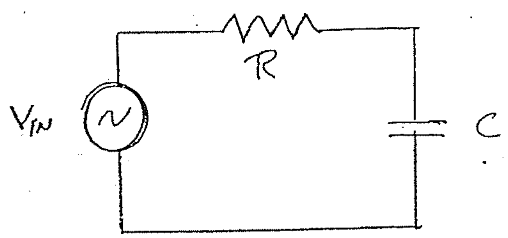
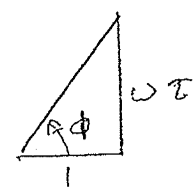
E-84 Problem Set #6 Key

Problems from Chapter 5

1. a. Show that for a sinusoidal input voltage v_{IN} = **A**$\sin\omega t$ the **steady state** solution for the voltage across the capacitor in the R-C circuit shown below is

$$v_{ss}(t) = \frac{A\sin(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}} = \frac{A\sin\omega(t - t_{lag})}{\sqrt{1+\omega^2\tau^2}} \quad \text{where} \quad \phi = \tan^{-1}\omega\tau \ \text{and} \ \tau = RC$$





Thus the voltage on the capacitor oscillates at the same frequency as the source, but with a smaller amplitude and a time lag, $t_{lag} = \phi/\omega$.

An easy way to get the particular (steady state) solution is to use circuit theory with phasors, so that

$\underline{V}_C = \underline{V}Z_C/(Z_C + R) = \underline{V}/(1 + j\omega RC) = V/(1 + \omega^2\tau^2)^{1/2}\underline{/-\phi}$
where τ = RC and $\phi = \tan^{-1}\omega RC$

Hence for v(t) = A$\sin\omega t$,

$$v_{ss}(t) = \frac{A\sin(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}} = \frac{A\sin\omega(t - t_{lag})}{\sqrt{1+\omega^2\tau^2}} \quad \text{where} \quad \phi = \tan^{-1}\omega\tau \ \text{and} \ \tau = RC\text{, and}$$
$t_{lag} = \phi/\omega = (1/\omega)\cdot\tan^{-1}\omega RC$

The same result is obtained by solving for the particular solution to the governing equation for this system:

$iR + v_c = v_{IN}$ or $RC(dv_c dt) + v_c = A\sin\omega t$

b. Show that for low frequency oscillations, when $\omega\tau \ll 1$, the time lag $t_{lag} \approx \tau$, while for high frequency oscillations when $\omega\tau \gg 1$, the time lag $t_{lag} \approx T/4$, where $T = 1/f = 2\pi/\omega$ is the period of the oscillations.

We are given

$$v_{ss}(t) = \frac{A\sin(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}} = \frac{A\sin\omega(t - t_{lag})}{\sqrt{1+\omega^2\tau^2}} \quad \textbf{where} \ t_{lag} = \phi/\omega \ \textbf{and} \ \phi = \tan^{-1}(\omega\tau)$$

Note that
$\tan^{-1}x \approx x$ for $x << 1$, and $\tan^{-1}x \approx \pi/2$ for $x >> 1$. Hence for $\omega\tau << 1$, $\phi \approx \omega\tau$ so that the time lag $t_{lag} \approx \tau$.
For $\omega\tau >> 1$, $\phi \approx \pi/2$ so that the time lag $t_{lag} \approx \pi/2\omega = T/4$, where $T = 1/f = 2\pi/\omega$ is the period of the oscillations.

c. Show that the complete solution, including the homogeneous (or transient) term is

$v_C(t) = v_{ss}(t) + Ce^{-t/\tau}$, where, for $v_C(0) = 0$, $C = (A\sin\phi)/(1 + \omega^2\tau^2)^{1/2}$

As shown in the notes, the homogenous solution for the R-C circuit is $v_{CH} = Ce^{-t/\tau}$. To determine C we use the initial condition for the entire solution, $v_C(0) = 0$, to find:

$$0 = \frac{-A\sin(\phi)}{\sqrt{1+\omega^2\tau^2}} + C \text{ so that } C = \frac{A\sin(\phi)}{\sqrt{1+\omega^2\tau^2}}$$

3. a. Prove that for an underdamped series R-L-C circuit with $V(t) = \sin\omega_o t$ and $v_c = dv_c/dt = 0$ at $t = 0$ the solution for the capacitor voltage is:

$$v_C(t) = Q\left[e^{-\alpha t}\left(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right) - \cos\omega_0 t\right], \text{ where } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_0}{2\alpha}$$

Thus as expected from the results in Chapter 4 for a band pass filter, the steady state amplitude of these oscillations is Q (the quality factor for the filter). Note that for a high Q system, the damping is minimal, so that $\alpha << \omega_o$ and $\beta \approx \omega_0$, so that

$$v_C(t) \approx -Q(1 - e^{-\alpha t})\cos\omega_0 t,$$

Once again, we use the fact that the particular (steady state) solution is the solution obtained from circuit theory, where
$\underline{V}_C = \underline{V}/(1 + j\omega RC - \omega^2 LC)$
so that at resonance ($\omega_0 = (LC)^{-1/2}$)
$\underline{V}_C = \underline{V}/(j\omega_0 RC) = -j\underline{V}Q$ with $Q = (L/C)^{1/2}\bullet(1/R)$
Hence for $v(t) = \sin\omega_0 t$,
$V_{CP}(t) = Q\sin(\omega_0 t - .5\pi) = -Q\cos\omega_0 t$.

Next, as shown in the notes, the solution to the homogenous equation (Eq. 5.25) is
$V_{CH}(t) = e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$, so that
(1) $V_C(t) = = e^{-\alpha t}(A\cos\beta t + B\sin\beta t) - Q\cos\omega_0 t$.

To find A and B we use the conditions that at $t = 0$, $v_C(0) = 0$ and $dv_C/dt = 0$. Thus we have
$V_C(0) = A - Q = 0$, so that $A = Q$.
Also
$(dv_C/dt)_{t=0} = -\alpha A + \beta B = 0$, so that $B = \alpha A/\beta = \alpha Q/\beta$
Substituting these values into (1) we find

$$v_C(t) = Q[e^{-\alpha t}(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t) - \cos\omega_0 t]$$

For a high Q system, $(\alpha/\beta) << 1$ and $\beta \approx \omega_0$, so that $v_C(t)$ reduces to $v_C(t) = Q\cos\omega_0 t[e^{-\alpha t} - 1]$, the result given in the problem statement.

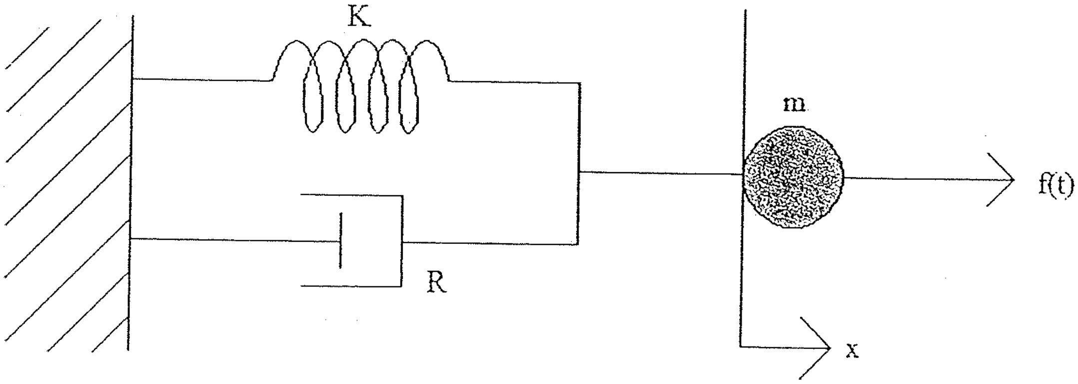
b. Show that for a high Q system it takes about Q cycles before the full amplitude of the steady state solution is achieved. Hint: show that after Q cycles, $\alpha t = \pi$, so that $e^{-\alpha t}$ becomes negligible ($e^{-\pi} = .0432$).

As shown in the notes, $\alpha = \omega_0/2Q$. Hence $e^{-\alpha t} = \exp{-\omega_0 t/2Q}$. Thus the exponential falls to $e^{-\pi}$, when $\pi = \omega_0 t/2Q$, or $t = 2\pi Q/\omega_0 = Q/f_0 = QT$, where T is the period of the oscillations.

6. The governing equation for the mechanical system with a mass, spring and dashpot (damper) shown below has the form

$$m\ddot{x} + R\dot{x} + \kappa x = f(t)$$

where m is the mass (kg), R is the viscous resistance (N-s/m), κ is the spring constant (N/m), and f is the force (N).



When a force of 1 N is applied, the displacement is found undergo about 20 oscillations in 20 s before "reaching" an equilibrium displacement of 1 mm. (By reaching equilibrium we mean the amplitude of the oscillations drops to about $e^{-\pi}$ compared to the amplitude of the first oscillation.) Show **that m ≈ 25.3 kg, R ≈ 7.9 N-s/m, and κ ≈ 10^3 N/m.**
Hint: A system that takes 20 oscillations to damp out has $\beta >> \alpha$, so that $\omega_0 \approx \beta$.

From the data given,
$x_{Eq} = 10^{-3}$ m; Q = 20; and $T = 1/f = 2\pi/\omega_0 = 1$ s, so that $\omega_0 = 2\pi$.

To find m, R, and κ, note first that
At equlibrium, **$\kappa = 1/x_{Eq} = 10^3$ N/m**

Dividing the governing equation by m, and comparing the homogenous equation (when f = 0) with Eq. 5.22, we see that **$\kappa/m = \omega_0^2$** so that **$m = \kappa/\omega_0^2 = 10^3/4\pi^2 = 25.3$ kg.**

From the same equation we also see that
$R/m = \omega_0/Q$, so that **$R = m\omega_0/Q = 25.3 \cdot 2\pi/20 = 7.9$ N-s/m**