Multiple Integrals

MATH 104

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Contents

1 Double and Iterated Integrals over Rectangles

1.1 Double Integrals

We begin our investigations of double integrals by considering the simplest type of region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R,

$$R: \quad a \leq x \leq b, \ c \leq y \leq d$$

.

We subdivide R into small rectangles using a network of lines parallel to the x- and y-axes. The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a **partition** of R. A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$ where ΔA_k is the area of the kth small rectangle.

To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k, and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The **norm** of a partition P is written $\|R\|$, is the largest width or height of any rectangle in the partition. If $\|R\| = 0.1$ then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of $\boldsymbol{R}$ goes to zero, written $\|P\| \to 0$.

The resulting limit is written as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

as $\|P\| \to 0$ and the rectangles get narrow and short, their number n increases, so we can also write the limit as

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

with understanding that $\|P\| \to 0$, and hence $\Delta A_k \to 0$ as $n \to \infty$.

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\int\int_R f(x, y) \mathrm{d}A \quad \text{or} \quad \int\int_R f(x, y) \mathrm{d}x\mathrm{d}y$$

1.2 Double Integrals as Volumes

When $f(x, y)$ is a positive function over a rectangular area R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$. Each term $f(X_K, y_k)\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid that stands directly above the base ΔA_k.

1.3 Fubini's Theorem for Calculating Double Integrals

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2, 0 \leq y \leq 1$ in the xy-plane. If we apply the method of slicing, with slices perpendicular to the x-axis, then the volume is

$$\int_{x=0}^{x=2} A(x)\mathrm{d}x$$

,

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(X) = \int_{y=0}^{y=1} (4 - x - y)\mathrm{d}y$$

,

which is the area under the curve in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining the equations the volume of the entire solid is

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\mathrm{d}y\mathrm{d}x$$

.

Fubini's Theorem First Form

If $f(x, y)$ is continuous throughout the rectangular region $R : a \leq x \leq b, c \leq y \leq d$ then

$$\iint_R f(x, y)dA = \int_c^d \int_a^b f(x, y)dxdy = \int_a^b \int_c^d f(x, y)dydx$$

2 Double Integrals over General Regions

2.1 Double Integrals over Bounded, Nonrectangular Regions

Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b, g_1(x) \leq y g_2(x)$, with g_1 and g_2 continuous on $[a, b]$ then,

$$\iint_R f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

2. If R is defined by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$$

2.2 Finding the Limits of Integration

Using Vertical Cross-Sections

1. Sketch. Sketch the region of integration and label the bounding curves.

2. Find the y-limits of integration. Imagine a vertical line cuttting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are y-limits of integration and are usually functions of x.

3. *Find the x-limits of the integration.* Choose x-limits that include all the vertical lines through R. The integral shown in the figure is

2.3 Properties of Double Integrals

If $f(x,y)$and $g(x,y)$ are continuous on the bounded region R, then the following properties hold.

1. Constant Multiple:
$$\iint_R cf(x,y)\mathrm{d}A = c\cdot\iint_R f(x,y)\mathrm{d}A$$
2. Sum and difference:
$$\iint_R (f(x,y)\pm g(x,y))\mathrm{d}A = \iint_R f(x,y)\mathrm{d}A \pm \iint_R g(x,y)\mathrm{d}A$$
3. Domination:
   a)
$$\iint_R f(x,y)\mathrm{d}A \geq 0 \quad \text{if} \quad f(x,y)\geq 0 \textit{ on } R$$
   b)
$$\iint_R f(x,y)\mathrm{d}A \geq \iint_R g(x,y)\mathrm{d}A \quad \text{if} \quad f(x,y)\geq g(x,y) \textit{ on } R$$
4. Additivity: If R is the union of two non overlapping Regions R_1 and R_2, then
$$\iint_R f(x,y)\mathrm{d}A = \iint_{R_1} f(x,y)\mathrm{d}A + \iint_{R_2} f(x,y)\mathrm{d}A$$

3 Area by Double Integration

3.1 Areas of Bounded Regions in the Plane

If we take $f(x,y)=1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k,y_k)\Delta A_k = \sum_{k=1}^{n} \Delta A_k.$$

DEFINITION(S)

The **area** of a closed, bounded plane region R is

$$A = \iint_R \mathrm{d}A$$

3.2 Average Value

Average Value of f over R

$$\frac{1}{\text{area of R}} \iint_R f \mathrm{d}A$$

4 Double Integrals In Polar Form

4.1 Integrals In Polar Coordinates

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is "polar rectangle" whose sides have constant r- and θ-values. To avoid ambiguities when describing the region of integration with polar coordinates, we use polar coordinate points (r, θ) where $r \geq 0$.

Suppose a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq \alpha$ for every value of θ between α and β. Then R lies in a fan shaped region Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$ where $0 \leq \beta - \alpha \leq 2\pi$

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, \Delta 2r, \ldots, m\Delta r$ where $\Delta r = \frac{a}{m}$.The rays are given by

$$\theta = \alpha, \quad \theta = \alpha + \Delta\theta, \quad \theta = \alpha + 2\Delta\theta, \quad \ldots, \theta = \alpha + m'\Delta\theta = \beta$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into smaller patches called called "polar rectangles"

We number the polar rectangles that lie inside R, calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$ where ΔA_n. We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k. We form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k$$

.

If f is continuous throughout R, this sum will approach a limit as we define the grid to make Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R. In symbols,

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\mathrm{d}A$$

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of ΔR and $\Delta\theta$. For convenience we choose r_k to be the average of the radii of the inner and outer arcs bounding the kth polar rectangle ΔA_k. The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$. The radius of the outer arc is $r_k + (\Delta r/2)$

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2}\theta r^2$$

as can be seen by multiplying πr^2, the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner Radius:} \quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

$$\text{Inner Radius:} \quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta$$

Therefore

$$\begin{aligned}\Delta A_k &= \text{area of the large sector} - \text{area of the small sector} \\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}\left(2r_k\Delta r\right) = r_k\Delta r\Delta\theta.\end{aligned}$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect tot r and θ as

$$\iint_R f(r,\theta)dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta)rdrd\theta$$

4.2 Finding Limits of Integration

1. Sketch. Sketch the region and label the bounding curves.
2. Find the r limits of the integration.
3. Find θ-limits of the integration. Find the smallest and largest θ-values that bound R.

Area in Polar Coordinates

The area of a a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R rdrd\theta$$

4.3 Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$ and replace $dxdy$ by $rdrd\theta$ in the cartesian integral. Then supply polar limits of integration for the boundry of R. The cartesian integral becomes

$$\iint_R f(x,y)dxdy = \iint_G f(r\cos\theta, r\sin\theta)rdrd\theta$$

5 Triple Integrals in Rectangular Coordinates

6 Volume of a Region in Space

If F is the constant function whose value is 1, then the sums reduce to

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k = \sum_{k=1}^{n} 1 \cdot \Delta V_k = \sum_{k=1}^{n} \Delta V_k$$

As Δx_k, Δy_k, and Δz_k approach zero, the cells ΔV_k become small and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral.

$$\lim_{n\to\infty} \Delta V_k = \iiiint_D dV$$

DEFINITION(S)

The **volume** of a solid region D in space is

$$V = \iiint_D \mathrm{d}V$$

6.1 Finding limits of Integration in the order dz dy dx

We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of the integration for these integrals.

To evaluate

$$\iiint_D F(x,y,z)dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x. The limits of integration are found by projecting D onto the xy-plane and then projecting the resulting region onto the x-axis.

1. Sketch. Sketch the region D along with its shadow R in the xy-plane. Label the upper and lower bounding surfaces of D and the bounding curves of R.
2. Find the z-limits of the integration. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.

3. Find the y-limits of the integration. For each value of x, the line segment in R that lies above x is bounded below by the curve $y = g_1(x)$ and above by $y = g_2(x)$. These are the y-limits of integration.

4. Find the x-limits of the integration. Choose x-limits that include all the vertical lines through R. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)dzdydx$$

6.2 Average Value of a Function in Space

The average value of a function $F(x, y, z)$ over a solid region D is

$$\boxed{\frac{1}{\text{Volume of D}} \iiint_D F(x,y,z)dV}$$