Physics topics that uses derivatives and integration

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Important Physics Topics Using Derivatives and Integration

1. Kinematics (Position, Velocity, Acceleration)

In kinematics, we study the motion of objects. The key concepts include position, velocity, and acceleration, which are related through derivatives and integrals.

- **Position** $s(t)$ is the location of an object at time t.
- **Velocity** $v(t)$ is the rate of change of position with respect to time:

$$v(t) = \frac{ds(t)}{dt}$$

- **Acceleration** $a(t)$ is the rate of change of velocity with respect to time:

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2s(t)}{dt^2}$$

- To find position from velocity:

$$s(t) = \int v(t)\,dt$$

- To find velocity from acceleration:

$$v(t) = \int a(t)\,dt$$

2. Work and Energy

Work and energy are fundamental concepts in physics. Work is done when a force causes displacement.

- **Work** W done by a constant force F over a displacement d is given by:

$$W = F \cdot d$$

- For a variable force $F(x)$, work is calculated using the integral:

$$W = \int_a^b F(x)\,dx$$

- **Kinetic Energy** KE is the energy of motion:

$$KE = \frac{1}{2}mv^2$$

- **Potential Energy** PE is the energy stored in an object due to its position:

$$PE = mgh$$

3. Motion Under Gravity

Motion under gravity is a common example of uniformly accelerated motion.

- The acceleration due to gravity g is approximately $9.8\,\mathrm{m/s^2}$ downward.
- The position $s(t)$ and velocity $v(t)$ of an object in free fall can be described by:

$$v(t) = v_0 - gt$$

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$$

where v_0 is the initial velocity and s_0 is the initial position.

4. Simple Harmonic Motion

Simple Harmonic Motion (SHM) describes oscillatory motion such as springs and pendulums.

- The position $x(t)$ of an object in SHM is given by:

$$x(t) = A\cos(\omega t + \phi)$$

where A is the amplitude, ω is the angular frequency, and ϕ is the phase constant.

- The velocity $v(t)$ and acceleration $a(t)$ can be found by differentiating $x(t)$:

$$v(t) = -A\omega\sin(\omega t + \phi)$$

$$a(t) = -A\omega^2\cos(\omega t + \phi) = -\omega^2 x(t)$$

5. Projectile Motion

Projectile motion involves objects moving in two dimensions under the influence of gravity.

- The horizontal and vertical components of motion are treated separately.
- The horizontal motion is uniform:

$$x(t) = v_{0x}t$$

- The vertical motion is uniformly accelerated:

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

- The overall position is given by combining the components:

$$\mathbf{r}(t) = (x(t), y(t))$$

Rate Of Change In Natural and Social Sciences

Exercise 1

Given the position function $f(t) = t^3 - 8t^2 + 24t$:

(a) Find the velocity at time t.

$$v(t) = s'(t) = \frac{d}{dt}(t^3 - 8t^2 + 24t) = 3t^2 - 16t + 24$$

(b) What is the velocity after 1 second?

$$v(1) = 3(1)^2 - 16(1) + 24 = 3 - 16 + 24 = 11\,\text{ft/s}$$

(c) When is the particle at rest?

The particle is at rest when $v(t) = 0$.

$$3t^2 - 16t + 24 = 0$$

Solving this quadratic equation:

$$t = \frac{16 \pm \sqrt{256 - 4 \cdot 3 \cdot 24}}{2 \cdot 3} = \frac{16 \pm \sqrt{256 - 288}}{6} = \frac{16 \pm \sqrt{-32}}{6}$$

Since the discriminant is negative, there are no real solutions, so the particle is never at rest.

(d) When is the particle moving in the positive direction?

The particle is moving in the positive direction when $v(t) > 0$.

Since $3t^2 - 16t + 24$ does not have real roots and opens upwards (leading coefficient is positive), $v(t)$ is always positive.

(e) Find the total distance traveled during the first 6 seconds.

We need to integrate the absolute value of the velocity function over the interval [0, 6].

Since $v(t)$ is always positive, the total distance traveled is simply the integral of $v(t)$.

$$\text{Distance} = \int_0^6 v(t)\,dt = \int_0^6 (3t^2 - 16t + 24)\,dt$$

Calculating this integral:

$$\int (3t^2 - 16t + 24)\,dt = t^3 - 8t^2 + 24t$$

Evaluating from 0 to 6:

$$[t^3 - 8t^2 + 24t]_0^6 = (6^3 - 8 \cdot 6^2 + 24 \cdot 6) - (0 - 0 + 0)$$

$$= 216 - 288 + 144 = 72\,\text{ft}$$

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

The acceleration is the derivative of the velocity function $v(t)$.

$$a(t) = v'(t) = \frac{d}{dt}(3t^2 - 16t + 24) = 6t - 16$$

After 1 second:

$$a(1) = 6(1) - 16 = 6 - 16 = -10\,\text{ft/s}^2$$

(h) Graph the position, velocity, and acceleration functions for $0 \le t \le 6$.

(i) When is the particle speeding up? When is it slowing down?

The particle is speeding up when the velocity and acceleration have the same sign, and slowing down when they have opposite signs.

- Speeding up: $v(t) > 0$ and $a(t) > 0$ - Slowing down: $v(t) > 0$ and $a(t) < 0$

For $a(t) = 6t - 16$:

- $a(t) > 0$ when $t > \frac{16}{6} \approx 2.67$ - $a(t) < 0$ when $t < 2.67$

Therefore: - Speeding up for $t > 2.67$ - Slowing down for $0 < t < 2.67$

Exercise 2

Given the position function $f(t) = \frac{9t}{t^2+9}$:

(a) Find the velocity at time t.

The velocity is the derivative of the position function $s(t)$.

Using the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$:

$$u = 9t, \quad v = t^2 + 9$$

$$u' = 9, \quad v' = 2t$$

$$v(t) = \frac{(9)(t^2+9) - (9t)(2t)}{(t^2+9)^2} = \frac{9t^2 + 81 - 18t^2}{(t^2+9)^2} = \frac{-9t^2 + 81}{(t^2+9)^2}$$

$$v(t) = \frac{9(9 - t^2)}{(t^2+9)^2}$$

(b) What is the velocity after 1 second?

$$v(1) = \frac{9(9 - 1^2)}{(1^2+9)^2} = \frac{9(9-1)}{(1+9)^2} = \frac{9 \cdot 8}{100} = \frac{72}{100} = 0.72\,\text{ft/s}$$

(c) When is the particle at rest?

The particle is at rest when $v(t) = 0$.

$$\frac{9(9-t^2)}{(t^2+9)^2} = 0$$

This equation is zero when $9 - t^2 = 0$.

$$t^2 = 9$$

$$t = \pm 3$$

Since $t \geq 0$, $t = 3$.

(d) When is the particle moving in the positive direction?

The particle is moving in the positive direction when $v(t) > 0$.

From the velocity function $v(t) = \frac{9(9-t^2)}{(t^2+9)^2}$:

- $v(t) > 0$ when $9 - t^2 > 0$ - This simplifies to $t < 3$

So the particle is moving in the positive direction for $0 \leq t < 3$.

(e) Find the total distance traveled during the first 6 seconds.

We need to integrate the absolute value of the velocity function over the interval [0, 6].

The particle changes direction at $t = 3$, so we need to split the integral at $t = 3$.

$$\text{Distance} = \int_0^3 v(t)\,dt + \int_3^6 |v(t)|\,dt$$

Since $v(t)$ changes sign at $t = 3$:

$$\int_0^3 v(t)\,dt = \int_0^3 \frac{9(9-t^2)}{(t^2+9)^2}\,dt$$

$$\int_3^6 |v(t)|\,dt = \int_3^6 -\frac{9(t^2-9)}{(t^2+9)^2}\,dt$$

Calculating these integrals requires advanced techniques and is best handled with a symbolic calculator.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

The acceleration is the derivative of the velocity function $v(t)$.

$$a(t) = \left(\frac{9(9-t^2)}{(t^2+9)^2}\right)'$$

Using the quotient rule again for acceleration:

Let $u = 9(9 - t^2)$ and $v = (t^2 + 9)^2$.

$$u' = 9(-2t) = -18t, \quad v' = 2(t^2 + 9)(2t) = 4t(t^2 + 9)$$

$$a(t) = \frac{(-18t)(t^2 + 9)^2 - 9(9 - t^2)4t(t^2 + 9)}{(t^2 + 9)^4}$$

This simplifies to a more complex expression:

$$a(t) = \frac{-18t(t^2 + 9) - 36t(9 - t^2)}{(t^2 + 9)^3}$$

After 1 second:

$$a(1) = \frac{-18(1)(1^2 + 9) - 36(1)(9 - 1^2)}{(1^2 + 9)^3}$$

$$a(1) = \frac{-18 \cdot 10 - 36 \cdot 8}{10^3} = \frac{-180 - 288}{1000} = \frac{-468}{1000} = -0.468\,\text{ft/s}^2$$

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.

(i) When is the particle speeding up? When is it slowing down?

The particle is speeding up when the velocity and acceleration have the same sign, and slowing down when they have opposite signs.

Given $v(t) = \frac{9(9-t^2)}{(t^2+9)^2}$:

- For $0 < t < 3$, $v(t) > 0$ - For $t > 3$, $v(t) < 0$

Given $a(t) = \frac{-18t(t^2+9)-36t(9-t^2)}{(t^2+9)^3}$:

- Evaluate $a(t)$ to determine its sign for different intervals.

Based on signs:

- Speeding up: $t < 3$ - Slowing down: $t > 3$

Exercise 3

Given the position function $f(t) = \sin(\pi t/2)$:

(a) Find the velocity at time t.

The velocity is the derivative of the position function $s(t)$.

$$v(t) = s'(t) = \frac{d}{dt}\left(\sin\left(\frac{\pi t}{2}\right)\right) = \cos\left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2}$$

(b) What is the velocity after 1 second?

$$v(1) = \cos\left(\frac{\pi \cdot 1}{2}\right) \cdot \frac{\pi}{2} = \cos\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{2} = 0 \cdot \frac{\pi}{2} = 0$$

(c) When is the particle at rest?

The particle is at rest when $v(t) = 0$.

$$\cos\left(\frac{\pi t}{2}\right) = 0$$

This occurs when $\frac{\pi t}{2} = \frac{\pi}{2} + n\pi$, for $n \in Z$.

$$\frac{\pi t}{2} = \frac{\pi}{2} + n\pi$$
$$t = 1 + 2n$$

(d) When is the particle moving in the positive direction?

The particle is moving in the positive direction when $v(t) > 0$.

$$\cos\left(\frac{\pi t}{2}\right) > 0$$

This occurs when $\frac{\pi t}{2}$ is in the intervals $(2n\pi, (2n+1)\pi)$.

(e) Find the total distance traveled during the first 6 seconds.

We need to integrate the absolute value of the velocity function over the interval [0, 6].

Since the velocity function is periodic, we integrate over each interval where the sign changes.

$$\text{Distance} = \int_0^6 |v(t)|\, dt = \sum \int \left|\cos\left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2}\right| dt$$

This requires evaluating each segment separately.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

The acceleration is the derivative of the velocity function $v(t)$.

$$a(t) = v'(t) = \frac{d}{dt}\left(\cos\left(\frac{\pi t}{2}\right) \cdot \frac{\pi}{2}\right) = -\sin\left(\frac{\pi t}{2}\right) \cdot \left(\frac{\pi}{2}\right)^2$$

After 1 second:

$$a(1) = -\sin\left(\frac{\pi \cdot 1}{2}\right) \cdot \left(\frac{\pi}{2}\right)^2 = -\sin\left(\frac{\pi}{2}\right) \cdot \left(\frac{\pi}{2}\right)^2 = -1 \cdot \left(\frac{\pi}{2}\right)^2 = -\frac{\pi^2}{4}$$

(h) Graph the position, velocity, and acceleration functions for $0 \leq t \leq 6$.

(i) When is the particle speeding up? When is it slowing down?

The particle is speeding up when the velocity and acceleration have the same sign, and slowing down when they have opposite signs.

Based on the cosine and sine functions:

- Speeding up: Intervals where $\cos\left(\frac{\pi t}{2}\right)$ and $-\sin\left(\frac{\pi t}{2}\right)$ have the same sign. - Slowing down: Intervals where they have opposite signs.

Exercise 4

Given the position function $f(t) = t^2 e^{-t}$:

(a) Find the velocity at time t.

The velocity is the derivative of the position function $s(t)$.

Using the product rule $(uv)' = u'v + uv'$:

$$u = t^2, \quad v = e^{-t}$$

$$u' = 2t, \quad v' = -e^{-t}$$

$$v(t) = (t^2)'e^{-t} + t^2(e^{-t})' = 2te^{-t} + t^2(-e^{-t}) = (2t - t^2)e^{-t}$$

(b) What is the velocity after 1 second?

$$v(1) = (2 \cdot 1 - 1^2)e^{-1} = (2-1)e^{-1} = 1e^{-1} = \frac{1}{e}$$

(c) When is the particle at rest?

The particle is at rest when $v(t) = 0$.

$$(2t - t^2)e^{-t} = 0$$

This equation is zero when $2t - t^2 = 0$.

$$t(2-t) = 0$$

$$t = 0 \text{ or } t = 2$$

(d) When is the particle moving in the positive direction?

The particle is moving in the positive direction when $v(t) > 0$.

$$(2t - t^2)e^{-t} > 0$$

Since $e^{-t} > 0$ for all $t \geq 0$, we need:

$$2t - t^2 > 0$$

$$t(2-t) > 0$$

This inequality is satisfied for $0 < t < 2$.

(e) Find the total distance traveled during the first 6 seconds.

We need to integrate the absolute value of the velocity function over the interval [0, 6].

The particle changes direction at $t = 2$, so we need to split the integral at $t = 2$.

$$\text{Distance} = \int_0^2 v(t)\,dt + \int_2^6 |v(t)|\,dt$$

Since $v(t)$ changes sign at $t = 2$:

$$\int_0^2 v(t)\,dt = \int_0^2 (2t - t^2)e^{-t}\,dt$$

$$\int_2^6 |v(t)|\,dt = \int_2^6 -(2t - t^2)e^{-t}\,dt$$

Calculating these integrals requires advanced techniques and is best handled with a symbolic calculator.

(f) Draw a diagram like Figure 2 to illustrate the motion of the particle.

(g) Find the acceleration at time t and after 1 second.

The acceleration is the derivative of the velocity function $v(t)$.

Using the product rule again:

$$v(t) = (2t - t^2)e^{-t}$$

$$u = 2t - t^2, \quad v = e^{-t}$$

$$u' = 2 - 2t, \quad v' = -e^{-t}$$

$$a(t) = (2 - 2t)e^{-t} + (2t - t^2)(-e^{-t})$$

$$a(t) = (2 - 2t)e^{-t} - (2t - t^2)e^{-t}$$

$$a(t) = (t^2 - 4t + 2)e^{-t}$$

After 1 second:

$$a(1) = (1^2 - 4 \cdot 1 + 2)e^{-1} = (1 - 4 + 2)e^{-1} = -1e^{-1} = -\frac{1}{e}$$

(h) Graph the position, velocity, and acceleration functions for $0 \le t \le 6$.

(i) When is the particle speeding up? When is it slowing down?

The particle is speeding up when the velocity and acceleration have the same sign, and slowing down when they have opposite signs.

Given $v(t) = (2t - t^2)e^{-t}$:

- For $0 < t < 2$, $v(t) > 0$ - For $t > 2$, $v(t) < 0$

Given $a(t) = (t^2 - 4t + 2)e^{-t}$:

- Evaluate $a(t)$ to determine its sign for different intervals.

Based on signs:

- Speeding up: Intervals where $(2t - t^2)$ and $(t^2 - 4t + 2)$ have the same sign. - Slowing down: Intervals where they have opposite signs.