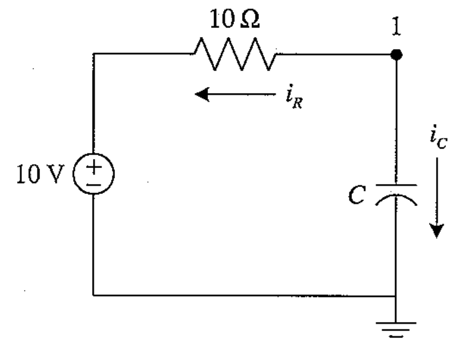
E59 Fall 2008
Problem Set #8 Solution

6-2.

(a) When the switch is at position **1**, the circuit can be represented by



To determine the voltage drop across capacitor, apply KCL at 1

$$i_R + i_C = 0$$

In terms of node voltages,

$$\frac{1}{R}(v_1 - 10) + C\frac{d}{dt}(v_1 - 0) = 0$$

Rearrange, note that $v_1 = v_C$, and express the governing equation in canonical form, one obtains

$$\frac{d}{dt}v_C(t) + \frac{1}{RC}v_C(t) = \frac{1}{RC}10$$

Solving, one finds

$$v_C(t) = 10(1 - e^{-\frac{t}{10C}})\ \mathrm{V}$$

whose steady state solution is

$$(v_C)_{ss} = \lim_{t\to\infty} v_C(t) = 10\ \mathrm{V}$$

Thus, to charge up to 95% of steady state value within 15 μs requires

$$v_C(t = 15\ \mu\mathrm{s}) = 95\%(v_C)_{ss}$$

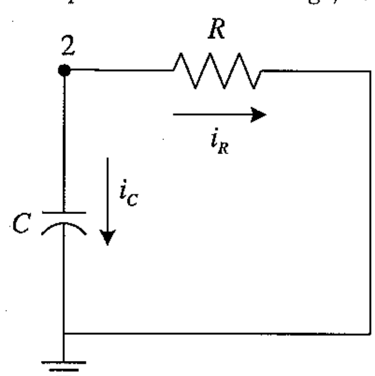
or

$$10(1 - e^{-\frac{15\ \mu\mathrm{s}}{10C}}) = 0.95 \times 10$$

Solving for C, one finds

$$\boxed{C = 0.5\ \mu\mathrm{F}}$$

(b) Switch is now at position 2; the capacitor has initial charge, but no input,



Applying KCL at node 2 yields

$$i_C + i_R = 0$$

or in terms of node voltages,

$$C\frac{d}{dt}(v_2 - 0) + \frac{1}{R}(v_2 - 0) = 0$$

Rearrange, note that $v_2 = v_C$, and express the governing equation in canonical form, one obtains

$$\frac{d}{dt}v_C(t) + \frac{1}{\tau}v_C(t) = 0$$

where $\tau = RC$ and R is to be determined. Solution to the above equation is

$$v_C(t) = v_C(0)e^{-t/\tau}$$

The capacitor voltage discharges to 63% of its initial voltage within 1 μs. Thus,

$$v_C(0)e^{-\frac{1\mu s}{RC}} = 63\% v_C(0)$$

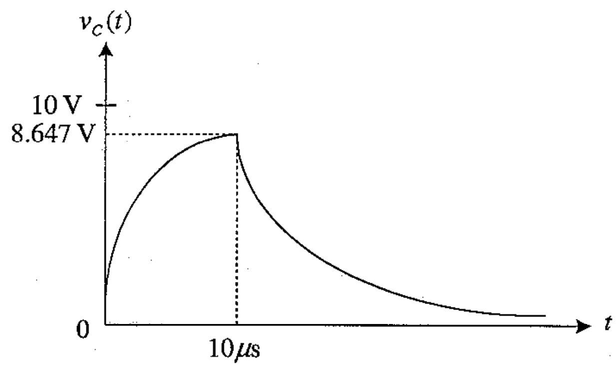
Solving for RC, one finds

$$RC = 2.164 \times 10^{-6}\ \text{s}$$

Because $C = 0.5\mu$F from part (a),

$$\boxed{R = 4.329\ \Omega}$$

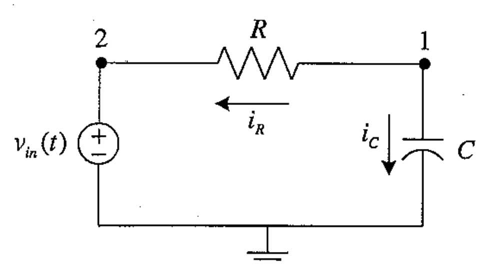
(c) The desired plot is shown below



The voltage drop across the capacitor right before the switch is flipped to position 2 is calculated below:

$$v_C(t = 10\mu s) = 10(1 - e^{-\frac{10\mu s}{10\ \Omega \cdot 0.5\mu F}}) = 8.647\ \text{V}$$

6-3. Consider the following RC circuit:



To determine the governing equation for the voltage drop across the capacitor, apply KCL at node 1, which yields

$$i_R + i_C = 0$$

or in terms of node voltages,

$$\frac{1}{R}(v_1 - v_{\text{in}}) + C\frac{d}{dt}(v_1 - 0) = 0$$

In canonical form, the above equation becomes

$$\frac{d}{dt}v_C + \frac{1}{RC}v_C = \frac{1}{RC}v_{\text{in}}$$

For a step input of $v_{\text{in}}(t) = V_{\text{in}}$, the voltage drop is given by

$$v_C(t) = V_{\text{in}}(1 - e^{-\frac{t}{RC}})$$

Charging up to 95% of its steady state value in less than 15 μs implies that

$$V_{\text{in}}(1 - e^{-\frac{15\mu s}{RC}}) \geq 0.95V_{\text{in}}$$

Solving for RC, one obtains

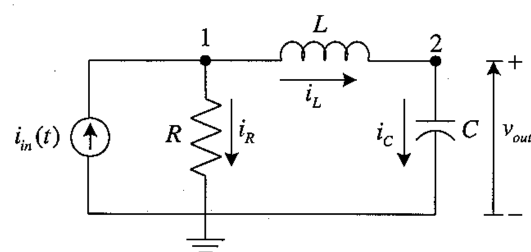
$$RC \leq 3.34 \times 10^{-6} \text{ s}$$

Given $C = 15$ pF, one finds

$$\boxed{R \leq 222 \text{ k}\Omega}$$

Correction: RC<=10^12/3=333 kilo Ohms

6-5. Consider the circuit shown below



To determine the voltage drop across the capacitor, first apply KCL at node 1

$$-i_{in}(t) + i_R + i_L = 0$$

or in terms of node voltages

$$\frac{1}{R}(v_1 - 0) + \frac{1}{L}\int_{-\infty}^{t}(v_1 - v_2)dt = i_{in}(t) \tag{4}$$

Applying KCL at node 2 gives

$$i_C(t) - i_L(t) = 0$$

or

$$C\frac{d}{dt}(v_2 - 0) - \frac{1}{L}\int_{-\infty}^{t}(v_1 - v_2)dt = 0 \tag{5}$$

Because $v_{\text{out}} = v_2(t)$ in Eqs. (4) and (5), one needs to eliminate v_1. From (5), one finds

$$C\ddot{v}_2 = \frac{1}{L}(v_1 - v_2)$$

Thus,

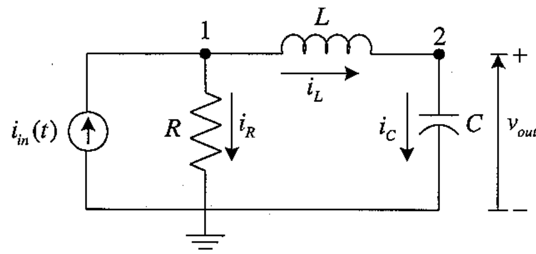
$$v_1 = LC\ddot{v}_2 + v_2 \tag{6}$$

Substitute the above into (4), one gets

$$\frac{1}{R}(LC\ddot{v}_2 + v_2) + \frac{1}{L}\int_{-\infty}^{t} LCv_2 dt = i_{in}(t)$$

Correction: v_2 in the integral should have two dots on top for 2nd order derivative.

6-5. Consider the circuit shown below



To determine the voltage drop across the capacitor, first apply KCL at node 1

$$-i_{in}(t) + i_R + i_L = 0$$

or in terms of node voltages

$$\frac{1}{R}(v_1 - 0) + \frac{1}{L}\int_{-\infty}^{t}(v_1 - v_2)dt = i_{in}(t) \tag{4}$$

Applying KCL at node 2 gives

$$i_C(t) - i_L(t) = 0$$

or

$$C\frac{d}{dt}(v_2 - 0) - \frac{1}{L}\int_{-\infty}^{t}(v_1 - v_2)dt = 0 \tag{5}$$

Because $v_{\text{out}} = v_2(t)$ in Eqs. (4) and (5), one needs to eliminate v_1. From (5), one finds

$$C\ddot{v}_2 = \frac{1}{L}(v_1 - v_2)$$

Thus,

$$v_1 = LC\ddot{v}_2 + v_2 \tag{6}$$

Substitute the above into (4), one gets

$$\frac{1}{R}(LC\ddot{v}_2 + v_2) + \frac{1}{L}\int_{-\infty}^{t} LCv_2 dt = i_{in}(t)$$

or

$$\frac{LC}{R}\ddot{v}_2 + C\dot{v}_2 + \frac{1}{R}v_2 = i_{in}(t)$$

In canonical form, the governing equation for the voltage drop across the capacitor is

$$\ddot{v}_{out} + \frac{R}{L}\dot{v}_{out} + \frac{1}{LC}v_{out} = \frac{R}{LC}i_{in}(t)$$

Thus,

$$\omega_n = \sqrt{\frac{1}{LC}} \quad \text{and} \quad 2\zeta\omega_n = \frac{R}{L}$$

Solving for R, one has

$$R = 2\zeta\omega_n L = 2\zeta\sqrt{\frac{L}{C}} \tag{7}$$

The percent overshoot constraint is given by

$$M_p = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\% < 10\%$$

Solving for ζ, one finds

$$\zeta > 0.592$$

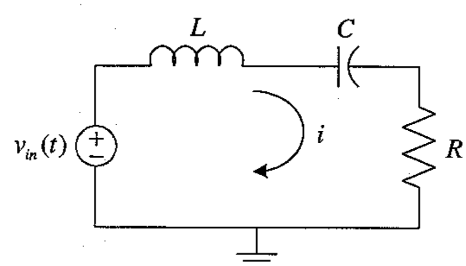
Moreover, since the system is underdamped, $\zeta < 1$. Thus,

$$1 > \zeta > 0.592$$

Substitute into Eq. (7), one gets

$$\boxed{44.72\Omega > R > 26.44\Omega}$$

6-6. Consider the following circuit



Apply KVL, one finds

$$v_L + v_C + v_R = v_{in}(t)$$

or

$$L\frac{di}{dt} + v_C + i_R = v_{in}(t) \tag{8}$$

But

$$i = i_C = C\frac{dv_C}{dt} \tag{9}$$

Substitute Eq. (9) into (8), one gets

$$LC\frac{d^2v_C}{dt^2} + CR\frac{dv_C}{dt} + v_C = v_{in}(t)$$

or in canonical form,

$$\frac{d^2v_C}{dt^2} + 2\zeta\omega_n\frac{dv_C}{dt} + \omega_n^2 v_C = \omega_n^2 v_{in}(t)$$

where

$$\omega_n^2 = \frac{1}{LC} \quad \text{and} \quad 2\zeta\omega_n = \frac{R}{L}$$

The transient specifications are given by

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1 \times 10^{-6} \text{ s} \tag{10}$$

$$t_s = \frac{-\ln(\epsilon\sqrt{1-\zeta^2})}{\zeta\omega_n} = 10 \times 10^{-6} \text{ s} \;\; \text{for } \epsilon = 2\% \tag{11}$$

The natural frequency can be expressed in terms of the damping factor as follows

$$\omega_n = \frac{\pi}{1 \times 10^{-6}\sqrt{1-\zeta^2}} \tag{12}$$

Substitute (12) into (11) and solve for ζ, one finds $\zeta \approx 0.1238$, from which one gets $\omega_n \approx 3.19 \times 10^6$ rad/sec. Thus,

$$\boxed{L = \frac{R}{2\zeta\omega_n} = \frac{1000\Omega}{2(0.1238)(3.19 \times 10^6)} = 1.266 \times 10^{-3} \text{ H}}$$

$$\boxed{C = \frac{1}{L\omega_n^2} = 7.911 \times 10^{-11} \text{ F}}$$

6-7.

(a) Apply KCL, one finds

$$i_s(t) = i_R + i_L + i_C = \frac{1}{R}v_C + i_L + C\frac{dv_c}{dt} \tag{13}$$

The voltage drop the capacitor can be expressed in terms of the current through the inductor using the following

$$i_L = \frac{1}{L}\int_{-\infty}^{t} v_c d\tau \Longrightarrow v_C = L\frac{di_L}{dt} \tag{14}$$

Substitute Eq. (14) into (13), one gets

$$\boxed{CL\frac{d^2i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = i_s(t)}$$

(b) In canonical form, the above equation becomes

$$\frac{d^2i_L}{dt^2} + 2\zeta\omega_n\frac{di_L}{dt} + \omega_n^2 i_L = \frac{1}{CL}i_s(t)$$

where

$$\omega_n = \frac{1}{\sqrt{CL}} \quad \text{and} \quad \zeta = \frac{1}{2\omega_n}\frac{1}{RC}$$

- From the given plot, the maximum percent overshoot is

$$M_p = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}\ \% \approx \frac{1.45-1}{1} \times 100\% = 0.45$$

  Solving for ζ, one gets $\zeta \approx 0.246$.
- The peak time from the plot is

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \approx (0.32\ \text{sec}) \times 10^{-4}$$

  Solving for ω_n, one obtains $\omega_n \approx 1.013 \times 10^5$ rad/s.

Thus,

$$2\zeta\omega_n = \frac{1}{RC} \Longrightarrow C = \frac{1}{R2\zeta\omega_n} = \frac{1}{(20)(2)(0.246)(1.013 \times 10^5)}$$

$$\boxed{C = 1.003 \times 10^{-6}\ \text{F} = 1\ \mu\text{F}}$$

and

$$\omega_n^2 = \frac{L}{C} \Longrightarrow L = \frac{1}{\omega_n^2 C}$$

$$\boxed{L = 9.713 \times 10^{-5}\ \text{H}}$$