21st Nov 2023

Homework 3

P-5.8 :

The output of the FIR filter is given by the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$$

In our case, from the graph we notice. $k \in [0,4]$ such that $h[k] \neq 0$

$$\Rightarrow y[n] = \sum_{k=0}^{4} h[k]x[n-k]$$

From graph: $h[0]=3$; $h[1]=7$; $h[2]=13$
$h[3]=9$; $h[4]=5$.

$$\Rightarrow y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] +$$
$$+ b_3 x[n-3] + b_4 x[n-4]$$

$$\Rightarrow \quad y[n] = 3x[n] + 7x[n-1] + 13x[n-2] +$$
$$+ 9x[n-3] + 5x[n-4]$$

$b_0 = 3$; $b_1 = 7$; $b_2 = 13$; $b_3 = 9$; $b_4 = 5$.

Input $x[n] = \{2, 1, -1\}$ is applied $\Rightarrow$

$$\Rightarrow \quad x[0] = 2$$
$$x[1] = 1$$
$$x[2] = -1$$

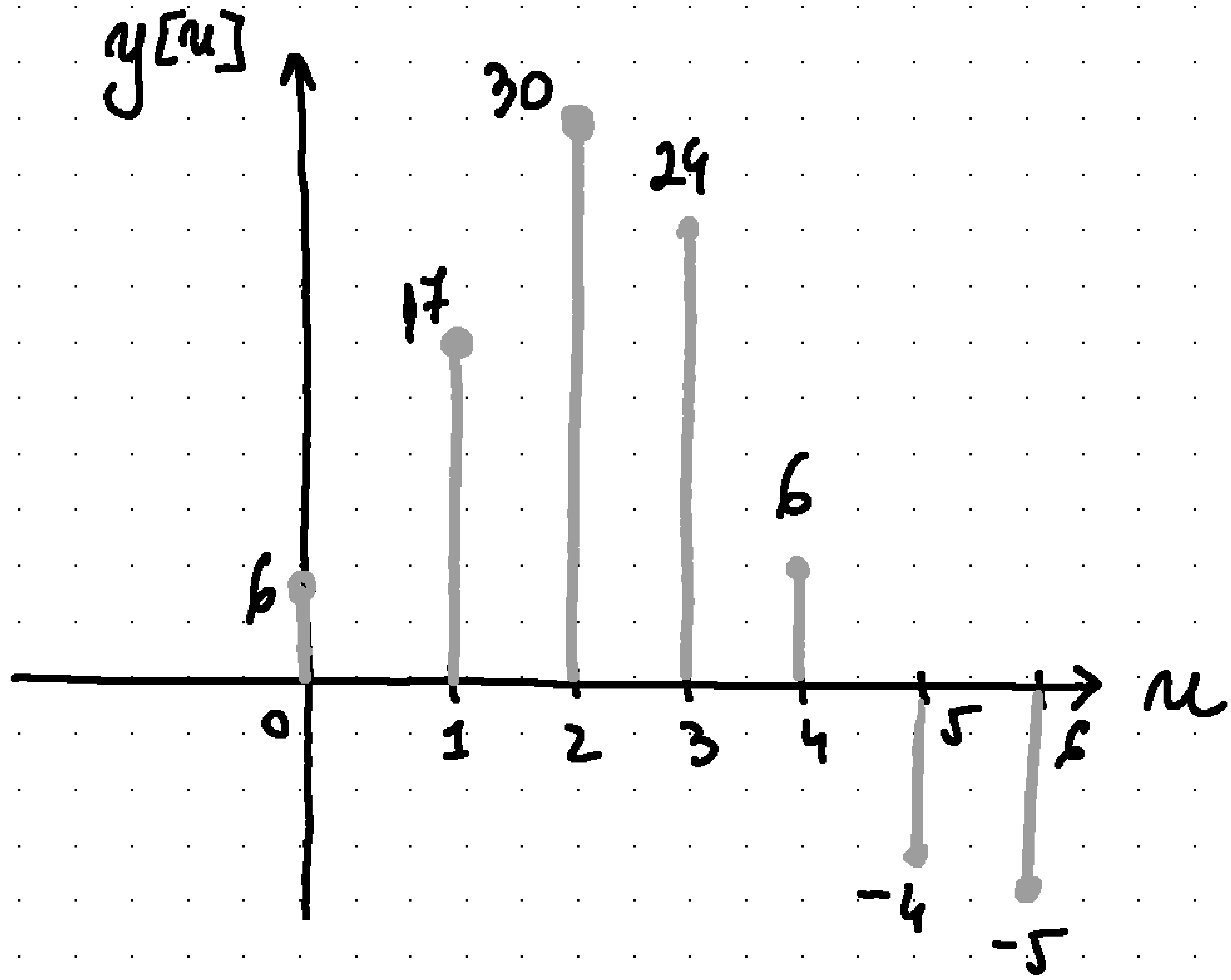
n	0	1	2	3	4	5	6
$h[0]x[n] = 3x[n]$	6	3	-3				
$h[1]x[n-1] = 7x[n-1]$		14	7	-7			
$h[2]x[n-2] = 13x[n-2]$			26	13	-13		
$h[3]x[n-3] = 9x[n-3]$				18	9	-9	
$h[4]x[n-4] = 5x[n-4]$					10	5	-5
$y[n] = \ldots$	6	17	30	24	6	-4	-5

2) $y[n] = 6\delta[0] + 17\delta[n-1] + 30\delta[n-2] +$

$+ 24\delta[n-3] + 6\delta[n-4] - 4\delta[n-5] - 5\delta[n-6]$.

$y[0] = 6$; $y[1] = 17$; $y[2] = 30$;

$y[3] = 24$; $y[4] = 6$; $y[5] = -4$; $y[6] = -5$.



P-5.13 LTI system : linear + time-invariant.

$x_1[n] = u[n] \rightsquigarrow$ output $y_1[n]$

$$y_1[n] = \delta[n] + 2\delta[n-1] - \delta[n-2] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 2, & n = 1 \\ -1, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

time delay

New input: $x_2[n] = \underline{3u[n]} - 6\underline{u[n-2]}$

Apply time-invariance property: (LTI)

$x_1[n] = u[n] \rightsquigarrow y_1[n]$

$x_1[n-2] = u[n-2] \rightsquigarrow \underline{y_1[n-2]}$.

Apply linearity property: (LTI)

$x_1[n] = u[n] \rightsquigarrow y_1[n]$

$3x_1[n] = 3u[n] \rightsquigarrow \underline{3y_1[n]}$.

$$x_2[n] = 3u[n] - 6u[n-2] \rightsquigarrow y_2[n] = 3y_1[n] - 6y_1[n-2]$$

$$y_1[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$$

$$\Rightarrow y_2[n] = 3(\delta[n] + 2\delta[n-1] - \delta[n-2]) -$$
$$- 6(\delta[n-2] + 2\delta[n-3] - \delta[n-4])$$

$$y_2[n] = 3\delta[n] + 6\delta[n-1] - \underline{3\delta[n-2]} - 6\underline{\delta[n-2]} -$$
$$- 12\delta[n-3] + 6[\delta-4]$$

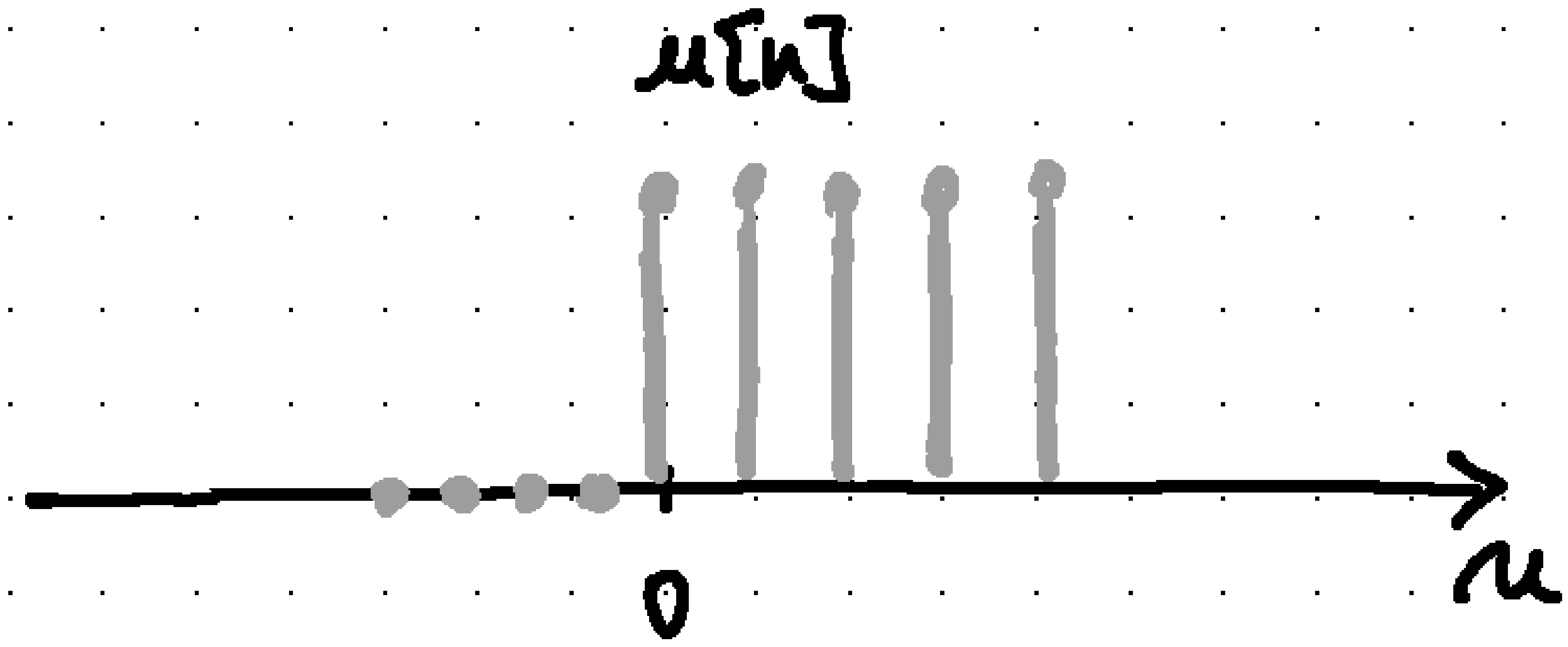
$$\boxed{y_2[n] = 3\delta[n] + 6\delta[n-1] - 9\delta[n-2] - 12\delta[n-3] + 6[\delta-4]}$$

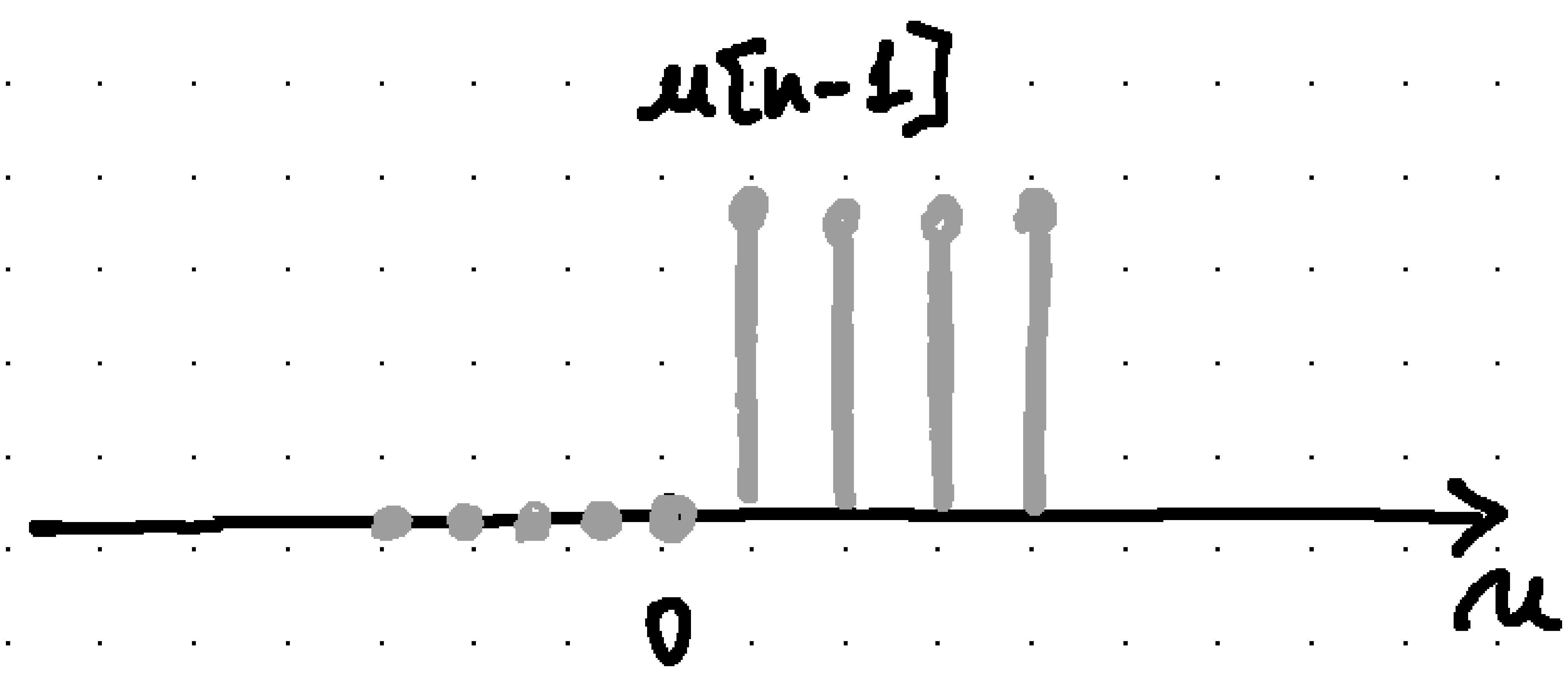
List of values:

n	<0	0	1	2	3	4	>4
$y_2[n]$	0	3	6	-9	-12	6	0

b)

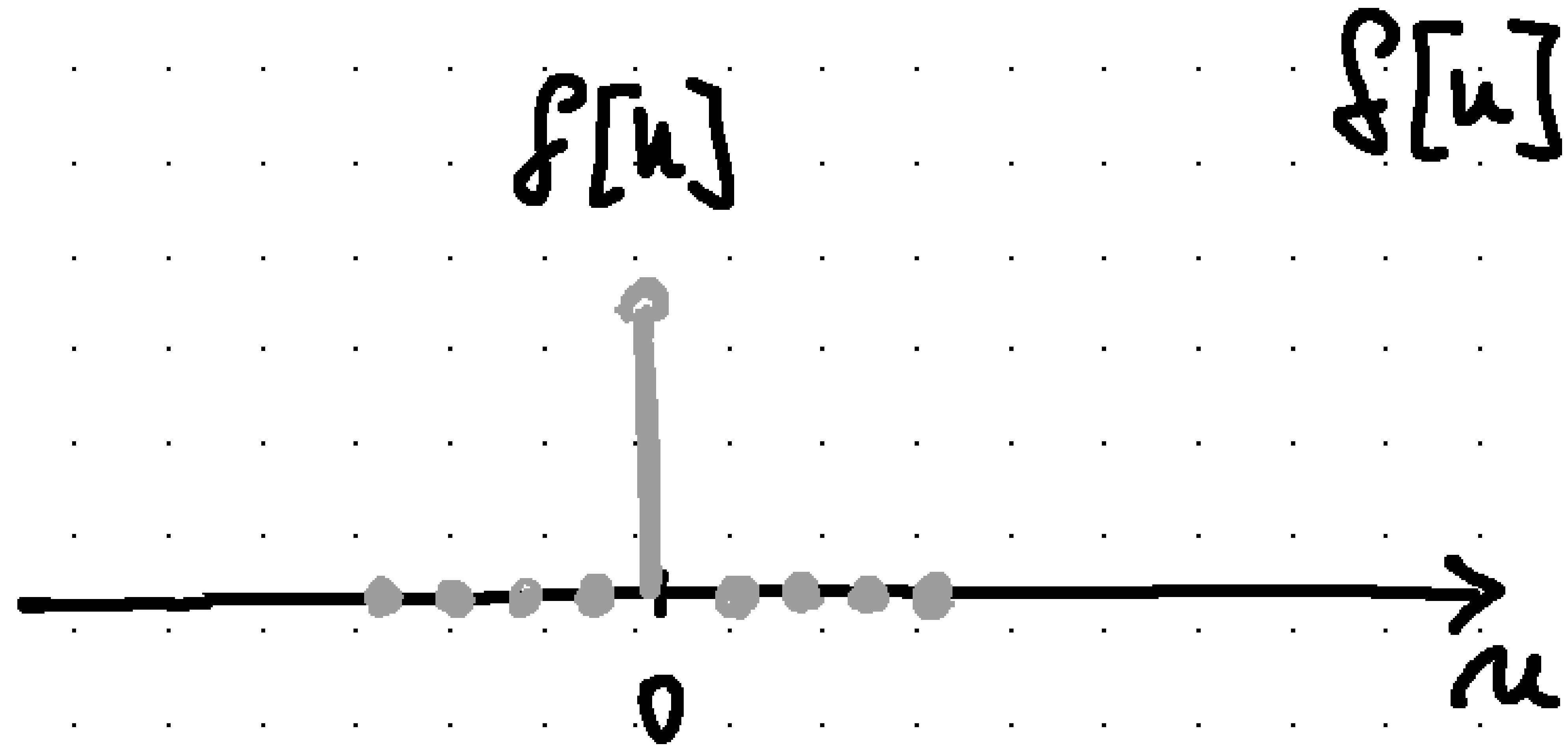
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \Rightarrow u[n-1] = \begin{cases} 1, & (n-1) \geq 0, n \geq 1 \\ 0, & (n-1) < 0, n < 1 \end{cases}$$





subtract : $u[n] - u[n-1]$:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$



$$\delta[n] = A u[n] + B u[n-1]$$

$$\delta[n] = u[n] - u[n-1]$$

$\Rightarrow \boxed{A = 1;\ B = -1}$

c) Use $\delta[n]$ as input to get $h[n]$

$$\delta[n] \longrightarrow \boxed{h[n]} \longrightarrow h[n] \quad \Rightarrow$$

$$\delta[n] = u[n] - u[n-1]$$

$$\Rightarrow u[n] - u[n-1] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{h[n]}$$

$$x_1 = u[n] \rightsquigarrow y_1[n]$$

$$x_1 = u[n-1] \underset{LTI}{\rightsquigarrow} y_1[n-1]$$

$\Rightarrow$ the LTI corresponding output

$$\Rightarrow y[n] = y_1[n] - y_1[n-1] = \delta[n] + 2\delta[n-1] - \delta[n-2] - (\delta[n-1] + 2[n-2] - \delta[n-3])$$

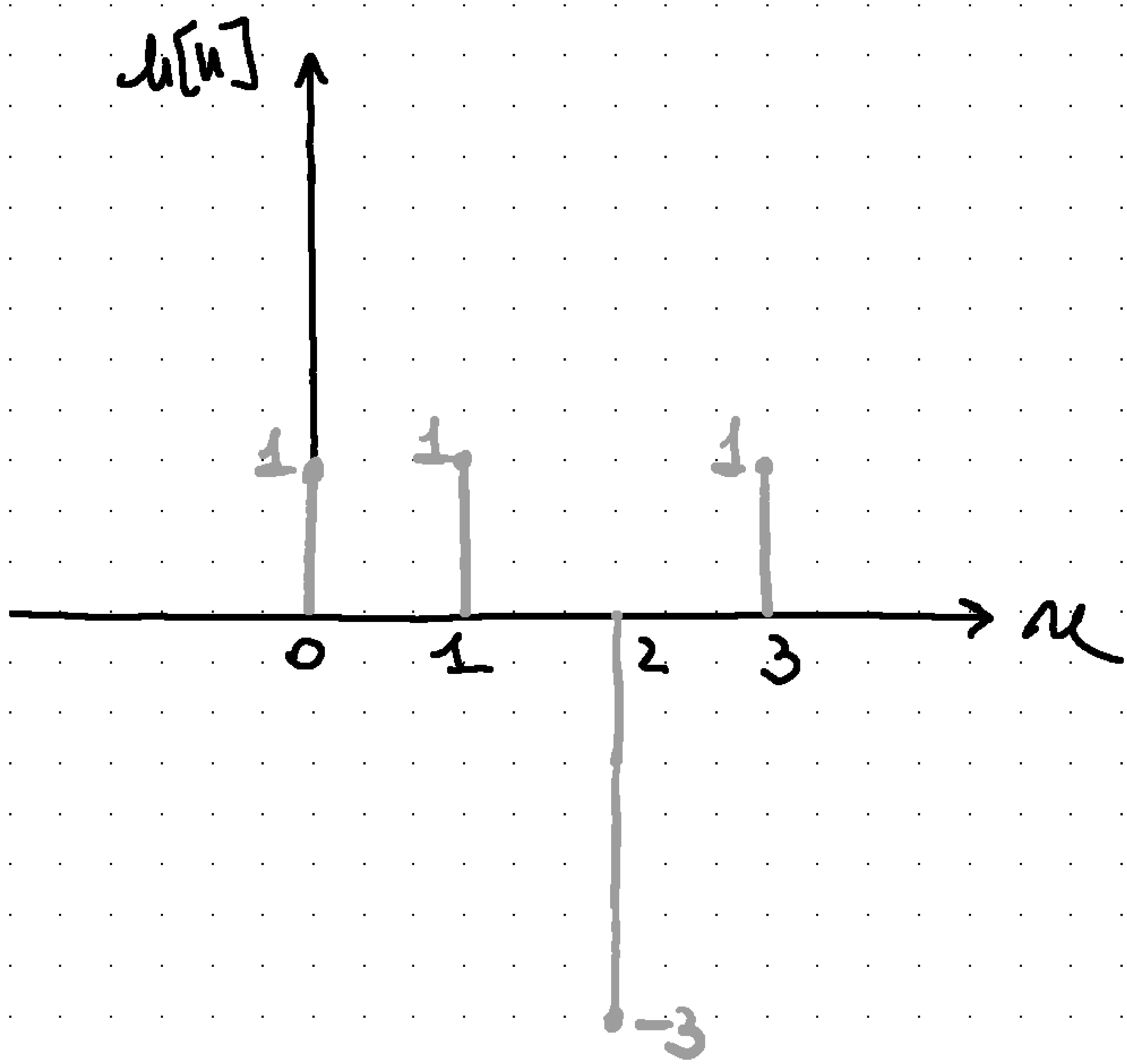
$$y[n] = \delta[n] + 2\delta[n-1] - \delta[n-2] - \delta[n-1] - 2\delta[n-2] + \delta[n-3]$$

$$y[n] = \delta[n] + \delta[n-1] - 3\delta[n-2] + \delta[n-3]$$

$$h[n] = y[n] \quad \Rightarrow$$

$h[n] = \delta[n] + \delta[n-1] - 3\delta[n-2] + \delta[n-3]$

$h[n] = \{1, 1, -3, 1\}$



P- 6.13 :

(a) Frequency response:

$$H(j\hat{\omega}) = \sum_{K=0}^{M} h[K] e^{-j\hat{\omega}K}$$

output : $y[n] = \sum_{K=-\infty}^{\infty} h[K] x[n-K] = h[n] * x[n]$

$S_1 : y_1[n] = x_1[n] + x_1[n-1]$

$\Rightarrow b_K = \{1, 1\}$

$$\Rightarrow H(j\hat{\omega}) = \sum_{K=0}^{1} h[K] e^{-j\hat{\omega}K} = e^0 + e^{-j\hat{\omega}} = 1 + e^{-j\hat{\omega}}$$

$S_2 : y_2 = x_2[n] - x_2[n-2]$

$\Rightarrow b_K = \{1, 0, -1\}$

$$\Rightarrow H(j\hat{\omega}) = \sum_{K=0}^{2} h[K] e^{-j\hat{\omega}K} = e^0 + 0 \cdot e^{-j\hat{\omega}} - e^{-j\hat{\omega}2} =$$

$$= 1 - e^{-j\hat{\omega}2}$$

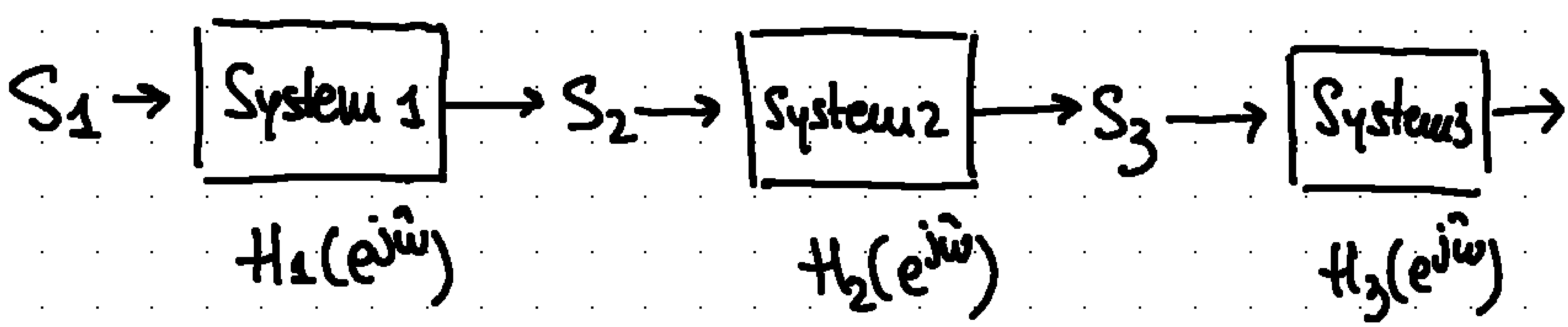
$S_3 : y_3[n] = x_3[n-2] - x_3[n-3]$.

$\Rightarrow b_k = \{0, 0, 1, -1\}$

$$\Rightarrow H(j\hat{\omega}) = \sum_{k=0}^{3} h[k] e^{-j\hat{\omega}k} = 1 \cdot e^{-j\hat{\omega}2} - 1 \cdot e^{-j\hat{\omega}3} =$$

$$= e^{-j\hat{\omega}2} - e^{-j\hat{\omega}3}$$

(b) In the frequency domain, the convolution in the time domain becomes a multiplication.



Equivalent frequency response:

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) \cdot H_2(e^{j\hat{\omega}}) \cdot H_3(e^{j\hat{\omega}})$$

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) \cdot H_2(e^{j\hat{\omega}}) \cdot H_3(e^{j\hat{\omega}})$$

From (a):

$$H_1 = 1 + e^{-j\hat{\omega}}$$
$$H_2 = 1 - e^{-j\hat{\omega}2}$$
$$H_3 = e^{-j\hat{\omega}2} - e^{-j\hat{\omega}3}$$

$\Longrightarrow$

Equivalent overall frequency response:

$$H(e^{j\hat{\omega}}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\hat{\omega}2})(e^{-j\hat{\omega}2} - e^{-j\hat{\omega}3})$$

c) According to the task:

$$x[n] = x_1[n]$$
$$y[n] = y_3[n]$$

$$x[n] = x_1[n] \rightarrow \boxed{S_1} \xrightarrow[x_2[n]]{y_1[n]} \boxed{S_2} \xrightarrow[x_3[n]]{y_2[n]} \boxed{S_3} \longrightarrow y_3[n] = y[n]$$

$$x_1[n] = x[n] \Rightarrow y_1[n] = x[n] + x[n-1]$$

$$y_1[n] = x_2[n]$$

$$y_2[n] = y_1[n] - y_1[n-2] \Rightarrow$$

$$y_2[n] = x[n] + x[n-1] - x[n-2] - x[n-3]$$

$$y_2[n] = x_3[n].$$

$$y[n] = y_3[n] = y_2[n-2] - y_2[n-3]$$

$$y[n] = x[n-2] + x[n-3] - x[n-4] - x[n-5] -$$
$$-\left(x[n-3] + x[n-4] - x[n-5] - x[n-6]\right)$$

$$y[n] = x[n-2] + \cancel{x[n-3]} - x[n-4] - \cancel{x[n-5]} -$$
$$- \cancel{x[n-3]} - x[n-4] + \cancel{x[n-5]} + x[n-6]$$

$$\boxed{y[n] = x[n-2] - 2x[n-4] + x[n-6]}$$

P - 6.15

c. $x(t) \rightarrow \boxed{\text{C-to-D}} \xrightarrow{x[n]} \boxed{\text{LTI}} \xrightarrow{y[n]} \boxed{\text{C-to-D}} \rightarrow y(t)$

$H(e^{j\hat{\omega}}), h[n]$

$f_s = 1/T_s$ $\qquad\qquad$ $f_s = 1/T_s$

$f_s = 1000$ samples/sec.

$$x(t) = 10 + 8\cos(200\pi t) + 6\cos(500\pi t + \pi/4)$$

$$x[n] = 10 + 8\cos\left(\frac{200\pi}{1000}n\right) + 6\cos\left(\frac{500\pi}{1000}n + \pi/4\right)$$

$$\boxed{x[n] = 10 + 8\cos(0.2\pi n) + 6\cos(0.5\pi n + \pi/4)}$$

↪ The input signal is a Sinusoidal Signal.

The impulse response : $h[n] = 1/4 \sum_{k=0}^{3} \delta[n-k]$

$$h[n] = \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2] + \frac{1}{4}\delta[n-3].$$

FIR = LTI system

- $A\cos(\hat{\omega}n+\phi) \xrightarrow{FIR} A|He^{j\hat{\omega}}|\cos[\hat{\omega}n+\phi+\angle H(e^{j\hat{\omega}})$
- $H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$

$$\boxed{y[n] = 10H(e^{j0}) + 8|He^{j0.2\pi}|\cos(0.2\pi n + \angle H(e^{j0.2\pi})) + 6|He^{j0.5\pi}|\cos(0.5\pi n + \pi/4 + \angle H(e^{j0.5\pi}))}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{3} h[k]e^{-j\hat{\omega}k} = \frac{1}{4} + \frac{1}{4}e^{-j\hat{\omega}} + \frac{1}{4}e^{-j\hat{\omega}2} + \frac{1}{4}e^{-j\hat{\omega}3}$$

$$= \frac{1}{4}\left(1 + e^{-j\hat{\omega}} + e^{-j\hat{\omega}2} + e^{-j\hat{\omega}3}\right) =$$

$$= \frac{1}{4}e^{-j\frac{3}{2}\hat{\omega}}\left(e^{j3/2\hat{\omega}} + e^{j1/2\hat{\omega}} + e^{j-1/2\hat{\omega}} + e^{-j3/2\hat{\omega}}\right) \overset{\text{Euler Inverse}}{=} \text{Theorem}$$

$$= 1/4\, e^{-j3/2\hat{\omega}}\left(2\cos(3/2\,\hat{\omega}) + 2\cos(1/2\,\hat{\omega})\right) =$$

$$= 1/2\, e^{-j3/2\hat{\omega}}\left(\cos(3/2\,\hat{\omega}) + \cos(1/2\,\hat{\omega})\right)$$

$$H(e^{j\hat{\omega}}) = 1/2\, e^{-j3/2\hat{\omega}} \left(\cos(3/2\,\hat{\omega}) + \cos(1/2\,\hat{\omega}) \right)$$

$$= 2\cos\frac{3/2\,\hat{\omega} + 1/2\,\hat{\omega}}{2} \cos\frac{3/2\,\hat{\omega} - 1/2\,\hat{\omega}}{2} =$$

$$= 2\cos\hat{\omega}\cos\hat{\omega}/2 .$$

$$\boxed{H(e^{j\hat{\omega}}) = e^{-j3/2\hat{\omega}}\cos\hat{\omega}\cos\hat{\omega}/2}$$ // frequency response formula

$$H(e^{j0}) = 1 \cdot \cos 0 \cdot \cos 0 = 1.$$

$$H(e^{j0.2\pi}) = e^{-j3/2 \cdot \frac{2}{10}\pi} \cdot \cos(0.2\pi)\cos(0.1\pi) = 0.769\, e^{-0.3\pi}$$

$\hookrightarrow$ $|H(e^{j0.2\pi})| = 0.769$; $\angle H(e^{j0.2\pi}) = -0.3\pi$

$$H(e^{j0.5\pi}) = e^{-j3/2 \cdot \frac{5}{10}\pi} \underbrace{\cos(0.5\pi)}_{=0} \cos(0.25\pi) = 0$$

$\Rightarrow y[n] = 10 \cdot 1 + 8 \cdot 0.769 (\cos(0.2\pi n - 0.3\pi)) + 0$

$$\boxed{y[n] = 10 + 6.152 \cos(0.2\pi n - 0.3\pi)}$$

Now we go to the time domain.

$$y(t) = y[n]\Big|_{\substack{n = t/T_s \\ T_s = 1/f_s}} \Rightarrow y(t) = y[n]\Big|_{n = f_s t}$$

$$y(t) = 10 + 6.152 \cos(0.2\pi f_s t - 0.3\pi) \Rightarrow$$

$f_s = 1000$ samples/sec

Final answer:

$$\boxed{y(t) = 10 + 6.152 \cos(200\pi t - 0.3\pi)}$$