Section 1.1

j

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1 Problem 1

1.1 a)

$f : A \to B$
$A_0 \subset A$
$B_0 \subset B$
$A_0 \subset f^{-1}(f(A_0))$
$a \in A_0$
$f(a) = b \in B_0$
$f^{-1}(B_0) = \{x|f(x) \in B_0\}$ $\to$DEFINITION OF INVERSE
$f^{-1}(f(A_0)) = \{x|f(x) \in f(A_0)\}$
$a \in f^{-1}(f(A_0))$ We know this is true because we have previously defined $f(x) \in f(A_0)$, and by following the definition of the inverse we find that the general "x" can also be set equivalent to $f^{-1}(f(A_0))$, or more specifically to this problem, "a" can be set equivalent.
Because $a \in A_0$, and a is the statement above, $A_0 \subset f^{-1}(f(A_0))$

$f^{-1}(f(A_0) \subset A_0$ if f is injective.
$a \in f^{-1}(f(A_0))$
$a \in \{x|f(x) \in f(A_0)\}$
$f(a) \in f(A_0) \leftrightarrow f(a) = f(x)$ for $x \in A_0$
$a = x$
Injective means the function is one-to-one, and since $f(a) = f(b)$, $a = b$.
$a \in A_0$
Because $a \in A_0$, and $a \in f^{-1}(f(A_0))$, $A_0 \subset f^{-1}(f(A_0))$

1.2 b)

$f : A \to B$
$A_0 \subset A$
$B_0 \subset B$
$f^{-1}(f(B_0)) \subset B_0$

$a \in A_0$
$b \in B_0$
$f(a) = b \in B_0$
$f^{-1}(B_0) = \{x|f(x) \in B_0\}$ DEFINITION OF INVERSE
$f^{-1}(f(B_0)) = \{x|f(x) \in f(B_0)\}$
$f(b) \in f(B_0)$
$b \in f^{-1}(f(B_0))$ For clarification on why you can do this, refer to my reasoning for Problem 1a.
Because $b \in B_0$ and the statement above applies to b, we can conclude that $B_0 \in f^{-1}(f(B_0))$

2 Problem 2

2.1 a)

$f : A \to B$
$B_0 \subset B_1 \to f^{-1}(B_0) \subset f^{-1}(B_1)$
$B_0 \subset B_1$
$b \in B_0$
$b \in B_1$
$a \in A_0$
$a \in A_1$
$f(a) \in b$
$f(a) \in B_0$
$f(a) \in B_1$
$a \in f^{-1}(B_0)$ (We proved this in Problem 1)
$a \in f^{-1}(B_1)$
Restating: $B_0 \subset B_1$
$f^{-1}(B_0) \subset f^{-1}(B_1)$

2.2 b)

$f : A \to B$
$A_0 \in A$ and $A_1 \in A$
$B_0 \in B$ and $B_1 \in B$
$f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$
Definition $\to f^{-1}(B_0 \cup B_1) = \{a|f(a) \in (B_0 \cup B_1)\}$
$\{a|(f(a) \in B_0)$ or $(f(a) \in B_1)\}$
$\{a|f(a) \in B_0\} \cup \{a|f(a) \in B_1\}$
$f^{-1}(B_0) \cup f^{-1}(B_1)$ (Reversible!)

2.3 c)

$f : A \to B$
$A_0 \in A$ and $A_1 \in A$

$B_0 \in B$ and $B_1 \in B$
$f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$
Definition $\rightarrow f^{-1}(B_0 \cap B_1) = \{a|f(a) \in (B_0 \cap B_1)\}$
$\{a|(f(a) \in B_0)$ and $(f(a) \in B_1)\}$
$\{a|f(a) \in B_0\} \cap \{a|f(a) \in B_1\}$
$f^{-1}(B_0) \cap f^{-1}(B_1)$ (Reversible!)

2.4 d)

$f : A \rightarrow B$
$A_0 \in A$ and $A_1 \in A$
$B_0 \in B$ and $B_1 \in B$
$f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$
Definition $\rightarrow f^{-1}(B_0 - B_1) = \{a|f(a) \in (B_0 - B_1)\}$
$\{a|(f(a) \in B_0)$ and not $(f(a) \in B_1)\}$
$\{a|f(a) \in B_0\} - \{a|f(a) \in B_1\}$
$f^{-1}(B_0) - f^{-1}(B_1)$ (Reversible!)

2.5 e)

$f : A \rightarrow B$
$A_0 \subset A_1 \rightarrow f(A_0) \subset f(A_1)$
$A_0 \subset A_1$
$a \in A_0$
$a \in A_1$
$b_0 \in f(A_0)$
$b_0 \in f(A_1)$
$B_0 \in B$
$B_1 \in B$
$f(A_0) \in B$
$f(A_1) \in B$
$b_0 \in f(A_0)$
$b_0 \in f(A_1)$
$f(A_0) \subset f(A_1)$

2.6 f)

$f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$
$f : A \rightarrow B$
$A_0 \in A$ and $A_1 \in A$
$B_0 \in B$ and $B_1 \in B$
$f(A_0) \subset B$ $f(A_1) \subset B$ Definition $\rightarrow f^{-1}(B_0 \cup B_1) = \{a|f(a) \in (B_0 \cup B_1)\}$
$\{a|(f(a) \in B_0)$ or $(f(a) \in B_1)\}$
$\{a|f(a) \in B_0\} \cup \{a|f(a) \in B_1\}$
$f^{-1}(B_0) \cup f^{-1}(B_1)$

2.7 g)

$f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$
$yinf(A_0) \cap f(A_1)$
$x \in (A_0 \cap A_1)$
$f(x) = y$
$x \in A_0 \cap A_1$
$x \in A_0 andx \in A_1$
$f(x) \in f(A_0)$ and $f(x) \in f(A_1)$
$y = f(A_0)$ and $y \in f(A_1)$
$f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$

Reversible for Injectivity::
$y \in f(A_0) \cap f(A_1)$
$a_0 \in A_0$ and $a_1 \in A_1$
$f(a_0) = f(a_1) = y$
Injectivity: $f(a_0) = f(a_1) \to a_0 = a_1$
$\therefore a_0 \in A_1$
$\therefore f(a_0) \in f(A_0)$ and $f(a_0) \in f(A_1)$
$f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$
$f(A_0) \cap f(A_1) = f(A_0 \cap A_1)$

2.8 h)

$y \in f(A_0 - A_1)$
$\exists x | x \in A_0 - A_1$
$f(x) = y$
$x \in A_0$
$x \notin A_1$
$\therefore f(x) \in f(A_0)$
$\therefore f(x) \notin f(A_1)$
$f(x) \in f(A_0) - f(A_1)$
$f(A_0 - A_1) \subset f(A_0) - f(A_1)$
Reversible for Injectivity:
$y \in f(A_0 - A_1)$
$\exists x | f(x) \in f(A_0) - f(A_1)$
$f(x) = y$
Since f is injective $\to$ x-value is different for each instance.
$x \notin A_1 \to f(x) \in f(A_0 - A_1)$
$f(A_0 - A_1) = f(A_0) - f(A_1)$

3 Problem 3

3.1 b)

$f^{-1}(\bigcup_{B\in\mathcal{B}} B) = \{x|f(x) \in \bigcup_{B\in\mathcal{B}} B\}$
$f^{-1}(\bigcup_{B\in\mathcal{B}} B) = \{x|f(x) \in B_0\}...\{x|f(x) \in B_n\}$
$f^{-1}(\bigcup_{B\in\mathcal{B}} B) = \bigcup_{B\in\mathcal{B}}$

3.2 c)

$f^{-1}(\bigcap_{B\in\mathcal{B}} B) = \{x|f(x) \in \bigcap_{B\in\mathcal{B}} B\}$
$f^{-1}(\bigcap_{B\in\mathcal{B}} B) = \{x|f(x) \in B_0\}...\{x|f(x) \in B_n\}$
$f^{-1}(\bigcap_{B\in\mathcal{B}} B) = \bigcap_{B\in\mathcal{B}}$

3.3 f)

let $x \in f(\bigcup_{A\in\mathcal{A}} A)$
$\exists a|a \in \bigcup_{A\in\mathcal{A}} A$ and $f(a) = x$
$a \in A_0$
$f(a) \in f(A_0)$
$f(a) \in \bigcup_{A\in\mathcal{A}} f(A)$
This is also reversible for injectivity, so: $f(\bigcup_{A\in\mathcal{A}} A) = \bigcup_{A\in\mathcal{A}} f(A)$

3.4 g)

let $x \in f(\bigcap_{A\in\mathcal{A}} A)$
$\exists a|a \in \bigcap_{A\in\mathcal{A}} A$ and $f(a) = x$
$a \in A_0$
$f(a) \in f(A_0)$
$f(a) \in \bigcap_{A\in\mathcal{A}} f(A)$
This is also reversible for injectivity, so: $f(\bigcap_{A\in\mathcal{A}} A) = \bigcap_{A\in\mathcal{A}} f(A)$

4 Problem 4

4.1 a)

$f : A \to B$
$g : B \to C$
$C_0 \subset C$
$(g \circ f)^{-1}(C_0) = f^{-1}(g^{-1}(C_0))$
$a \in (g \circ f)^{-1}(C_0)$
$(g \circ f)(a) \in C_0$

$g(f(a)) \in C_0$
$f(a) \in g^{-1}(C0)$
$a \in f^{-1}(g^{-1}(C_0)$

4.2 b)

$f : A \to B$
$g : B \to C$
$g(f(a))$
$a \in A$
$a_1 \in A$
$g(f(a)) = g(b) \in C$
$g(f(a_1)) = g(b) \in C$

4.3 c)

f has to be injective in each case, but that doesn' necessarily mean that g has to be injective at the same time.

4.4 d)

There is a case where $f(x)$ and $g(x)$ are surjective but at the same time $g(f(x))$ is not.
If $\exists c | c \in C$ and such that $g(f(A))$ does not map from any group of elments into it since $f(x)$ has to be surjective.
$f(A) \to (maps)B$
Since $g(x)$ is surjective, $g(f(A)) \to (maps)$ to each C
This means that $g(f(x))$ has to be surjective by definition of surjectivity.

4.5 e)

Although g is surjective, f does not necessarily have to be surjective as a result.

4.6 f)

If f and g are both bijective, then $g \circ f$ will also resultingly be bijective.
Because $g \circ f$ us bijective, g must be surjective and f must be injective in order to include both injectivity and surjectivity to meet the definition of bijectivity.

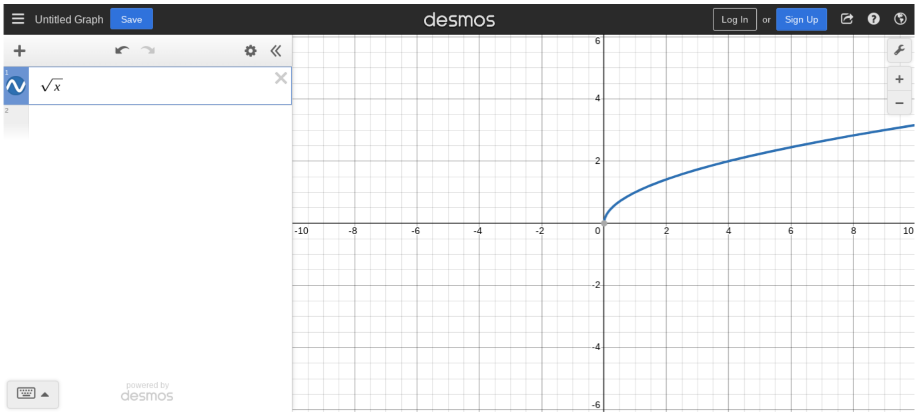
5 Problem 5

5.1 a)

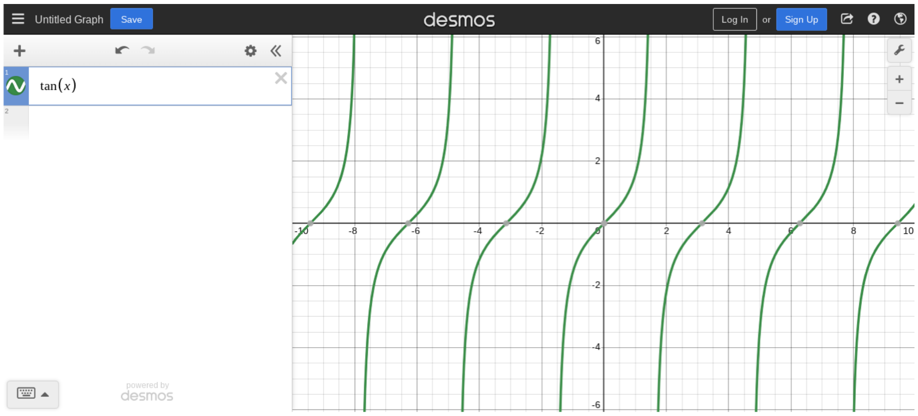
$g(f(a)) = a$
Injective $\to f(a) = f(b)$

$a = b$
if $f(a_1) = f(a_2) \rightarrow a_1 = a_2$
$g(f(a_1)) = g(f(a_2))$
$a_1 = a_2$
$f(h(b_1)) = b_1$
$\exists a | f(a) = b_1$
$a \in A$
if $b \in B$
$\exists x | f(x) = b$
$x \in A$

5.2 b)



5.3 c)



5.4 d)

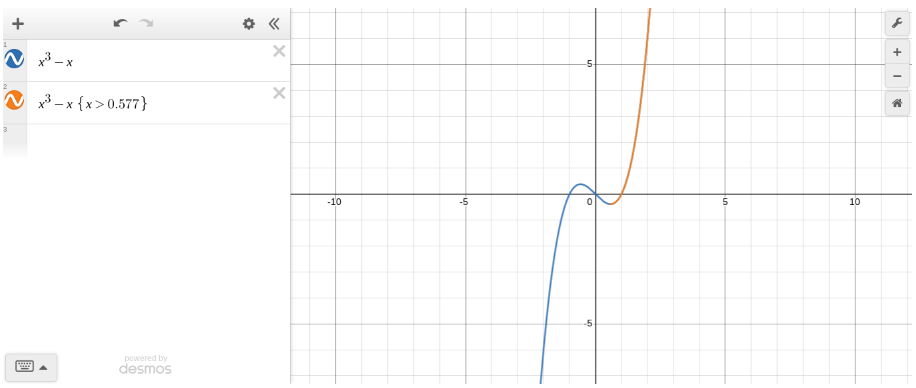
Since the domains of two functions don't necessarily need to be the same, you can get different left inverses by having functions differ in the set of B that is simultaneously not in the range of f.
However, you can only apply this when the function is not surjective.

As for more than one right inverses, the reason is the same but you just need some $b \in B$ such that $f^{-1}(y) >1$.

5.5 e)

Earlier we proved that a left inverse indicates injectivity and a right inverse indicates surjectivity.
We also already proved that if you have a function that is injective and surjective is also bijective.
Therefore, if a function has a left inverse it is injective, and if a function has a right inverse it's surjective, so the function f is injective.
Also, because g and h are respective inverses, $f-1$ has to be equal to it's own inverses.

6 Problem 6



In this graph, we are keeping the function the same, but restricting it at a minimum such that the function conforms to the rules of surjectivity and injectivity (definition of bijectivity).