Vector Valued Functions and Motions In Space

MATH 104

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1 Vector Valued Functions and Motions In Space

When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I.

$$x = f(t),\ y = f(t),\ z = f(t),\ t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t)), t \in I$, make up the **curve** in space that we call the particle's **path**. The equations and interval parametrize the curve.

A curve in space can also be represented in vector form. The vector

$$r(t) = \vec{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time t is the particle's position vector. The functions are **component functions** of the position vector. We think of the particle's path as the curve traced bt **r** during the time interval I.

A **vector-valued function** or **vector function** on a domain set D is a rule that assigns a vector in space to each element in D.

Real-valued functions are called scalar functions to distinguish them from vector functions. The components of **r** are the scalar functions of t. The domain of a vector-valued function is the common domain of its components.

EXAMPLE: Graph the vector function

$$r(t) = cos(t)\mathbf{i} + sin(t)\mathbf{j} + t\mathbf{k}$$

This vector function $\mathbf{r(}$t$\mathbf{)}$ is defined for all real values of t. The curve traced by $\mathbf{r}$ winds around the circular cylinder$x^2 + y^2 = 1$. The curve lied on the cylinder because $\mathbf{i}$ and $\mathbf{j}$ components of $\mathbf{r}$, being x- and y- coordinates of the tip of $\mathbf{r}$ satisfy the cylinder's equation:

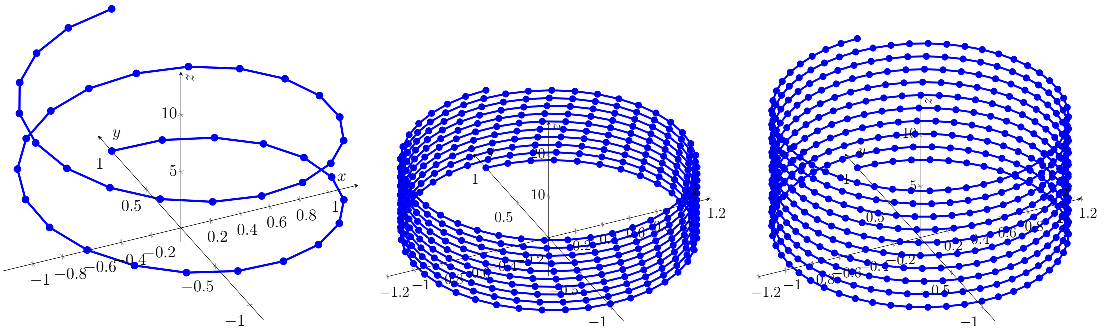
$$x^2 + y^2 = (\cos(t))^2 + (\sin(t))^2 = 1$$

The curve rises as the **k**-component $z = t$ increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a **helix**.

The equations

$$x = cos(t), y = sin(t), z = t$$

parametrize the helix.



(a) $r(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$

(b) $r(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 0.3t\mathbf{k}$

(c) $r(t) = \cos(5t)\mathbf{i} + \sin(5t)\mathbf{j} + t\mathbf{k}$

Figure 1: Multiple helix plots side by side

1.1 Limits and Continuity

DEFINITION

Let $\mathbf{r}(\mathbf{t}) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domaiN D, and $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$ then it can be shown that $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ precisely When

$$\lim_{t \to t_0} f(t) = L_1, \lim_{t \to t_0} G(t) = L_2, \lim_{t \to t_0} h(t) = L_3$$

EXAMPLE: If $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ then

$$\lim_{t \to \frac{\pi}{4}} \mathbf{r}(t) = \left(\lim_{t \to \frac{\pi}{4}} \cos(t)\right)\mathbf{i} + \left(\lim_{t \to \frac{\pi}{4}} \sin(t)\right)\mathbf{j} + \left(\lim_{t \to \frac{\pi}{4}} t\right)\mathbf{k} \tag{1}$$

$$= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k} \tag{2}$$

DEFINITION

A vector function $\mathbf{r}(t)$ is **continious at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continious** if it is continious at every point in its domain.

1.2 Derivatives and Motion

Suppose that $\mathbf{r}(\mathbf{t}) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is the position vector of a particle moving along a curve in space and that f, g, h, are differentiable functions of t. Then the difference between the particle's position at time t and time $t + \Delta t$ is the vector

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \left[\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}\right]\mathbf{i} + \left[\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}\right]\mathbf{j} + \left[\lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t}\right]\mathbf{k}$$

$$= \left[\frac{\mathrm{d}f}{\mathrm{d}t}\right]\mathbf{i} + \left[\frac{\mathrm{d}g}{\mathrm{d}t}\right]\mathbf{j} + \left[\frac{\mathrm{d}h}{\mathrm{d}t}\right]\mathbf{k}$$

DEFINITION

The vector function $\mathbf{r(t)} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative at t** if f, g and h have Derivatives at t. The derivative is the vector function

$$\mathbf{r}' = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}g}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}h}{\mathrm{d}t}\mathbf{k}$$

A vector function $\mathbf{r}$ is **differentiable** if it is differentiable at every point of its domain. The curve traced by $\mathbf{r}$ is **smooth** if $\mathrm{d}\mathbf{r}/\mathrm{d}t$ is **continious** and **NEVER 0**. A curve that is made up of finite number of smooth curves pieced together in a continious fashion is called **piecewise smooth**.

DEFINITION

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space then,

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$, when it exists, it is the particle's **acceleration vector**.

Differentiation Rules for Vector Functions

1. Dot Product Rule: $\frac{d}{dt}\left[\mathbf{u}(t) \cdot \mathbf{v}(t)\right] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{v}'(t) \cdot \mathbf{u}(t)$
2. Cross Product Rule: $\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{v}'(t) \times \mathbf{u}(t)$

1.3 Vector Functions of Constant Lenght

If $\mathbf{r}$ is a differentiable vector function of t and the lenght of $\mathbf{r}(t)$ is consistant, then

$$\mathbf{r} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 0$$

2 Integrals of Vector Functions; Projectile Motion

2.1 Integrals of Vector Functions

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval I if $\mathrm{d}\mathbf{R}/\mathrm{dt} = \mathbf{r}$ at each point of I. The set of all antiderivatives on $\mathbf{r}$ on I is the **indefinite integral** of $\mathbf{r}$ on I.

DEFINITION

The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$ denoted by $\int \mathbf{r}(t)$

$$\int \mathbf{r}(t)\mathrm{d}t = \mathbf{R}(t) + \mathbf{C}.$$

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DEFINITION

If the components of $\mathbf{r}(\mathbf{t}) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$ and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\mathrm{d}t = \left(\int_a^b f(t)\mathrm{d}t\right)\mathbf{i} + \left(\int_a^b g(t)\mathrm{d}t\right)\mathbf{j} + \left(\int_a^b h(t)\mathrm{d}t\right)\mathbf{k}$$

2.2 The Vector and Parametric Equations for Ideal Projectile Motion

If v_0 makes an angle α with the horizontal, then

$$\mathbf{v}_0 = (|\mathbf{v}_0|\cos\alpha)\mathbf{i} + (|\mathbf{v}_0|\sin\alpha)\mathbf{j}$$

Newton's second law of motion says that the force acting on the projectile is equal to the projectile's m times its acceleration ($\mathrm{m}\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$). If force is solely the gravitational force $-mg\mathbf{j}$ then,

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -mg\mathbf{j} \quad \text{and} \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -g\mathbf{j}$$

where g is the acceleration due to gravity. We find $\mathbf{r}$ as a function of t by solving the initial value problem.

$$\text{Differential equation: } \tfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -g\mathbf{j}$$

$$\text{Initial Conditions: } r = r_0 \text{ and } \tfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = v_0 \text{ when } t = 0$$

The first integration gives

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -(gt)\mathbf{j} + v_0$$

And the second integration gives

$$r = -\frac{1}{2}gt^2\mathbf{j} + v_0 t + r_0$$

Substituting values of v_0 and r_0 from given initial conditions gives

$$r = -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + 0$$

Ideal Projectile Motion Equation

$$r = (v_0\cos\alpha)t\mathbf{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}$$

Height, Flight Time and Range for Ideal Projectile Motion

$$\text{Maximum Height:}\quad y_{\max} = \frac{(v_0\sin\alpha)^2}{2g}$$

$$\text{Flight Time:}\quad t = \frac{2v_0\sin\alpha}{g}$$

$$\text{Range:}\quad R = \frac{{v_0}^2}{g}\sin(2\alpha)$$

3 Arc Lenght in Space

3.1 Arc Lenght Along a Space Curve

DEFINITION

The **lenght** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + \mathbf{z}(t)$ $a \geq t \geq b$, that is traced exactly once as t increases from $t = a$ to $t = b$ is

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}$$

The squere root in the definition of lenght is $|\mathbf{v}|$, the lenght of velocity vector $\mathrm{d}\mathbf{r}/\mathrm{d}t$. This enables us to write the formula for lenght in a shorter way.

Arc Lenght Formula

$$L = \int_a^b |\mathbf{v}|\mathrm{d}t$$

If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on c and a "directed distance"

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)\mathrm{d}\tau|$$

measured along C from the base point. This is the arc lenght formula for plane curves that have no z-component. If $t > t_0$, $s(t)$ is the distance along the curve from $P(t_0)$ to $P(t)$. If $t < t_0$ is the negative of the distance. Each value of s determines a point on C, and this parametrizes C with respect to s. We call s an **arc lenght parametrer** for the curve.

Arc Lenght with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} = \int_{t_0}^t |\mathbf{v}(\tau)|\mathrm{d}\tau$$

If a curve $\mathrm{r}(t)$ is already given in terms of some parameter t and $s(t)$ is the arc lenght function given, then we may be able to solve for t as a function of $s : t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for $t : \mathbf{r} = \mathbf{r}(t(s))$. The new parametrization identifies a point on the curve with its directed distance along the curve from the base point.

EXAMPLE: This is an example for which we can actually find the arc lenght parametrization of a curve. If $t_0 = 0$, the arc lenght parameter along the helix

$$\mathbf{r}(t) = (\cos(t))\mathbf{i} + (\sin(t))\mathbf{j} + t\mathbf{k}$$

from t_0 to t is

$$\begin{aligned} s(t) &= \int_{t_0}^{t} |\mathbf{v}(\tau)| \mathrm{d}\tau \\ &= \int_{t_0}^{t} \sqrt{2} \mathrm{d}\tau \\ &= \sqrt{2}t \end{aligned}$$

Solving this equation for t gives $t = \frac{s}{\sqrt{2}}$. Substituting into the position vector $\mathbf{r}$ gives the following arc lenght parametrization for the helix

$$\mathbf{r}(t(s)) = \left(\cos\left(\frac{s}{\sqrt{2}}\right)\right)\mathbf{i} + \left(\sin\left(\frac{s}{\sqrt{2}}\right)\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}$$

3.2 Speed on a Smooth Curve

$$\boxed{\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{v}(t)|}$$

The equation says that the speed with which a particle moves along its path is the magnitude of $\mathbf{v}$. Although the base point $P(t_0)$ plays a role in defining s in the arc lenght formula with base point it plays no role here. The rate at whichn a moving particle covers distance along its path is independent of how far it is away from the base point.

Notice that $\frac{\mathrm{d}s}{\mathrm{d}t} > 0$ since, by definition $|\mathbf{v}|$ is never zero for a smooth curve.

3.3 Unit Tangent Vector

We already know that the velocity vector $v = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ is tangent to the curve $\mathbf{r}(t)$ and the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the smooth curve, called the **unit tangent vector**. The unit tangent vector $\mathbf{T}$ is a differentiable function of t whenever $\mathbf{v}$ is a differentiable function of$t$.

4 Curvature and Normal Vectors of a Curve

4.1 Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\mathbf{T} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}$ turns as the curve bends. Since $\mathbf{T}$ is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the *curvature*. The traditional symbol for the curvature function is the Greek κ ("kappa").

DEFINITION

If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$$

If $\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large. If $\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller.

If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter other than the arc length parameter s, we can calculate the curvature as

$$\begin{aligned}\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| &= \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}\right| \\ &= \frac{1}{\left|\frac{\mathrm{d}s}{\mathrm{d}t}\right|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| \\ &= \frac{1}{|v|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|\end{aligned}$$

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|$$

Where $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

EXAMPLE: Here we find the curvature of a circle. We begin with the parametrization

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$

of a circle of radius a. Then,

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2} = a$$

From this we find

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{v}}{|v|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \\ \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} &= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j} \\ \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| &= 1 \end{aligned}$$

Hence, for any value of the parameter t the curvature of the circle is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| = \frac{1}{a}$$

Among the vectors orthogonal to the unit tangent vector $\mathbf{T}$, there is one of particular significance because it points int the direction which the space curve is turning. Since $\mathbf{T}$ has constant lenght, the derivative $\mathrm{d}\mathbf{T}/\mathrm{d}s$ is orthogonal to $\mathbf{T}$. Therefore id we divide $\mathrm{d}\mathbf{T}/\mathrm{d}s$ by its lenght κ we obtain a unit vector $\mathbf{N}$ orthogonal to $\mathbf{T}$

DEFINITION

At point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}$$

The vector $\mathrm{d}\mathbf{T}/\mathrm{d}s$ points in the direction in which $\mathbf{T}$ turns as the curve bends. Therefore if we face in the direction of increasing arc lenght, the vector $\mathrm{d}\mathbf{T}/\mathrm{d}s$ points toward the right if $\mathbf{T}$ turns clockwise and toward the left if $\mathbf{T}$ turns counterclockwise. In other words, the principal normal vector $\mathbf{N}$ will point toward the concave side of the curve.

$$\begin{aligned} \mathbf{N} &= \frac{\mathrm{d}\mathbf{T}/\mathrm{d}s}{|\mathrm{d}\mathbf{T}/\mathrm{d}s|} \\ &= \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}\frac{\mathrm{d}\mathbf{t}/\mathrm{d}s}{|\mathrm{d}\mathbf{t}/\mathrm{d}s|} \\ &= \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|} \end{aligned}$$

Formula for Calculating N

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$

Notice that $T \cdot N = 0$ verifying that N is orthogonal to T

4.2 Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at the point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve

The **radius of curvature** of the curve at P is the radius of the circle of the curvature, which is

$$\text{Radius of curvature} = \quad \rho = \frac{1}{\kappa}$$

To find ρ we find κ and take the reciprocal. The **center of curvature** of curve at P is the center of the circle of the curvature.

EXAMPLE: Find and graph the osculating circle of the parabola $y = x^2$ at the origin.
Solution
We parametrize the parabola using the parameter $t = x$

$$r(t) = t\mathbf{i} + t^2\mathbf{j}$$

First we findd the curvature of the parabola (κ) at the origin.

$$v = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{i} + 2t\mathbf{j}$$
$$|v| = \sqrt{1^2 + 4t^2}$$

so that

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(1 + 4t^2\right)^{-1/2}\mathbf{i} + 2t\left(1 + 4t^2\right)^{-1/2}\mathbf{j}$$

From this we find

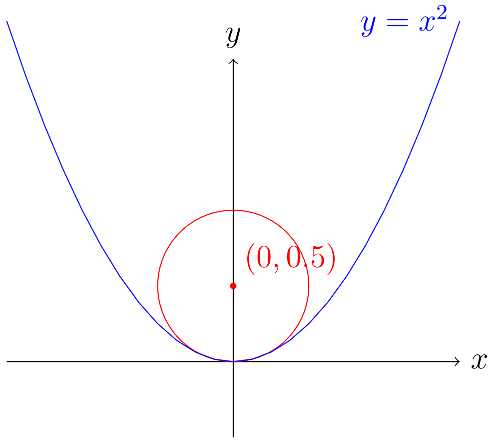
$$\frac{d\mathbf{T}}{dt} = -4t\left(1 + 4t^2\right)^{-3/2}\mathbf{i} + \left[2\left(1 + 4t^2\right)^{-1/2} - 8t^2\left(1 + 4t^2\right)^{-3/2}\right]\mathbf{j}$$

At the origin $t = 0$, so the curvature is

$$\begin{aligned}\kappa(0) &= \frac{1}{|\mathbf{v}(0)|}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}(0)\\ &= \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}|\\ &= (1)\sqrt{0^2 + 2^2} = 2\end{aligned}$$

Therefore the radius of curvature is $1/\kappa = 1/2$. At the origin we have $t = 0$ and $\mathbf{T} = \mathbf{i}$, so $\mathbf{N} = \mathbf{j}$. Thus the center of circle is (0,0.5). The equation of the osculating circle is

$$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$



4.3 Curvature and Normal Vectors for Space Curves

If a smooth curve in the space is specified by the position vector $\mathbf{r}(t)$ as a function of some paramete t, and if s is the arc lenght parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $\mathrm{d}\mathbf{r}/\mathrm{d}s = \mathbf{v}/|\mathbf{v}|$. The **curvature** in space is defined to be

$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|$$

just as for plane curves. The vector $\mathrm{d}\mathbf{T}/\mathrm{d}s$ is orthogonal to $\mathbf{T}$, and we define the principal unit normal to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{T}/\mathrm{d}t}{|\mathrm{d}\mathbf{T}/\mathrm{d}t|}$$