Sad Attempts of Conveying My Understanding(Calculus)

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1 Introduction

In this booklet, I will use various definitions for jargons one needs to know in a Single Variable Calculus class or Calculus I for short. I will use 3 different books' definitions for each term then my own personal interpretation in a more normal language, but it is only to help as a guide as to what the various texts' definitions are trying to convey to the student.

As predicted, my knowledge grew and now everything needs to be rewritten.

Notation and terminology one will be seeing frequently.

1. **Function Notation** - $f(x)$, which denotes a function f applied to an argument x.
2. **Limit Notation** - $\lim_{x \to a} f(x)$ represents the limit of $f(x)$ as x approaches a.
3. **Derivative Notation** - $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$ for derivatives.
4. **Integral Notation** - $\int f(x)\,dx$ used for *indefinite integrals*, $\int_a^b f(x)\,dx$ for *definite integrals*.

Part I: Continuity

A function f is defined to be continuous at x_0 if the following three conditions hold:

1. $f(x_0)$ is defined;
2. $\lim_{x \to x_0} f(x)$ exists;
3. $\lim_{x \to x_0} f(x) = f(x_0)$.

(Frank Ayres, Elliott Mendelson, Schaum's Outline of Calculus)

> Their definition isn't clear if you don't already know limits, henceforth they used an example to help strengthen said definition they gave.

DEFINITION f is "continuous at $x = a$" if $f(a)$ is defined and $f(x) \to f(a)$ as $x \to a$. If f is continuous at every point where it is defined, it is a continuous function. *(Gilbert Strang, Calculus)*

What I personally understand

Continuity relies upon a defined function for if said function is not defined then it can be said there isn't anything per se. When looking at the word itself continuity (continue) from basic common sense I could place into mathematical terms that for me to continue on a path there mustn't be anything interrupting said path as such if I came from the right or left side I would meet at the same point every time.

What does "$f(x_0)$ is defined" mean? There must be value at x_0. Therefore, $f(x)$ cannot be "undefined".

What does "$\lim_{x \to x_0} f(x)$ exists" mean? In simple terms, as you walk along the graph of $f(x)$, you should be able to predict exactly where you'll be when you reach without any surprises or abrupt changes.

What does "$\lim_{x \to x_0} f(x) = f(x_0)$" mean? As you approach or get very close to something there aren't any jumps, gaps, shifts at what you're approaching or near what you're approaching.

Part II: Limits

If f is a function, then we say:

A is the limit of $f(x)$ as x approaches a.

if the value of $f(x)$ gets arbitrarily close to A as x approaches a.

This is written in mathematical notation as:

$lim_{x \to a} f(x) = A$

(Frank Ayres, Elliott Mendelson, Schaum's Outline of Calculus)

What I personally understand

Imagine you're walking towards a specific spot marked on the ground—let's call this spot 'a'. As you get closer to 'a', you're supposed to pick up a card from a series of boxes placed along your path. Each box contains a number. The rule is that the numbers on these cards should get closer and closer to a certain number, let's call it 'B', as you approach the spot 'a'.

The idea of a limit is essentially like saying, As I step closer and closer to the spot 'a', the numbers I pick up are nearly the same as 'B', and just before I step on 'a', I am almost certain that the next number I would pick up if I could continue would be 'B'.

1.1 Numerical Example

The numerical example of what was said would be going from 5 to 2. You essentially say $4 \to 3 \to 2$. But keep in mind that there are numbers between $4 \to 5$. Numbers such as $4.1 \to 4.2 \to 4.3 \to 4.4 \to$ etc.

1.1.1 A next illustration

Lets look at the following linear function $f(x) = x + 1$ whatever value you put into x, the output is plus 1.

1.1.2 Specific Points and Calculation

Starting at $x = 1$:

$$f(1) = 1 + 1 = 2$$

Increment x by 1 to $x = 2$:

$$f(2) = 2 + 1 = 3$$

Increment x to $x = 3$:

$$f(3) = 3 + 1 = 4$$

1.1.3 Questions and Detailed Steps

Here are the questions we should be asking ourselves:

˜**What happens as x approaches 3?**

As x increases from 1 to 3, you see the function's output increases from 2 to 4. The example can be broken down into even finer steps to show the gradual progression:

$$\begin{aligned} x = 2.5 :& \quad f(2.5) = 2.5 + 1 = 3.5 \\ x = 2.75 :& \quad f(2.75) = 2.75 + 1 = 3.75 \\ x = 2.9 :& \quad f(2.9) = 2.9 + 1 = 3.9 \\ x = 2.99 :& \quad f(2.99) = 2.99 + 1 = 3.99 \end{aligned}$$

One can see that as x gets closer to 3, $f(x)$ gets closer to 4. The progression shows how $f(x)$ approaches its limit at $x = 3$. This example is very simple and doesn't showcase everything, but it fundamentally demonstrates what is happening.

1.2 Putting Continuity Together

Putting what was said and continuity together:

One can check that $f(x)$ at $x = 3$, $f(3) = 4$, which is the same as the limit of $f(x)$ as x approaches 3.

Part II. Derivative

The derivative of a function describes the function's instantaneous rate of change at a certain point. Another common interpretation is that the derivative gives us the slope of the line tangent to the function's graph at that point.

Symbols

Being straightforward, there isn't much to define in regards to the derivative, so we focus on symbols.

- f' - which is called f prime / taking the first derivative.
- $\frac{dy}{dx}$ - which means we are differentiating y with respect to x.

Things to Note

There is also the 3rd and 4th derivatives. Anything past 2nd is called higher-order derivatives. But you won't see them being represented as f with multiple apostrophes. That becomes annoying and hard to keep track of.

Leibniz Notation

Leibniz notation is written using dy and dx as you see above. So using Leibniz notation it would be:

- First derivative: $\frac{dy}{dx}$
- Second derivative: $\frac{d^2y}{dx^2}$
- Third derivative: $\frac{d^3y}{dx^3}$

Lagrange's Notation

- First derivative: y'
- Second derivative: y''
- Third derivative: y'''

More Things to Note About Each Derivative

- We take the **First Derivative** to find the rate of change, for example understanding motion, growth & decay in real-life scenarios. Slope of the tangent line helps us approximate the function near that point. Identifying extrema which is finding the local minima and maxima of functions by identifying where their derivative is zero (critical points).
- We take the **Second Derivative** to find the concavity & inflection points. A positive second derivative indicates that the graph is concaving upwards. If a positive second derivative means concaving up, the negative second derivative must indicate that the function is concaving downwards. The inflection points refer to where the concavity changes, which is where the second derivative is zero or undefined. The second derivative of position with respect to time is acceleration, indicating how the rate of change over velocity varies over time.
- We take the **Third Derivative** to measure the rate of change of acceleration (jerk). Aerospace engineers and mechanical engineers would need to evaluate that to see how smoothly the vehicle changes its speed.

Being Aware

- **Differentiability -** Not every function is differentiable everywhere. Points where a function fails to be differentiable could include corners, cusps, or vertical tangents.

Explicit Differentiation

When we talk about a function where you can see the output (dependent variable) clearly written as a formula of the input (independent variable), we say it's explicit.

For example: $y = f(x) = x^2 + 3x - 5$. It is evident that y is explicitly defined as a function of x. So one could proceed and differentiate by using the standard rules e.g. power rule, chain rule, quotient rule, product rule.

Implicit Differentiation

These are situations where both variables are mixed together in the equation, and you can't see the output variable by itself on one side. For example, in the equation $x^2 + y^2 = 25$, the approach would be to differentiate both sides with respect to x.

$$\begin{aligned}
&\text{Step 1:} \quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25) \\
&\text{Step 2:} \quad 2x + 2y\frac{dy}{dx} = 0 \\
&\text{Step 3:} \quad 2y\frac{dy}{dx} = -2x \\
&\text{Step 4:} \quad \frac{dy}{dx} = -\frac{x}{y}
\end{aligned}$$

Note: Both sides of the equation are treated with respect to the differentiation variable, usually x, and derivatives of y are handled using the chain rule.

Examples of Local Minima and Maxima

The function is given below:

$$f(x) = x^3 - 3x^2 + 2$$

First order of business is finding the first derivative:

$$f'(x) = 3x^2 - 6x$$

Set derivative equal to zero to find critical points:

$$3x^2 - 6x = 0$$
$$x(3x - 6) = 0$$
$$x = 0 \text{ or } x = 2$$

For $x = 0$:

$$f''(x) = 6x - 6$$
$$f''(0) = -6$$
$$\text{Since } f''(0) < 0, \text{ that tells me that at } x = 0 \text{ there is a local maxima.}$$
$$f(0) = 2$$

For $x = 2$:

$$f''(x) = 6x - 6$$
$$f''(2) = 6$$
$$\text{Since } f''(2) > 0, \text{ that tells me that at } x = 2 \text{ there is a local minima.}$$
$$f(2) = -2$$

Part III. Integration

Integration is a fundamental concept in calculus that deals with finding the accumulation of quantities and the areas under curves. There are two main types of integration: indefinite and definite integration.

Indefinite Integration

Indefinite integration is the process of finding the antiderivative of a function. If $f(x)$ is a continuous function, the indefinite integral of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. This is written as:

$$\int f(x)\,dx = F(x) + C$$

Where:

- $\int$ is the integral sign.
- $f(x)$ is the integrand, the function being integrated.
- dx indicates the variable of integration.
- $F(x)$ is the antiderivative or the indefinite integral of $f(x)$.
- C is the constant of integration, representing an arbitrary constant.

Definite Integration

Definite integration is the process of finding the accumulated value of a function over a specific interval $[a, b]$. If $f(x)$ is a continuous function on the interval $[a, b]$, the definite integral of $f(x)$ from a to b is given by:

$$\int_a^b f(x)\,dx$$

This represents the net area under the curve $f(x)$ from $x = a$ to $x = b$. The definite integral can be evaluated using the Fundamental Theorem of Calculus, which states that if $F(x)$ is an antiderivative of $f(x)$, then:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Where:

- $\int_a^b$ indicates integration from a to b.
- $F(b)$ is the value of the antiderivative at the upper limit b.
- $F(a)$ is the value of the antiderivative at the lower limit a.

Informally and Simply Put

Antiderivative is a function that "undoes" the process of differentiation. If you start with a function and differentiate it, you get another function. The antiderivative is the reverse of this process. It's like asking, "What function did I start with before I took the derivative?"

For example, if you know that the derivative of $F(x)$ is $f(x)$, then $F(x)$ is an antiderivative of $f(x)$. There can be many antiderivatives of a function because you can always add a constant to $F(x)$ and still get a valid antiderivative, since the derivative of a constant is zero. This is why we include $+C$ in the result, where C is any constant.

Riemann Sum

A method for approximating the area under a curve.

Imagine you have a wiggly line (a function) on a graph, and you want to find out how much space is underneath that line between two points on the x-axis. Instead of trying to calculate this area exactly, you break it down into smaller, more manageable pieces.

- Chop the space between your two points on the x-axis into a bunch of smaller intervals. How many intervals you choose is up to you, but more intervals usually give a better approximation. That process is called dividing the interval.
- Over each smaller interval, you draw a rectangle. The height of each rectangle is determined by the value of the function at a specific point within that interval. You can choose different points for this, such as: the left endpoint of the interval, right endpoint of the interval, and midpoint of the interval.
- To find the area of a rectangle, you multiply its height (the function's value at your chosen point) by its width (the width of the interval).
- Then sum the areas of all these rectangles. This total gives you an approximation of the area under the curve.

Example:

$$f(x) = x^2 \text{ from } [1, 3]$$

Split the interval $[1, 3]$ into 4 parts: $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$.

Use the right endpoint of the interval for the height of each rectangle to keep things simple: $f(1.5)$, $f(2)$, $f(2.5)$, $f(3)$.

- Width of each interval is 0.5.
- Heights: $f(1.5) = 1.5^2 = 2.25$, $f(2) = 2^2 = 4$, $f(2.5) = 2.5^2 = 6.25$, $f(3) = 3^2 = 9$.
- Areas: $0.5 \times 2.25 = 1.125$, $0.5 \times 4 = 2$, $0.5 \times 6.25 = 3.125$, $0.5 \times 9 = 4.5$.
- Total area $\approx 1.125 + 2 + 3.125 + 4.5 = 10.75$.

Examples of Basic Integrals: Polynomials

$$
\begin{aligned}
\int x^4\,dx &= \frac{x^5}{5} + C \quad \text{[The Reverse Power Rule]} \\
\int (2x^3 + 5x^2)\,dx &= \int 2x^3\,dx + \int 5x^2\,dx \quad \text{[Integral of a sum of polynomials]} \\
&= \frac{2x^4}{4} + \frac{5x^3}{3} + C \\
&= \frac{x^4}{2} + \frac{5x^3}{3} + C \\
\int (4x^2 - 3x + 7)\,dx &= \int 4x^2\,dx - \int 3x\,dx + \int 7\,dx \quad \text{[Integral of a polynomial with a constant]} \\
&= \frac{4x^3}{3} - \frac{3x^2}{2} + 7x + C \\
\int (6x^5 + 3x^3 - x^2 + 2)\,dx &= \int 6x^5\,dx + \int 3x^3\,dx - \int x^2\,dx + \int 2\,dx \\
&= \frac{6x^6}{6} + \frac{3x^4}{4} - \frac{x^3}{3} + 2x + C \\
&= x^6 + \frac{3x^4}{4} - \frac{x^3}{3} + 2x + C \\
\int (x^4 - 3x^2 + 2x + 5)\,dx &= \int x^4\,dx - \int 3x^2\,dx + \int 2x\,dx + \int 5\,dx \\
&= \frac{x^5}{5} - x^3 + x^2 + 5x + C \\
\int \left(\frac{1}{2}x^3 - \frac{3}{4}x + \frac{5}{6}\right)\,dx &= \frac{1}{2}\int x^3\,dx - \frac{3}{4}\int x\,dx + \frac{5}{6}\int 1\,dx \\
&= \frac{1}{2}\cdot\frac{x^4}{4} - \frac{3}{4}\cdot\frac{x^2}{2} + \frac{5}{6}x + C \\
&= \frac{x^4}{8} - \frac{3x^2}{8} + \frac{5x}{6} + C \\
\int (7x^6 - 4x^4 + 3x^2 - 6)\,dx &= \int 7x^6\,dx - \int 4x^4\,dx + \int 3x^2\,dx - \int 6\,dx \\
&= \frac{7x^7}{7} - \frac{4x^5}{5} + x^3 - 6x + C \\
&= x^7 - \frac{4x^5}{5} + x^3 - 6x + C
\end{aligned}
$$

Basic Trig Integrals

$$\int \sin(x)\,dx = -\cos(x) + C$$
$$\int \cos(x)\,dx = \sin(x) + C$$
$$\int \cot(x)\,dx = \ln|\sin(x)| + C$$
$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$
$$\int \csc(x)\,dx = -\ln|\csc(x) + \cot(x)| + C$$
$$\int \sec^2(x)\,dx = \tan(x) + C$$
$$\int \csc^2(x)\,dx = -\cot(x) + C$$
$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$
$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

Riemann Sums Example

Left Riemann Sum for $f(x)$ on $[a, b]$:

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

Right Riemann Sum for $f(x)$ on $[a, b]$:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

Physics Problem

Given velocity $v(t) = \frac{d}{dt}x(t)$ find position $x(t)$:

$$x(t) = \int v(t)\,dt + C$$

Example: If $v(t) = 5t^2$, then:

$$x(t) = \int 5t^2\,dt = \frac{5t^3}{3} + C$$