2020-340-hw1

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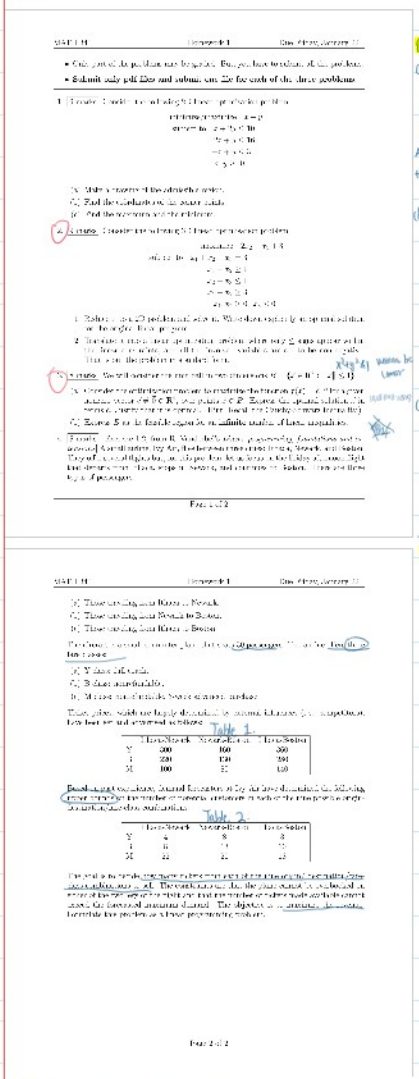

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Q1: $J(x,y) = x+y$

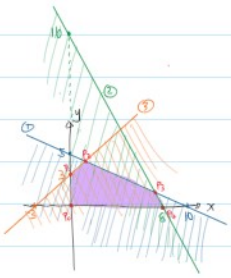
(a) admissible region:

① $x+2y=10$, $f(0,0)=0 \le 10$ (below)

② $2x+y=16$, $f(0,0)=0 \le 16$ (below)

③ $-x+y=3$, $f(0,0)=0 \le 3$ (below)

After drawing, the admissible area is the purple area.



(b) the corner points:

As shown in the figure in (a),

$P_0 = (0,0)$; $P_1 = (0,3)$; $P_4 = (8,0)$

P_2: $\begin{cases} x+2y=10 \\ -x+y=3 \Rightarrow y-3=x \end{cases} \Rightarrow 10-2y = y-3 \Rightarrow \begin{cases} y = \frac{13}{3} \\ x = \frac{4}{3} \end{cases} = P_2 = (\frac{4}{3}, \frac{13}{3}) = (1.33, 4.33)$

P_3: $\begin{cases} x+2y=10 \Rightarrow 2x+4y=20 \\ 2x+y=16 \end{cases} \Rightarrow 20-4y = 16-y \Rightarrow \begin{cases} y = \frac{4}{3} \\ x = 10-2y = \frac{22}{3} \end{cases} = P_3 = (\frac{22}{3}, \frac{4}{3}) = (7.33, 1.33)$

(c) Find the max/min:

$J(0,0) = 0+0 = 0$

$J(0,3) = 0+3 = 3$

$J(1.33, 4.33) = 1.33 + 4.33 = 5.66$

$J(7.33, 1.33) = 7.33 + 1.33 = 8.66$

$J(8,0) = 8+0 = 8$

$\Rightarrow \begin{cases} Max = J(7.33, 1.33) = \boxed{8.66} \\ Min = J(0,0) = \boxed{0} \end{cases}$

Q2: 1. Since $x_3 \le 0$, we define new $x_3 = |original\ x_3|$. Therefore, the question becomes:

maximize: $2x_2 + x_3 + 3 = z$, subject to
(change the signs before each x_3)

$$\begin{cases} x_1 + x_2 - x_3 = 3 \\ x_1 - x_2 \ge 1 \\ x_2 + x_3 \le 1 \\ x_1 + x_2 \ge 3 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

$x_1 = 3 - x_2 + x_3$

$$\therefore \begin{cases} 3 - x_2 + x_3 - x_2 \ge 1 \\ -x_2 - x_3 \ge -1 \\ 3 - x_2 + x_3 + x_2 \ge 3 \\ x_2 \ge 0, \ x_3 \ge 0 \end{cases} \Rightarrow \begin{cases} -3 + 2x_2 - x_3 \le -1 \\ x_2 + x_3 \le 1 \\ -3 - x_3 \le -3 \\ x_2 \ge 0, x_3 \ge 0 \end{cases} \Rightarrow \begin{cases} w_1 = 2 - 2x_2 + x_3 \\ w_2 = 1 - x_2 - x_3 \\ w_3 = x_3 \\ x_2, x_3, w_1, w_2, w_3 \ge 0 \end{cases}$$

max: $2x_2 + x_3 + 3 = z$

Admissible solution 1: $x_2 = 0$, $x_3 = 0$, $w_1 = 2$, $w_2 = 1$, $w_3 = 0$

$\Rightarrow \boxed{z = 3}$

Now, keep $x_3 = 0$, increase x_2.

$w_1 = 2 - 2x_2$; $w_2 = 1 - x_2$; $w_3 = 0$

$$\begin{cases} 2 - 2x_2 \ge 0 \\ 1 - x_2 \ge 0 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 \le 1 \\ x_3 = 0 \end{cases}$$

Admissible solution 2: $x_2 = 1$, $x_3 = 0$, $w_1 = 0$, $w_2 = 0$, $w_3 = 0$

$\Rightarrow \boxed{z = 5}$ (better than before)

Now: $z = 2x_2 + x_3 + 3$

subject to: $\begin{cases} w_1 = 2 - 2x_2 + x_3 \\ w_2 = 1 - x_2 - x_3 \\ w_3 = x_3 \\ x_2, x_3, w_1, w_2, w_3 \ge 0 \end{cases}$ (*) $\Rightarrow x_2 = 1 - \frac{1}{2}w_1 + \frac{1}{2}x_3$

$$\begin{cases} x_2 = 1 - \frac{1}{2}w_1 + \frac{1}{2}x_3 \\ w_2 = 1 - x_2 - x_3 \\ w_3 = x_3 \\ x_2, x_3, w_1, w_2, w_3 \ge 0 \end{cases}$$

put x_2 into w_2: $\Rightarrow \begin{cases} x_2 = 1 - \frac{1}{2}w_1 + \frac{1}{2}x_3 \\ w_2 = \frac{1}{2}w_1 - \frac{3}{2}x_3 \\ w_3 = x_3 \\ x_2, x_3, w_1, w_2, w_3 \ge 0 \end{cases}$

$\therefore z = 2x_2 + x_3 + 3 = 2 - w_1 + x_3 + x_3 + 3 = 5 - w_1 + 2x_3$

To maximum z, we assume $w_1 = 0$.

Then: $\begin{cases} x_2 = 1 + \frac{1}{2}x_3 \ge 0 \\ w_2 = -\frac{3}{2}x_3 \ge 0 \end{cases} \Rightarrow \begin{cases} x_3 = 0 \end{cases}$

Thus, $z = 5 - 0 + 2 \cdot 0 = 5$ (the same as previous)

$\therefore$ The optimal solution of maximum z is $\boxed{z = 5}$

with the admissible solution: $\boxed{x_2 = 1, x_3 = 0, w_1 = w_2 = w_3 = 0, x_1 = 2}$

2. In this part we still use the new $x_3 = |original\ x_3|$.

when LHS = RHS, if and only if LHS $\le$ RHS and LHS $\ge$ RHS. Thus:

$\Rightarrow$ maximize: $2x_2 + x_3 + 3$

subject to: $\begin{cases} x_1 + x_2 - x_3 \le 3 \\ x_1 + x_2 - x_3 \ge 3 \\ x_1 - x_2 \ge 1 \\ x_2 + x_3 \le 1 \\ x_1 + x_2 \ge 3 \\ x_1, x_2, x_3 \ge 0 \end{cases} \xrightarrow{\text{standard form}} \boxed{\begin{cases} x_1 + x_2 - x_3 \le 3 \\ -x_1 - x_2 + x_3 \le 3 \\ -x_1 + x_2 \le 1 \\ x_2 + x_3 \le 1 \\ -x_1 - x_2 \le 3 \\ x_1, x_2, x_3 \ge 0 \end{cases}}$

Q3: unit ball: $B = \{\vec{x} \in R^2 : \|\vec{x}\| \le 1\}$

(a) $f(x) = \vec{c} \cdot \vec{x}$

$f(x) = \|\vec{c}\| \, \|\vec{x}\| \cos\theta$ $\quad \|\vec{x}\| = \sqrt{x^2+y^2} \le 1$

Since $-1 \le \cos\theta \le 1$,

To maximize $f(x)$, set $\cos\theta = 1$. ($\therefore \theta = 0°$)

Then $f(x) = \|\vec{c}\| \, \|\vec{x}\|$ now.

Also, $\|\vec{x}\| \le 1$

to maximize $f(x)$, we set $\|\vec{x}\| = 1$.

Therefore, the maximum $\boxed{f(x) = \|\vec{c}\|}$

Because $\|\vec{x}\| = 1$, which means $\vec{x} = (1,0)$ or $(0,1)$.

$\vec{x}$ lies on the x-axis or y-axis.

Also, $\cos\theta = 1$, $\Rightarrow \theta = 0°$, the direction of $\vec{c}$ is the same as the direction of $\vec{x}$. (either in $\hat{x}$ or $\hat{y}$)

Also, since $f(x) = \vec{c} \cdot \vec{x} = \|\vec{c}\|$ ← magnitude of $\vec{c}$

which means $\vec{x}$ is the unit vector of $\vec{c}$.

$\therefore \boxed{\vec{x} = \frac{\vec{c}}{\|\vec{c}\|}}$

To prove max $f(x) = \|\vec{c}\|$, we set $\vec{c} = (u,v)$, $\vec{x} = (x,y)$.

And recall Cauchy-Schwarz inequality:

$f(x) = \vec{c} \cdot \vec{x} = ux + vy = \sqrt{u^2+v^2} \cdot \sqrt{x^2+y^2} \cdot \cos\theta$

$|\sqrt{u^2+v^2} \cdot \sqrt{x^2+y^2} \cdot \cos\theta| \le \sqrt{u^2+v^2} \cdot \sqrt{x^2+y^2}$ (since $\cos\theta \in [-1,1]$)

$\Rightarrow |ux+vy| \le \sqrt{u^2+v^2} \cdot \sqrt{x^2+y^2}$

$\|\vec{c}\| \cdot \|\vec{x}\|$ optimal, $\cos\theta = 1$

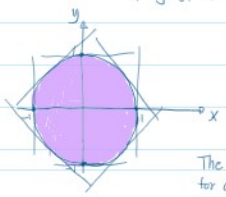
which satisfies the Cauchy-Schwarz inequality:

$\Rightarrow (ux+vy)^2 \le (u^2+v^2) \cdot (x^2+y^2)$

(b) Since B is a unit ball and $\{\vec{x} \in R^2 : \|\vec{x}\| \le 1\}$

subject to $x^2 + y^2 \le 1$

$\Rightarrow y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$



when $y=0$, $x = \pm 1$

when $x=0$, $y = \pm 1$

The linear inequalities are described as:

B: $x \cdot \cos\theta + y \sin\theta \le 1$

$x, y \ge 0$; $\theta \in [0, 2\pi]$

The feasible region of B for a infinite number of linear inequalities is as show on the left.

The slopes (linear inequalities) of the boundary change continuously. Therefore, it has infinite number of linear inequalities.

Q4: Let "1" denotes Ithaca – Newark

"2" denotes Newark – Boston

"3" denotes Ithaca – Boston

Also, set: "y_i" to be the number of passengers of the corresponding Y travel.

"b_i" to be the number of passengers of the corresponding B travel.

"m_i" to be the number of passengers of the corresponding M travel.

Therefore, from the Table 1, the max revenue is:

$$\boxed{\text{max revenue} = z = \begin{cases} 300 \cdot y_1 + 160\, y_2 + 360\, y_3 + \\ 220\, b_1 + 130\, b_2 + 280\, b_3 + \\ 100\, m_1 + 80\, m_2 + 140\, m_3 \end{cases}}$$

In addition, the airplane has 30 seats, and the Table 2 provides the upper bound of the passengers.

Therefore, z is subjected to the constraints below:

$$\begin{cases} y_1 + y_2 + y_3 + b_1 + b_2 + b_3 + m_1 + m_2 + m_3 \le 30 \quad // \text{max is 30 people} \\ 0 \le y_1 \le 4; \quad 0 \le y_2 \le 8; \quad 0 \le y_3 \le 3 \\ 0 \le b_1 \le 8; \quad 0 \le b_2 \le 13; \quad 0 \le b_3 \le 10 \\ 0 \le m_1 \le 22; \quad 0 \le m_2 \le 20; \quad 0 \le m_3 \le 18 \\ // \text{ according to the upper bounds in Table 2.} \\ \text{And passengers cannot be less than 0.} \end{cases}$$