# Lecture 3-W3

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$\max\ z = -3x_3 + x_4$ ← objective function

subject to: ?

$$\begin{cases} -3x_3 + 3x_4 \le 6 \\ -8x_3 + 4x_4 \le 4 \\ x_3 , x_4 \ge 0 \end{cases}$$

$\max\ z = -3x_3 + x_4$ non-basic variables

subject to: basic

$$\begin{cases} x_1 = 6 + 3x_3 - x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ \underbrace{x_1 \ldots x_4 \ge 0}_{\text{admissibility}} \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

augmented problem,

a dictionary

Step 0: moving to the augmented form

· Once we get here, a few things can happen.

**ex1.**

$\max\ z = -3x_3 - x_4$ ← objective function

subject to:

$$\begin{cases} -3x_3 + x_4 \le 6 \\ -8x_3 + 4x_4 \le 4 \\ x_3 , x_4 \ge 0 \end{cases} \Rightarrow \begin{cases} x_1 = 6 + 3x_3 - x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \ldots x_4 \ge 0 \end{cases}$$

All variables appear in the objective function have a negative sign.

$\therefore\ x_3 = x_4 = 0,\ x_1 = 6,\ x_2 = 4$ ← admissible solution ($z=0$)

But since $x_3$ and $x_4$ appear with negative sign, even slightly increasing $x_3$, $x_4$, the $z$ would drop value. $\Rightarrow \boxed{\max\ z = 0}$

**ex2.**

$\max\ z = -3x_3$

$$\begin{cases} x_1 = 6 + 3x_3 - x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \ldots x_4 \ge 0 \end{cases}$$

In this case, I'm forced to take $x_3 = 0$, leading to $\max\ z = 0$.

I have multiple max points. $x_3 = 0$, $x_4 \ge 0$. Provided that $x_1, x_2 \ge 0$

$$\Rightarrow \begin{matrix} x_1 = 6 - x_4 \\ x_2 = 4 - 4x_4 \end{matrix} \Rightarrow \begin{cases} 6 - x_4 \ge 0 \\ 4 - 4x_4 \ge 0 \end{cases} \Rightarrow 0 \le x_4 \le 1$$

$$\vec{x}_i = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 6-t \\ 4-4t \\ 0 \\ t \end{bmatrix} \quad 0 \le t \le 1$$

family of admissible vectors giving the maximum value.

**ex3.**

$\max\ z = -3x_3 + x_4$

$$\begin{cases} x_1 = 6 + 3x_3 - 3 \cdot x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \ldots x_4 \ge 0 \end{cases}$$

In this case, $x_4$ appears with the positive sign in the objective function, so I can increase the value of $x_4$ in the initial admissible solution: $x_3 = x_4 = 0$, $x_1 = 6$, $x_2 = 4$.

and increase the value of z

$$\left|\begin{array}{l} x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \dots x_4 \geq 0 \end{array}\right.$$

of $x_4$ in the initial admissible solution: $x_3 = x_4 = 0$
$x_1 = 6,\ x_2 = 4.$

and increase the value of $z$.

How much to increase $x_4$?

$$\begin{cases} x_1 = 6 - 3x_4 \geq 0 \\ x_2 = 4 - 4x_4 \geq 0 \end{cases} \Rightarrow \begin{cases} \frac{6}{3} \geq x_4 \\ 1 \geq x_4 \end{cases} \Rightarrow \begin{cases} 2 \geq x_4 \\ 1 \geq x_4 \end{cases} \Rightarrow \boxed{0 \leq x_4 \leq 1}$$

$x_3 = 0,\ x_4 = 1,\ x_1 = 6 - 3 = 3,\ x_2 = 4 - 4 = 0$

max $x_3$, $x_2 = 0$, So I can use these as independent variables.

$$x_2 = 4 + 8x_3 - 4x_4$$
$$\Rightarrow x_4 = 1 + 2 \cdot x_3 - \frac{x_2}{4} = \boxed{1 - 0.25\,x_2 + 2x_3}$$

$$x_1 = 6 + 3 \cdot x_3 - 3 \cdot x_4$$
$$\Rightarrow \ = 6 + 3x_3 - 3(1 - 0.25x_2 + 2x_3)$$
$$= \boxed{3 + 0.75x_2 - 3x_3}$$

new objective function $z = -3x_3 + x_4$

$z = 1 - 0.25x_2 + 2x_3 - 3x_3$

$z = 1 - 0.25x_2 - x_3$ ← all variables have negative sign.

subject to:
$$\begin{cases} x_4 = 1 - 0.25x_2 + 2x_3 \\ x_1 = 3 + 0.75x_2 - 3x_3 \\ x_1 \dots x_4 \geq 0 \end{cases}$$

So $\boxed{\text{in this case, the maximum } z = 1 \text{ and } x_2, x_3 = 0,\ x_4 = 1,\ x_1 = 3}$

↑ admissible vector that gives the max value.

**ex4.** max $z = +3x_3 - x_4$

subject to:
$$\begin{cases} x_1 = 6 + 3x_3 - 3x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \cdots x_4 \geq 0 \end{cases}$$

$x_3 = x_4 = 0$ admissible before
$x_1 = 6,\ x_2 = 4$

In this case if increase the value of $x_3$, I improve the objective function.

$$\begin{cases} x_1 = 6 + 3x_3 \geq 0 \\ x_2 = 4 + 8x_3 \geq 0 \end{cases}$$
$$\Downarrow$$

← move negative, change signs.

$$\begin{cases} 6 \geq -3x_3 \\ 4 \geq -8x_3 \end{cases} \Rightarrow \begin{cases} -2 \leq x_3 \\ -\frac{1}{2} \leq x_3 \end{cases} \Rightarrow$$ no problem since $x_3 \geq 0$.
Thus, any positive $x_3$ is okay.

$$\therefore \boxed{\max z = \infty}$$

**ex 5** max: $z = -2x_1 - x_2 + 1$

subject to:

admissible region
$$\begin{cases} -x_1 + x_2 \leq -1 \\ -x_1 - 2x_2 \leq -2 \\ x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases} \Rightarrow \begin{cases} w_1 = -1 + x_1 - x_2 \\ w_2 = -2 + x_1 + 2x_2 \\ w_3 = 1 - x_2 \\ w_1, w_2, w_3, x_1, x_2 \geq 0 \end{cases}$$

max $z = 1$ ? no. if $x_1, x_2 = 0$, $w_1, w_2 < 0$

Only the other hand, $x_1 = x_2 = 0$ lead to $w_1 = -1$, $w_2 = -2$, $w_3 = 1$. NOT admissible!

region $\left| \begin{cases} x_2 \le 1 \\ x_1, x_2 \ge 0 \end{cases} \right.$ $\left| \begin{matrix} w_3 = 1 - x_2 \\ w_1, w_2, w_3, x_1, x_2 \ge 0 \end{matrix} \right.$

max $z = 1$ ? no. $\boxed{\text{if } x_1, x_2 = 0,\ w_1, w_2 < 0}$

ex 5' max $\zeta = -\lambda$ Auxiliary Problem (for initial solutions, always set the independent variables 0) — S.R. Liu

$$\begin{cases} w_1 = -1 + x_1 - x_2 + \lambda = -1 \\ w_2 = -2 + x_1 + 2x_2 + \lambda = -2 \\ w_3 = 1 - x_2 + \lambda \\ \lambda, w_1, w_2, w_3, x_1, x_2 \ge 0 \end{cases}$$

We pick the WORST condition.

The new problem has non-empty admissible region.

- Setting $\lambda = 0$ in any dictionary of this problem I get back a dictionary for the original problem.
- max $\zeta = 0$ $\Leftrightarrow$ there is one admissible solution for the original problem.

Take $x_1 = x_2 = 0$ and $\lambda \gg 0$

$$w_2 = -2 + x_1 + 2x_2 + \lambda$$
$$\Downarrow$$
$$\boxed{\lambda = 2 - x_1 - 2x_2 + w_2}$$

O.F $= \zeta = -2 + x_1 + 2x_2 - w_2 = -\lambda$

subject to:

$$\begin{cases} w_1 = -1 + \cancel{x_1} - x_2 + (2 - \cancel{x_1} - 2x_2 + w_2) \\ \lambda = 2 - x_1 - 2x_2 + w_3 \\ w_3 = 1 - x_2 + (2 - x_1 - 2x_2 + w_3) \\ w_i \ge 0,\ x_i \ge 0,\ \lambda \ge 0 \end{cases}$$

Maximum $-2 + x_1 + 2x_2 - w_2$

subject to: $\begin{cases} w_1 = 1 - 3x_2 + w_2 \\ \lambda = 2 - x_1 - 2x_2 + w_2 \\ w_3 = 3 - x_1 - 3x_2 + w_2 \\ w_i \ge 0,\ x_i \ge 0,\ \lambda \ge 0 \end{cases}$

Claim: max $\zeta = 0$
$\Updownarrow$
the admissible region is non-empty for the original problem

Suppose that $\begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \\ \lambda \end{bmatrix}$ is max, and admissible $x_i \ge 0$, $w_i \ge 0$, $\zeta \ge 0$

max $\zeta = 0$.

max $\underset{0}{\underset{\parallel}{\zeta}} = \zeta(x_1, x_2, w_1, w_2, w_3, \underset{-\lambda}{\underset{\parallel}{\lambda}})$

$\Rightarrow \lambda = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \\ \lambda \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ w_1 \\ w_2 \\ w_3 \\ 0 \end{bmatrix}$$

$$\begin{cases} w_1 = -1 + x_1 - x_2 + \cancel{\lambda} \ge 0 \\ w_2 = -2 + x_1 + 2x_2 + \cancel{\lambda} \ge 0 \\ w_3 = 1 - x_2 + \cancel{\lambda} \ge 0 \\ \lambda, w_1, w_2, w_3, x_1, x_2 \ge 0 \end{cases}$$

## Cont'd Jan 28th.

$$x_1 = x_2 = w_2 = 0$$
$$\Rightarrow w_1 = 1,\ \lambda = 2,\ w_3 = 3$$

Substitude $x_2$. $(w_2 = 0)$ $(x_1 = 0)$ since the coeffienre is 2.

Maximum $-2 + x_1 + 2x_2 - w_2$

subject to: $\begin{cases} w_1 = 1 - 3x_2 \\ \lambda = 2 - 2x_2 \\ w_3 = 3 - 3x_2 \\ w_i \ge 0,\ x_i \ge 0,\ \lambda \ge 0 \end{cases}$

$$\Downarrow$$

$$\begin{cases} 1 \ge 3x_2 \\ 2 \ge 2x_2 \\ 3 \ge 3x_2 \end{cases} \Rightarrow \left\{ x_2 \le \frac{1}{3} \right.$$

So I flip $x_2$ and $w_1$ ☆ ? why not $x_1, \lambda$, $x_2, \lambda$, or $x_1, w_1$

maximize $\zeta = -\lambda$

when ? $\lambda$ always on RHS ?

$$\begin{cases} x_2 = \frac{1}{3} - \frac{1}{3}w_1 + 0.33w_2 \\ x_1 = 1.33 - \lambda + 0.67w_1 + 0.33w_2 \\ w_3 = 0.67 + \lambda + 0.33w_1 - 0.33w_2 \\ w_i \ge 0,\ x_i \ge 0,\ \lambda \ge 0 \end{cases}$$

$\lambda$ always on RHS ?

$w_3 = 0.67 + \lambda + 0.33w_1 - 0.33w_2$

$w_i \geq 0, \ x_i \geq 0, \ \lambda \geq 0.$

$\Rightarrow$ max $z = 0$.
$w_1 = w_2 = 0$,
$x_2 = 0.33$, $x_1 = 1.33$, $w_3 = 0.67$

is admissible for the original problem.

setting $\lambda = 0$, we get a dictionary for the original problem.

max $z = -2x_1 - x_2 + 1$

$x_2 = 0.33 - 0.33w_1 + 0.33w_2$

$x_1 = 1.33 + 0.67w_1 + 0.33w_2$

$w_3 = 0.67 + 0.33w_1 - 0.33w_2$

$w_i \geq 0, \ x_i \geq 0,$

$z = -2x_1 - x_2 + 1$

$= -3 - w_1 - w_2$

So maximum $z = -3$. and $x_2 = 0.33$, $x_1 = 1.33$ is a maximum solution