common derivatives

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Derivatives of Trigonometric Functions

1. Derivative of $\sin(x)$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

Explanation:
To find the derivative of $\sin(x)$, we use the limit definition of the derivative:

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Using the trigonometric identity for the sine of a sum, we get:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Substitute this into the limit:

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

This separates into two limits:

$$\frac{d}{dx}(\sin(x)) = \sin(x)\left(\lim_{h \to 0} \frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\lim_{h \to 0} \frac{\sin(h)}{h}\right)$$

Using the known limits $\lim_{h \to 0} \frac{\cos(h)-1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$:

$$\frac{d}{dx}(\sin(x)) = \sin(x) \cdot 0 + \cos(x) \cdot 1$$

Thus:

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

2. Derivative of $\cos(x)$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Explanation:
Similar to the sine function, we use the limit definition:

$$\frac{d}{dx}(\cos(x)) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Using the trigonometric identity for the cosine of a sum:

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$$

Substitute this into the limit:

$$\frac{d}{dx}(\cos(x)) = \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

This separates into two limits:

$$\frac{d}{dx}(\cos(x)) = \cos(x)\left(\lim_{h\to 0} \frac{\cos(h)-1}{h}\right) - \sin(x)\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)$$

Using the known limits:

$$\frac{d}{dx}(\cos(x)) = \cos(x)\cdot 0 - \sin(x)\cdot 1$$

Thus:

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

3. Derivative of $\tan(x)$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

Explanation:
Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, we can use the quotient rule:

$$\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin(x))}{\cos^2(x)}$$

This simplifies to:

$$\frac{d}{dx}(\tan(x)) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Using the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$:

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)} = \sec^2(x)$$

4. Derivative of $\cot(x)$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Explanation:
Since $\cot(x) = \frac{\cos(x)}{\sin(x)}$, we use the quotient rule:

$$\frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{\sin(x)\cdot(-\sin(x)) - \cos(x)\cdot\cos(x)}{\sin^2(x)}$$

This simplifies to:

$$\frac{d}{dx}(\cot(x)) = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

Using the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$:

$$\frac{d}{dx}(\cot(x)) = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

5. Derivative of $\sec(x)$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

Explanation:
Since $\sec(x) = \frac{1}{\cos(x)}$, we use the chain rule:

$$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \frac{0 \cdot \cos(x) - 1 \cdot (-\sin(x))}{\cos^2(x)}$$

This simplifies to:

$$\frac{d}{dx}(\sec(x)) = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} = \tan(x) \cdot \sec(x)$$

Thus:

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

6. Derivative of $\csc(x)$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

Explanation:
Since $\csc(x) = \frac{1}{\sin(x)}$, we use the chain rule:

$$\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = \frac{0 \cdot \sin(x) - 1 \cdot \cos(x)}{\sin^2(x)}$$

This simplifies to:

$$\frac{d}{dx}(\csc(x)) = \frac{-\cos(x)}{\sin^2(x)} = -\cot(x) \cdot \csc(x)$$

Thus:

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

Chain Rule Variants

The chain rule applied to some specific functions.

1. Power of a Function

$$\frac{d}{dx}\left([f(x)]^n\right) = n[f(x)]^{n-1}f'(x)$$

Purpose: To find the derivative of a function raised to a power.
Use Case: Differentiating functions like $(3x^2+2)^5$.
Example:

$$y = (3x^2+2)^5$$

$$\frac{dy}{dx} = 5(3x^2+2)^4 \cdot 6x = 30x(3x^2+2)^4$$

2. Exponential Function

$$\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

Purpose: To differentiate an exponential function with a variable exponent.
Use Case: Differentiating functions like e^{x^2}.
Example:

$$y = e^{x^2}$$

$$\frac{dy}{dx} = 2xe^{x^2}$$

3. Natural Logarithm

$$\frac{d}{dx}\left(\ln[f(x)]\right) = \frac{f'(x)}{f(x)}$$

Purpose: To find the derivative of the natural logarithm of a function.
Use Case: Differentiating functions like $\ln(x^3+4)$.
Example:

$$y = \ln(x^3+4)$$

$$\frac{dy}{dx} = \frac{3x^2}{x^3+4}$$

4. Sine of a Function

$$\frac{d}{dx}\left(\sin[f(x)]\right) = f'(x)\cos[f(x)]$$

Purpose: To differentiate the sine of a function.
Use Case: Differentiating functions like $\sin(3x+1)$.
Example:

$$y = \sin(3x+1)$$

$$\frac{dy}{dx} = 3\cos(3x+1)$$

5. Cosine of a Function

$$\frac{d}{dx}\left(\cos[f(x)]\right) = -f'(x)\sin[f(x)]$$

Purpose: To find the derivative of the cosine of a function.
Use Case: Differentiating functions like $\cos(2x-5)$.
Example:

$$y = \cos(2x-5)$$
$$\frac{dy}{dx} = -2\sin(2x-5)$$

6. Tangent of a Function

$$\frac{d}{dx}\left(\tan[f(x)]\right) = f'(x)\sec^2[f(x)]$$

Purpose: To differentiate the tangent of a function.
Use Case: Differentiating functions like $\tan(x^2+1)$.
Example:

$$y = \tan(x^2+1)$$
$$\frac{dy}{dx} = 2x\sec^2(x^2+1)$$

7. Secant of a Function

$$\frac{d}{dx}\left(\sec[f(x)]\right) = f'(x)\sec[f(x)]\tan[f(x)]$$

Purpose: To find the derivative of the secant of a function.
Use Case: Differentiating functions like $\sec(4x)$.
Example:

$$y = \sec(4x)$$
$$\frac{dy}{dx} = 4\sec(4x)\tan(4x)$$

8. Inverse Tangent of a Function

$$\frac{d}{dx}\left(\tan^{-1}[f(x)]\right) = \frac{f'(x)}{1+[f(x)]^2}$$

Purpose: To differentiate the inverse tangent (arctangent) of a function.
Use Case: Differentiating functions like $\tan^{-1}(x^3)$.
Example:

$$y = \tan^{-1}(x^3)$$
$$\frac{dy}{dx} = \frac{3x^2}{1+x^6}$$

Complex Calculus Problems

Example 1: Find the derivative of $y = (2x^3+3x^2-5)^{10}$.
Using the power rule variant:

$$y = (2x^3+3x^2-5)^{10}$$
$$\frac{dy}{dx} = 10(2x^3+3x^2-5)^9 \cdot (6x^2+6x)$$
$$= 60x(2x^3+3x^2-5)^9(x+1)$$

Example 2: Differentiate $y = e^{\sin(x^2)}$.
Using the exponential and sine rules:

$$\begin{aligned} y &= e^{\sin(x^2)} \\ \frac{dy}{dx} &= e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x \\ &= 2xe^{\sin(x^2)} \cos(x^2) \end{aligned}$$

Example 3: Find the derivative of $y = \ln(\cos(3x^2 + 1))$.
Using the logarithm and cosine rules:

$$\begin{aligned} y &= \ln(\cos(3x^2 + 1)) \\ \frac{dy}{dx} &= \frac{1}{\cos(3x^2 + 1)} \cdot (-\sin(3x^2 + 1)) \cdot 6x \\ &= \frac{-6x \sin(3x^2 + 1)}{\cos(3x^2 + 1)} \\ &= -6x \tan(3x^2 + 1) \end{aligned}$$

Example 4: Differentiate $y = \tan^{-1}(e^{x^2})$.
Using the inverse tangent and exponential rules:

$$\begin{aligned} y &= \tan^{-1}(e^{x^2}) \\ \frac{dy}{dx} &= \frac{e^{x^2} \cdot 2x}{1 + e^{2x^2}} \\ &= \frac{2xe^{x^2}}{1 + e^{2x^2}} \end{aligned}$$

Implicit Differentiation - Practice Problems

For problems 1 – 3 do each of the following:

- (a) Find y' by solving the equation for y and differentiating directly.
- (b) Find y' by implicit differentiation.
- (c) Check that the derivatives in (a) and (b) are the same.

1. $\frac{x}{y^3} = 1$

(a) Solve for y' by solving the equation for y and differentiating directly:
Solve for y:

$$y^3 = x$$
$$y = x^{1/3}$$

Differentiate directly:

$$y' = \frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3}$$

(b) Find y' by implicit differentiation:
Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\frac{x}{y^3}\right) = \frac{d}{dx}(1)$$

Use the quotient rule on the left side:

$$\frac{y^3 \cdot 1 - x \cdot 3y^2y'}{y^6} = 0$$

Simplify:

$$\frac{y^3 - 3xy^2y'}{y^6} = 0$$
$$y^3 - 3xy^2y' = 0$$
$$y^3 = 3xy^2y'$$
$$y' = \frac{y}{3x}$$

Substitute $y = x^{1/3}$:

$$y' = \frac{x^{1/3}}{3x} = \frac{1}{3}x^{-2/3}$$

(c) Check that the derivatives in (a) and (b) are the same:

$$\frac{1}{3}x^{-2/3}$$

Both methods yield the same derivative.

2. $x^2 + y^3 = 4$

(a) Solve for y' by solving the equation for y and differentiating directly:
Solve for y:

$$y = \sqrt[3]{4 - x^2}$$

Differentiate directly:

$$y' = \frac{d}{dx}\left((4 - x^2)^{1/3}\right)$$

Using the chain rule:

$$y' = \frac{1}{3}(4 - x^2)^{-2/3} \cdot (-2x)$$

$$y' = -\frac{2x}{3(4 - x^2)^{2/3}}$$

(b) Find y' by implicit differentiation:
Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(4)$$

Apply the power rule:

$$2x + 3y^2 y' = 0$$

Solve for y':

$$y' = -\frac{2x}{3y^2}$$

(c) Check that the derivatives in (a) and (b) are the same: Substitute $y = \sqrt[3]{4 - x^2}$ into the implicit result:

$$y' = -\frac{2x}{3(\sqrt[3]{4 - x^2})^2} = -\frac{2x}{3(4 - x^2)^{2/3}}$$

Both methods yield the same derivative.

3. $x^2 + y^2 = 2$

(a) Solve for y' by solving the equation for y and differentiating directly:
Solve for y:

$$y = \sqrt{2 - x^2}$$

Differentiate directly:

$$y' = \frac{d}{dx}(\sqrt{2 - x^2})$$

Using the chain rule:

$$y' = \frac{1}{2\sqrt{2 - x^2}} \cdot (-2x)$$

$$y' = \frac{-x}{\sqrt{2 - x^2}}$$

(b) Find y' by implicit differentiation:
Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(2)$$

Apply the power rule:

$$2x + 2yy' = 0$$

Solve for y':

$$y' = -\frac{x}{y}$$

(c) Check that the derivatives in (a) and (b) are the same: Substitute $y = \sqrt{2-x^2}$ into the implicit result:

$$y' = -\frac{x}{\sqrt{2-x^2}}$$

Both methods yield the same derivative.

4. $2y^3 + 4x^2 - y = x^6$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(2y^3 + 4x^2 - y) = \frac{d}{dx}(x^6)$$

Apply the power rule and chain rule:

$$6y^2y' + 8x - y' = 6x^5$$

Combine like terms:

$$6y^2y' - y' = 6x^5 - 8x$$
$$y'(6y^2 - 1) = 6x^5 - 8x$$
$$y' = \frac{6x^5 - 8x}{6y^2 - 1}$$

5. $7y^4 + \sin(3x) = 12 - y^4$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(7y^4 + \sin(3x)) = \frac{d}{dx}(12 - y^4)$$

Apply the power rule and chain rule:

$$28y^3y' + 3\cos(3x) = -4y^3y'$$

Combine like terms:

$$28y^3y' + 4y^3y' = -3\cos(3x)$$
$$y'(28y^3 + 4y^3) = -3\cos(3x)$$
$$y' = -\frac{3\cos(3x)}{32y^3}$$

6. $e^x - \sin(y) = x$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(e^x - \sin(y)) = \frac{d}{dx}(x)$$

Apply the chain rule:

$$e^x - \cos(y)y' = 1$$

Solve for y':

$$-\cos(y)y' = 1 - e^x$$
$$y' = \frac{e^x - 1}{\cos(y)}$$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(4x^2y^7 - 2x) = \frac{d}{dx}(x^5 + 4y^3)$$

Apply the product rule and chain rule:

$$8xy^7 + 4x^2 \cdot 7y^6y' - 2 = 5x^4 + 12y^2y'$$

Combine like terms:

$$8xy^7 - 2 = 5x^4 + 12y^2y' - 28x^2y^6y'$$
$$8xy^7 - 2 = 5x^4 + y'(12y^2 - 28x^2y^6)$$
$$y'(12y^2 - 28x^2y^6) = 8xy^7 - 2 - 5x^4$$
$$y' = \frac{8xy^7 - 2 - 5x^4}{12y^2 - 28x^2y^6}$$

8. $\cos(x^2 + 2y) + xe^{y^2} = 1$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\cos(x^2 + 2y) + xe^{y^2}\right) = \frac{d}{dx}(1)$$

Apply the chain rule and product rule:

$$-\sin(x^2 + 2y) \cdot \frac{d}{dx}(x^2 + 2y) + \frac{d}{dx}(xe^{y^2}) = 0$$

$$-\sin(x^2 + 2y) \cdot (2x + 2y\frac{dy}{dx}) + e^{y^2} + x \cdot e^{y^2} \cdot 2y\frac{dy}{dx} = 0$$

Combine like terms:

$$-\sin(x^2 + 2y) \cdot 2x - \sin(x^2 + 2y) \cdot 2y\frac{dy}{dx} + e^{y^2} + 2xye^{y^2}\frac{dy}{dx} = 0$$

$$-\sin(x^2 + 2y) \cdot 2x + e^{y^2} + \left(-2y\sin(x^2 + 2y) + 2xye^{y^2}\right)\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$e^{y^2} - 2x\sin(x^2 + 2y) = \left(2ye^{y^2}x - 2y\sin(x^2 + 2y)\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{e^{y^2} - 2x\sin(x^2 + 2y)}{2ye^{y^2}x - 2y\sin(x^2 + 2y)}$$

Simplifying further:

$$\frac{dy}{dx} = \frac{e^{y^2} - 2x\sin(x^2 + 2y)}{2y(e^{y^2}x - \sin(x^2 + 2y))}$$

9. $\tan(2x^4y) = 3x + y^2$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\tan(2x^4y)\right) = \frac{d}{dx}(3x + y^2)$$

Apply the chain rule and product rule:

$$\sec^2(2x^4y) \cdot \frac{d}{dx}(2x^4y) = 3 + 2y\frac{dy}{dx}$$

$$\sec^2(2x^4y) \cdot (8x^3y + 2x^4\frac{dy}{dx}) = 3 + 2y\frac{dy}{dx}$$

Combine like terms:

$$8x^3y\sec^2(2x^4y) + 2x^4\sec^2(2x^4y)\frac{dy}{dx} = 3 + 2y\frac{dy}{dx}$$

$$8x^3y\sec^2(2x^4y) = 3 + 2y\frac{dy}{dx} - 2x^4\sec^2(2x^4y)\frac{dy}{dx}$$

Factor out $\frac{dy}{dx}$:

$$8x^3y\sec^2(2x^4y) = 3 + \left(2y - 2x^4\sec^2(2x^4y)\right)\frac{dy}{dx}$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{8x^3y\sec^2(2x^4y) - 3}{2y - 2x^4\sec^2(2x^4y)}$$

Simplifying further:

$$\frac{dy}{dx} = \frac{8x^3y\sec^2(2x^4y) - 3}{2(y - x^4\sec^2(2x^4y))}$$

For problems 10 and 11, find the equation of the tangent line at the given point.

10. $x^4 + y^2 = 3$ **at** $(1, -\sqrt{2})$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x^4 + y^2) = \frac{d}{dx}(3)$$

$$4x^3 + 2yy' = 0$$

Solve for y':

$$y' = -\frac{4x^3}{2y} = -\frac{2x^3}{y}$$

At the point $(1, -\sqrt{2})$:

$$y' = -\frac{2(1)^3}{-\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

The slope of the tangent line is $\sqrt{2}$.
Using the point-slope form:

$$y - (-\sqrt{2}) = \sqrt{2}(x - 1)$$

$$y + \sqrt{2} = \sqrt{2}x - \sqrt{2}$$

$$y = \sqrt{2}x - 2\sqrt{2}$$

11. $y^2 e^{2x} = 3y + x^2$ at $(0, 3)$

Differentiate both sides with respect to x:

$$\frac{d}{dx}(y^2 e^{2x}) = \frac{d}{dx}(3y + x^2)$$

Apply the product rule and chain rule:

$$2yy'e^{2x} + y^2 \cdot 2e^{2x} = 3y' + 2x$$
$$2yy'e^{2x} + 2y^2 e^{2x} = 3y' + 2x$$

Combine like terms:

$$y'(2ye^{2x} - 3) = 2x - 2y^2 e^{2x}$$
$$y' = \frac{2x - 2y^2 e^{2x}}{2ye^{2x} - 3}$$

At the point $(0, 3)$:

$$y' = \frac{2(0) - 2(3)^2 e^{2(0)}}{2(3)e^{2(0)} - 3} = \frac{-18}{6 - 3} = -6$$

The slope of the tangent line is -6.
Using the point-slope form:

$$y - 3 = -6(x - 0)$$
$$y = -6x + 3$$

For problems 12 and 13, assume that $x = x(t)$, $y = y(t)$, and $z = z(t)$, then differentiate the given equation with respect to t.

12. $x^2 - y^3 + z^4 = 1$

Differentiate both sides with respect to t:

$$\frac{d}{dt}(x^2 - y^3 + z^4) = \frac{d}{dt}(1)$$

Apply the chain rule:

$$2x\frac{dx}{dt} - 3y^2\frac{dy}{dt} + 4z^3\frac{dz}{dt} = 0$$

13. $x^2 \cos(y) = \sin(y^3 + 4z)$

Differentiate both sides with respect to t:

$$\frac{d}{dt}(x^2 \cos(y)) = \frac{d}{dt}(\sin(y^3 + 4z))$$

Apply the product rule and chain rule on the left side and the chain rule on the right side:

$$\frac{d}{dt}(x^2 \cos(y)) = \frac{d}{dt}(x^2) \cdot \cos(y) + x^2 \cdot \frac{d}{dt}(\cos(y))$$
$$= 2x\cos(y)\frac{dx}{dt} - x^2 \sin(y)\frac{dy}{dt}$$

And for the right side:

$$\frac{d}{dt}(\sin(y^3 + 4z)) = \cos(y^3 + 4z) \cdot \frac{d}{dt}(y^3 + 4z)$$
$$= \cos(y^3 + 4z) \cdot (3y^2\frac{dy}{dt} + 4\frac{dz}{dt})$$

Putting it all together:

$$2x\cos(y)\frac{dx}{dt} - x^2 \sin(y)\frac{dy}{dt} = \cos(y^3 + 4z)(3y^2\frac{dy}{dt} + 4\frac{dz}{dt})$$