9th Dec 2023

Homework 6

7-9.9 Step response $\Rightarrow$ input $= u(t)$.

$$h(t) = 5u(t-1) - 5u(t-4)$$

We know that: 1) $u(t) * u(t) = t\,u(t)$.

2) $u(t-d) * u(t) = u(t) * \delta(t-d) * u(t) =$
$= t(u(t)) * \delta(t-d) = (t-d)u(t-d)$ *

a) $y(t) = u(t) * h(t)$

$$y(t) = u(t) * (5u(t-1) - 5u(t-4))$$

$$y(t) = 5(u(t) * (u(t-1) - u(t-4))) =$$

$$y(t) = 5(u(t) * u(t-1) - u(t)u(t-4))$$

Apply *:

$$y(t) = 5((t-1)u(t-1) - (t-4)u(t-4))$$

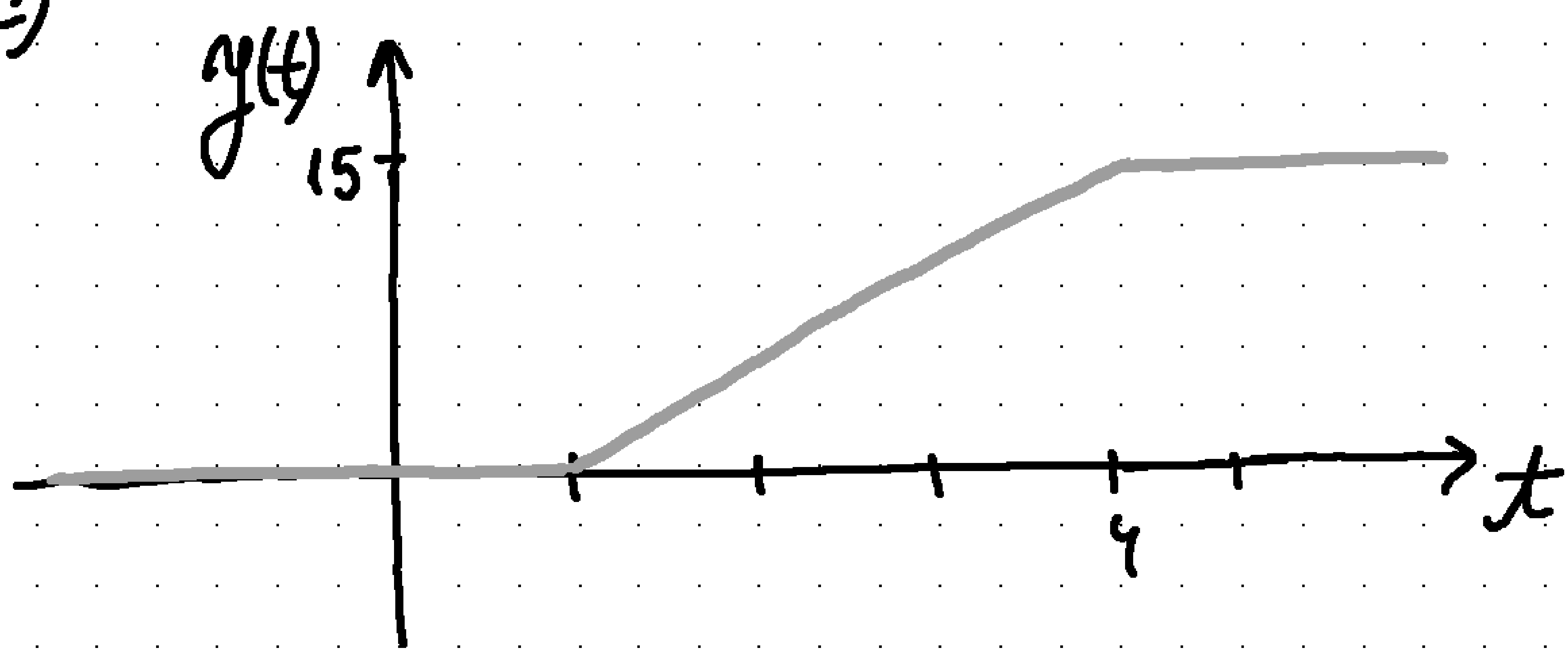
b)

$$y(t) = \begin{cases} 0 & , t < 1 \\ 5(t-1) & , t \in [1,4) \\ 5(t-1) - 5(t-4) & , t \geq 4 \end{cases} =$$

$\hookrightarrow \cancel{5t} - 5 - \cancel{5t} + 20 = 15$

$$= \begin{cases} 0 , & t < 1 \\ 5(t-1) , & t \in [1,4) \\ 15 , & t \geq 4 \end{cases}$$

c)



P - 9.15

a) The LTI system is stable if and only if:

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < C < +\infty$$

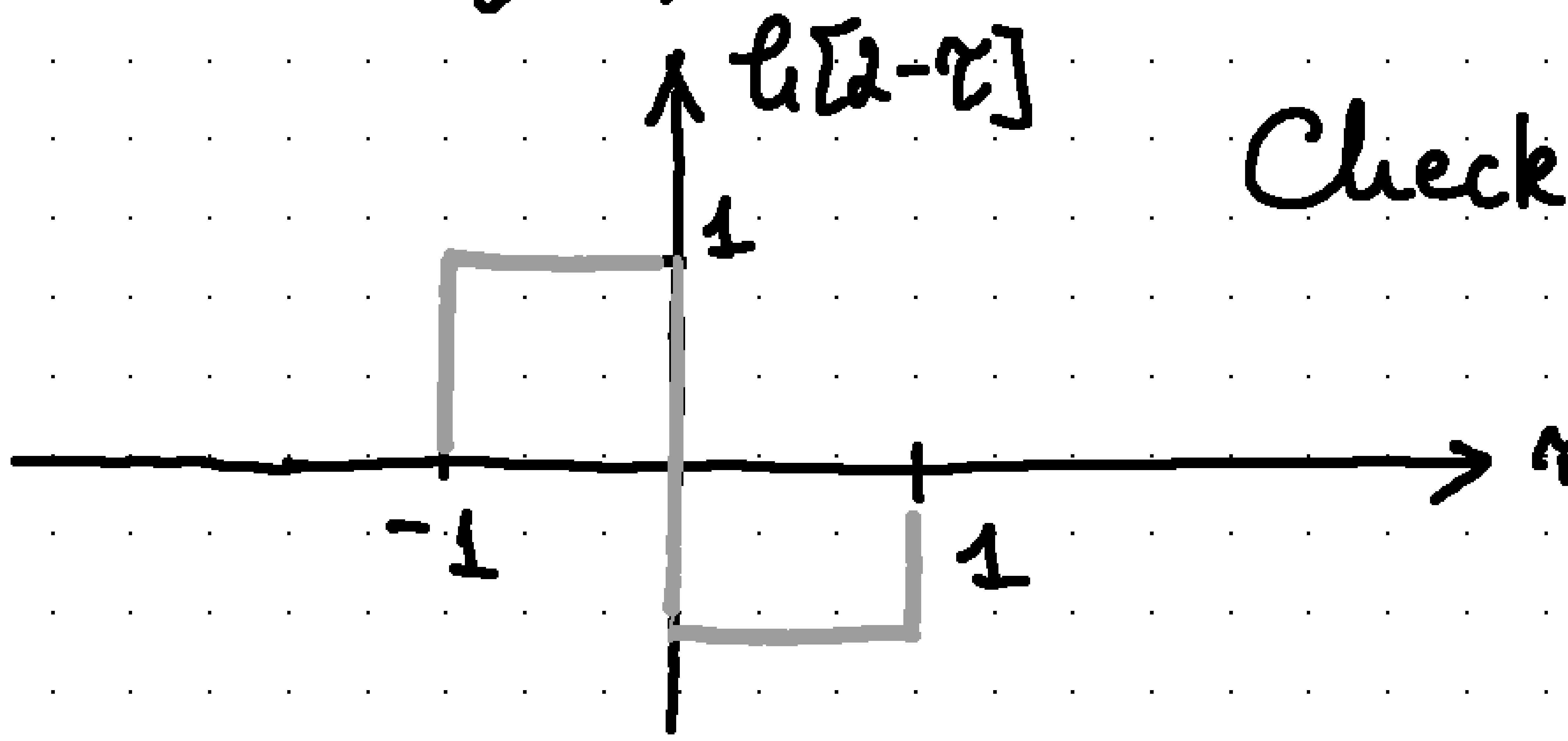
=> In our case:

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_{1}^{3} 1\, d\tau = [\tau]_1^3 = 3 - 1 = 2 < \infty$$

=> The system is stable.

b) $t \in [1;3] \Rightarrow [2-\tau] \in -[1;-1] = [-1;1]$.

Shift the graph, where $\tau \in [-1,1]$.



Check: $\tau = -1$

$h(2-(-1)) = 1$

$h(3) = 1$

(correct)

c) LTI system $\Rightarrow$ output is described by a convolution integral.

$$y(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau .$$

$$\Rightarrow \; y(2) = \int_{-\infty}^{\infty} x(\tau)\, h(2-\tau)\, d\tau$$

$x(\tau) = u(\tau)$ // task

$$y(2) = \int_{-\infty}^{\infty} u(\tau)\, h(2-\tau)\, d\tau = \int_{0}^{\infty} h(2-\tau)\, d\tau$$

$h(2-\tau) > 0$ iff $\tau \in [-1;1]$
$\tau > 0$

$$y(2) = \int_{0}^{1} h(2-\tau)\, d\tau$$

$\tau \in [0,1]$ from graph $\Rightarrow h(2-\tau) = -1$

$$y(2) = \int_{0}^{1} -1\, d\tau = [-\tau]_0^1 \Rightarrow \boxed{y(2) = -1}$$

d) $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$.

$x(\tau) = u(\tau)$

$y(t) = \int_{0}^{\infty} h(t-\tau) = 0 ; \quad t \in [T_1, T_2]$

$y(t) = 0$ iff $h(t-\tau) = 0$ Based on the graph, we shift $h(t)$ by τ

$\Rightarrow h(t-\tau) = 0$

$t \in [2,3] \Rightarrow h(t) = 1 \Rightarrow t-\tau \in [2-\tau, 3-\tau]$

$\Rightarrow t < 2-\tau$ and $t < 3-\tau$ such that

$$\begin{cases} T_1 = 2-\tau \\ T_2 = 3-\tau \end{cases}$$

P-10.5 :

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad a_k = \frac{10 \sin(\pi k/4)}{\pi k}.$$

a) from figure : $T_0 = 8$.

$$\omega_0 = 2\pi f_0$$

$$T_0 = \frac{1}{f_0} \Rightarrow f_0 = \frac{1}{T_0} = \frac{1}{8} \Rightarrow \omega_0 = 2\pi \cdot \frac{1}{8_4}$$

$$\boxed{\omega_0 = \pi/4 \text{ rad}}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi k f_0 t} dt$$

$$a_k = \frac{1}{8} \int_{-4}^{4} x(t) e^{-j2\pi k 1/8 t} dt$$ // eliminate portions where integral is zero

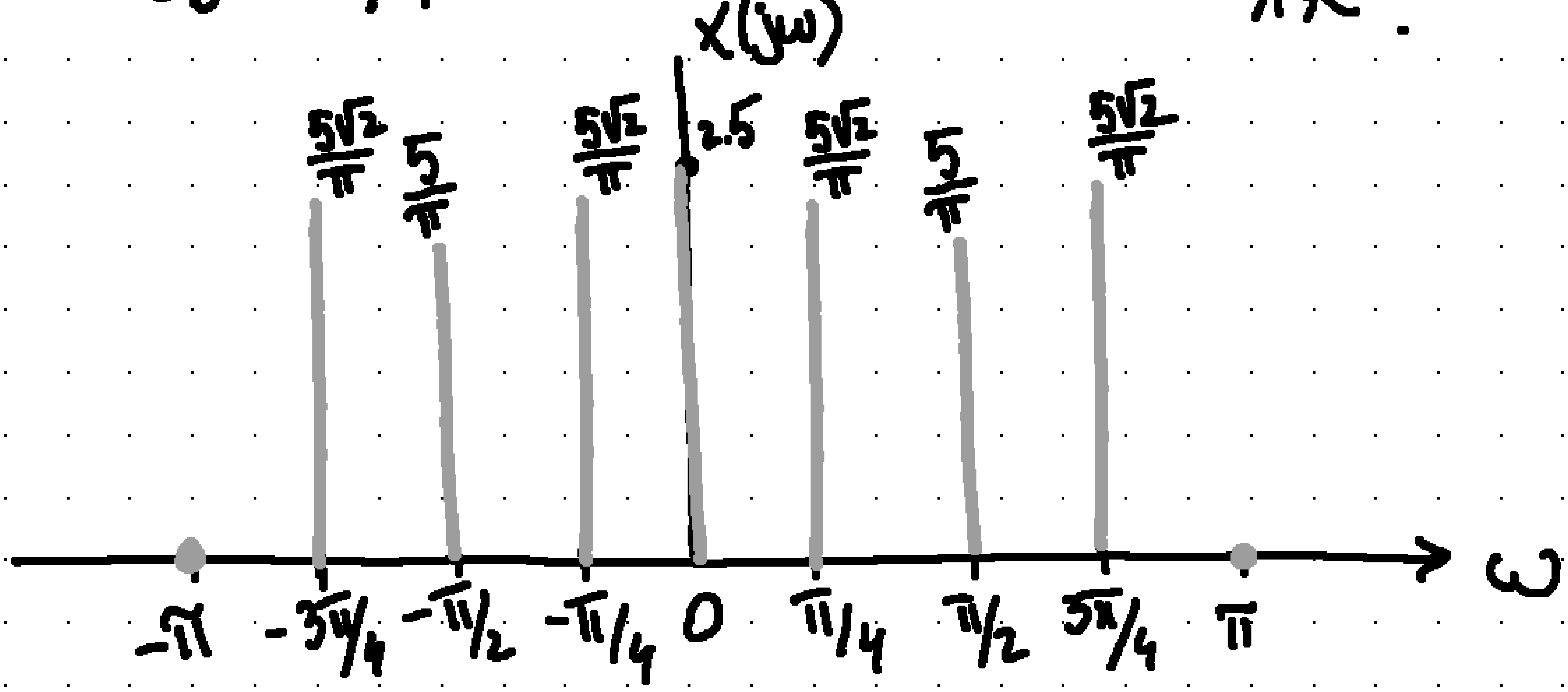
$$a_k = \frac{1}{8} \int_{-1}^{1} 10 e^{-j\pi/4 kt} dt$$

b) $-4\omega_0 \le \omega \le 4\omega_0$

$\omega = \pi/4\, K$. $\Longrightarrow$ $-\pi \le \pi/4\, K \le \pi$

$\omega_0 = \pi/4$

$$a_K = \frac{10\sin(\pi K/4)}{\pi K}.$$



$K=0 \Rightarrow a_0 = \dfrac{10\sin(\pi\cdot 0/4)}{\pi\cdot 0}$ $\underline{\underline{\text{division by zero}}}$ do a limit

$$\Rightarrow a_0 = \lim_{K\to 0} \frac{10}{4}\underbrace{\frac{\sin(\pi K/4)}{\pi K/4}}_{\to 1} = \frac{10}{4}. \quad (\omega=0)$$

$$\Rightarrow a_1 = \frac{10\sin(\pi/4)}{\pi} = \frac{10\,\sqrt{2}/2}{\pi} = \frac{5\sqrt{2}}{\pi} = a_{-1} \quad (\omega=\pi/4)$$

$$\Rightarrow a_2 = \frac{\overset{5}{\cancel{10}}\sin(\pi/2)}{\cancel{2}\pi} = \frac{5\cdot 1}{\pi} = \frac{5}{\pi} = a_{-2} \quad (\omega=\pi/2)$$

$$\Rightarrow a_3 = \frac{10\sin 3\pi/4}{3\pi} = \frac{\overset{5}{\cancel{10}}\,\sqrt{2}/\cancel{2}}{3\pi} = \frac{5\sqrt{2}}{3\pi} = a_{-3} \quad (\omega=3\pi/4)$$

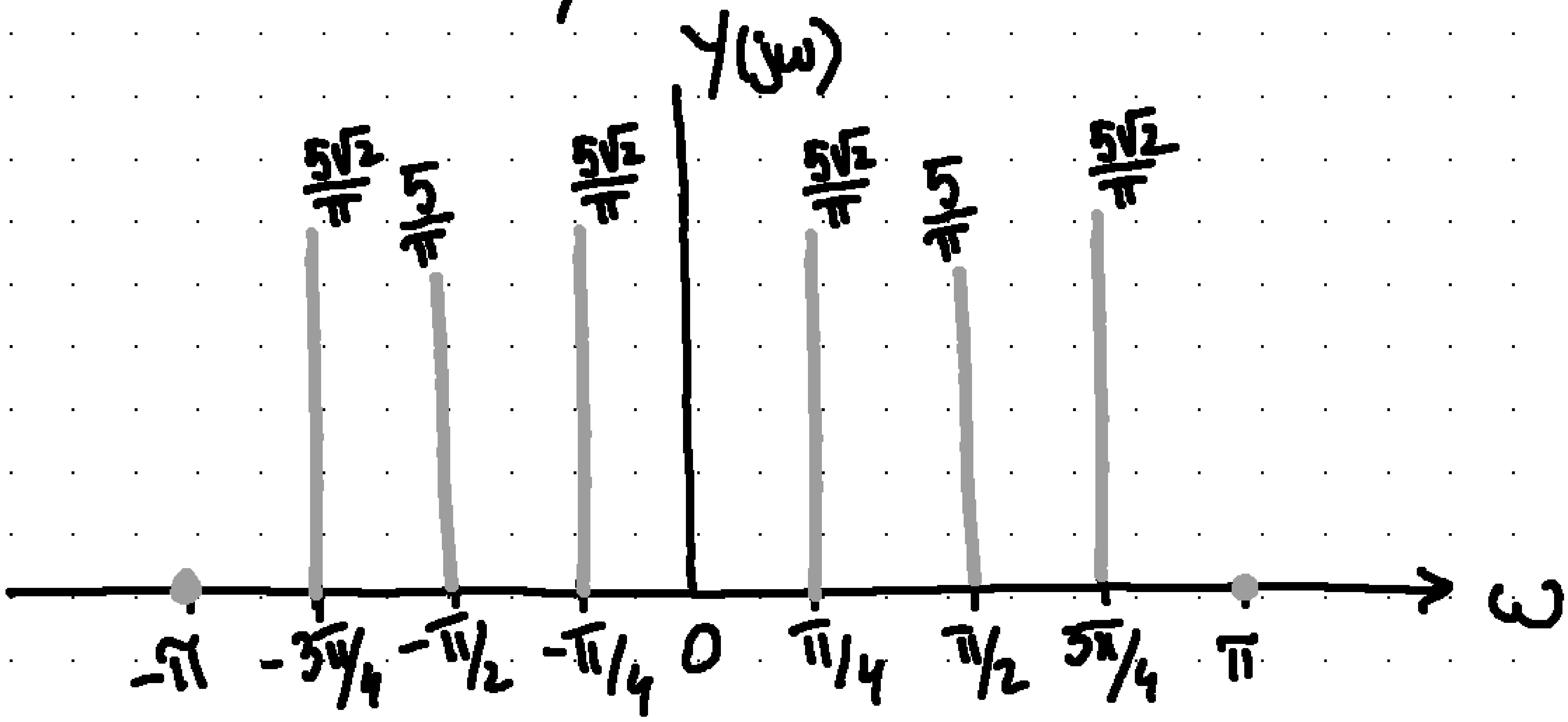
$$\Rightarrow a_4 = \frac{10\sin\pi}{4\pi} = 0 = a_{-4}. \quad (\omega=\pi)$$

c) ideal highpass filter

$$H(j\omega) = \begin{cases} 0 & ; |\omega| < \pi/8 \\ 1 & ; |\omega| \geq \pi/8 \end{cases} \Rightarrow \omega_{co} = \pi/8$$

$$Y(j\omega) = X(j\omega)H(j\omega)$$

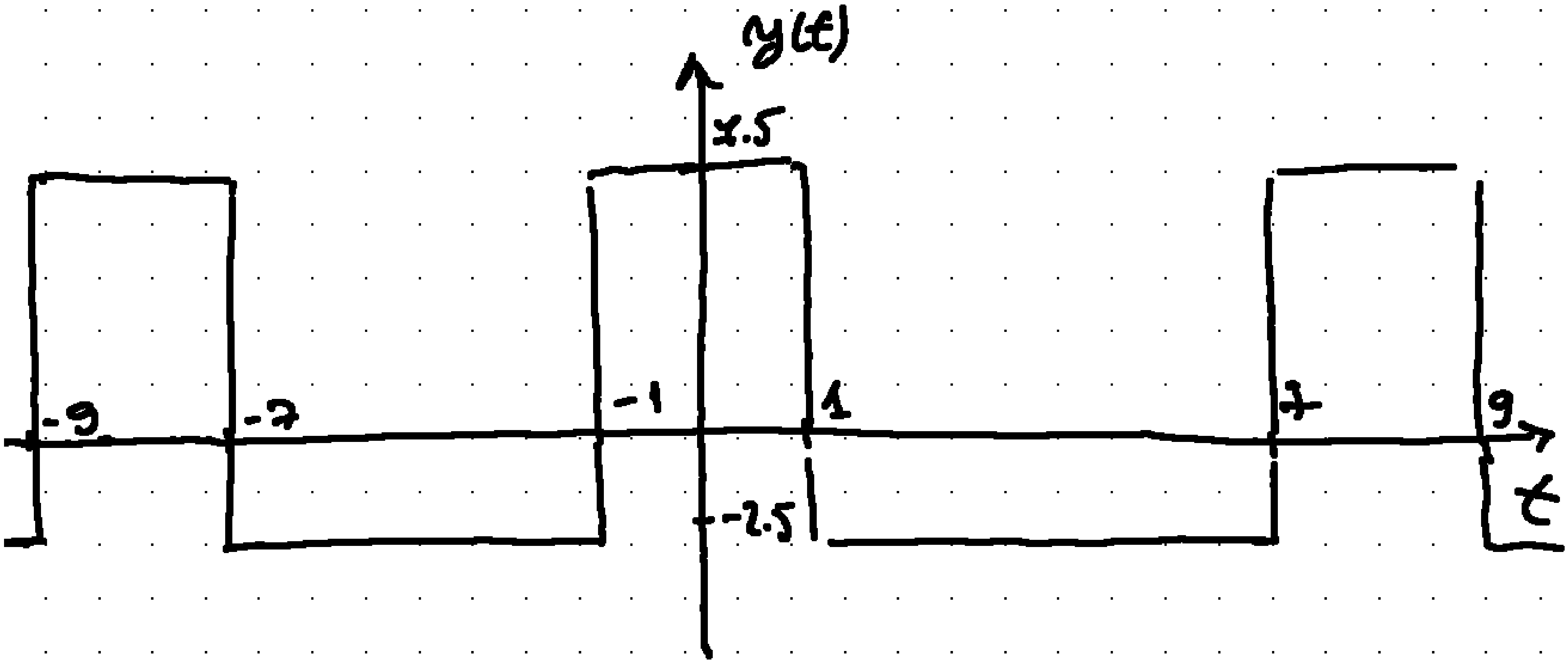
$\Rightarrow$ from previous task in b) we exclude the DC component.



e) Switch to time domain $\Rightarrow$ $y(t) = x(t) - a_0$

$= x(t) - 2.5$.

$\hookrightarrow$ Shift down the graph in Fig P-10.5 by 2.5.



d) ideal lowpass filter

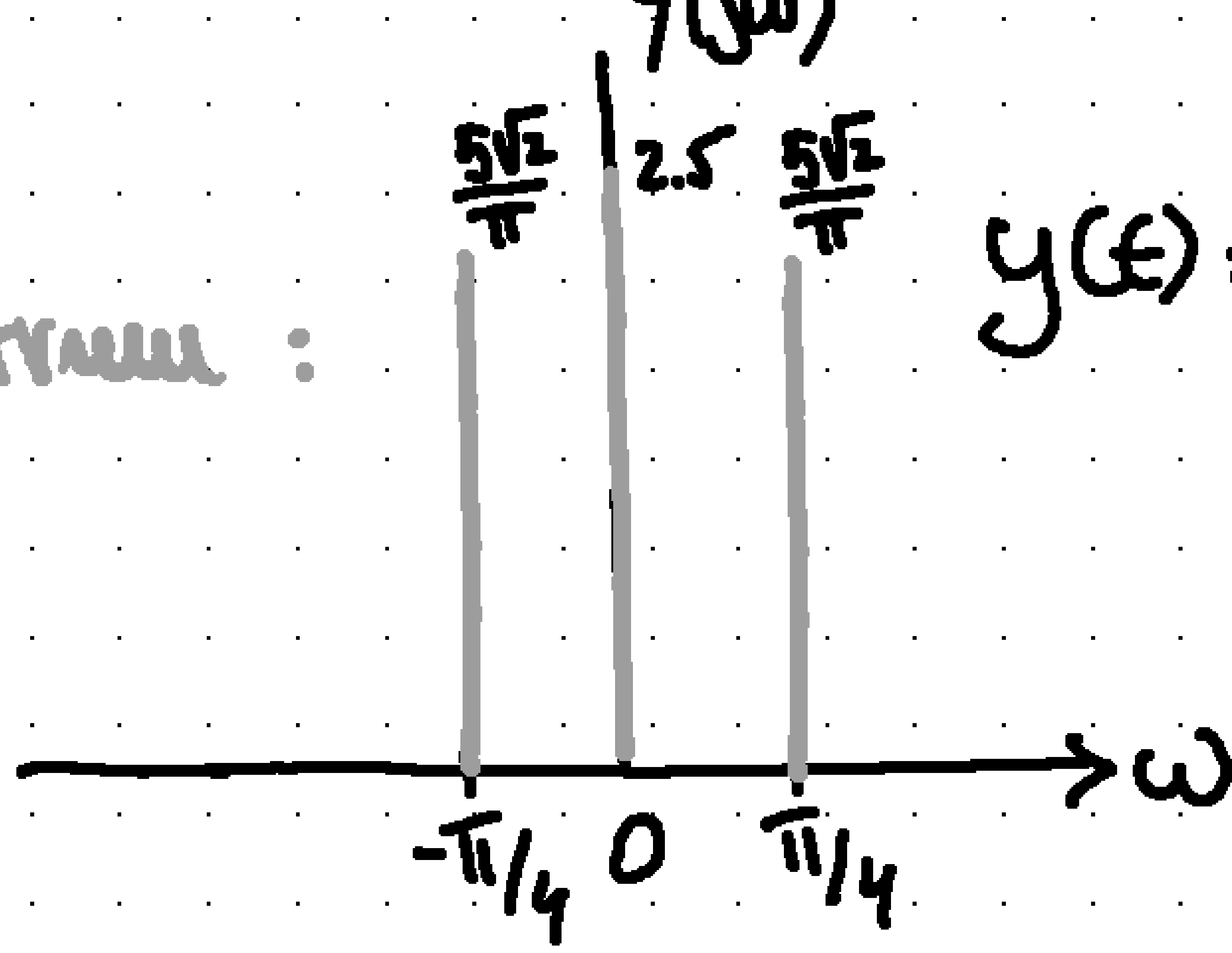
$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_{co} \\ 0, & |\omega| > \omega_{co} \end{cases}.$$

$$y(t) = A + B\cos(\omega_0 t + \phi) \quad , A, B \neq 0$$

$y(t)$: DC component + one two-sided spectrum frequency = 2 components.

$\Rightarrow \omega_{co} = \pi/2$

The Spectrum :



$$y(t) = 2.5 + \frac{10\sqrt{2}}{\pi}\cos(\pi/4\, t).$$

e) $H(j\omega) = 1 - e^{-j2\omega}$; Use Fourier Transf.

$$H(j\omega) \overset{\mathcal{F}}{\longleftrightarrow} h(t)$$

$$1 - e^{-j2\omega} \overset{\mathcal{F}}{\longleftrightarrow} \delta(t) - \delta(t-2)$$

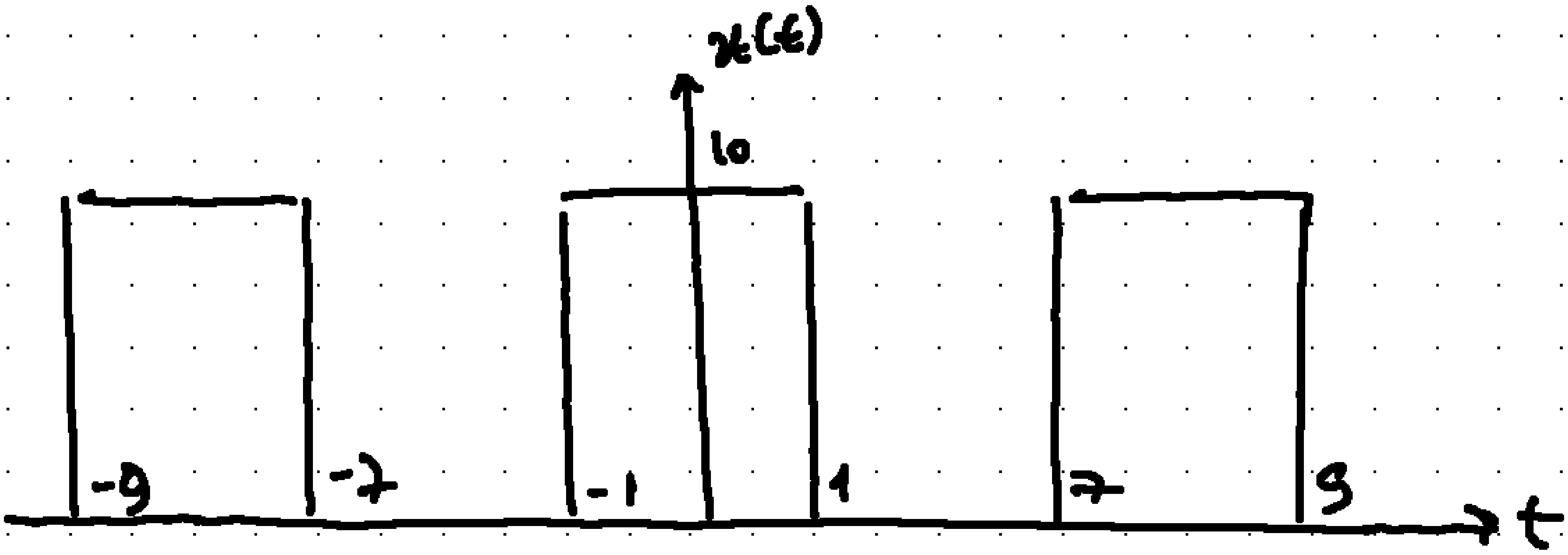
$$y(t) = x(t) * h(t).$$

$$y(t) = x(t) * (\delta(t) - \delta(t-2))$$

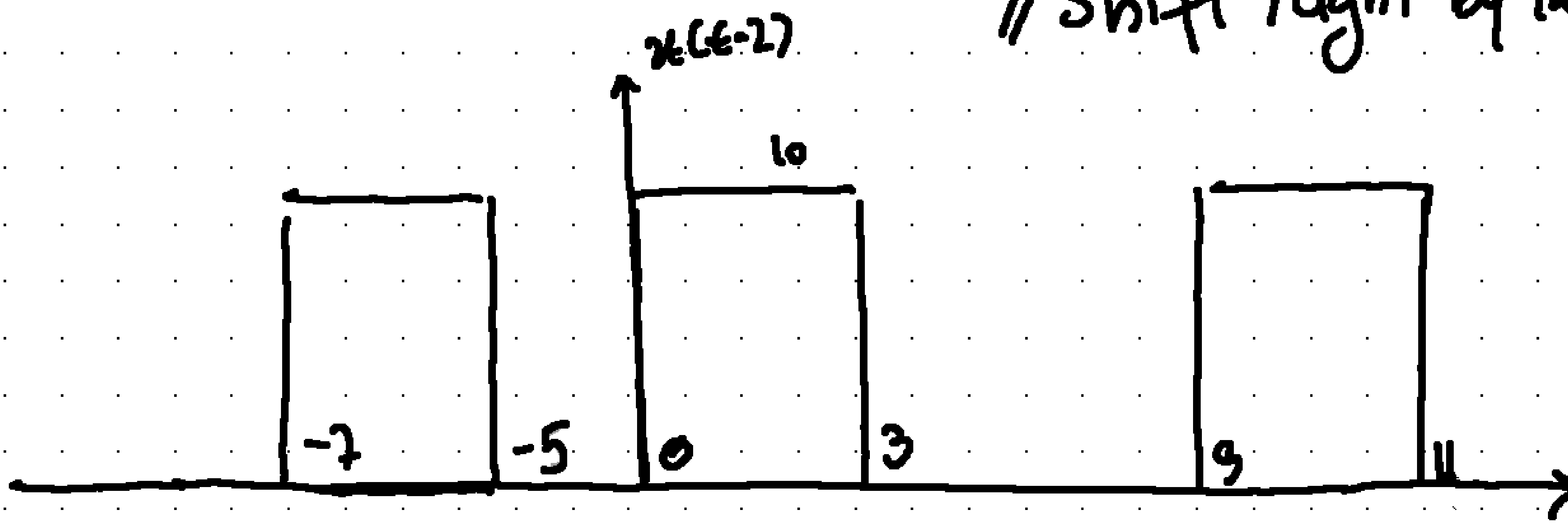
$$y(t) = x(t) * \delta(t) - x(t) * \delta(t-2)$$

$$\boxed{y(t) = x(t) - x(t-2)}$$

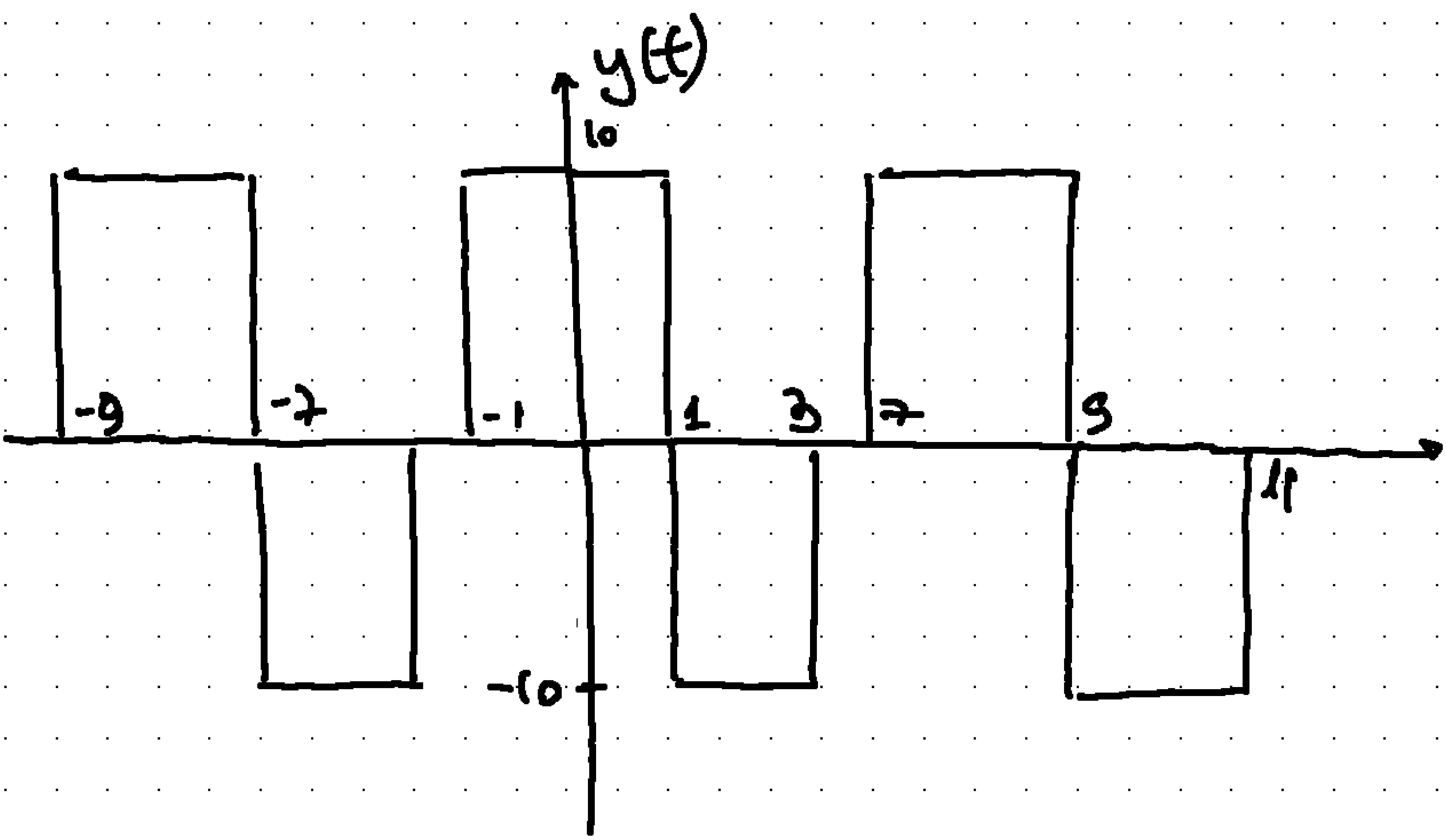
Plot: next page



// Shift right by two



// subtract



P-11.12

$$h(t) = \frac{4\sin(\omega_{co}t)}{\pi t} \quad ; \; T_0 = 1.$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

a) Input signal periodic $\Rightarrow$

$$\Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi \delta(\omega - k\omega_0)$$

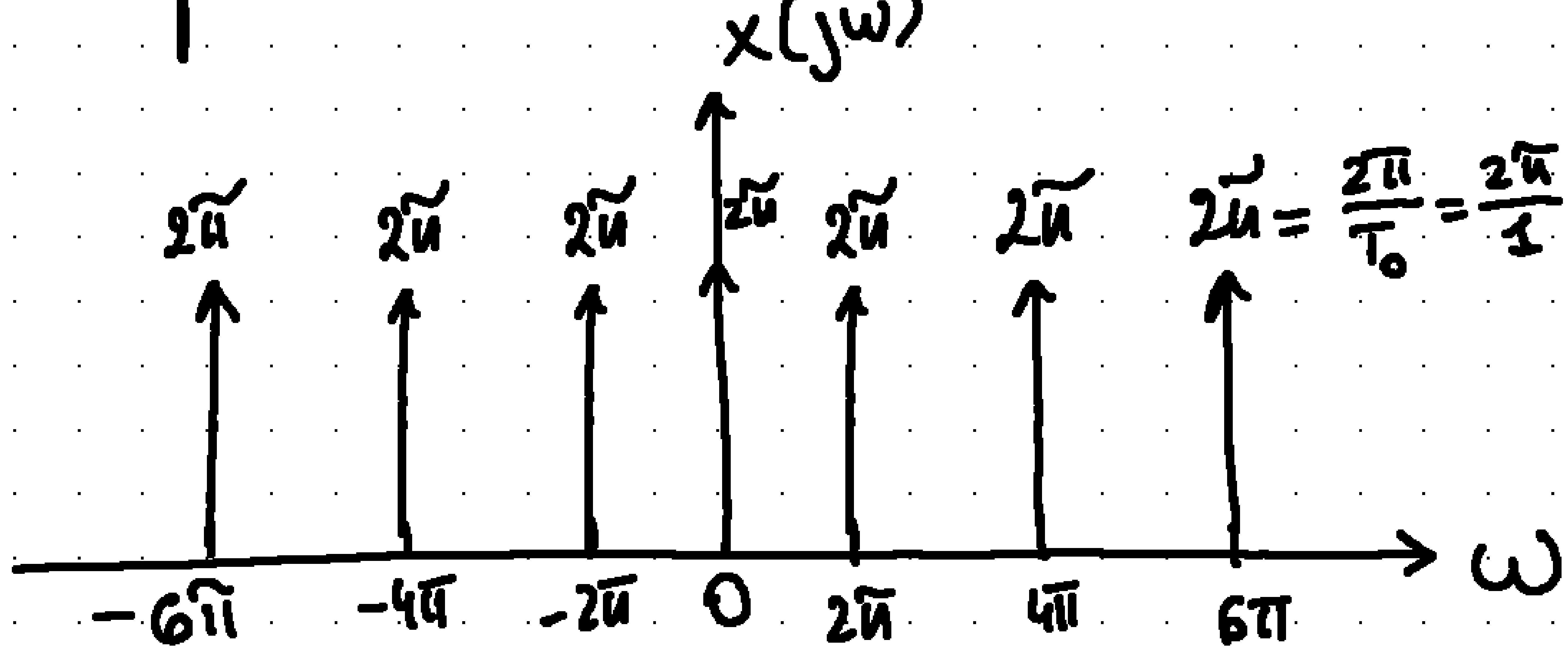
$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk2\pi t/T_0} dt = \frac{1}{T_0} = \frac{1}{1} = 1.$$

$$\Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 1 \cdot 2\pi \delta(\omega - k\omega_0) =$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \quad \left[\begin{array}{l} \omega_0 = 2\pi f_0 \\ T_0 = \frac{1}{f_0} \Rightarrow f_0 = \frac{1}{T_0} = \frac{1}{1} = 1 \; / \Rightarrow \omega_0 = 2\pi \end{array} \right]$$

$$\boxed{X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).}$$

The plot:

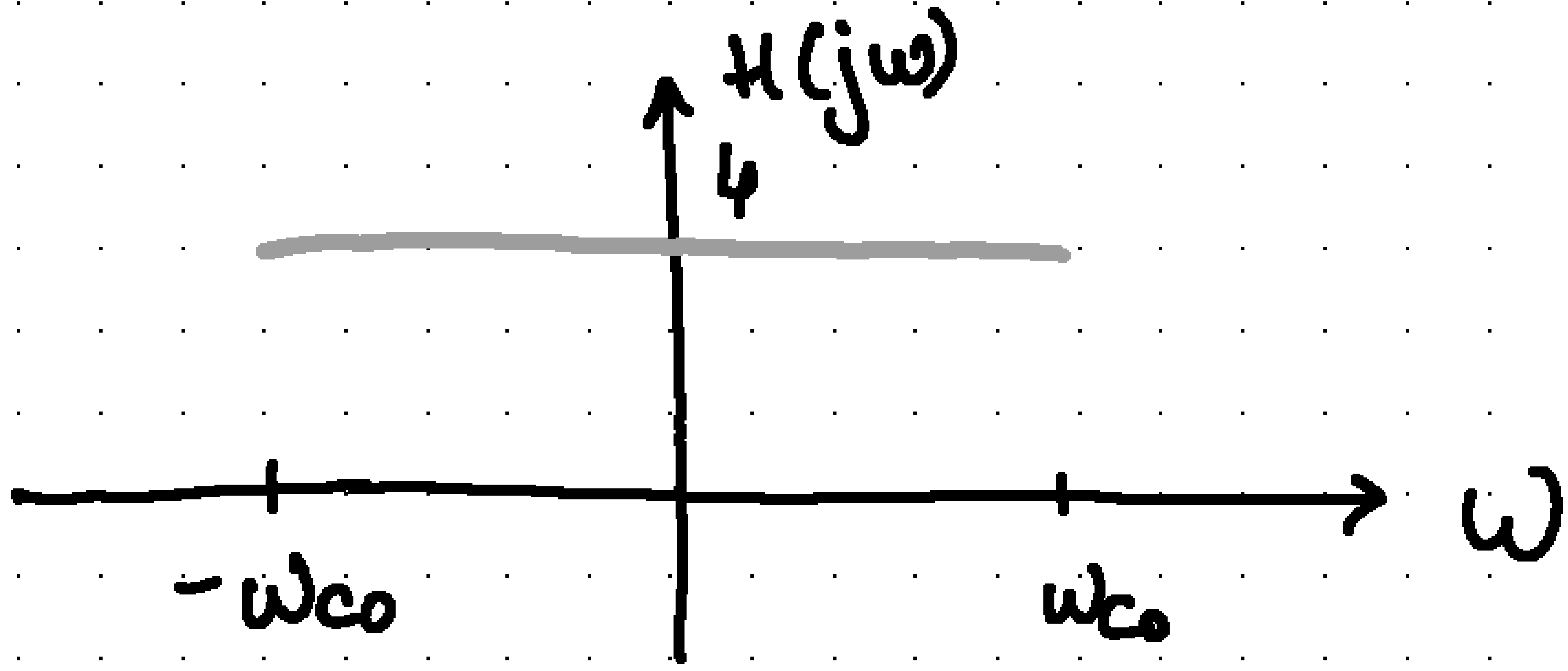


b) $H(j\omega) = \mathcal{F}\left\{ \frac{4\sin(\omega_{co}t)}{\pi t} \right\} =$

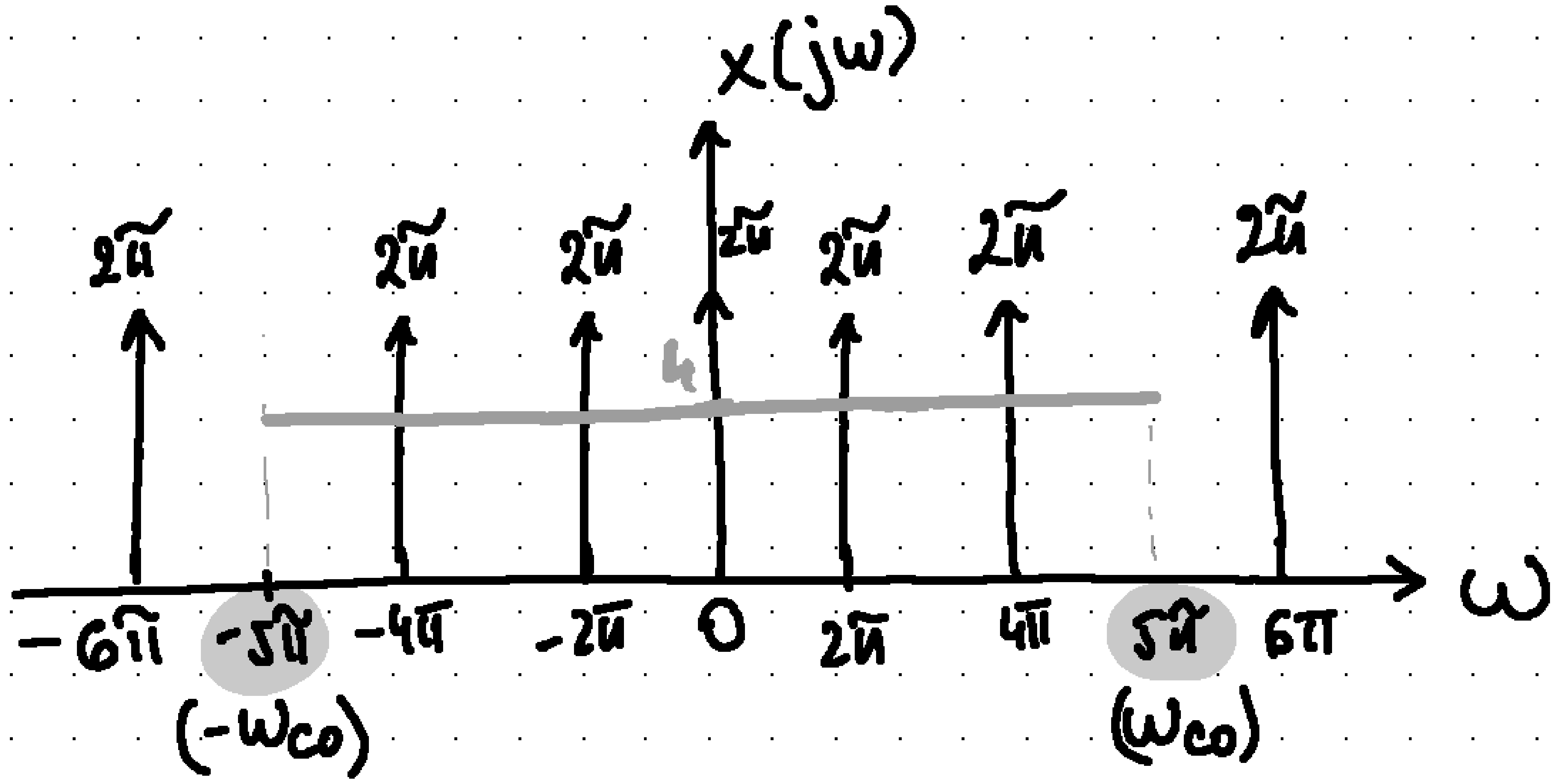
$= 4\left(u(\omega+\omega_{co}) - u(\omega-\omega_{co}) \right)$

- $u(\omega+\omega_{co}) = \begin{cases} 1 \; ; & \omega+\omega_{co} \geq 0 \Rightarrow \omega \geq -\omega_{co} \\ 0 \; ; & \omega+\omega_{co} < 0 \Rightarrow \omega < -\omega_{co} \end{cases}$

- $u(\omega-\omega_{co}) = \begin{cases} 1 \; ; & \omega-\omega_{co} \geq 0 \Rightarrow \omega \geq \omega_{co} \\ 0 \; ; & \omega-\omega_{co} < 0 \Rightarrow \omega < \omega_{co}. \end{cases}$



$\Rightarrow$ The graph:



c) $\omega_{co} = 5\pi$ $\quad y(t) = ?$
input: $x(t)$

$Y(j\omega) = X(j\omega)H(j\omega) \Rightarrow y(t) = x(t) * h(t)$.

$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$.

$H(j\omega) = h\left(u(\omega + \omega_{co}) - u(\omega - \omega_{co})\right)$

$\omega_{co} = 5\pi$: Take from graph $\omega_0 \geq 5\pi$;
otherwise $H(j\omega) = 0$.

$2\pi k = \{-4\pi, -2\pi, 0, 2\pi, 4\pi\} \Rightarrow$
$k = \{-2, -1, 0, 1, 2\}$.

$$\Rightarrow Y(j\omega) = 4\cdot 2\pi\left(\delta(\omega-4\pi) + \delta(\omega-2\pi) + \delta(\omega) + \delta(\omega+2\pi) + \delta(\omega+4\pi)\right)$$

$$Y(j\omega) = 4\cdot 2\pi(\omega-4\pi) + 4\cdot 2\pi\,\delta(\omega-2\pi) + + 4\cdot 2\pi\,\delta(\omega) + 4\cdot 2\pi(\omega+2\pi) + 4\cdot 2\pi\,\delta(\omega+4\pi))$$

$$\boxed{e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega-\omega_0)}$$

$$y(t) = 4e^{-j4\pi t} + 4^{-j2\pi t} + 4^{0} + 4e^{j2\pi t} + 4e^{j4\pi t}$$

[Apply Euler's Inverse Th]

$$\boxed{y(t) = 4 + 8\cos(2\pi t) + 8\cos(4\pi t)}$$

d) $\omega_{co} = ?$ s.t $y(t) = C$. $C = ?$

We notice from c) the only constant is 4 for $\underline{\omega_0 = 0}$. ($C = 4$)

Choose $\omega_{co} \in (-2\pi, 2\pi)$ ~ only one pulse in the plot in this interval $\Rightarrow$ the one for $\underline{\omega_0 = 0}$

$$Y(j\omega) = 8\pi\delta(\omega) \Rightarrow y(t) = 4; \ C = 4.$$