Lecture 2 The Simplex Method

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$\max\ 5x_1 + 4x_2 + 3x_3 \quad \Rightarrow \quad \max\ z = 5x_1 + 4x_2 + 3x_3$

subject to: $\qquad\qquad$ subject to:

$$\begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases} \qquad \begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ \underbrace{w_1, w_2, w_3, x_1, x_2, x_3 \geq 0}_{\text{constraints}} \end{cases}$$

x_1, x_2, x_3 are independent variables. w_1, w_2, w_3 are dependent variables

$x_1 = 0 \quad x_2 = 0, \quad x_3 = 0$

$w_1 = 5 - 0 - 0 - 0 = 5 \geq 0$

$w_2 = 11 - 0 - 0 - 0 = 11 \geq 0$

$w_3 = 8 - 0 - 0 - 0 = 8 \geq 0$

$$\boxed{\begin{array}{l} x_1 = 0,\ x_2 = 0,\ x_3 = 0 \\ w_1 = 5,\ w_2 = 11,\ w_3 = 8 \end{array}}$$

is admissible (可接受的) solution.

i.e. solution with non-negative components

for this solution $\boxed{z = 0}$.

$z = 5x_1 + 4x_2 + 3x_3$ ← objective function

Now say I increase the value of x_1 and keep $x_2, x_3 = 0$. Then z gets larger and close to the maximum value.

On the other hand, if I increase it too much, the w_i's maybe negative.

since $x_2 = x_3 = 0$,

$w_1 = 5 - 2x_1; \quad w_2 = 11 - 4x_1; \quad w_3 = 8 - 3x_1; \quad \{w_1, w_2, w_3 \geq 0\}$

$$\begin{cases} 5 - 2x_1 \geq 0 \\ 11 - 4x_1 \geq 0 \\ 8 - 3x_1 \geq 0 \end{cases} \Rightarrow \begin{cases} 5/2 \geq x_1 \\ 11/4 \geq x_1 \\ 8/3 \geq x_1 \end{cases} \Rightarrow \boxed{x_1 = \tfrac{5}{2}}$$

$$\boxed{\begin{array}{l} \text{when } x_1 = \tfrac{5}{2},\ x_2 = x_3 = 0 \\ w_1 = 5 - 2\cdot\tfrac{5}{2} - 0 - 0 = 0 \\ w_2 = 11 - 4\cdot 5/2 - 0 - 0 = 1 \\ w_3 = 8 - 3\cdot\tfrac{5}{2} - 0 - 0 = \tfrac{1}{2} \end{array}}$$

new admissible solution

for this solution

$z = 5\cdot\tfrac{5}{2} + 4\cdot 0 + 3\cdot 0 = \boxed{12.5}$ better because $z_{new} = 12.5 > 0$

objective fuction is

subject to:

$$\begin{cases} w_1 = 5 - 2x_1 - 3x_2 - x_3 \quad (*) \\ w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \end{cases}$$

$(*)\ w_1 = 5 - 2x_1 - 3x_2 - x_3$

$\Updownarrow$

$2x_1 = 5 - w_1 - 3x_2 - x_3$

$$\begin{cases} W_2 = 11 - 4X_1 - X_2 - 2X_3 \\ W_3 = 8 - 3X_1 - 4X_2 - 2X_3 \\ W_1, W_2, W_3, X_1, X_2, X_3 \geq 0 \end{cases}$$

constraints

$$2X_1 = 5 - W_1 - 3X_2 - X_3$$

$$X_1 = \frac{5}{2} - \frac{1}{2}W_1 - \frac{3}{2}X_2 - \frac{1}{2}X_3$$

$$\begin{cases} X_1 = \frac{5}{2} - \frac{1}{2}W_1 - \frac{3}{2}X_2 - \frac{1}{2}X_3 \\ W_2 = 11 - 4X_1 - X_2 - 2X_3 \\ W_3 = 8 - 3X_1 - 4X_2 - 2X_3 \\ W_1, W_2, W_3, X_1, X_2, X_3 \geq 0 \end{cases}$$

put X_1 into W_2, W_3

dep

indep

$$\begin{cases} X_1 = \frac{5}{2} - \frac{1}{2}W_1 - \frac{3}{2}X_2 - \frac{1}{2}X_3 \\ W_2 = 1 + 2W_1 + 5X_2 \\ W_3 = \frac{1}{2} + \frac{3}{2}W_1 + \frac{X_2}{2} - \frac{X_3}{2} \\ W_1, W_2, W_3, X_1, X_2, X_3 \geq 0 \end{cases}$$

so, $W_1 = X_2 = 0$

$$\therefore\ z = 5\cdot\left[\frac{5}{2} - \frac{W_1}{2} - \frac{3X_2}{2} - \frac{X_3}{2}\right] + 4X_2 + 3X_3$$

$$= \frac{25}{2} - \frac{5W_1}{2} - \frac{15X_2}{2} - \frac{5X_3}{2} + 4X_2 + 3X_3 = 12.5 - 2.5W_1 - 3.5X_2 + 0.5X_3$$

new objective function

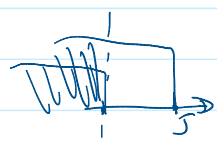
max

We play with X_3.

$$\begin{cases} X_1 = \frac{5}{2} - \frac{1}{2}X_3 \geq 0 \\ \cancel{W_2 = 1} \text{ ignore} \\ W_3 = \frac{1}{2} - \frac{1}{2}X_3 \geq 0 \end{cases} \Rightarrow \begin{cases} 5 \geq X_3 \\ 1 \geq X_3 \end{cases} \Rightarrow \boxed{X_3 = 1}$$

optimal choice

$\Rightarrow X_1 = 2$



$$\therefore\ \boxed{\begin{matrix} W_1 = 0, & X_2 = 0, & W_3 = 0 \\ X_1 = 2, & X_3 = 1, & W_2 = 1 \end{matrix}}$$

newnew admissible solution

$z = 12.5 + 0.5\cdot 1 = 13$ newnew solution

Now I should flip W_3 and X_3 in equations.

? why

$$W_3 = \frac{1}{2} + \frac{3W_1}{2} + \frac{X_2}{2} - \frac{X_3}{2}$$

$$\Downarrow$$

$$X_3 = 1 - 3W_1 + X_2 - 2W_3$$

substitude X_3 into X_1 and W_2:

$$X_1 = 2 - 2W_1 - 2X_2 + W_3$$

$$W_2 = 1 + 2W_1 + 5X_2$$

original X_1 ✓ Yes

$$\begin{cases} X_1 = 2 - 2W_1 - 2X_2 + W_3 \end{cases}$$

changing the objective function

original x_1 ✓ Yes

$$\Rightarrow \begin{cases} x_1 = 2 - 2w_1 - 2x_2 + w_3 \\ w_2 = 1 + 2w_1 + 5x_2 \\ x_3 = 1 - 3w_1 + x_2 - 2w_3 \\ w_{123},\ x_{123} \geq 0 \end{cases}$$

changing the objective function I got to:

$\max\ z = 13 - w_1 - 3x_2 - w_3$

In this formation of the problem the only thing I can do is to set:

$$w_1 = x_2 = w_3 = 0$$

$$x_1 = 2$$

$$w_2 = 1 \quad \Rightarrow \quad z = 13$$

$$x_3 = 1$$

$x_1 = 2,\ x_2 = 0,\ x_3 = 1$ is the optimum admissible solution corresponding to value of $\boxed{z = 13}$ max!

Another example: Jan 21st.

substitute positive signs.

$\max: z = 13 + 3x_5 - 6x_4 + 6x_6$ ← objective function

subject to:

$$\begin{cases} x_4 + 2x_5 \leq 4 \\ x_4 - x_5 \leq 1 \\ x_5 + 2x_6 - 2x_4 \leq 1 \\ x_4, x_5, x_6 \geq 0 \end{cases} \Rightarrow \begin{cases} x_1 = 4 - x_4 - 2x_5 \\ x_2 = 1 - x_4 + x_5 \\ x_3 = 1 - x_5 - 2x_6 + 2x_4 \\ x_1, x_2, \dots x_6 \geq 0 \end{cases}$$

$x_4, x_5, x_6 = 0$, $x_1 = 4,\ x_2 = 1,\ x_3 = 1$, $\Rightarrow z = 13$

$$x_3 = 1 - x_5 - 2x_6 + 2x_4$$

$$x_5 = 1 - x_3 - 2x_6 + 2x_4$$

$$\Rightarrow \begin{cases} x_1 = 4 - x_4 - 2 + 2x_3 + 4x_6 - 4x_4 \\ x_2 = 1 - x_4 + 1 - x_3 - 2x_6 + 2x_4 \\ x_5 = 1 - x_3 - 2x_6 + 2x_4 \end{cases} \Rightarrow \begin{cases} x_1 = 2 - 5x_4 + 2x_3 + 4x_6 \\ x_2 = 2 + x_4 - x_3 - 2x_6 \\ x_5 = 1 - x_3 - 2x_6 + 2x_4 \end{cases}$$

$$z = 13 + 3(1 - x_3 - \cancel{2x_6} + \cancel{2x_4}) - \cancel{6x_4} + \cancel{6x_6} = 16 - 3x_3$$

subject to $\begin{cases} x_1 = 2 - 5x_4 + 2x_3 + 4x_6 \\ x_2 = 2 + x_4 - x_3 - 2x_6 \\ x_5 = 1 - x_3 - 2x_6 + 2x_4 \\ x_1, \cdots x_5 \geq 0 \end{cases}$

assume $x_3 = 0$. if we wanna maximize.

$\therefore$ $z = 16$ there are no restrictions on the other variables.

So if we choose $x_4, x_6 \geq 0$. and so that,

$$\begin{cases} x_1 = 2 - 5x_4 \cancel{+2x_3} + 4x_6 \\ x_2 = 2 + x_4 \cancel{-x_3} - 2x_6 \\ x_5 = 1 \cancel{-x_3} - 2x_6 + 2x_4 \end{cases} \Rightarrow \begin{cases} x_1 = 2 - 5x_4 + 4x_6 \geq 0 \\ x_2 = 2 + x_4 - 2x_6 \geq 0 \\ x_5 = 1 - 2x_6 + 2x_4 \geq 0 \\ x_4, x_6 \geq 0 \end{cases}$$

So for all x_4 and x_4, solving this system, I get a different maximum point.

All optimum admissible vectors are of the form (x_4, x_5, x_6)

$= (x_4, 1 - 2x_6 + 2x_4, x_6)$

So the max $= 16$ and there is a whole 2D family leads to the solution.

Unboundness max: $z = x_2 + 2x_1$

subject to:

$$\begin{cases} -0.5x_2 - 1.5x_1 \leq 2.5 \\ x_1 - x_2 \leq 12 \\ -0.5x_2 - 0.5x_1 \leq 2.5 \\ x_1, x_2 \geq 0 \end{cases} \Rightarrow \begin{cases} x_3 = 2.5 + 0.5x_2 + 1.5x_1 \\ x_4 = 12 - x_1 + x_2 \\ x_5 = 2.5 + 0.5x_2 + 0.5x_1 \\ x_1, x_2 \geq 0 \end{cases} \quad (*)$$

$$(*) \begin{cases} x_2 = 2x_3 - 3x_1 - 5 \\ x_4 = 7 - 4x_1 + 2x_3 \\ x_5 = x_3 - x_1 \\ x_1, x_2 \geq 0 \\ \cdots x_5 \end{cases} \Rightarrow z = 2x_3 - x_1 - 5$$

assume $x_1 = 0$

$$\begin{cases} x_2 = 2x_3 - 5 \geqslant 0 \\ x_4 = 7 + 2x_3 \geqslant 0 \\ x_5 = x_3 \\ x_1, x_2 \geqslant 0 \\ \cdots x_5 \end{cases} \Rightarrow \begin{cases} x_3 \geqslant \frac{5}{2} \\ x_3 \geqslant -\frac{7}{2} \\ x_3 \geqslant 0 \end{cases}$$

$\therefore$ maximum doesn't exist.