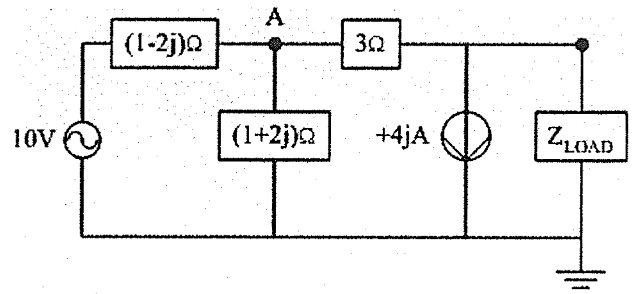
E-84 Problem Set #5 Key

Ch 3 #11.

11. Show that for the circuit shown below: Z_{th} = 5.5 Ω and $\underline{V}_{th} = 13\underline{/-67.4^\circ}$ V



Hint: first find $\underline{V}_A$ using node analysis with the load removed.

To calculate Z_{Th} replace the current source with an open circuit and the voltage source with a short circuit. That makes the impedance from the top of the load to ground 3 Ω plus the other two impedance going in parallel to ground. Hence

Z_{Th} = 3 + (1 + 2j)(1 – 2j)/(1 + 2j + 1 – 2j) = 3 + 5/2 = <u>5.5 Ω</u>

To calculate V_{Th} we remove the load and calculate V_A using node analysis at A: We have

$(10 - V_A)/(1 - 2j) = V_a/(1 + 2j) + 4j$ so that collecting terms,

$$\frac{10}{1-2j} - 4j = V_A\left(\frac{1}{1+2j} + \frac{1}{1-2j}\right) = \frac{2V_A}{5}$$

, so that (multiplying the first term by (1 + 2j)/(1 + 2j) we have;

$2((1 + 2j) - 4j = 2V_A/5$ from which we easily find $\underline{V_A = 5\ V}$.

But with Z_{LOAD} removed, $V_{Th} = V_A - 12j$ (where the second term is the voltage drop across the 3 Ω resistor. Hence

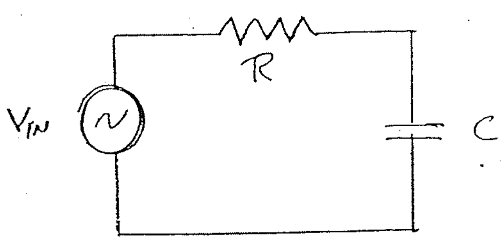
$\underline{V_{Th} = 5 - 12j}$ (which also equals $13\underline{/\tan^{-1}(-12/5)} = 13\underline{/-67.4^\circ}$

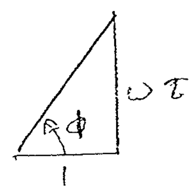
E-84 Problem Set #5 Key-continued

Chapter 5

1. **a.** Show that for a sinusoidal input voltage $v_{IN} = A\sin\omega t$ the **steady state** solution for the voltage across the capacitor in the R-C circuit shown below is

$$v_{ss}(t) = \frac{A\sin(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}} \quad \text{where } \phi = \tan^{-1}\omega\tau \text{ and } \tau = RC$$





Thus the voltage on the capacitor oscillates at the same frequency as the source, but with a smaller amplitude and a time lag, $t_{lag} = \phi/\omega$.
b. Show that for low frequency oscillations, when $\omega\tau << 1$, the time lag $t_{lag} \approx \tau$, while for high frequency oscillations when $\omega\tau >> 1$, the time lag $t_{lag} \approx T/4$, where $T = 1/f = 2\pi/\omega$ is the period of the oscillations.
c. Show that the complete solution, including the homogeneous (or transient) term is

$v_C(t) = v_{ss}(t) + Ce^{-t/\tau}$, where, for $v_C(0) = 0$, $C = (A\sin\phi)/(1+\omega^2\tau^2)^{1/2}$

a. The governing equation for this system is:

$$V_{IN} = iR + v_C = RC\frac{dv_C}{dt} + v_C = \tau\dot{v}_C + v_C$$

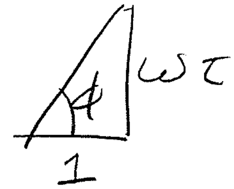
If we try the particular solution

$$v_{ss}(t) = \frac{A\sin(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}} \qquad (\tau = RC)$$

Then

$$\dot{v}_{ss} = \frac{A\omega\cos(\omega t - \phi)}{\sqrt{1+\omega^2\tau^2}}$$

However for $\phi = \tan^{-1}\omega\tau$



$$\therefore \frac{1}{\sqrt{1+\omega^2\tau^2}} = \cos\phi$$

$$\frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}} = \sin\phi$$

Hence

$$v_{ss} + \tau\dot{v}_{ss} = A\left[\cos\phi\sin(\omega t-\phi) + \sin\phi\cos(\omega t-\phi)\right]$$

$$= \sin(\omega t-\phi+\phi) = \sin\omega t\,!$$

(using $\sin a\cos b + \cos a\sin b = \sin(a+b)$)

$$\therefore v_{ss} + \tau\dot{v}_{ss} = \underline{A\sin\omega t = v_{IN}}$$

b. $v_{ss} = \dfrac{A\sin[\omega(t-t_{lag})]}{\sqrt{1+\omega^2\tau^2}}$ where $\underline{t_{lag} = \dfrac{\phi}{\omega}}$

$$\therefore t_{lag} = \frac{\tan^{-1}(\omega\tau)}{\omega}$$

(i) for $\omega\tau \ll 1$, $\tan^{-1}\omega\tau \cong \omega\tau$

$$\therefore t_{lag} \cong \frac{\omega\tau}{\omega} = \tau$$

(ii) for $\omega\tau \gg 1$, $\tan^{-1}(\omega\tau) \cong \frac{\pi}{2}$

$\therefore$ $t_{lag} \cong \frac{\pi}{2\omega}$ But $\omega = 2\pi f = \frac{2\pi}{T}$

$\therefore$ $t_{lag} \cong \frac{\pi T}{2 \cdot 2\pi} = \underline{\underline{\frac{T}{4}}}$

c) As shown in notes $v_c(t) = v_{ss}(t) + v_H(t)$ where

$$\tau \dot{v}_H + v_H = 0$$

$\therefore$ Try $v_H = Ce^{st} \Rightarrow \dot{v}_H = sCe^{st}$
so that we get

$$Ce^{st}[s\tau + 1] = 0 \Rightarrow \underline{\underline{s = -\frac{1}{\tau}}}$$

$\therefore$ $v_H = Ce^{-t/\tau}$

& $v_c(t) = Ce^{-t/\tau} + \frac{A\sin(\omega t - \theta)}{\sqrt{1+\omega^2\tau^2}}$

To find C, suppose $v_c(0) = 0$:

Then $0 = C + \frac{A\sin(-\theta)}{\sqrt{1+\omega^2\tau^2}} \Rightarrow C = \frac{A\sin\theta}{\sqrt{1+\omega^2\tau^2}}$

BST (5)
PS#5

3. **a.** Prove that for an underdamped series R-L-C circuit, with $v(t) = \sin\omega_0 t$ and $v_C(0) = \dot{v}_C(0) = 0$

$$v_C(t) = Q\left[e^{-\alpha t}\left(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right) - \cos\omega_0 t\right], \text{ where } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_0}{2\alpha}$$

Thus as expected from the results in Chapter 4 for a band pass filter, the steady state amplitude of these oscillations is Q (the quality factor for the filter). Note that for a high Q system, the damping is minimal, so that $\alpha << \omega_o$ and $\beta \approx \omega_0$, so that

$$v_C(t) \approx -Q(1 - e^{-\alpha t})\cos\omega_0 t,$$

b. Show that for a high Q system it takes about Q cycles before the full amplitude of the steady state solution is achieved. Hint: show that after Q cycles, $\alpha t = \pi$, so that $e^{-\alpha t}$ becomes negligible.

a) First as shown in notes the homogeneous soln to the equation

$$(1)\quad v(t) = \frac{1}{\omega_0^2}\ddot{v}_c + \frac{2\alpha}{\omega_0^2}\dot{v}_c + v_c$$

is $v_c = e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$

For $v_p(t)$ try $v_p = C\sin(\omega_0 t + \phi)$ — to match $v(t)$ & we set in (1)

$$\sin\omega_0 t = -\cancel{C\sin(\omega_0 t + \phi)} + \frac{2\alpha}{\omega_0}C\cos(\omega_0 t + \phi) + \cancel{C\sin(\omega_0 t}$$

Hence we let $\phi = +90°$ so that $\cos(\omega_0 t + \phi) = -\sin\omega_0 t$

or

$$\cancel{\sin\omega_0 t}^{1} = -\frac{2\alpha C}{\omega_0}\cancel{\sin\omega_0 t} \Rightarrow \boxed{C = -\frac{\omega_0}{2\alpha} = -Q}$$

∴ before apply BC's we have (using $v_p = -Q\sin(\omega_0 t + 90°) = -Q\cos\omega_0 t$)

(2) $v_c(t) = e^{-\alpha t}(A\cos\beta t + B\sin\beta t) - Q\cos\omega_0 t$

To find A & B we use the initial conditions

$v_c(0) = A - Q = 0 \Rightarrow \boxed{A = Q}$

$\dot{v}_c(0) = -\alpha A + B\beta = 0 \Rightarrow \boxed{B = \frac{\alpha A}{\beta} = \frac{\alpha Q}{\beta}}$

∴ Substituting these values into (2),

$v_c(t) = Q\left[e^{-\alpha t}\left(\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right) - \cos\omega_0 t\right]$

For a high Q system $\frac{\alpha}{\beta} \ll 1$ and $\beta \cong \omega_0$

∴ to a good approximation

$v_c(t) = Q\cos\omega_0 t\,(e^{-\alpha t} - 1)$

b) Recall that $2\alpha = \frac{\omega_0}{Q} = \frac{2\pi f_0}{Q} = \frac{2\pi}{QT}$

where $T = \frac{1}{f_0}$ is the period for an oscillation

∴

$\alpha = \frac{\pi}{QT}$ and for $t_0 = QT$, $\alpha t_0 = \pi$

∴ After Q cycles $t = t_0 = QT$ and

$e^{-\alpha t_0} = e^{-\pi} = .0432 \Rightarrow \boxed{v_c(t) \cong -Q\cos\omega_0 t}$

BJT ⑦
PS#5

4. **a.** Show that to a good approximation if a voltage V is applied at t' for a very short time dt', the voltage across the capacitor for an R-C circuit with time constant τ is

$$v_C(t) \approx (Vdt'/\tau)e^{-(t-t')/\tau}$$

b. If the same short pulse at t' is applied to an L-C circuit with resonant frequency ω_0, show that

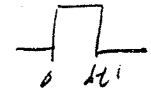
$$v_C(t) \approx (V\omega_0 dt')\sin\omega_0(t-t')$$

These results are called the impulse response of these systems, and Vdt' is the "strength" of the impulse.

These results are easily extended to a "series of impulses" with strength V(t')dt', to show that for a circuit driven by an arbitrary function V(t'), the voltage across the capacitor is given by

$$v_C(t) = (1/\tau)\int_0^t V(t')e^{-(t-t')/\tau}dt', \text{ for an R-C circuit, and}$$

$$v_C(t) = \omega_0\int_0^t V(t')\sin\omega_0(t-t')dt', \text{ for an L-C circuit, and}$$

a) For a pulse of length dt';  we've shown

$$v_C(t) = V[\cancel{1}-e^{-t/\tau}] - V[\cancel{1}-e^{-(t-dt')/\tau}]$$

$$= Ve^{-t/\tau}[e^{dt'/\tau}-1] \cong Ve^{-t/\tau}[\cancel{1}+\frac{dt'}{\tau}\cancel{-1}]$$

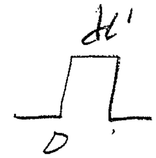
$$= \frac{Ve^{-t/\tau}dt'}{\tau} \quad \text{(pulse applied at } t=0)$$

If pulse is applied at $\underline{t'}$ we get the same result but with $t-t'$ replacing t; i.e.

$$v_C(t) = \begin{cases} 0, & t<t' \\ \frac{V}{\tau}e^{-(t-t')/\tau}dt', & \text{for } t>t' \end{cases}$$

b) For  , we've shown that

for $v(t) = V$; $v_C(t) = V[1-\cos\omega_0 t]$ (Eq. (5.24))

$\therefore$ for $v(t) \rightarrow$ 

$$V_c(t) = V[1-\cos\omega_0 t] - V[1-\cos\omega_0(t-dt')]$$

$$= V(\cos\omega_0(t-dt') - \cos\omega_0 t)$$

But $\cos\omega_0(t-dt') = \cos\omega_0 t \cos\omega_0 dt' + \sin\omega_0 t \sin\omega_0 dt'$

and as $dt' \to 0$: $\cos\omega_0 dt' \cong 1$ while $\sin\omega_0 dt' \cong \omega_0 dt'$

$\therefore$ we get

$$v_c(t) \cong V[\cancel{\cos\omega_0 t} + \omega_0 dt' \sin\omega_0 t - \cancel{\cos\omega_0 t}]$$

$$= V\omega_0 dt' \sin\omega_0 t$$

If the short pulse is applied at t' we get the same result (for $t>t'$) but with $t \to (t-t')$: i.e.

$$v_c(t) = \begin{cases} 0, & t<t' \\ V\omega_0 \sin\omega_0(t-t')\, dt' \end{cases}$$

Hence for an arbitrary $v(t')$ we have

for RC: $v_c(t) = \dfrac{1}{\tau}\displaystyle\int_0^t v(t') e^{-(t-t')/\tau} dt'$

for RL: $v_c(t) = \omega_0 \displaystyle\int_0^t v(t') \sin\omega_0(t-t') dt'$