

Infinite Sequences and Series

MATH 104
Rüzgar ERİK

Contents

1	Sequences	2
1.1	Convergence and Divergence	2
1.2	Calculating Limits of Sequences	4
1.3	Recursive Definitions	7
2	Infinite Series	8
2.1	Geometric Series	9
2.2	The nth Term Test for a Divergent Series	11
2.3	The Integral Test	12

1 Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

in a given order. Each of a_1, a_2, a_3 represents a number. These are the terms of the sequence.

1.1 Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index n increases. This happens in the sequence

$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$$

whose terms approach 0 as n gets large, and in the sequence

$$\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, 1 - \frac{1}{n}, \dots\}$$

whose terms approach 1. On the other hand sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{n}, \dots\}$$

have terms that get larger than any number as n increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}\}$$

bounce back and forth between 1 and -1, never converging to a single value.

DEFINITION

The sequence $\{a_n\}$ **converges** to the number L if for every positive number ε there corresponds an integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n > N \quad (1)$$

If no such number L exists, we say that $\{a_n\}$ **diverges**. If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$, or simply $a_n \rightarrow L$ and call L the limit of the sequence.

It says that if we go far enough in the sequence, by taking the index n to be larger than some value N , the difference between a_n and the limit of the sequence becomes less than any preselected number $\varepsilon > 0$.

EXAMPLE: Show that **(a)** $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ and **(b)** $\lim_{n \rightarrow \infty} k = k$

(a) Let $\varepsilon > 0$ be given. We must show that there exists an integer N such that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon \quad \text{whenever} \quad n > N$$

The inequality $\left| \frac{1}{n} - 0 \right| < \varepsilon$ will hold if $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$.

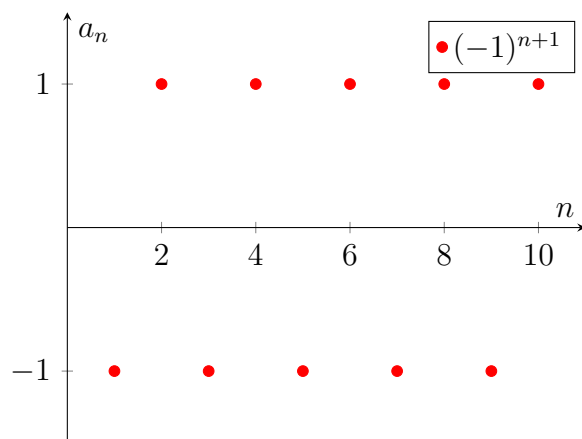
If N is any integer greater than $\frac{1}{\varepsilon}$, the inequality will hold for all $n > N$. This proves that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

(b) Let $\varepsilon > 0$ be given. We must show how there exists an integer N such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad n > N$$

Since $k - k = 0$, we can use any positive integer for N and the inequality will hold.

EXAMPLE: Show that sequence $\{1, -1, 1, -1, 1, -1, \dots, (-1)^{n+1}\}$ diverges



Suppose the sequence converges to some number L . Then the numbers in the sequence eventually get arbitrarily close to the limit L . This can't happen if they keep oscillating between 1 and -1. We can see this by choosing $\varepsilon = \frac{1}{2}$ in the definition of the limit. Then all terms of a_n of the sequence with index n larger than some N must lie within $\varepsilon = \frac{1}{2}$ of L . Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\varepsilon = \frac{1}{2}$ of L . It follows that $|L - 1| < \frac{1}{2}$, or equivalently, $\frac{1}{2} < L < \frac{3}{2}$. Likewise, the number -1 appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < \frac{1}{2}$, or equivalently, $-\frac{3}{2} < L < -\frac{1}{2}$. But the number L cannot lie in both of the intervals because they have no overlap. Therefore no such limit L exists and so the sequence diverges. Note that the same argument works for any positive number ε smaller than 1 not just $\frac{1}{2}$.

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As n increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$\lim_{n \rightarrow \infty} \sqrt{n} = \infty$$

In writing infinity as the limit of the sequence, we are not saying that the differences between the terms a_n and ∞ become small as n increases. Nor we are asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that a_n eventually gets and stays larger than any fixed number as n gets large. The terms of sequence might also decrease to negative infinity.

DEFINITION

The sequence a_n **diverges to infinity** if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If this condition holds we write

$$\lim_{n \rightarrow \infty} a_n = \infty \quad \text{or} \quad a_n \rightarrow \infty$$

Similarly, if for every number m there is an integer N such that for all $n > N$ we have a $a_n < m$ we say $\{a_n\}$ diverges to negative infinity and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \quad \text{or} \quad a_n \rightarrow -\infty$$

1.2 Calculating Limits of Sequences

THEOREM

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$.

1. Sum Rule: $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$
2. Difference Rule: $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$
3. Constant Multiple Rule: $\lim_{n \rightarrow \infty} (k \cdot b_n) = k \cdot B$ (any number k)
4. Product Rule: $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$
5. Quotient Rule: $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

One consequence of the theorem is that every nonzero multiple of a divergent sequence **diverges**. Suppose, to the contrary, that ca_n converges for some number $c \neq 0$. Then, by taking $k=1/c$ in the Constant Multiple Rule, we can see the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

THEOREM

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N , and if $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ then $\lim_{n \rightarrow \infty} b_n = L$ also.

EXAMPLE:

1. $\frac{\cos n}{n} \rightarrow 0$ because $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$
2. $\frac{1}{2^n} \rightarrow 0$ because $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$
3. $(-1)^n \frac{1}{n} \rightarrow 0$ because $\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$
4. If $|a_n| \rightarrow 0$, then $a_n \rightarrow 0$ because $-|a_n| \leq a_n \leq |a_n|$

THEOREM

Let $\{a_n\}$ be a sequence of numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all $\{a_n\}$, then $f(a_n) \rightarrow f(L)$

EXAMPLE: Show that $\sqrt{\frac{n+1}{n}} \rightarrow 1$

We know that $\frac{n+1}{n} \rightarrow 1$. Taking $f(x) = \sqrt{x}$ and $L=1$ theorem gives $\sqrt{\frac{n+1}{n}} \rightarrow \sqrt{1}$

EXAMPLE: The sequence $\{\frac{1}{n}\}$ converges to 0. By taking $a_n = \frac{1}{n}$, $f(x) = 2^x$ and $L = 0$ we can see that $2^{\frac{1}{n}} = \frac{1}{n} \rightarrow f(L) = 2^0 = 1$

THEOREM

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n \rightarrow \infty} a_n = L \text{ whenever } \lim_{x \rightarrow \infty} f(x) = L$$

EXAMPLE: Show that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$. The function $\frac{\ln x}{x}$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ will equal $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ if the latter exists. A single application of L'Hôpital's Rule shows that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$$

We conclude that $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

EXAMPLE: Does the sequence whose n th term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n \rightarrow \infty} a_n$. The limit leads to the indeterminate form 1^∞ . We can apply L'Hôpital's Rule if we change the form to $\infty \cdot 0$ by taking the natural logarithm of a_n .

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n \ln\left(\frac{n+1}{n-1}\right)$$

Then

$$= \lim_{n \rightarrow \infty} \ln a_n = \lim_{n \rightarrow \infty} n \ln\left(\frac{n+1}{n-1}\right) \quad (\infty \cdot 0 \text{ form})$$

$$= \lim_{n \rightarrow \infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{n \rightarrow \infty} \frac{-2/(n^2-1)}{-1/n^2} \quad \text{L'Hôpital's Rule}$$

$$= \lim_{n \rightarrow \infty} \frac{2n^2}{n^2-1} = 2.$$

Since $\ln a_n \rightarrow 2$ and $f(x) = e^x$ is continuous,

$$a_n = e^{\ln a_n} \rightarrow e^2$$

The sequence $\{a_n\}$ **converges** to e^2 .

THEOREM

- | | |
|--|---|
| 1. $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$ | 4. $\lim_{n \rightarrow \infty} x^n = 0 \quad (x < 1)$ |
| 2. $\lim_{n \rightarrow \infty} x^{1/n} = 1 \quad (x > 0)$ | 5. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$ |
| 3. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ | 6. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$ |

EXAMPLE:

a) $\frac{\ln n^2}{n} = \frac{2 \ln n}{n} = 2 \cdot 0 = 0$

d) $\left(-\frac{1}{2}\right)^n \rightarrow 0$

b) $\sqrt[n]{n^2} = n^{2/n} = 1$

e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \rightarrow e^{-2}$

c) $\sqrt[n]{3n} = 3^{1/n} (n^{1/n} \rightarrow 1 \cdot 1 = 1)$

f) $\frac{100^n}{n!} \rightarrow 0$

1.3 Recursive Definitions

So far we have calculated a_n directly from the value of n . But sequences are often defined recursively by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

EXAMPLE:

- a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$ define the sequence $1, 2, 3, \dots, n$ of positive integers.
- b) The statements $a_1 = 1$ and $a_n = a_{n-1} \cdot n$ for $n > 1$ define the sequence $1, 2, 6, 24, \dots, n!$ of factorials.
- c) The statements $a_1 = 1$ and $a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ define the sequence $1, 1, 2, 3, 5, \dots$ of Fibonacci Numbers.
- d) As we can see by applying Newton's method, the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2) / (\cos x_n - 2x_n)]$ for $n > 0$ define a sequence that, when converges, gives a solution to the equation $\sin x - x^2 = 0$

DEFINITION

A sequence a_n is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n . The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence a_n is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n . The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an unbounded sequence.

!Convergent Sequences Are Bounded!

EXAMPLE:

- a) The sequence $1, 2, 3, \dots, n, \dots$ has no upper bound because it eventually surpasses every number M . However, it is bounded below by every real number less than or equal to 1. The number $m=1$ is the greatest lower bound of the sequence.
- b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is bounded above by every real number greater than or equal to 1. The upper bound $M = 1$ is the least upper bound. The sequence is also bounded below by every number less than or equal to $\frac{1}{2}$, which is its greatest lower bound.

DEFINITION

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all n . That is $a_1 \leq a_2 \leq a_3 \leq \dots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all n . The sequence is **monotonic** if it is either nondecreasing or nonincreasing.

EXAMPLE:

- a) The sequence $1, 2, 3, \dots, n, \dots$ is nondecreasing.
- b) The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$ is nondecreasing.
- c) The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, \dots$ is nonincreasing.
- d) The constant sequence $3, 3, 3, \dots, 3, \dots$ is both nondecreasing and nonincreasing.
- e) The sequence $1, -1, 1, -1, 1, -1, 1, -1$ **NON MONOTONIC**.

A nondecreasing sequence that is bounded from above always has a least upper bound. Likewise, a nonincreasing sequence bounded from below always has a greatest lower bound. These results are based on the *completeness property* of the real numbers.

THEOREM

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

2 Infinite Series

An infinite series is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

The sum of the first n terms is an ordinary finite sum and can be calculated by normal addition. It is called the *n th partial sum*. As n gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of sequence approach a limit.

DEFINITION

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the **nth term** of the series. The sequence s_n defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k \quad \vdots$$

2.1 Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots \quad r=1/2, a=1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(\frac{-1}{3}\right)^{n-1} \quad r=-1/3, a=1$$

If $r=1$, the n th partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = n \cdot a$$

and the series diverges because $\lim_{n \rightarrow \infty} s_n = \pm \infty$ depending on the sign of a . If $r = -1$ the series diverges because the n th partial sums alternate between a and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \text{write the } n\text{th partial sum.}$$

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \quad \text{multiply } s_n \text{ by } r$$

$$s_n - rs_n = a - ar^n \quad \text{Subtract } rs_n \text{ from } s_n. \text{ Most of the terms on the right cancel.}$$

$$s_n(1 - r) = a(1 - r^n) \quad \text{Factor.}$$

$$s_n = \frac{a(1 - r^n)}{1 - r} \quad \text{We can solve for } s_n \text{ if } r \neq 1$$

If $|r| < 1$, then $r^n \rightarrow 0$ as $n \rightarrow \infty$ so $s_n \rightarrow \frac{a}{(1-r)}$. On the other hand, if $|r| > 1$, then $|r_n| \rightarrow \infty$ and the series **diverges**.

If $|r| < 1$, the geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ converges to $a/(1-r)$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| > 1$ the series diverges.

The formula $a/(1-r)$ for the sum of geometric series applies only when the summation index begins with $n=1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ or with the index $n=0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$

EXAMPLE: The geometric series with $a=1/9$ and $r=1/3$ is

$$\sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1-(1/3)} = \frac{1}{6}$$

EXAMPLE: The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} \cdots$$

is a geometric series with $a=5$ and $r = -1/4$. It converges to

$$\frac{a}{1-r} = \frac{5}{1+(1/4)} = 4$$

EXAMPLE:

Find the sum of the "telescoping" series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. The key observation is partial fraction decomposition.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Cancelling adjacent terms of the opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}$$

we now see that $s_k \rightarrow 1$ as $k \rightarrow \infty$. The series converges and its sum is 1

2.2 The nth Term Test for a Divergent Series

One reason that a series may fail to converge is that its terms don't become small.

EXAMPLE: The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of n terms is greater than n .

We now show that $\lim_{n \rightarrow \infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges.

THEOREM

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \rightarrow 0$.

The nth-Term Test for Divergence $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n$ fails to exist or different from zero.

EXAMPLE:

- a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \rightarrow \infty$
- b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \rightarrow 1$
- c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n \rightarrow \infty} (-1)^{n+1}$ does not exist.
- d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$

THEOREM

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series then

1. Sum rule: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. Difference Rule: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. Constant Multiple Rule: $\sum k a_n = k \sum a_n = kA$

1. Every **nonzero** constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge

CAUTION Remember that $\sum(a_n + b_n)$ can converge even if both $\sum a_n$ and $\sum b_n$ diverge. For example, $\sum a_n = 1 + 1 + 1 + \dots$ and $\sum b_n = (-1) + (-1) + (-1) + \dots$ diverge whereas $\sum(a_n + b_n)$ converges to 0.

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n - h$. To lower the starting value of the index h units, replace the n in the formula for a_n by $n + h$.

2.3 The Integral Test**Nondecreasing Partial Sums**

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all n . Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$ so

$$s_1 \leq s_2 \leq s_3 \leq \dots \leq s_n \leq s_{n+1}$$

Since the partial sums form a nondecreasing sequence, the Monotonic sequence theorem gives the following result.

Corollary

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

The Integral Test

We now introduce the integral Test with a series that is related to the harmonic series but whose n th term is $1/n^2$ instead of $1/n$.

THEOREM

Let a_n be a sequence of positive terms. Suppose $a_n = f(n)$ where f is a continuous, positive, decreasing function of x for $x \geq N$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x) dx$ both converge or both diverge.

Bounds For The Remainder In The Integral

Suppose $\{a_n\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$ and that $\sum a_n$ converges to S . Then the remainder $R_n = S - s_n$

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

If we add the partial sum s_n to each side of inequalities we get,

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq s_n + \int_n^{\infty} f(x) dx$$

THEOREM

Direct Comparison Test

Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$

1. If $\sum b_n$ converges then also $\sum a_n$
2. If $\sum a_n$ diverges then $\sum b_n$ also diverges.

The Limit Comparison Test

THEOREM

Suppose that $a_n > 0$ and $b > 0$ for all $n \geq N$

1. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges then $\sum a_n$ converges.
3. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges then $\sum a_n$ diverges.

DEFINITION

A series $\sum a_n$ **converges absolutely** or its absolutely convergent if the corresponding series of absolute values $\sum |a_n|$ converges.

THEOREM

The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

The Ratio Test**THEOREM**

Let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

Then, the series converges absolutely if $\rho < 1$ the series diverges if $\rho > 1$ or ρ is infinite the test is inconclusive if $\rho = 1$

The Root Test**THEOREM**

Let $\sum a_n$ be any series and suppose that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \rho$$

- a The series converges if $\rho < 1$
- b The series diverges if $\rho > 1$
- c **The Test Is inconclusive If $\rho = 1$**

Alternating Series and Conditional Convergence

THEOREM

The Alternating Series Test The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$$

converges if the following conditions are satisfied:

1. The u_n 's are all positive.
2. The u_n 's are eventually nondecreasing: $u_n \geq u_{n+1}$ for all $n \geq N$.
3. The limit of u_n as n approaches infinity is 0: $\lim_{n \rightarrow \infty} u_n = 0$.