Lecture 5-W4

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Pivot Rule:

Max $-2+x_1+2x_2-w_2$

subject to:
$$\begin{cases} w_1 = 1-3x_2+w_2 \\ \lambda = 2-x_1-2x_2+w_2 \\ w_3 = 3-x_1-3x_2+w_2 \\ w_i \geq 0,\ x_i \geq 0,\ \lambda \geq 0 \end{cases}$$

non-basic variables = independent variables.
basic variables = dependent variables.

While iterating the basic step of the simplex Method,
We operate two choices:

(1) A variable with positive coefficient from the objective function.

(2) a basic variable to substitude.

(a) a variable goes from non-basic to basic. – Entering variable
(b) a variable goes from basic to non-basic – Leaving variable

A computer needs a set of specific instructions. So we are required to choose a rule that specifies how to make this decision. Differents rules lead to different alogrithm.
A pivoting rule is a rule specifing how to choose the leaving and the entering variable at each iteration of the simplex method.

- Anstee's Rule
- Blond's Rule

} pivoting rule. only for positive-coefficient variables in the objective function.

Rmk: A good pivoting rule is a rule for which the termination of the simplex method is guaranteed.

(1) Anstee's rule: (Not the best) rule of thumb / in exam. use this

- choose the entering variable with the largest possible coefficient. If there is a tie, then choose the smallest subscript.
- (choose the dependent var to flip): If there is to choose the leaving (Y) variable, just choose the one with the smallest subscript.

(2) Blond's rule: good rule to get termination

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- everytime you have to choose a variable x_k, just choose the one with the smallest subscript k.

Example:

$$\max \ \zeta = x_1 - 2x_2 - 2x_4$$

subject to

$$\begin{cases} w_1 = -0.5x_1 + 3.5x_2 + 2x_3 - 4x_4 \\ w_2 = -0.5x_1 + x_2 + 0.5x_3 - 0.5x_4 \\ w_3 = 1 - x_1 \\ x_i \geq 0, \quad w_i \geq 0 \end{cases}$$

$$(w_1 \cdots w_m,\ x_1 \cdots x_n)$$
$$\parallel$$
$$(x_1 \cdots x_m,\ x_{m+1} \cdots x_{m+n})$$

$$\boxed{\begin{aligned} &x_1 = x_2 = x_3 = x_4 = 0 \\ &w_1 = 0,\ w_2 = 0,\ w_3 = 1 \end{aligned}}$$

← This is admissible so I don't need to run the auxiliary problem.

I increase the value of x_1, (choosing x_1 as entering variable)

$$\begin{cases} w_1 = -0.5x_1 \geq 0 \\ w_2 = -0.5x_1 \geq 0 \\ w_3 = 1 - x_1 \geq 0 \end{cases} \qquad \begin{cases} x_1 \leq 0 \\ x_1 \leq 0 \\ x_1 \leq 1 \end{cases} \Rightarrow \boxed{x_1 = 0}$$

no gain in the objective function but maybe I change dictionary and things can be improved in the next step of the algorithm.

For Anstee and Blond's rules, we choose w_1 as leaving variable.

pivot, x_1 enters and w_1 leaves bringing us to:

$$\begin{array}{rl} \zeta = & -2w_1 + 5x_2 + 4x_3 - 10x_4 \\ \hline x_1 = & -2w_1 + 7x_2 + 4x_3 - 8x_4 \\ w_2 = & w_1 - 2.5x_2 - 1.5x_3 + 3.5x_4 \\ w_3 = & 1 + 2w_1 - 7x_2 - 4x_3 + 8x_4 . \end{array}$$

For the second iteration, x_2 enters and w_2 leaves bringing us to: $w_2 \Rightarrow x_1$

$$\begin{array}{rl} \zeta = & -2w_2 + x_3 - 3x_4 \\ \hline x_1 = & 0.8w_1 - 2.8w_2 - 0.2x_3 + 1.8x_4 \\ x_2 = & 0.4w_1 - 0.4w_2 - 0.6x_3 + 1.4x_4 \\ w_3 = & 1 - 0.8w_1 + 2.8w_2 + 0.2x_3 - 1.8x_4 . \end{array}$$

For the third iteration, x_3 enters and x_1 leaves:

$$\begin{array}{rl} \zeta = & 4w_1 - 16w_2 - 5x_1 + 6x_4 \\ \hline x_3 = & 4w_1 - 14w_2 - 5x_1 + 9x_4 \\ x_2 = & -2w_1 + 8w_2 + 3x_1 - 4x_4 \\ w_3 = & 1 - x_1 . \end{array}$$

For the fourth iteration, x_4 enters and x_2 leaves:

only substitude with the sharpest equation!
In this case, w_3 is excluded.

$$\begin{aligned} x_2 &= -2w_1 + 8w_2 + 3x_1 - 4x_4 \\ w_3 &= 1 \qquad\qquad\qquad - x_1. \end{aligned}$$

For the fourth iteration, x_4 enters and x_2 leaves:

$$\begin{array}{rl} \zeta = & w_1 - 4w_2 - 0.5x_1 - 1.5x_2 \\ \hline x_3 = & -0.5w_1 + 4w_2 + 1.75x_1 - 2.25x_2 \\ x_4 = & -0.5w_1 + 2w_2 + 0.75x_1 - 0.25x_2 \\ w_3 = 1 & \qquad\qquad - x_1. \end{array}$$

In the fifth iteration, w_1 enters and x_3 leaves:

$$\begin{array}{rl} \zeta = & -2x_3 + 4w_2 + 3x_1 - 6x_2 \\ \hline w_1 = & -2x_3 + 8w_2 + 3.5x_1 - 4.5x_2 \\ x_4 = & x_3 - 2w_2 - x_1 + 2x_2 \\ w_3 = 1 & \qquad\qquad - x_1. \end{array}$$

Lastly, for the sixth iteration, w_2 enters and x_4 leaves:

$$\begin{array}{rl} \zeta = & -2x_4 + x_1 - 2x_2 \\ \hline w_1 = & 2x_3 - 4x_4 - 0.5x_1 + 3.5x_2 \\ w_2 = & 0.5x_3 - 0.5x_4 - 0.5x_1 + x_2 \\ w_3 = 1 & \qquad\qquad - x_1. \end{array}$$

→ The same objective function as before.

Rmk: if the algorithm does not terminate then we enter an implicte cycle.

$x_1 \cdots x_n$ independent.
$w_1 \cdots w_m$ dependent.
$n = \#$ of variables
$m = \#$ of constraints.

Every configuration of the simplex algorithm is usiuasully determined by the choice of the n independent variables.

$$\underbrace{w_1 \cdots w_m \; x_1 \cdots x_n}_{\text{pick } n \text{ of them}} \Rightarrow \binom{n+m}{n} = \underbrace{\frac{(n+m)!}{n! \cdot m!}}$$

number of all possible configurations we can hit with the simplex method.

This is a large number but finite: If I keep iterating forever I must come back on some of the configurations:

Long story short:

THEOREM: if we make decisions with Bland's rule, cycling cannot happen.

Matrix Notation

$$\max \ z = \overset{\text{scalar product}}{c_0} + \overbrace{c_1 x_1 + \cdots + c_n \cdot x_n}^{\text{linear}}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \cdots + v_n \cdot w_n .$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Rightarrow z = c_0 + \vec{x} \cdot \vec{c}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$\vec{v} = \vec{w}$ means $v_1 = w_1$, $v_2 = w_2$, $\vdots$

assume $\vec{v} \le \vec{w}$ if $v_1 \le w_1$, $v_2 \le w_2$

$$\begin{cases} a_{11} x_1 + \cdots + a_{1n} \cdot x_n \le b_1 \\ a_{21} x_1 + \cdots + a_{2n} \cdot x_n \le b_2 \\ \quad \vdots \qquad\qquad \vdots \\ a_{m1} \cdot x_1 + \cdots + a_{mn} \cdot x_n \le b_m \\ x_1 \cdots x_n \geq 0 . \end{cases} \Rightarrow \vec{A} \cdot \vec{x} \le \vec{b}$$

$\vec{x} \geq 0$

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{\vec{A}} \cdot \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}}_{\vec{x}}$$

$$\Rightarrow O \cdot F = \max z = c_0 + \vec{c} \cdot \vec{x}$$

subject to $\begin{cases} \vec{A} \cdot \vec{x} \le \vec{b} \\ \vec{x} \geq 0 \end{cases}$

Feb 4^{th}. 2021

Step 0: introduce aditional variables $w_1 \cdots w_m$.

$$\begin{cases} w_1 = b_1 - a_{11} x_1 - \cdots a_{1n} \cdot x_n \\ w_2 = b_2 - a_{21} x_1 - \cdots a_{2n} \cdot x_n \\ \qquad \vdots \\ w_m = b_m - a_{m1} x_1 - \cdots a_{mn} \cdot x_n \\ \qquad x_1 \cdots x_n \geq 0 \end{cases}$$

$$\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$\text{Max } z = c_0 + \vec{c} \cdot \vec{x}$$

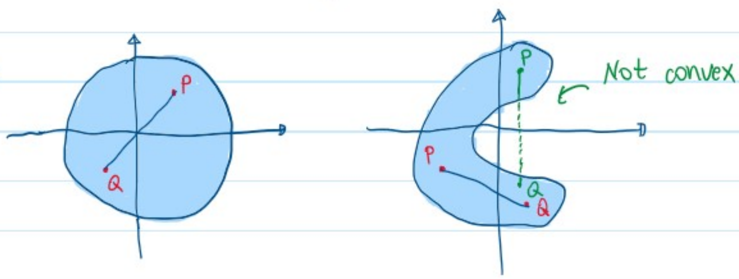
$$\begin{cases} \vec{w} = \vec{b} - A \cdot \vec{x} \\ \vec{w} \geq 0 , \quad \vec{x} \geq 0 \end{cases}$$

Rmk: since there are $\binom{n+m}{n}$ dictionaris for a problem in n variables on m constraints the simplex method needs at most $\binom{n+m}{n}$ steps to end.
There are examples where exactly $\binom{n+m}{n}$ steps are required. (worst sinario).

Convexity

$S \subseteq \mathbb{R}^n$ (vector) is convex if however I choose two points $P, Q \in S$ the line segment ℓ joining P and Q is totally contained in S, ie. $\ell \subseteq S$.

ex.



Rmk: the points in the line segment jointing P and Q can be writen as $t \cdot P + (1-t) Q$, $0 \le t \le 1$

$\forall$ = for all
$\exists$ = there exist.
$S \subseteq \mathbb{R}^n$ is convex if $\forall P, Q \in S$ we have $tP + (1-t)Q \in S$ $\forall$ $0 \le t \le 1$