METIS

Lesson 9: Calculus

Introduction

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Lecture Overview:

Goals of the lecture:

1. Line Equation
2. Functions
3. Derivatives and its properties
4. Maximum and Minimum
5. Partial Derivatives and Gradients
6. Anti-derivatives (Integral) and its properties
7. Area Under the Curve

Line Equation

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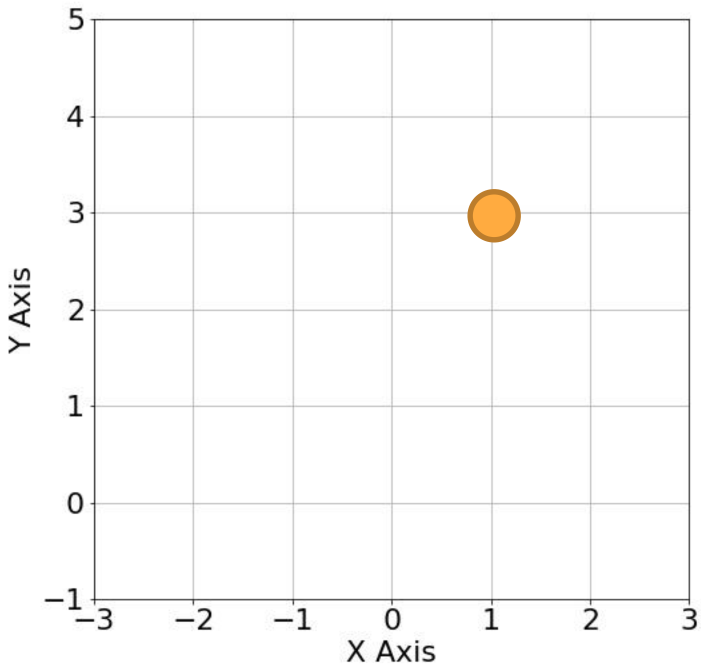
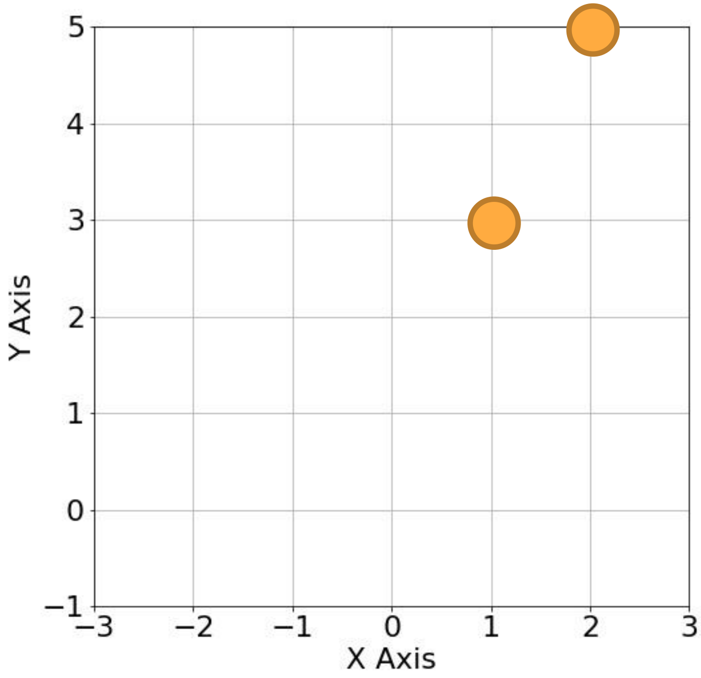
Coordinates in 2 Dimensions

- $p_1 = (1,3)$



Coordinates in 2 Dimensions

- $p_1 = (1,3)$
- $p_2 = (2,5)$



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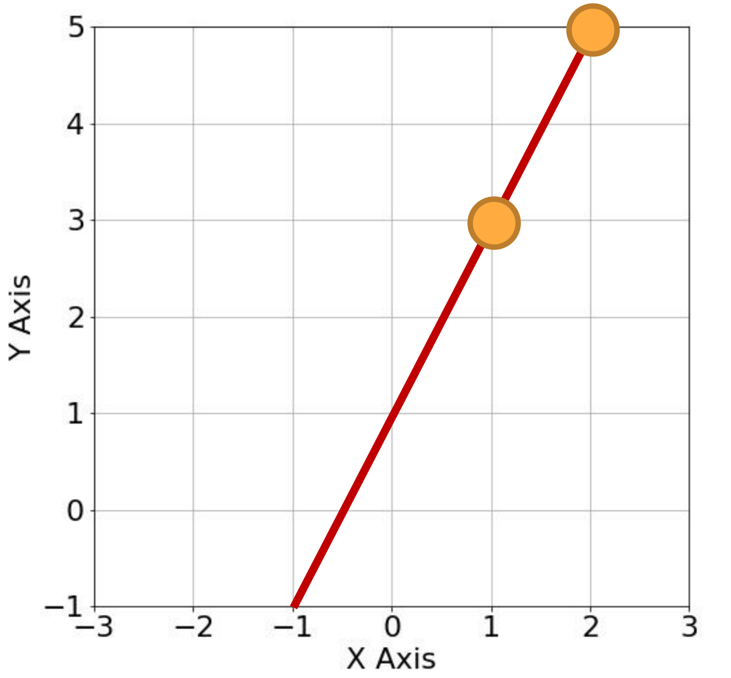
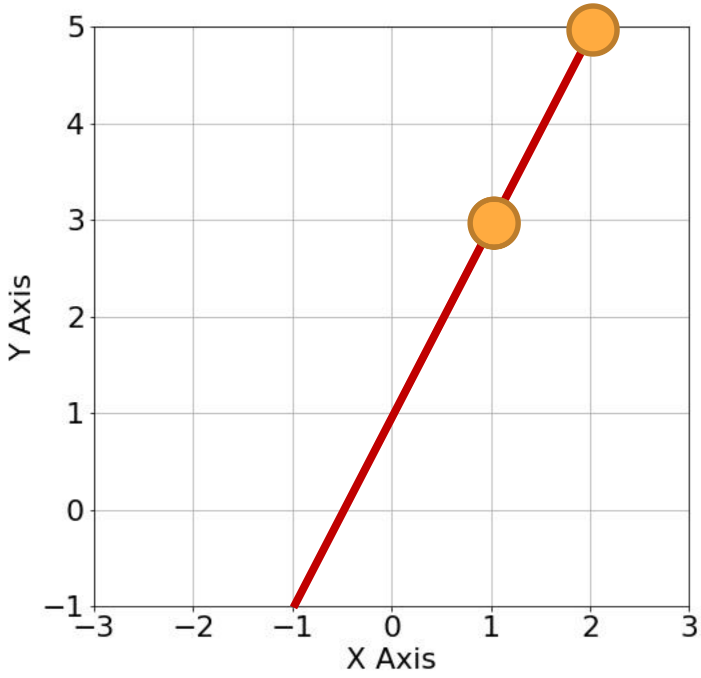


Line Equation

- $p_1 = (1,3)$
- $p_2 = (2,5)$

$$y = mx + b$$



Line Equation

- $p_1 = (1,3)$
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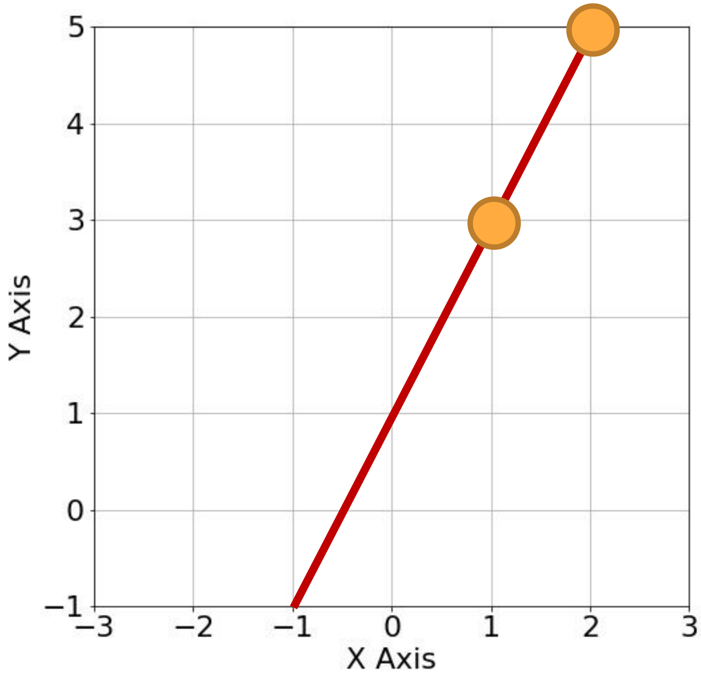
$$y = mx + b$$
$$y = 2x + 1$$



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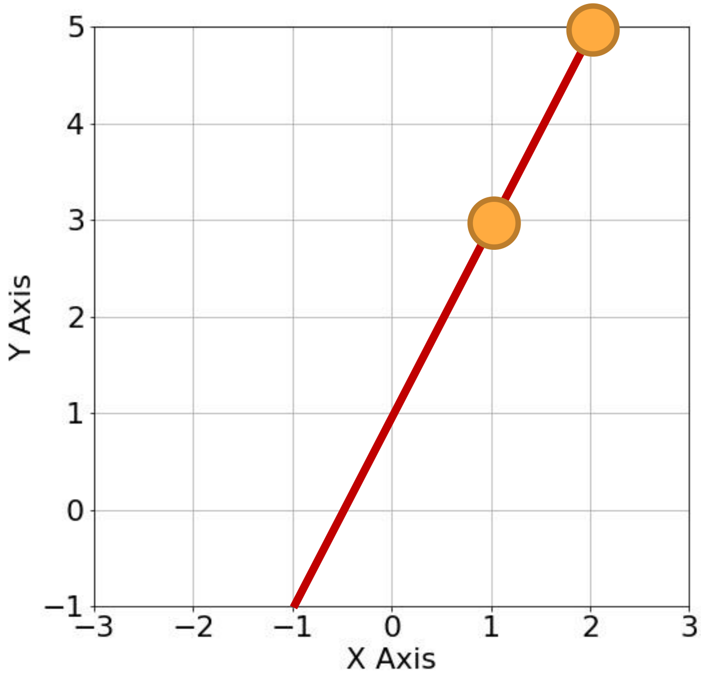
$$y = 2 \cdot 1 + 1 = 3$$



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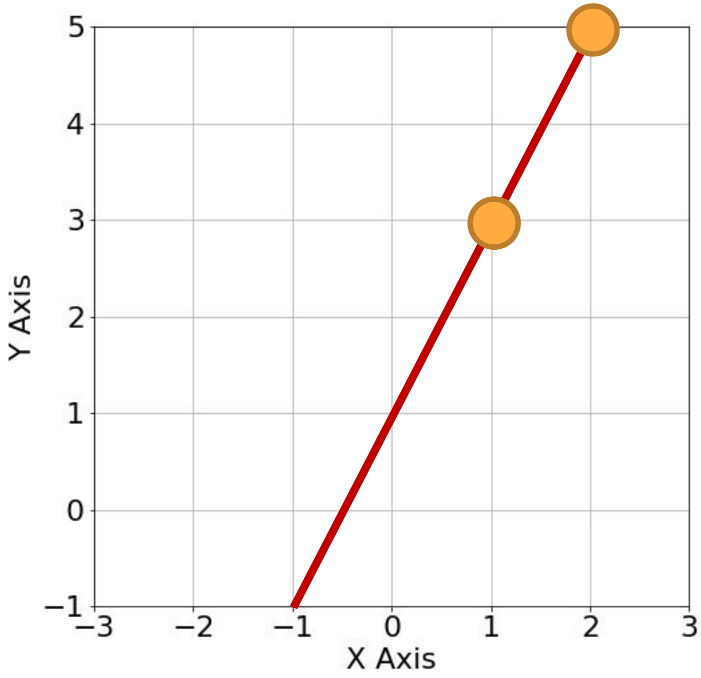

What is “b”?

“b” a.k.a. intercept:

Point where a line crosses the y-axis

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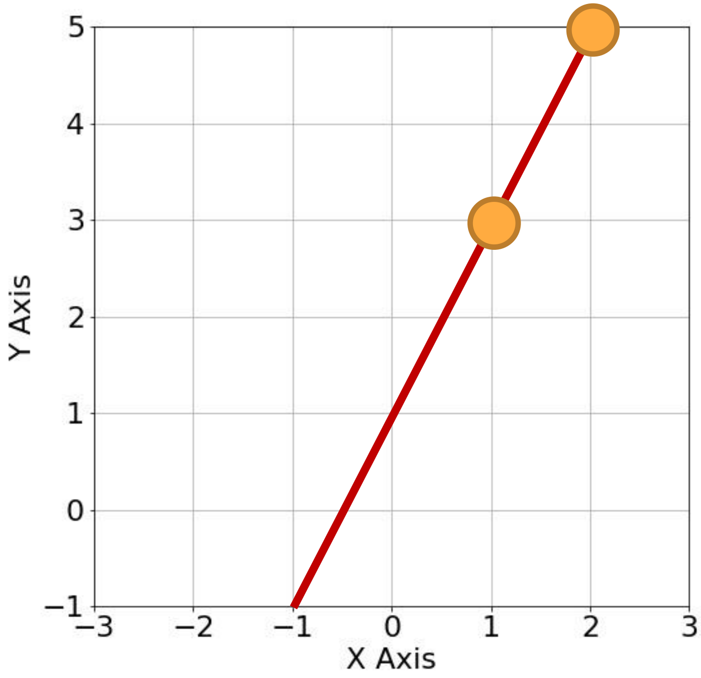

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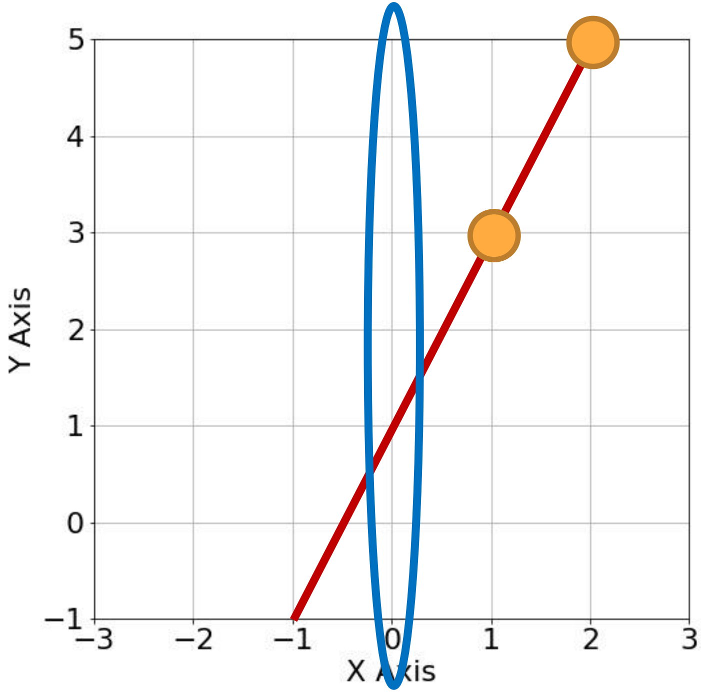

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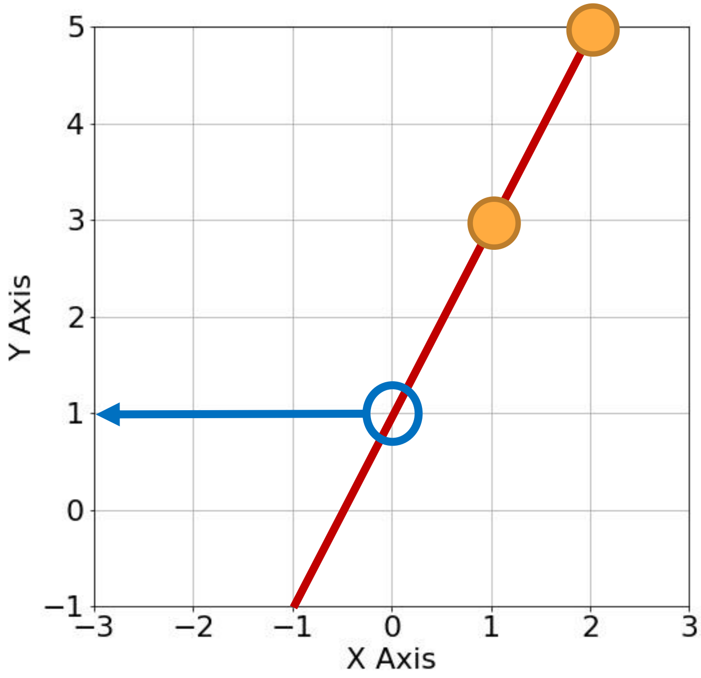
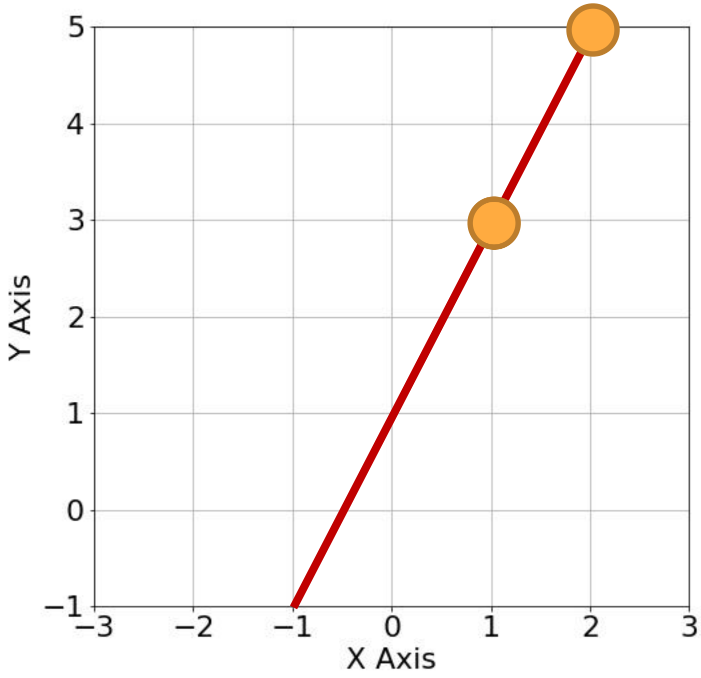


What is “m”?

“m” a.k.a. slope:

Indicates how steep the line is

$$y = \color{#3A9AD9}{m}x + b$$
$$y = 2x + 1$$



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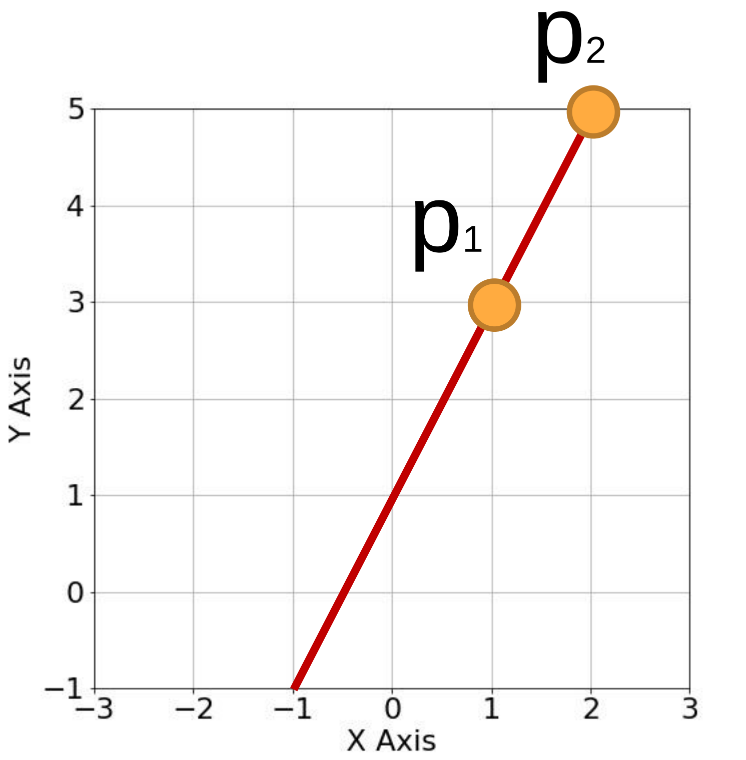

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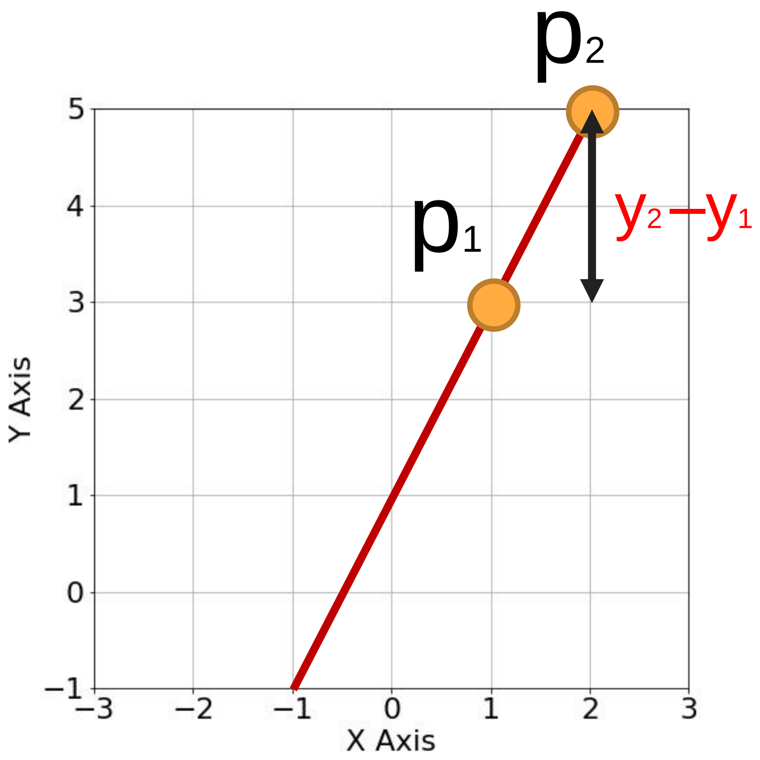

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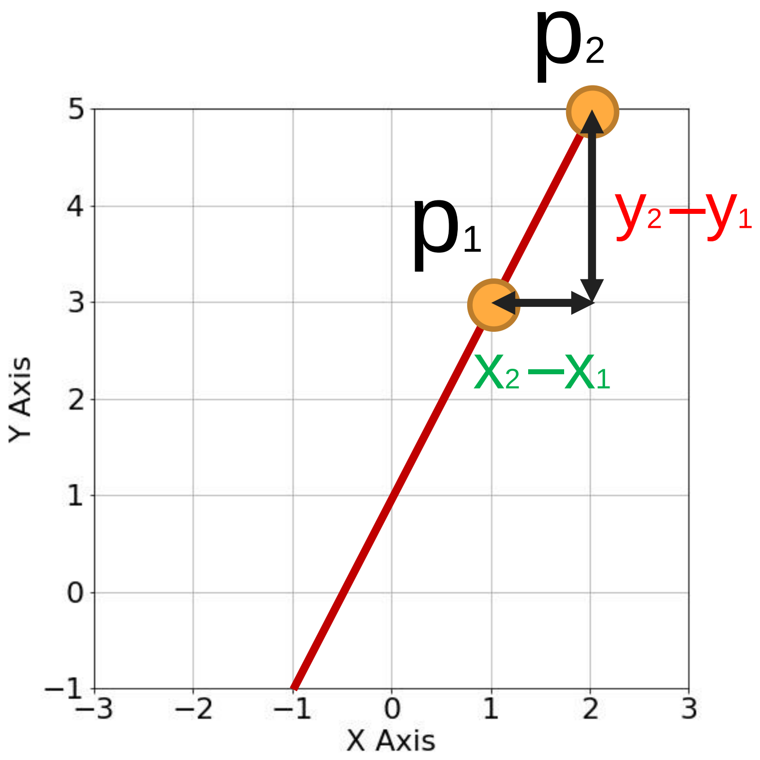

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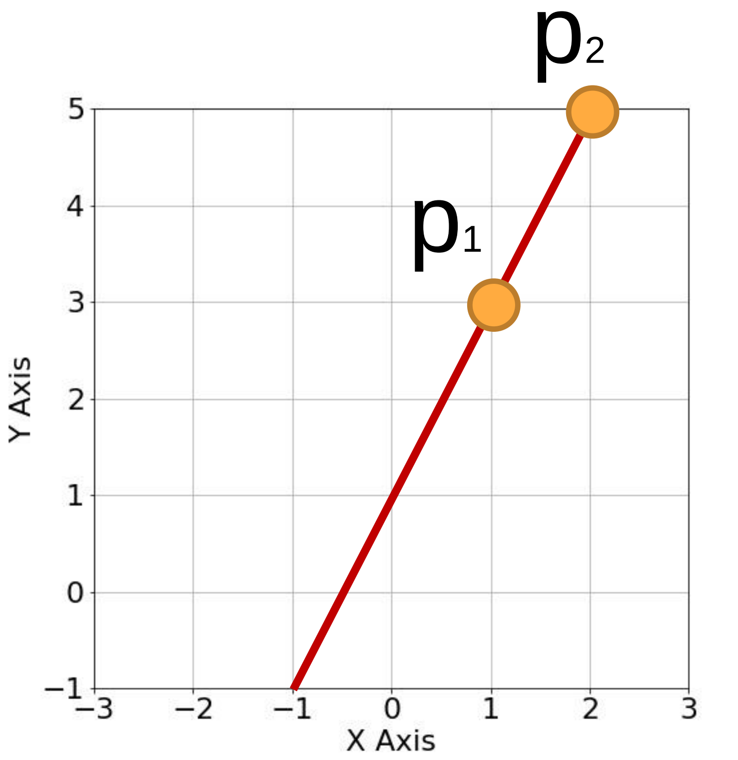

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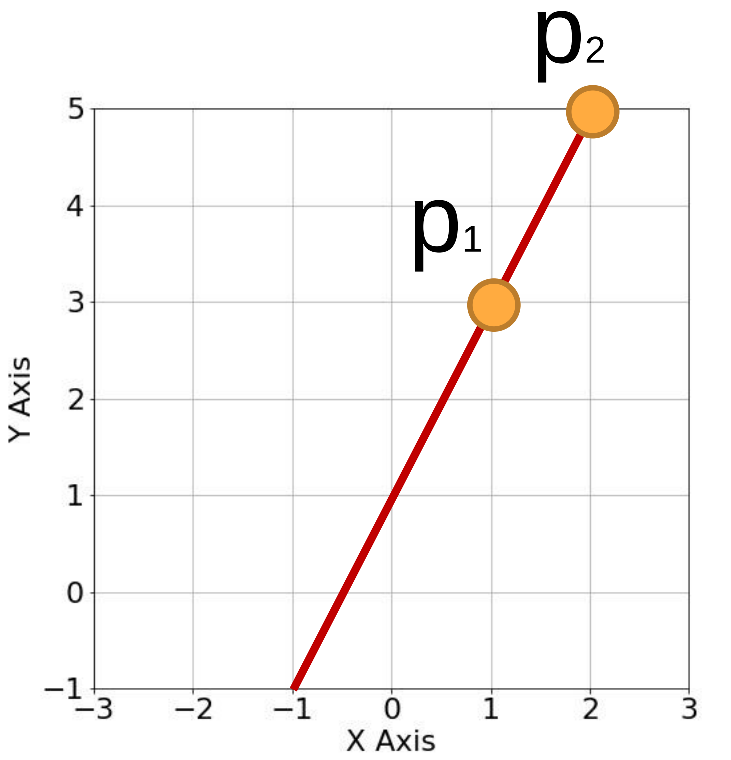
$$y = 2x + 1$$



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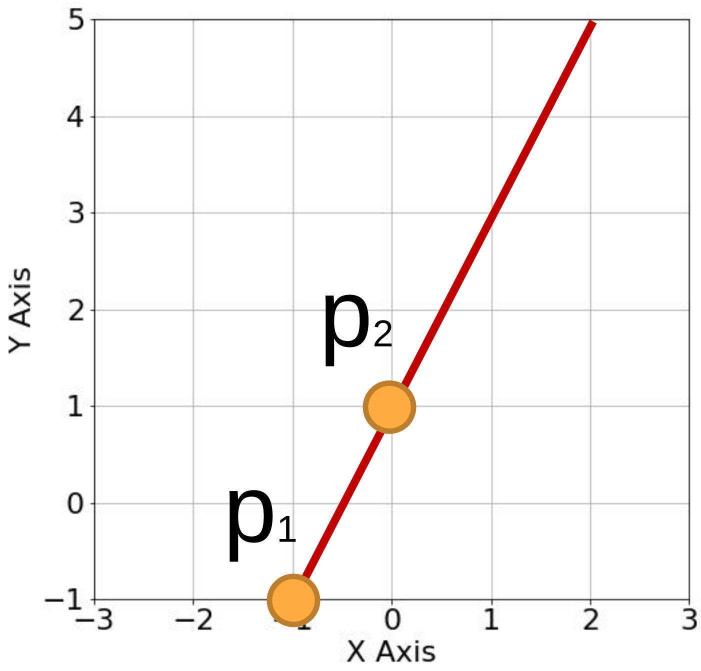

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$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

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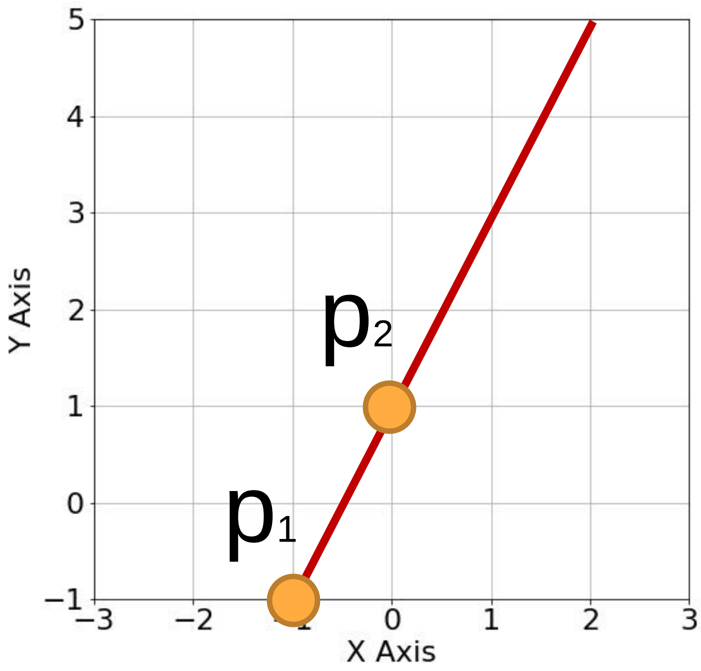

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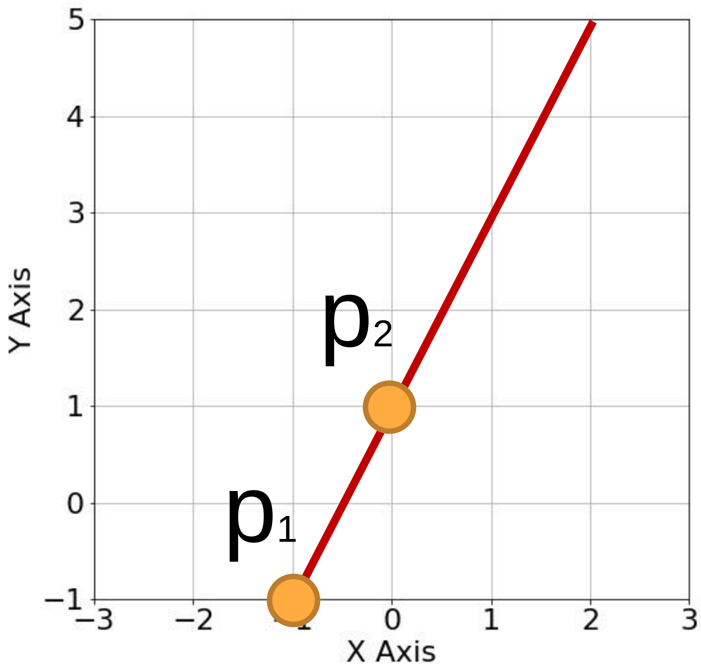

Problem 1:

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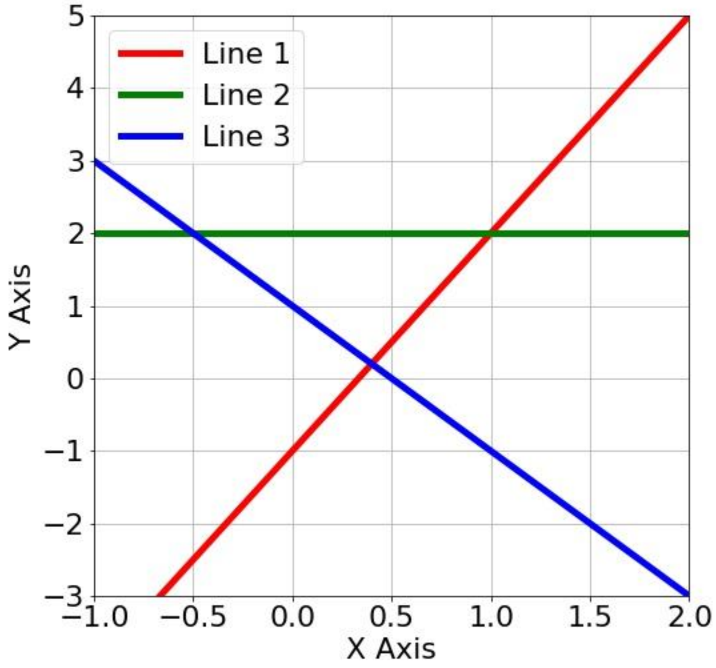
Calculate the line equation for the following lines. Helper equations:

$$y = mx + b$$

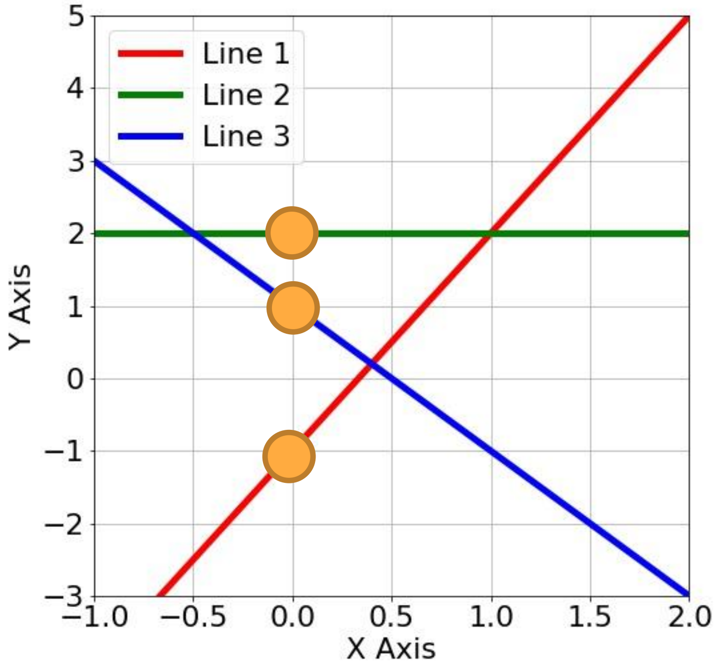
$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$



Problem 1:

Let's first extract the intercept:

$$y = mx + b = mx - 1$$
$$y = mx + b = mx + 2$$
$$y = mx + b = mx + 1$$



Exercise

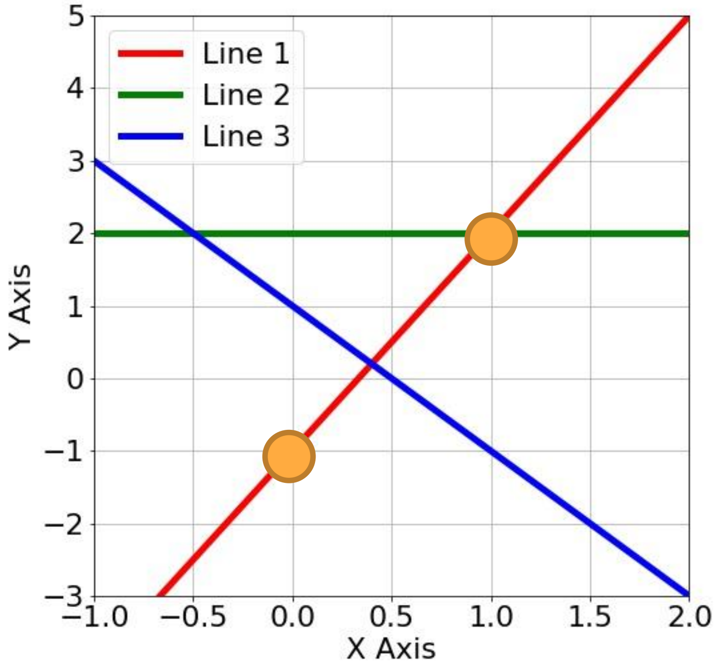
Let's extract the slope:

$$y = mx + b = 3x - 1$$

$$y = mx + b = mx + 2$$

$$y = mx + b = mx + 1$$

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(2 - (-1))}{(1 - 0)} = 3$$



Exercise

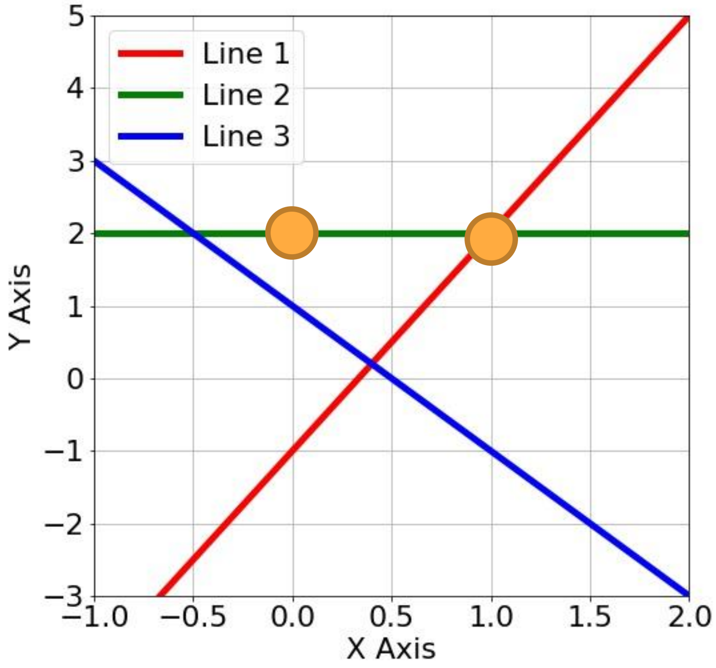
Let's extract the slope:

$$y = mx + b = 3x - 1$$

$$y = mx + b = 2$$

$$y = mx + b = mx + 1$$

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(2-2)}{(1-0)} = 0$$



Exercise

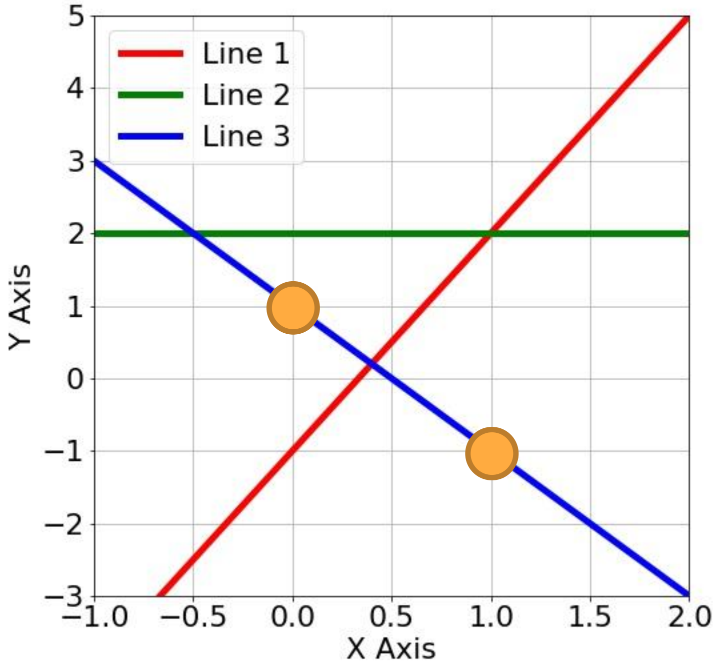
Let’s extract the slope:

$$y = mx + b = 3x - 1$$

$$y = mx + b = 2$$

$$y = mx + b = -2x + 1$$

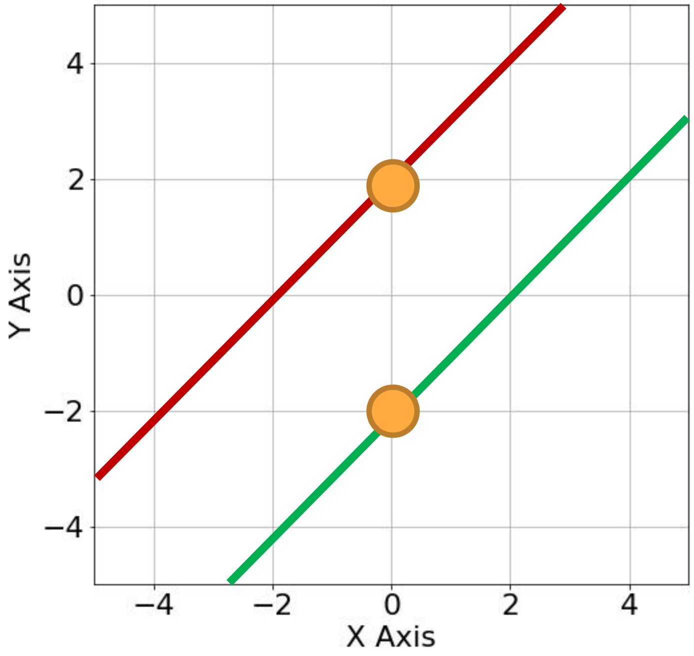
$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(-1 - 1)}{(1 - 0)} = -2$$



A few observations

Parallel lines have the same slope, but different intercept.

$$y = 1x + 2$$
$$y = 1x - 2$$

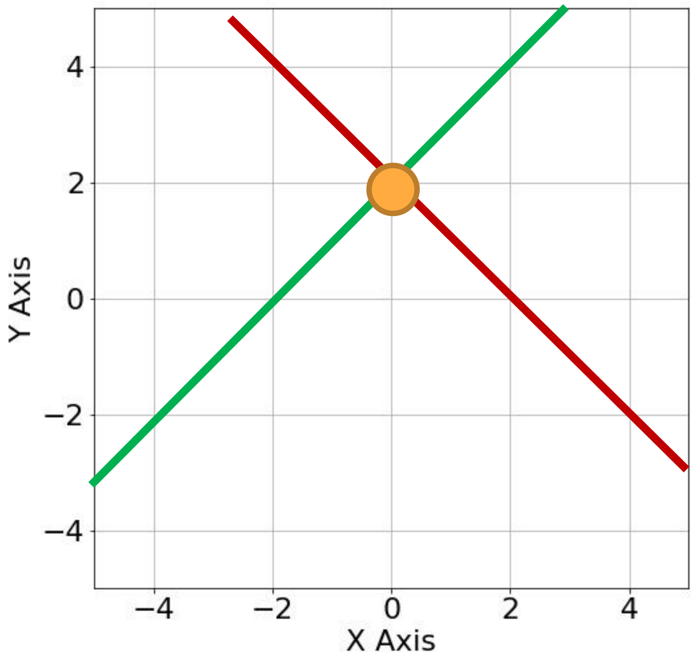


A few observations

Lines that cross the y-axis at the same point have the same intercept, but different slope.

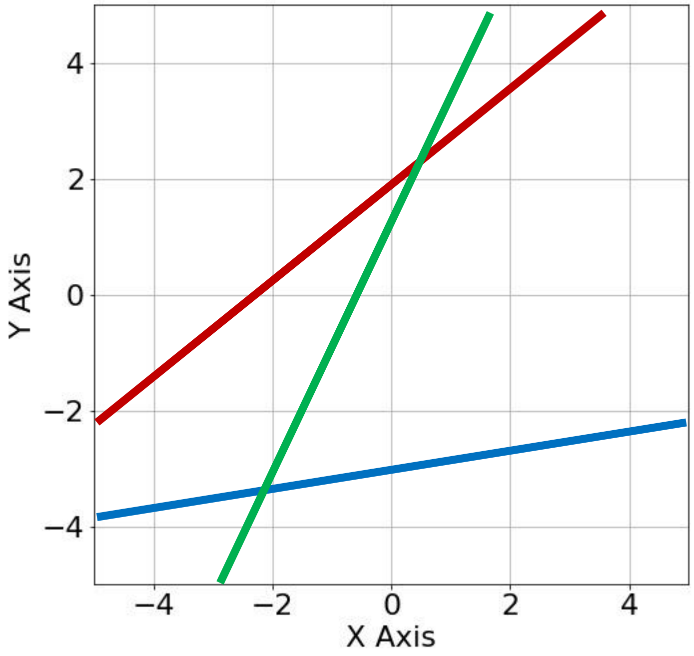
$$y = 1x + 2$$

$$y = -1x + 2$$



A few observations

All these lines have positive slope.



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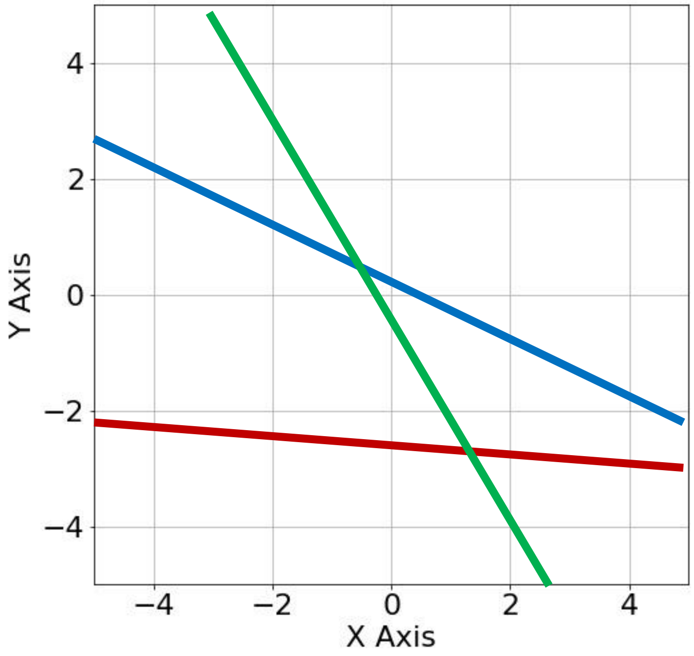
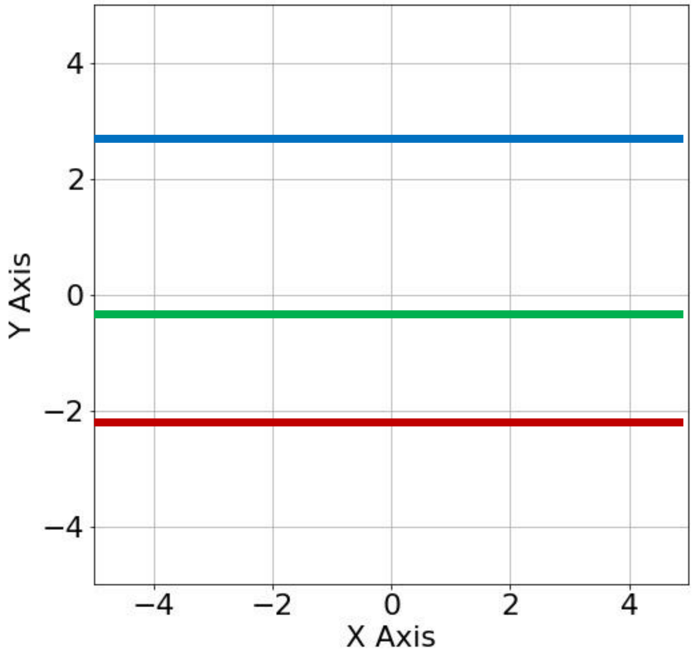


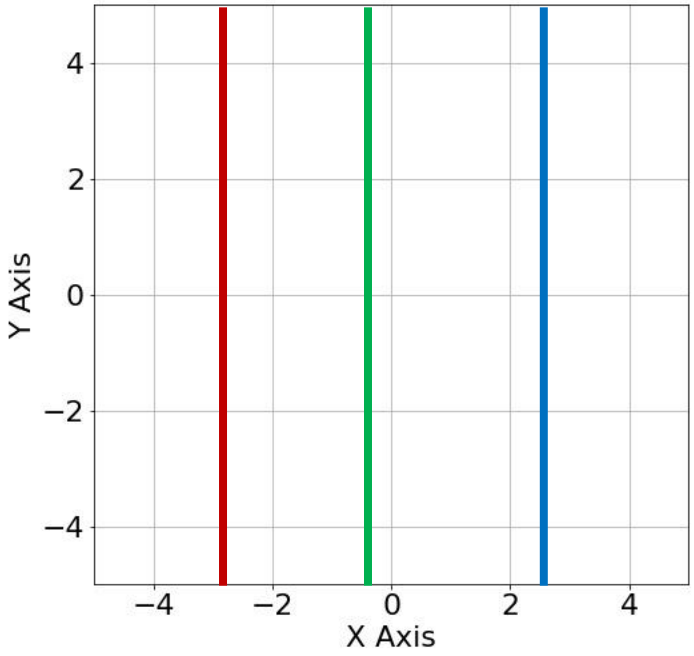
A few observations

These lines have a slope of 0.



A few observations

These lines have a slope of infinity.



Derivatives

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Derivative = Slope

Functions

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Definition:

A function is a rule that assigns to each element from a set (for example x) a unique value (for example y)

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$$f(x) = sin(x) \qquad f(x_1, x_2) = 3x_1 + 2x_2$$

Independent Variables

Definition:

A variable whose variation **does not depend** on that of another

$$f(x) = y = 2x + 1 \qquad f(x) = x^2$$

$$f(x) = sin(x) \qquad f(x_1, x_2) = 3x_1 + 2x_2$$

Dependent Variables

Definition:

A variable whose variation **depends** on that of another

$$f(x) = y = 2x + 1$$

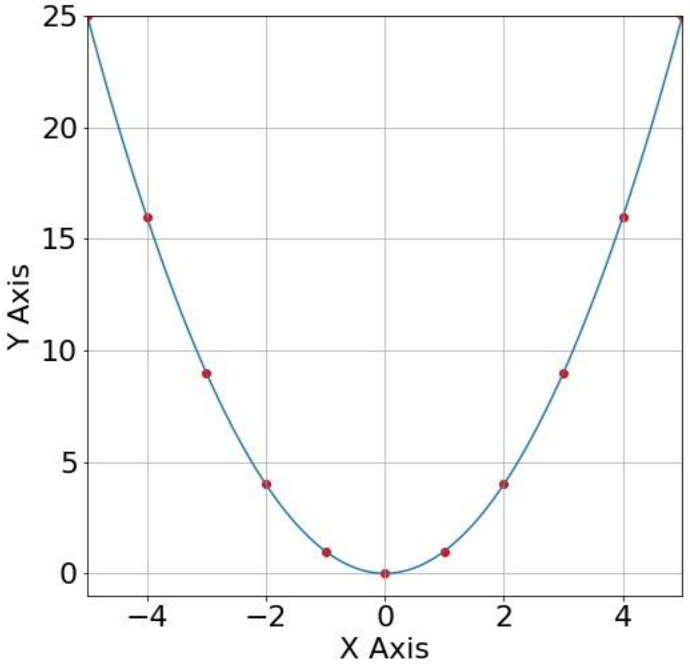
$$f(x) = x^2$$

$$f(x) = sin(x)$$

$$f(x_1, x_2) = 3x_1 + 2x_2$$

Plotting a Function

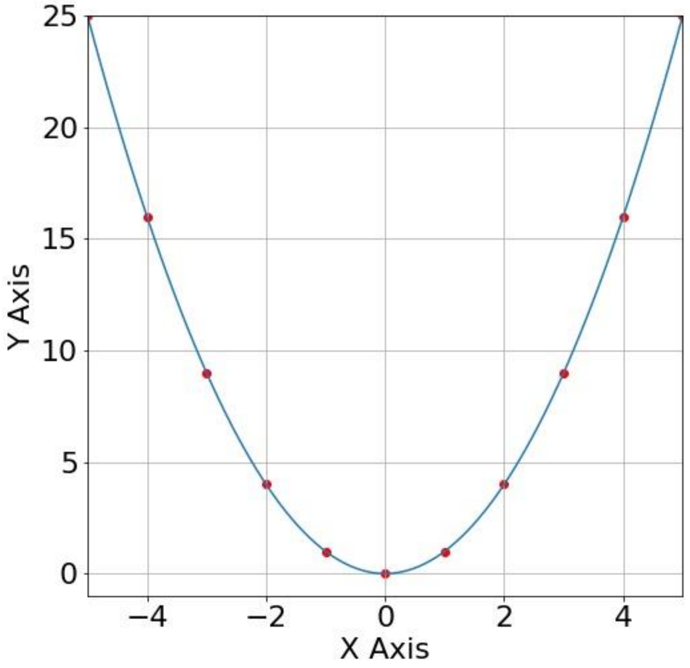
$$f(x) = x^2$$



Plotting a Function

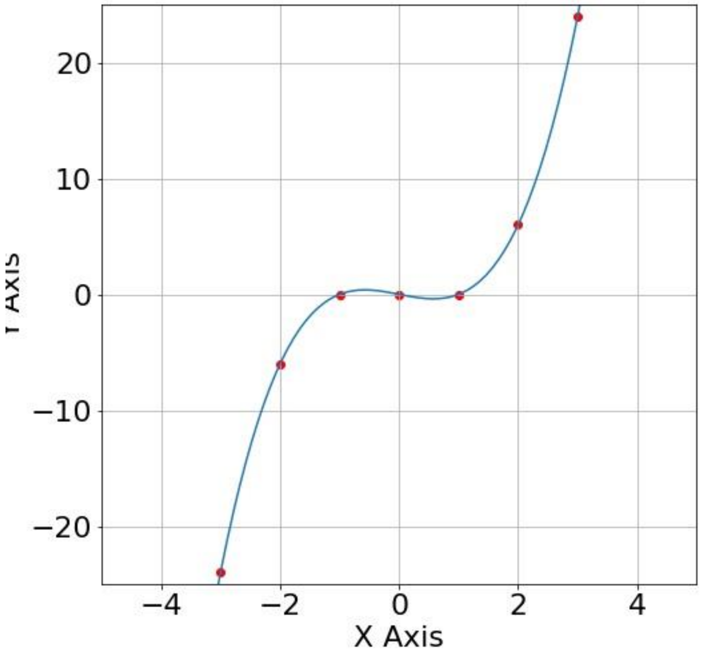
$$f(x) = x^2$$

$$f(-4) = 16$$
$$f(-3) = 9$$
$$f(-2) = 4$$
$$\ldots$$



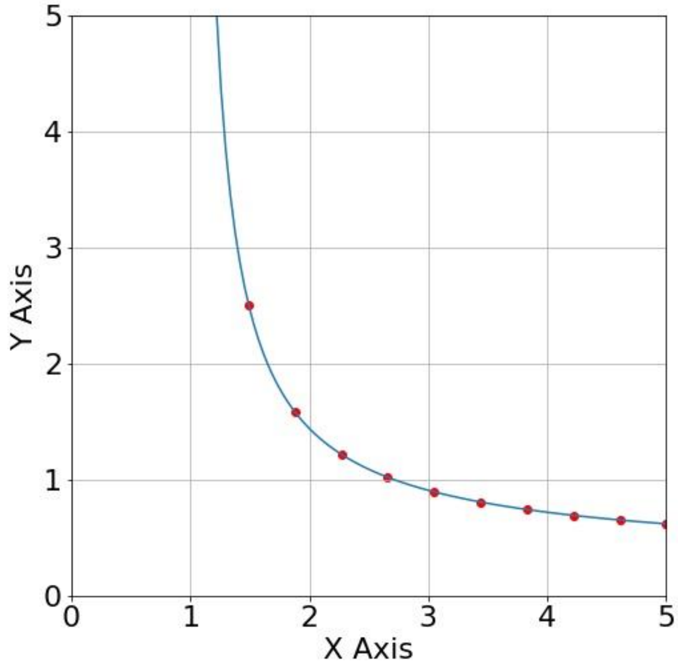
Plotting a Function

$$f(x) = x^3 - x$$



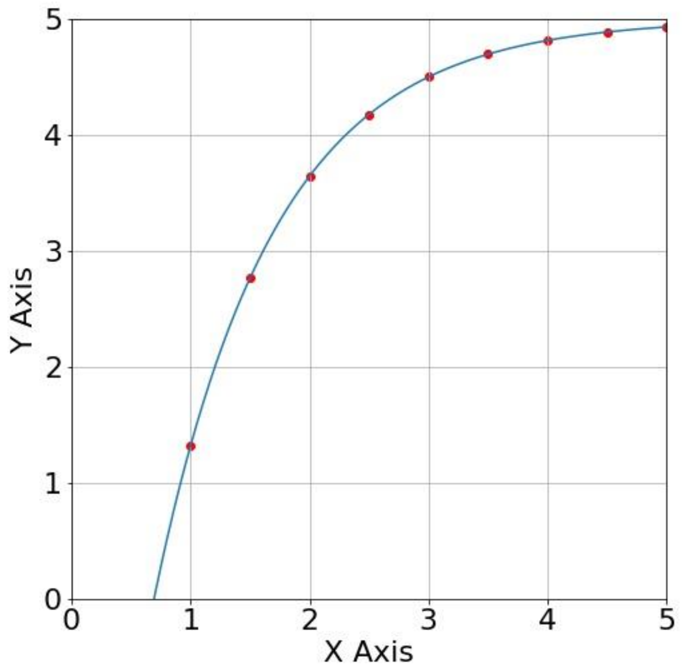
Plotting a Function

$$f(x) = \frac{1}{ln(x)}$$



Plotting a Function

$$f(x) = 5 - 10 \cdot e^{-x}$$



Problem 2:

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$$f(x) = 2x^2 - 0.5x^3 - 2$$

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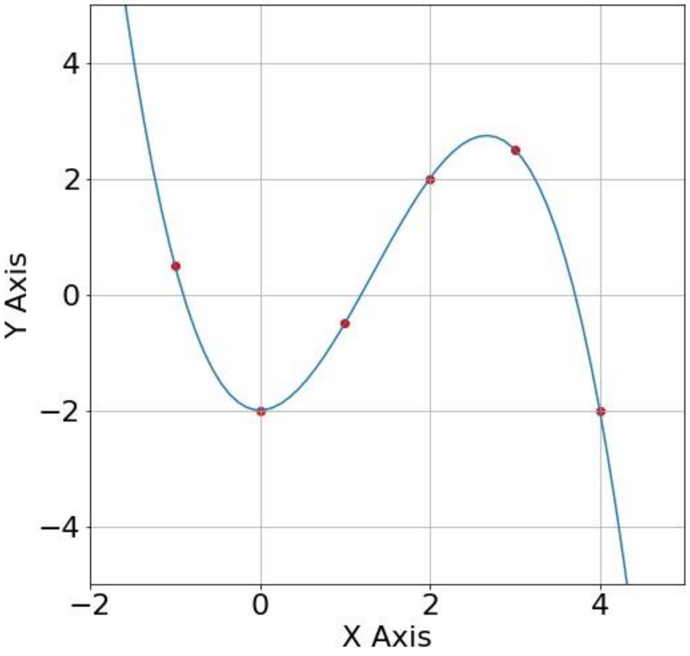
$$f(x) = 2x^2 - 0.5x^3 - 2$$
$$f(-1) = 0.5$$
$$f(0) = -2$$
$$f(1) = -0.5$$
$$f(2) = 2$$



Derivatives

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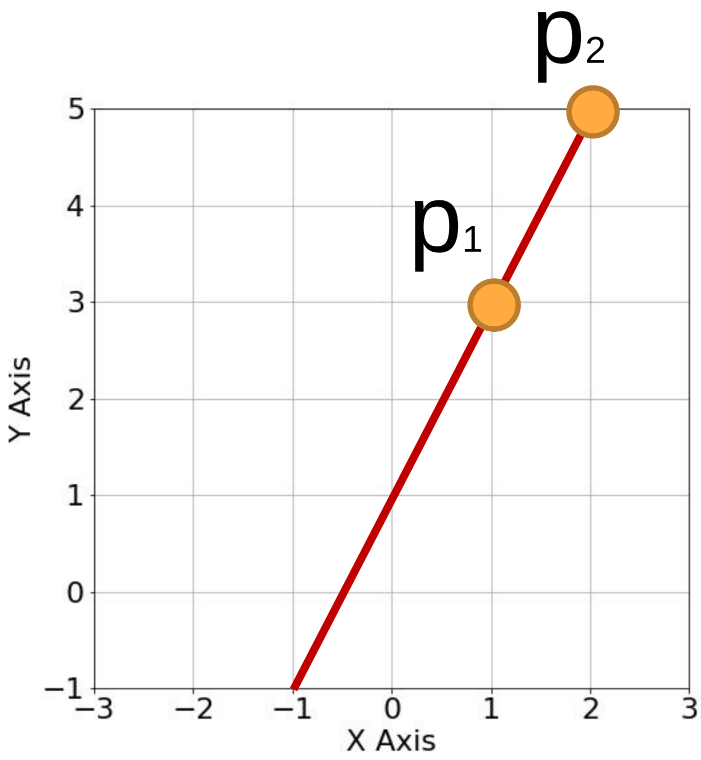
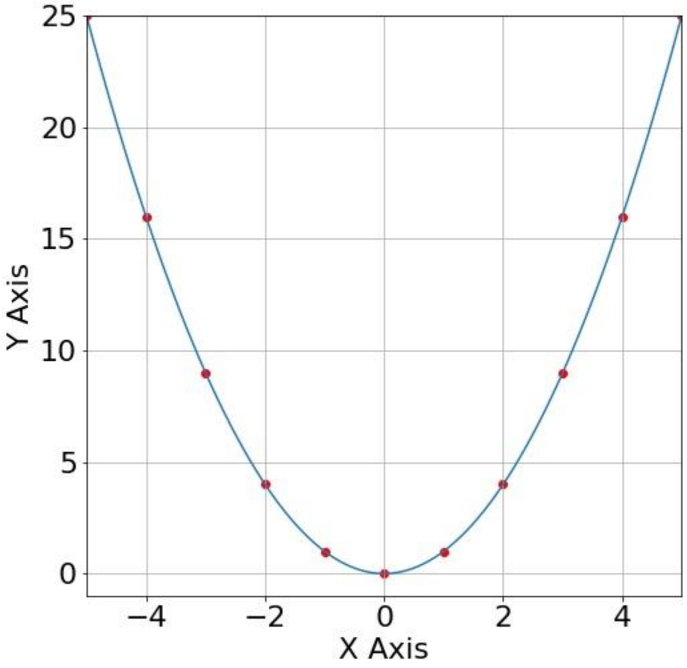


Derivative Notation

$$f'(x) = \frac{d}{dx} f(x)$$

Derivative of a Function

$$f(x) = x^2$$



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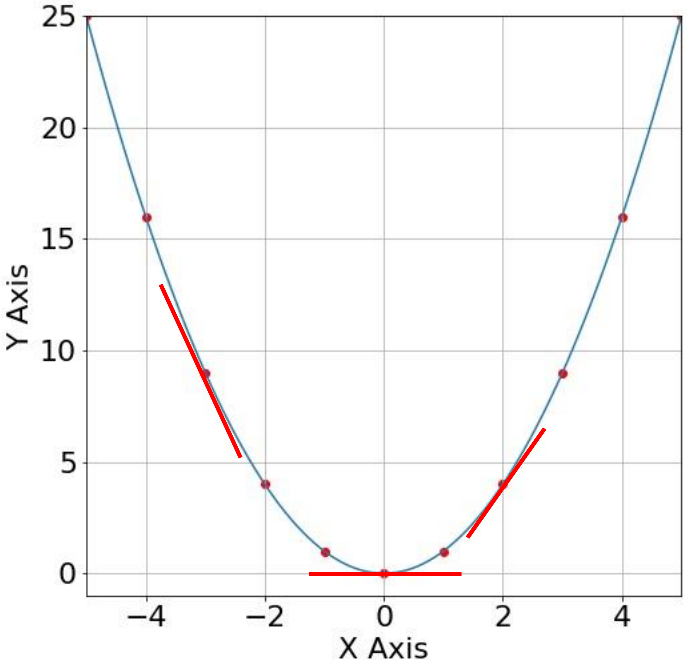

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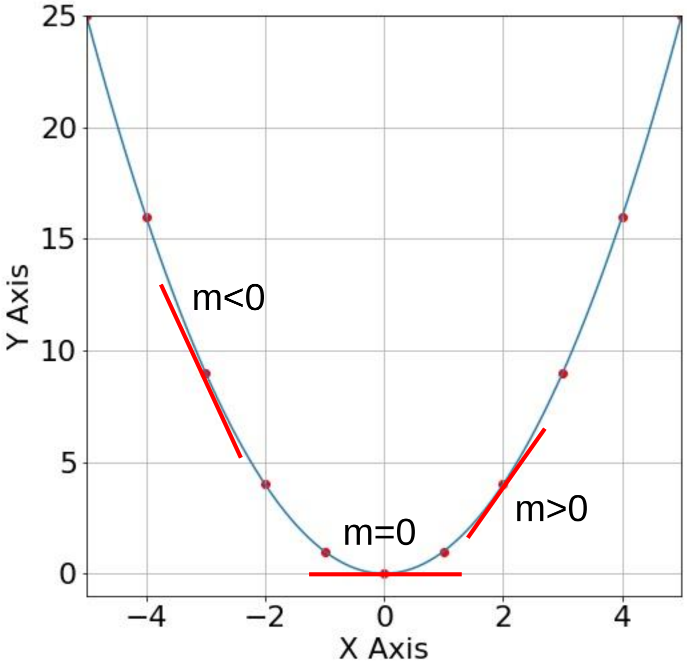

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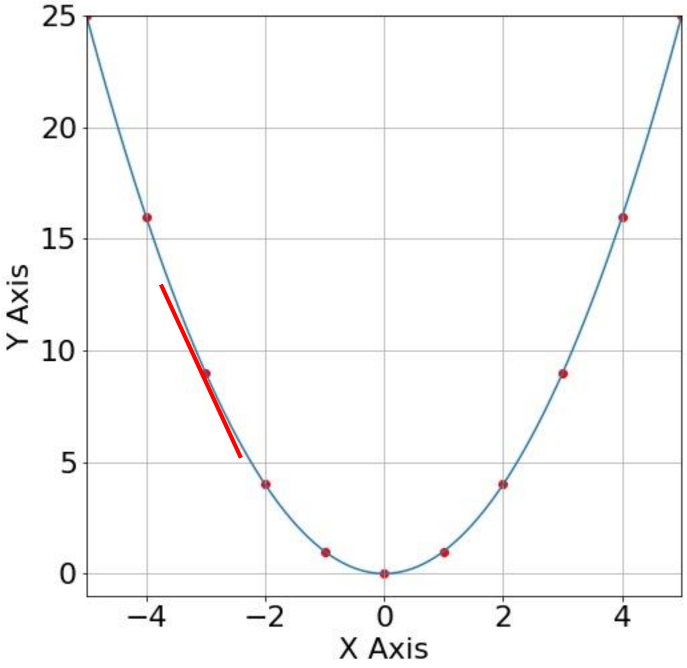

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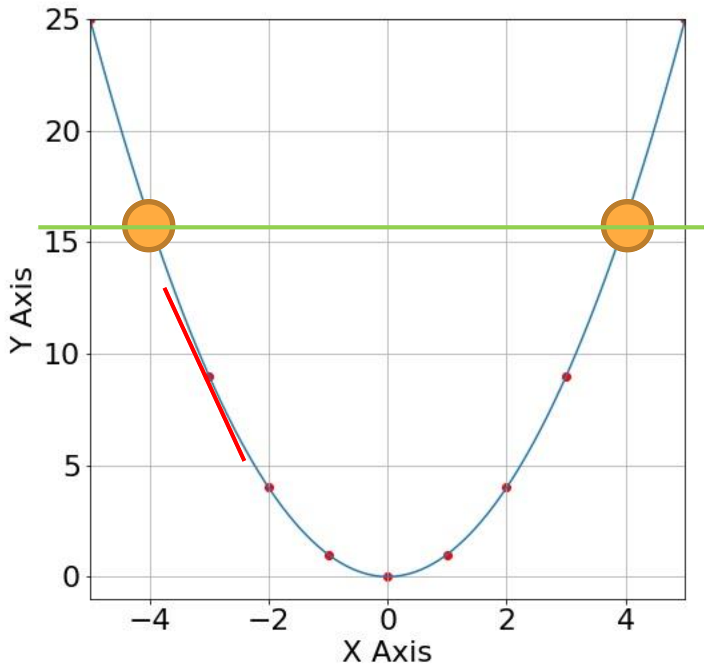

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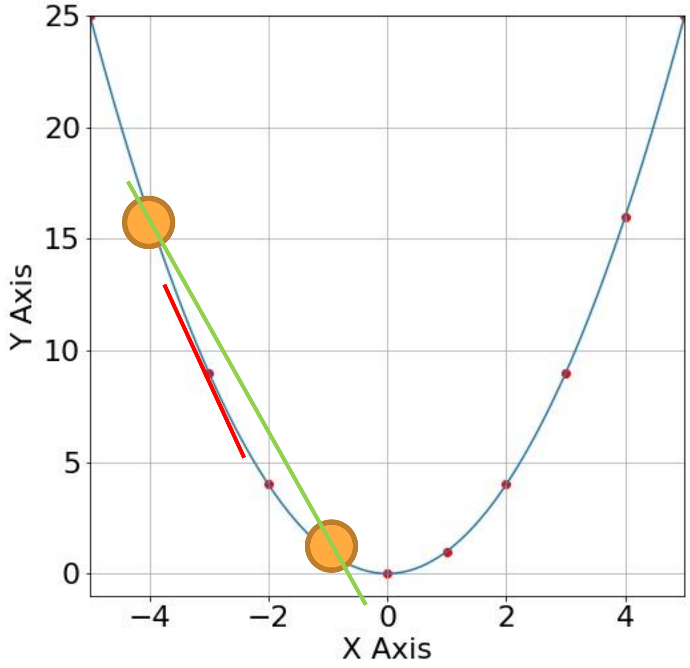

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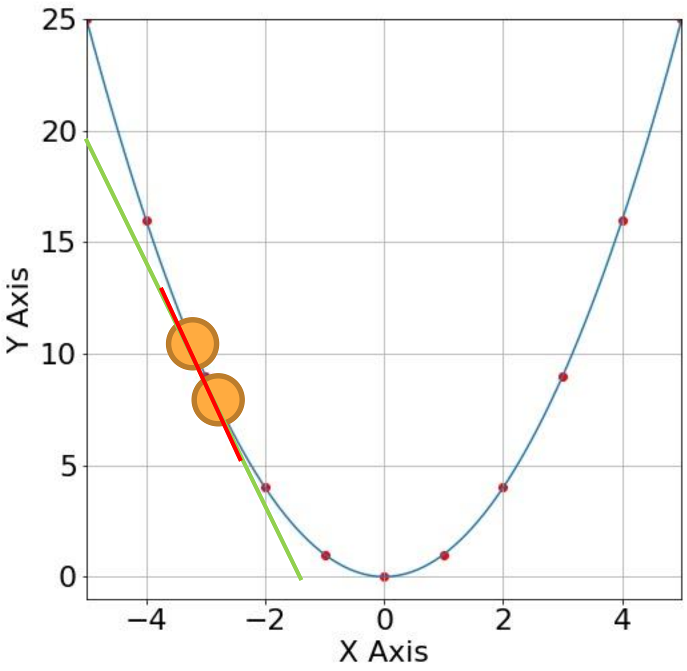

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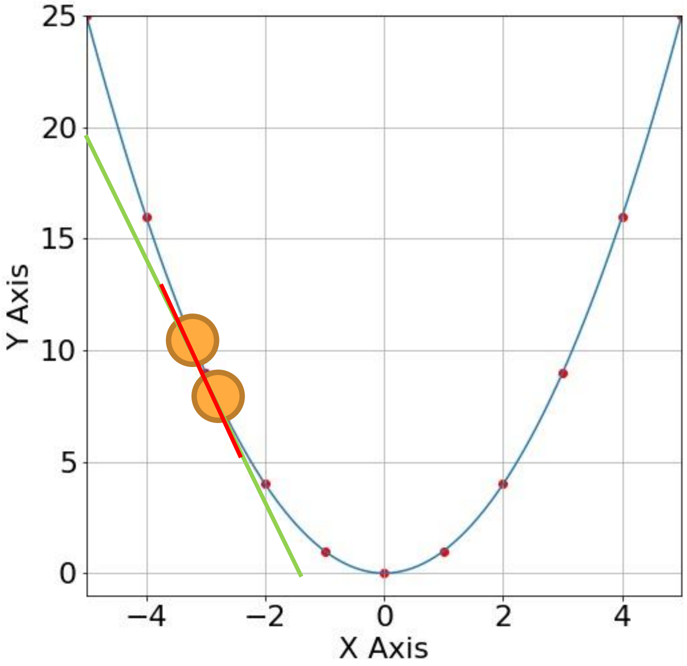

Derivative of a Function

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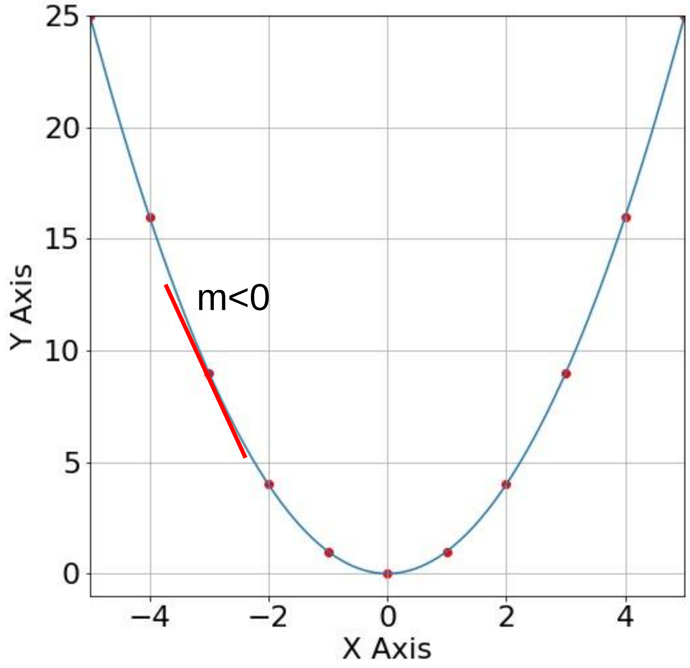

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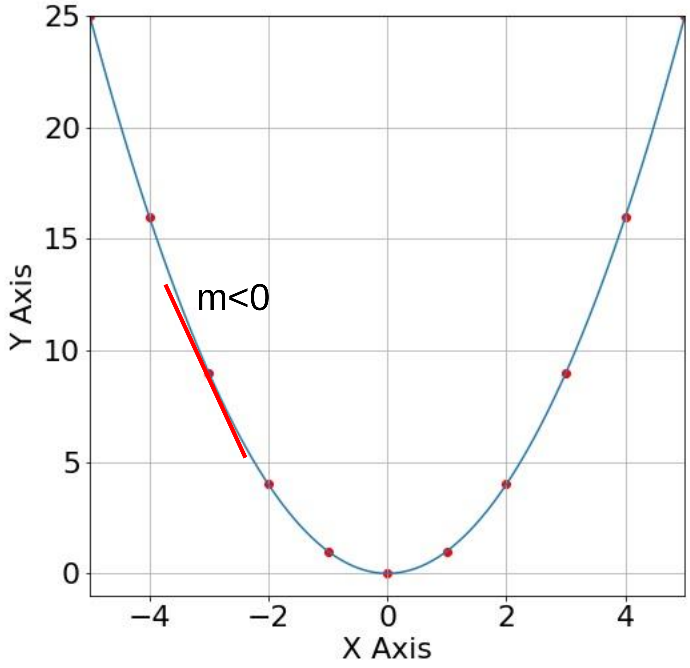

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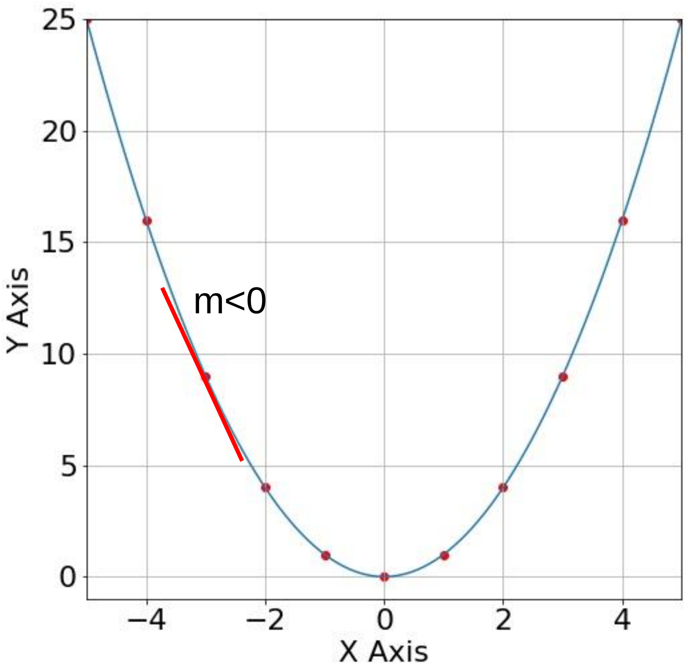
$$x_2 = x_1 + h$$



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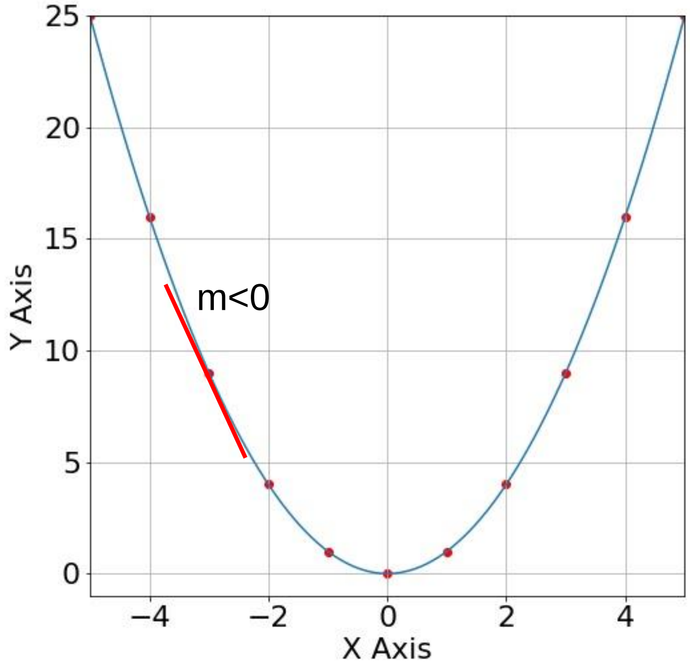
$$x_2 = x_1 + h$$

$$m = \frac{\left(f(x_1 + h) - f(x_1)\right)}{(x_1 + h - x_1)}$$



Derivative of a Function

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$m = \frac{\left(f(x_2) - f(x_1)\right)}{(x_2 - x_1)}$$

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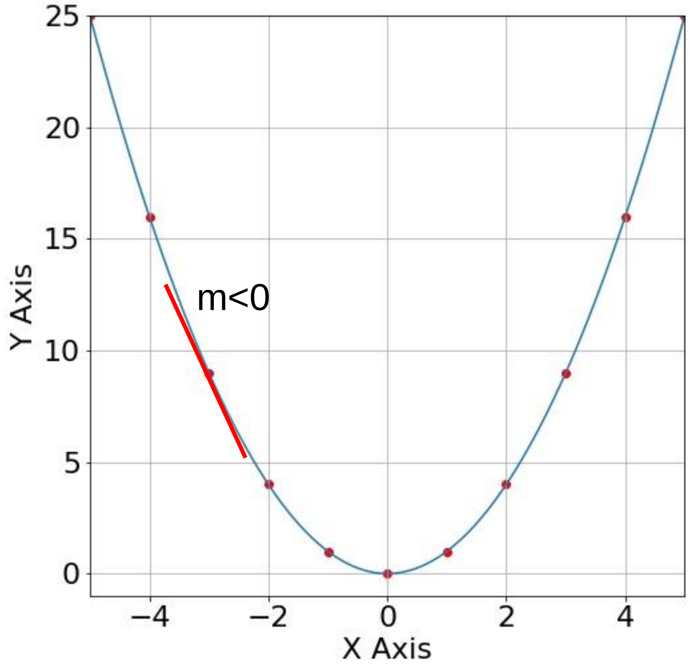
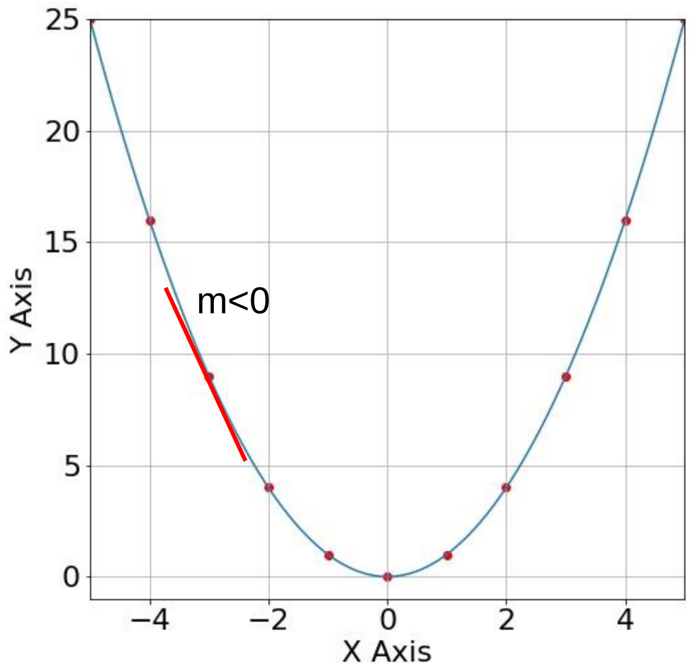


Derivative of a Function

$$m = \frac{\big(f(x_1 + h) - f(x_1)\big)}{h}$$



Derivative of a Function

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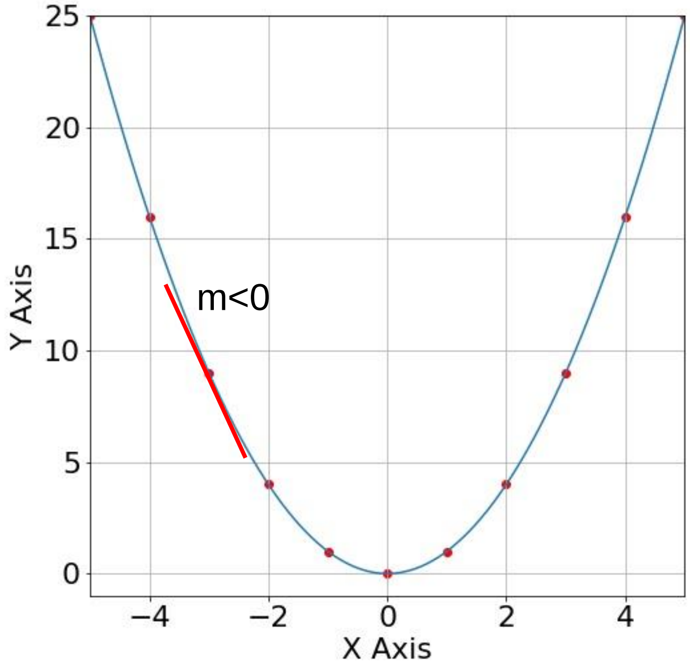

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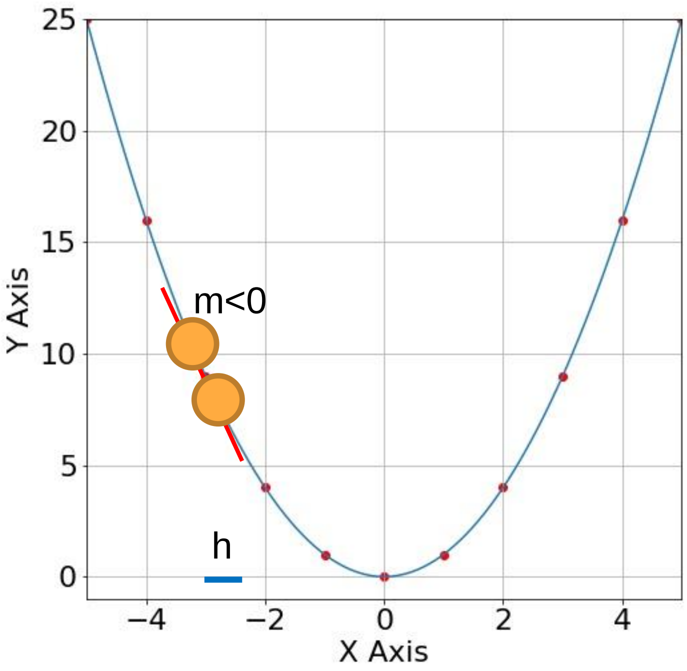
Assume h = 0.1

$$m = \frac{f(-3 + 0.1) - f(-3)}{0.1} = \frac{8.41 - 9}{0.1} = -5.1$$



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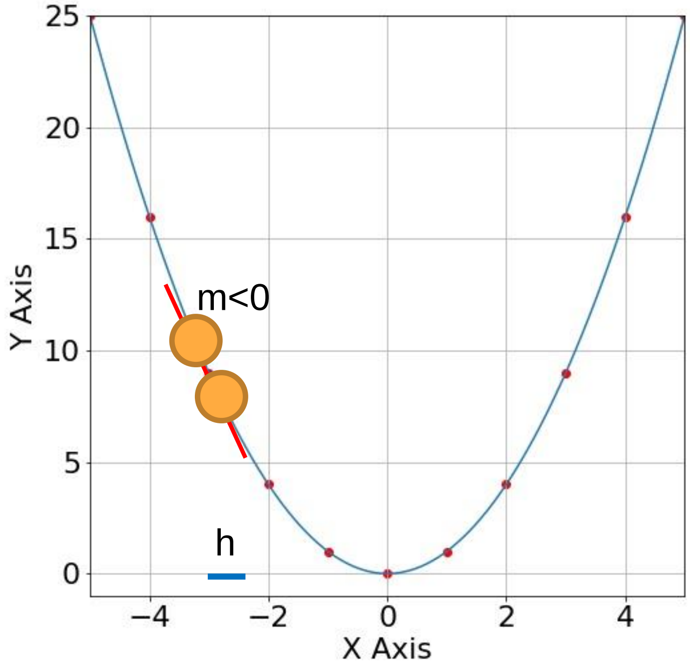
Assume h = 0.01

$$m = \frac{f(-3 + 0.01) - f(-3)}{0.01} = \frac{8.94 - 9}{0.01} = -5.99$$



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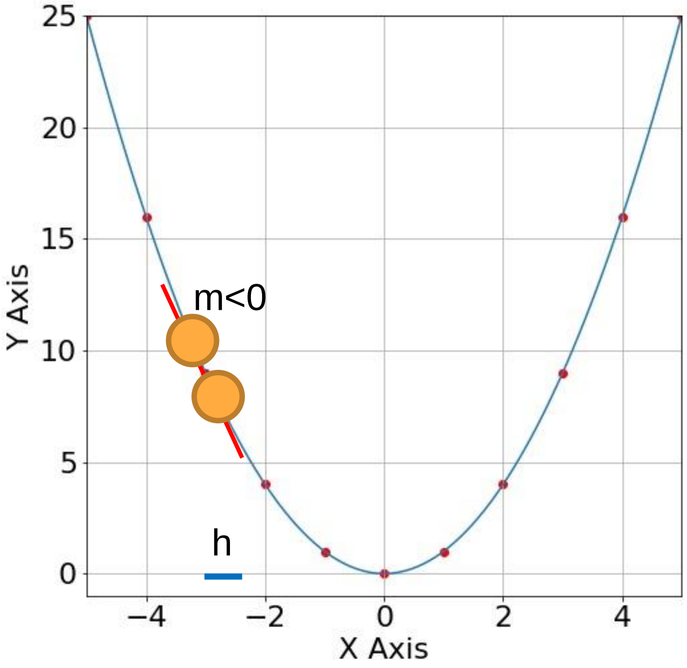

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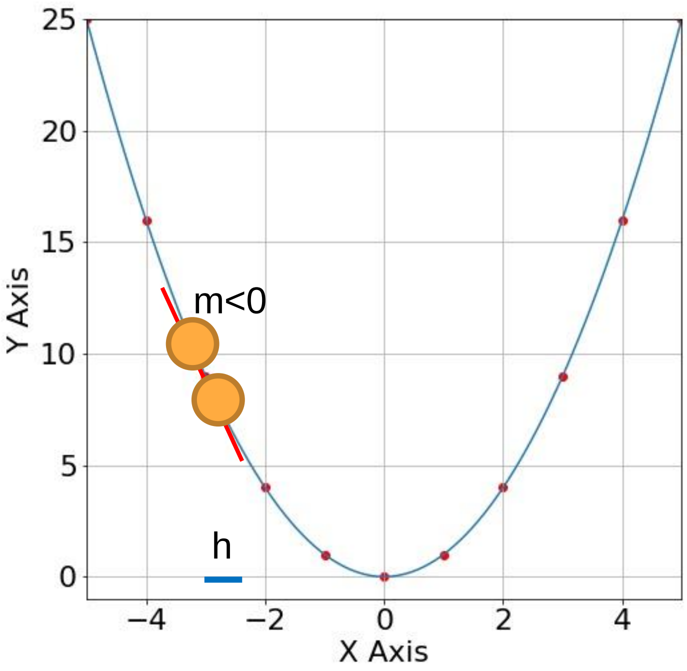

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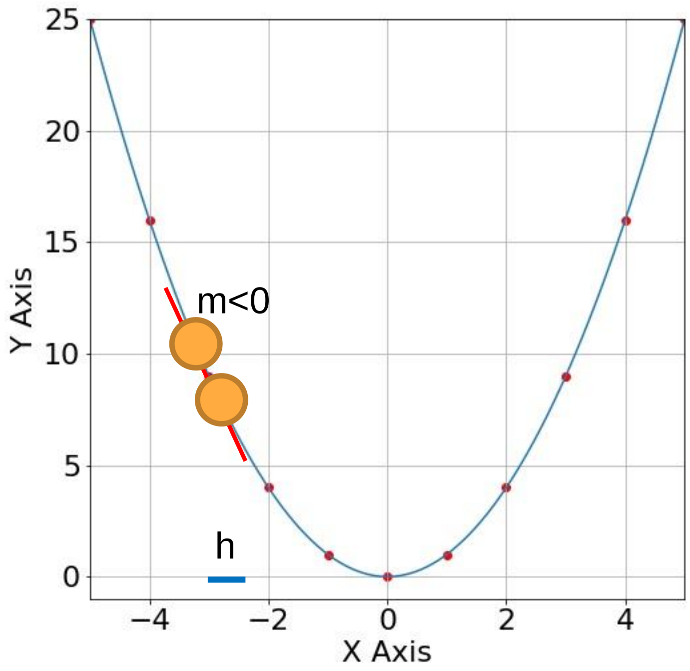

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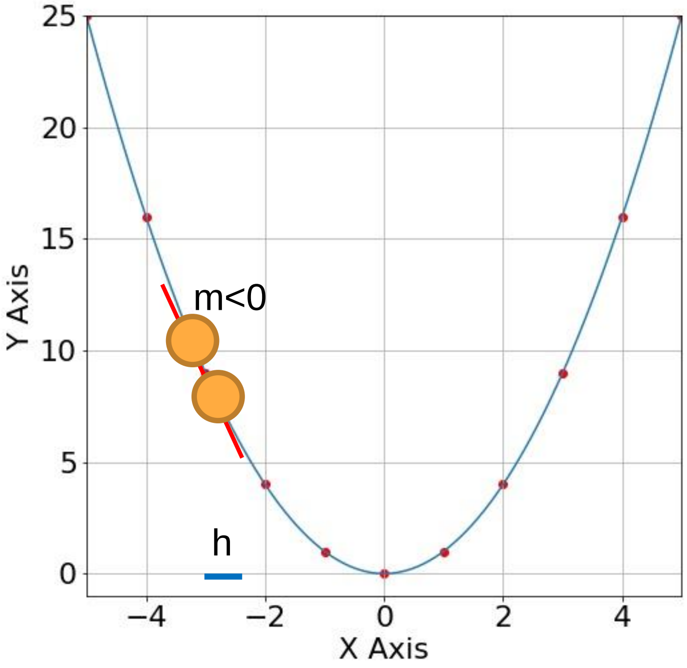

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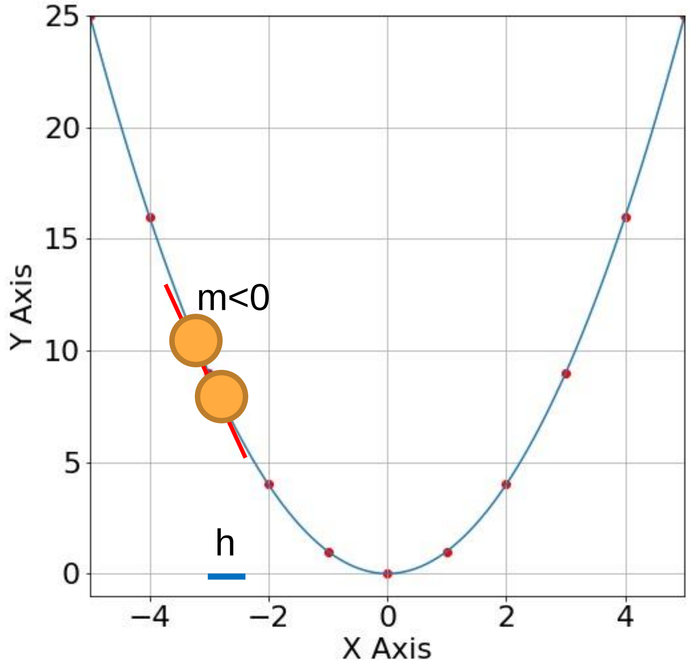

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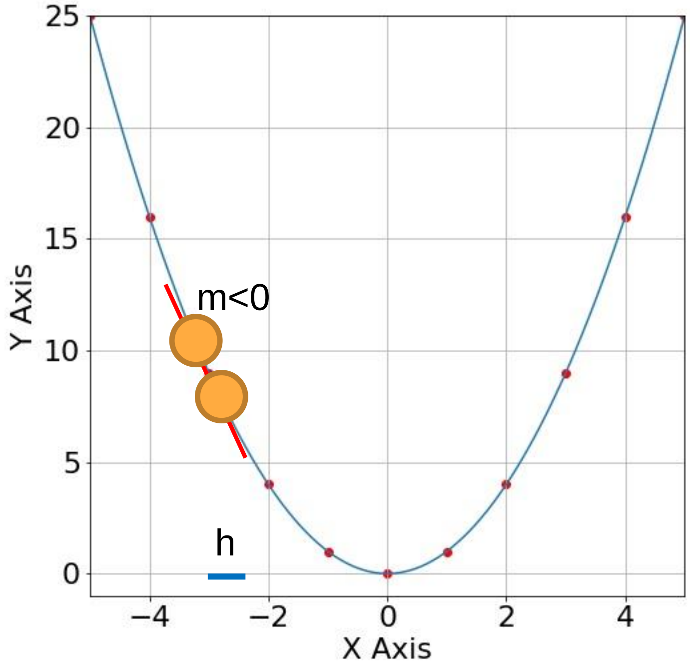
$$lim_{h\to 0}\frac{(-3+h)^2 - 9}{h} =$$



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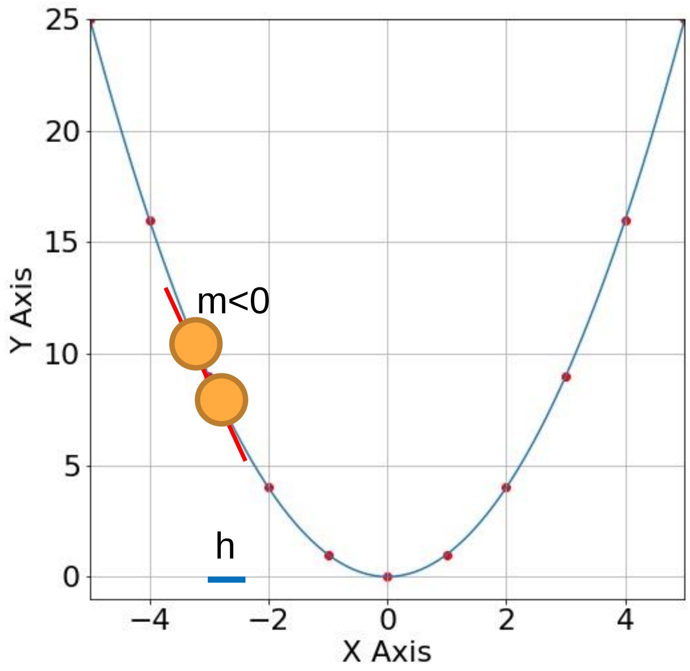

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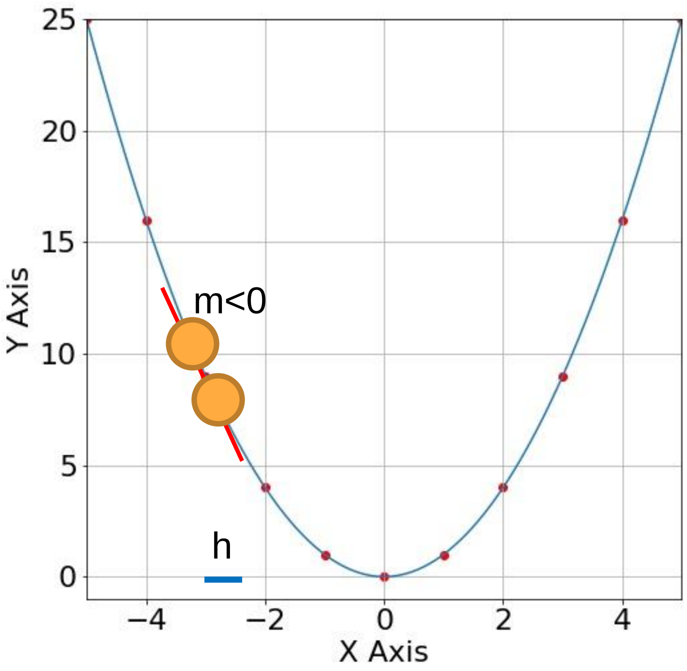

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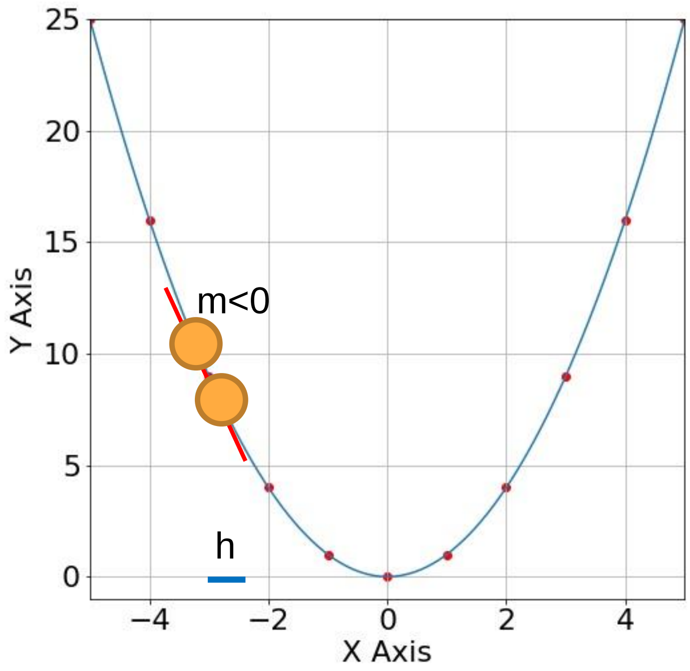
$$lim_{h \to 0} \frac{h^2 - 6h}{h} =$$

$$lim_{h \to 0} \frac{h - 6}{1} = \frac{0 - 6}{1} = -6$$

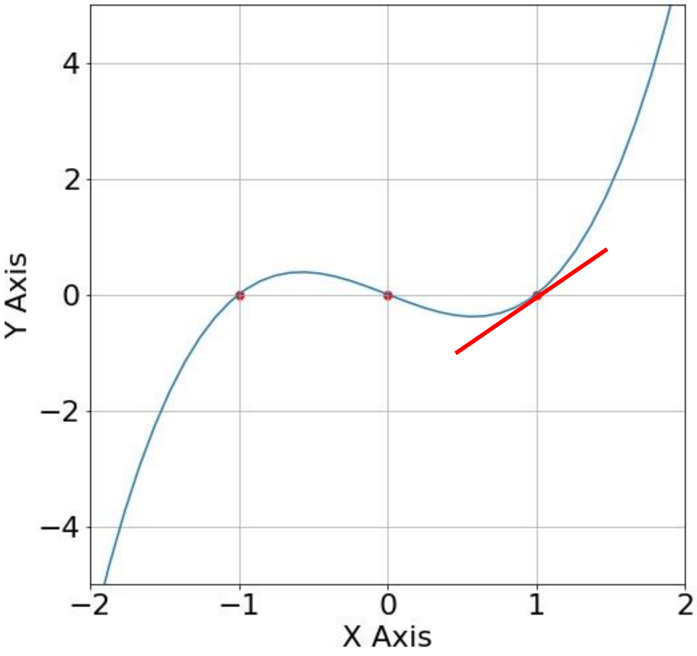


Problem 3:

Problem 3: Calculate f '(x) at x_1=1

$$f(x) = x^3 - x$$

$$f'(x_1) = lim_{h\to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$



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$$f(x) = x^3 - x$$

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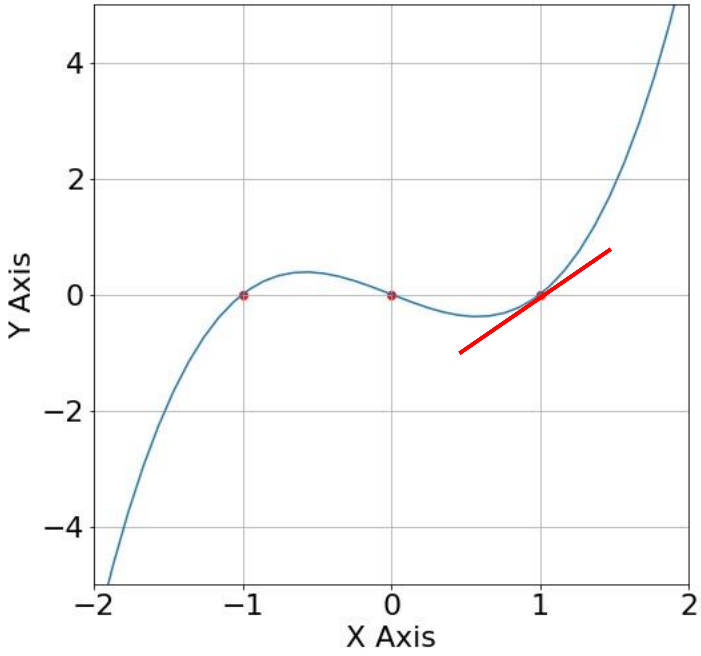

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$$f(x) = x^3 - x$$

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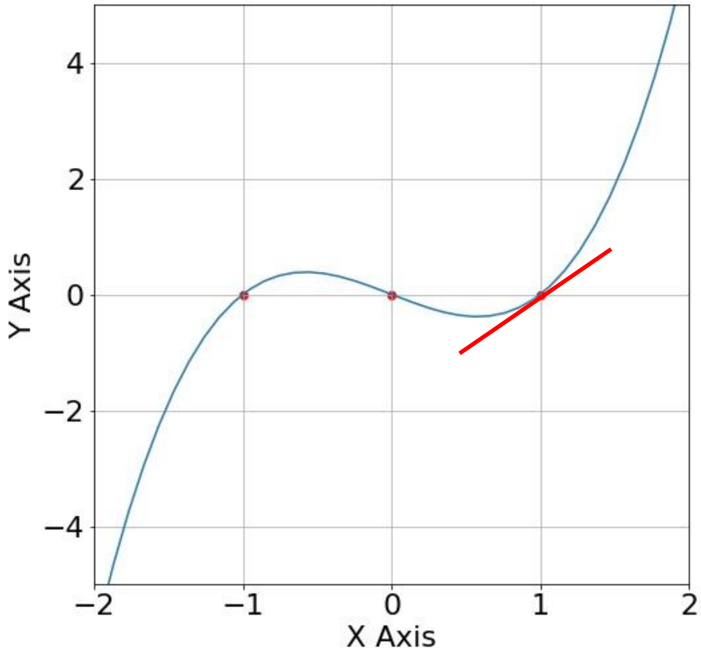

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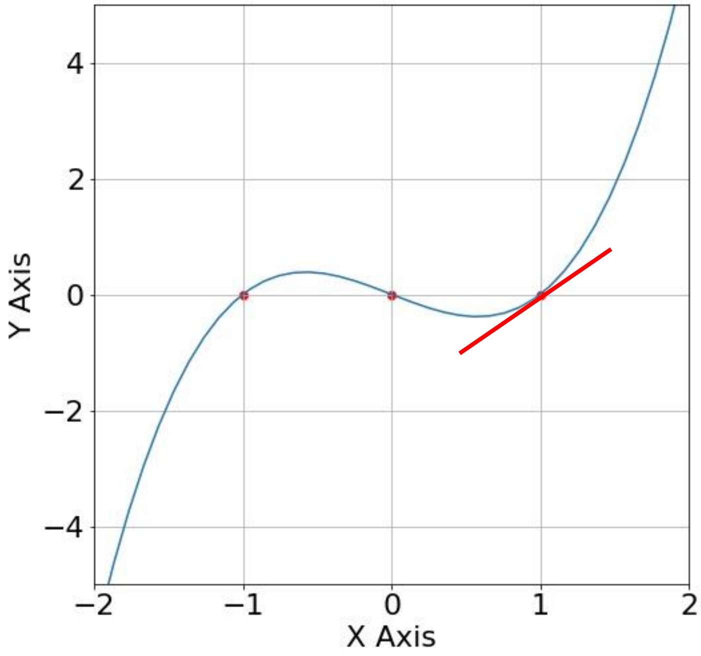

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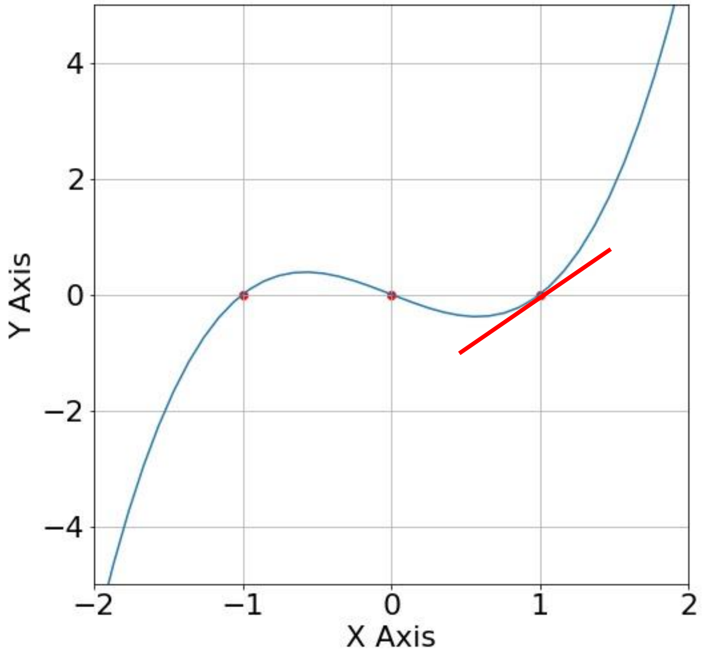

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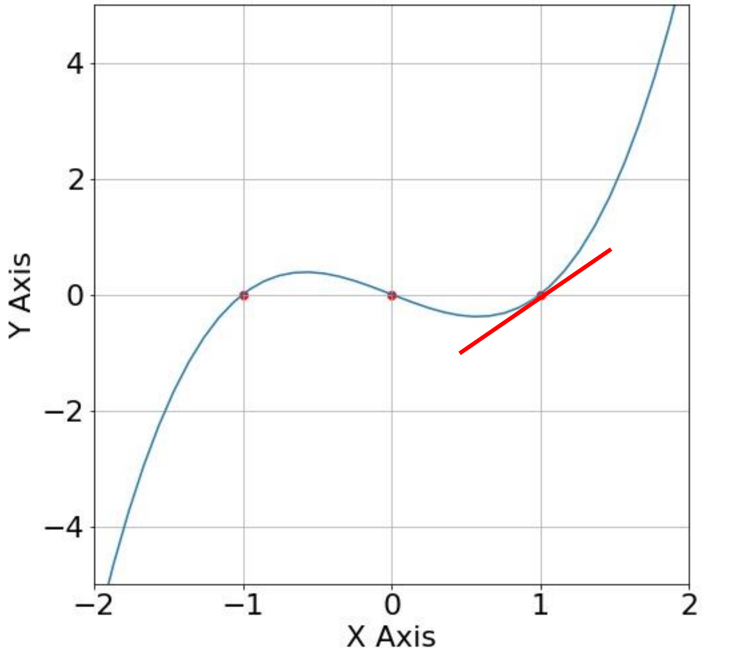
$$lim_{h\to 0} \frac{h^2 + 3h + 2}{1} = \frac{0 + 0 + 2}{1} = 2$$



Continuous Functions

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Continuous Functions

Definition:

A **continuous function** is a **function** for which sufficiently small changes in the input result in arbitrarily small changes in the output.

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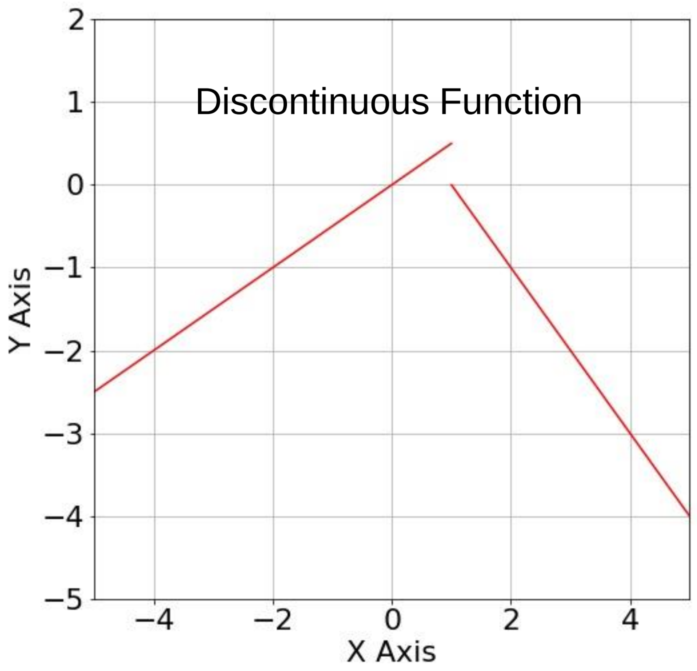
$$f(x) = \frac{x}{2} \; for \; x < 1$$

$$f(x) = -x + 1 \; for \; x \geq 1$$



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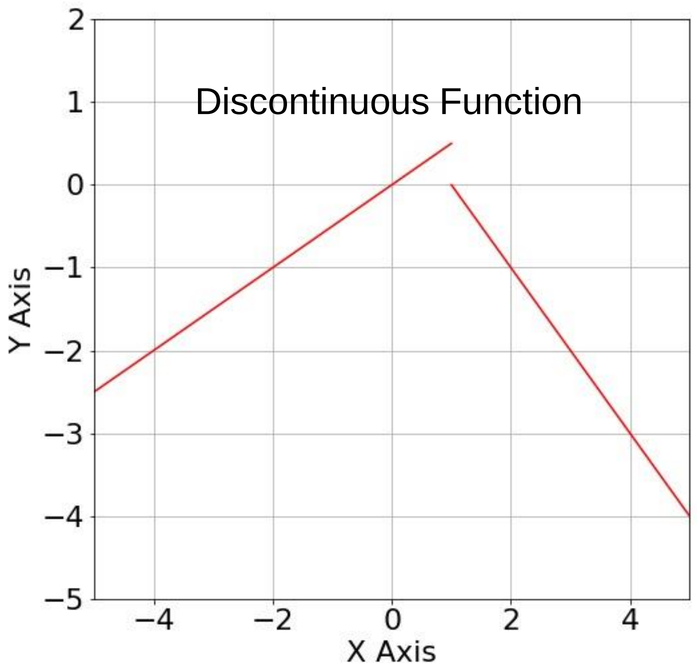


Derivatives exist only on continuous functions

Derivatives of Common Functions

METIS

Polynomials

$$f(x) = ax^n$$

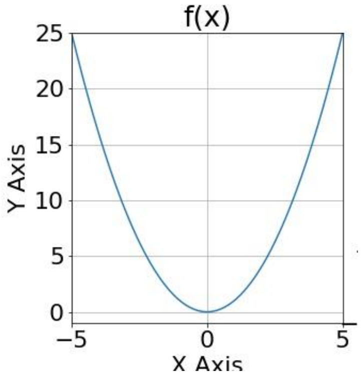
$$f'(x) = anx^{n-1}$$

Polynomials

$$f(x) = ax^n$$

$$f'(x) = anx^{n-1}$$

$$f(x) = x^2$$



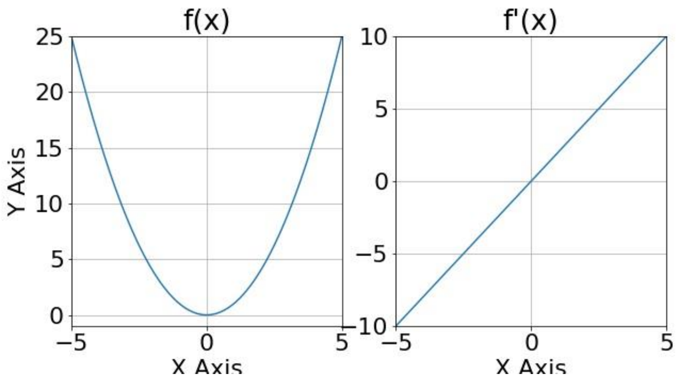
Polynomials

$$f(x) = ax^n$$

$$f'(x) = anx^{n-1}$$

$$f(x) = x^2$$

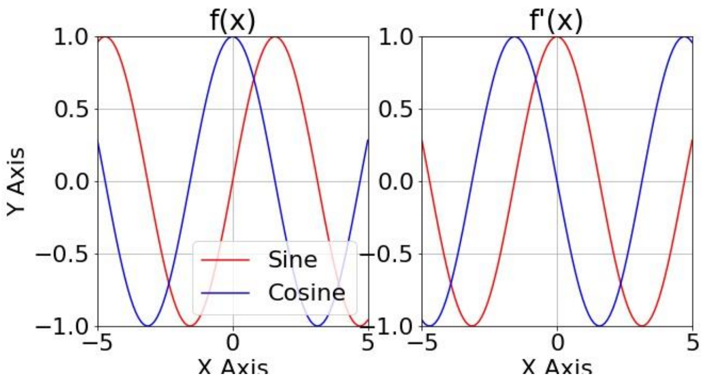
$$f'(x) = 2x^{2-1} = 2x$$



Trigonometric Functions

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$



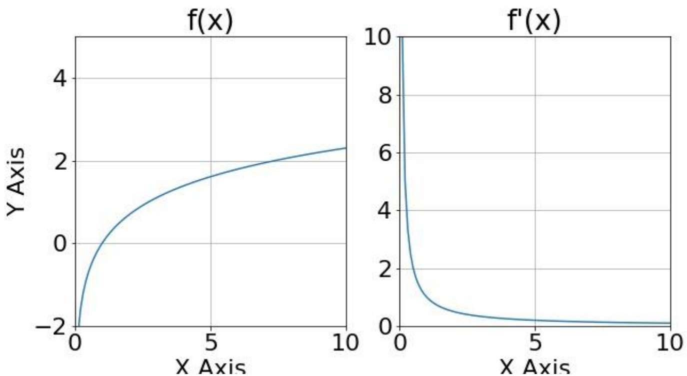
$$f(x) = \cos(x)$$

$$f'(x) = -\sin(x)$$

Logarithms Functions

$$f(x) = \ln(x)$$

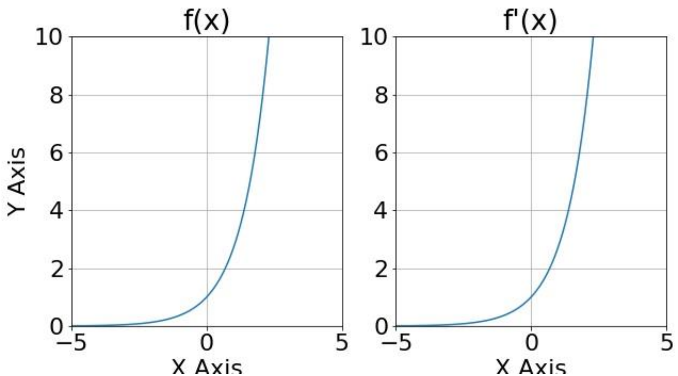
$$f'(x) = \frac{1}{x}$$



Exponential Functions

$$f(x) = e^x$$

$$f'(x) = e^x$$



Common Derivatives (Cheat Sheet)

Polynomials

$$\frac{d}{dx}(ax^n) = a \cdot nx^{n-1}$$

Radicals

$$\frac{d}{dx}\sqrt[m]{x^n} = \frac{d}{dx}\left(x^{\frac{n}{m}}\right) = \frac{n}{m}x^{\frac{n}{m}-1}$$

Exponentials

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$$

Logarithms

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\log_b(x) = \frac{1}{\ln(b)x}$$

Common Derivatives (Cheat Sheet)

Trigonometric

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x) = -\frac{1}{\sin^2(x)}$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) = \frac{\sin(x)}{\cos^2(x)}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) = -\frac{\cos(x)}{\sin^2(x)}$$

Common Derivatives (Cheat Sheet)

Inverse Trigonometric

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Problem 4:

Problem 4: Calculate f ’(x)

$$f(x) = 2 \cdot x^{23}$$

Problem 4:

$$\frac{d}{dx}(ax^n) = a \cdot nx^{n-1}$$

$f(x) = 2 \cdot x^{23}$

$a = 2$

$n = 23$

$f'(x) = 2 \cdot 23x^{23-1}$

$f'(x) = 46x^{22}$

Problem 5:

Problem 5: Calculate f '(x)

$$f(x) = 7^x$$

Problem 5:

$$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$$

$f(x) = 7^x \qquad a = 7$

$f'(x) = \ln(7) \cdot 7^x$

$f'(x) = 1.94 \cdot 7^x$

Problem 6:

Problem 6: Calculate f ’(x)

$$f(x) = \sqrt[3]{x^7}$$

Exercise: Calculate the derivative

$$\frac{d}{dx}\sqrt[m]{x^n} = \frac{d}{dx}\left(x^{\frac{n}{m}}\right) = \frac{n}{m}x^{\frac{n}{m}-1}$$

$$f(x) = \sqrt[3]{x^7}$$

$$f(x) = x^{7/3}$$

$$f'(x) = \frac{7}{3}x^{7/3-1}$$

$$f'(x) = \frac{7}{3}x^{\frac{7-3}{3}} = \frac{7}{3}x^{4/3} = \frac{7}{3}\sqrt[3]{x^4}$$

Rules for Derivatives

METIS

Rules for Derivatives

Definition:

Addition: $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

Multiplication: $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Composition: $\left(f(g(x))\right)' = f'\left(g(x)\right) \cdot g'(x)$

Example for Addition-Subtraction

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}h(x) = \frac{d}{dx}(\sin(x) + x^2)$$

Example for Addition-Subtraction

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

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$$= \frac{d}{dx}(\sin(x)) + \frac{d}{dx}(x^2)$$

Example for Addition-Subtraction

$$\big(f(x) \pm g(x)\big)' = f'(x) \pm g'(x)$$

$$\frac{d}{dx}h(x) = \frac{d}{dx}(\sin(x) + x^2)$$

$$= \frac{d}{dx}(\sin(x)) + \frac{d}{dx}(x^2) = \cos(x) + 2x$$

Example for Products

$$\big(f(x)\cdot g(x)\big)' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

$$\frac{d}{dx}h(x) = \frac{d}{dx}(x\cdot cos(x))$$

Example for Products

$$\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\frac{d}{dx} h(x) = \frac{d}{dx}(x \cdot cos(x)) = \frac{d}{dx}(x) \cdot \cos(x) + x \cdot \frac{d}{dx}(\cos(x))$$

Example for Products

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$$= 1\cdot\cos(x) - x\cdot\sin(x)$$

Example for Composition (Chain Rule)

$$\left(f\left(g(x)\right)\right)' = f'\left(g(x)\right) \cdot g'(x)$$

Example for Composition (Chain Rule)

$$\big(f(g(x))\big)' = f'\big(g(x)\big) \cdot g'(x)$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}(\sin(x^3 - x^2))$$

Example for Composition (Chain Rule)

$$\big(f(g(x))\big)' = f'\big(g(x)\big) \cdot g'(x)$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}(\sin(x^3 - x^2)) = \frac{d}{ds}(\sin(g(x)))\frac{d}{dx}(x^3 - x^2)$$

Example for Composition (Chain Rule)

$$\big(f(g(x))\big)' = f'\big(g(x)\big) \cdot g'(x)$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}(\sin(x^3 - x^2)) = \frac{d}{ds}(\sin(g(x))) \frac{d}{dx}(x^3 - x^2)$$

$$= \cos(x^3 - x^2)\,(3x^2 - 2x)$$

Problem 7:

Problem 7: Calculate h ’(x)

$$h(x) = \frac{x}{x^2 - 2}$$

Problem 7:

Problem 7: Calculate h ’(x)

$$h(x) = \frac{x}{x^2-2}$$

$$f(x) = x$$

$$f'(x) = 1$$

$$g(x) = \frac{1}{x^2-2} = (x^2-2)^{-1}$$

$$g'(x) = -(x^2-2)^{-2} \cdot 2x = -2x(x^2-2)^{-2}$$

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) =$$

$$1 \cdot \frac{1}{x^2-2} + x \cdot (-2x(x^2-2)^{-2}) = \frac{1}{(x^2-2)} - \frac{2x^2}{(x^2-2)^2} = \frac{x^2-2-2x^2}{(x^2-2)^2} = \frac{-x^2-2}{(x^2-2)^2}$$

Problem 8:

Problem 8: Calculate m '(x)

$m(x) = x \cdot \ln(\cos x)$

Problem 8:

Problem 8: Calculate m ’(x)

$$m(x) = \underset{\substack{\uparrow \\ f(x)}}{x} \cdot \underset{\substack{\uparrow \\ g(h(x))}}{\ln(\cos x)}$$

$$h(x) = \cos x$$
$$h'(x) = -\sin x$$

$$f(x) = x$$
$$f'(x) = 1$$

$$g(h(x)) = \ln(\cos x)$$

$$g(h(x))' = g'(h(x))\, h'(x) =$$
$$\frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

$$m'(x) = f'(x)\, g(h(x)) + f(x) \cdot g(h(x))' = 1 \cdot \ln(\cos x) + x \cdot (-\tan x) =$$
$$\ln(\cos x) - x \cdot \tan x$$

Problem 9:

Problem 9: Calculate m ’(x)

$$m(x): \ln\left(e^x - x\right)$$

Problem 9:

Problem 9: Calculate m ’(x)

$$m(x) = \ln(e^x - x)$$

$$f(g(x)) \qquad g(x) = e^x - x$$
$$g'(x) = e^x - 1$$

$$f(g(x))' = f'(g(x)) \cdot g'(x) = \frac{1}{e^x - x} \cdot (e^x - 1) = \frac{e^x - 1}{e^x - x}$$

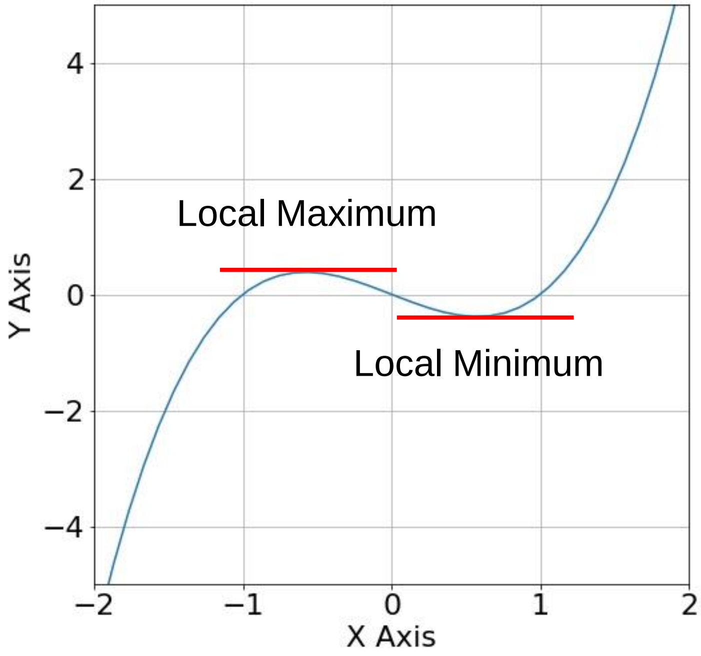
Maximum and Minimum

METIS

Finding Maximum and Minimum

$$f(x) = x^3 - x$$

For which values of x do we have a maximum and/or minimum?

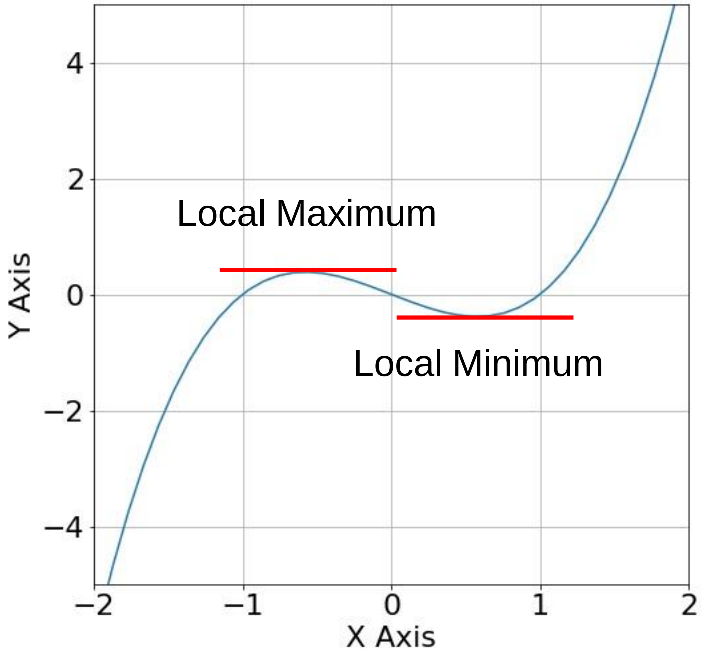


Finding Maximum and Minimum

$$f(x) = x^3 - x$$

For which values of x do we have a maximum and/or minimum?

$$f'(x) = 3x^2 - 1 = 0$$



Finding Maximum and Minimum

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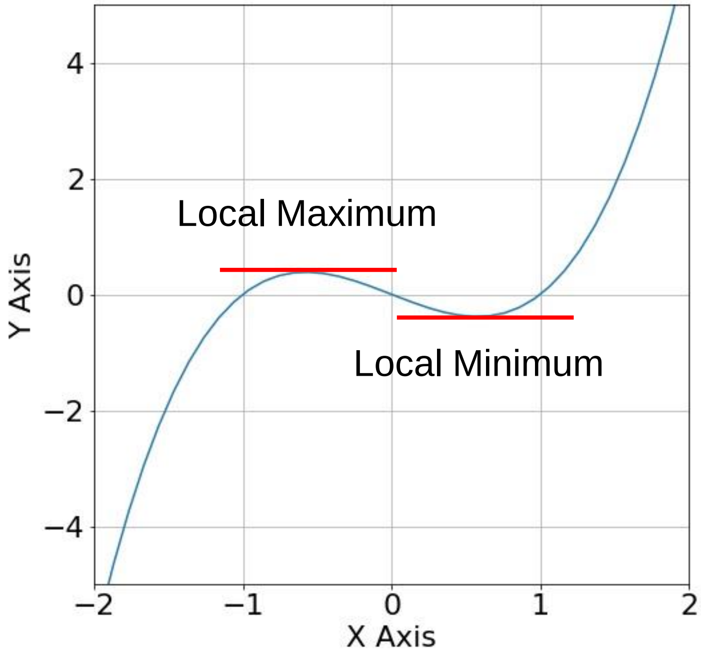

Finding Maximum and Minimum

$$f(x) = x^3 - x$$

For which values of x do we have a maximum and/or minimum?

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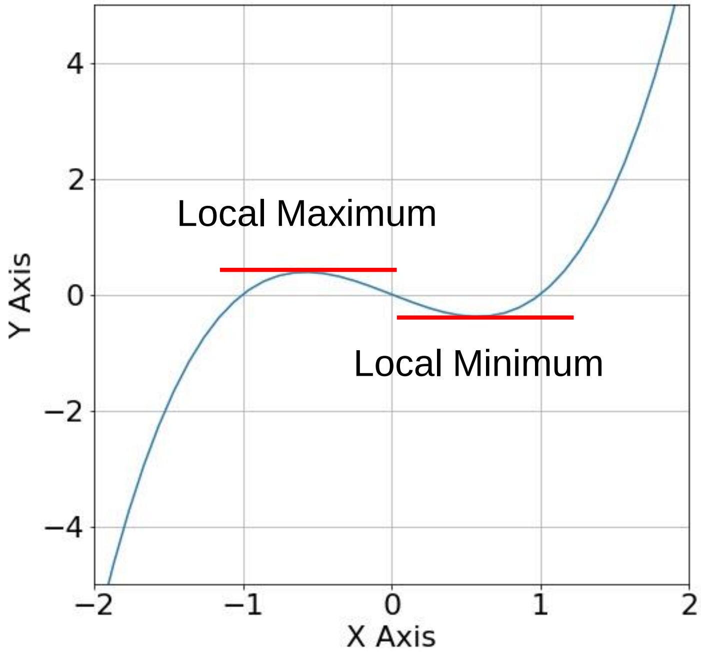
$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$



Finding Maximum and Minimum

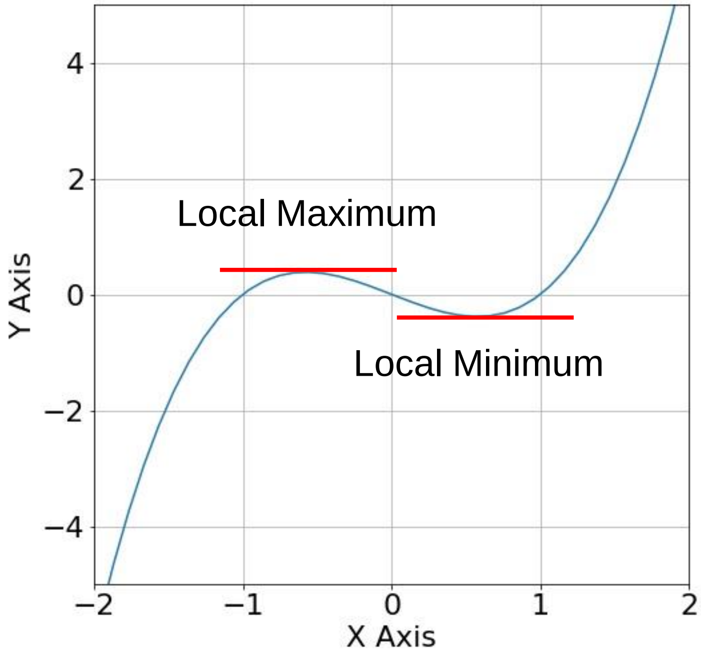
$$f(x) = x^3 - x$$

For which values of x do we have a maximum and/or minimum?

$$f'(x) = 3x^2 - 1 = 0$$

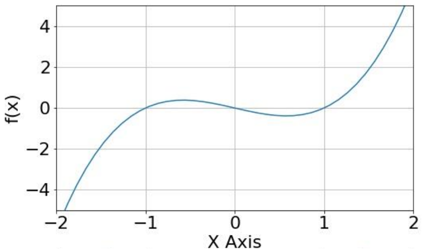
$$3x^2 = 1$$

$$x^2 = \frac{1}{3} \qquad x = \pm\sqrt{\frac{1}{3}} = \pm 0.54$$



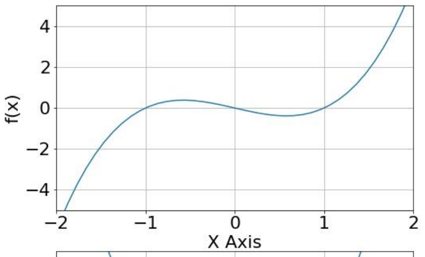
Concavity-Convexity

$$f(x) = x^3 - x$$

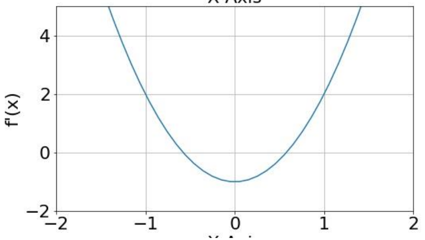


Concavity-Convexity

$$f(x) = x^3 - x$$



$$f'(x) = 3x^2 - 1$$

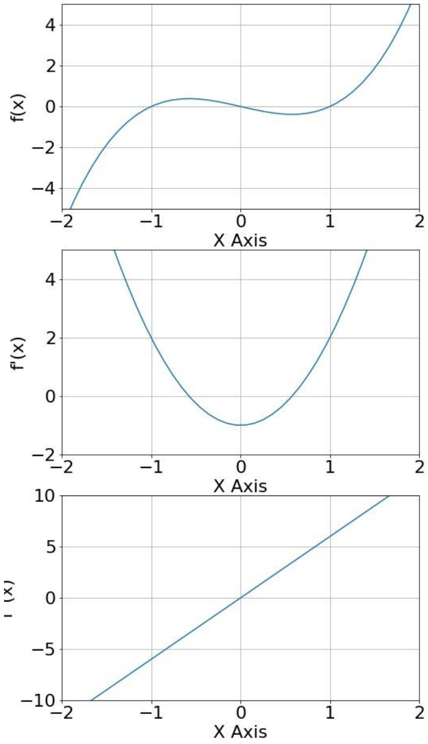


Concavity-Convexity

$$f(x) = x^3 - x$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x$$



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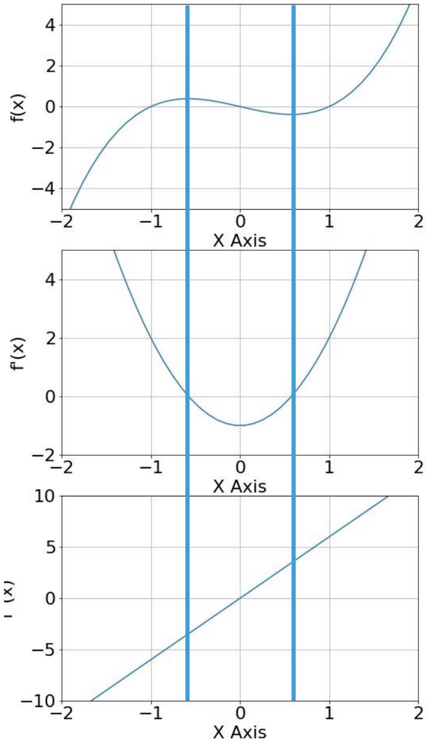
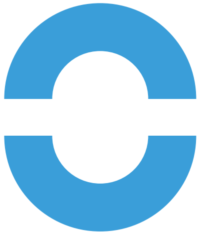




Concave shapes f ''(x) < 0

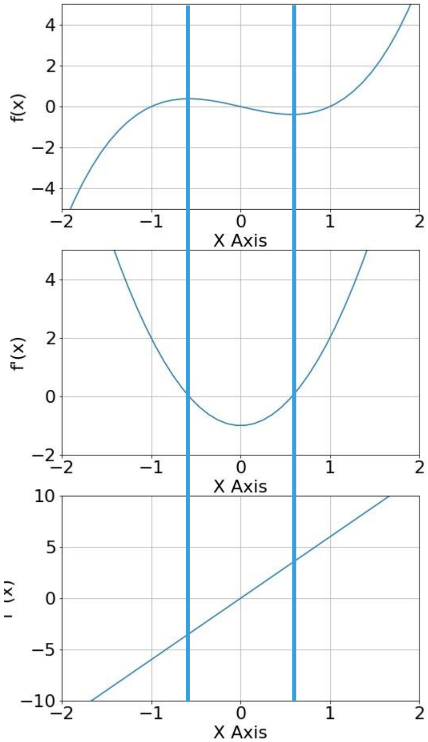
Convex shapes f ''(x) > 0

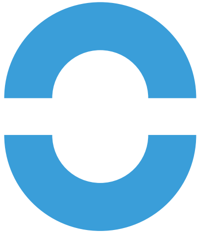
Concavity-Convexity

$$f(x) = x^3 - x$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x$$





Concave shapes f ''(x) < 0

Convex shapes f ''(x) > 0

Definition:

Maximum: f '(x) = 0 and f ''(x) < 0

Minimum: f '(x) = 0 and f ''(x) > 0

Neither: f '(x) = 0 and f ''(x) = 0

Problem 10:

Problem 10:

Find the point where there is a maximum or minimum, and determine if it is a maximum or minimum.

$$f(x) = 60x - x^2$$

Problem 10:

Problem 10:

Find the point where there is a maximum or minimum, and determine if it is a maximum or minimum.

$$f(x) = 60x - x^2$$

$$f'(x) = 60 - 2x = 0$$

$$60 = 2x$$

$$x = 30$$

$$f''(x) = -2$$

$-2 < 0 \Rightarrow$ Concave

At $x = 30$ there is a Maximum

Partial Derivatives and Gradients

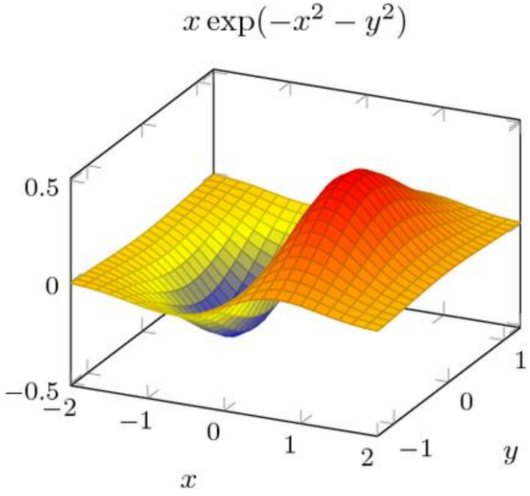
METIS

Partial Derivatives

$$f(x, y) = x^2 - xy$$

$$\frac{\partial}{\partial x} f(x, y) = 2x - y$$

$$\frac{\partial}{\partial y} f(x, y) = -x$$

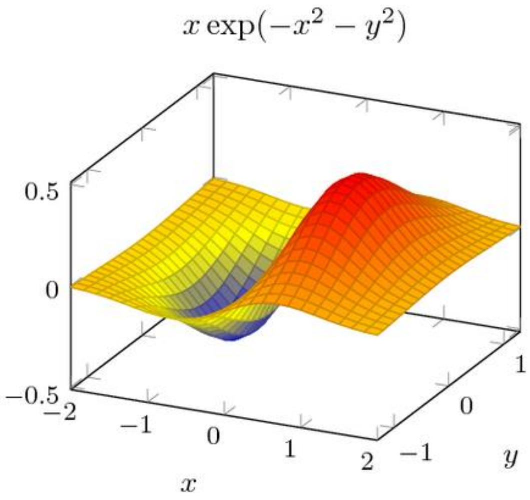


Gradient

$$f(x,y) = x^2 - xy$$

$$\nabla f(x,y) = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

$$\nabla f(x,y) = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$$



Summary

Operator	Symbol	Example
Derivative	$\frac{d}{dx}$	$\frac{d}{dx}x^3 = 3x^2$
Partial Derivative	$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial x}x^3y = 3x^2y$
Gradient	∇	$\nabla x^3y = \begin{bmatrix} 3x^2y \\ x^3 \end{bmatrix}$

Problem 11:

Problem 11: Calculate the gradient.

$$f(x_1, x_2) = x_1 \ln(x_2) + \sin(x_1)$$

Problem 11:

Problem 11: Calculate the gradient.

$$f(x_1, x_2) = x_1 \ln(x_2) + \sin(x_1)$$

$$\frac{\partial f}{\partial x_1} = 1 \cdot \ln(x_2) + \cos(x_1) \qquad \frac{\partial f}{\partial x_2} = \frac{x_1}{x_2}$$

$$\nabla f = \begin{bmatrix} \ln(x_2) + \cos(x_1) \\ x_1 / x_2 \end{bmatrix}$$

Integrals

METIS

Indefinite Integrals

Definition:

Indefinite integrals are the anti-derivatives

$$\int f(x)dx = F(x) + C$$

$$\frac{d}{dx}(F(x) + C) = f(x)$$

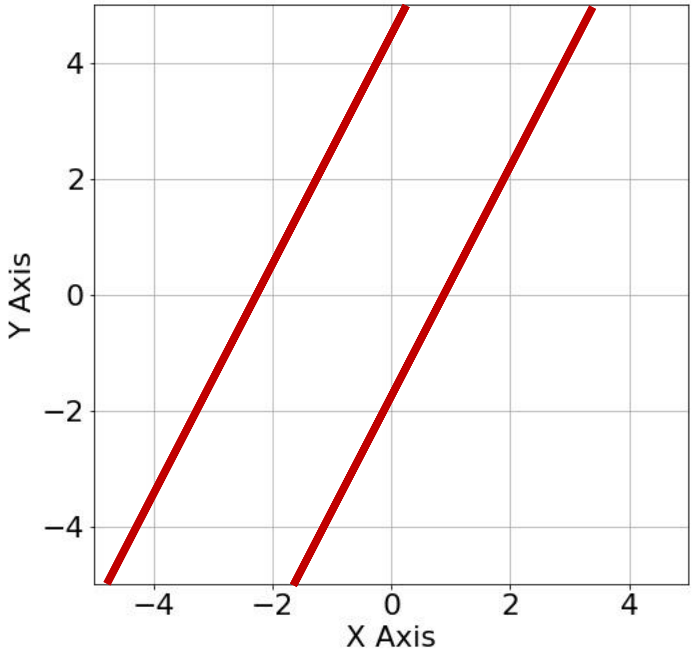
Constant C

Definition:

Indefinite integrals are the anti-derivatives

$$\int f(x)dx = F(x) + C$$

$$\frac{d}{dx}(F(x) + C) = f(x)$$



Integrals (Cheat Sheet)

Polynomials

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Radicals

$$\int \sqrt[m]{x^n} \, dx = \int x^{\frac{n}{m}} \, dx = \frac{x^{\frac{n}{m}+1}}{\frac{n}{m}+1} + C$$

Exponentials

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln(a)} + C$$

Logarithms

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

Integrals (Cheat Sheet)

Trigonometric

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

Integrals (Cheat Sheet)

Inverse Trigonometric

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Example: Indefinite Integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^{20} dx = \frac{x^{20+1}}{20+1} + C = \frac{x^{21}}{21} + C$$

Rules for Integrals

METIS

Rules for Integrals

Definition:

Addition:

$$\int f(x) \pm g(x)\, dx \ = \int f(x)dx \pm \int g(x)dx$$

Multiplication:

$$\int cf(x)dx = c \int f(x)dx$$

Rules for Integrals (Advanced)

Definition:

Integration by parts:

$$\int u(x)v(x)\,dx\ = u(x)\int v(x)dx - \int u'(x)(\int v(x)dx)dx$$

Integration by substitution:

$$\int f(u)du = \int f(g(x))g'(x)dx$$

Problem 12:

Problem 12: Calculate the integral.

$$\int x^3 - 2x \, dx$$

Problem 12:

Problem 12: Calculate the integral.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int x^3 - 2x \, dx$$

$$\int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4}$$

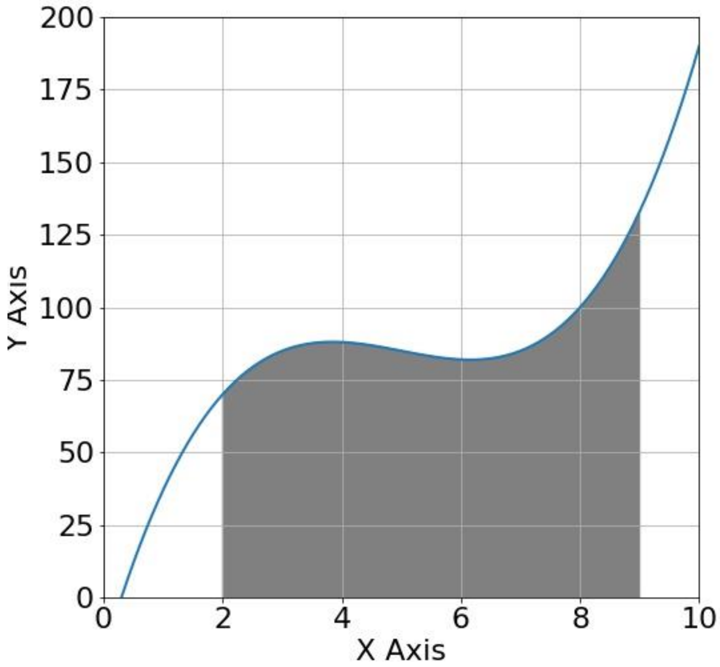
$$\int -2x \, dx = -2 \int x \, dx = -2 \frac{x^{1+1}}{1+1} = -2\frac{x^2}{2} = -x^2$$

$$\int x^3 - 2x \, dx = \frac{x^4}{4} - x^2 + C$$

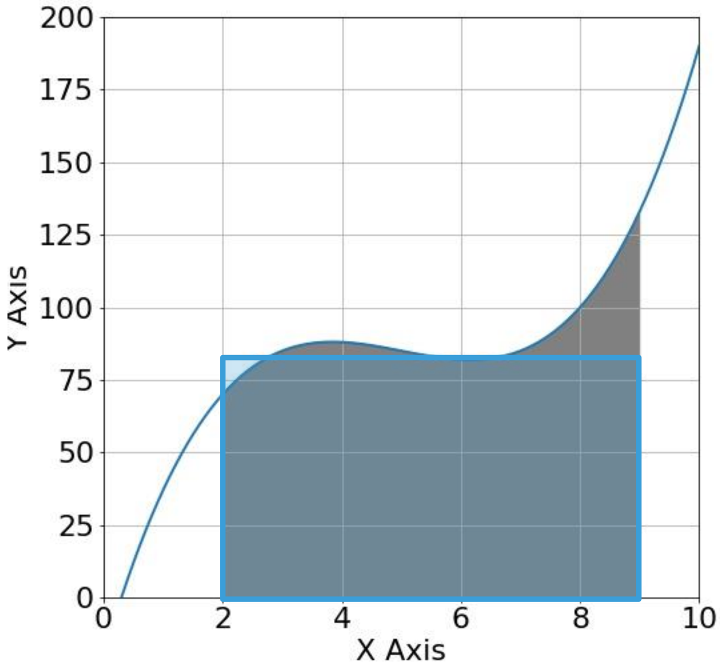
Area Under the Curve

METIS

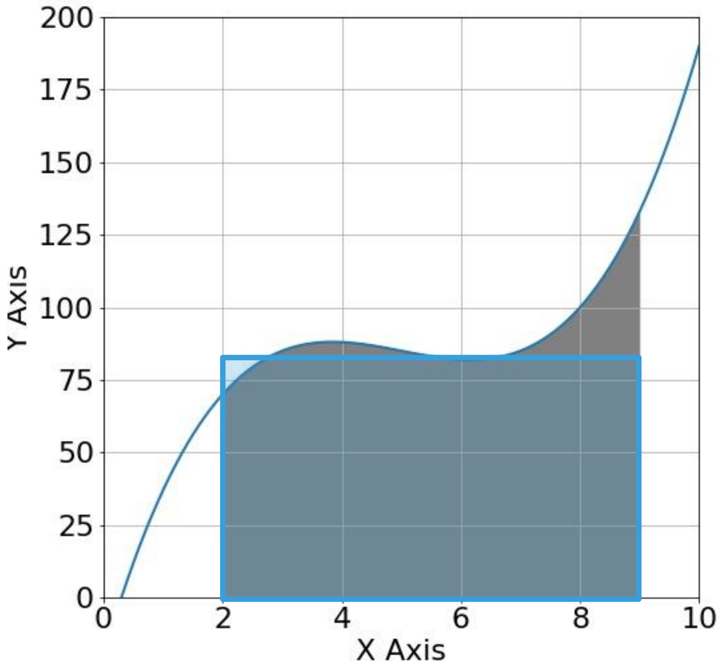
Area Under the Curve



Area Under the Curve

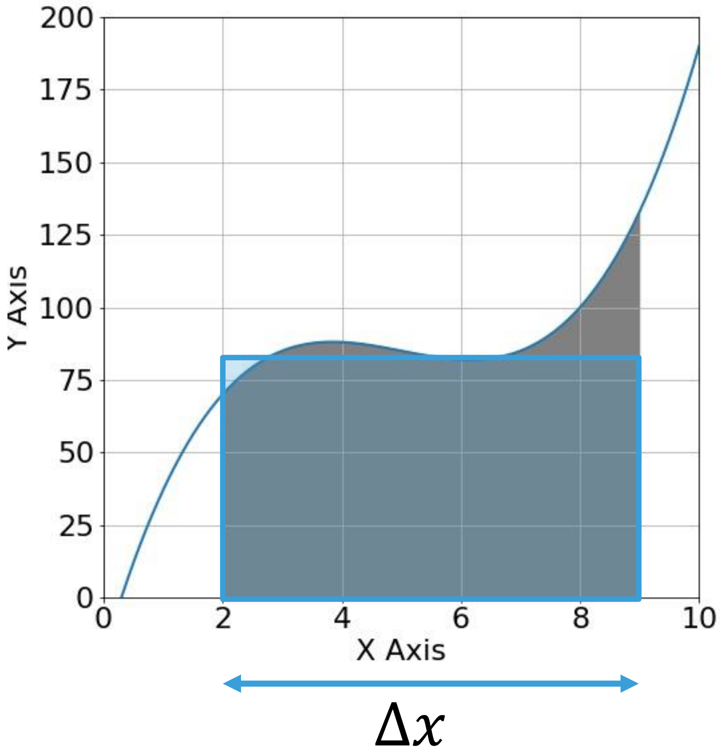


Area Under the Curve



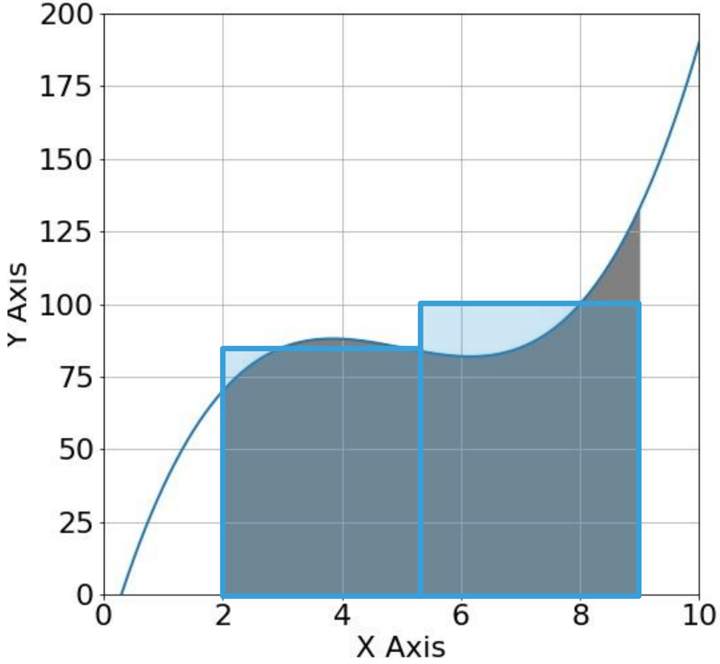
$$A = f(x)\Delta x$$

Area Under the Curve

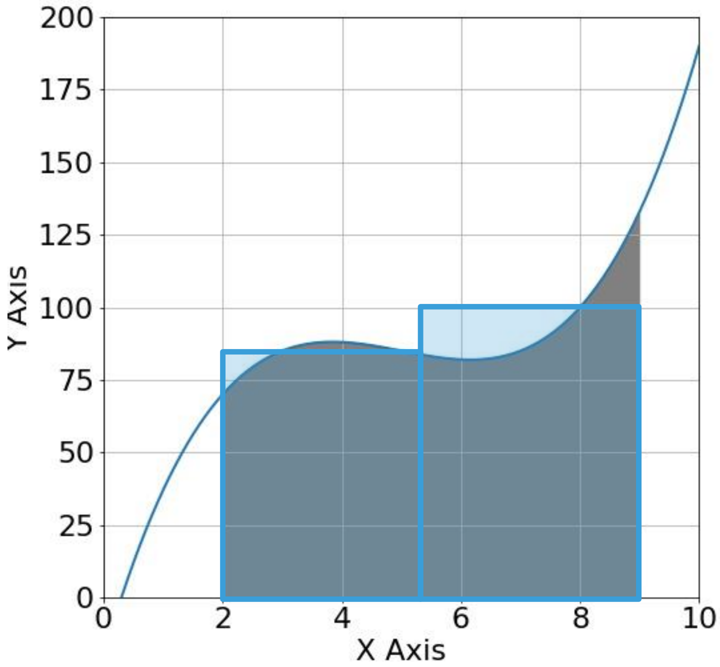


$$A = f(x)\Delta x$$

Area Under the Curve

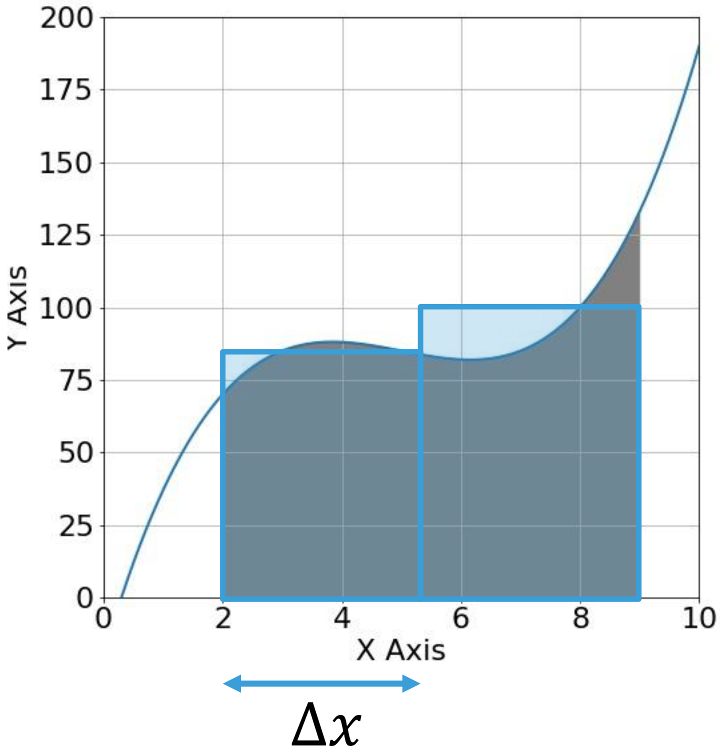


Area Under the Curve



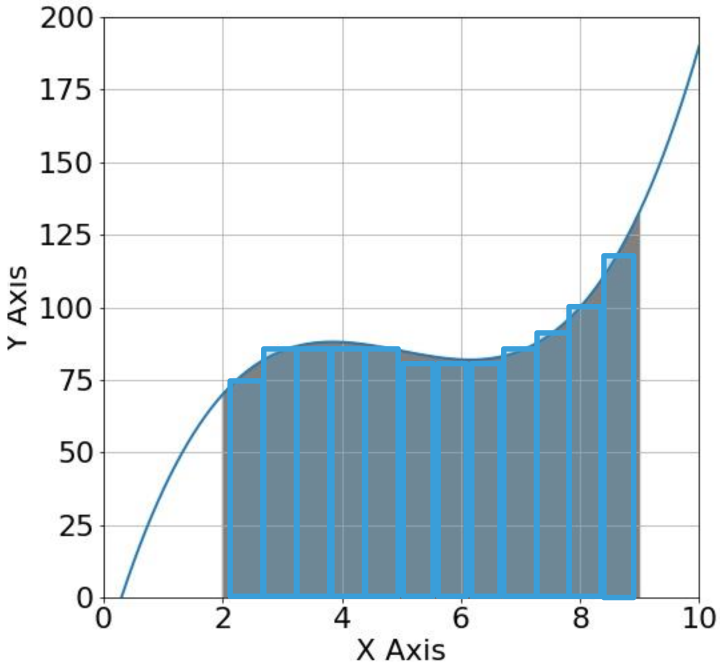
$$A =$$
$$f(x_1)\Delta x + f(x_2)\Delta x$$

Area Under the Curve

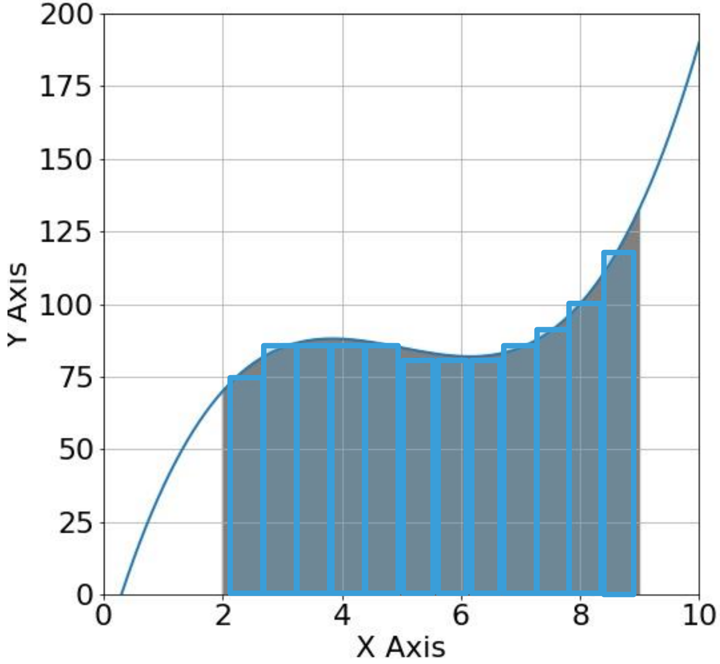


$$A = f(x_1)\Delta x + f(x_2)\Delta x$$

Area Under the Curve

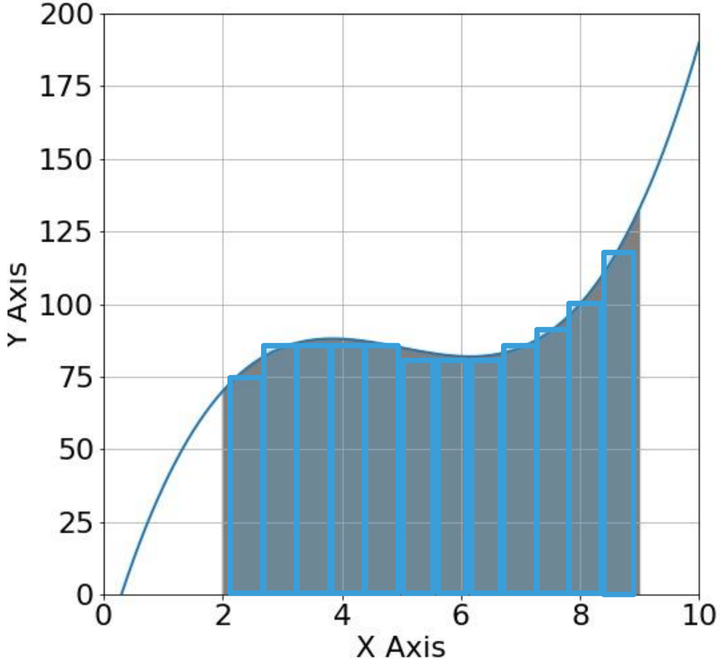


Area Under the Curve



$$A = \sum_{i=1}^{k} f(x_i)\Delta x$$

Area Under the Curve

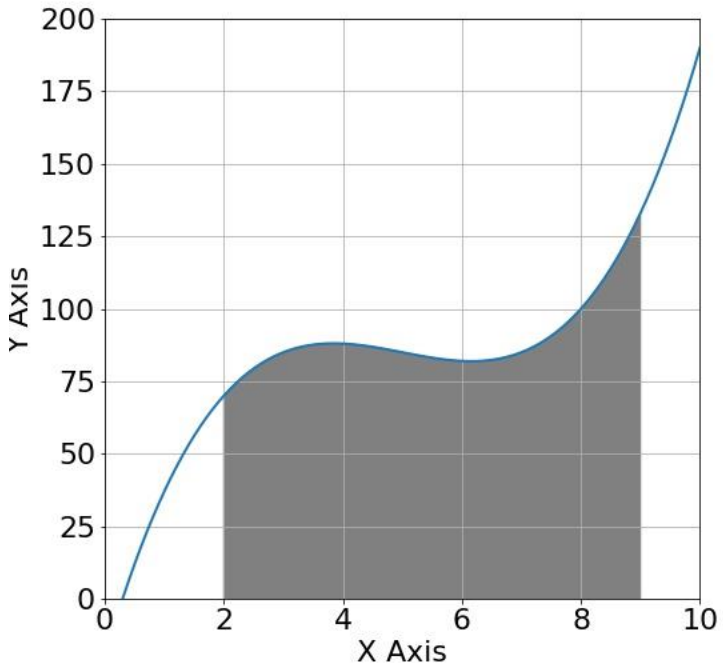


Δx

$$A = \sum_{i=1}^{k} f(x_i)\Delta x$$

Area Under the Curve

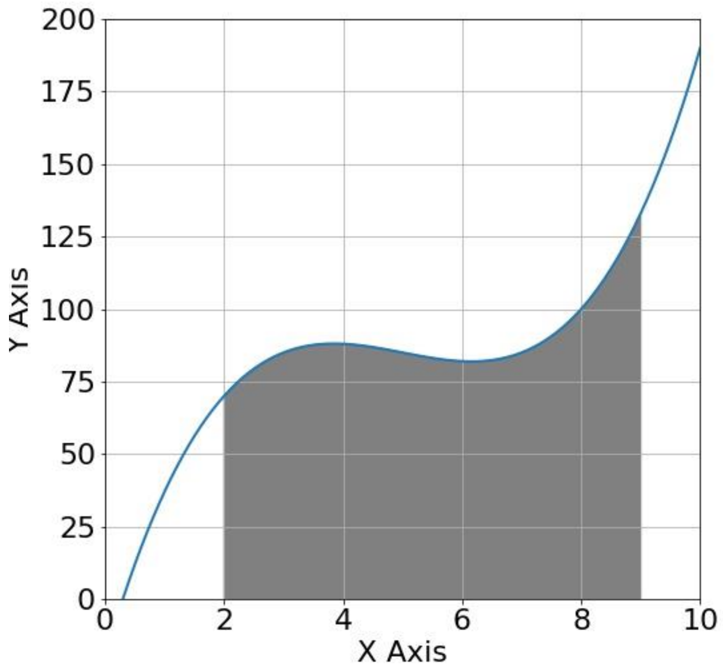
$$A = lim_{k \to \infty} \sum_{i=1}^{k} f(x_i) \Delta x =$$



Area Under the Curve

$$A = lim_{k\to\infty}\sum_{i=1}^{k} f(x_i)\Delta x =$$

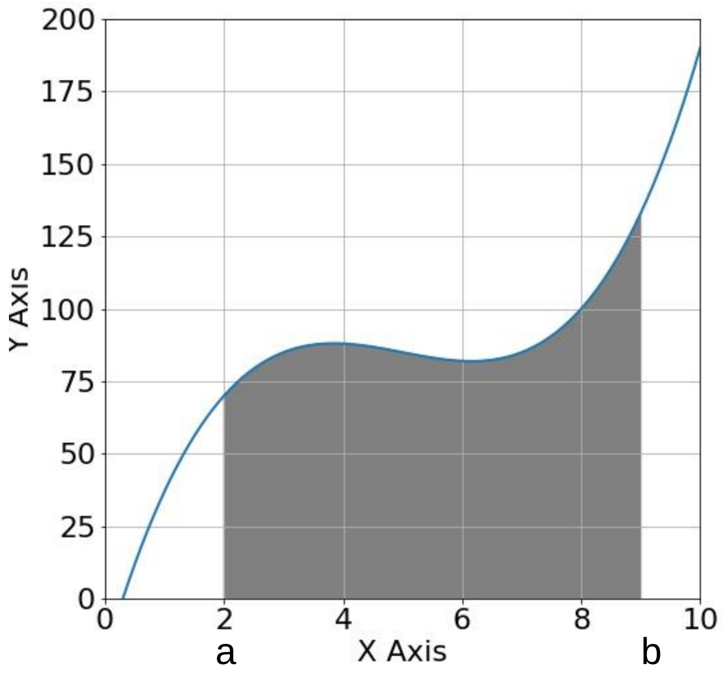
$$\int_a^b f(x_i)dx =$$



Area Under the Curve

$$A = lim_{k \to \infty} \sum_{i=1}^{k} f(x_i) \Delta x =$$

$$\int_a^b f(x_i) dx =$$

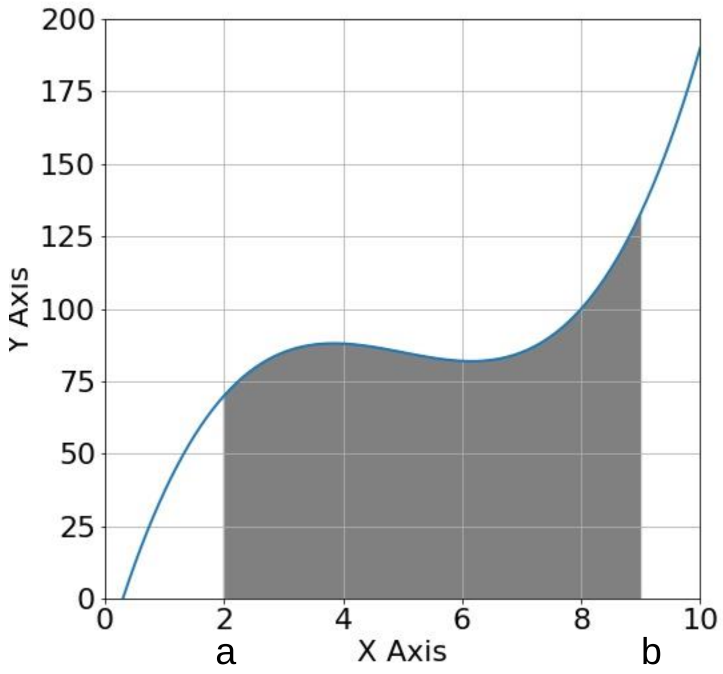


Area Under the Curve

$$A = lim_{k\to\infty}\sum_{i=1}^{k} f(x_i)\Delta x =$$

$$\int_a^b f(x_i)dx =$$

$$F(b) - F(a)$$



Area Under the Curve

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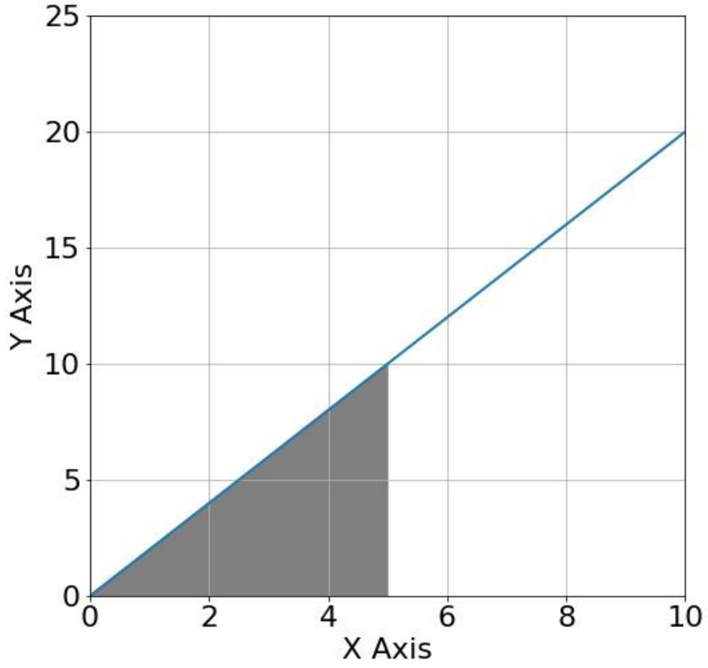
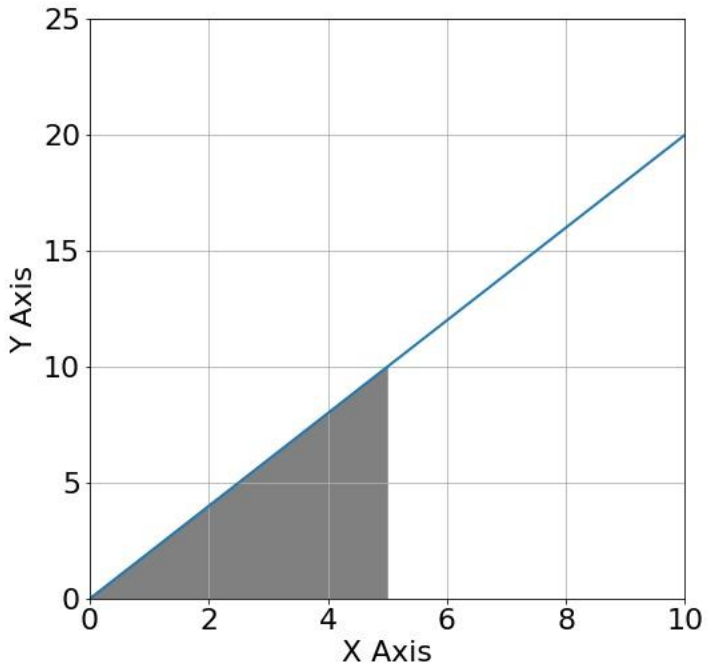


Area Under the Curve

$$\int_a^b f(x_i)dx =$$

$$F(b) - F(a)$$

$$A = \frac{b \cdot h}{2} = \frac{5 \cdot 10}{2} = 25$$



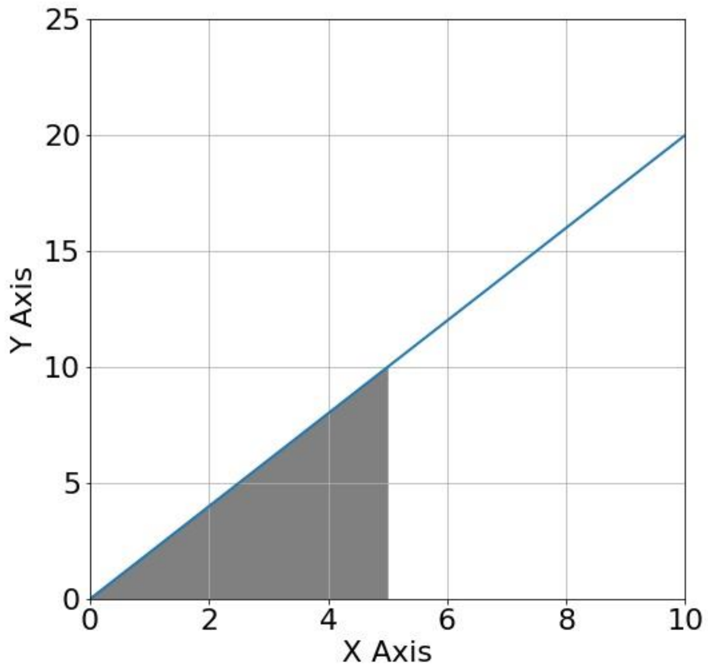
Area Under the Curve

$$\int_a^b f(x_i)dx =$$

$$F(b) - F(a)$$

$$A = \frac{b \cdot h}{2} = \frac{5 \cdot 10}{2} = 25$$

$$\int_0^5 2x dx = x^2 = 5^2 - 0^2 = 25$$



Problem 13:

Problem 13: Calculate the AUC.

$$\int_0^2 (x-3)(x-5)\,dx$$

Problem 13:

Problem 13: Calculate the AUC.

$$\int_a^b f(x_i)dx =$$

$$F(b) - F(a)$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int_0^2 (x-3)(x-5)\,dx$$

$$\int_0^2 x^2 - 8x + 15\,dx = \frac{x^3}{3} - \frac{8x^2}{2} + 15x \Big|_0^2 =$$

$$\frac{2^3}{3} - \frac{8 \cdot 2^2}{2} + 15 \cdot 2 + C - \left(\frac{0^3}{3} - \frac{8 \cdot 0^2}{2} + 15 \cdot 0 + C\right) =$$

$$\frac{8}{3} - 16 + 30 = 14 + \frac{8}{3}$$

QUESTIONS?