

Invariance Principle

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Outline

Introduction

Problems

Parity, Mod 9, Coloring
Monovariants

Introduction

- ▶ The invariance principle is a problem solving strategy.
- ▶ When there is repetition, look for something which does not change (PSS).

Example

A circle is divided into 6 sectors. The numbers 1, 0, 1, 0, 0, and 0 are written in that order. You can increase two adjacent numbers by 1. Is it possible to make all numbers equal? (PSS, wording by Howard Halim from UTS).

Parity (Even and Odd)

P1

Let n be a positive integer. The numbers $1, 2, 3, \dots, 2n$ are written on a blackboard. A person takes any two numbers on the board a, b , erases them, and writes the number $|a - b|$ on the board. The person keeps doing this until there is only one number left on the board. Prove this number must be odd (PSS).

- Note that $|a - b|$ means the positive difference between a and b .

P2

Many handshakes are exchanged at a big international congress. Show that at any moment, there is an even number of people who have exchanged an odd number of handshakes (Howard Halim).

Sum of Digits

P3

Each of the numbers from 1 to 10^6 is repeatedly replaced by the sum of its digits, until we end up with a list of 10^6 one-digit numbers. Will there be more 1's or 2's in this list (Howard Halim)?

Coloring

P4

Given a 8 by 8 chessboard, is it possible to completely cover the board with 2 by 1 dominos if two opposite corners of the chessboard are removed?

Monovariants

- ▶ Some trend that changes one way (ex. increasing only).

Example

In a school, each student has at most 3 enemies. Can a teacher split the students into two rooms such that each student has at most one enemy in his or her room (PSS)?

P5

There are n red points and n blue points in the plane, with no 3 points collinear. Show that we can draw n segments joining the blue points to the red points such that no pair of segments intersect.