INVARIANTS AND EXTREMES

PIETRO KREITLON CAROLINO

Today we're going to do some problems for their own sake. Whereas in previous meetings our problems were intended as a guide to some larger mathematical vista, today they are meant to show you problem-solving strategies that will help you along any path you travel.

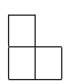
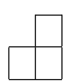
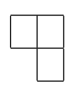
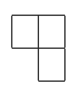
Invariance.

The problems below deal with processes that consist of the repetition of simples steps many many times. One of the best (and in fact only) ways to analyze and understand such processes is to find some feature of them that does not change as the steps are performed. Locating such **invariant** (that is, unchanging) features is more art than science, and is best learned by solving many problems.

1. Danny writes the numbers 1 through 2015 on a blackboard. He then chooses two of the numbers, say a and b, erases them both, and writes down $a - b$. He does this again and again until only one number is left. Can Danny choose his numbers carefully so that the number left at the end is 1?

2*. You are looking under a microscope and there are three kinds of bacteria: green, yellow, and blue. You are studying their unusual reproductive dynamics. You've noticed that sometimes a green bacterium divides producing one yellow and one blue specimen. Other times, three yellow and one blue specimen combine to produce two green ones. You leave the lab one night knowing that under the microscope there are 5 green, 7 yellow, and 3 blue bacteria. You arrive in the morning to find 8 green, 2 yellow, and 3 blue. Could this have happened naturally, or did someone tamper with your research?

3. You have an 8×8 chessboard and the numbers 1 through 64 are written in order in the squares. You are allowed the following moves: choose an L-shape consisting of three squares, like

 or  or  or 

and add 1 to each number in the L-shape. Is there a sequence of moves that will make all the numbers on the board be 99?

4*. The Fibonacci sequence starts with 1,1 and then each term is the sum of the two preceding terms. Thus the first few terms are 1, 1, 2, 3, 5, 8, 13, ...

What is the greatest common divisor (gcd) of two consecutive numbers on the Fibonacci sequence? (Hint: the gcd satisfies the following property: if $a > b$ then $\gcd(a, b) = \gcd(a - b, b)$. For instance, $\gcd(23, 7) = \gcd(23 - 7, 7) = \gcd(16, 7)$.)

5.** The numbers 1 through 10 are written in order on a blackboard. You are allowed two pick any two numbers and switch their positions. Can you put the numbers in reverse order using exactly 20 such switches?