Worded Questions Calc

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Key Considerations for Related Rates Problems

When solving related rates problems, there are a few critical aspects to keep in mind:

Identify the Variables

Determine which quantities are changing and assign variables to them. Clearly define what each variable represents.

Understand the Relationships

Find the equation that relates the variables. This could be geometric formulas (like the Pythagorean theorem or volume formulas) or physical laws (like Boyle's law). Sometimes, you may need to use additional relationships, like similar triangles, to express one variable in terms of another.

Differentiate with Respect to Time

Implicit differentiation is your primary tool. Differentiate the equation that relates the variables with respect to time t. Use the chain rule to differentiate each term, remembering that each variable is a function of time.

Substitute Known Values and Rates

After differentiating, substitute the known values for the variables and their rates of change. Pay careful attention to the units and ensure they are consistent.

Solve for the Desired Rate

Isolate the rate you need to find (usually $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dr}{dt}$, etc.). Simplify the equation to solve for this rate.

Practical Tips

- Carefully Read the Problem: Ensure you understand what is given and what you need to find. Identify if quantities are increasing or decreasing and note the units.
- Draw a Diagram: For geometric problems, drawing a diagram can help visualize the relationships between variables. Label the diagram with the known quantities and variables.
- Keep Track of Signs: Rates of change can be positive or negative. A negative rate indicates a decreasing quantity. Be consistent with the signs when substituting values into your differentiated equation.
- Use Consistent Units: Ensure all quantities are in compatible units before substituting into equations.

Displacement in Simple Terms

Displacement is a vector quantity that refers to the change in position of an object. It is the straight-line distance in a specific direction from the starting point to the ending point. In simple terms, displacement tells you "how far out of place" an object is, considering both the distance and the direction.

For example, if you walk 3 meters north and then 4 meters east, your total displacement is the straight-line distance from your starting point to your ending point, considering the direction.

Using Integration to Find Displacement

When dealing with motion, especially in physics, we often use integration to find displacement when we know the velocity function.

Velocity is the rate of change of position with respect to time. If $v(t)$ is the velocity function, the displacement $s(t)$ from time $t = a$ to $t = b$ can be found using the integral of the velocity function:

$$s(t) = \int_a^b v(t)\, dt$$

This integral gives us the total change in position, or the displacement, over the time interval from $t = a$ to $t = b$.

Example

Suppose the velocity of a car is given by $v(t) = 3t^2$ meters per second, where t is in seconds. To find the displacement of the car from $t = 0$ to $t = 2$ seconds, we integrate the velocity function:

$$s(t) = \int_0^2 3t^2\, dt$$

Evaluate the integral:

$$s(t) = t^3\Big|_0^2 = 2^3 - 0^3 = 8 \text{ meters}$$

So, the displacement of the car over the first 2 seconds is 8 meters.

Using Derivatives to Find Displacement

While integration helps us find displacement from velocity, derivatives are used to find velocity and acceleration from displacement.

Position function $s(t)$ gives the position of an object at time t. The **velocity** $v(t)$ is the derivative of the position function:

$$v(t) = \frac{ds(t)}{dt}$$

The **acceleration** $a(t)$ is the derivative of the velocity function, which is the second derivative of the position function:

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2s(t)}{dt^2}$$

Example

If the position of a particle is given by $s(t) = t^3 - 6t^2 + 9t$, we can find the velocity by differentiating $s(t)$:

$$v(t) = \frac{ds(t)}{dt} = 3t^2 - 12t + 9$$

We can find the acceleration by differentiating the velocity function:

$$a(t) = \frac{dv(t)}{dt} = 6t - 12$$

Summary

- **Displacement**: The change in position of an object in a specific direction.
- **Integration**: Used to find displacement from velocity.

$$s(t) = \int v(t)\, dt$$

- **Derivatives**: Used to find velocity and acceleration from position.

$$v(t) = \frac{ds(t)}{dt}$$

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2s(t)}{dt^2}$$

Problem Solutions

Problem 1

A kite is flying at a height of 40 ft. A boy is flying it so that it is moving horizontally at a rate of 3 ft/sec. If the string is taut, at what rate is the string being paid out when the length of the string released is 50 ft?

Solution

$$y = 40 \text{ ft (constant)}, \quad x = \text{horizontal distance}, \quad s = \text{string length}$$

$$\frac{dx}{dt} = 3 \text{ ft/sec}$$

$$s^2 = x^2 + y^2$$

Differentiate with respect to t:

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

Substitute known values:

$$50^2 = x^2 + 40^2 \implies 2500 = x^2 + 1600 \implies x = 30 \text{ ft}$$

$$2(50)\frac{ds}{dt} = 2(30)(3)$$

$$100\frac{ds}{dt} = 180 \implies \frac{ds}{dt} = 1.8 \text{ ft/sec}$$

Problem 2

A spherical balloon is being inflated so that its volume is increasing at the rate of 5 ft^3/min. At what rate is the diameter increasing when the diameter is 12 ft?

Solution

$$V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dt} = 5 \text{ ft}^3/\text{min}, \quad d = 12 \text{ ft} \implies r = 6 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute known values:

$$5 = 4\pi(6^2)\frac{dr}{dt}$$

$$5 = 144\pi \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{5}{144\pi} \text{ ft/min}$$

Since $d = 2r$:

$$\frac{dd}{dt} = 2 \cdot \frac{5}{144\pi} = \frac{10}{144\pi} = \frac{5}{72\pi} \text{ ft/min}$$

Problem 3

A spherical snowball is being made so that its volume is increasing at the rate of 8 ft^3/min. Find the rate at which the radius is increasing when the snowball is 4 ft in diameter.

Solution

$$V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dt} = 8 \text{ ft}^3/\text{min}, \quad d = 4 \text{ ft} \implies r = 2 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute known values:

$$8 = 4\pi(2^2)\frac{dr}{dt}$$

$$8 = 16\pi\frac{dr}{dt} \implies \frac{dr}{dt} = \frac{8}{16\pi} = \frac{1}{2\pi} \text{ ft/min}$$

Problem 4

Suppose that when the diameter is 6 ft the snowball in Exercise 3 stopped growing and started to melt at the rate of $\frac{dV}{dt} = -3 \text{ ft}^3/\text{min}$. Find the rate at which the radius is changing when the radius is 3 ft.

Solution

$$V = \frac{4}{3}\pi r^3, \quad \frac{dV}{dt} = -3 \text{ ft}^3/\text{min}, \quad r = 3 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute known values:

$$-3 = 4\pi(3^2)\frac{dr}{dt}$$

$$-3 = 36\pi\frac{dr}{dt} \implies \frac{dr}{dt} = \frac{-3}{36\pi} = \frac{-1}{12\pi} \text{ ft/min}$$

Problem 5

Sand is being dropped at the rate of 10 ft^3/min onto a conical pile. If the height of the pile is always twice the base radius, at what rate is the height increasing when the pile is 8 ft high?

Solution

$$V = \frac{\pi h^3}{12}, \quad \frac{dV}{dt} = 10 \text{ ft}^3/\text{min}, \quad h = 8 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Substitute known values:

$$10 = \frac{\pi(8^2)}{4}\frac{dh}{dt}$$

$$10 = 16\pi\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{10}{16\pi} = \frac{5}{8\pi} \text{ ft/min}$$

Problem 6

A light is hung 15 ft above a straight horizontal path. If a man 6 ft tall is walking away from the light at the rate of 5 ft/sec, how fast is his shadow lengthening?

Solution

$$\frac{15}{x+s} = \frac{6}{s}$$

$$15s = 6(x+s)$$

$$15s = 6x + 6s$$

$$9s = 6x \implies s = \frac{2}{3}x$$

Differentiate with respect to t:

$$\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt}$$

Substitute known values:

$$\frac{ds}{dt} = \frac{2}{3} \cdot 5 = \frac{10}{3} \text{ ft/sec}$$

Problem 7

In Exercise 6, at what rate is the tip of the man's shadow moving?

Solution

$$\frac{dx}{dt} = 5 \text{ ft/sec}, \quad \frac{ds}{dt} = \frac{10}{3} \text{ ft/sec}$$

$$\text{Rate of tip of shadow} = \frac{dx}{dt} + \frac{ds}{dt} = 5 + \frac{10}{3} = \frac{15}{3} + \frac{10}{3} = \frac{25}{3} \text{ ft/sec}$$

Problem 8

A man 6 ft tall is walking toward a building at the rate of 5 ft/sec. If there is a light on the ground 50 ft from the building, how fast is the man's shadow on the building growing shorter when he is 30 ft from the building?

Solution

$$\frac{6}{x} = \frac{s}{50}$$

$$6 \cdot 50 = s \cdot x \implies s = \frac{300}{x}$$

Differentiate with respect to t:

$$\frac{ds}{dt} = 300 \left(-\frac{1}{x^2} \right) \frac{dx}{dt}$$

Substitute known values:

$$\frac{ds}{dt} = 300 \left(-\frac{1}{30^2} \right) (-5) = \frac{300}{900} \cdot 5 = \frac{5}{3} \text{ ft/sec}$$

Problem 9

A water tank in the form of an inverted cone is being emptied at the rate of 6 ft^3/min. The altitude of the cone is 24 ft, and the base radius is 12 ft. Find how fast the water level is lowering when the water is 10 ft deep.

Solution

$$V = \frac{\pi h^3}{12}, \quad \frac{dV}{dt} = -6 \text{ ft}^3/\text{min}, \quad h = 10 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

Substitute known values:

$$-6 = \frac{\pi(10^2)}{4}\frac{dh}{dt}$$

$$-6 = 25\pi\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{-6}{25\pi} \text{ ft/min}$$

Problem 10

A trough is 12 ft long and its ends are in the form of inverted isosceles triangles having an altitude of 3 ft and a base of 3 ft. Water is flowing into the trough at the rate of 2 ft^3/min. How fast is the water level rising when the water is 1 ft deep?

Solution

$$V = 6h^2, \quad \frac{dV}{dt} = 2 \text{ ft}^3/\text{min}, \quad h = 1 \text{ ft}$$

Differentiate with respect to t:

$$\frac{dV}{dt} = 12h\frac{dh}{dt}$$

Substitute known values:

$$2 = 12 \cdot 1 \cdot \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{2}{12} = \frac{1}{6} \text{ ft/min}$$

Problem 11

Boyle's law for the expansion of gas is $PV = C$, where P is the number of pounds per square unit of pressure, V is the number of cubic units of volume of the gas, and C is a constant. At a certain instant, the pressure is 3000 lb/ft$^{\mathbf{2}}$, the volume is 5 ft$^{\mathbf{3}}$, and the volume is increasing at the rate of 3 ft$^{\mathbf{3}}$/min. Find the rate of change of the pressure at this instant.

Solution

$$P = 3000 \text{ lb/ft}^{\mathbf{2}}, \quad V = 5 \text{ ft}^{\mathbf{3}}, \quad \frac{dV}{dt} = 3 \text{ ft}^{\mathbf{3}}\text{/min}$$

Differentiate with respect to t:

$$P\frac{dV}{dt} + V\frac{dP}{dt} = 0$$

Substitute known values:

$$3000 \cdot 3 + 5 \cdot \frac{dP}{dt} = 0$$

$$9000 + 5 \cdot \frac{dP}{dt} = 0$$

$$5 \cdot \frac{dP}{dt} = -9000 \implies \frac{dP}{dt} = -1800 \text{ lb/ft}^{\mathbf{2}}\text{/min}$$

Problem 12

The adiabatic law (no gain or loss of heat) for the expansion of air is $PV^{1.4} = C$, where P is the number of pounds per square unit of pressure, V is the number of cubic units of volume, and C is a constant. At a specific instant, the pressure is 40 lb/in^2 and is increasing at the rate of 8 lb/in^2 each second. What is the rate of change of volume at this instant?

Solution

$$P = 40 \text{ lb/in}^2, \quad \frac{dP}{dt} = 8 \text{ lb/in}^2\text{/sec}$$

Differentiate with respect to t:

$$P \cdot 1.4V^{0.4}\frac{dV}{dt} + V^{1.4}\frac{dP}{dt} = 0$$

Substitute known values:

$$40 \cdot 1.4V^{0.4}\frac{dV}{dt} + V^{1.4} \cdot 8 = 0$$

$$56V^{0.4}\frac{dV}{dt} = -8V^{1.4}$$

Divide both sides by $V^{0.4}$:

$$56\frac{dV}{dt} = -8V$$

$$\frac{dV}{dt} = \frac{-8V}{56} = -\frac{V}{7}$$