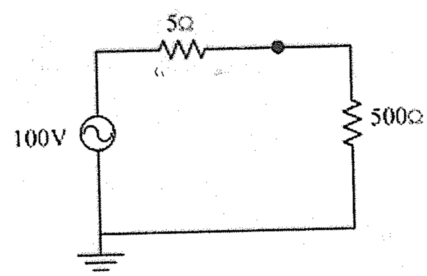
E-84 Electric & Magnetic Circuits & Devices — Sam Tanenbaum
Solution Key for PS #4

1. Design **two** types of matching circuits that maximize the power to the load in the 60 Hz circuit shown below:



a. First use a **transformer** to achieve maximum load power. Assume the transformer is ideal, specify the turns ratio, and draw the complete circuit.

If a transformer is inserted between the 5 Ω resistor and the 500 Ω load resistor, the equivalent resistance for the load becomes:

$$R_{eq} = (N_1/N_2)^2 500$$

To maximize the power to the load, we must have R_{eq} = 5 Ω.

Hence we want the turns ratio for the transformer to be:

$(N_1/N_2) = (5/500)^{1/2} = 1/10$.

Thus for example we might have $N_1 = 5$ and $N_2 = 50$.

b. Next use a design with an **inductor and two capacitors** to maximize load power. Specify values for the capacitors and the inductor and draw the complete circuit.

Using the **geometric mean method on pg 69**, we use a circuit like the one shown after Eq. (4.4), where both capacitors and the inductor have impedances with magnitude X. where

$$R_{eq} = X^2 R_L$$

To maximize the power to the load, we must have R_{eq} = 5 Ω.

Hence we want **X** to be given by:

$X = (5 \bullet 500)^{1/2} = 50$.

For the Capacitors, $X = 1/\omega C$, so that **$C = 1/\omega X$.** Hence for each capacitor

$$C = 1/2\pi \bullet 60 \bullet 50 = 53 \bullet 10^{-6}\ \text{Fd}$$

For the Inductor, $X = \omega L$, so that **$L = X/\omega$.** Hence for the inductor

$$L = 50/2\pi \bullet 60 = .133\ \text{H}$$

c. Show that the power delivered to the load increases from **19.6 W** with no matching circuit to **500 W** with matching.

From the voltage divider formula, the voltage across the load is:

$V_L = 100 \bullet (500/505) = 99.0$ **V, so that**

$P_{Load} = V^2/R = 99^2/500 = 19.6$ **W**

When the matching circuit is added the voltage across the load soars. However, we can also note that we can also think of the circuit as being unchanged except

that the effective resistance drops to 5 Ω. Using this value in the circuit above, from the voltage divider formula,
$V_L = 100 \cdot (5/10) = 50$ V, so that
$P_{Load} = V^2/R_{Ef} = 50^2/5 = 500$ W

2. Recall that for a linear system with input voltage V_i and output voltage V_0, if **V_i = $A\sin\omega t$**, then in steady state,

$V_0(t) = |T| A\sin(\omega t + \phi)$, where **$V_0/V_i = |T| \underline{/\phi^\circ}$.**

What is the steady state output $V_0(t)$ for an R-C **low pass filter with cutoff frequency ω_c**, when

a. $\omega = 0.1\,\omega_c$,

From Eq (4.10),
$V_0/V_i = 1/(1 + j\omega/\omega_c) = [1]/[1 + (\omega/\omega_c)^2]^{1/2}\underline{/\tan^{-1}(-\omega/\omega_c)}$
Thus for $\omega = 0.1\,\omega_c$,
$|T| = 1/1.01^{1/2} = .995$, and $\phi = \tan^{-1}(-.1) = -5.7^\circ = -0.0997$ radians, so that
$V_0(t) = .995A\sin(\omega t - .0997)$.

b. $\omega = \omega_c$,

As in (a), for $\omega = \omega_c$,
$|T| = 1/2^{1/2} = .707$, and $\phi = \tan^{-1}(-1) = -45^\circ = -0.785$ radians, so that
$V_0(t) = .707A\sin(\omega t - .785)$.

c. $\omega = 10\,\omega_c$,

As in (a), for $\omega = 10\omega_c$,
$|T| = 1/101^{1/2} = .0995$, and $\phi = \tan^{-1}(-10) = -93.7^\circ = -1.47$ radians, so that
$V_0(t) = .0995A\sin(\omega t - 1.47)$.

d. Show that if **$\omega = \omega_c = 2\pi$ rad/s**, (i.e f = 1 Hz),the peaks of the output occur 1/8 second after the peaks of the input voltage.

From (b), if we factor out **ω**, we obtain,
$V_0(t) = .707A\sin(\omega t - .785) = .707A\sin[\omega(t - t_{delay})]$, where the time delay,
$t_{delay} = .785/\omega$ where, for **$\omega = 2\pi$ rad/s,**
$t_{delay} = .785/2\pi = .125$ s. Hence the output lags by 1/8 second.

5. Prove that $C = 1/R_0\omega_0$ and $L = R_0/\omega_0$ and find values for L and C required to have Q = .01 for a 60 Hz. blocking circuit when R = 1000 Ω. Is this practical?

Recall from pg 77 that **$R_0 = (L/C)^{1/2}$, while $\omega_0 = 1/(LC)^{1/2}$**
Hence $R_0 \cdot \omega_0 = 1/C$, so that $\underline{C = 1/R_0\omega_0}$.
In addition, again using the definitions of **R_0 and ω_0**, **$\underline{R_0/\omega_0 = L}$**

To calculate L and C, in terms of Q and R, recall from pg 78 that
$Q = (L/C)^{1/2}/R$. Hence for **$Q = .01$, $R = 1{,}000$**, and **$\omega_0 = 2\pi \cdot 60$**

$(L/C)^{1/2} = 10$, and $1/(LC)^{1/2} = 120 \cdot \pi$. Multiplying these results together, we obtain: $1/C = 1200\pi$, so that
$C = .000265 = 265\ \mu F$, while **$L = 100C = 26.5\ mH$**
These are capacitors and inductors that are available.

7. A step-down transformer reduces the 60 Hz voltage at a substation from 100,000 V to 10,000 V.

a. If $N_1 = 300$ turns, show that the **rms** value for Φ is .884 Webers.

Recall that $V_1 = d\psi/dt = N_1 d\Phi/dt$, so that for the primary side of the transformer:
$10^5 = 300 j\omega \underline{\Phi}$, where $\omega = 2\pi \cdot 60$.
Solving for Φ, we get
$\Phi = 100/36\pi = .884$ W

b. If the maximum value for B in the core is 1 T, show that the cross-section area of the core must be 1.25 m^2.

Since $\Phi = BA$, and Φ is an oscillating function, $\Phi_{max} = 2^{1/2}(.884)$. **Since $B_{max} = 1$ T,**
$A_{Min} = 2^{1/2}(.884) = 1.25\ m^2$.

8. Find values for R and L in the band-pass filter described by Eq. (4.14) when:

$C = 0.1\ nF = 10^{-10}$ F
$f_0 = 100\ MHz = 10^8$ Hz
Bandwidth, $BW = 200\ kHz = 2 \cdot 10^5$ Hz.

To find L, recall that $\omega_0^2 = 1/LC$ so that for the given values of f_0 and C,
$L = 10^{10}/4\pi^2 \cdot 10^{16} = \underline{.0253 \cdot 10^{-6}\ H}$

Next recall that $Q = f_0/BW = (1/R)(L/C)^{1/2}$. Hence for the given values of f_0, BW, L, and C, $Q = 500$ so that
$R = (1/Q) \cdot (L/C)^{1/2} = (1/500) \cdot (.025 \cdot 10^{-6}/10^{-10})^{1/2} = \underline{.0318\ \Omega}$
(This is a very small resistor, so that these may not be a practical set of values!)

12. The circuit shown on the next page consists of two band-stop filters in series. To obtain an expression for V_0/V_i it is convenient to use the Thevenin equivalent circuit shown on the right.

a. Show that $V_{Th} = V_i R_1/(R_1 + Z_1)$ and $Z_{Th} = Z_2 + R_1 || Z_1$, where Z_1 and Z_2 are the impedances of the first and second pairs of inductors and capacitors in parallel.

V_{Th} is the output voltage when R_2 is removed. In this case, no current can flow through Z_2 and we can use the voltage divider formula to directly obtain:

$$V_{Th} = V_i R_1/(R_1 + Z_1)$$

Z_{Th} is the impedance from the output to ground when R_2 is removed and V_i is replaced by a short circuit. This places R_1 in parallel with Z_1, so that

$$Z_{Th} = Z_2 + R_1 || Z_1$$

b. Use the Thevenin equivalent circuit to derive an expression for V_0/V_i, and show that if all of the relevant Q values are small,

$|V_0/V_i| \approx 0$, for either resonance frequency, $\omega_{01} = (1/L_1C_1)^{1/2}$ **or** $\omega_{02} = 1/L_2C_2)^{1/2}$ and

$|V_0/V_i| \approx 1$, for all other frequencies.

From the Thevenin Equivalent Circuit, we see that

$V_0 = V_{Th}R_2/(R_2 + Z_{Th})$, so that

$$V_0/V_i = R_1R_2/(R_1 + Z_1)(R_2 + Z_2 + R_1 || Z_1)$$
$$= R_1R_2/(R_1R_2 + R_1Z_2 + R_2Z_1 + Z_1Z_2 + R_1Z_1)$$
$$= 1/[1 + (Z_2/R_2) + (Z_1/R_1) + (Z_1/R_2) + (Z_1Z_2/R_1R_2)]$$
$$\rightarrow 0 \text{ when either } Z_1 \text{ or } Z_2 \rightarrow \infty$$

This shows that there is no output voltage when the input frequency is either

$\omega_{01} = (1/L_1C_1)^{1/2}$ **(which makes Z_1 infinite) or**

$\omega_{02} = 1/L_2C_2)^{1/2}$ **(which makes Z_2 infinite)**

This result applies regardless of the Q values.

Away from the resonant frequencies,

$|Z_1| < R_{01} = (L_1/C_1)^{1/2}$ and $|Z_2| < R_{02} = (L_2/C_2)^{1/2}$. **Hence**

$$1 > V_0/V_i > 1/[1 + (R_{02}/R_2) + (R_{01}/R_1) + (R_{01}/R_2) + (R_{01}R_{02}/R_1R_2)]$$

But for a resonant circuit, **$Q = R_0/R$.** Hence, if all the **Q values are small,** away from resonance,

$$V_0/V_i \approx 1$$