Determinants of a Matrix

Shinboon Sangnoom

May 2025

1 Introduction

Sudoku is a now very popular puzzle consisting mainly of a 9×9 grid divided into 9 smaller 3×3 grids. The goal of the game is with some clues already given, fill the rest of the grid so that each row and column contain each number from 1 to 9 and each 3×3 compartment contain each numbers 1 to 9. For example, the left one is an example of an initial state of the puzzle, and the right is its solved state:

	2		5		1		9	
8			2		3			6
	3			6			7	
		1				6		
5	4						1	9
		2				7		
	9			3			8	
2			8		4			7
	1		9		7		6	

Unsolved Sudoku Board

4	2	*6*	5	*7*	1	*3*	9	*8*
8	*5*	*7*	2	*9*	3	*1*	*4*	6
1	3	*9*	*4*	6	*8*	*2*	7	*5*
9	*7*	1	*3*	*8*	*5*	6	*2*	*4*
5	4	*3*	*7*	*2*	*6*	*8*	1	9
6	*8*	2	*1*	*4*	*9*	7	*5*	*3*
7	9	*4*	*6*	3	*2*	*5*	8	*1*
2	*6*	*5*	8	*1*	4	*9*	*3*	7
3	1	*8*	9	*5*	7	*4*	6	*2*

Solved Sudoku Board

Interestingly, its geometry makes a great resemblance to matrices. I shall establish that the completed form of the example sudoku board can be represented as the following matrix:

$$\begin{bmatrix} 4 & 2 & 6 & 5 & 7 & 1 & 3 & 9 & 8 \\ 8 & 5 & 7 & 2 & 9 & 3 & 1 & 4 & 6 \\ 1 & 3 & 9 & 4 & 6 & 8 & 2 & 7 & 5 \\ 9 & 7 & 1 & 3 & 8 & 5 & 6 & 2 & 4 \\ 5 & 4 & 3 & 7 & 2 & 6 & 8 & 1 & 9 \\ 6 & 8 & 2 & 1 & 4 & 9 & 7 & 5 & 3 \\ 7 & 9 & 4 & 6 & 3 & 2 & 5 & 8 & 1 \\ 2 & 6 & 5 & 8 & 1 & 4 & 9 & 3 & 7 \\ 3 & 1 & 8 & 9 & 5 & 7 & 4 & 6 & 2 \end{bmatrix}$$

This paper will focus on finding the determinants of such matrices.

2 Determinants

Although it would be a great pain for one to calculate the determinant of these matrices manually, it is trivial work for one to write some code to find that the determinant of the matrix given earlier is -24163920, or even simpler, plug the matrix into an online calculator and let that do all the work for you. It would be a much more interesting case to work for a general case with questions such as whether any of them will yield a determinant of 0.

There are 2 main ways to approach this question. The first one would be to consider the board as a block matrix with a few choices for how to section it. The second one is to view the board as a 9-dimensional basis made out of 9 vectors.

3 Breaking the Fourth Wall

Due to the fact that this is a final project meant to be done in a few weeks, there was simply not enough time to see through either approach the whole way and work out the question this paper centers around.

4 Block Matrix Approach

One of the most intuitive way ways is to consider the sudoku board to be a 3×3 block matrix:[**taboga˙determinant˙2021**]

$$\begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix}$$

And each block is a 3×3 matrix, for example:

$$\begin{bmatrix} 4 & 2 & 6 \\ 8 & 5 & 7 \\ 1 & 3 & 9 \end{bmatrix}$$

Although approaches for block matrices deal with 2×2 block matrices, and anything bigger than that is reformatted into a 2×2 matrix. There are 2 options for how we reformat our matrix:

$$\begin{bmatrix} A & K \\ L & N \end{bmatrix} \text{ or } \begin{bmatrix} N & L \\ K & J \end{bmatrix}$$

with N being our 2×2 block of 3×3 grids. The rest is considering how establishing one block affects the possibilities of other blocks.

To try to limit the possibilities of determinants, I have calculated the determinants of all of the $9! = 362,880$ permutations. There are 635 possible determinants in total ranging from 0 to ±332.

One interesting property of the 3×3 grid is that since swapping rows multiplies the determinant by -1, transposing has no effect on the determinant, and column swaps can be described as row swaps in the transposed matrix, there are 72 permutations of a matrix that yield $|det(A)|$ determinant with half having $+n$ as their determinant, and the other half with $-n$ as their determinant.

To consider the possible permutations a block can have, one may check which numbers on which rows or columns hasn't yet been used and check for ones that are valid. Although if done by a computer, this no different from brute forcing through every single possibility, but the amount of computation needed is greatly reduced with the following formula:[2]

$$det(\begin{bmatrix} A & K \\ L & N \end{bmatrix}) = det(A)det(N - LA^{-1}K)$$

if A is invertible or

$$det(\begin{bmatrix} A & K \\ L & N \end{bmatrix}) = det(N)det(A - KN^{-1}L)$$

if N is invertible. If neither are invertible, there are a few options: 1) Rearrange the blocks in the other way, 2) Calculate $det(M^TM)$ instead for $det(M)^2$.

Calculating the determinants of sudoku boards this way reduces the amount of calculations required by turning it into a multiplication of a precalculated block, which only takes a few seconds for python code to do for all of the permutations, and a few matrix calculations reducing a great amount into a 3×3 matrix that we can rapidly calculate its determinant.

Unfortunately, there was not enough time to go through all of the boards.

5 9 Vectors Approach

With the rule where each row and column can only have a number once, one may also simply view the matrix version of a sudoku as a composition of 9 vectors:

$$\begin{bmatrix} \begin{pmatrix} 4 \\ 8 \\ 1 \\ 9 \\ 5 \\ 6 \\ 7 \\ 2 \\ 3 \end{pmatrix} & \begin{pmatrix} 2 \\ 5 \\ 3 \\ 7 \\ 4 \\ 8 \\ 9 \\ 6 \\ 1 \end{pmatrix} & \begin{pmatrix} 6 \\ 7 \\ 9 \\ 1 \\ 3 \\ 2 \\ 4 \\ 5 \\ 8 \end{pmatrix} & \begin{pmatrix} 5 \\ 2 \\ 4 \\ 3 \\ 7 \\ 1 \\ 6 \\ 8 \\ 9 \end{pmatrix} & \begin{pmatrix} 7 \\ 9 \\ 6 \\ 8 \\ 2 \\ 4 \\ 3 \\ 1 \\ 5 \end{pmatrix} & \begin{pmatrix} 1 \\ 3 \\ 8 \\ 5 \\ 6 \\ 9 \\ 2 \\ 4 \\ 7 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \\ 2 \\ 6 \\ 8 \\ 7 \\ 5 \\ 9 \\ 4 \end{pmatrix} & \begin{pmatrix} 9 \\ 4 \\ 7 \\ 2 \\ 1 \\ 5 \\ 3 \\ 8 \\ 6 \end{pmatrix} & \begin{pmatrix} 8 \\ 6 \\ 5 \\ 4 \\ 9 \\ 3 \\ 1 \\ 7 \\ 2 \end{pmatrix} \end{bmatrix}$$

With this view, one may guess that it's determinant is probably not 0. I don't have the time to prove you wrong, but I will leave that as a hypothesis. With this approach, it becomes more obvious that one can brute force through all of the permutations, slightly reducing the time to generate all of the permutations from 9^{81} ($\approx 2\times10^{77}$) to $9!^9$ ($\approx 10^{50}$), the same amount as the going through all possible block permutations.

It is important to note that calculating a 9×9 matrix is very computationally expensive; it took around 0.39 seconds for my me to calculate the determinant of each 9×9 matrix using cofactor expansion. It is very advisable to find ways to save time calculating them using faster algorithms and only calculating valid sudoku boards such as LU decomposition or calculating them as block matrices as described earlier, especially when there are a total of $\approx 6.7\times10^{21}$ valid ones.[1]

Unfortunately, there was not enough time to see this through.

6 Code

All of the code is available on Github: `https://github.com/Rvtar/Sudoku-Permutations` as a Jupyter Notebook file. Note that the all of the code is quite generalized and more so proofs of concept as I did not have time to dig further into this topic. I may come back to this later and add more stuff.

7 Conclusion

A sudoku board can be viewed as a 9×9 matrix. One way to interpret it is as a 3×3 block matrix of 3×3 matrices, and the other way as a composition of 9 vectors. In order to calculate the determinant of a sudoku board, both views may be utilized in their own ways, but methods involved may be put together to optimize the overall process.

References

[1] Ed Russel and Frazer Jarvis. "Mathematics of Sudoku II". In: (2006).

[2] Macro Taboga. *Determinant of a block matrix.* 2021. URL: `https://www.statlect.com/matrix-algebra/determinant-of-block-matrix` (visited on 05/15/2025).