HW 8

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W1

Part (a)

$$\begin{aligned} Q^n &= (P^{-1}RP)^n \\ &= P^{-1}R(PP^{-1})R(PP^{-1})\cdots RP \\ &= P^{-1}R^nP \end{aligned}$$

Part (b)

$$\begin{aligned} e^{tQ} &= \sum_{k=0}^{\infty}\frac{1}{k!}(tQ)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}t^k(P^{-1}RP)^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}t^kP^{-1}R^kP \\ &= P^{-1}(\sum_{k=0}^{\infty}\frac{1}{k!}t^kR^k)P \\ &= P^{-1}e^{tR}P \end{aligned}$$

W2

$$Q = \lim_{t\searrow 0}\frac{P_t - I}{t} = \frac{d}{dt}\Big|_{t=0}P_t$$

W3

$$\begin{aligned} \pi Q &= 0 \\ \lim_{t\to\infty} p_t(i,j) &= \pi(j) \end{aligned}$$

P1

Part (a)

The probability is $\binom{2}{1} * \frac{1}{2} * \frac{1}{2} = \frac{1}{2}$

Part (b)

Let S denote the total spend of customers, let $X_{[t_1 - t_2]}$ denote the number of customers during time interval $[t_1 - t_2]$, let Y denote per customer's spend.

$$\begin{aligned} ES &= ES_{11am-1pm} + ES_{\text{other time}} \\ &= E[X_{11am-1pm} Y_{11am-1pm}] + E[X_{\text{other time}} Y_{\text{other time}}] \\ &= E[X_{11am-1pm}]E[Y_{11am-1pm}] + E[X_{\text{other time}}]E[Y_{\text{other time}}] \\ &= \$1250 \end{aligned}$$

Due to the independent increments probability, $X_{8am-11am}, X_{11am-1pm}, X_{1pm-4pm}$ are mutually independent. And assuming the unit time in this question is hour.

Therefore:

$$\begin{aligned} var(S) &= var(S_{11am-1pm}) + var(S_{\text{other time}}) \\ &= E[X_{11am-1pm}]var(Y_{11am-1pm}) + var(X_{11am-1pm})E[Y_{11am-1pm}]^2 \\ &\cdots + E[X_{\text{other time}}]var(Y_{\text{other time}}) + var(X_{\text{other time}})E[Y_{\text{other time}}]^2 \\ &= 25 * 50 + 2\lambda * 20^2 + 75 * 20 + 6\lambda * 10^2 \\ &= 2750 + 1400\lambda \\ &= 20250 \end{aligned}$$

P2

Part (a)

Let S denote the total size of the translated signals for a day, let Y_i denote the size of ith successfully translated signal. Due to the thining property, the rate $\lambda'(s)$ for successfully translated signal in each period is:

$$\lambda'(s) = \begin{cases} 240s & s \in [0, 0.5] \\ 120 & s \in [0.5, 1] \\ 60 & s \in [1, 2] \end{cases}$$

$$\begin{aligned} ES &= E\tilde{N}EY_i \\ &= 1.5 * 12 \int_0^2 \lambda'(s)ds \\ &= 2700\text{MB} \end{aligned}$$

Part (b)

$$\lim_{t \to \infty} t^{-1} \tilde{N}_t = \frac{1}{\mu} = \frac{1}{75}$$

Part (c)

$$\begin{aligned}
\rho &= \lim_{t \to \infty} \frac{\tilde{N}_t}{N_t} \\
&= \frac{E[\lambda'(s)]}{E[\lambda(s)]} \\
&= \frac{5}{6}
\end{aligned}$$

Part (d)

$$\begin{aligned}
P(\tilde{T}_1 \in [0,1] | \tilde{N}_2 = 1) &= P(\tilde{T}_1 \in [0, 0.5] | \tilde{N}_2 = 1) + P(\tilde{T}_1 \in [0.5, 1] | \tilde{N}_2 = 1) \\
&= \frac{30}{150} + \frac{60}{150} \\
&= \frac{3}{5}
\end{aligned}$$

Part (e)

During each period, if $p(s) = 1$ covers time $[0, 1]$, which relates to the highest $\lambda(s)$, ρ_τ would be lergest. Let $\{N_t^*\}$ denote the modified process.

In this setting:

$$\begin{aligned}
\rho_\tau &= \lim_{t \to \infty} \frac{N_t^*}{N_t} \\
&= \frac{E[\lambda^*(s)]}{E[\lambda(s)]} \\
&= \frac{165}{180} = \frac{11}{12}
\end{aligned}$$

P3

Part (a)

Let $\tau_i = \tau_i^1 + \tau_i^2 + \tau_i^3$ be the ith interval of renewal process. Let x_t^1, x_t^2, x_t^3 denote the holding time for 3 kids by time t. We have,

$$\lim_{t \to \infty} \frac{x_t^k}{t} = \frac{E[\tau_1^k]}{E[\tau_1]}$$

Therefore,

$$E[r_i^1] = E[m] = \frac{1}{36}\sum_{i=1}^{6}(2*i*(i-1)+i) = \frac{161}{36}$$
$$E[r_i^2] = E[n] = \frac{1}{21}\sum_{i=1}^{6}(i*(6-i)+i) = \frac{91}{36}$$
$$E[r_i^3] = \frac{1}{2}(E[n]+E[m]) = \frac{126}{36}$$

$$\lim_{t\to\infty}\frac{x_t^1}{t} = \frac{E[\tau_1^1]}{E[\tau_1]} = \frac{161}{378}$$
$$\lim_{t\to\infty}\frac{x_t^2}{t} = \frac{E[\tau_1^2]}{E[\tau_1]} = \frac{91}{378}$$
$$\lim_{t\to\infty}\frac{x_t^3}{t} = \frac{E[\tau_1^3]}{E[\tau_1]} = \frac{1}{3}$$

Therefore, the asymptotic fraction of possession time for each child for Plan A is: $[\frac{161}{378}, \frac{91}{378}, \frac{1}{3}]$

Part (b)

Let $\tilde{\tau_i} = 1\cdot\{D=1\}\tau_i^1 + 1\cdot\{D=2\}\tau_i^2 + 1\cdot\{D=3\}\tau_i^3$

r_i is randomly assigned as 1 out of r_i^1, r_i^2, r_i^3, we have

$$E[1\cdot\{D=1\}\tau_i^1] = \frac{1}{3}E[r_i^1]$$
$$E[1\cdot\{D=2\}\tau_i^2] = \frac{1}{3}E[r_i^2]$$
$$E[1\cdot\{D=2\}\tau_i^3] = \frac{1}{3}E[r_i^3]$$

Therefore, the asymptotic fraction of possession time for each child would not change from Plan A. It is $[\frac{161}{378}, \frac{91}{378}, \frac{1}{3}]$

Part (c)

Let $\tilde{\tau_i} = 1\cdot\{D=1\}(\tau_i^2+\tau_i^3) + 1\cdot\{D=2\}(\tau_i^1+\tau_i^3) + 1\cdot\{D=3\}(\tau_i^1+\tau_i^2)$

$$\begin{aligned}\tilde{\tau_i} &= 1\cdot\{D=1\}(\tau_i^2+\tau_i^3) + 1\cdot\{D=2\}(\tau_i^1+\tau_i^3) + 1\cdot\{D=3\}(\tau_i^1+\tau_i^2)\\ &= (1\cdot\{D=2\}+1\cdot\{D=3\})\tau_i^1 + (1\cdot\{D=1\}+1\cdot\{D=3\})\tau_i^2 + (1\cdot\{D=1\}+1\cdot\{D=2\})\tau_i^3\end{aligned}$$

Therefore,

$$E[1\cdot\{D=1\}\tau_i^1] = \frac{2}{3}E[r_i^1]$$
$$E[1\cdot\{D=2\}\tau_i^2] = \frac{2}{3}E[r_i^2]$$
$$E[1\cdot\{D=2\}\tau_i^3] = \frac{2}{3}E[r_i^3]$$

Therefore, the asymptotic fraction of possession time for each child would not change from Plan A. It is $[\frac{161}{378}, \frac{91}{378}, \frac{1}{3}]$

Part (d)

Denote the rate for 1,2,3 as p_1, p_2, p_3, as the probability for the dice shows up 1,2,3 respectively. We have:

$$\begin{cases} p_1 * \frac{161}{378} = p_2 * \frac{91}{378} = p_3 * \frac{1}{3} \\ p_1 + p_2 + p_3 = 1 \end{cases} \Rightarrow [p_1, p_2, p_3] = [0.247, 0.437, 0.316]$$

P4

Part (a)

In 36 mins, the expected number of customer is $10 * 36/60 = 6$. Therefore, the fraction of the customers actually goes to hilton is $\frac{7}{13}$.

Part (b)

The expected interval between arrival is $1/\lambda = 0.1$ hour. Therefore, the average amount of time a person who goes to Hilton has to wait is $0.1 * (6+5+4+3+2+1)/7 = 0.3$ hour.

P5

Part (a)

Let μ denote the mean of distribution G. The dealer's rate is $\frac{1}{\frac{1}{\lambda}+\mu}$

Part (b)

The fraction of customers are lost is $\frac{\mu}{\frac{1}{\lambda}+\mu}$

P6

Part (a)

The fraction that Duke has the ball is $\frac{2}{2+6} = \frac{1}{4}$

Part (b)

Duke score: $60 \times \frac{1}{4} \div 2 \times \frac{1}{4} = \frac{15}{8}$

Miami score: $60 \times \frac{3}{4} \div 6 \times 1 = \frac{15}{2}$