Necklaces, Permutations, and Periodic Critical Orbits of Quadratic Polynomials

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1 Introduction

A *binary necklace* is binary string of length n which is equal up to cyclic shifting. An example is given in Fig. 1.

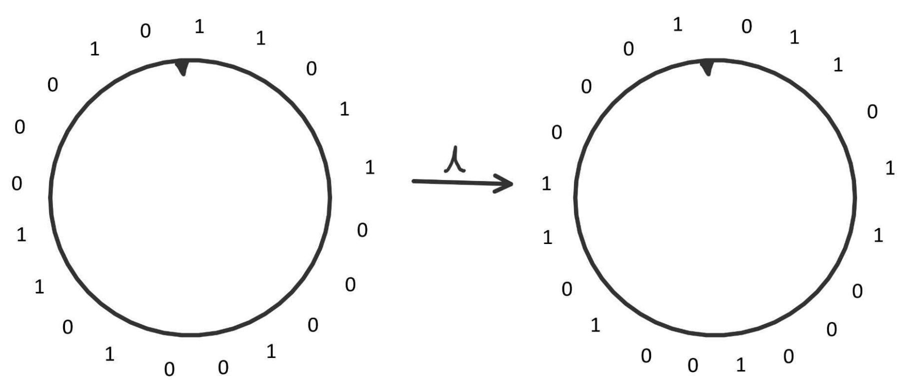

Figure 1: The two binary necklaces shown are equivalent since they are related by a shift.

We call a necklace of length n *primitive* if there is no cyclic shift of size smaller than n that gives the same binary string. We define $N^+(n)$ the set of primitive binary necklaces of length n with even number of ones and $N^-(n)$ the set of primitive binary necklaces of length n with odd number of ones. The set $\widetilde{N}^+(n)$ is the union of the sets $N^+(n)$ and $N^-(n/2)$ (when n is even). We define $\bar{N}(n)$ to be the set of primitive binary necklaces of length n up to inversion. For example, the necklaces 0111 and 1000 are members of the same equivalence class in $\bar{N}(4)$.
We will define a map from $\widetilde{N}^+(n) \to \bar{N}(n)$ and show that it is bijective. For a sequence $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \{\pm 1\}^{\mathbb{N}}$, let $\mathrm{rev}(\epsilon) = (\epsilon_n, \epsilon_{n-1}, \cdots, \epsilon_1)$ be its reverse sequence. Suppose we are given a necklace ν in $\widetilde{N}^+(n)$ represented by a binary string $s = (a_1 a_2 \cdots a_n)$. (If $\nu \in N^-(n/2)$, we simply repeat any representative string twice to get a string of length n.)
We define $\omega(s) = \epsilon$, where $\epsilon_1 = (-1)^{a_1}$ and $\epsilon_{i+1} = (-1)^{a_1 + \cdots + a_{i+1}}$ for $i \in \{1, 2, ..., n-1\}$. Let $\bar{\omega}(s) = \mathrm{rev}(\omega(s))$. Now convert $\bar{\omega}(s)$ back to a binary string s' by the correspondence $+1 \mapsto 0$, $-1 \mapsto 1$. Let $\varphi : \bar{\omega}(s) \to s'$. We set $\Xi(s) = s'$.

Example 1.0.1. We take the necklace $s = (0011) \in \widetilde{N}^+(n)$. Then $\omega(s) = (1, 1, -1, 1)$ and its reverse sequence is $\mathrm{rev}(\omega(s)) = (1, -1, 1, 1)$. Then $\varphi(1, -1, 1, 1) = (0, 1, 0, 0) \in \bar{N}(n)$.

Theorem A.
(1). The map Ξ is a well defined function from $\widetilde{N}^+(n)$ to $\bar{N}(n)$
(2). The induced map $\Xi : \widetilde{N}^+(n) \to \bar{N}(n)$ is a bijection.

We also have another bijection as follows. We define a related map $\Xi' : \widetilde{N}^+(n) \to \bar{N}(n)$ and show that it is a bijection. Given a necklace $\nu = (a_1, a_2, \cdots, a_n) \in \widetilde{N}^+(n)$ where we repeat the string twice if it the necklace belongs to $N^-(n/2)$, consider the corresponding sequence $\epsilon := \omega(\nu) \in E$. Let E denote the set $\{\pm 1\}^{\mathbb{N}}$ of sequences $\epsilon = (\epsilon_1, \epsilon_2 ...)$ with $\epsilon_i \in \{\pm 1\}$ for all i. Following Weiss-Rogers [5], the *twisted shift operator* F: $E \to E$ is defined by

$$F(\epsilon_1, \epsilon_2, \epsilon_3, \cdots) = (\epsilon_1\epsilon_2, \epsilon_1\epsilon_3, \epsilon_1\epsilon_4, \cdots).$$

Iterating the twisted shift operator F gives an F-orbit of size n, let this be C.

$$C = \{\epsilon, F(\epsilon), F^2(\epsilon), \cdots, F^{n-1}(\epsilon)\}$$

Let $\mathrm{rev}(C)$ denote the set of all reversed strings of the orbit, that is

$$\mathrm{rev}(C) := \{\mathrm{rev}(\epsilon), \mathrm{rev}(F(\epsilon)), \mathrm{rev}(F^2(\epsilon)), \cdots, \mathrm{rev}(F^{n-1}(\epsilon))\}.$$

Let $\epsilon' \in E$ be the lexicographically minimal element of $\mathrm{rev}(C)$, truncated to an n-tuple $(\epsilon'_1, \epsilon'_2, \cdots, \epsilon'_n) \in \{\pm 1\}^n$. Now we convert ϵ' to a binary string s' by the usual correspondence $s' = \varphi(\epsilon')$. We set $\Xi'(\nu) := s'$

Example 1.0.2. We take the necklace $s = (01) \in \widetilde{N}^+(4)$. Since $(01) \in N^-(n/2)$ we repeat the string twice to get (0101). Then $\omega(0101) = (1, -1, -1, 1)$. We then consider the F-cycle of $(1, -1, -1, 1)$.

$$C = \{(1, -1, -1, 1), (-1, -1, 1, 1)\}$$

Reversing all elements of the cycle we get that $\mathrm{rev}(C) = \{(1, -1, -1, 1), (1, 1, -1, -1)\}$, and the lexicographically minimal element of $\mathrm{rev}(C)$ is $(1, -1, -1, 1)$. Then $\varphi(1, -1, -1, 1) = (0, 1, 1, 0) \in \bar{N}(n)$.

Theorem B. The map $\Xi' : \widetilde{N}^+(n) \to \bar{N}(n)$ is a bijection from $\widetilde{N}^+(n)$ to $\bar{N}(n)$.

1.1 Background and motivation

Consider $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^2 + c$. This defines dynamical system $z \mapsto f(z)$. We will look at the orbits

$$0, f(0), f(f(0)), f(f(f(0))), ...$$

We say that 0 is periodic if $f^n(0) = 0$ for some $n \geq 1$. If 0 is periodic, it has period n if $f^n(0) = 0$ and $f^i(0) \neq 0$ for $i < n$. Let us define the n^{th} *Gleason polynomial*, whose roots correspond to parameters c such that the critical point 0 is periodic of exact period n under iteration of $z^2 + c$.

Example 1.1.1. The first few Gleason polynomials are given below:

$$\begin{aligned} G_1(c) &= c \\ G_2(c) &= \frac{c^2 + c}{G_1(c)} = c + 1 \\ G_3(c) &= \frac{(c^2 + c)^2 + c}{G_1(c)} = c^3 + 2c^2 + c + 1 \\ G_4(c) &= \frac{((c^2 + c)^2 + c)^2 + c}{G_1(c)G_2(c)} = c^6 + 3c^5 + 3c^4 + 3c^3 + 2c^2 + 1 \end{aligned}$$

We have the following surprising result:

Theorem 1.1.2. *(Buff–Floyd–Koch–Parry [2, Theorem 1.5]) The number of real roots of n^{th} Gleason polynomial is equal to the number of its irreducible factors over $\mathbb{F}_2$ for all $n \in \mathbb{N}$.*

Example 1.1.3. $G_3(c) = c^3 + 2c^2 + c + 1$ is irreducible over $\mathbb{F}_2$, and has one real root, $G_4(c)$ has two factors $(c^4 + c + 1)(c^2 + c + 1)$ and two real roots.

Buff–Floyd–Koch–Parry establish this bijection by showing that the two sets are of the same size. Our work gives an explicit bijection through a series of bijections between different sets. We define the following:

(M1) *Real numbers c for which the critical orbit $z^2 + c$ has period n, i.e., real roots of G_n.*
(M2) *Irreducible factors of G_n over $\mathbb{F}_2$.*
(P1) *The set CUP(n) of cyclic unimodal permutations of $[n] := \{1, ..., n\}$.*
(P2) *The set of CGP(n) of cyclic Gilbreath permutations of $[n]$.*
(N1) *The set $N^-(n)$ of primitive binary necklaces of length n with an odd number of 1's.*
(N2) *The set $\widetilde{N}^+(n)$ of binary necklaces which are either primitive of length n with an even number of 1's or primitive length $n/2$ with an odd number of 1's.*
(N3) *The set $\bar{N}(n)$ of equivalence classes of primitive binary necklaces of length n under inversion (sending 0's to 1's and vice-versa).*
(I1) *The set $I^-(n)$ of non-centered irreducible polynomials of degree n over $\mathbb{F}_2$.*
(I2) *The set $\widetilde{I}^+(n)$ of centered irreducible polynomials of degree n over $\mathbb{F}_2$, together with the set of non-centered irreducible polynomials of degree $n/2$ over $\mathbb{F}_2$.*

Here CUP(n) is defined in Definition 4.0.1. In this paper we are not concerned with cyclic Gilbreath permutations or centered (and non-centered) irreducible polynomials. It is known that these sets all have the same cardinality, namely $\gamma_n = \frac{1}{2n} \sum_{m|n, m \text{ odd}} \mu(m) 2^{\frac{n}{m}}$. These sets arise naturally in abstract algebra, complex dynamics, enumerative combinatorics, and card tricks, as well as in other settings. We briefly summarise the known explicit bijections, demonstrated in Fig. 2.

The bijection between (P1) and (P2) is well known, a permutation σ of $[n]$ is unimodal if and only if its inverse σ^{-1} is Gilbreath. Milnor and Thurston [4] gave an explicit bijection between (M1) and (P1). The authors of [2] show that the sets (M2) and (I2) coincide. A bijection between (P1) and (N1) follows in a straightforward way from the work of Weiss and Rogers [5], and this is made explicit in [1]. To the best of our knowledge, all other explicit bijections are currently unknown. Theorem A and Theorem B together give an explicit bijection between (N2) and (N3), and hence an explicit bijection between (M1) and (M2), which was previously unknown.

In Section 4 we make progress on an explicit bijection between (P1) and (N2), and we suspect adapting the explicit inverse function of the Weiss–Rogers map given in [1] may show that the map $\widetilde{\Phi}^+$ is invertible. This conjecture, combined with the other results in this paper and previous work, suffices to establish explicit bijections between (M1), (M2), (P1), (N1), (N2), (N3) (where N2 to N3 is given by Theorem A and Theorem B), (I1), and (I2).

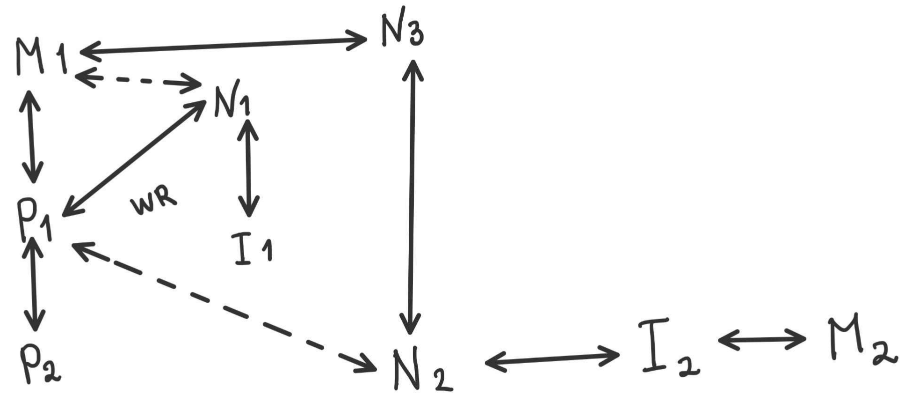

Figure 2: The diagram showing bijections between certain sets

1.2 Acknowledgements

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2 The twisted shift operator

2.1 Twisted shift operator and signature

Definition 2.1.1 (The ω function)**.** For some binary sequence $s = (a_1, a_2, ...a_n)$, where $a_i \in \{0, 1\}$, the function

$$\omega : \{0,1\}^{\mathbb{N}} \to \{\pm 1\}^{\mathbb{N}}$$

is defined as $\omega(s) = \epsilon$, where $\epsilon = (\epsilon_1, \epsilon_2 ... \epsilon_n)$ with

$$\epsilon_i = (-1)^{\sum_{s=1}^{i} a_s}$$

for all $i = \{1, 2...\}$.

Proposition 2.1.2. *The map $\omega : \{0,1\}^{\mathbb{N}} \to \{\pm 1\}^{\mathbb{N}}$ has a two sided inverse ω^{-1}.*

Proof. Note that $\omega : (a_1, a_2, \cdots, a_n) \to ((-1)^{a_1}, (-1)^{a_1+a_2}, \cdots, (-1)^{a_1+\cdots+a_n})$, define a map ω^{-1} as follows:

$$\omega^{-1} : (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \to (a_1, a_2, \cdots, a_n), \;\; a_k = \begin{cases} 0 & \text{if } \epsilon_k/\epsilon_{k-1} = 1 \\ 1 & \text{if } \epsilon_k/\epsilon_{k-1} = -1 \end{cases}$$

Let $\epsilon_0 = 1$ by convention. We claim that ω^{-1} is the two-sided inverse of ω. First we show $\omega^{-1} \circ \omega = \mathrm{Id}$.

$$\omega^{-1} \circ \omega(a_1, a_2, \cdots, a_n) = \omega^{-1}((-1)^{a_1}, (-1)^{a_1+a_2}, \cdots, (-1)^{a_1+\cdots+a_n}) = (b_1, b_2, \cdots, b_n)$$

$$\text{For } b_k = \begin{cases} 0 & \text{if } \epsilon_k/\epsilon_{k-1} = 1 \\ 1 & \text{if } \epsilon_k/\epsilon_{k-1} = -1 \end{cases} \Rightarrow b_k = \begin{cases} 0 & \text{if } a_k = 0 \\ 1 & \text{if } a_k = 1 \end{cases} \Rightarrow (b_1, b_2, \cdots, b_n) = (a_1, a_2, \cdots, a_n)$$

Hence we have that $\omega^{-1} \circ \omega = \mathrm{Id}$. We further show that $\omega \circ \omega^{-1} = \mathrm{Id}$. Let

$$(\omega \circ \omega^{-1})(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = (\epsilon'_1 \epsilon'_2, \cdots, \epsilon'_n).$$

$$(\omega \circ \omega^{-1})(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = \omega(a_1, a_2, \cdots, a_n) \text{ , where } a_k = \begin{cases} 0 & \text{if } \epsilon_k/\epsilon_{k-1} = 1 \\ 1 & \text{if } \epsilon_k/\epsilon_{k-1} = -1 \end{cases}$$

$$\omega(a_1, a_2, \cdots, a_n) = ((-1)^{a_1}, (-1)^{a_1+a_2}, \cdots, (-1)^{a_1+\cdots+a_n}) \implies \epsilon'_k = (-1)^{a_1+a_2+\cdots+a_k}$$

We wish to show that $\epsilon'_k = \epsilon_k$ for each $k : 1 \le k \le n$, we proceed by induction on k. Note that $\epsilon'_1 = \epsilon_1$ so the base case holds. Now suppose that $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{k-1}) = (\epsilon'_1, \epsilon'_2, \cdots, \epsilon'_{k-1})$. We go through the cases.

$$(\epsilon_{k-1}, \epsilon_k) = (1, -1) \implies a_k = 1 \implies \epsilon'_k = -\epsilon_{k-1} = -1$$

$$(\epsilon_{k-1}, \epsilon_k) = (-1, -1) \implies a_k = 0 \implies \epsilon'_k = \epsilon_{k-1} = -1$$

$$(\epsilon_{k-1}, \epsilon_k) = (1, 1) \implies a_k = 0 \implies \epsilon'_k = \epsilon_{k-1} = 1$$

$$(\epsilon_{k-1}, \epsilon_k) = (-1, 1) \implies a_k = 1 \implies \epsilon'_k = -\epsilon_{k-1} = 1$$

Hence we have that $\epsilon'_k = \epsilon_k$ for each $k : 1 \le k \le n$ and so ω^{-1} is a two-sided inverse to ω. □

Definition 2.1.3 (Twisted shift operator)**.** Let E denote the set $\{\pm 1\}^{\mathbb{N}}$ of sequences $\epsilon = (\epsilon_1, \epsilon_2...)$ with $\epsilon_i \in \{\pm 1\}$ for all i. The *twisted shift operator* F: $E \to E$ is defined by

$$F(\epsilon)_i = \epsilon_1 \cdot \epsilon_{i+1}$$

for all $i = \{1, 2, ...\}$.

We write $\mathrm{Per}_n(F)$ for the set of elements of E such that $F^n(\epsilon) = \epsilon$. It is easy to check that if $\epsilon \in \mathrm{Per}_n(F)$ then ϵ is determined by its first n coordinates, and so we write elements of $\mathrm{Per}_n(F)$ as n-tuples $(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ instead of infinite strings.

Proposition 2.1.4. *Let $\lambda : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ denote the left shift operator. Then $F \circ \omega = \omega \circ \lambda$.*

Proof. We simply show the function compositions agree on any string $s \in \{0,1\}^{\mathbb{N}}$. Let $s = (a_1, a_2, \cdots)$.

$$(F \circ \omega)(a_1, a_2, a_3, \cdots) = F((-1)^{a_1}, (-1)^{a_1+a_2}, (-1)^{a_1+a_2+a_3}, \cdots) = ((-1)^{a_2}, (-1)^{a_2+a_3}, \cdots)$$

$$(\omega \circ \lambda)(a_1, a_2, \cdots) = \omega(a_2, a_2, a_3, \cdots) = ((-1)^{a_2}, (-1)^{a_2+a_3}, (-1)^{a_2+a_3+a_4}, \cdots)$$

So we're done. □

Corollary 2.1.5. *Let $\lambda : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ denote the left shift operator. Then $F^k \circ \omega = \omega \circ \lambda^k$.*

Proposition 2.1.6. *Let the F-orbit of a point ϵ be denoted $Orb_F(\epsilon)$, then the nth coordinate ϵ_n is independent of the choice of $\epsilon_i \in Orb_F(\epsilon)$.*

Proof. Let $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \in \mathrm{Per}_n(F)$, where we denote ϵ with its first n coordinates as they uniquely determine the rest of the string. It follows from an elementary induction on k that

$$F^k(\epsilon) = (\chi_k \epsilon_{k+1}, \chi_k \epsilon_{k+2}, \chi_k \epsilon_{k+3}, \cdots)$$

$$\text{for } \chi_k = \left(\prod_{i=1}^{k} \epsilon_i^{2^{k-i}} \right) = \left(\prod_{i=1}^{k-1} \epsilon_i^{2^{k-i-1}} \right)^2 \cdot \epsilon_k = \epsilon_k$$

The base case holds from the definition of F. Now suppose that the result holds for $k-1$. Then

$$F^k(\epsilon) = (F \circ F^{k-1})(\epsilon) = F(\epsilon_{k-1}\epsilon_k, \epsilon_{k-1}\epsilon_{k+1}, \cdots) = (\epsilon_{k-1}^2 \epsilon_k \epsilon_{k+1}, \epsilon_{k-1}^2 \epsilon_k \epsilon_{k+2}, \cdots) = (\epsilon_k \epsilon_{k+1}, \epsilon_k \epsilon_{k+2}, \cdots).$$

Hence, this completes the induction.
Setting $k = n$ and using the fact that $\epsilon \in Per_n(F) \iff F^{\circ n}(\epsilon) = \epsilon$, we see that

$$\epsilon = (\chi_n \epsilon_{n+1}, \chi_n \epsilon_{n+2}, \chi_n \epsilon_{n+3}, \cdots) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n).$$

Hence we have that $\epsilon_n \epsilon_{n+k} = \epsilon_k$ for $1 \le k \le n$. Now we can show all elements of the same F-orbit have the same nth coordinate. Iterating F and looking at the nth coordinate of $F^{\circ k}(\epsilon)$, we wish to show

$$\epsilon_n = \epsilon_1 \epsilon_{n+1} = \epsilon_1^2 \epsilon_2 \epsilon_{n+2} = \epsilon_1^4 \epsilon_2^2 \epsilon_3 \epsilon_{n+3} = \cdots \iff \epsilon_n \epsilon_{n+k} = \epsilon_k \text{ for } 1 \le k < n.$$

But we've already shown this to be true, so we're done. □

Definition 2.1.7. The n^{th} coordinate of ϵ_n is called the *signature* of $Orb_F(\epsilon)$ (or ϵ). We say it is *odd* or *even* if the signature is -1 or $+1$ respectively.

Let $\mathrm{Per}_n^-(F)$ denote the elements of $\mathrm{Per}_n(F)$ with an odd signature, and similarly define $\mathrm{Per}_n^+(F)$ for elements of $\mathrm{Per}_n(F)$ with an even signature.

Let $E^-(n)$ denote the elements of primitive period n under the iteration of F with an odd signature, and similarly define $E^+(n)$ for elements of primitive period n under the iteration of F with an even signature.

Example 2.1.8. There are two elements in $E^-(4)$, namely $(1,1,1,-1)$ and $(1,-1,1,-1)$. The two corresponding F-orbits are

$$(1,1,1,-1) \mapsto (1,1,-1,-1) \mapsto (1,-1,-1,-1) \mapsto (-1,-1,-1,-1),$$

$$(1,-1,1,-1) \mapsto (-1,1,-1,-1) \mapsto (-1,1,1,-1) \mapsto (-1,-1,1,-1).$$

We see that the n^{th} element of the string is independent of the element of the F-orbit chosen, namely the signatures of all elements of each orbit are odd.

Let us say that $N^-(n)$ is the set of primitive binary necklaces of length n with an odd number of 1's and $N^+(n)$ is the set of primitive binary necklaces of length n with an even number of 1's. We will also consider $E^-(n)$ and $E^+(n)$, sets of $\epsilon \in \{\pm 1\}^{\mathbb{N}}$ with primitive period n under F, with $\epsilon_n = -1$ and $\epsilon_n = 1$ respectively. We will call ϵ_n an *odd* signature if $\epsilon_n = -1$, and *even* if $\epsilon_n = 1$.

Proposition 2.1.9. *(1) If $\nu \in N^-(n)$, the cycle generated by $\epsilon = \omega(\nu)$ belongs to $E^-(n)$.*
(2) The induced map $\omega^- : N^-(n) \to E^-(n)$ is a bijection.

Proof. Recall that if $\nu \in N^-(n)$ then ν is a primitive binary necklace with an odd number of ones. Let $\nu = (a_1, a_2, \cdots, a_n)$. We have that $s = \sum_{i=1}^{n} a_i \equiv 1 \mod 2$. Hence $(-1)^s = -1$. We have that

$$\omega(\nu) = ((-1)^{a_1}, (-1)^{a_1+a_2}, \cdots, (-1)^s) = ((-1)^{a_1}, (-1)^{a_1+a_2}, \cdots, -1),$$

and so $\omega(\nu)$ has an odd signature. We now claim that $\omega(\nu)$ has primitive period n under F. By $\nu \in N^-(n)$ we have $\lambda^n(\nu) = \nu$ and for any k such that $1 \leq k < n$ we have $\lambda^k(\nu) \neq \nu$. Note by an induction on k with our base case being Section 2.1 we have that

$$(F^k \circ \omega)(\nu) = (\omega \circ \lambda^k)(\nu).$$

For the inductive step suppose the result holds for $k-1$ we have that

$$(F^k \circ \omega)(\nu) = (F \circ F^{k-1} \circ \omega)(\nu) = (F \circ \omega \circ \lambda^{k-1})(\nu) = ((F \circ \omega) \circ \lambda^{k-1})(\nu) = ((\omega \circ \lambda) \circ \lambda^{k-1})(\nu) = (\omega \circ \lambda^k)(\nu).$$

And so by induction the result holds. Then for $k = n$, we have $F^n(\omega(\nu)) = \omega(\lambda^n(\nu)) = \omega(\nu)$. Hence $\omega(\nu) \in \mathrm{Per}_n(F)$. We now want to show for $1 \leq k < n$ that $(F^k \circ \omega)(\nu) \neq \omega(\nu)$. For the sake of contradiction suppose that for some k such that $1 \leq k < n$ we have $(F^k \circ \omega)(\nu) = \omega(\nu)$. Then,

$$\begin{aligned} \omega(\nu) &= (\omega \circ \lambda^k)(\nu) \\ \implies (\omega^{-1} \circ \omega)(\nu) &= (\omega^{-1} \circ \omega \circ \lambda^k)(\nu) \\ \implies \nu &= \lambda^k(\nu) \end{aligned}$$

But ν is a primitive binary necklace and so $\nu \neq \lambda^k(\nu)$ for $1 \leq k < n$. This is a contradiction. Hence $\omega(\nu)$ has primitive period n under F and an odd signature and so $\omega(\nu) \in E^-(n)$. Then, to show that $\omega : \nu \to \omega(\nu)$ is a bijection $N^-(n) \to E^-(n)$ first note that by Proposition 2.1.2 we have that the map is injective. We can now show for all $\epsilon \in E^-(n)$ that $\omega^{-1}(\epsilon) \in N^-(n)$. Note that from the analogous identity $\omega^{-1} \circ F = \lambda \circ \omega^{-1}$, which follows from Section 2.1 we may similarly show that

$$(\lambda^k \circ \omega^{-1})(\epsilon) = (\omega^{-1} \circ F^k)(\epsilon).$$

We wish to show that $\omega^{-1}(\epsilon)$ has an odd number of ones, note that this follows from an induction on the length of the string ϵ. Suppose for all strings ϵ of length $n-1$ that if $\epsilon \in E^{\pm}(n-1)$ then $\omega^{-1}(\nu) \in N^{\pm}(n-1)$. Then, note by definition of ω^{-1} that adding a ± 1 to a string ϵ of length $n-1$ to make a string of length n (which we will call ϵ') will change the number of ones in the binary string $\omega^{-1}(\epsilon)$ only if the signatures of ϵ and ϵ' are different. From here the inductive step follows and the result holds for all n, with our base case holding for $n = 1$ trivially from the definition of ω^{-1}. Note the case where ϵ' is not primitive is irrelevant.

Suppose for contradiction that $\omega^{-1}(\epsilon)$ is not primitive. The same approach as for ω works, yielding a contradiction by implying that ϵ does not have a primitive period n under F. This gives us that for all $\epsilon \in E^-(n)$ that $\omega^{-1}(\epsilon) \in N^-(n)$. This suffices to show that the map $\omega^- : N^-(n) \to E^-(n)$ is a bijection. □

Example 2.1.10. For $n = 4$, the $\nu_1 = (0, 0, 0, 1) \in N^-(4)$, or $\nu_2 = (0, 1, 1, 1) \in N^-(4)$.

F-cycles:

$$\omega(\nu_1) = (1, 1, 1, -1) \to (1, 1, -1, -1) \to (1, -1 - 1 - 1) \to (-1, -1 - 1 - 1)$$

$$\omega(\nu_2) = (1, -1, 1, -1) \to (-1, 1, -1, -1) \to (-1, 1, 1, -1) \to (-1, -1, 1, -1)$$

In both cases we can see every element has odd signature, and primitive period 4, so belongs to $E^-(4)$.

2.2 Extended twisted shift operator

In order to extend a bijection from $\widetilde{N}^+(n)$ to $\bar{N}(n)$, we define an extended twist operator. Let $\widetilde{E}$ denote a product $E \times \{\pm 1\}$.

$$\widetilde{F}(\epsilon, \chi) = (F(\epsilon), \epsilon_1 \chi)$$

Sometimes we will also denote elements of $\widetilde{E}$ corresponding to ϵ_- and ϵ_+ by $(\epsilon, -1)$ and $(\epsilon, 1)$ respectively.

We will also give a definition of the set $\mathrm{Per}_n(F)$ which is a set of elements of E with $F^n(\epsilon) = \epsilon$.

Proposition 2.2.1. *A point $\tilde{\epsilon} = (\epsilon, \chi) \in \widetilde{E}$ with $\epsilon \in Per_n(F)$ and having even signature has primitive period n for $\widetilde{F}$ if and only if either $\epsilon \in E^+(n)$ or $\epsilon \in E^-(n/2)$. (The latter case occurs only when n is even).*

Proof. First, we will prove that if $\tilde{\epsilon}$ has primitive period n for $\widetilde{F}$ then $\epsilon \in E^+(n)$ or $\epsilon \in E^-(n/2)$. Consider the $\widetilde{F}$-cycle for point $\tilde{\epsilon} = (\epsilon, \chi) = (\epsilon_1, \epsilon_2 ..., \epsilon_n, \chi)$:

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1 \chi) \to ... \to (F^i(\epsilon), \epsilon_1^2 \epsilon_2^2 ... \epsilon_{i-1}^2 \epsilon_i \chi) \to ... \to (F^n(\epsilon), \epsilon_1^2 \epsilon_2^2 ... \epsilon_{n-1}^2 \epsilon_n \chi)$$

which is equivalent to:

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1 \chi) \to ... \to (F^i(\epsilon), \epsilon_i \chi) \to ... \to (F^n(\epsilon), \epsilon_n \chi)$$

Let us take the primitive period s of ϵ for F. $F^s(\epsilon) = \epsilon$ and $F^n(\epsilon) = \epsilon$, so $s|n$, and say $n = s \cdot k$. Suppose for a contradiction that $k \geq 3$. Then look at (ϵ, χ), $(F^k(\epsilon), \epsilon_k \chi)$, $(F^{2k}(\epsilon), \epsilon_{2k} \chi)$. We know that $\epsilon = F^k(\epsilon) = F^{2k}(\epsilon)$, and as signature can take only values 1 or -1, there will be two equal elements in the cycle, which contradicts that $\tilde{\epsilon}$ has primitive period n for $\tilde{F}$. So $k = 1$ or $k = 2$.
If $k = 1$, then $s = n$. As ϵ has a primitive signature, $\epsilon \in E^+(n)$ by definition.
If $k = 2$, $s = n/2$. Its $\widetilde{F}$-cycle is:

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1 \chi) \to ... \to (F^{n/2-1}(\epsilon), \epsilon_{n/2-1} \chi) \to (\epsilon, \epsilon_{n/2} \chi) \to (F(\epsilon), \epsilon_{n/2+1} \chi) \to ... \to (F^{n/2-1}(\epsilon), \epsilon_{n-1} \chi)$$

Because the $\widetilde{F}$-cycle has primitive period n, all its elements are different, so $\epsilon_i \chi = -\epsilon_{i+n/2} \chi$ and so $\epsilon_i = -\epsilon_{i+n/2}$, for all $i = \{1, ..., n/2\}$. In particular, $\chi = -\epsilon_{n/2} \chi$, so $\epsilon_{n/2} = -1$. That means $\epsilon = (\epsilon_1, \epsilon_2, ... \epsilon_{n/2}, -\epsilon_1, -\epsilon_2, ... - \epsilon_{n/2})$.
Say $\epsilon' = (\epsilon_1, \epsilon_2, ... \epsilon_{n/2})$, and assume for contradiction that its primitive period is $p \neq n/2$. Then, because ϵ has primitive period $n/2$, $F^{n/2}(\epsilon') = \epsilon'$. So $p|(n/2) \iff n/2 = p \cdot t$. Write $\epsilon' = (E_1, E_2, E_3, ..., E_t, (E_1', ..., E_t'))$, where $E_i = (\epsilon_{ip}, \epsilon_{ip+1} ..., \epsilon_{(i+1)p-1})$ for all $i = \{1, 2, ..., t\}$ and $(E_1', ..., E_t')$ is a continuation of ϵ' we will use for a shift.
Let us determine what $E_1', E_2', ... E_t'$ are equal to. We have

$$F^{n/2}(\epsilon') = (\epsilon_{n/2} E_1', \epsilon_{n/2} E_2', ..., \epsilon_{n/2} E_{t-1}', \epsilon_{n/2} E_t') = (-E_1', -E_2', ..., -E_t').$$

So $E_i' = -E_i$ for $i = \{1, ... t\}$. $F^t(\epsilon') = (\epsilon_p E_2, \epsilon_p E_3, ..., \epsilon_p E_k, \epsilon_p E_1', ..., \epsilon_p E_t')$. That means that $E_i = \epsilon_p E_{i+1}$ for $i = \{1, ..., t-1\}$, $E_t = \epsilon_p E_1' = -\epsilon_p E_1$ and $E_i' = \epsilon_p E_{i+1}' = -\epsilon_p E_{i+1}$ for $i = \{1, ..., t-1\}$.
Now, let us consider

$$\begin{aligned}
F^p(\epsilon) &= F^p(E_1, E_2, ... E_t, -E_1, -E_2, ..., -E_t) \\
&= (\epsilon_p E_2, ... \epsilon_p E_k, -\epsilon_p E_1, -\epsilon_p E_2, ... - \epsilon_p E_t, \epsilon_p E_1) \\
&= (E_1, E_2, ... E_t, E_1', ... E_t') \\
&= (E_1, E_2, ... E_t, -E_1, ... - E_t) \\
&= (\epsilon', -\epsilon') = \epsilon
\end{aligned}$$

But ϵ has primitive period $n/2$, so $(n/2)|p$ and recalling that $p|(n/2)$, then $p = n$. This contradicts our assumption, so finally, we can say that ϵ' has primitive period $n/2$ and so $\epsilon \in E^-(n/2)$.

Now let us prove that if $\epsilon \in E^+(n)$ or $\epsilon \in E^-(n/2)$ then ϵ has primitive period n for the $\tilde{F}$-cycle. For $\epsilon \in E^+(n)$, we have $\widetilde{F}$-cycle:

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1 \chi) \to ... \to (F^i(\epsilon), \epsilon_i \chi) \to ... \to (F^n(\epsilon), \epsilon_n \chi)$$

And as ϵ has primitive period n, ϵ, $F(\epsilon)$, ... $F^{n-1}(\epsilon)$ are all different. As $F^n(\epsilon) = \epsilon$, and the signature of n^{th} element in the cycle is $\epsilon_n \chi = \chi$, so $\tilde{\epsilon}$ has a primitive period n for $\widetilde{F}$.
If $\epsilon \in E^-(n/2)$, then $\epsilon_{n/2} = -1$, and its $\widetilde{F}$ cycle is as following:

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1 \chi) \to ... \to (F^{n/2-1}(\epsilon), \epsilon_{n/2-1} \chi) \to (\epsilon, -\chi) \to (F(\epsilon), -\epsilon_1 \chi) \to ... \to (F^{n/2-1}(\epsilon), -\epsilon_{n/2-1} \chi)$$

$\epsilon, F(\epsilon) ..., F^{n/2-1}(\epsilon)$ are all different, and the sign of the signature in the second part of the cycle changes, so we have n different elements in the $\widetilde{F}$ cycle. Also $(\epsilon, \chi) = (F^n(\epsilon), -\epsilon_{n/2} \chi)$, so $\tilde{\epsilon}$ has primitive period n for $\widetilde{F}$. □

We will say that set $\widetilde{N}^+(n)$ is the set of binary necklaces which are either primitive of length n with an even number of 1's or primitive length n/2 with an odd number of 1's. Let us denote the $\widetilde{E}^+$ the union of $E^+(n)$ and when n is even $E^-(n/2)$.

Example 2.2.2. Let us check the if direction. For $n = 4$, $\epsilon_1 = (1, 1, -1, 1) \in E^+(4)$ or $\epsilon_2 = (1, -1, -1, 1) \in E^-(2)$.
$\widetilde{F}-$cycles for $\chi = +1$:

$$(1, 1, -1, 1)_+ \to (1, -1, 1, 1)_+ \to (-1, 1, 1, 1)_+ \to (-1, -1, -1, 1)_-$$

$$(1, -1, -1, 1)_+ \to (-1, -1, 1, 1)_+ \to (1, -1, -1, 1)_- \to (-1, -1, 1, 1)_- \to (1, -1, -1, 1)_+$$

So ϵ has primitive period n for $\widetilde{F}$-cycle for both cases.

Proposition 2.2.3. *(1) If $\nu \in \widetilde{N}^+(n)$, the F-cycle generated by $\omega(\nu)$ belongs to $\widetilde{E}^+(n)$.*
(2) The induced map $\omega^+ \colon \widetilde{N}^+(n) \to \widetilde{E}^+(n)$ is a bijection.

Proof. (1) To show that the F-cycle generated by $\omega(\nu)$ is in $\widetilde{E}^+(n)$, it is sufficient to show that every element of the F-cycle belongs to $\widetilde{E}^+(n)$, i.e. has primitive period n, and signature 1, or has primitive period $n/2$, and signature -1.
(1.1) $\nu \in N^+(n)$

$$F^k(\omega(\nu)) = F^t(\omega(\nu)) \underset{Corollary\ 2.1.5}{\iff} \omega(\lambda^k(\nu)) = \omega(\lambda^t(\nu)) \underset{Proposition\ 2.1.2}{\iff} \lambda^k(\nu) = \lambda^t(\nu)$$

So as ν has a primitive period n, F also has primitive period n. Also, $F^i(\omega(\nu)) = F^i(\epsilon_1, \epsilon_2, ...\epsilon_n) = (\epsilon_i\epsilon_{i+1}, \epsilon_i\epsilon_{i+2}, ...\epsilon_i\epsilon_{i-1}, \epsilon_i^2)$, so it also has an even signature (as $\epsilon_i^2 = 1$) for $i \geq 1$, and $\omega(\nu)$ has even signature by assumption. So $F^i(\omega(\nu)) \in E^+(n)$ for all $i = \{1, 2, ...n\}$.
(1.2) $\nu \in N^-(n/2)$
We know that $\omega(\nu) = (\epsilon_1, \epsilon_2, ...\epsilon_{n/2}, -\epsilon_1, -\epsilon_2, ...\epsilon_{n/2})$. Analogously to (1.1) F has primitive period $n/2$.

$$F^i(\omega(\nu)) = F^i(\epsilon_1, \epsilon_2, ...\epsilon_{n/2}, -\epsilon_1, -\epsilon_2, ...\epsilon_{n/2}) = (\epsilon_i\epsilon_{i+1}, \epsilon_i\epsilon_{i+2}, ...\epsilon_{n/2}, -\epsilon_i\epsilon_1, ... -\epsilon_i\epsilon_i, ... -\epsilon_i\epsilon_{n/2}, \epsilon_i\epsilon_1, ...\epsilon_i\epsilon_i)$$

Notice, that the $(n/2)^{th}$ element of $F^i(\omega(\nu))$ is $-\epsilon_i^2 = -1$, which is its signature, so $F^i(\omega(\nu)) \in E^-$ for all $i = \{1, ...n/2\}$.

(2) All elements of the F-cycle generated by $\omega(\nu)$ correspond to the same element in $\widetilde{E}^+$ up to shift, as $F^k(\omega(\nu)) = \lambda^k(\omega(\nu))$. By *Proposition* 2.1.2, there exists the inverse of ω, so the ω^+ inverse also exist:

$$\epsilon \underset{\omega^-}{\to} \nu$$

$\nu \in \widetilde{N}^+(n)$ because it has the same primitive period as ϵ, and if $\epsilon \in E^-(n/2)$ then $\epsilon_{n/2} = -1$ and ν has odd number of 1's, so $\nu \in N^-(n/2)$, or if $\epsilon \in E^+(n)$ then $\epsilon_n = 1$ and ν has even number of 1's, so $\nu \in N^+(n)$. And so ω^+ is bijective.

□

Example 2.2.4. Take $n = 4$, $\nu_1 = (0, 0, 1, 1) \in N^+(4)$ or $\nu_2 = (0, 1, 0, 1) \in N^-(2)$.
F-cycles:

$$\omega(0, 0, 1, 1) = (1, 1, -1, 1) \to (1, -1, 1, 1) \to (-1, 1, 1, 1) \to (-1, -1, -1, 1)$$

$$\omega(0, 1, 0, 1) = (1, -1, -1, 1) \to (-1, -1, 1, 1)$$

The first cycle has even signature and primitive period 4, so belongs to $\widetilde{E}^+(4)$. The second cycle has an odd signature ($\epsilon_2 = -1$) and primitive period 2, so is in $E^-(2)$.

Lemma 2.2.5. *Let the $\widetilde{F}$-orbit of $(\epsilon, -)$ be denoted as $Orb_{\widetilde{F}}(\epsilon_-)$, and similarly define $Orb_{\widetilde{F}}(\epsilon_+)$. We have that for any $\epsilon \in E^+(n)$ that $Orb_{\widetilde{F}}(\epsilon_-)$ and $Orb_{\widetilde{F}}(\epsilon_+)$ are disjoint.*

Proof. Note that by definition no two elements of the same $\widetilde{F}$-orbit are the same. Suppose that $\epsilon_1 \in \mathrm{Orb}_{\widetilde{F}}(\epsilon_-)$ and $\epsilon_1 \in \mathrm{Orb}_{\widetilde{F}}(\epsilon_+)$. By comparing the first coordinates which are of the form $F^k(\epsilon)$, we have that ϵ_1 appears in the same position of the $\widetilde{F}$-orbit. However, note that the signs of the second coordinate χ are swapped in $\mathrm{Orb}_{\widetilde{F}}(\epsilon_-)$ and $\mathrm{Orb}_{\widetilde{F}}(\epsilon_+)$. Hence the two strings cannot be the same, so we're done.

□

Example 2.2.6. For $n = 4$, and $\epsilon = (1, 1, 1, -1)$ we have the following two orbits.

$$\mathrm{Orb}_{\widetilde{F}}(\epsilon_-) = (1, 1, 1, -1)_-, (1, 1, -1, -1)_-, (1, -1, -1, -1)_-, (-1, -1, -1, -1)_-$$

$$\mathrm{Orb}_{\widetilde{F}}(\epsilon_+) = (1, 1, 1, -1)_+, (1, 1, -1, -1)_+, (1, -1, -1, -1)_+, (-1, -1, -1, -1)_+$$

For $\epsilon = (1, -1, 1, -1)$ we have the following two orbits.

$$\mathrm{Orb}_{\widetilde{F}}(\epsilon_-) = (1, -1, 1, -1)_-, (-1, 1, -1, -1)_-, (-1, 1, 1, -1)_+ (-1, -1, 1, -1)_-$$

$$\mathrm{Orb}_{\widetilde{F}}(\epsilon_+) = (1, -1, 1, -1)_+, (-1, 1, -1, -1)_+, (-1, 1, 1, -1)_-, (-1, -1, 1, -1)_+$$

Here it is clear that each pair of orbits is disjoint, and the above proof reduces to saying that signs are swapped for each pair of elements in the same position of each orbit.

2.3 F-cycles to inversion classes

In order to connect the information encoded within the extra $\{\pm 1\}$ coordinate of elements of $\widetilde{E}$ and necklace inversion we define the function $H : \widetilde{E}(n) \to \{0,1\}^n$ as follows

$$H : (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)_\chi \to (a_1, a_2, \cdots, a_n), \text{ for } a_k = \begin{cases} 0 & \text{if } \chi\epsilon_k = 1 \\ 1 & \text{if } \chi\epsilon_k = -1. \end{cases}$$

This function has the key property that $\varphi \circ H = \lambda \circ \varphi$ and so can be used to biject from $\widetilde{F}$-orbits in $\widetilde{E}$ to equivalence classes of necklaces under shifting. We prove some important properties of H in the following lemmas.

Lemma 2.3.1. *Let $\epsilon \in E^+(n)$. The function $H : Orb_{\widetilde{F}}(\epsilon, -) \cup Orb_{\widetilde{F}}(\epsilon, +) \to \{0,1\}^n$ is injective.*

Proof. We proceed by contradiction. Suppose that $H(\epsilon_1) = H(\epsilon_2)$ for $\epsilon_1 \neq \epsilon_2$. Either ϵ_1 and ϵ_2 lie in the same $\widetilde{F}$-orbit, or they lie in different orbits - note that no elements of $\widetilde{E}$ lie in both orbits by Lemma 2.2.5.

Suppose $H(\epsilon_1) = H(\epsilon_2)$. We may write $\epsilon_1 = (\epsilon_1', \chi_1)$ and $\epsilon_2 = (\epsilon_2', \chi_2)$. If $\chi_1 = \chi_2$ this forces (by definition of H) that $\epsilon_1' = \epsilon_2'$. But then $\epsilon_1 = \epsilon_2$ and so by Lemma 2.2.5 the claim holds.

If $\chi_1 = -\chi_2$ then this forces ϵ_1' and ϵ_2' to be inverses of each other, i.e. mapping $1 \mapsto -1$ and $-1 \mapsto 1$ gives ϵ_1' from ϵ_2' and vice versa. We note that this forces the signature of ϵ_1' and ϵ_2' have opposite signatures since their nth coordinates of each string are inverted.

So one of the two strings ϵ_1,ϵ_2 has an odd signature. But then note both strings lie in the F-cycle of ϵ, which has an even signature. This gives us a contradiction by Definition 2.1.7 □

This lemma can be adapted to the case when $\epsilon \in E^-(n/2)$ where we repeat the string twice to produce a string of length n. Let ϵ' be the string produced by concatenating the string ϵ twice. A similar approach to the above lemma yields the following.

Lemma 2.3.2. *Let $\epsilon \in E^-(n/2)$. The function $H : Orb_{\widetilde{F}}(\epsilon', -) \to \{0,1\}^n$ is injective*

Proof. We proceed the same way. If two elements of the orbit have the same image under H then they must be of the form $(\overline{\epsilon_1}, \chi_1)$ and $(\epsilon_1, -\chi_1)$ respectively, else they are equal (where $\overline{\epsilon}$ is the inverse of ϵ). But then noting that $\overline{\epsilon_1}$ and ϵ have opposite signature this forces one of the signatures to be odd which yields a contradiction by Definition 2.1.7 since ϵ' has even signature. □

Lemma 2.3.3. *Let $\epsilon \in E^+(n)$. The image of $Orb_{\widetilde{F}}(\epsilon, -) \cup Orb_{\widetilde{F}}(\epsilon, +)$ under H lies in the same equivalence class of necklaces up to shifts and inversion.*

Proof.

$$\begin{aligned} \mathrm{Orb}_{\widetilde{F}}(\epsilon, -) &= \{(\epsilon, -1),\ \widetilde{F}(\epsilon, -1),\ \widetilde{F}^{\circ 2}(\epsilon, -1),\ \cdots,\ \widetilde{F}^{\circ n}(\epsilon, -1)\} \\ \mathrm{Orb}_{\widetilde{F}}(\epsilon, +) &= \{(\epsilon, +1),\ \widetilde{F}(\epsilon, +1),\ \widetilde{F}^{\circ 2}(\epsilon, +1),\ \cdots,\ \widetilde{F}^{\circ n}(\epsilon, +1)\} \end{aligned}$$

We note that $(H \circ \widetilde{F}^k)(\epsilon, -1) = \overline{(H \circ \widetilde{F}^k)(\epsilon, +1)}$, and so the images of two elements in the same position of each $\widetilde{F}$-orbit lie in the same equivalence class under inversion.

It suffices to show that for all elements of the same $\widetilde{F}$-orbit they are sent to the same equivalence cycle up to shifts and inversion, and then combining this with the above fact will yield the result. This can be shown by the following claim:

Claim: if $\epsilon_k \in orb_{\widetilde{F}}(\epsilon, \chi)$, then $H \circ \widetilde{F}(\epsilon_k) = \lambda \circ H(\epsilon_k)$.

We may check this holds explicitly. Let $\epsilon_k = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)_\chi$ and $H(\epsilon_k) = (a_1 a_2 a_3 \cdots a_n)$. If $\epsilon_1 = 1$ then F acts on ϵ as a left shift (since here ϵ has an even signature. This preserves the sign of χ when we apply $\widetilde{F}$. Hence we have that:

$$(H \circ \widetilde{F})(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)_\chi = (a_2 a_3 \cdots a_n a_1) = \lambda(a_1 a_2 \cdots a_n) = (\lambda \circ H)(\epsilon)$$

We consider the case when $\epsilon_1 = -1$, here we have that under F the string ϵ is left shifted and inverted. Note the signature of $F(\epsilon)$ is 1 $(= -\epsilon_1)$. Then note that under $\widetilde{F}$ the sign of χ inverts, and so under H the string is inverted again to give a binary necklace. Hence in this case, $(H \circ \widetilde{F})(\epsilon, \chi) = (\lambda \circ H)(\epsilon, \chi)$. The claim follows and it follows all elements in the same $\widetilde{F}$-orbit are sent to the same equivalence class under shifting, so the result follows. □

Example 2.3.4. For $n = 4$, $\epsilon = (1, 1, -1, 1) \in E^+(4)$.

$$Orb_{\widetilde{F}}((1,1,-1,1),-1) = \{((1,1,-1,1),-1),((1,-1,1,1),-1),((-1,1,1,1),-1),((-1,-1,-1,1),1)\}$$

$$Orb_{\widetilde{F}}((1,1,-1,1),1) = \{((1,1,-1,1),1),((1,-1,1,1),1),((-1,1,1,1),1),((-1,-1,-1,1),-1)\}$$

Under H it is:

$$\{(1,1,0,1),(1,0,1,1),(0,1,1,1),(1,1,1,0)\}$$

$$\{(0,0,1,0),(0,1,0,0),(1,0,0,0),(0,0,0,1)\}$$

In each set we have the same strings up to the shift, and both sets are equivalent up to inversion.

3 The main bijections

3.1 Proof of Theorem A

We give a proof of the main theorems.

Proof of Theorem A. The map $\omega^+ : \widetilde{N}^+(n) \to \widetilde{E}^+(n)$ defined by $s \to \omega(s)$ is a bijection by Proposition 2.2.3. Then, by $F \circ \omega = \omega \circ \lambda$ we have that necklaces s_1, s_2 in $\widetilde{N}^+(n)$ that are equivalent up to cyclic shifts are mapped to the same F-orbit in $\widetilde{E}^+(n)$ under the map ω^+.

We have shown necklaces equivalent up to cyclic shifts are mapped to the same F-orbit in $\widetilde{E}^+(n)$. Hence, for Ξ to be well defined we want to show that the elements of the same F-orbit in $\widetilde{E}^+(n)$ are mapped to the same equivalence class of necklaces of length n (up to inversion) under the map Φ given by $\Phi = \text{rev} \circ \varphi$. Note that the order in which we compose the maps rev and φ does not change the map Ξ.

It is sufficient to show that the image of $\omega(s)$ and $F(\omega(s))$ under Φ belong to the same equivalence class of necklaces under inversion, and so it follows that the image of any two members of the same F-orbit lie in the same equivalence class. We verify this directly. We split into two cases based on whether $\epsilon_1 = 1$ or $\epsilon_1 = -1$.

If $s \in N^+(n)$ then $\omega(s) = (\epsilon_1\epsilon_2 \cdots \epsilon_n)$ and $\epsilon_n = 1$. Let $\varphi(\epsilon) = (a_1, a_2, \cdots, a_n)$.
If $\epsilon_1 = 1$ then $F(\epsilon) = (\epsilon_2 \cdots \epsilon_n\epsilon_1)$ then $\varphi(F(\epsilon)) = (a_n, a_1, \cdots a_{n-1})$
If $\epsilon_1 = -1$ then $F(\epsilon) = (-\epsilon_2 \cdots - \epsilon_n - \epsilon_1)$ then $\varphi(F(\epsilon)) = (a'_n, a'_1, \cdots a'_{n-1})$, for $a'_i = 1 - a_i$

If $s \in N^-(n/2)$ then $\omega(s) = (\epsilon_1 \cdots \epsilon_{n/2} - \epsilon_1 \cdots - \epsilon_{n/2})$ and $\epsilon_n = 1$. Let $\varphi(\epsilon) = (a_1, \cdots, a_{n/2}, a'_1, \cdots, a'_{n/2})$.
If $\epsilon_1 = 1$ then $F(\epsilon) = (\epsilon_2 \cdots \epsilon_{n/2} - \epsilon_1 \cdots - \epsilon_{n/2}\epsilon_1)$ then $\varphi(F(\epsilon)) = (a_2, \cdots a_{n/2}, a'_1, \cdots a'_{n/2}a_1)$
If $\epsilon_1 = -1$ then $F(\epsilon) = (-\epsilon_2 \cdots - \epsilon_{n/2}\epsilon_1 \cdots \epsilon_{n/2}, -\epsilon_1)$ then $\varphi(F(\epsilon)) = (a'_2, \cdots a'_{n/2}, a_1, \cdots a_{n/2}a'_1)$

Then, in the case where $\epsilon_1 = 1$ we have that $\Phi(\epsilon)$ and $(\Phi \circ F)(\epsilon)$ differ by a cyclic shift, and so lie in the same equivalence class of necklaces. In the case where $\epsilon_1 = -1$ we have that $\Phi(\epsilon)$ and $(\Phi \circ F)(\epsilon)$ differ by a cyclic shift followed by an inversion, and so both $\Phi(\epsilon)$ and $(\Phi \circ F)(\epsilon)$ lie in the same equivalence class up to shifts and inversion in either case. Hence, two equivalent necklaces in $\widetilde{N}^+(n)$ are mapped to equivalent binary necklaces of length n up to shifts and inversion under Ξ.

We now wish to show that $s' \in \bar{N}(n)$. We split into two cases, depending on whether $\omega(s) = \epsilon \in E^+(n)$ or if $\omega(s) = \epsilon \in E^-(n/2)$. The main idea of the proof is to use the fact that if $\epsilon \in \widetilde{E}^+(n)$ then ϵ has primitive period n under $\widetilde{F}$ (by Proposition 2.2.1) in order to construct a set of distinct elements in the same equivalence class of $\Xi(s) = s'$ by using the function H: $\widetilde{E}(n) \to \{0, 1\}^n$. Note that a binary necklace ν is primitive if and only if its reverse necklace rev(ν) is primitive. Hence, it is sufficient to show that the modified map $\widetilde{\Xi} = \phi \circ \omega^+ = \text{rev} \circ \Xi$ is a function from $\widetilde{N}^+(n) \to \bar{N}(n)$.

Case 1: $\omega(s) = \epsilon \in E^+(n)$

By Proposition 2.2.1 we have that the $\widetilde{F}$-orbit of (ϵ, χ) has size n. We denote the orbit of the point (ϵ, χ) under $\widetilde{F}$ by $\text{Orb}_{\widetilde{F}}(\epsilon, \chi)$. By Lemma 2.2.5 we have that $\text{Orb}_{\widetilde{F}}(\epsilon, -)$ and $\text{Orb}_{\widetilde{F}}(\epsilon, -)$ are disjoint. By Lemma 2.3.1 we have that the map $H : \text{Orb}_{\widetilde{F}}(\epsilon, -) \cup \text{Orb}_{\widetilde{F}}(\epsilon, +) \to \{0, 1\}^n$ is injective. Furthermore, by Lemma 2.3.3 we have that the elements of the image of $\text{Orb}_{\widetilde{F}}(\epsilon, -) \cup \text{Orb}_{\widetilde{F}}(\epsilon, +)$ under H belong to the same equivalence class of necklaces up to shifts and inversion. Furthermore, note that s' lies in this equivalence class since $s' = H(\epsilon, +)$ and $(\epsilon, +) \in \text{Orb}_{\widetilde{F}}(\epsilon, +)$. By combining the results of Proposition 2.2.1, Lemma 2.2.5, Lemma 2.3.1, and Lemma 2.3.3 we have that the size of the equivalence class of s' is at least $n + n = 2n$.

Now suppose for the sake of contradiction that s' is not a primitive binary necklace, and so we have that $\lambda^k(s') = s'$ for some $k : 1 \le k < n$. Then it is clear that the size of the equivalence class of s' up to shifts and inversion has size at most $2k$. However, $|\mathcal{C}_{s'}| \ge 2n$ and $|\mathcal{C}_{s'}| \le 2k \implies k \ge n$ which is a contradiction, hence forcing s' to be a primitive binary necklace.

Case 2: $\omega(s) = \epsilon \in E^-(n/2)$

By Proposition 2.2.1 we have that $\text{Orb}_{\widetilde{F}}(\epsilon, \chi)$ has size n. A similar approach to Lemma 2.3.1 shows that the map H induces an injection $\text{Orb}_{\widetilde{F}}(\epsilon, -) \hookrightarrow \{0, 1\}^n$ stated in Lemma 2.3.2. Furthermore, by Lemma 2.3.3 we have that the elements of the image of $\mathrm{Orb}_{\widetilde{F}}(\epsilon,-)$ under H belong to the same equivalence class of necklaces up to shifts and inversion. We note that the image of $\mathrm{Orb}_{\widetilde{F}}(\epsilon,-)$ under H lies in the same equivalence class since s' as $H(\epsilon,-)$ is the inverse of s'. Hence we have found n elements in the same equivalence class as s' by combining Proposition 2.2.1, Proposition 2.2.3, Lemma 2.3.2, Lemma 2.3.3 .

Note since $\epsilon \in E^-(n/2)$ we may write $\varphi(\epsilon) = (a_1, \cdots, a_{n/2}, a'_1, \cdots, a'_{n/2}) = s'$ and so it is easy to check that $\lambda^{n/2}(s')$ is equal to the inverse of s'. Hence we have that the size of the equivalence class of s' is precisely the primitive period of s' under λ. Then, note that since s' is a binary necklace of length n we have that $|\mathcal{C}_{s'}| \leq n$ and also that $|\mathcal{C}_{s'}| \geq n$ hence s' has primitive period n under λ and so s' is a primitive binary necklace.

This suffices to show that the map Ξ is a well defined function from $\widetilde{N}^+(n)$ to $\bar{N}(n)$

We now wish to show that Ξ induces a bijection. We first show that Ξ is invertible. Note that $\Xi = \phi \circ \mathrm{rev} \circ \omega$ and each function in the composition is invertible - ϕ and rev are trivially invertible while ω is invertible by Proposition 2.1.2. By composing each of the inverses we have $\Xi^{-1} = \omega^{-1} \circ \ \mathrm{rev} \circ \phi^{-1}$, hence forcing Ξ to be injective. Combining this with the fact $\mid \widetilde{N}^+(n) \mid = \mid \bar{N}(n) \mid = \gamma_n$ this gives us that Ξ is injective and surjective and so $\Xi : \widetilde{N}^+(n) \to \bar{N}(n)$ is a bijection.. $\square$

We define a related map $\Xi' : \widetilde{N}^+(n) \to \bar{N}(n)$ and show that it is a bijection. Given a necklace $\nu = (a_1, a_2, \cdots, a_n) \in \widetilde{N}^+(n)$, consider the corresponding sequence $\epsilon := \omega(\nu) \in E$. Iterating the twisted shift operator F gives an F-orbit of size n, let this be C, where

$$C = \{\epsilon, F(\epsilon), F^2(\epsilon), \cdots, F^{n-1}(\epsilon)\}.$$

Let $\mathrm{rev}(C)$ denote the set of all reversed strings of the orbit, that is

$$\mathrm{rev}(C) := \{\mathrm{rev}(\epsilon), \mathrm{rev}(F(\epsilon)), \mathrm{rev}(F^2(\epsilon)), \cdots, \mathrm{rev}(F^{n-1}(\epsilon))\}.$$

Let $\epsilon' \in E$ be the lexicographically minimal element of $\mathrm{rev}(C)$, truncated to an n-tuple $(\epsilon'_1, \epsilon'_2, \cdots, \epsilon'_n) \in \{\pm 1\}^n$. Now we convert ϵ' to a binary string s' by the usual correspondence $s' = \varphi(\epsilon')$. We set $\Xi'(\nu) := s'$

3.2 Proof of Theorem B

To prove theorem Theorem B we first note a consequence of theorem Theorem A.

Lemma 3.2.1. *If $\nu \in \widetilde{N}^+(n)$, and $\omega(\nu) = \epsilon$ then for any $\epsilon' \in Orb_F(\epsilon)$ we have that $\varphi(\epsilon')$ is a primitive binary necklace.*

Proof. Given a necklace $\nu \in \widetilde{N}^+(n)$ we have that (by Proposition 2.2.3) ω^+ induces a bijection between F-orbits in $\widetilde{E}^+(n)$ and necklaces in $\widetilde{N}^+(n)$. Take an $\epsilon' \in \mathrm{Orb}_F(\epsilon)$, then we have (by Section 2.1) that $\varphi(\epsilon') = \varphi \circ \omega^k(\nu)$ for some $k : 1 \leq k < n$. Then note that in Theorem A we show that $\varphi \circ \omega$ induces a bijection from $\widetilde{N}^+(n) \to \bar{N}(n)$, and so it follows that $\varphi(\epsilon')$ is a primitive binary necklace. $\square$

Proof of Theorem B. First we show that the map Ξ' is well defined. By Section 2.1 we have that $F \circ \omega = \omega \circ \lambda$ and so equivalent necklaces up to shifting in $\widetilde{N}^+(n)$ are mapped to the same F-cycle and hence mapped to the same element of $\{0,1\}^n$.
The map is invertible since we have the following inverse map, where ω^{-1} is given explicitly in Proposition 2.1.2.

$$(\Xi')^{-1} : s' \xrightarrow{\varphi^{-1}} \epsilon' \xrightarrow{\mathrm{rev}} \epsilon \xrightarrow{\omega^{-1}} \nu$$

Hence we have that Ξ' is well-defined and is invertible. Hence the map is injective.

We wish to show $s' \in \bar{N}(n)$. We look at the map slightly differently. We again take the lexicographically minimal element ϵ' of $\mathrm{rev}(C)$. Then we swap the order of the maps rev and φ. Note that the two maps we swap commute and so changing their order of composition does not change the map Ξ'. We define $s'_1 := \varphi(\epsilon')$

By using the fact that a necklace is primitive if and only if its reverse is primitive, it is sufficient to show that $s'_1 \in \bar{N}(n)$. Note that ϵ' is an element of $\mathrm{Orb}_F(\epsilon)$ and so by Lemma 3.2.1 so we have that $s'_1 = \varphi(\epsilon')$ is a primitive, binary necklace. If we are able to show that under Ξ' exactly one member of each equivalence class in $\bar{N}(n)$ is targeted it follows that Ξ' induces a bijection between $\widetilde{N}^+(n)$ and $\bar{N}(n)$.

Claim: The image of $\widetilde{N}^+(n)$ under Ξ' contains exactly one member of each equivalence class in $\bar{N}(n)$.

It is sufficient to show that if s and t lie in the same equivalence class of $\bar{N}(n)$ then $\varphi^{-1}(s)$ and $\varphi^{-1}(t)$ lie in the same F-orbit in $\widetilde{E}^+(n)$, since then if $s \neq t$ in $\{0,1\}^n$ then by φ^{-1} injective we have that $\varphi^{-1}(s) \neq \varphi^{-1}(t)$ and both lie in the same F-orbit and so one is not lexicographically minimal and so we are done -since then one of s or t could not possibly be in the image of Ξ'.

In Theorem A we show that Φ induces a bijection between F-orbits in $\widetilde{E}^+(n)$ and equivalence classes in $\bar{N}(n)$. Let $\widetilde{E}^+(n)/F$ denote the set of F-robits in $\widetilde{E}^+(n)$ and let the representative of each orbit be the ϵ' in each orbit such that when all elements of the F-orbit are reversed, $\mathrm{rev}(\epsilon')$ is lexicographically minimal.

Then the image of the representatives under φ is exactly one element from each equivalence class in $\bar{N}(n)$. Then this map is invertible (and hence injective) and so is a bijection as $|\widetilde{N}^+(n)| = |\widetilde{E}^+(n)/F| = |\bar{N}(n)| = \gamma_n$. so our map is injective and surjective, where the second equality follows from Proposition 2.2.3 and the last is known. Hence is s and t lie in the same equivalence class in $\bar{N}(n)$ it follows that $\varphi^{-1}(s)$ and $\varphi^{-1}(t)$ lie in the same F-orbit in $\widetilde{E}^+(n)$. Hence the claim follows and so the theorem follows. □

4 Further Bijections

Weiss and Rogers do not consider the case of even periodic cycles in their paper. We gain information about even periodic cycles by considering cycles generated by $\widetilde{F}$ on elements of $\widetilde{E}$.

Definition 4.0.1. We call a permutation σ of $[n]$ unimodal if there exists some $m \in [n]$ such that $\sigma(i) > \sigma(j) > \sigma(m)$ for all $1 \leq i < j < m$ and $\sigma(m) < \sigma(i) < \sigma(j)$ for all $m < i < j \leq n$.

The map $\widetilde{\Phi}^+$: $Orb_F(\epsilon) \to S_n$ (group of permutations) is defined in the following way. Given a cycle $C \in \widetilde{E}^+(n)$ and an element $\epsilon \in C$, the cycle $\widetilde{C}$ generated by $\widetilde{\epsilon} := \epsilon_-$ has primitive period n for $\widetilde{F}$. Ordering the elements of $\widetilde{C}$ lexicographically defines an order-preserving bijection ι: $\widetilde{C} \to [n]$. There is a unique permutation σ of $[n]$ such that $\iota \circ \widetilde{F} = \sigma \circ \iota$. This permutation does not depend on the choice of ϵ. We denote the map taking C to the corresponding permutation σ by $\widetilde{\Phi}^+$.

Example 4.0.2. Let $n = 4$. There are two necklaces in $\widetilde{N}^+(n)$, corresponding to 0110 and 01. The map ω takes 0110 to $(1, -1, 1, 1)$. The $\widetilde{F}$-orbit of $(1, -1, 1, 1)_-$ is the 4-cycle

$$(1, 1, 1, -1)_- \mapsto (1, 1, -1, -1)_- \mapsto (1, -1, -1, -1)_+ \mapsto (-1, -1, -1, -1)_- .$$

Since

$$(-1, -1, -1, -1)_+ < (-1, 1, 1, 1)_- < (1, -1, 1, 1)_- < (1, 1, -1, 1)_- ,$$

$\widetilde{\Phi}^+(1, -1, 1, 1)$ is the cyclic unimodal permutation $(3214) = (1432)$.
The map ω takes 01 to $(1, -1, -1, 1)$. The $\widetilde{F}$-orbit of $(1, -1, -1, 1)_-$ is the 4-cycle

$$(1, -1, -1, 1)_- \mapsto (-1, -1, 1, 1)_- \mapsto (1, -1, -1, 1)_+ \mapsto (-1, -1, 1, 1)_+ .$$

Since

$$(-1, -1, 1, 1)_- \mapsto (-1, -1, 1, 1)_+ \mapsto (1, -1, -1, 1)_- \mapsto (1, -1, -1, 1)_+ ,$$

$\widetilde{\Phi}^+(1, -1, -1, 1)$ is the cyclic unimodal permutation $(3141) = (1423)$.

Lemma 4.0.3. *If $C \in \widetilde{E}^+(n)$ then $\widetilde{\Phi}^+(C) \in CUP(n)$*

Proof. We split into two cases, considering when C is generated from an element of $E^+(n)$ or an element of $E^-(n/2)$. We consider the element of C mapped to the lexicographically minimal element of the cycle under F.

Case 1: $\omega(s) = \epsilon \in E^+(n)$

Note that in this case since ϵ has primitive period n under F that it is not needed to consider the extra χ of elements in the cycle when ordering lexicographically. Let ϵ' be the element of C such that the image of ϵ' under F is lexicographically minimal. This element ϵ' will be of the following form:

$$\epsilon' = 1\underbrace{\text{-1 -1 -1 \ldots -1 -1}}_{\text{maximal}}\epsilon_k \ldots \epsilon_n \text{ OR } \epsilon = -1\underbrace{1\ 1\ 1 \ldots 1\ 1}_{\text{maximal}}\epsilon_k \ldots \epsilon_n$$

where the consecutive block of the same character is of the maximal length out of all elements of the cycle. We then consider two elements ϵ_1 and ϵ_2 lexicographically smaller than ϵ', we wish to show that the permutation induced by F is decreasing here, so $F(\epsilon_1) <_{\text{lex}} F(\epsilon_2)$.

Note that the first character of ϵ_1 and ϵ_2 must be -1, else they would be lexicographically larger than ϵ' in either case. Then let

$$\epsilon_1 = (-1, \epsilon_1, \epsilon_2, \cdots, \epsilon_k, -1, \cdots, \epsilon_n)$$
$$\epsilon_2 = (-1, \epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1, \cdots, \epsilon_n).$$

Now consider the image of ϵ_1 and ϵ_2 under F.

$$F(\epsilon_1) = (-\epsilon_1, -\epsilon_2, \cdots, -\epsilon_k, 1, \cdots, \epsilon_n)$$
$$F(\epsilon_2) = (-\epsilon_1, -\epsilon_2, \cdots, -\epsilon_k, -1, \cdots, \epsilon_n).$$

Hence we have that for $\epsilon_1 < \epsilon_2 < \epsilon'$ that $F(\epsilon_1) > F(\epsilon_2)$. So the permutation induced is decreasing up to the position ϵ' in the cycle. We now want to show that the permutation is increasing for elements of the cycle lexicographically larger than ϵ'. Take elements ϵ_1 and ϵ_2 in the cycle C such that $\epsilon' < \epsilon_1 < \epsilon_2$. It then follows that both strings must start with a 1 else if they are smaller than ϵ'. Then let

$$\epsilon_1 = (1, \epsilon_1, \epsilon_2, \cdots, \epsilon_k, -1, \cdots, \epsilon_n) \Longrightarrow F(\epsilon_1) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_k, -1, \cdots, \epsilon_n).$$

$$\epsilon_2 = (1, \epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1, \cdots, \epsilon_n) \Longrightarrow F(\epsilon_2) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_k, 1, \cdots, \epsilon_n).$$

Hence we have that $F(\epsilon_1) < F(\epsilon_2)$ and so the permutation is increasing for elements of the cycle larger than ϵ'. We note that ϵ' was the element of the cycle C such that its image under F was lexicographically minimal, and so it follows that the permutation induced is unimodal. Note that by definition of the permutation, we have that $\widetilde{\Phi}^+(C)$ is cyclic and so $\widetilde{\Phi}^+(C) \in \mathrm{CUP}(n)$.

Case 2: $\omega(s) = \epsilon \in E^-(n/2)$

In this case the primitive period of ϵ under F is $n/2$ and so, as shown in Proposition 2.2.3 the cycle C is of the form

$$(\epsilon, \chi) \to (F(\epsilon), \epsilon_1\chi) \to ... \to (F^{n/2-1}(\epsilon), \epsilon_{n/2-1}\chi) \to (\epsilon, \epsilon_{n/2}\chi) \to (F(\epsilon), \epsilon_1\chi) \to ... \to (F^{n/2-1}(\epsilon), \epsilon_{n/2}\chi)$$

Note that the elements of the cycle in positions i and $i + n/2$ are consecutive when ordered lexicographically as they differ only by their χ coordinate. A similar method to the first case shows that the permutation induced on the F-cycle $\epsilon \to F(\epsilon) \to ... \to F^{n/2-1}(\epsilon) \to \epsilon$ is cyclic unimodal.

First we find the form of the element ϵ' in the F-cycle of ϵ such that its image under F is minimal. Similarly, we have that

$$\epsilon' = 1\underbrace{\text{-1 -1 -1 ...-1 -1}}_{\text{maximal}}\epsilon_k \ldots \epsilon_n \text{ OR } \epsilon = -1\underbrace{1\ 1\ 1\ \ldots 1\ 1}_{\text{maximal}}\epsilon_k \ldots \epsilon_n$$

The elements $(\epsilon', -)$ and $(\epsilon', +)$ are such that their images under C are the minimal and second smallest in some order, where the order does not matter - in either case it does not contradict unimodality of the permutation.

It is now sufficient to show that if $(\epsilon_i, -) < (\epsilon_i, +) < (\epsilon', -)$ then $\widetilde{F}(\epsilon_i, -) > \widetilde{F}(\epsilon_i, +)$ and if $(\epsilon', +) < (\epsilon_i, -) < (\epsilon_i, +)$ then $\widetilde{F}(\epsilon_i, -) > \widetilde{F}(\epsilon_i, +)$. Note that if both elements are less than $(\epsilon', -)$ they must have their strings beginning with a -1, and so we can check that the signs of the χ coordinate of each element invert and so give the required result, and noting that if both elements are greater than $(\epsilon', +)$ then their strings must begin with a 1 yields the required result. Comparing consecutive elements with different strings follows from the fact that the F-cycle of ϵ is cyclic unimodal.

Hence we are done.

□

Conjecture. *The map $\widetilde{\Phi}^+$ induces a bijection from $\widetilde{N}^+(n)$ to CUP(n)*

This conjecture, combined with the other results in this paper and previous work, suffices to establish explicit bijections between (M1), (M2), (P1), (N1), (N2), (N3) (where N2 to N3 is given by Theorem A and Theorem B), as shown by Fig. 1.

References

[1] Kassie Archer and Sergi Elizalde. Cyclic permutations realized by signed shifts. *J. Comb.,* 5(1):1-30 2014.

[2] Xavier Buff, William Floyd, Sarah Koch, and Walter Parry. Factoring Gleason polynomials modulo 2. *J. Théor. Nombres Bordeaux,* 34(3):787-812, 2022.

[3] John H. Hubbard and Dierk Schleicher. The spider algorithm. In *Complex dynamical systems (Cincinnati, OH, 1994)*, volume 49 of *Proc. Sympos. Appl. Math.*, pages 155−180. Amer. Math. Soc., Providence, RI, 1994.

[4] John Milnor and William Thurston. On iterated maps of the interval. In *Dynamical systems (College Park, MD,* 1986 − 87, volume 1342 of *Lecture Notes in Math.*, pages 465−563. Springer, Berlin, 1988.

[5] A. Weiss and T. D. Rogers. The number of orientation reversing cycles in the quadratic map. In *Oscillations, bifurcation and chaos (Toronto, Ont., 1986)*, volume 8 of *CMS Conf. Proc.*, pages 703−711. Amer. Math. Soc., Providence, RI, 1987.