Lecture 6-W4

2021年2月4日 星期四 10:07

Duality Theory

$$\max \quad Z = 4\cdot x_1 + x_2 + 3\cdot x_3$$

subject to: $\begin{cases} x_1 + 4x_2 \leq 1 \\ 3\cdot x_1 - x_2 + x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

take constraints ① $\times 2$:

$$2\cdot(x_1 + 4\cdot x_2) \leq 1\cdot 2$$

$+$ ------ ② $\times 3$:

$$3(3x_1 - x_2 + x_3) \leq 3\cdot 3$$

$$\Rightarrow \quad 11\cdot x_1 + 5x_2 + 3\cdot x_3 \leq 11$$

(x_1, x_2, x_3) is admissible, then

$$11x_1 + 5x_2 + 3x_3 \leq 11$$
$$\rotatebox{90}{$\leq$}$$
$$4x_1 + x_2 + 3x_3 = Z$$
$$\Downarrow$$
$$Z \leq 11. \Rightarrow \text{maximum } \boxed{Z \leq 11}$$

What's special about factor 2 and 3?

To understand, we play the same game with two parameters y_1 and y_2.

$$y_1(x_1 + 4x_2) \leq y_1\cdot 1$$
$$+\ y_2(3x_1 - x_2 + x_3) \leq y_2\cdot 3$$

$$\Rightarrow (y_1 + 3y_2)x_1 + (4y_1 - y_2)\cdot x_2 + y_2\cdot x_3 \leq y_1 + 3y_2$$
$$\rotatebox{90}{$\leq$}$$
$$4x_1 + x_2 + 3x_3 = Z$$

to get this I need

$$y_1 + 3y_2 \geq 4$$
$$4y_1 - y_2 \geq 1$$
$$y_2 \geq 3$$

(y_1, y_2 are positive)

otherwise the inequalities flip!

$\Rightarrow$ y_1 and y_2 satisfy

$$\begin{cases} y_1 + 3y_2 \geq 4 \\ 4y_1 - y_2 \geq 1 \\ y_2 \geq 3 \\ y_1, y_2 \geq 0 \end{cases}$$

this is satisfied

$$y_1 + 3y_2 \geq Z$$
$$\Downarrow$$

$$y_1 + 3y_2 \geq z$$
$$\Downarrow$$
$$\boxed{y_1 + 3y_2 \geq \max z}$$

Each choice of y_1 and y_2 is going to generate a different estimate:

$$\max z \leq y_1 + 3y_2$$

what is the best possible estimate ? $\boxed{\max z \leq \min \delta}$

Ans : $\leftarrow$ Dual Optimization problem

$$\text{minimum} \quad y_1 + 3y_2 = \delta$$

$$\text{subject to} \begin{cases} y_1 + 3y_2 \geq 4 \\ 4y_1 - y_2 \geq 1 \\ y_2 \geq 3 \\ y_1, y_2 \geq 0 \end{cases}$$

Linear optimazation problem

$$\max z \leq \min \delta$$

Rmk: $\min \delta = \max -\delta$

$$\delta(x_{min}) \leq \delta(x) \leq \delta(x_{max})$$
$$\Downarrow$$
$$-\delta(x_{min}) \geq -\delta(x) \geq -\delta(x_{max})$$

meaning that:

- x_{min} is the max of $-\delta$
- x_{max} is the min of $-\delta$
- $\min \delta = \max -\delta$; $\min -\delta = \max \delta$

ex 3: $\max z = -3x_3 + x_4$

$$\text{S.T} \Rightarrow \begin{cases} x_1 = 6 + 3x_3 - 3x_4 \\ x_2 = 4 + 8x_3 - 4x_4 \\ x_1 \cdots x_4 \geq 0 \end{cases}$$

ex: $\max z = \cancel{c_0} \, c_1 \cdot x_1 \cdots + c_n \cdot x_n + c_0$

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n} \cdot x_n \leq b_1 \\ a_{21}x_1 + \cdots + a_{2n} \cdot x_n \leq b_2 \\ \quad \vdots \qquad\qquad \vdots \\ a_{m1} \cdot x_1 + \cdots + a_{mn} \cdot x_n \leq b_m \\ x_1 \cdots x_n \geq 0 \end{cases} \Rightarrow \begin{aligned} & y_1 \cdots y_m \geq 0 \\ & y_1 (a_{11} x_1 + \cdots a_{1n} \cdot x_n) \leq y_1 b_1 \\ + \; & y_2 (a_{21} \cdot x_1 + \cdots a_{2n} \cdot x_n) \leq y_2 \cdot b_2 \\ & \vdots \\ + \; & y_m (a_{m1} \cdot x_1 + \cdots a_{mn} \cdot x_n) \leq y_m \cdot b_m \end{aligned}$$

$$\Rightarrow \left(\sum_{j=1}^{m} y_j a_{j1}\right) \cdot x_1 + \cdots + \left(\sum_{j=1}^{m} y_j \cdot a_{jn}\right) x_n = \text{LHS}$$

$$\text{RHS} = y_1 \cdot b_1 + y_2 b_2 + \cdots y_m \cdot b_m$$

$$\left(\sum_{j=1}^{m} y_j a_{j1}\right) \cdot x_1 + \cdots + \left(\sum_{j=1}^{m} y_j \cdot a_{jn}\right) x_n + c_0 \leq \sum_{i=1}^{m} y_i \cdot b_j + c_0$$

$$\left(\sum_{j=1}^{m} y_j a_{j1}\right)\cdot x_1 + \cdots + \left(\sum_{j=1}^{m} y_j \cdot a_{jn}\right) x_n \; {\color{red}+c_0} \le \sum_{j=1}^{m} y_j \cdot b_j \; {\color{red}+c_0}$$

$\vee$? what about c_0 ? No c_0 ✓

$$c_0 + c_1 x_1 + \cdots + c_n x_n$$

$$\begin{cases} \sum_{j=1}^{m} y_j a_{j1} \geq c_1 \\ \vdots \\ \sum_{j=1}^{m} y_j \cdot a_{jn} \geq c_n \\ y_1 \dots y_m \geq 0 \end{cases}$$

and However we choose the admissible vectors $(x_1 \cdots x_n)$ and $(y_1 \cdots y_m)$, we have

$$z = \sum_{i=1}^{n} c_i \cdot x_i \le \sum_{j=1}^{m} b_j y_j = \delta$$

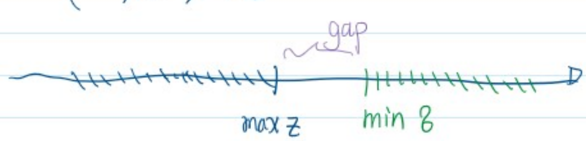
In particular $X = X_{max}$, $Y = Y_{min}$ we get

$$\boxed{\max z = z(X_{max}) \le \delta(Y_{min}) = \min \delta}$$ weak duality theorem.

This leads to

$$\begin{cases} \sum_{j=1}^{m} y_j a_{j1} \geq c_1 \\ \vdots \\ \sum_{j=1}^{m} y_j \cdot a_{jn} \geq c_n \\ y_1 \dots y_m \geq 0 \end{cases} \qquad \min \delta = \sum_{j=1}^{m} b_j \cdot y_j + c_0$$

Duality Problem



Duality theorem: There is no gap: $\max z = \min \delta$