# Lecture 1

2021年1月11日 星期一 14:54

$J(x_1, \dots x_n) \quad S \subseteq R^n$

$M = \max_{\vec{x} \in S} J(\vec{x}) \quad , \quad (\vec{x} = (x_1, \dots x_n))$

$m = \min_{\vec{x} \in S} J(\vec{x}) \quad ,$

$$S = \left\{ \vec{x} \in R^n \;\middle|\; \begin{array}{l} f_i(\vec{x}) = 0 \\ g_j(\vec{x}) \leq 0 \\ h_x(\vec{x}) \geq 0 \end{array} \right\}$$

LHS = RHS

if and only if.

LHS ≥ RHS and

LHS ≤ RHS

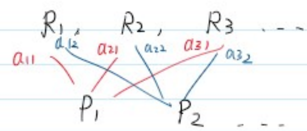
ie. $S$ is the set of points $\vec{x} =$ ?

$f_1(x_1, \dots, x_n) = 0$
$f_p(x_1, \dots, x_n) = 0$
$g_1(x_1, \dots, x_n) \leq 0$
$g_p(x_1, \dots, x_n) \leq 0$
$h_1(x_1, \dots, x_n) \geq 0$
$h_p(x_1, \dots, x_n) \geq 0$

$\boxed{\text{example:}}$ there is a company using different products $P_1 \dots P_n$; Let $R_1 \dots R_n$ be the row material needed in production.

$x_j$ = amount of $P_j$ produced.

$x_j \geq 0 \quad , \quad j = 1, \dots n$




$b_i$ = amount of row material $R_i$ available.

$\rho_i$ = $\boxed{\text{price unit}}$ of row material $R_i$.

$a_{ij}$ = quantity of row material $R_i$ need to produce $P_i$

cost of $P_j$ per unit $= \sum_{i=1} \overset{\text{row}}{a_{ij}} \cdot \rho_i$

**Assumption**: the production fascilities are small to the market size, so they cannot change the prices according to the rules of supply & demend.

Viewpoint of the manage.

$\sum_i a_{ij} \cdot \rho_i$ cost of prod unit.

sigma → $\sigma_j$ = market price per unit of $P_j$
$x_j$ = amount of $P_j$ produced (supply)

Revenue for $P_j$:

$(\text{supply}) \cdot (\text{price}) = \sigma_j \cdot x_j$

cost of production for $P_j$:

$\underbrace{(\sum_i a_{ij} \cdot \rho_i)}_{\text{cost of unit}} \cdot \underbrace{x_j}_{\text{units produced}}$

$(\underbrace{\sum_i a_{ij} \cdot \rho_i}_{\text{cost / unit}}) \cdot \underbrace{x_j}_{\text{units produced}}$

profit coming from $P_j$:

$$\underbrace{\delta_j x_j}_{\text{revenue}} - (\underbrace{\sum_i a_{ij} \cdot \rho_i) \cdot x_j}_{\text{cost}}$$

$$= (\underbrace{\delta_j - \sum_i a_{ij} \cdot \rho_i}_{c_j}) \cdot x_j .$$

$\Rightarrow$ Total profit:

$$\boxed{Z = C_1 \cdot X_1 + C_2 \cdot X_2 + \cdots + C_n \cdot X_n}$$

Would like to max $Z$. However, there are some constraints:

the amount of row material $R_i$ needed to produce $P_1 \dots P_n$ in quantity $x_1 \dots x_n$

$\sum_j a_{ij} \cdot x_j \le b_j \leftarrow$ amount of row material $R_i$ avaliable.

Max $z = c_1 \cdot x_1 + \cdots + c_n \cdot x_n$

$$\begin{cases} \sum_{j=1}^{n} a_{1j} \cdot x_j \le b_1 \\ \sum_{j=1}^{n} a_{2j} \cdot x_j \le b_2 \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \le b_m \\ x_1, \dots, x_n \ge 0 \end{cases}$$

$\boxed{\text{example:}}$ a school manager has to design a diet for the student population.

$X_1, X_2, X_3 \cdots X_n$. different foods from where to compose a meal.

$P_i$ = the price of $X_i$ per ounce of serving

$X_1$ = pasta

$P_1$ = 2 cents per ounce

? $X_2$ = apple

$P_2$ = 1 cent per once

Each $X_j$ has a different nutritional profile.

eg. $a_{1j}$ calories,

$a_{2j}$ carbs.

$\vdots$

(per ounce of serving)

Each student has a daily minimum requirements of $c_1$ calories, $c_2$ carbs ...

| Nutritions | food index ↑ Food ($X_1, X_2, X_n$) | minimun daily required |
|---|---|---|
| 1 cal | $a_{11}, a_{12} \cdots a_{1n}$ | $C_1$ |
| 2 carb | $a_{21}, a_{22} \cdots a_{2n}$ | $C_2$ |
| 3 | $a_{31}, a_{32} \cdots a_{3n}$ | $C_3$ |
| $\vdots$ | | $\vdots$ |
| m | $a_{m1}, a_{m2} \cdots a_{mn}$ | $C_m$ |

\$ of a meal

cost = $z = P_1 \cdot X_1 + P_2 \cdot X_2 + \cdots + P_n \cdot X_n$

$x_j$ = amount of food $X_j$ he is going to use in the menu.

Would like to minimize $z$: $*$ $X$ food ; $x$ amount.

$$\underbrace{a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} \cdot x_n}_{\text{amount of cal in a meal made of } X_1 \dots X_n}$$

$$\underbrace{a_{11}\cdot x_1 + a_{12}\cdot x_2 + \cdots + a_{1n}\cdot x_n}$$

amount of cal in a meal made of $x_1 \cdots x_n$

$$\underbrace{a_{21}\cdot x_1 + a_{22}\cdot x_2 + \cdots + a_{2n}\cdot x_n}$$

amount of carb in a meal made of $x_1 \cdots x_n$

Consider that the school has budget of $M$ thousand CAD per school day.
also, impose the condition:

$$p_1\cdot x_1 + \cdots + p_n\, x_n \le \frac{M-C}{N}$$

$C$ = cost of meal
$N$ = number of student

This leads to another opitimazation problem

$$\min\ Z = p_1\cdot x_1 + \cdots + p_n\cdot x_n$$

$$\begin{cases} a_{11}\cdot x_1 + a_{12}\cdot x_2 \cdots + a_{1n}\cdot x_n \ge c_1 \\ \vdots \\ a_{m1}\cdot x_{m1} + a_{m2}\cdot x_2 + \cdots a_{mn}\cdot x_n \ge c_m. \\ p_1\cdot x_1 + \cdots + p_n\cdot x_n \le \frac{M-C}{N} \\ x_1 \cdots x_n \ge 0 \end{cases}$$

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Jan 14th.

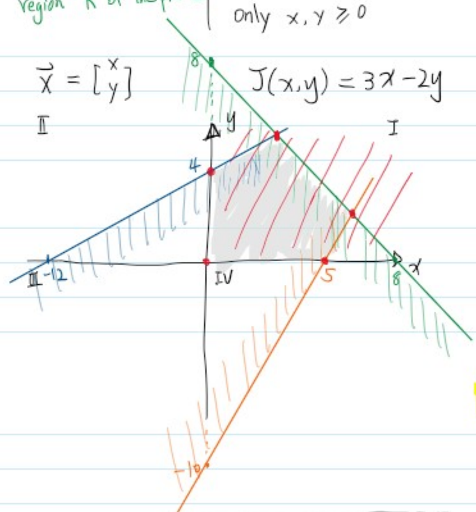
## 2D Example

min/max $J(x,y)$ ← $J$: some functions (no need to be linear)

subject to: $\begin{cases} -x+3y \le 12 \\ 2x - y \le 10 \\ x+y \le 8 \\ \text{only } x, y \ge 0 \end{cases}$

this defines a region $R$ of the plane →

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad J(x,y) = 3x - 2y$




Lines: $-x+3y = 12$ $\quad (x - 3y = -12)$ ↓

find the location of the origin to find $\ge, \le$

$2x - y = 10$

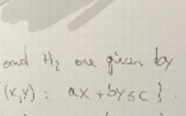
origin: $2(0) - (0) \le 10$ ↑

$x + y = 8$

$0 + 0 \le 8$ ↓

Rmark:

Rmrk: $ax + by = c$
defines a line tha cuts the plane in two half spaces $H_1$ and $H_2$




$H_1$ and $H_2$ are given by
$\{(x,y) : ax + by \le c\}$.
$\{(x,y) : ax + by \ge c\}$

**Claim:** $J$ attains its max/min at the corner points.

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$J(\vec{x})$


$\vec{x} = t\vec{P} + (1-t)\vec{Q}$

$0 \le t \le 1$

↳ $J(x,y) = 3x - 2y$

vertixes: $(0,0)$, $(0,4)$, $(5,0)$

$-x+3y = 12$ & $x+y = 8 \Rightarrow (3,5)$

**Step 1:** the min/max is on the boundary.

($\vec{x}$ is a point)


$\therefore\ \vec{x} = t\vec{P} + (1-t)\vec{Q}$

$J(\vec{x}) = J(t\vec{P} + (1-t)\vec{Q})$

vertixes: $(0,0)$, $(0,4)$, $(5,0)$

$-x+3y=12$ & $x+y=8 \Rightarrow (3,5)$

$2x-y=10$ & $x+y=8 \Rightarrow (6,2)$

$J(0,0)=0$

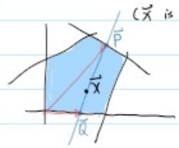
$J(0,4)=-8$

$J(3,5)=-1$

$J(6,2)=14$

$J(5,0)=15$

$\boxed{\max J = 15, \quad \min J = -8.}$ solution



($\vec{x}$ is a point)

$\therefore \vec{x} = t\cdot\vec{p} + (1-t)\vec{Q}$

$J(\vec{x}) = J(t\vec{p}+(1-t)\vec{Q})$

$= t\cdot J(\vec{p}) + (1-t)J(\vec{Q})$

$f(t) = t\cdot J(\vec{p}) + (1-t)J(\vec{Q})$ is a function of $t$.

$J(\vec{p})$ and $J(\vec{Q})$ are just numbers.

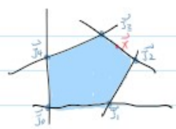
So max/min $f(t)$, $0\le t\le 1$ is either $f(0)$ or $f(1)$

$f(0) = J(\vec{Q})$

$f(1) = J(\vec{p})$

Thus when I ristrinct $J$ to the segement joining two boundarys points I get min/max on the boundary points.

Step 2: indeed it's on the vertives



$\vec{x} = t\cdot\vec{J_3} + (1-t)\vec{J_2}$

$J(\vec{x}) = t\cdot J(\vec{J_3}) + J(\vec{J_2})\cdot(1-t)$

from step 1, I conclude that min/max on boundarys. so it may be two end points ?

## Standard form

min/max $J(x_1 \cdots x_n) = c_1x_1 + \cdots + c_nx_n$.

subject to:
$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n \overset{?}{\underset{\ge}{\le}} b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \overset{?}{\underset{\ge}{\le}} b_n \end{cases}$$

Note that:

$a_1x_1 + \cdots + a_nx_n \ge b$ or $a_1x_1 + \cdots + a_nx_n \le b$

$\Downarrow \qquad \Downarrow$

$(-a_1)x_1 + \cdots + (-a_n)x_n \le -b \qquad (-a_1)x_1 + \cdots + (-a_n)x_n \ge -b$

So I can get rid of $\ge$ and turn it to $\ge$.

$a_1x_1 + \cdots + a_nx_n \le b \quad (*)$

$w_1 = b - a_1x_1 - \cdots - a_nx_n$

For $(*)$ to be true should impose $w_1 \ge 0$

Example
$$\begin{cases} x_1+x_2\le 2 \\ 2x_1+2x_2\ge 9 \\ x_1, x_2\ge 0 \end{cases} \Rightarrow \begin{cases} x_1+x_2\le 2 \\ -2x_1-2x_2\le -9 \\ x_1, x_2\ge 0 \end{cases}$$

standard form: when only "$\le$" appear

$$\begin{cases} w_1 = 2-x_1-x_2, \ w_1\ge 0 \\ -2x_1-2x_2\le -9 \\ x_1, x_2\ge 0 \end{cases} \Rightarrow \begin{cases} w_1 = 2-x_1-x_2, \ w_1\ge 0 \\ w_2 = -9+2x_1+2x_2, \ w_2\ge 0 \\ x_1, x_2\ge 0 \end{cases}$$

min/max $J(x_1, \ldots x_n) = c_1x_1 + \cdots + c_n\cdot x_n$ subject to:
$$\Rightarrow \begin{cases} w_1 = 2-x_1-x_2, \\ w_2 = -9+2x_1+2x_2, \\ x_1, x_2, w_1, w_2\ge 0 \end{cases}$$

This is what we call a linear problem in standard form.

dictionary formulation of the optimazation problem

Example cont'd:

Max $3x_1+2x_2$.

$\{ x_1+x_2\le 2$

turn to in standard form.

Example cont'd.

$$\text{Max } 3x_1 + 2x_2.$$

$$\begin{cases} x_1 + x_2 \le 2 \\ 2x_1 + 2x_2 = 9 \\ x_1, x_2 \ge 0 \end{cases}$$

turn to in standard form.

Rmk: $A = B$.
if and only if $A \ge B$ and $A \le B$ ☆

$2x_1 + 2x_2 = 9$ if only and if: $2x_1 + 2x_2 \ge 9$
$2x_1 + 2x_2 \le 9$

$\le$ $\boxed{max}$: $3x_1 + 2x_2$

$$\Rightarrow \begin{cases} x_1 + x_2 \le 2 \\ 2x_1 + 2x_2 \ge 9 \\ 2x_1 + 2x_2 \le 9 \\ x_1, x_2 \ge 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 \le 2 \\ -2x_1 - 2x_2 \le -9 \\ 2x_1 + 2x_2 \le 9 \\ x_1, x_2 \ge 0 \end{cases}$$ ★ standard form

Rmk: in $\boxed{min}$ ($\ge$) problem, we reverse the inequalities.

min:

$$\Rightarrow \begin{cases} -x_1 - x_2 \ge -2 \\ 2x_1 + 2x_2 \ge 9 \\ -2x_1 - 2x_2 \ge -9 \\ x_1, x_2 \ge 0 \end{cases}$$ ? standard form.