UNIVERSITY OF TORONTO

Statistical Sciences

Tutorial - Week 2

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July 13, 2023

Outline

- Limit Theorems
- Modelling
- Estimators

Limit Theorems - LLN

- Chebyshev's Inequality
- Weak Law of Large Numbers
- Strong Law of Large Numbers

Limit Theorems - CLT

- Central Limit Theorem

Exercise - MIPS 13.9

Let $X_1, X_2, \ldots$.be an independent sequence of $U(-1, 1)$ random variables and let $T_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. It is claimed that for some a and any $\epsilon > 0$ $\lim_{n\to\infty} P(|T_n - a| > \epsilon) = 0$.

a. Explain how this could be true.
b. Determine a.

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Exercise - MIPS 14.5

Let X be a $Bin(n, p)$ distributed random variable. Show that the random variable $\frac{X - np}{\sqrt{np(1-p)}}$ has a distribution that is approximately standard normal.

Exercise - MIPS 14.8

Let $X_1, X_2, \ldots$be a sequence of independent $N(0,1)$ distributed random variables. For $n = 1, 2, \ldots,$ let Y_n be the random variable, defined by $Y_n = X_1^2 + \ldots + X_n^2$

a. Show that $E(X_i^2) = 1$
b. One can show—using integration by parts—that $E(X_i^4) = 3$. Deduce from this that $Var(X_i^2) = 2$.
c. Use the central limit theorem to approximate $P(Y_{100} > 110)$.

Modelling

- Parametric model: model distribution, model parameters
- Sample statistics and realizations
- Linear Regression $Y_i = \alpha + \beta x_i + U_i$ where $U_1, \ldots, U_n$ are independent random variables with mean zero and same variance :
  - Regression line, intercept & slope, response variables (dependent variables), explanatory variables (independent variables).

Estimators

- Estimate: function of dataset
- Estimator: function of random sample
- Bias, Unbias
- Consistency (Converge in probability)

Exercise - MIPS 19.1

Suppose our dataset is a realization of a random sample $X_1, X_2, \ldots, X_n$ from a uniform distribution on the interval $[-\theta, \theta]$, where θ is unknown.

a. Show that $T = \frac{3}{n}(X_1^2 + \ldots + X_n^2)$ is an unbiased estimator for θ^2

b. Is $\sqrt{T}$ also unbiased estimator for θ? If not, large whether is has positive or negative bias.

Exercise - MIPS 19.8

Recall a linear regression model without intercept $Y_i = \beta x_i + U_i$ for $i = 1, 2, \ldots, n$, where $U_1, U_2, \ldots, U_n$ are independent random variables with $E[U_i] = 0$ and $Var(U_i) = \sigma^2$. We discussed three estimators for the parameter β:

- $B_1 = \frac{1}{n}\left(\frac{Y_1}{x_1} + \ldots + \frac{Y_n}{x_n}\right)$
- $B_2 = \frac{Y_1+\ldots+Y_n}{x_1+\ldots+x_n}$
- $B_3 = \frac{x_1Y_1+\ldots+x_nY_n}{x_1^2+\ldots+x_n^2}$

Show that all three estimators are unbiased for β.