Quadcopter : Dynamics and Control

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Contents

What is Quadcopter ?

- Quadcopter is a multirotor helicopter that is lifted and propelled by four rotors[1].

Figure 1: DJI Phantom Drone

X and + Configuration and basic Movements

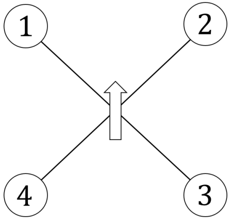


Figure 2: Cross Configuration

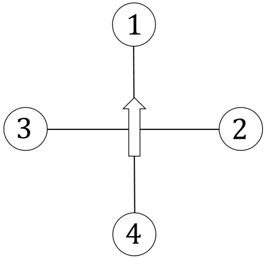


Figure 3: Plus configuration

- Hovering, Up and Down movement
- Forward & Backward movement
- Left & Right movement
- Yawing movement

Assumptions

- The body reference frame is attached to the C.O.M of quadcotper
- Quadcopter system is rigid
- The body frame is coincide with the body principle axes of inertia
- Thrust generated by motor is proportional to the square of the rotor's speed
- External force such as wind forces and drag forces are not considered

FBD & Euler Angle Representation

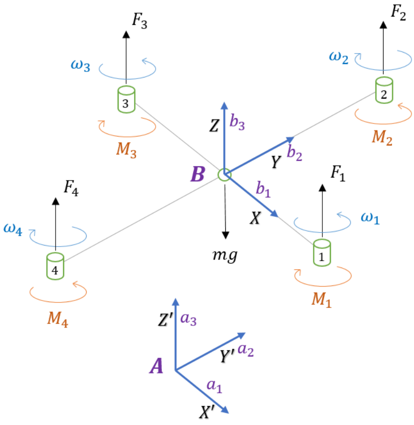


Figure 4: Free Body Diagram

ZXY Euler Angle Representation

$$R_1 = Rot(x, \phi), \quad Roll = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix}$$

$$R_2 = Rot(y, \theta), \quad Pitch = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

$$R_3 = Rot(z, \psi), \quad Yaw = \begin{bmatrix} cos\psi & -sin\psi & 0 \\ sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{AB} = R_3 R_1 R_2$$

$$R = R_3 R_1 R_2 \therefore \qquad R = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ s\psi c\theta + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\theta s\phi c\psi \\ -s\theta c\phi & s\phi & c\phi c\theta \end{bmatrix}$$

$$c \triangleq cos, \; s \triangleq sin$$

EOM

Newton- Euler Equations

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\0\\-mg\end{bmatrix} + R_{AB}\begin{bmatrix}0\\0\\F_1+F_2+F_3+F_4\end{bmatrix} \tag{1}$$

$$I\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}L(F_2-F_4)\\ L(F_2-F_4)\\ M_1-M_2+M_3-M_4\end{bmatrix} - \begin{bmatrix}0 & -r & q\\ r & 0 & -p\\ -q & p & 0\end{bmatrix}\begin{bmatrix}I_{11}p\\ I_{22}q\\ I_{33}r\end{bmatrix} \tag{2}$$

Where,

$$\begin{bmatrix}p\\q\\r\end{bmatrix} = \begin{bmatrix}cos\theta & 0 & -sin\theta cos\theta\\ 0 & 1 & sin\phi\\ sin\theta & 0 & cos\theta cos\phi\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} \tag{3}$$

FBD & EOM of planner Quadcopter

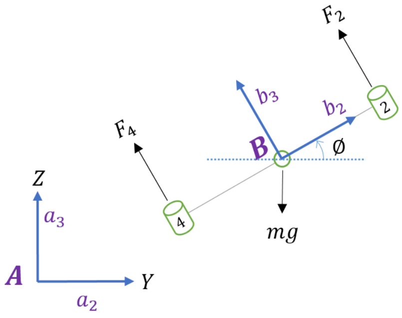


Figure 5: Planner Quadcopter

- Only consider YZ plane
- No Yaw and Pitch motion

EOM of Planner Quadcopter

$$m\ddot{y} = -u_1 \, sin\phi \tag{4}$$

$$m\ddot{z} = -mg + u_1 \, cos\phi \tag{5}$$

$$I_{11}\ddot{\phi} = u_2 \tag{6}$$

Where,

$u_1 = F_2 + F_4$, $u_2 = (F_2 - F_4)l$

State Space model

$$X = \begin{bmatrix} x \\ y \\ \phi \\ \dot{x} \\ \dot{y} \\ \dot{\phi} \end{bmatrix}, \ \dot{X} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \dot{z} \\ \dot{\phi} \\ 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{\sin\phi}{m} & 0 \\ \frac{\cos\phi}{m} & 0 \\ 0 & \frac{1}{I_{11}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7}$$

- Dynamics are Non-linear
- Need to find u_1 and u_2 to follow a certain trajectory
- Under actuated system (Inputs are less then ouputs)

Linearized Dynamical Model

Linearization about hover equilibrium point

$$y, z, \phi = 0$$
$$u_1 = mg$$
$$u_2 = 0$$

i.e. $sin\phi \approx \phi$ and $cos\phi \approx 1$

Linearized EOMs

$$\ddot{y} = -g\,\phi \tag{8}$$
$$\ddot{z} = -g + \frac{u_1}{m} \tag{9}$$
$$\ddot{\phi} = \frac{u_2}{I_{11}} \tag{10}$$

Where,
$u_1 = F_2 + F_4$, $u_2 = (F_2 - F_4)l$

Trajectory Tracking

- Two inputs can track two desired trajectory

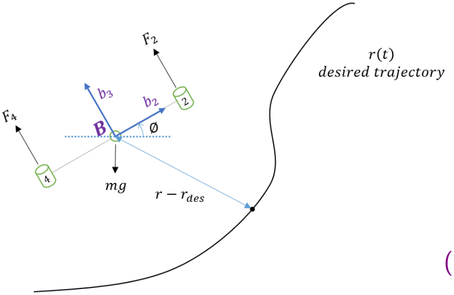


Figure 6: Trajectory Tracking

- Position and Velocity Error

$$e_p = r - r_{des}$$
$$e_v = \dot{r} - \dot{r}_{des}$$

- Want to Decrease Error exponentially to zero

$$(\ddot{r} - \ddot{r}_{des}) + K_v(\dot{r} - \dot{r}_{des}) + K_p(r - r_{des}) = 0$$

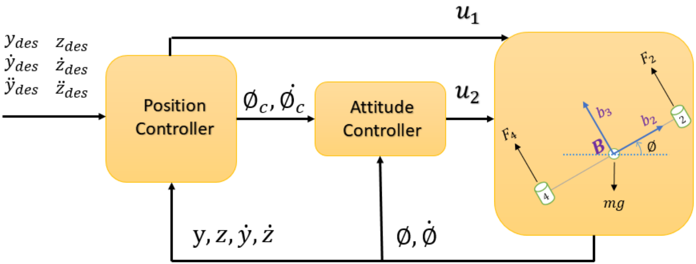
- Positive values of K_p and K_v guaranteed that error goes exponential to zero

Nested Control Structure

$$(\ddot{y} - \ddot{y}_{des}) + K_{v_y}(\dot{y} - \dot{y}_{des}) + K_{p_y}(y - y_{des}) = 0$$

$$(\ddot{z} - \ddot{z}_{des}) + K_{v_z}(\dot{z} - \dot{z}_{des}) + K_{p_z}(z - z_{des}) = 0$$

$$(\ddot{\phi} - \ddot{\phi}_c) + K_{v_\phi}(\dot{z} - \dot{\phi}_c) + K_{p_\phi}(z - \phi_c) = 0$$



$$\ddot{y} = -g\,\phi$$

$$\ddot{z} = -g + \frac{u_1}{m}$$

$$\ddot{\phi} = \frac{u_2}{I_{11}}$$

Control Equations

- Commanded ϕ_c

$$\phi_c = -\frac{1}{g}(\ddot{y}_{des} + K_{v_y}(\dot{y}_{des} - \dot{y}) + K_{p_y}(y_{des} - y)) \tag{11}$$

- Input u_1 and u_2

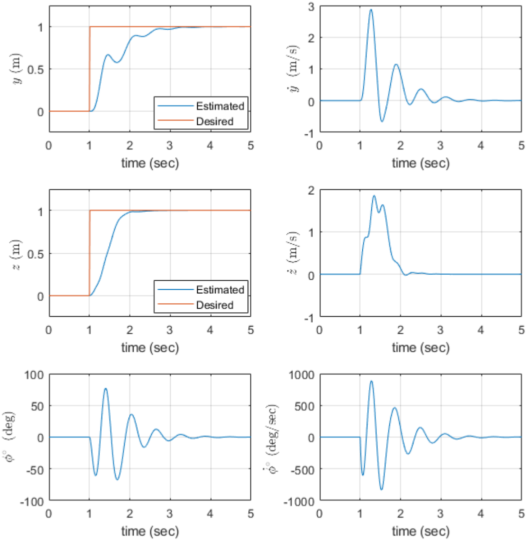
$$u_1 = m(g + \ddot{z}_{des} + K_{v_z}(\dot{z}_{des} - \dot{z}) + K_{p_z}(z_{des} - z)) \tag{12}$$

$$u_2 = I_{11}(\ddot{\phi}_c + K_{v_\phi}(\dot{\phi}_c - \dot{\phi}) + K_{p_\phi}(\phi_c - \phi)) \tag{13}$$

- Need to tune PD gains

Results

Parameter	Value
m	0.5 kg
I_{11}	0.05 kgm^2
l	0.02 m



EOM of Quadcopter

Newton- Euler Equations

$$m\ddot{x} = (cos\psi\, sin\theta + cos\theta\, sin\phi\, sin\psi)u_1 \tag{14}$$

$$m\ddot{y} = (sin\psi\, sin\theta - cos\theta\, sin\phi\, cos\psi)u_1 \tag{15}$$

$$m\ddot{z} = -mg + (cos\phi\, cos\theta)u_1 \tag{16}$$

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = u_2 - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_{11} p \\ I_{22} q \\ I_{33} r \end{bmatrix} \tag{17}$$

$Where,$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & -sin\theta cos\theta \\ 0 & 1 & sin\phi \\ sin\theta & 0 & cos\theta cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix},\ u_2 = \begin{bmatrix} L(F_2 - F_4) \\ L(F_2 - F_4) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} \tag{18}$$

Linearization Dynamical Model

Linearization about hover equilibrium point

$$x, y, z = 0$$
$$\phi, \theta = 0$$
$$\psi = \psi_0$$
$$u_1 = mg$$
$$u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

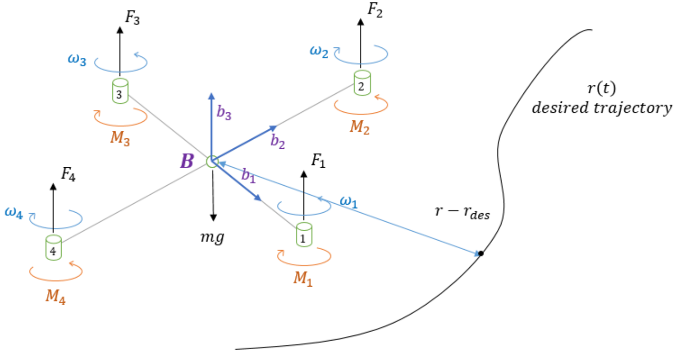
i.e.
$sin\phi \approx \phi$ and $cos\phi \approx 1$
$sin\theta \approx \theta$ and $cos\theta \approx 1$

Linearized EOMs

$$\ddot{x} = (\theta\, cos\psi_0 + \phi\, sin\psi_0)g \quad (19)$$
$$\ddot{y} = (\theta\, sin\psi_0 - \phi\, cos\psi_0)g \quad (20)$$
$$\ddot{z} = -g + \frac{u_1}{m} \quad (21)$$
$$I \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = u_2 \quad (22)$$

- Under actuated system
- Four inputs can track 4 desired trajectories

Trajectory Tracking in 3 Dimensions



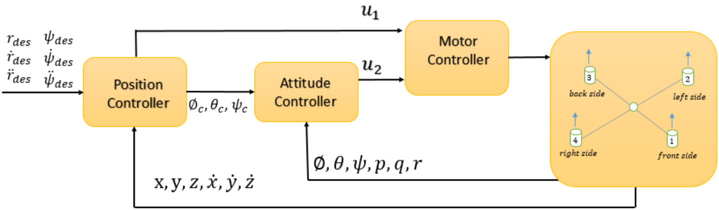
$$(\ddot{x} - \ddot{x}_{des}) + K_{vx}(\dot{x} - \dot{x}_{des}) + K_{p_x}(x - x_{des}) = 0 \quad (23)$$

$$(\ddot{y} - \ddot{y}_{des}) + K_{vy}(\dot{y} - \dot{y}_{des}) + K_{p_y}(y - y_{des}) = 0 \quad (24)$$

$$(\ddot{z} - \ddot{z}_{des}) + K_{vz}(\dot{z} - \dot{z}_{des}) + K_{p_z}(z - z_{des}) = 0 \quad (25)$$

$$(\ddot{\psi} - \ddot{\psi}_{des}) + K_{v\psi}(\dot{\psi} - \dot{\psi}_{des}) + K_{p_\psi}(\psi - \psi_{des}) = 0 \quad (26)$$

Nested Control Structure



$$u_2 = \begin{bmatrix} K_{p_\phi}(\phi_c - \phi) + K_{v\phi}(p_c - p) \\ K_{p_\theta}(\theta_c - \theta) + K_{v\theta}(q_c - q) \\ K_{p_\psi}(\psi_c - \psi) + K_{v\psi}(r_c - r) \end{bmatrix}$$

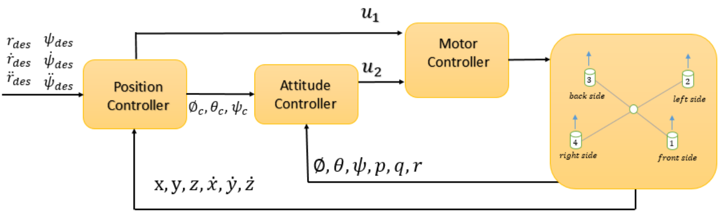
From eq.(23) and eq.(24),

$$\ddot{x}_c = \ddot{x}_{des} + K_{vx}(\dot{x} - \dot{x}_{des}) + K_{p_x}(x - x_{des})$$
$$\ddot{y}_c = \ddot{y}_{des} + K_{vy}(\dot{y} - \dot{y}_{des}) + K_{p_y}(y - y_{des})$$

From eq.(19) and eq.(20),

$$\theta_c = \frac{1}{g}(\ddot{y}_c sin\psi_{des} + \ddot{x}_c cos\psi_{des})$$
$$\phi_c = \frac{1}{g}(\ddot{x}_c sin\psi_{des} - \ddot{y}_c cos\psi_{des})$$
$$\psi_c = \psi_{des}$$

Nested Control Structure



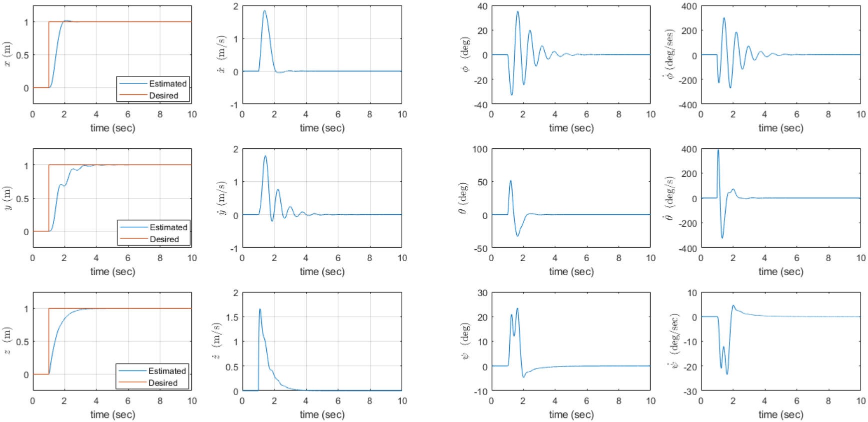
From eq.(21) and eq.(25),

$$u_1 = m(g + \ddot{z}_{des} + K_{v_z}(\dot{z}_{des} - \dot{z}) + K_{p_z}(z_{des} - z)) \tag{27}$$

Results

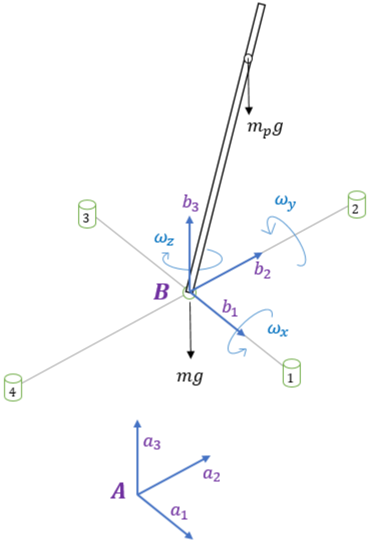
Parameter	Value
m	1 kg
I_{11}, I_{22}	0.05 kg m^2
I_{33}	0.001 kg m^2
L	0.02 m

Results



x (m)
ẋ (m/s)
φ (deg)
φ̇ (deg/ses)
y (m)
ẏ (m/s)
θ (deg)
θ̇ (deg/s)
z (m)
ż (m/s)
ψ (deg)
ψ̇ (deg/sec)
time (sec)
Estimated
Desired

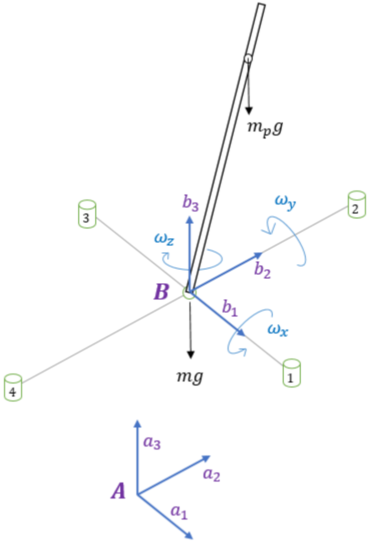
Inverted Pendulum on Quadcopter



Assumption

- Mass of Pendulum is small compared to mass of Quadcopter
- Quadcopter Dynamics do not depend on pendulum
- Spherical joint is Friction less and located at COM of Quadcopter
- Ignore rotational acceleration dynamics thus vehicle body rate can directly control i.e. $\omega_x, \omega_y, \omega_z$
- Zero Yawing

EOM of pendulum



Position of Pendulum

$$x_p = x + r \tag{28}$$

$$y_p = y + s \tag{29}$$

$$z_p = \sqrt{L^2 - r^2 - s^2} \tag{30}$$

r - distance of the COM of pendulum to COM of quadcopter along x axis
s - distance of the COM of pendulum to COM of quadcopter along y axis

EOM of pendulum

Lagrangian of pendulum,

$$\mathcal{L} = \frac{1}{2}\left((\dot{x}+\dot{r})^2 + (\dot{y}+\dot{s})^2 + (\dot{z} - \frac{r\dot{r}+s\dot{s}}{L})^2\right) - g(z+z_p) \tag{31}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0 \tag{32}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) - \frac{\partial \mathcal{L}}{\partial s} = 0 \tag{33}$$

EOM of entire system

For Quad copter

$$\mathrm{m}\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}0\\ 0\\ -mg\end{bmatrix} + \mathrm{R}_{AB}\begin{bmatrix}0\\ 0\\ u_1\end{bmatrix}, \begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix} cos\theta & 0 & sin\theta \\ tan\phi\, sin\theta & 1 & tan\phi\, cos\theta \\ -\frac{sin\theta}{cos\phi} & 0 & \frac{cos\theta}{cos\phi}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

For Pendulum

$$\ddot{r} = \frac{1}{(L^2-s^2){z_p}^2}\left(-r^4\ddot{x} - (L^2-s^2)^2\ddot{x} - 2r^2(s\dot{r}\dot{s} + (\text{-}\mathrm{L}^2 + s^2)\ddot{x}) + r^3(\dot{s}^2 + s\ddot{s} - z_p(g+\ddot{z})) + r(-L^2 s\ddot{s} + s^3\ddot{s} + s^2(-\dot{r}^2 - \dot{s}^2 + z_p(g+\ddot{z}))\right)$$

$$\ddot{s} = \frac{1}{(L^2-r^2){z_p}^2}\left(-s^4\ddot{x} - (L^2-r^2)^2\ddot{x} - 2s^2(r\dot{r}\dot{s} + (-L^2 + r^2)\ddot{x}) + s^3(\dot{r}^2 + r\ddot{r} - z_p(g+\ddot{z})) + s(-L^2 r\ddot{r} + r^3\ddot{r} + r^2(-\dot{s}^2 - \dot{r}^2 + z_p(g+\ddot{z}))\right)$$

Linearized Dynamical Model

Linearization about hover equilibrium point

$$x, y, z, r, s = 0$$
$$\phi, \theta, \psi = 0$$
$$u_1 = (m + m_p)g$$
$$u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e.
$sin\phi \approx \phi$ and $cos\phi \approx 1$
$sin\theta \approx \theta$ and $cos\theta \approx 1$
$sin\psi \approx \psi$ and $cos\psi \approx 1$

Linearized EOMs

$$\ddot{x} = \theta g \tag{34}$$
$$\ddot{y} = -\phi g \tag{35}$$
$$\ddot{z} = \frac{u1}{m} \tag{36}$$
$$\ddot{r} = r\frac{g}{L} - \theta g \tag{37}$$
$$\ddot{s} = s\frac{g}{L} + \phi g \tag{38}$$
$$\dot{\theta} = \omega_y \tag{39}$$
$$\dot{\phi} = \omega_x \tag{40}$$

LQR based controller

Subsystem 1

$$\ddot{x} = \theta g$$
$$\ddot{r} = r\frac{g}{L} - \theta g$$
$$\dot{\theta} = \omega_y$$

Subsystem 2

$$\ddot{y} = -\phi g$$
$$\ddot{s} = s\frac{g}{L} + \phi g$$
$$\dot{\phi} = \omega_x$$

Subsystem 3

$$\ddot{z} = \frac{u1}{m}$$

State-space model

$$X_1 = [x \quad \theta \quad r \quad \dot{x} \quad \dot{r}]^T$$
$$\dot{X}_1 = A_1 X_1 + B_1 \omega_y$$

State-space model

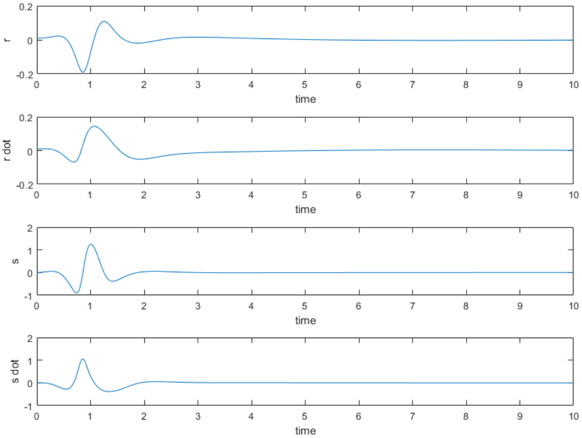
$$X_2 = [y \quad \phi \quad s \quad \dot{y} \quad \dot{s}]^T$$
$$\dot{X}_2 = A_2 X_2 + B_2 \omega_x$$

State-space model

$$X_3 = [z \quad \dot{z}]$$
$$\dot{X}_3 = A_3 X_3 + B_3 u_1$$

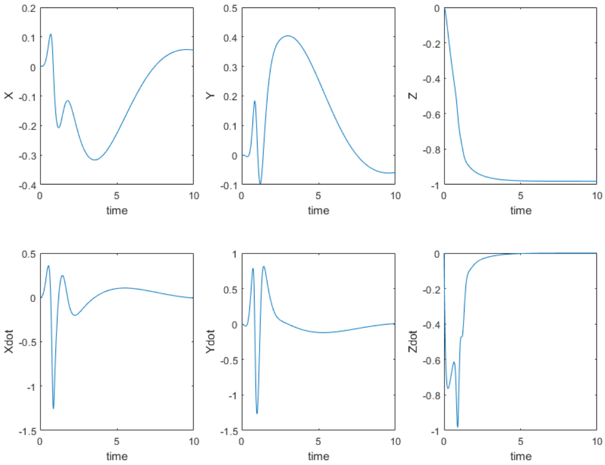
Results

- Pendulum states



Results

- Transnational states



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- Thank You.