term inside the brackets?

Example: Direct Variations: Cantilever Under Self-Weight



the above cantilever beam of length L, height t and width b is composed of a material with constant density ρ and is unloaded except for its own mass due to gravity g

this beam will bend under its own self-weight

what is the deflected shape $u(x)$ that it will attain in equilibrium conditions?

Example: Direct Variations: Cantilever Under Self-Weight

because this is an equilibrium problem, use the theorem of minimum potential energy to solve it

for convenience use:

$$A = bt \qquad I = \frac{bt^3}{12}$$

the potential energy $\mathscr{V}$ is composed of the internal strain energy $\mathscr{U}$ due to bending and the external gravitational potential $\mathscr{E}$ associated with the displaced weight of the beam:

$$\mathscr{U} = \int_0^L \frac{1}{2} M\kappa \, \mathrm{d}x = \int_0^L \frac{1}{2} EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \mathrm{d}x$$

where $M(x)$ is the bending moment and $\kappa(x)$ is the local curvature; $M = EI\kappa$

$$\mathscr{E} = -\int_0^L \rho g A u \, \mathrm{d}x$$

Example: Direct Variations: Cantilever Under Self-Weight

combine these into a single statement of the total potential energy in the system:

$$\mathscr{V} = \int_0^L \left(\frac{1}{2} EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 - \rho g A u \right) \mathrm{d}x$$

the goal is to find the function $u(x)$ that minimizes $\mathscr{V}$; what method can be used to do this? can this be solved by the methods of calculus you have already learned?

this type of problem that variational calculus is intended to solve

Example: Direct Variations: Cantilever Under Self-Weight

take the first variation of $\mathscr{V}$ with respect to the displacement u:

$$\delta\mathscr{V} = \mathscr{V}(u + \delta u) - \mathscr{V}(u) = \delta\mathscr{U} + \delta\mathscr{E} = \mathscr{U}(u + \delta u) - \mathscr{U}(u) + \mathscr{E}(u + \delta u) - \mathscr{E}(u)$$

find the first variation of the internal strain energy:

$$\begin{aligned}
\delta\mathscr{U} = \mathscr{U}(u + \delta u) - \mathscr{U}(u) &= \frac{1}{2} EI \int_0^L \left(\left(\frac{\mathrm{d}^2 (u + \delta u)}{\mathrm{d}x^2} \right)^2 - \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \right) \mathrm{d}x \\
&= \frac{1}{2} EI \int_0^L \left(\left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 + 2 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right)^2 - \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \right)^2 \right) \mathrm{d}x \\
&= EI \int_0^L \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) \mathrm{d}x
\end{aligned}$$

Example: Direct Variations: Cantilever Under Self-Weight

next, find the first variation of the external potential energy:

$$\delta\mathscr{E} = \mathscr{E}(u+\delta u) - \mathscr{E}(u) = -\rho g A \int_0^L (u + \delta u - u)\ \mathrm{d}x$$

$$= -\rho g A \int_0^L \delta u \, \mathrm{d}x$$

hence, the first variation of the total potential energy is:

$$\delta\mathscr{V} = \int_0^L \left(EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) - \rho g A \delta u \right) \mathrm{d}x$$

the goal is to find the function $u(x)$ that makes the first variation of $\mathscr{V}$ equal to zero

Example: Direct Variations: Cantilever Under Self-Weight

integrate twice by parts the term containing the two second derivatives:

$$\begin{aligned}
\delta\mathscr{V} &= \int_0^L \left(EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}^2 \delta u}{\mathrm{d}x^2} \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L + \int_0^L \left(-EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(-EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \delta u \right) - \rho g A \delta u \right) \mathrm{d}x \\
&= EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \right) - \rho g A \right) \delta u \, \mathrm{d}x
\end{aligned}$$

Example: Direct Variations: Cantilever Under Self-Weight

we have the first variation of the total potential energy:

$$\delta\mathscr{V} = EI \left(\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \frac{\mathrm{d}\delta u}{\mathrm{d}x} \right) \Bigg|_0^L - EI \left(\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} \delta u \right) \Bigg|_0^L + \int_0^L \left(EI \left(\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \right) - \rho g A \right) \delta u \, \mathrm{d}x$$

look at the two boundary terms generated by the integrations by parts; in each of the four cases that must be evaluated, one of the factors is zero because of the physical boundary conditions of the problem: at $x = 0$, δu and $\mathrm{d}\delta u/\,\mathrm{d}x$ are zero, while at $x = L$, $\mathrm{d}^2 u/\,\mathrm{d}x^2$ and $\mathrm{d}^3 u/\,\mathrm{d}x^3$ are zero hence all of the boundary terms are zero

what remains is:

$$\delta\mathscr{V} = \int_0^L \left(EI \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \rho g A \right) \delta u \, \mathrm{d}x = 0$$

in order for this always to be true, given the arbitrariness on δu, the bracketed part of the integrand must be zero:

$$EI \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - \rho g A = 0$$

Example: Direct Variations: Cantilever Under Self-Weight

the equation remaining to solve is an ordinary differential equation that we can solve by repeated integration; by replacing $\rho g A$ with w, this would become the Euler-Bernoulli beam equation:

$$EI \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = \rho g A$$

integrating four times and solving for the constants of integration using the physical boundary conditions generates:

$$u(x) = \frac{\rho g A}{EI} \left(\frac{x^4}{24} - \frac{L x^3}{6} + \frac{L^2 x^2}{4} \right)$$

where the maximum deflection, at $x = L$ is:

$$u(L) = \frac{\rho g A L^4}{8EI} = \frac{3 \rho g L^4}{2Et^2}$$

3.3 Variational Methods

Variational Calculus

the goal of structural analysis is to find conditions of minimum total potential energy: this is minimising a functional (a function of a function), so the techniques of variational calculus are extremely important

the notation used in the description of virtual work (δu_i, δE etc) can be transferred directly to the notation of variational calculus: instead of referring to virtual quantities, refer to variations of a function or a functional

stated more rigorously, if there is a function u defined between points a and b, with known values $u(a)$ and $u(b)$, the first variation of u is δu, where δu has the following properties:

- δu can be decomposed into the product $\epsilon \eta$
- ϵ is an infinitesimally small quantity

- η is an arbitrary function with $\eta(a) = \eta(b) = 0$

at this point, the function u may be unknown; u may be scalar-valued, vector-valued, or higher-order-valued

Variational Calculus

the archetypal problem in variational calculus is the brachistochrone problem: given two points $(a, u(a))$ and $(b, u(b))$ in a gravitational field, what path would a mass take to minimise the time to move from $(a, u(a))$ to $(b, u(b))$ if it were to be accelerated only by gravity?

more generally, variational calculus looks to solve problems of minimisation of functionals; for example, consider the functional $J[u]$:

$$J[u] = \int_a^b F(x, u, u')\,\mathrm{d}x$$

where the function u satisfies the boundary conditions $u(a) = u_a$ and $u(b) = u_b$

this functional is of the same form as the total potential energy

to make progress, employ the assumption that F is continuous and differentiable with respect to x, u and u' up to second order partial derivatives, and that such derivatives are continuous

Variational Calculus

there is an infinitely large family of functions $u(x)$ which satisfy the boundary conditions and continuity requirements set out; however, only one of these functions – call it $y(x)$ – also minimises the functional $J[u]$

the goal of variational calculus is to find this function

assume that the problem of minimising $J[u]$ has the solution $y(x)$ which is a member of the family of functions $u(x)$ which satisfy the boundary conditions and continuity requirements; if so, then:

$$J[y] \leq J[u]$$

create the function $\eta(x)$ with the properties that $\eta(x)$, $\eta'(x)$ and $\eta''(x)$ are continuous on $a \leq x \leq b$ and that $\eta(a) = \eta(b) = 0$

any function $u(x)$ that is reasonably close to $y(x)$ can be expressed as:

$$u(x) = y(x) + \epsilon\eta(x)$$

for some $\eta(x)$ and ϵ

Variational Calculus

introduce the function:

$$\phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F[x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)]\,\mathrm{d}x$$

since $y(x)$ is a particular function, assumed to be known, $\phi(\epsilon)$ is a function only of ϵ for a given $\eta(x)$

as a consequence of $J[y]$ minimising J, it must be that:

$$\phi(0) \le \phi(\epsilon)$$

which implies:

$$\phi'(0) = 0$$

where the prime indicates differentiation with respect to ϵ

Variational Calculus

differentiate under the integral:

$$\begin{aligned}\phi'(\epsilon) &= \int_a^b \left[F_u\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right)\eta(x)\right.\\ &+ \left. F_{u'}\left(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x)\right)\eta'(x)\right]\,\mathrm{d}x\end{aligned}$$

where:

$$F_u = \frac{\partial F}{\partial u}; \quad F_{u'} = \frac{\partial F}{\partial u'}$$

more compactly, after removing the explicit functional dependances:

$$\phi'(\epsilon) = \int_a^b \left(F_u\eta + F_{u'}\eta'\right)\,\mathrm{d}x$$

integrating the second term by parts produces:

$$\phi'(\epsilon) = \int_a^b \left[F_u - \frac{\mathrm{d}}{\mathrm{d}x}F_{u'}\right]\eta(x)\,\mathrm{d}x + F_{u'}\eta(x)\Big|_a^b$$

because $\eta(a) = \eta(b) = 0$, the last term vanishes

Variational Calculus

this produces the equation:

$$0 = \phi'(0) = \int_a^b \left[F_y\left(x, y, y'\right) - \frac{\mathrm{d}}{\mathrm{d}x}F_{y'}\left(x, y, y'\right)\right]\eta(x)\,\mathrm{d}x$$

where, because $\epsilon = 0$, the relevant member of the family u of functions is now y, the solution to the minimisation problem; the notation changes to reflect this the fundamental lemma of variational calculus states: *If $\psi(x)$ is a continuous function on $a \leq x \leq b$ and the relation*

$$\int_a^b \psi(x)\eta(x)\,\mathrm{d}x = 0$$

holds for all functions $\eta(x)$ which vanish at a and b and are continuous with continuous derivatives, then:

$$\psi(x) \equiv 0$$

the consequence of this lemma is that the term in the integrand multiplied by $\eta(x)$ must be uniformly zero

Variational Calculus

this leads directly to Euler's equation of variational calculus:

$$F_y\,(x, y, y') - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'}\,(x, y, y') = 0$$

writing this in expanded form gives:

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\frac{\partial^2 F}{\partial y' \partial y'} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial^2 F}{\partial y' \partial y} - \frac{\partial^2 F}{\partial y' \partial x} = 0$$

the goal is to find the function $y(x)$ which minimises $J[u]$; the satisfaction of Euler's equation is necessary for the function $y(x)$ to provide a local extremum of $J[u]$

Variational Calculus

to return to the more common notation, it is customary to call the function $\epsilon\eta(x)$ the first variation of $u(x)$ and write:

$$\delta u(x) = \epsilon\eta(x)$$

similarly, the first variation of the functional $J[u]$ is:

$$\delta J = \epsilon\phi'(\epsilon)$$

this notation is analogous to differential notation, where the differential of f, $\mathrm{d}f = \epsilon f'(x)$

Variational Calculus and Minimum Potential Energy

the goal of structural analysis is to find deformed configurations that minimise the total potential energy of a system

align the physical problem with the mathematical structure of variational calculus: the functional to be minimised is the total potential energy $\mathscr{V}$

the total potential energy arises because of the deformations u_i, so $\mathscr{V} = \mathscr{V}(u_i)$; note that the deformations are vector quantities the deformations form a field: they vary with position, so $u_i = u_i(x)$, where x is a position in space

the total potential energy is the integral over the body of some set of functions that relate the displacements u_i to the strain energy and the external potential energy

because the total potential energy is a function of deformation, which is a function of position, the total potential energy is a functional

Example: Variational Calculus

as an example, take the functional:

$$J[u] = \int_0^1 \left(1 + (u')^2\right) \, \mathrm{d}x$$

with boundary conditions:

$$u(0) = 0 \qquad u(1) = 1$$

find the function y that minimises J

use Euler's equation where:

$$F = 1 + (u')^2$$

hence:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = 0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(2u'\right) = 0$$

Example: Variational Calculus

evaluating the previous expression:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(2u'\right) = -2u'' = 0$$

and integrate; after solving for the constants, it is the straight line:

$$y = x$$

what this result states is that the minimum distance between points $(0, 0)$ and $(1, 1)$ is a straight line (note that the initial integrand F is the square of the distance along a line)

Example: Variational Calculus

find the function y that minimises the functional:

$$J[u] = \int_0^{\pi/2} \left((u')^2 - u^2\right) \, \mathrm{d}x$$

with boundary conditions:

$$u(0) = 0 \qquad u\left(\frac{\pi}{2}\right) = 1$$

the integrand is:

$$F = (u')^2 - u^2$$

and applying Euler's equation gives:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} = -2u - 2u'' = 0$$

this is $y = \sin(x)$

Variations with Higher Derivatives

it may be necessary to find minima of functionals of the form:

$$J[u] = \int_a^b F\left(x, u, u', u''\right)\,\mathrm{d}x$$

there is a family of functions u with continuous derivatives and known values of the function u and its first derivative u' at the boundaries

construct the arbitrary function $\eta(x)$ which has continuous derivatives and both η and its η' are zero at the boundaries

to find the function $y(x)$ that minimises $J[u]$, form the function $u(x) = y(x) + \epsilon\eta(x)$

Variations with Higher Derivatives

this generates the functional:

$$\phi(\epsilon) = J[y + \epsilon\eta] = \int_a^b F\left(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta''\right)\,\mathrm{d}x$$

again, differentiate under the integral, integrate by parts and remove terms equal to zero:

$$0 = \phi'(\epsilon) = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial u''}\right)\eta\,\mathrm{d}x$$

in general for an integrand F that is a function of the derivatives of y up to order n, the following holds:

$$\sum_{\nu=0}^{n} (-1)^\nu \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left(\frac{\partial F}{\partial u^{(\nu)}}\right) = 0$$

with this it is possible to handle variational problems involving derivatives of any order; needless to say, solving the resulting differential equation is not necessarily easy

Example: Cantilever Under Self-Weight

recall the potential energy of a cantilever loaded only by its own self-weight:

$$\mathscr{V} = \int_0^L \left(\frac{1}{2}EI\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 - \rho g A u\right)\mathrm{d}x$$

here:

$$F = \frac{1}{2}EI\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 - \rho gAu$$

based on the last slide, this version of the Euler-Lagrange equation for functionals involving second derivatives can be employed:

$$\frac{\partial F}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial u'} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial u''} = 0$$

Example: Cantilever Under Self-Weight

there are no terms involving u', but u and u'' appear:

$$\frac{\partial F}{\partial u} = -\rho gA \qquad\qquad \frac{\partial F}{\partial u''} = EIu''$$

evaluating the term involving u'' in the Euler-Lagrange equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}EIu'' = EIu''''$$

and hence the totality of the Euler-Lagrange equation applied to the cantilever loaded by its self-weight is:

$$EIu'''' - \rho gA = 0$$

this, once again, is the fundamental expression for Euler-Bernoulli beam theory

3.4 Variational Equations of Motion

Variational Equations of Motion

for small displacements, the equation of motion in Eulerian coordinates is:

$$\sigma_{ji,j} + B_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

consider virtual displacements δu_i which are zero over the parts of the surface where the displacements are specified; if so, the external virtual work done by the surface tractions and the body forces is:

$$\delta\mathscr{E} = \int_V B_i\delta u_i \,\mathrm{d}V + \int_S T_i\delta u_i \,\mathrm{d}S$$

transform the surface integral into a volume integral using Cauchy's Law and Gauss's theorem:

$$\begin{aligned}\int_S T_i \delta u_i \,\mathrm{d}S &= \int_S \sigma_{ji} \nu_j \delta u_i \,\mathrm{d}S = \int_V \left(\sigma_{ji} \delta u_i\right)_{,j} \,\mathrm{d}V \\ &= \int_V \sigma_{ji,j} \delta u_i \,\mathrm{d}V + \int_V \sigma_{ji} \delta u_{i,j} \,\mathrm{d}V\end{aligned}$$

Variational Equations of Motion

substituting in the equation of motion:

$$\int_V \sigma_{ji,j} \delta u_i \,\mathrm{d}V + \int_V \sigma_{ji} \delta u_{i,j} \,\mathrm{d}V = \int_V \left(\rho \frac{\partial^2 u_i}{\partial t^2} - B_i\right) \delta u_i \,\mathrm{d}V + \int_V \sigma_{ji} \delta \epsilon_{ij} \,\mathrm{d}V$$

combining all of this gives:

$$\int_V \sigma_{ij} \delta \epsilon_{ij} \,\mathrm{d}V = \int_V \left(B_i - \rho \frac{\partial^2 u_i}{\partial t^2}\right) \delta u_i \,\mathrm{d}V + \int_S T_i \delta u_i \,\mathrm{d}S$$

this is the variational equation of motion

for an elastic system with a strain energy density $\mathscr{W}$:

$$\delta \int_V \mathscr{W} \,\mathrm{d}V = \int_V \left(B_i - \rho \frac{\partial^2 u_i}{\partial t^2}\right) \delta u_i \,\mathrm{d}V + \int_S T_i \delta u_i \,\mathrm{d}S$$

Variational Equations of Motion

integrate this over an arbitrary period of time, from t_0 to t_1:

$$\int_{t_0}^{t_1} \int_V \delta \mathscr{W} \,\mathrm{d}V \,\mathrm{d}t = \int_{t_0}^{t_1} \left[\int_V B_i \delta u_i \,\mathrm{d}V + \int_S T_i \delta u_i \,\mathrm{d}S - \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \,\mathrm{d}V\right] \mathrm{d}t$$

take the acceleration term, reverse the order of integration and integrate by parts to get:

$$\int_{t_0}^{t_1} \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \,\mathrm{d}V \,\mathrm{d}t = \int_V \rho \frac{\partial u_i}{\partial t} \delta u_i \,\mathrm{d}V \Big|_{t_0}^{t_1} - \int_V \int_{t_0}^{t_1} \frac{\partial u_i}{\partial t} \left(\rho \frac{\partial \delta u_i}{\partial t} + \frac{\partial \rho}{\partial t} \delta u_i\right) \mathrm{d}t \,\mathrm{d}V$$

require that $\delta u_i(t_0) = \delta u_i(t_1) = 0$ and take note of the density partial derivative, which by the equation of continuity becomes a partial derivative of velocity: this makes that term of higher order and hence negligible

Variational Equations of Motion

the result is that:

$$\int_{t_0}^{t_1} \int_V \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i \, \mathrm{d}V \, \mathrm{d}t = -\int_{t_0}^{t_1} \int_V \rho \frac{\partial u_i}{\partial t} \frac{\partial \delta u_i}{\partial t} \, \mathrm{d}V \, \mathrm{d}t = -\int_{t_0}^{t_1} \delta \mathscr{K} \, \mathrm{d}t$$

which is the first variation of the kinetic energy of the moving body:

$$\mathscr{K} = \frac{1}{2} \int_V \rho \left(\frac{\partial u_i}{\partial t} \right)^2 \mathrm{d}V$$

recall the definition of the variation of potential energy:

$$\delta \mathscr{V} = \int_V \delta \mathscr{W} \, \mathrm{d}V - \int_V B_i \delta u_i \, \mathrm{d}V - \int_S T_i \delta u_i \, \mathrm{d}S$$

combine everything to get:

$$0 = \int_{t_0}^{t_1} \delta \mathscr{K} - \delta \mathscr{V} \, \mathrm{d}t$$

Lagrangian and Hamiltonian

the main result is:

$$0 = \int_{t_0}^{t_1} \delta \mathscr{K} - \delta \mathscr{V} \, \mathrm{d}t$$

where the Lagrangian is:

$$\mathscr{L} = \mathscr{K} - \mathscr{V}$$

and the Hamiltonian is:

$$\mathscr{A} = \int_{t_0}^{t_1} \mathscr{L} \, \mathrm{d}t$$

manipulation of the variational equation of motion shows that

$$\delta \mathscr{A} = \int_{t_0}^{t_1} \delta \mathscr{K} - \delta \mathscr{V} \, \mathrm{d}t = 0$$

that is, the variation of the Hamiltonian is zero for an admissible motion

Hamilton's Principle

Hamilton's principle states:

> the time integral of the Lagrangian function between two arbitrary times is an extremum for the "actual" motion with respect to all admissible virtual displacements which vanish at times t_0 and t_1 and on the parts of the boundary with displacements prescribed

3.5 Strong Forms and Weak Forms

Strong Form of the Equilibrium Equations

the standard ("strong") forms for the equilibrium equations in linearised elasticity are:

1. equilibrium:
$$\sigma_{ji,j} + B_i = 0$$
2. constitutive law:
$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$
3. strain - displacement relation:
$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$
4. traction boundary conditions:
$$\sigma_{ji}\nu_j = T_i \;\text{ on }\; S_\sigma$$
5. displacement boundary conditions:
$$u_i = \bar{u} \;\text{ on }\; S_u$$

combine 1, 2 and 3 to get:

$$\frac{1}{2}\left(C_{ijkl}\left(u_{k,l} + u_{l,k}\right)\right)_{,j} + B_i = 0$$

Weak Form of the Equilibrium Equations

take the equilibrium equation of the strong form as expressed above, multiply it by a "weight function" δw_i and integrate over the volume:

$$\int_V \sigma_{ji,j}\delta w_i \,\mathrm{d}V + \int_V B_i\delta w_i \,\mathrm{d}V = 0$$

using Gauss's theorem, transform the first integral into:

$$\int_V \sigma_{ji,j}\delta w_i \,\mathrm{d}V = \int_S \left(\delta w_i\sigma_{ji}\right)\nu_j \,\mathrm{d}S - \int_V \delta w_{i,j}\sigma_{ji} \,\mathrm{d}V$$

substituting, this generates:

$$\int_V \delta w_{i,j}\sigma_{ji} \,\mathrm{d}V = \int_V \delta w_i B_i \,\mathrm{d}V + \int_S \left(\delta w_i\sigma_{ji}\right)\nu_j \,\mathrm{d}S$$

which can be written as:

$$\int_V \delta w_{i,j}C_{ijkl}\epsilon_{kl} \,\mathrm{d}V = \int_V \delta w_i B_i \,\mathrm{d}V + \int_S \left(\delta w_i C_{ijkl}\epsilon_{kl}\right)\nu_j \,\mathrm{d}S$$

Why Do This??

replacing the strains with the derivatives of displacement:

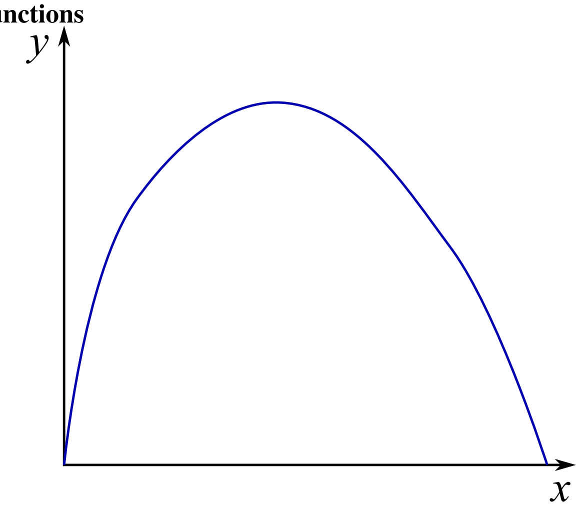
$$\frac{1}{2}\int_V \delta w_{i,j} C_{ijkl}\left(u_{k,l}+u_{l,k}\right)\,\mathrm{d}V = \int_V \delta w_i B_i\,\mathrm{d}V + \frac{1}{2}\int_S \left(\delta w_i C_{ijkl}\left(u_{k,l}+u_{l,k}\right)\right)\nu_j\,\mathrm{d}S$$

clearly this is a legitimate mathematical operation: multiplying by the variation of w_i, which is some arbitrary function, and then integrating over the domain should still produce zero if the original function is zero everywhere

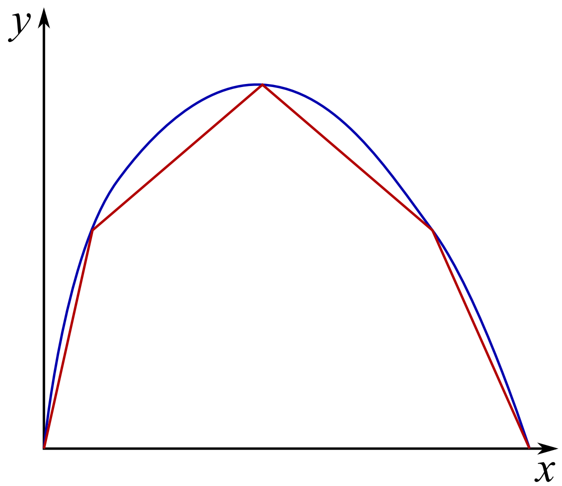
so why do this?

examine the order of the derivatives in the strong and weak forms, and recall that strain is a derivative of displacement: what benefit could this have?

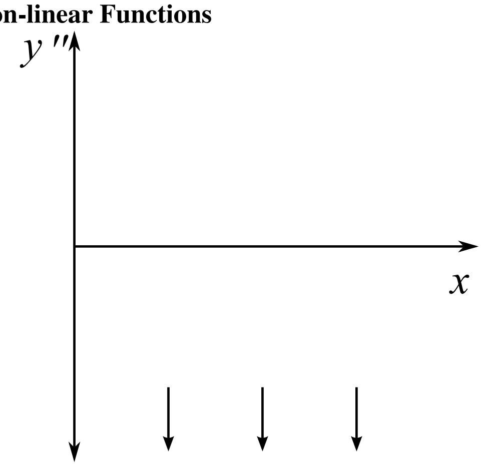
Non-linear Functions



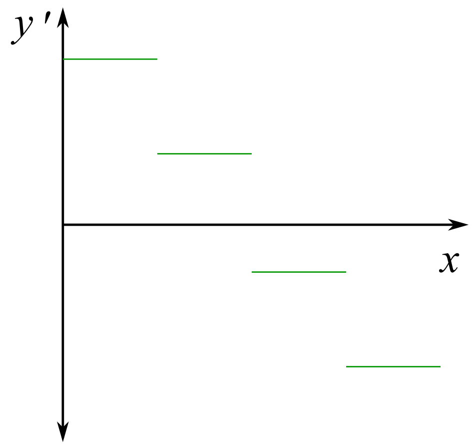
Approximating Non-linear Functions



Approximating Non-linear Functions



Approximating Non-linear Functions



The Value of the Weak Form

the key advantage to the weak form is that there are no derivatives higher than the first

to approximate a displacement function with a set of piecewise continuous functions, the first derivative can be integrated, but the second derivative is problematic

the weak form allows much looser requirements on the smoothness of the functions used to approximate the actual solution: this makes finite element analysis possible

3.6 Linking Energy Methods to Finite Elements

Background to Finite Element Methods

classical elasticity methods, based upon the principles discovered by Newton and Hooke and extended by Euler and Cauchy, provide elegant, exact and complete solutions to a very limited number of problems; a more general method for solving solids problems is desirable

by approaching problems in solids mechanics from the perspective of energy balances, it is possible to change the formulation of the problem to permit approximate solutions which satisfy (approximately) the equations of equilibrium

the finite element method is an example of a *Galerkin* method of solution; the fundamental concept of finite elements is that the problem domain is broken into discrete parts (finite elements) and the discretisation is performed by a conversion to the weak form and the use of linear interpolants both as the trial functions and the weight functions

Background to Finite Element Methods

in practice, the mechanics of the finite element method are generally hidden behind the interface of a commercial software package

Richard Courant published the “first” paper in finite element analysis 1943 wherein he used triangular elements to solve some vibrational problems

a stronger mathematical footing was found when it was shown that for benign conditions, the finite element method would always converge to the correct solution provided the number of elements went to infinity

the falling cost of computer power makes finite element analysis very widely available, and very widely used

today, billions of dollars are spent annually on finite element analysis, and many major commercial software packages are available, including ABAQUS, ANSYS, HyperWorks, Comsol and LS-DYNA

Overview of Finite Element Methods

there are a standard set of steps involved in the finite element method, all of which will be covered, at least briefly, in AER 1403

before beginning to solve any problem using the finite element methods, it is essential to ensure that the problem is properly formulated: for static finite element methods the system must be in equilibrium and must have sufficient, but not excessive, boundary conditions

the problem is discretized through a process called *meshing*, wherein the problem domain is divided into elements that are connected at nodes

in general, elements should be as close as possible to squares (for quadrilateral elements) or equilateral triangles (for triangular elements) as possible

however, for non-regular problem domains this must be only approximate

Overview of Finite Element Methods

within each element, the displacement is approximated by interpolation from the displacements at the nodes; in AER 1403 the interpolations will be linear but in principle they could be of higher order

the interpolants are called *shape functions*

using the weak form of the equilibrium equations and the approximations given by the shape functions, the equilibrium expression for each element is constructed

integration of the weak form generates the element stiffness matrix

assembly of the stiffness matrices for all the elements in the system creates the global structural stiffness matrix, which should be symmetric, invertable and positive definite

the solution of this matrix accompanied by the vector of applied forces generates the vector of approximate nodal displacements

3.7 1-D Element

Finite Elements in One Dimension

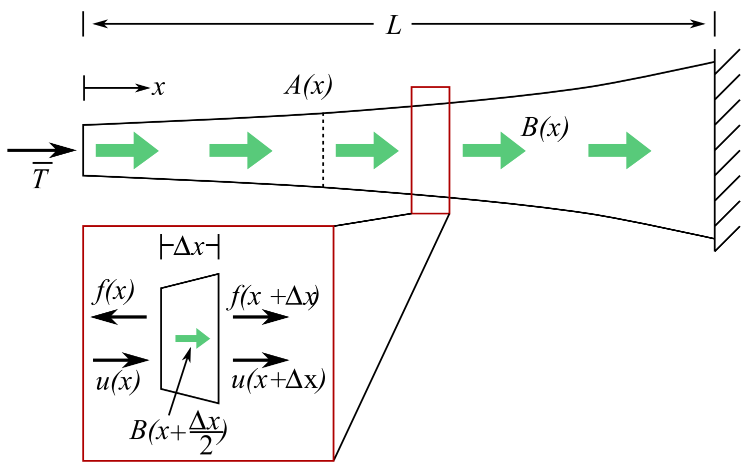
take a one-dimensional beam which is oriented parallel to the x-axis, which has varying area $A(x)$, Young's modulus E, and has body force $B(x)$ acting in the x-direction

the beam is fixed at $x = L$, and has traction $\bar{T}$ acting at $x = 0$ which has units of force per area

all displacements are in the x-direction

equilibrium of a small element of the beam is shown in the inset

Finite Elements in One Dimension



Finite Elements in One Dimension

the bar must satisfy several conditions:

1. the forces must be in equilibrium
2. the material must satisfy Hooke's Law: $\sigma(x) = E\epsilon(x)$
3. the displacement field must be smooth and compatible
4. the kinematics must obey the one-dimensional strain - displacement relation: $\epsilon(x) = \mathrm{d}u/\mathrm{d}x$

Strong Form in One Dimension

the governing equation for equilibrium of the bar is determined from the local equilibrium of an element, where $f(x)$ is the internal force on a cross-section and $B(x)$ is the force per unit length:

$$-f(x) + B\left(x + \frac{\Delta x}{2}\right)\Delta x + f(x + \Delta x) = 0$$

rearranging terms and dividing by Δx:

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}+B\left(x+\frac{\Delta x}{2}\right)=0$$

taking the limit as Δx goes to zero gives the one-dimensional analogue to the equation of equilibrium:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}+B(x)=0$$

Strong Form in One Dimension

in one dimension, define stress as force per area:

$$\sigma(x)=\frac{f(x)}{A(x)}$$

coupled with the constitutive law and the strain - displacement relation this gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+B=0$$

which is a second-order ordinary differential equation

to solve this, the boundary conditions at the two ends of the beam are needed; for this case they are:

$$\sigma(0)=-\bar{T}$$
$$u(L)=0$$

these final three expressions make up the strong form in one dimension

Weak Form in One Dimension

to develop the weak form in one dimension, take the equilibrium equation and the traction boundary condition, multiply them by the "weight function" δw and integrate the equilibrium equation over the domain:

$$\int_0^L \delta w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+B\right)\,\mathrm{d}x=0$$
$$\left(\delta wA\left(E\frac{\mathrm{d}u}{\mathrm{d}x}+\bar{T}\right)\right)_{x=0}=0$$

note that δw is restricted to be zero at $x=L$

integrate by parts:

$$\left[\delta wAE\frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L-\int_0^L\frac{\mathrm{d}\delta w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x+\int_0^L\delta wB\,\mathrm{d}x=0$$

and evaluate:

$$(\delta wA\sigma)_{x=L}-(\delta wA\sigma)_{x=0}-\int_0^L\frac{\mathrm{d}\delta w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x+\int_0^L\delta wB\,\mathrm{d}x=0$$

Weak Form in One Dimension

the first term is zero because $\delta w(L) = 0$, and $\sigma(0) = -\bar{T}$:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta w B\,\mathrm{d}x + \left(\delta w A\bar{T}\right)_{x=0}$$

this is the weak form of the governing equations, and the function $u(x)$ with $u(L) = 0$ which satisfies the weak form for all δw with $\delta w(L) = 0$, is the solution to the equation

because the displacement boundary condition must be satisfied by any possible solution $u(x)$, it is called an *essential* boundary condition; the traction boundary condition is satisfied within the weak form and is called a *natural* boundary condition

Equivalence of the Weak and Strong Forms

this is the weak form:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta w B\,\mathrm{d}x + \left(\delta w A\bar{T}\right)_{x=0}$$

it is necessary to show that the function $u(x)$ which satisfies the weak form is also the solution to the strong form by reversing the order of the analysis and showing that no information is lost or added

integration by parts shows us that:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \left[\delta w AE \frac{\mathrm{d}u}{\mathrm{d}x}\right]_0^L - \int_0^L \delta w \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

substituting into the weak form and collecting integrals generates:

$$\int_0^L \delta w \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right)\mathrm{d}x + \delta w A\left(\bar{T} + \sigma\right)_{x=0} = 0$$

Equivalence of the Weak and Strong Forms

because δw is an arbitrary function, it can be anything; set it to:

$$\delta w = \psi(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right)$$

where $\psi(x)$ is smooth, positive, and vanishes on the boundaries

this yields:

$$\int_0^L \psi\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + B\right)^2 \mathrm{d}x = 0$$

and the boundary term is zero because of how δw has been defined because the integrand is the product of a positive function ψ and a quadratic term, the integral must always be positive, unless the integrand is zero, which requires:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+B=0$$

and the strong form of the equilibrium equation has been recovered

Equivalence of the Weak and Strong Forms

return a couple of steps and using an alternate δw generates:

$$\delta wA\left(\bar{T}+\sigma\right)_{x=0}=0$$

the only way this can always be true is if

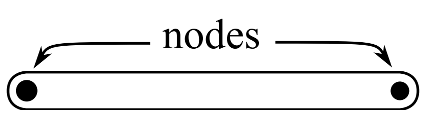
$$\sigma(0)=-\bar{T}$$

all aspects of the strong form have been recovered, and hence a solution to the weak form must also be a solution to the strong form

Approximating the Solution and Weight Functions

a key aspect of finite element analysis is that the actual solution (displacements, in the case of solid mechanics) is approximated by dividing the structure into pieces (the "finite elements") and solving approximately in each element

a one-dimensional system can be divided into one-dimensional *elements*; at each end of an element it will be connected to other elements or a boundary at a location called a *node*



call the displacements at the nodes d_1 and d_2 and approximate the displacement at any position in the element based on displacements at the nodes, so $u(x) = F_1(x, d_1) + F_2(x, d_2)$, where F_1 and F_2 are interpolating functions

Approximating the Solution and Weight Functions

the easiest approximation is linear; use a system of local coordinates such that node 1 is at position $x_1 = 0$ and node 2 is at position $x_2 = \ell$ (the length of the element is $\ell = x_2 - x_1$)

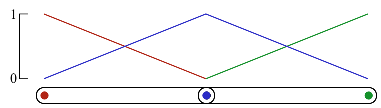
create a local position variable x which then ranges from zero to ℓ

a linear approximation of the displacements throughout the element, using only the nodal displacements d_1 and d_2, is:

$$u(x) = d_1 \frac{\ell - x}{\ell} + d_2 \frac{x}{\ell}$$

this linearisation has the property that the displacement at node 1 has no effect on the displacement at node 2 (and *vice versa*)

this figure shows the regions of *influence* of the displacements at the nodes; note that at any location the sum of the influence lines is 1



Approximating the Solution and Weight Functions

these linear functions are going to be used to approximate the actual solution within each element

note that it is very easy to take the derivatives of these functions, which are what will be necessary for the integrations of the stiffness matrices:

$$u'(x) = \frac{1}{\ell}(d_2 - d_1)$$

the weight functions δw are also approximated by the same interpolating function as the solution functions: the values of the weight functions at the nodes are known $(\delta w_1, \delta w_2)$ and are used to approximate the value of the weight function anywhere in the element

$$\delta w = \delta w_1 \frac{\ell - x}{\ell} + \delta w_2 \frac{x}{\ell}$$

finite element methods which use the same approximation for the solution ('trial') and the weight ('test') function are called "Galerkin methods"

Discretisation of the Weak Form

for convenience and compactness, write the equations in vector - matrix form

the solution function $u(x)$ for the displacements is approximated by:

$$u(x) \approx \begin{bmatrix} \frac{\ell-x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \mathbf{Nd}$$

where **N** is the element shape function and **d** is the vector of nodal displacements

select a value $0 \le x \le \ell$ and evaluate the shape functions at x to get an approximate displacement $u(x)$ at x using the nodal displacements d_1 and d_2

the derivative of the solution function, $u'(x)$ is:

$$u'(x) \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \mathbf{Hd}$$

for the weight function δw and its derivative:

$$\delta w \approx \begin{bmatrix} \frac{\ell - x}{\ell} & \frac{x}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{Nw} \qquad \delta w' \approx \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \mathbf{Hw}$$

Discretisation of the Weak Form

begin with the weak form:

$$\int_0^L \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \delta w B\,\mathrm{d}x + \left(\delta w A\bar{T}\right)_{x=0}$$

decompose the system into elements and replace the integral over the whole domain with a sum of the integrals over each element:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \frac{\mathrm{d}\delta w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x - \int_{x_1}^{x_2} \delta w B\,\mathrm{d}x - \left(\delta w A\bar{T}\right)_{x=0} \right) = 0$$

replace the displacement $u(x)$ and the weight function δw with their approximate matrix representations:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \mathbf{w}^T \mathbf{H}^T AE\mathbf{Hd}\,\mathrm{d}x - \int_{x_1}^{x_2} \mathbf{w}^T \mathbf{N}^T B\,\mathrm{d}x - \left(\mathbf{w}^T \mathbf{N}^T A\bar{T}\right)_{x=0} \right) = 0$$

by interpreting δw as a variation of displacement, this generates the approximate, discretized one-dimensional theorem of minimum potential energy

Discretisation of the Weak Form

since $\mathbf{w}$ is an arbitrary function, it can be removed from the integrations; similarly, $\mathbf{d}$ is not a function of x, so it can come outside the first integral:

$$\sum^{\text{elems}} \mathbf{w}^T \left(\int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\,\mathrm{d}x\mathbf{d} - \int_{x_1}^{x_2} \mathbf{N}^T B\,\mathrm{d}x - \left(\mathbf{N}^T A\bar{T}\right)_{x=0} \right) = 0$$

the integral:

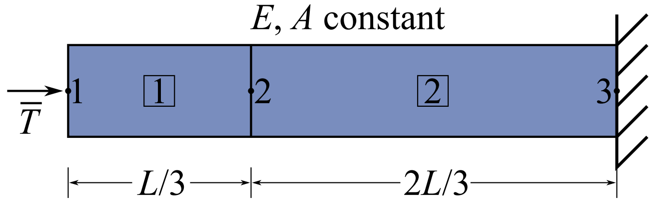
$$\mathbf{K} = \int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\,\mathrm{d}x$$

is the element stiffness matrix; assembling these for all of the elements generates the structural stiffness matrix

similarly, the external body and boundary forces are:

$$\mathbf{f}_\Omega = \int_{x_1}^{x_2} \mathbf{N}^T B\,\mathrm{d}x; \qquad \mathbf{f}_\Gamma = \left(\mathbf{N}^T A\bar{T}\right)_{x=0}$$

Example: One-Dimensional Finite Element



the bar shown above is one-dimensional: all forces and displacements are axial to the bar

it is fixed at the right end and loaded with a traction $\bar{T}$ at the free end

it is divided into two one-dimensional elements, associated with three nodes

use the finite element method to find the nodal displacements for this bar

Example: One-Dimensional Finite Element

evaluate the integral that provides the element stiffness matrix:

$$\mathbf{K}_{\text{elem}} = \int_{x_1}^{x_2} \mathbf{H}^T AE\mathbf{H}\, \mathrm{d}x$$

in this case, E and A are constant and come outside the integral; expressing $\mathbf{H}^T$ and $\mathbf{H}$ in full and using $x_1 = 0$ and $x_2 = \ell$ gives:

$$\begin{aligned}
\mathbf{K}_{\text{elem}} &= AE \int_0^{\ell} \begin{bmatrix} -\frac{1}{\ell} \\ \frac{1}{\ell} \end{bmatrix} \begin{bmatrix} -\frac{1}{\ell} & \frac{1}{\ell} \end{bmatrix} \mathrm{d}x \\
&= AE \int_0^{\ell} \begin{bmatrix} \frac{1}{\ell^2} & -\frac{1}{\ell^2} \\ -\frac{1}{\ell^2} & \frac{1}{\ell^2} \end{bmatrix} \mathrm{d}x \\
&= \frac{AE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

Example: One-Dimensional Finite Element

next, calculate the element stiffness matrices for each of the two elements:

$$\mathbf{K}_1 = \frac{3AE}{L} \cdot \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{K}_2 = \frac{3AE}{2L} \cdot \begin{array}{c} \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

combine the two element matrices into a single structural stiffness matrix:

$$\mathbf{K} = \frac{3AE}{L} \cdot \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Example: One-Dimensional Finite Element

determine the force vector and the displacement vector; first the force vector:

$$\sum^{\text{elems}} \left(\int_{x_1}^{x_2} \cancelto{0}{\mathbf{N}^T B}\, \mathrm{d}x - \left(\mathbf{N}^T A \bar{T} \right)_{x=0} \right)$$

the transpose of the element shape function, $\mathbf{N}^T$, is:

$$\mathbf{N}^T = \begin{bmatrix} \frac{\ell - x}{\ell} \\ \frac{x}{\ell} \end{bmatrix}$$

so for element 1

$$\mathbf{f}_1 = \left(\mathbf{N}^T A \bar{T} \right)_{x=0} = \begin{bmatrix} A\bar{T} \\ 0 \end{bmatrix}$$

and for the full force vector:

$$\mathbf{f} = \begin{bmatrix} A\bar{T} \\ 0 \\ f_3 \end{bmatrix}$$

Example: One-Dimensional Finite Element

the displacement vector is:

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix}$$

note that d_1, d_2 and f_3 are unknown and must be solved; for each pair d_i, f_i, one of the pair must be unknown

the problem to solve is:

$$\mathbf{f} = \mathbf{K}\mathbf{d}$$

Example: One-Dimensional Finite Element

part of the displacement vector, **d**, is zero, so partition the problem to solve the smallest linear system possible:

$$\begin{bmatrix} A\bar{T} \\ 0 \end{bmatrix} = \frac{3AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

solving:

$$d_1 = \frac{3}{2} d_2; \quad A\bar{T} = \frac{3AE}{L}\left(\frac{3}{2}d_2 - d_2\right) \Longrightarrow d_2 = \frac{2\bar{T}L}{3E}; \quad d_1 = \frac{\bar{T}L}{E}$$

Example: One-Dimensional Finite Element

if desired, solve for the reaction force, f_3, using the full displacement vector and the full stiffness matrix:

$$\begin{bmatrix} A\bar{T} \\ 0 \\ f_3 \end{bmatrix} = \frac{3AE}{L} \cdot \begin{bmatrix} 1 & -1 & 0 \\ -1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\bar{T}L}{E} \\ \frac{2\bar{T}L}{3E} \\ 0 \end{bmatrix}$$

$$f_3 = \frac{3AE}{L} \cdot 0 \cdot \frac{\bar{T}L}{E} + \frac{3AE}{L} \cdot -\frac{1}{2} \cdot \frac{2\bar{T}L}{3E} + \frac{3AE}{L} \cdot \frac{1}{2} \cdot 0 = -A\bar{T}$$

3.8 Two-Dimensional Solid Elements

Strong Form of Linear Elasticity in Two Dimensions

to develop a two-dimensional finite element use the strong form for a vector-valued function (displacement) in two dimensions

the strong form consists of:

1. equilibrium conditions
2. kinematic conditions that relate strain and displacement
3. a constitutive relation: linear elasticity
4. stress and traction relations to ensure that the stress state is consistent with the boundary conditions

this is identical in process to the one-dimensional elements: combining these four considerations creates the strong form

Strong Form of Linear Elasticity in Two Dimensions

the equilibrium relation is Euler's equation for equilibrium:

$$\sigma_{ji,j} + B_i = 0$$

writing this out in full for two dimensions gives:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + B_1 = 0$$
$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + B_2 = 0$$

taking advantage of the symmetry of σ_{ij} and the fact that $\sigma_{12} = \sigma_{21}$, write this in matrix form as:

$$\boldsymbol{\nabla}_S^T \boldsymbol{\sigma} + \mathbf{B} = \boldsymbol{\nabla}_S^T \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \mathbf{B} = 0$$

where $\boldsymbol{\nabla}_S^T$ is the transpose of the symmetric gradient matrix operator

Strong Form of Linear Elasticity in Two Dimensions

for a linear analysis, assume that there are neither material nor geometric non-linearities; the displacements and rotations therefore must be small

the kinematic relation between strain and displacement is the linearised form of the strains:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

using $\gamma_{12} = 2\epsilon_{12}$:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \gamma_{12} \end{bmatrix}^T = \boldsymbol{\nabla}_S \mathbf{u}$$

where the symmetric gradient matrix operator is:

$$\boldsymbol{\nabla}_S = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 \\ 0 & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \end{bmatrix}$$

and:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Strong Form of Linear Elasticity in Two Dimensions

the constitutive equation is Hooke's Law:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

using the vector representation of stress, for plane stress:

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

and for plane strain:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

Strong Form of Linear Elasticity in Two Dimensions

the traction on a surface is given by Cauchy's law:

$$T_i = \sigma_{ji}\nu_j$$

it is standard in the finite element literature, like the composites literature, to use a vector representation of stress:

$$\boldsymbol{\sigma}^T = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}] \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

using this, in vector notation the relation of the surface traction to the local stress state is:

$$\mathbf{T} = \boldsymbol{\sigma}\boldsymbol{\nu}$$

Strong Form of Linear Elasticity in Two Dimensions

the summary of the strong form in two dimensions is:

1. *equilibrium:*
$$\boldsymbol{\nabla}_S^T \boldsymbol{\sigma} + \mathbf{B} = 0$$

2. *strain-displacement:*
$$\boldsymbol{\epsilon} = \boldsymbol{\nabla}_S \mathbf{u}$$

3. *constitutive law:*
$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

4. *boundary conditions:*
$$\overline{\mathbf{T}} = \boldsymbol{\sigma}\boldsymbol{\nu}$$

combining 1, 2 and 3 produces:

$$\boldsymbol{\nabla}_S^T \left(\mathbf{D}\boldsymbol{\nabla}_S \mathbf{u}\right) + \mathbf{B} = 0$$

Weak Form of Linear Elasticity in Two Dimensions

convert the strong form to the weak form by multiplying the strong form by a weight function (a variation) and integrating by parts

the weight function must be vector-valued to be consistent with the displacement solution function $\mathbf{u}$:

$$\delta w_i = \delta \mathbf{w} = \delta \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

the weight function is zero at all surfaces with imposed displacement boundary conditions

this must be done for each dimension:

$$\int_\Omega \delta w_i \sigma_{ji,j} \,\mathrm{d}\Omega + \int_\Omega \delta w_i B_i \,\mathrm{d}\Omega = 0 \qquad \text{no summation on } i$$

$$\int_\Gamma \delta w_i \left(\overline{T_i} - \sigma_{ji}\nu_j\right) \mathrm{d}\Gamma = 0 \qquad \text{no summation on } i$$

Weak Form of Linear Elasticity in Two Dimensions

apply Gauss's divergence theorem, integrate by parts, evaluate boundary conditions and add the equations for each dimension together into a single equation:

$$\int_\Omega \delta w_{i,j}\sigma_{ij}\,\mathrm{d}\Omega = \int_\Gamma \delta w_i T_i\,\mathrm{d}\Gamma + \int_\Omega \delta w_i B_i\,\mathrm{d}\Omega$$

expand the integrand of the first integral as:

$$\begin{aligned}
\delta w_{i,j}\sigma_{ij} &= \frac{\partial\delta w_1}{\partial x_1}\sigma_{11} + \frac{\partial\delta w_1}{\partial x_2}\sigma_{12} + \frac{\partial\delta w_2}{\partial x_1}\sigma_{12} + \frac{\partial\delta w_2}{\partial x_2}\sigma_{22} \\
&= \begin{bmatrix} \frac{\partial\delta w_1}{\partial x_1} & \frac{\partial\delta w_2}{\partial x_2} & \left(\frac{\partial\delta w_1}{\partial x_2} + \frac{\partial\delta w_2}{\partial x_1}\right) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \\
&= (\boldsymbol{\nabla}_S\,\delta\mathbf{w})^T\,\boldsymbol{\sigma}
\end{aligned}$$

using this:

$$\int_\Omega (\boldsymbol{\nabla}_S\,\delta\mathbf{w})^T\,\boldsymbol{\sigma}\,\mathrm{d}\Omega = \int_\Gamma \delta\mathbf{w}^T\overline{\mathbf{T}}\,\mathrm{d}\Gamma + \int_\Omega \delta\mathbf{w}^T\mathbf{B}\,\mathrm{d}\Omega$$

Weak Form of Linear Elasticity in Two Dimensions

finally, employ Hooke's Law and the strain-displacement relation:

$$\int_\Omega (\boldsymbol{\nabla}_S\,\delta\mathbf{w})^T\,\mathbf{D}\boldsymbol{\nabla}_S\,\mathbf{u}\,\mathrm{d}\Omega = \int_\Gamma \delta\mathbf{w}^T\overline{\mathbf{T}}\,\mathrm{d}\Gamma + \int_\Omega \delta\mathbf{w}^T\mathbf{B}\,\mathrm{d}\Omega$$

this is the weak form of the two-dimensional linear elastic system for displacements

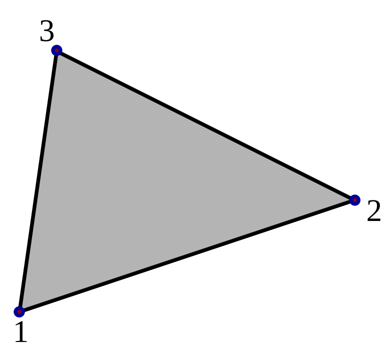
these make the implicit assumption that the system is in plane stress or plane strain and is one unit thick

it is this that will be used to perform the finite element discretisation

3.9 Two-Dimensional Solid Triangular Elements

Two-Dimensional Triangular Element

the simplest two-dimensional element is the three-node triangular element



a disadvantage is that the three-node triangular element is not very accurate: it tends to be much too stiff

note that the order of the nodes is *always* anticlockwise; because we are going to number the nodes 1 to 3, we will use x, y coordinates and u, v displacements to keep the notation clear

Two-Dimensional Triangular Element

to begin, consider a scalar function t that we wish to approximate within the triangular element; we want to make the approximation using three constants, to be consistent with the number of nodes in the element:

$$t\,(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \mathbf{p}\boldsymbol{\alpha}$$

here, $t(x, y)$ is the approximated value of t at some location (x, y) within the element

this approximation will be used to construct shape functions for a linear triangular element; use the term *linear* to refer to the order of the approximation functions (and hence the shape functions) that we use

Two-Dimensional Triangular Element

express the nodal values of the function t in terms of the constants α_i and the nodal locations:

$$t_1 = \alpha_0 + \alpha_1 x_1 + \alpha_2 y_1$$

$$t_2 = \alpha_0 + \alpha_1 x_2 + \alpha_2 y_2$$

$$t_3 = \alpha_0 + \alpha_1 x_3 + \alpha_2 y_3$$

here, x_1 is the x coordinate of node 1, x_2 is the x coordinate of node 2, and so on

rewrite this in vector - matrix form:

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \mathbf{t} = \mathbf{M}\boldsymbol{\alpha}$$

inverting this gives:

$$\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{t}$$

this is an expression for α that can be substituted into the equation for $t(x, y)$:

$$t\,(x, y) = \mathbf{pM}^{-1}\mathbf{t}$$

similar to the one-dimensional element, express the function t as a shape function multiplied by the nodal values of the function:

$$t\,(x, y) = \mathbf{Nt}$$

where:

$$\mathbf{N} = \mathbf{pM}^{-1}$$

this requires an expression for $\mathbf{M}^{-1}$:

$$\mathbf{M}^{-1} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

where:

$$2A = \det(\mathbf{M}) = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)$$

Triangular Element Shape Functions

evaluating the last expressions for $\mathbf{N}$ produces the three shape functions:

$$N_1 = \frac{1}{2A}\left((x_2y_3 - x_3y_2) + (y_2 - y_3)\,x + (x_3 - x_2)\,y\right)$$

$$N_2 = \frac{1}{2A}\left((x_3y_1 - x_1y_3) + (y_3 - y_1)\,x + (x_1 - x_3)\,y\right)$$

$$N_3 = \frac{1}{2A}\left((x_1y_2 - x_2y_1) + (y_1 - y_2)\,x + (x_2 - x_1)\,y\right)$$

the derivative of the shape functions is:

$$\mathbf{H} = \frac{1}{2A}\begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

note that $\mathbf{H}$ is not a function of x or y; this makes it very easy to integrate during the creation of the element stiffness matrix

Triangular Element - Vector Functions

note that this is only for a scalar function in two dimensions; displacement has two components in two dimensions, so it is a vector-valued function

for vector-valued functions, such as $\mathbf{u} = [u, v]^T$, both components must be interpolated and discretized

each component of $\mathbf{u}$ can be treated as an independent scalar variable: interpolate u and v separately

in principle, it is possible to use different shape functions for each component of $\mathbf{u}$; however, this is almost never done and the same shape functions are used for all of the components of the vector

for both components of $\mathbf{u}$, use linear interpolations and hence linear shape functions

Triangular Element - Vector Functions

each node has two components of displacement and hence two degrees of freedom, so the vector of nodal displacements has six components:

$$\mathbf{d} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{bmatrix}^T$$

the element shape function matrix is:

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

such that:

$$\mathbf{u}(x, y) \approx \mathbf{Nd}$$

recall the weak form of the governing equations:

$$\int_\Omega (\boldsymbol{\nabla}_S \delta\mathbf{w})^T\, \mathbf{D}\boldsymbol{\nabla}_S \mathbf{u}\, \mathrm{d}\Omega = \int_\Gamma \delta\mathbf{w}^T \overline{\mathbf{T}}\, \mathrm{d}\Gamma + \int_\Omega \delta\mathbf{w}^T \mathbf{B}\, \mathrm{d}\Omega$$

and replace the integrands with their discretised approximations

Triangular Element - Vector Functions

first, recall the appropriate treatment for the strain:

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{bmatrix} = \boldsymbol{\nabla}_S \mathbf{u} \approx \boldsymbol{\nabla}_S \mathbf{Nd} = \mathbf{Hd}$$

again, this requires a representation for the derivatives of $\mathbf{N}$:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

and $\mathbf{H}$ is the strain - displacement relation

Triangular Element - Vector Functions

additionally, this requires a representation for the weight function $\delta\mathbf{w}$;

employ a Galerkin method, and approximate it with $\delta\mathbf{w}(x, y) \approx \mathbf{N}\mathbf{w}$ where $\mathbf{w}$ is the value of the weight function at the nodes:

$$(\boldsymbol{\nabla}_S\,\mathbf{w})^T \approx (\boldsymbol{\nabla}_S\,\mathbf{N}\mathbf{w}) = (\mathbf{H}\mathbf{w})^T = \mathbf{w}^T\mathbf{H}^T$$

substitute the approximations into the weak form and discretize, element by element:

$$\mathbf{w}^T \sum^{\text{elems}} \left(\int_\Omega \mathbf{H}^T\mathbf{D}\mathbf{H}\, \mathrm{d}\Omega \mathbf{d} - \int_\Gamma \mathbf{N}^T \overline{T}\, \mathrm{d}\Gamma - \int_\Omega \mathbf{N}^T\mathbf{B}\, \mathrm{d}\Omega \right) = 0$$

the element stiffness matrix is:

$$\int_\Omega \mathbf{H}^T\mathbf{D}\mathbf{H}\, \mathrm{d}\Omega = \mathbf{K}$$

3.10 Gauss Quadrature

Gauss Quadrature

for the two-dimensional triangular element, the integrals that had to be evaluated were all integrals of constants, and hence were unproblematic

in general, the integrals that must be evaluated cannot be integrated in closed form

a fast, accurate numerical method to evaluate the necessary integrals is essential

Gauss quadrature is one of the most efficient methods available to integrate polynomial or nearly polynomial functions

Gauss quadrature will be the standard method in this course for evaluating the required integrals

Gauss Quadrature

Gauss quadrature is used to evaluate integrals:

$$I = \int_a^b f(x)\,\mathrm{d}x$$

where the function $f(x)$ is known and can be evaluated at any arbitrary point, but there is not have a conveniently available closed-form indefinite integral to evaluate

the Gauss quadrature formulae enable numerical integration of such integrals; the process involves evaluation of the function $f(x)$ at certain points, multiplying the function evaluations by weights, and then summing the resulting products

first, map the interval, a to b, over which the integral is to be evaluated onto a parent domain, which is always $\xi \in [-1, 1]$

Gauss Quadrature in One Dimension

the relation between the physical coordinate x and the parent coordinate ξ is:

$$x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a)$$

which can be transformed into:

$$x = a\frac{1-\xi}{2} + b\frac{1+\xi}{2} = aN_1(\xi) + bN_2(\xi)$$

which strongly resembles linear shape functions: Gauss quadrature and linear approximations are very closely related

Gauss Quadrature in One Dimension

from evaluating the derivatives:

$$\mathrm{d}x = \frac{1}{2}(b-a)\;\mathrm{d}\xi = \frac{\ell}{2}\,\mathrm{d}\xi = J\,\mathrm{d}\xi$$

where J is the Jacobian determinant given by $(b-a)/2$, which is also half the element length

write the integral as:

$$I = J\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi = J\hat{I}$$

it is this integral $\hat{I}$ that will be approximated by the Gauss quadrature procedure

Gauss Quadrature in One Dimension

approximate the function $\hat{I}$ using a sum of the products of evaluations of the function f and weights:

$$\hat{I} \approx W_1 f(\xi_1) + W_2 f(\xi_2) + \cdots + W_n f(\xi_n) = \begin{bmatrix} W_1 & W_2 & \ldots & W_n \end{bmatrix}\begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{bmatrix} \equiv \mathbf{W}^T\mathbf{f}$$

where W_i are the weight functions; the ξ_i are locations at which the function f is to be evaluated, not different coordinate axes

choose the weights W_i and the locations at which the function is evaluated ξ_i to integrate the highest possible polynomial exactly

the question is, how are the weights and the locations (known as integration points or Gauss points) determined?

Gauss Quadrature in One Dimension

first, approximate the function $f(\xi)$ by the $(m-1)^{\text{th}}$ order polynomial:

$$f(\xi) \approx \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \cdots = \begin{bmatrix} 1 & \xi & \xi^2 & \cdots & \xi^{m-1} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{bmatrix} \equiv \mathbf{p}(\xi)\boldsymbol{\alpha}$$

next, express the values of the coefficients α_i in terms of the values of the function $f(\xi)$ at the integration points:

$$\mathbf{f} \equiv \begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{bmatrix} = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{m-1} \\ 1 & \xi_2 & \xi_2^2 & \cdots & \xi_2^{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_n & \xi_n^2 & \cdots & \xi_n^{m-1} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_m \end{bmatrix} \equiv \mathbf{M}\boldsymbol{\alpha}$$

by combining these:

$$\hat{I} \approx \mathbf{W}^T\mathbf{M}\boldsymbol{\alpha}$$

Gauss Quadrature in One Dimension

this analysis provides the weights and integration points necessary to evaluate polynomials of a certain order exactly; the order of the polynomial that can be evaluated exactly is determined by the number of integration points

if n_{gp} is the number of Gauss integration points, the the order p of the polynomial that can be integrated exactly is:

$$p \le 2n_{gp} - 1$$

to find the weights and the Gauss points, integrate the polynomial $f(\xi)$:

$$\hat{I} = \int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx \int_{-1}^{1} \begin{bmatrix} 1 & \xi & \xi^2 & \cdots & \xi^{m-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \mathrm{d}\xi = \begin{bmatrix} \xi & \dfrac{\xi^2}{2} & \dfrac{\xi^3}{3} & \cdots \end{bmatrix}_{-1}^{1} \boldsymbol{\alpha}$$

or

$$\hat{I} \approx \begin{bmatrix} 2 & 0 & \dfrac{2}{3} & 0 & \cdots \end{bmatrix} \boldsymbol{\alpha} \equiv \hat{\mathbf{P}}\boldsymbol{\alpha}$$

Gauss Quadrature in One Dimension

the integral has been expressed in two ways:

$$\hat{I} \approx \mathbf{W}^T \mathbf{M} \boldsymbol{\alpha}$$

and

$$\hat{I} \approx \hat{\mathbf{P}}\boldsymbol{\alpha}$$

as well, $\hat{\mathbf{P}}$ has been determined through analytical integration

therefore:

$$\mathbf{W}^T \mathbf{M} = \hat{\mathbf{P}} \equiv \begin{bmatrix} 2 & 0 & \dfrac{2}{3} & 0 & \cdots \end{bmatrix} \Rightarrow \mathbf{M}^T \mathbf{W} = \hat{\mathbf{P}}^T$$

which is a system of non-linear equations for the Gauss points ($\mathbf{M}$) and the weights ($\mathbf{W}$)

Gauss Quadrature in One Dimension

solve this system for as many Gauss points as desired; the results are tabulated:

n_{gp}	Gauss point ξ_i	Weight W_i
1	0.0	2.0
2	± 0.5773502693	1.0
3	± 0.7745966692	0.5555555556
	0.0	0.8888888889
4	± 0.8611363116	0.3478548451
	± 0.3399810436	0.6521451549

Gauss Quadrature in Two Dimensions

this enables evaluation of integrals of functions that are polynomial or nearly polynomial, which includes all of the shape functions in this course

in two dimensions, there will be integrals of the form:

$$I = \int_\Omega f(x, y)\, \mathrm{d}\Omega$$

this requires conversion of the physical area, $\mathrm{d}\Omega$, to an area in the parent coordinates (and the reverse operation):

$$\mathrm{d}\Omega = \det \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ & \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \mathrm{d}\xi\, \mathrm{d}\eta = |\mathbf{J}|\ \mathrm{d}\xi\, \mathrm{d}\eta$$

where $|\mathbf{J}|$ is the determinant of the Jacobian matrix, often called the Jacobian determinant

Gauss Quadrature in Two Dimensions

re-express the integral as:

$$I = \int_{-1}^{1} \int_{-1}^{1} |\mathbf{J}(\xi, \eta)|\, f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

to evaluate this integral, carry out the integration over ξ through Gauss quadrature, in the same manner as in one dimension; this yields:

$$I = \int_{-1}^{1} \left(\int_{-1}^{1} |\mathbf{J}(\xi, \eta)|\, f(\xi, \eta)\, \mathrm{d}\xi \right) \mathrm{d}\eta \approx \int_{-1}^{1} \left(\sum_{i=1}^{n_{gp}} W_i\, |\mathbf{J}(\xi_i, \eta)|\, f(\xi_i, \eta) \right) \mathrm{d}\eta$$

finally, complete the integration by performing a Gauss quadrature over η:

$$I \approx \int_{-1}^{1} \left(\sum_{i=1}^{n_{gp}} W_i\, |\mathbf{J}(\xi_i, \eta)|\, f(\xi_i, \eta) \right) \mathrm{d}\eta \approx \sum_{j=1}^{n_{gp}} \left(\sum_{i=1}^{n_{gp}} W_j W_i \left|\mathbf{J}(\xi_i, \eta_j)\right| f(\xi_i, \eta_j) \right)$$

Gauss Quadrature in Two Dimensions

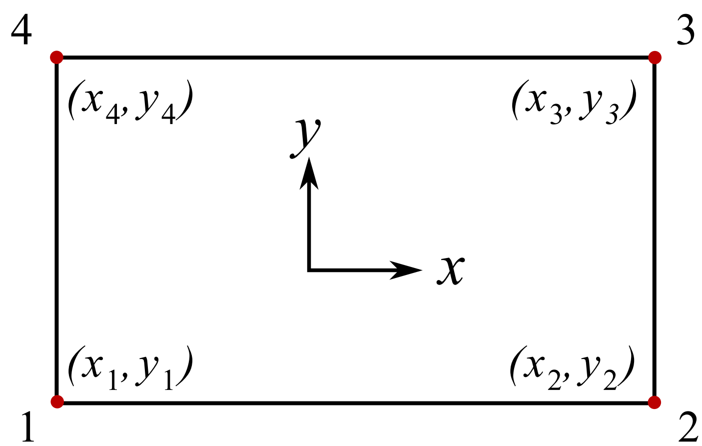
so the process for utilising Gauss quadrature is as follows:

- identify the function to be integrated
- determine how many Gauss points will be used and find the Gauss point locations and weights

- convert x (or whatever the variable of interest is) to ξ and get the Jacobian determinant
- evaluate the function at the Gauss points and multiply by the appropriate weights
- sum the product of the function evaluations and weight functions; this provides the approximated integral (exact for a polynomial given enough Gauss points)

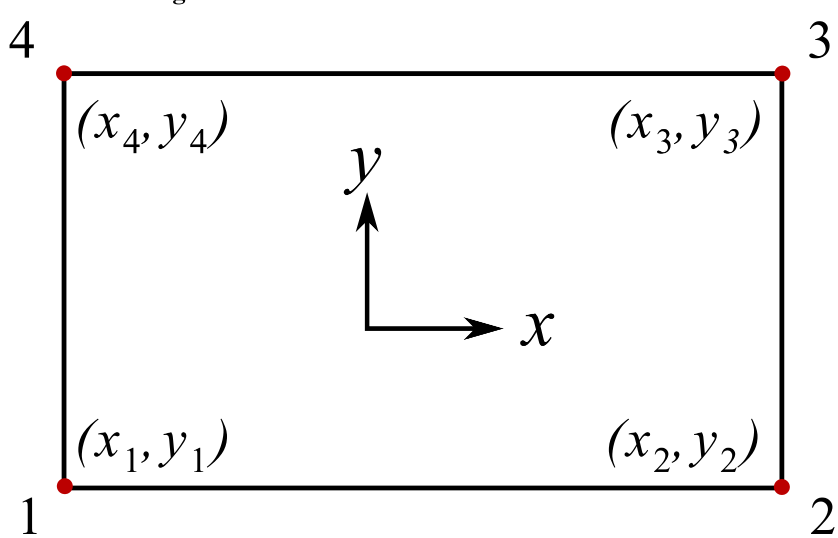
3.11 Two Dimensional Solid Quadrilateral Elements

Four-Node Rectangular Element



consider the four-node rectangular element above; again, the nodes are ordered anticlockwise

Four-Node Rectangular Element

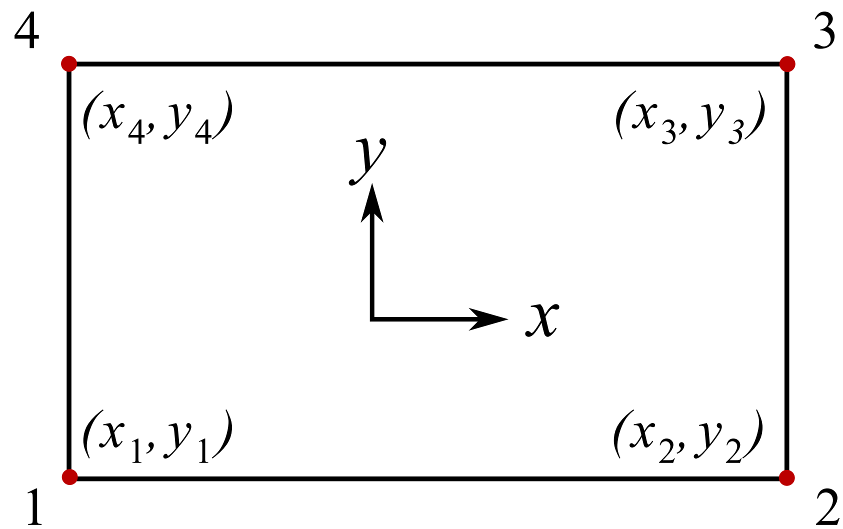


the three-node triangular element is the simplest element to derive and use in two dimensions, but it tends to be inaccurate because it models the strains as constant through the element

the weight functions for the triangular element had three constants: one associated with a constant term, and two associated with terms linear in x and y; this was consistent with the number of nodes in the element

a four-node element will need four constants; with what combination of x and y is the fourth constant associated?

Four-Node Rectangular Element



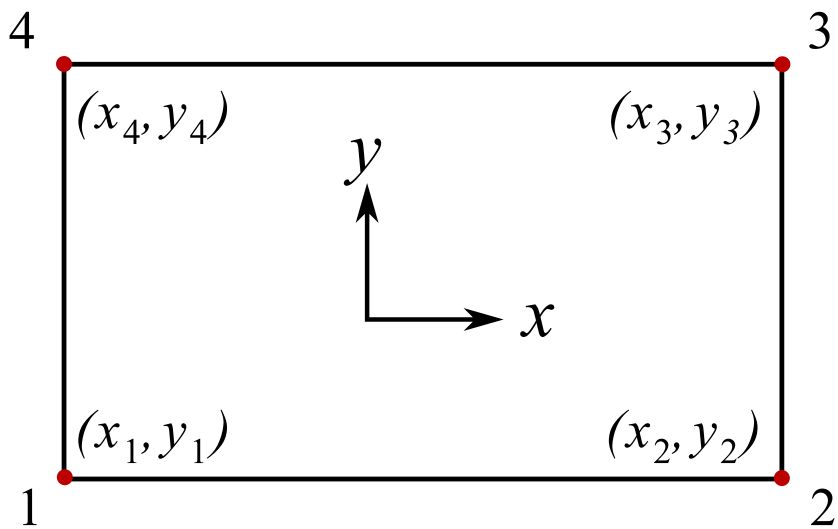
the answer to this comes from considering continuity between two elements

because the deformed geometry of each element is linked to the deflections at the nodes, the elements are automatically continuous at the nodes

however, continuity of displacements along the edges at which two elements meet is not necessarily ensured

consider the three-node triangle: the displacements along adjacent edges of separate elements were the same linear interpolations of the displacements at the nodes, which guaranteed continuity

Four-Node Rectangular Element



the same guarantee of continuity is necessary for rectangular elements, and to achieve this it is necessary to ensure linear interpolations along the sides of an element

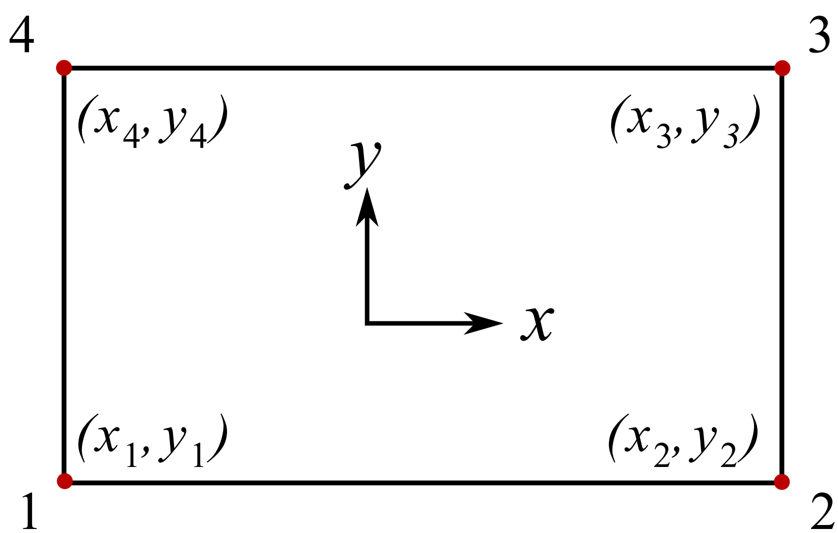
the shape function for the rectangular element will be:

$$u(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 f(x, y),$$

but what is $f(x, y)$ that will maintain linear interpolations along the edges?

the solution will come from the third row of Pascal's triangle

Four-Node Rectangular Element



the term x^2 will result in displacements that are quadratic in x along the edges between 1 and 2 and between 3 and 4; a similar argument can be made for y^2

instead, using the term xy, one of x or y will be constant along any boundary (provided the boundaries are aligned with the coordinate axes) and hence the variation will be linear

this is the necessary term, and hence:

$$u(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy$$

Four-Node Rectangular Element

the shape functions for the rectangular element are the next necessary component

the shape functions must obey the Kronecker delta rule: the displacement at one node has no effect on the displacement at other nodes; for example, $N_1(x_1, y_1) = 1$ while $N_1(x_2, y_2) = 0$

the four shape functions (which can be generated by multiplying the linear shape functions for a one-dimensional element in their various combinations) are:

$$N_1 = \frac{x - x_2}{x_1 - x_2}\frac{y - y_4}{y_1 - y_4} = \frac{1}{A}(x - x_2)(y - y_4)$$

$$N_2 = \frac{x - x_1}{x_2 - x_1}\frac{y - y_3}{y_2 - y_3} = -\frac{1}{A}(x - x_1)(y - y_3)$$

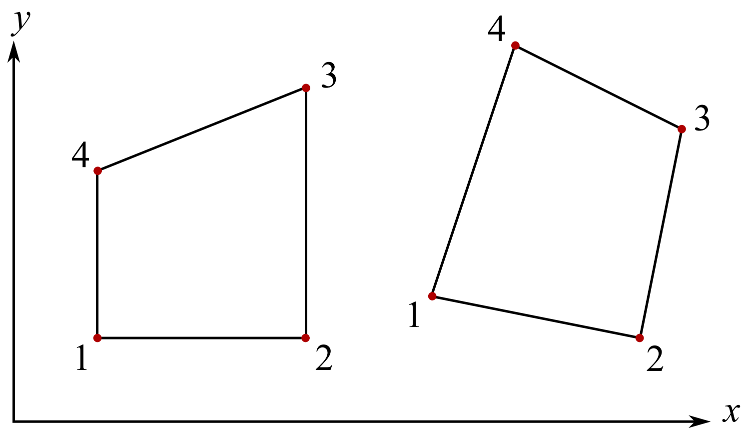
$$N_3 = \frac{x - x_4}{x_3 - x_4}\frac{y - y_2}{y_3 - y_2} = \frac{1}{A}(x - x_4)(y - y_2)$$

$$N_4 = \frac{x - x_3}{x_4 - x_3}\frac{y - y_1}{y_4 - y_1} = -\frac{1}{A}(x - x_3)(y - y_1)$$

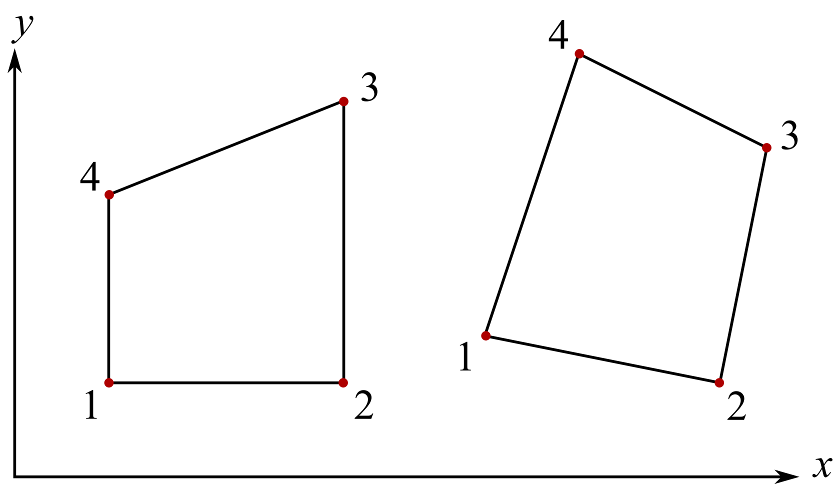
where A is the area of the element; note: $x_1 = x_4$, $x_2 = x_3$, $y_1 = y_2$, $y_3 = y_4$

Four-Node Rectangular Element

these shape functions will work for rectangles oriented with the coordinate axes; will it work for the elements below?



Isoparametric Element



the shape functions on the previous pages will not work for these elements because the element edges are not oriented parallel to the coordinate axes

consequently, the interpolations are cannot be linear along the edges that are not parallel to the coordinate axes

hence, two nodes are not sufficient to ensure continuity of displacement along the boundary between two elements

this means that there might be jumps in displacement between elements

Isoparametric Element

the development of isoparametric elements is a key advance that makes finite element methods so widely applicable isoparametric elements permit arbitrarily shaped and oriented elements to be used in simulations; this allows complicated boundaries to be modelled accurately

recall Gauss quadrature, where the domain of integration was parameterised using ξ

isoparametric elements employ the same approach

using a one-dimensional element as an example, map the physical coordinate x to the parent coordinate ξ:

$$x = x_1 \frac{1-\xi}{2} + x_2 \frac{1+\xi}{2}$$

the parent coordinate ξ has domain $\xi \in [-1, 1]$ while x_1 and x_2 are the physical coordinates of the nodes

this provides a parameterisation of position within a one-dimensional element

Isoparametric Element

staying with the one-dimensional case, this also provides a parameterisation of the value of some property (displacement u, as an example) within the element based on the values at the nodes, using u_1 and u_2 the values of the displacement at nodes 1 and 2

take the interpolation for displacement $u(x)$ in terms of the physical coordinate x, and substitute the expression for x in terms of the parent coordinate ξ:

$$\begin{aligned} u &= u_1 \frac{x - x_2}{x_1 - x_2} + u_2 \frac{x - x_1}{x_2 - x_1} \\ &= u_1 \frac{x_1 (1-\xi) + x_2 (1+\xi) - 2x_2}{2 (x_1 - x_2)} + u_2 \frac{x_1 (1-\xi) + x_2 (1+\xi) - 2x_1}{2 (x_2 - x_1)} \\ &= u_1 \frac{1-\xi}{2} + u_2 \frac{1+\xi}{2} \end{aligned}$$

this is the same as the mapping from the physical coordinates to the parent coordinates and is crucial: the linear approximation of the solution function u is identical to the map from the physical coordinate to the parent coordinate

this is the essential feature of an isoparametric element

Isoparametric Element

moving to two dimensions, the parent element is a four-node two-unit by two-unit square, with nodal coordinates given by:

Node i	ξ_i	η_1
1	-1	-1
2	1	-1
3	1	1
4	-1	1

the physical position within an element can be treated like a property that is interpolated from the nodal values

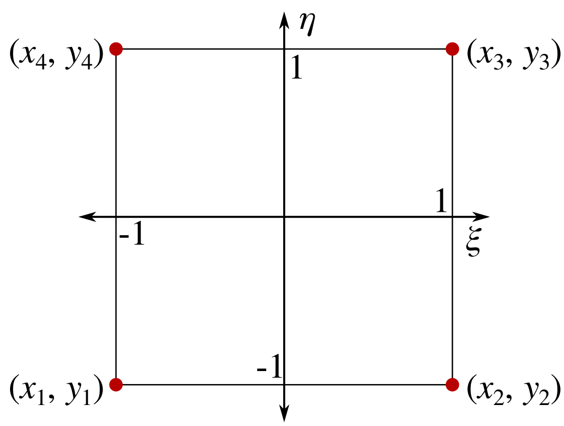
map the parent element to the physical element using the shape functions for the four-node rectangular element:

$$x(\xi, \eta) = \mathbf{N}^{4Q}\mathbf{x}; \qquad y(\xi, \eta) = \mathbf{N}^{4Q}\mathbf{y}$$

where:

$$\mathbf{x} = [x_1, x_2, x_3, x_4]^T \qquad \mathbf{y} = [y_1, y_2, y_3, y_4]^T$$

Isoparametric Element



Isoparametric Element

the shape functions for the parent element, $\mathbf{N}^{4Q}$, are given by:

$$N_1^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi_1\xi)(1 + \eta_1\eta)$$

$$N_2^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi_2\xi)(1 + \eta_2\eta)$$

$$N_3^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi_3\xi)(1 + \eta_3\eta)$$

$$N_4^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi_4\xi)(1 + \eta_4\eta)$$

these are obtained by replacing x and y in the shape functions for the four-node rectangular element with their parametric representations in terms of ξ and η

given values for $\begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}$, the physical nodal coordinates, the pair (x, y) can be determined by evaluating the shape functions at the desired parent coordinates (ξ, η) and multiplying by the physical nodal coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{N}^{4Q}(\xi, \eta) \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}$$

Isoparametric Element

the values of ξ_1, η_1 and the other nodal locations in the parent coordinate system are known:

$$N_1^{4Q}(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)$$

$$N_2^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta)$$

$$N_3^{4Q}(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$

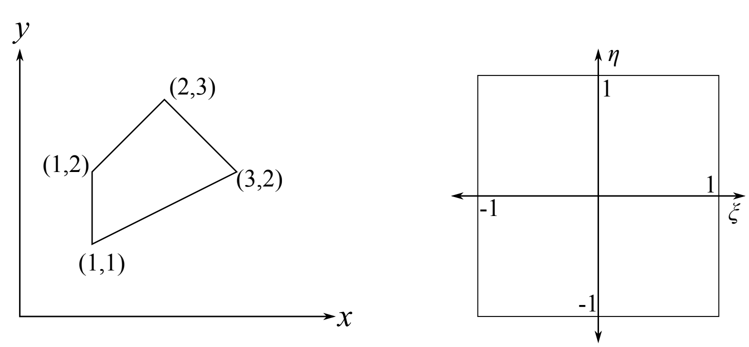
$$N_4^{4Q}(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

these are the shape functions for every four-noded quadrilateral element in the parent coordinate system

again, check that these shape functions obey the Kronecker delta rule

Isoparametric Element

take the element below on the left and map it onto the parent element on the right:



Isoparametric Element

generate the vectors of nodal positions:

$$\mathbf{x} = [1, 3, 2, 1]^T \qquad \mathbf{y} = [1, 2, 3, 2]^T$$

for any value of (ξ, η), where $-1 \le \xi \le 1$ and $-1 \le \eta \le 1$, evaluate the shape functions and then multiply by the nodal coordinates:

$$x(\xi, \eta) = \mathbf{N}^{4Q}\mathbf{x}; \qquad y(\xi, \eta) = \mathbf{N}^{4Q}\mathbf{y}$$

this gives the values of (x, y) in the physical coordinate system for the known values of (ξ, η) in the parent coordinate system

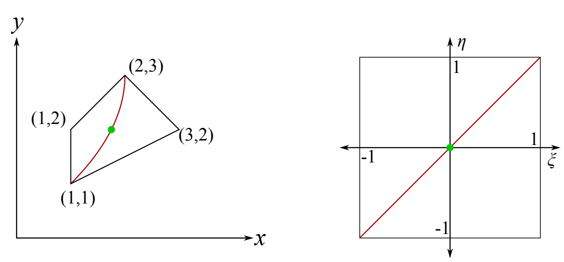
usually the mapping is from the parent element to the physical element; the mapping from physical to parent is typically only necessary for the nodes (the coordinates of the nodes of the parent element are known)

Isoparametric Element

in this example, $(\xi, \eta) = (0, 0)$ maps to $(x, y) = (1.75, 2)$; the values of the shape functions at this point are calculated by substituting $(\xi, \eta) = (0, 0)$ into the expressions for the shape functions to get:

$$\mathbf{N}^{4Q}(\xi = 0, \eta = 0) = [1/4, 1/4, 1/4, 1/4]$$

a straight line connecting nodes 1 and 3 in the parent element is mapped to a curved line in the physical element



straight lines on physical boundaries are mapped to straight lines in the parent element, but straight lines elsewhere in the physical element may be mapped to curved lines in the parent element

Isoparametric Element

the solution function is approximated by the same shape functions:

$$\mathbf{u}(\xi, \eta) \approx \mathbf{N}^{4Q}\mathbf{d}$$

and hence the element is isoparametric

the shape functions contain a constant term, a term linear in ξ, a term linear in η and a bilinear term $\xi\eta$

since the parent element is a square with sides aligned with the coordinate axes, linear interpolations are guaranteed along the element boundaries

isoparametric elements ensure continuity of displacement (or any other desired solution function) along the boundaries between elements regardless of the physical element shape

the next step is to calculate the stiffness matrices for isoparametric elements

Isoparametric Element

the element stiffness matrices require computation of integrals of the form:

$$\mathbf{K} = \int_{\Omega} \mathbf{H}^T \mathbf{D} \mathbf{H} \, \mathrm{d}\Omega$$

where $\mathbf{H}$ contains derivatives of the isoparametric shape functions

remember where $\mathbf{H}$ arises: in physical coordinates, strain is given by the equation:

$$\boldsymbol{\epsilon} = \boldsymbol{\nabla}\mathbf{u} \approx \mathbf{H}\mathbf{d} = \boldsymbol{\nabla}_S \mathbf{N}_I^{4Q}\mathbf{d}$$

hence

$$\mathbf{H} = \boldsymbol{\nabla}_S \mathbf{N}^{4Q} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} & 0 \\ 0 & N_1^{4Q} & 0 & N_2^{4Q} & 0 & N_3^{4Q} & 0 & N_4^{4Q} \end{bmatrix}$$

Isoparametric Element

evaluating the vector-matrix product, the strain-displacement relation $\mathbf{H} = \boldsymbol{\nabla}_S \mathbf{N}_I^{4Q}$ is:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial N_1^{4Q}}{\partial x} & 0 & \dfrac{\partial N_2^{4Q}}{\partial x} & 0 & \dfrac{\partial N_3^{4Q}}{\partial x} & 0 & \dfrac{\partial N_4^{4Q}}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1^{4Q}}{\partial y} & 0 & \dfrac{\partial N_2^{4Q}}{\partial y} & 0 & \dfrac{\partial N_3^{4Q}}{\partial y} & 0 & \dfrac{\partial N_4^{4Q}}{\partial y} \\ \dfrac{\partial N_1^{4Q}}{\partial y} & \dfrac{\partial N_1^{4Q}}{\partial x} & \dfrac{\partial N_2^{4Q}}{\partial y} & \dfrac{\partial N_2^{4Q}}{\partial x} & \dfrac{\partial N_3^{4Q}}{\partial y} & \dfrac{\partial N_3^{4Q}}{\partial x} & \dfrac{\partial N_4^{4Q}}{\partial y} & \dfrac{\partial N_4^{4Q}}{\partial x} \end{bmatrix}$$

the general rule is that the strain-displacement relation has dimensions (number of components of strain x number of degrees of freedom)

however, the shape functions are expressed in parent coordinates while the derivatives are with respect to the physical coordinates

so this must be transformed

Isoparametric Element

use the chain rule; for example, for the shape function for node 1:

$$\frac{\partial N_1^{4Q}}{\partial \xi} = \frac{\partial N_1^{4Q}}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_1^{4Q}}{\partial y}\frac{\partial y}{\partial \xi}$$

the collection of these relations can be written, where $I \in (1, 2, 3, 4)$ as:

$$\begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial \xi} \\[2ex] \dfrac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial x} \\[2ex] \dfrac{\partial N_I^{4Q}}{\partial y} \end{bmatrix}$$

this uses the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Isoparametric Element

what are needed are the partials with respect to the physical coordinates, so invert the Jacobian matrix:

$$\begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial x} \\[2ex] \dfrac{\partial N_I^{4Q}}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \dfrac{\partial N_I^{4Q}}{\partial \xi} \\[2ex] \dfrac{\partial N_I^{4Q}}{\partial \eta} \end{bmatrix}$$

in vector notation, this becomes:

$$\mathbf{H} = \boldsymbol{\nabla}_S \mathbf{N}_I^{4Q} = \mathbf{J}^{-1}\mathbf{G}\mathbf{N}_I^{4Q}$$

where:

$$\mathbf{G} = \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\[2ex] \dfrac{\partial}{\partial \eta} \end{bmatrix}$$

is the gradient operator in the parent coordinate system

Isoparametric Element

the mapping between physical coordinates and parent coordinates is:

$$x(\xi,\eta) = \mathbf{N}^{4Q}\mathbf{x}; \qquad y(\xi,\eta) = \mathbf{N}^{4Q}\mathbf{y}$$

this can be used to calculate the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

through the interpolations:

$$x(\xi,\eta) = \begin{bmatrix} N_1^{4Q} & N_2^{4Q} & N_3^{4Q} & N_4^{4Q} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$y(\xi,\eta) = \begin{bmatrix} N_1^{4Q} & N_2^{4Q} & N_3^{4Q} & N_4^{4Q} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Isoparametric Element

the nodal coordinates are not functions of the parent coordinates, but the shape functions are:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial N_1^{4Q}}{\partial \xi} & \dfrac{\partial N_2^{4Q}}{\partial \xi} & \dfrac{\partial N_3^{4Q}}{\partial \xi} & \dfrac{\partial N_4^{4Q}}{\partial \xi} \\ \dfrac{\partial N_1^{4Q}}{\partial \eta} & \dfrac{\partial N_2^{4Q}}{\partial \eta} & \dfrac{\partial N_3^{4Q}}{\partial \eta} & \dfrac{\partial N_4^{4Q}}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -1 - \eta \\ \xi - 1 & -1 - \xi & 1 + \xi & 1 - \xi \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

converting to vector notation, this is:

$$\mathbf{J} = \mathbf{G}\mathbf{N}^{4Q} \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix}$$

Isoparametric Element

this is a numerical process that enables the calculation of $\mathbf{J}$ for any location in the element

once $\mathbf{J}$ can be calculated, it can also be inverted to produce $\mathbf{J}^{-1}$, which is the mathematical object needed to calculate $\mathbf{H}$

as an intermediate step, write $\mathbf{H}^*$ in matrix form as:

$$\mathbf{H}^* = \mathbf{J}^{-1}\mathbf{G}\mathbf{N}^{4Q}$$

providing:

$$\mathbf{H}^* = \begin{bmatrix} \dfrac{\partial N_1^{4Q}}{\partial x} & \dfrac{\partial N_2^{4Q}}{\partial x} & \dfrac{\partial N_3^{4Q}}{\partial x} & \dfrac{\partial N_4^{4Q}}{\partial x} \\[2ex] \dfrac{\partial N_1^{4Q}}{\partial y} & \dfrac{\partial N_1^{4Q}}{\partial y} & \dfrac{\partial N_3^{4Q}}{\partial y} & \dfrac{\partial N_4^{4Q}}{\partial y} \end{bmatrix}$$

these are the derivatives of the shape functions with respect to the physical coordinates, but are not yet the strain-displacement relation $\mathbf{H}$

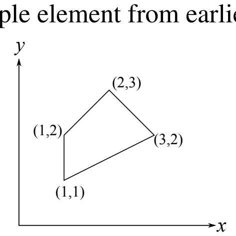
Isoparametric Element

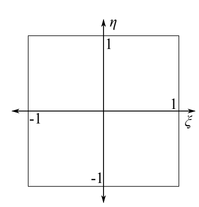
finally, the terms in $\mathbf{H}^*$ must be distributed into $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial N_1^{4Q}}{\partial x} & 0 & \dfrac{\partial N_2^{4Q}}{\partial x} & 0 & \dfrac{\partial N_3^{4Q}}{\partial x} & 0 & \dfrac{\partial N_4^{4Q}}{\partial x} & 0 \\[2ex] 0 & \dfrac{\partial N_1^{4Q}}{\partial y} & 0 & \dfrac{\partial N_2^{4Q}}{\partial y} & 0 & \dfrac{\partial N_3^{4Q}}{\partial y} & 0 & \dfrac{\partial N_4^{4Q}}{\partial y} \\[2ex] \dfrac{\partial N_1^{4Q}}{\partial y} & \dfrac{\partial N_1^{4Q}}{\partial x} & \dfrac{\partial N_2^{4Q}}{\partial y} & \dfrac{\partial N_2^{4Q}}{\partial x} & \dfrac{\partial N_3^{4Q}}{\partial y} & \dfrac{\partial N_3^{4Q}}{\partial x} & \dfrac{\partial N_4^{4Q}}{\partial y} & \dfrac{\partial N_4^{4Q}}{\partial x} \end{bmatrix}$$

Isoparametric Element

here is the example element from earlier





$$\mathbf{J} = \frac{1}{4}\begin{bmatrix} \eta - 1 & 1 - \eta & 1 + \eta & -1 - \eta \\ \xi - 1 & -1 - \xi & 1 + \xi & 1 - \xi \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 - \eta & 2 \\ -1 - \xi & 2 \end{bmatrix}$$

$$|\mathbf{J}| = \frac{16}{2\xi - 2\eta + 8}$$

$$\mathbf{J}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ 1+\xi & 3-\eta \end{bmatrix}$$

Isoparametric Element

the mapping from the physical element to the parent element must be unique and invertible; hence the determinant of the Jacobian matrix must be greater than zero for all x and y

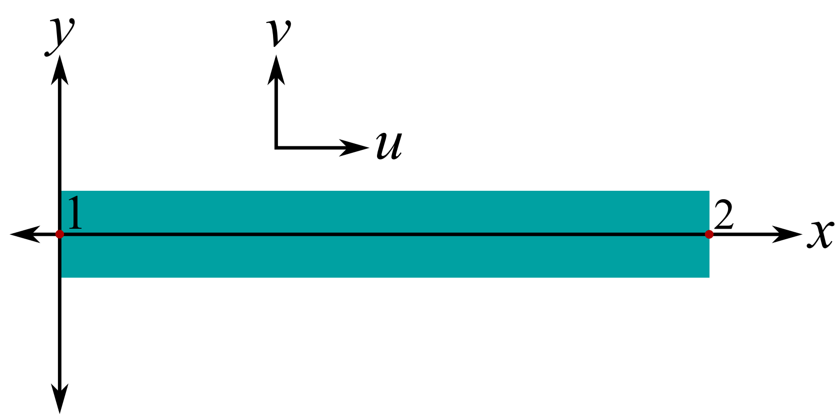
this can be shown to be the case for all physical quadrilateral elements where the angles at the nodes are less than 180°

note that the shape functions do not depend upon the physical element coordinates (nor therefore size); this is the explanation for the superscripted $4Q$: they are universal functions associated with this element type

however, the Jacobian matrix does depend upon the physical element coordinates

3.12 Beam Elements

Beam Elements



Beam Elements

another common finite element is the beam element, which can be used to model beams in bending

in order to capture bending behaviour accurately, higher order approximation is necessary, so this is an example of a higher order element

number the nodes at the end of the beam element 1 and 2

this derivation will use Bernoulli beam theory (Timoshenko beam theory could alternately be used if the additional complication is not prohibitive) remember the main assumptions of Bernoulli beam theory:

1. the beam lies along the x-axis
2. the deflections of interest, v, are normal to the beam, in the y direction
3. sections normal to the axis of the beam remain normal to the axis of the beam after bending; for symmetric beams, the axis (and the neutral axis of bending) is the mid-line

Beam Elements

the displacements of a point on the beam are given by u and v, aligned with the x and y axes

the normality assumption determines the u component of displacement through the thickness of the beam by:

$$u = -y \sin \theta(x)$$

where $\theta(x)$ is the rotation of the axis of the beam after deflection at location x

if the deflections are small, then the angle $\theta(x)$ is small and $\sin \theta \approx \theta$, making the angle of rotation correspond to the slope of the mid-line:

$$\theta(x) = \frac{\partial v}{\partial x}$$

combining these two equations gives:

$$u = -y \frac{\partial v}{\partial x}$$

Beam Elements

the expression for axial strain is:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -y \frac{\partial^2 v}{\partial x^2}$$

assume that v is only a function of x

consider a more general case when the mid-line is also stretched or compressed; by superposition:

$$u = u^M - y \frac{\partial v}{\partial x}$$

where u^M is the displacement at the mid-line

this generates three expressions for strain:

$$\epsilon_{xx} = \frac{\partial u^M}{\partial x} - y \frac{\partial^2 v}{\partial x^2} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} = 0 \qquad \epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Beam Elements

assume that Hookean linear elasticity holds; the stress-strain law is:

$$\sigma_{xx} = E\epsilon_{xx} = E\left(\frac{\mathrm{d}u^M}{\mathrm{d}x} - y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)$$

note that these are total derivatives, since both u^M and v are functions of x only

the stresses on the beam section can be related directly to the moment by integrating:

$$M = -\int_A y\sigma_{xx}\,\mathrm{d}A = -\int_A yE\left(\frac{\mathrm{d}u^M}{\mathrm{d}x} - y\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right)\mathrm{d}A = \int_A Ey^2\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\mathrm{d}A$$

recall that:

$$I = \int_A y^2\,\mathrm{d}A$$

and therefore:

$$M = EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = EI\kappa$$

which is a constitutive law

Beam Elements

the transverse equilibrium statement for a beam element is:

$$\left(V + \frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x\right) - V + p\left(\frac{\mathrm{d}x}{2}\right)\mathrm{d}x = 0$$

where V is the shear force and p is the distributed force (evaluated at the centre of the differential element)

in the limit, this gives:

$$\frac{\mathrm{d}V}{\mathrm{d}x} + p = 0$$

moment equilibrium states that:

$$\left(M + \frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x\right) - M + \left(V + \frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x\right)\mathrm{d}x + \frac{1}{2}\,\mathrm{d}x^2 p\left(x + \frac{\mathrm{d}x}{2}\right) = 0$$

which, in the limit, gives:

$$\frac{\mathrm{d}M}{\mathrm{d}x} + V = 0$$

Beam Elements

amalgamating all of these equations produces:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} - p = 0,$$

which, along with $M = EI\kappa$, is a strong form of the beam equations combining these generates the basic expression of Bernoulli beam theory:

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - p = 0$$

the above is a fourth order ordinary differential equation for the vertical displacement of the mid-line of the beam; note that this is of higher order than the equations for the one-dimensional or two-dimensional finite elements

Beam Elements - Boundary Conditions

boundary conditions are needed to solve this equation; there are several possibilities:

1. $v = \overline{v}$ on Γ_u (essential)
2. $\dfrac{\mathrm{d}v}{\mathrm{d}x} = -\overline{\theta}$ on Γ_θ (essential)
3. $EI\dfrac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \overline{M}$ on Γ_M (natural)
4. $-EI\dfrac{\mathrm{d}^3 v}{\mathrm{d}x^3} = \overline{V}$ on Γ_V (natural)

these can be combined in several ways:

1. at a free end with an applied load: $V = \overline{V}$, $M = \overline{M}$
2. at a simple support: $v = 0$, $M = 0$
3. at a clamped support: $v = 0$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$

an important consideration is that two conditions that are work conjugate cannot be specified at a single point on the boundary

Beam Elements - Weak Form

to find the weak form for the beam element, multiply the equilibrium equation and the natural boundary conditions by the weight function or its derivative and then integrate the equilibrium equation:

$$\int_\Omega \delta w\left(\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} - p\right)\mathrm{d}x = 0$$

$$\delta w\left(-EI\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} - \overline{V}\right)\bigg|_{\Gamma_V} = 0$$

$$\frac{\mathrm{d}\delta w}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \overline{M}\right)\bigg|_{\Gamma_M} = 0$$

these will lead to expressions for the weak form of the equilibrium equations for the beam element

Beam Elements - Weak Form

integrate by parts:

$$\int_\Omega \delta w \frac{\mathrm{d}^2 M}{\mathrm{d}x^2}\,\mathrm{d}x = \left(\delta w \frac{\mathrm{d}M}{\mathrm{d}x}\right)\Bigg|_{\Gamma_V} - \int_\Omega \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x = \left(-\delta w \overline{V}\right)\Big|_{\Gamma_V} - \int_\Omega \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x$$

this uses the divergence theorem to move from a volume integral over the body (Ω) to a surface integral over the boundary (Γ)

then integrate the second term above by parts:

$$\int_\Omega \frac{\mathrm{d}\delta w}{\mathrm{d}x}\frac{\mathrm{d}M}{\mathrm{d}x}\,\mathrm{d}x = \left(\frac{\mathrm{d}\delta w}{\mathrm{d}x}\overline{M}\right)\Bigg|_{\Gamma_M} - \int_\Omega \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} M\,\mathrm{d}x$$

Beam Elements - Weak Form

combine all of the above results and substitute back into the original weak form of the equilibrium equation to get:

$$\int_\Omega \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} M\,\mathrm{d}x = \int_\Omega \delta w p\,\mathrm{d}x + \left(\frac{\delta w}{\delta x}\overline{M}\right)\Bigg|_{\Gamma_M} + \left(-\delta w \overline{V}\right)\Big|_{\Gamma_V}$$

put Hooke's Law into this equation to produce:

$$\int_\Omega \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} EI \frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\,\mathrm{d}x = \int_\Omega \delta w p\,\mathrm{d}x + \left(\frac{\delta w}{\delta x}\overline{M}\right)\Bigg|_{\Gamma_M} + \left(-\delta w \overline{V}\right)\Big|_{\Gamma_V}$$

what impact does the nature of this equation have on the allowable functions we can use for δw and the approximation function?

Beam Elements - Discretization

this equation has higher continuity requirements than for the one-dimensional element: beam elements require both displacement continuity and slope continuity at the boundaries of the element and hence the approximation functions must be C^1

this will require additional degrees of freedom for the element: two displacement degrees of freedom (four if axial compression or stretching is allowed; here it is not) and two rotational degrees of freedom, one of each at each node:

$$\mathbf{d} = [v_1, \theta_1, v_2, \theta_2]^T$$

the nodal forces are work conjugate to the nodal displacements:

$$\mathbf{f} = [F_1, M_1, F_2, M_2]^T$$

Beam Elements - Discretization

the shape functions for the beam element weight function and solution function are:

$$N_1^v = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_1^\theta = \frac{\ell}{8}(1-\xi)^2(1+\xi)$$

$$N_2^v = \frac{1}{4}(1+\xi)^2(2-\xi)$$

$$N_2^\theta = \frac{\ell}{8}(1+\xi)^2(\xi-1)$$

in a local coordinate system with the origin at one end of the element with position parameterised by ξ, which is:

$$\xi = \frac{2x}{\ell} - 1$$

where ℓ is the length of the element and $-1 \leq \xi \leq 1$

check that the shape functions obey the Kronecker delta properties and note that this is again similar to mapping in Gauss quadrature

Beam Elements - Discretization

the shape functions will be used to interpolate both the solution and the weight function and the derivatives of both from the nodal values **d** and **w**:

$$v \approx \mathbf{Nd} \quad \delta w \approx \mathbf{Nw}$$

the derivatives are with respect to x so use the Jacobian to change variables:

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}\xi}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{2}{\ell}\frac{\mathrm{d}}{\mathrm{d}\xi}$$

the second derivative of the shape function is:

$$\frac{\mathrm{d}^2\mathbf{N}}{\mathrm{d}x^2} = \frac{1}{\ell}\left[\begin{array}{cccc} \frac{6\xi}{\ell} & 3\xi-1 & \frac{-6\xi}{\ell} & 3\xi+1 \end{array}\right] = \mathbf{B}$$

hence:

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \approx \mathbf{Bd} \quad \frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \approx \mathbf{Bw}$$

Beam Elements - Discrete Equations

in matrix form, the discrete equations are the same as for other elements:

$$\mathbf{Kd} = \mathbf{f} + \mathbf{r}$$

where **r** is the residual, which contains both the reaction forces and small errors due to approximation and discretization the stiffness matrix for the element is:

$$\mathbf{K} = \int_\Omega \mathbf{B}^T EI\mathbf{B}\, \mathrm{d}x$$

and the external force vector is:

$$\mathbf{f} = \int_\Omega \mathbf{N}^T p\, \mathrm{d}x + \left(\mathbf{N}^T \overline{V}\right)\Big|_{\Gamma_V} + \left(\frac{\mathrm{d}\mathbf{N}^T}{\mathrm{d}x}\overline{M}\right)\Bigg|_{\Gamma_M}$$

Beam Elements - Discrete Equations

if the stiffness E and second moment of area I are constant through the element, the stiffness matrix is given by:

$$\mathbf{K} = \frac{EI}{\ell^3}\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix}$$

and the nodal forces associated with a constant applied distributed load p:

$$\mathbf{f} = \int_\Omega \mathbf{N}^T p\, \mathrm{d}x = \int_0^\ell \mathbf{N}^T p\, \mathrm{d}x = \frac{p\ell}{12}\begin{bmatrix} 6 \\ \ell \\ 6 \\ -\ell \end{bmatrix}$$

3.13 Solution Methods

System Equations

the individual element stiffness matrices are combined into a global stiffness matrix through a systematic process

this process is described in the finite element literature as “scatter” and “gather”

distinguish between element matrices, which have no superscript, and global matrices, which have the superscript g

the element stiffness relation is:

$$\mathbf{f} = \mathbf{K}\mathbf{d}$$

whereas the global relation:

$$\mathbf{f}^g + \mathbf{r}^g = \mathbf{K}^g\mathbf{d}^g$$

is the equation that eventually needs solution

here, $\mathbf{r}^g$ is a reaction - residual term that contains the reaction forces and any small numerical inaccuracies due to the approximations

System Equations

take a one-dimensional system consisting of two two-noded elements

for element 1, the element relation is:

$$\mathbf{d} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

while the total system displacement is given by:

$$\mathbf{d}^g = \begin{bmatrix} u_1^g \\ u_2^g \\ u_3^g \end{bmatrix}$$

the relationship between the element and global displacements is:

$$\mathbf{d} = \mathbf{L}\mathbf{d}^g$$

where:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

System Equations

the matrix **L** is called the "gather" matrix because it collects the element terms from the global matrix

the gather matrix is different for each element, and relates the local element node numbering to the global node numbering

for the example, the gather matrix for element 2 is:

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

these matrices are very important because they connect not only the element displacements to the global displacements, but also the element stiffness matrices and element force vectors to their global counterparts

note, however, that actually constructing and employing these matrices in Matlab leads to very slow code

System Equations

for a single element:

$$\mathbf{K}\mathbf{L}\mathbf{d}^g = \mathbf{f}$$

examine the force vector; it can be transformed in a similar manner to the displacement vector:

$$\mathbf{f}^g = \begin{bmatrix} f_1^g \\ f_2^g \\ f_3^g \end{bmatrix}; \quad \mathbf{L}^T\mathbf{f} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

where $\mathbf{L}^T$ is called the "scatter" matrix

the global force vector can be assembled from the element force vectors:

$$\sum_{elements} \mathbf{L}^T\mathbf{f} = \mathbf{f}^g + \mathbf{r}^g$$

System Equations

use this to modify the global equation:

$$\sum_{elements} \left(\mathbf{L}^T\mathbf{K}\mathbf{L}\right)\mathbf{d}^g = \sum_{elements} \left(\mathbf{L}^T\mathbf{f}\right) = \mathbf{f}^g + \mathbf{r}^g$$

the key part of this is the stiffness matrix: the global stiffness matrix is:

$$\mathbf{K}^g = \sum_{elements} \mathbf{L}^T\mathbf{K}\mathbf{L}$$

this provides the final form to solve for the complete finite element system:

$$\mathbf{K}^g\mathbf{d}^g = \mathbf{f}^g + \mathbf{r}^g$$

Partitioned System

the final set of system equations typically has the form:

$$\begin{bmatrix} \vdots \\ f_7 \\ f_8 \end{bmatrix} = \left[\begin{array}{cc|cc} \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & K_{77} & K_{78} \\ \vdots & \vdots & K_{87} & K_{88} \end{array}\right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_7 \\ d_8 \end{bmatrix}$$

where many of the elements of the vector $\mathbf{d}$ are known to be zero

in that case, it is only necessary to solve the partitioned system:

$$\begin{bmatrix} P_7 \\ P_8 \end{bmatrix} = \begin{bmatrix} K_{77} & K_{78} \\ K_{87} & K_{88} \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \end{bmatrix}$$

Partitioned System

in general, it is possible to remove the elements of the displacement and nodal force vectors, and the corresponding rows and columns of the stiffness matrix, for elements of the displacement vector known to be zero from the boundary conditions

typically, the displacements that are zero due to the boundary conditions are not quite so easily arranged into a single group as in this example

clever node numbering can often achieve this, but in general it is necessary to use another algorithm to remove the matrix rows and columns that are unnecessary; note that, if element *i* in the displacement vector is zero because of the boundary conditions, in MATLAB the command:

```
K(i,:)  = []; K(:,i) = [];
```

will remove the relevant elements from the stiffness matrix **K**

the same operation must be carried out on the vectors of displacements and nodal forces as well

Element Load Vector

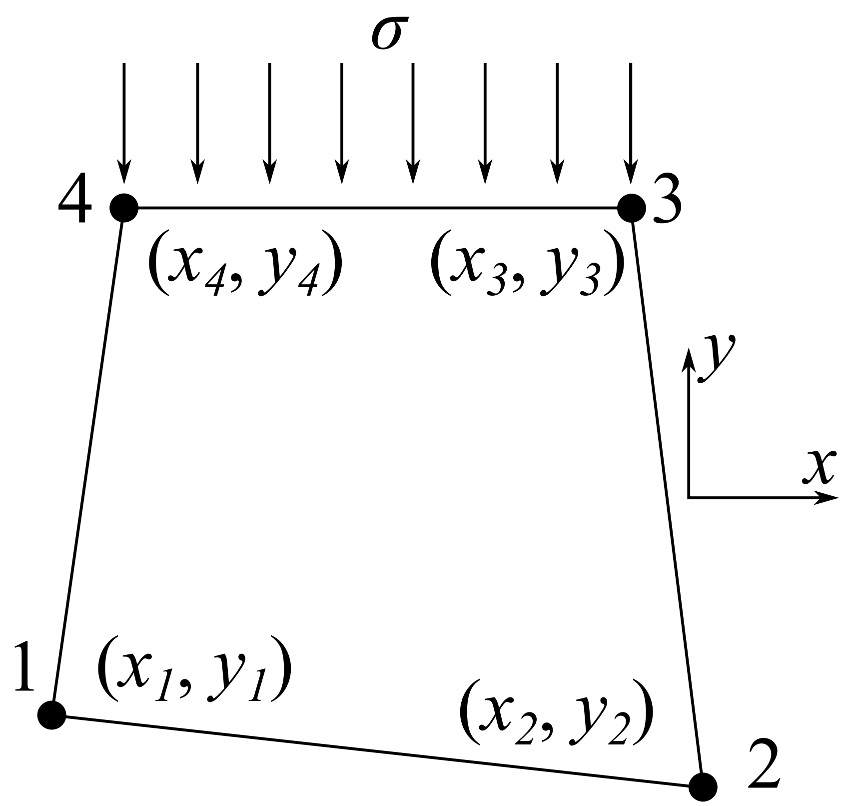
for each element, there is a load vector with size equal to the number of degrees of freedom in the element

for a two-dimensional four-node quadrilateral element with a vector-valued solution (for example, in stress analysis), there are eight degrees of freedom for each element, two at each node

this load vector represents the distribution of external forces acting on the element, distributed amongst the nodes; forces can only act at the nodes in the approximated, discretised system

the external forces can be either surface tractions or body forces

Element Load Vector



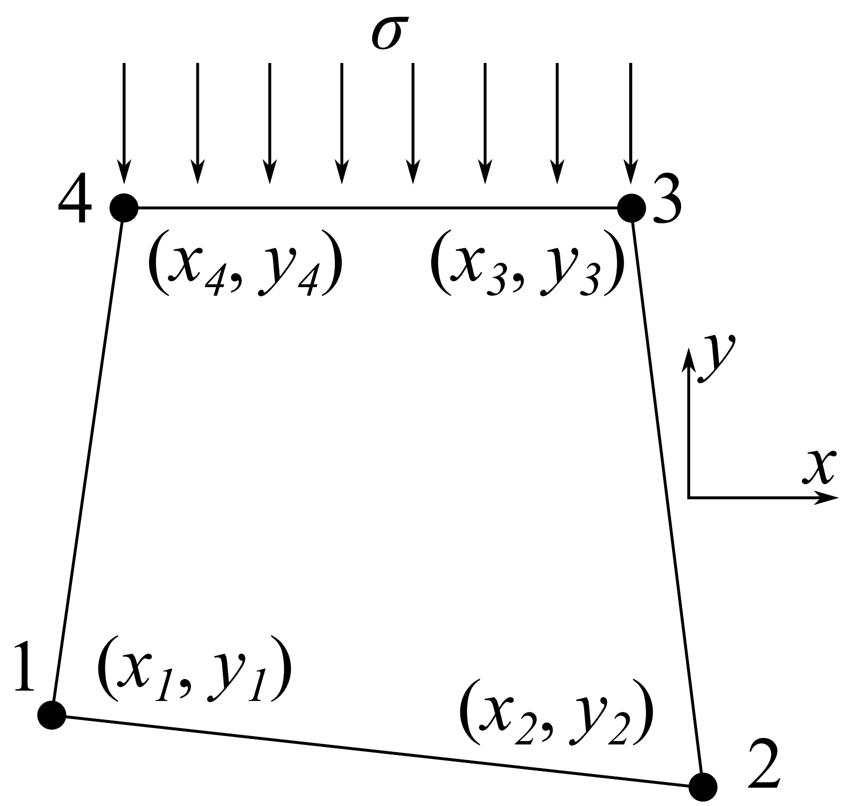
distributed loads, as seen to the left, must allocated to the element nodes

typically, if b is the thickness of the element, then the loads in the y-direction applied to nodes 3 and 4 would be equal to:

$$f_y = b\frac{(x_3 - x_4)}{2}\sigma$$

note that these are forces, not stresses or tractions

Element Load Vector

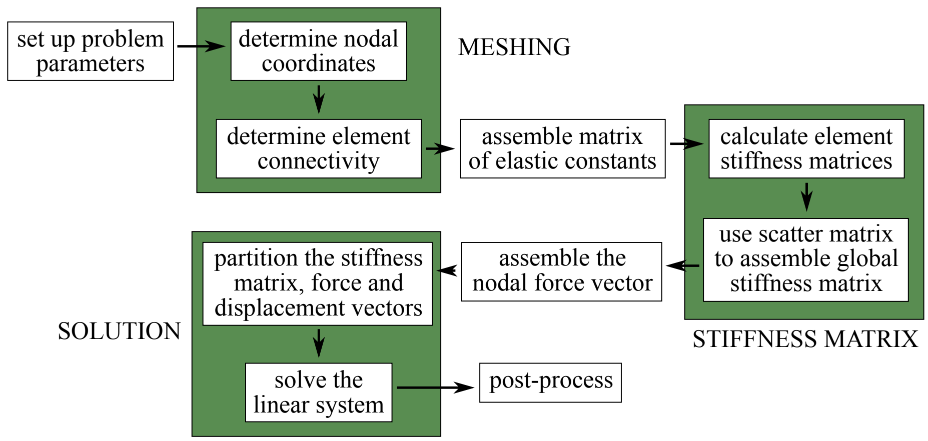


as a consequence, the nodal force vector for this element would be:

$$f = \begin{bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_y \\ 0 \\ f_y \end{bmatrix}$$

here, f_{1x} (for example) is the nodal force at node 1 in the x-direction

Structure of a Finite Element Code



Meshing

not much has been said about meshing; a good mesh is critical to the solution of a finite element problem

a "good" mesh is one that is sufficiently fine in areas where the strains are changing rapidly to capture the details of the material behaviour, coarse in regions where the strains are relatively constant, and with elements that are fairly regularly shaped: as close to square as possible for quadratic elements

to create a mesh, determine the locations of the nodes that will serve as the boundaries of the elements; it is generally a good idea to divide the area of interest into regions based upon an intuitive understanding of whether the strain gradients will be large or small

ideally, this could be done in an automated fashion but it usually requires substantial user intervention even with commercial packages

Mesh Convergence

it is always necessary to check that the mesh used for a finite element calculation achieves convergence

this means that increasing the number of elements does not have a significant effect on the final result

typically a convergence study would be performed by taking an existing finite element solution and comparing the solution to a new solution generated with four times (or sixteen times) as many elements

if there is minimal change between the two solutions (with minimal defined by the user depending upon the circumstances, but often by requiring that no displacement changes more than 0.5%), then the solution is considered to have converged with respect to the mesh

Meshing

in general, it is a good idea to store the coordinates of the nodes in separate vectors, that is:

$$\mathbf{x}(i) \quad \text{and} \quad \mathbf{y}(i)$$

where, for example, `y(10)` is the y-coordinate of node 10

the next step is to determine the nodes corresponding to the element vertices; remember to order them anticlockwise

these data are typically stored in matrix format, in a matrix **e** which has dimensions (number of elements x number of nodes per element); thus, in MATLAB,

```
x(e(12,2))
```

provides the x-coordinate of the second node of element 12

Meshing

most commercial finite element packages take care of the meshing of a finite element problem based upon geometric input and some guidance from the user on where elements should be finer or coarser

there are specialized meshing packages that can generate meshes from user-defined geometry and input, and are also able to work with output from CAD programs more easily than can finite element packages themselves

even with automated meshing, it is incumbent upon the user to check the mesh very carefully for problems; poor meshes can lead to very poor results with concomitant consequences for engineering decision-making

Stiffness Matrix

the calculation of the stiffness matrix is the heart of the finite element method, but actually takes much less code than meshing or post-processing do

for each element:

1. determine the element nodal coordinates, node numbers and degree of freedom numbers
2. assemble the scatter matrix
3. for each Gauss point in an element, the usual process is:
   (a) calculate the shape functions for that Gauss point
   (b) calculate the derivatives of the shape function at the Gauss point
   (c) calculate the Jacobian matrix and the Jacobian determinant
   (d) calculate the derivative matrix **H** for the Gauss point
   (e) calculate the element stiffness contribution for the Gauss point:

$$\mathbf{K_G} = W_i W_j \mathbf{H}^T(\xi_i, \eta_j)\mathbf{D}\mathbf{H}(\xi_i, \eta_j)\left|\mathbf{J}(\xi_i, \eta_j)\right|$$

(f) scatter the Gauss point stiffness contribution into the global matrix

it is often helpful to retain some of the intermediate calculations for use later during post-processing

Post-Processing

there is no limit to the amount of code that can be written to do post-processing of finite element results

the solution of the matrix equation is the vector of displacements of the nodes (the displacements known to be zero can be added back at this point)

using this vector, the new nodal locations can be calculated; these can be used to generate a deformed mesh, which is often shown (with the deformations magnified) in conjunction with the undeformed mesh

with the complete (non-partitioned) displacement vector, the complete matrix can be used to calculate the complete nodal force vector; subtracting the external force vector gives the vector of reaction forces plus the residual (which contains any error in the linear solution)

Post-Processing

using the nodal displacements and the **H** matrices, the strains can be calculated at each *Gauss point*; note that the strains at the nodes have not been calculated directly

strains elsewhere can be interpolated from the strains at the Gauss points

however, the Gauss points are typically not arranged in any orderly fashion if the mesh is non-uniform or contains elements with faces not parallel to the coordinate axes

this requires additional work to accomplish the interpolations

once the strains are known, the matrix of elastic constants can be used to determine the stresses, again at the Gauss points

the same comment about interpolation applies here

Post-Processing

other measures of stress and strain can be calculated; often, local principal stresses or von Mises stresses are of interest

often when looking at stress and strain, it is helpful to plot contours of stress and strain; Matlab has powerful tools to do this, but there are some restrictions on the form of the input matrices that are allowable

other quantities that may be of interest include vectors of displacements or contours of strain energy density