HW 6

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W1

Consider a sequence of i.i.d. random variables $X_1, X_2, \cdots$ and N be another independent $\mathbb{N}_0-$value RVs. We would like to consider the compound sum as:

$$Y = \sum_{k=1}^{N} X_k$$

Then we have

$$\begin{aligned}\mathbb{E}Y &= \mathbb{E}[N]\mathbb{E}[X_1] \\ \text{Var}[Y] &= \mathbb{E}[N]\text{Var}[X_1] + \text{Var}[N]\mathbb{E}[X_1]^2\end{aligned}$$

W2

Consider a marked Poisson process $\{N_t\}_{t\geq 0}$ (with rate λ), $\{X_k\}_{k\in\mathbb{N}_0}$, where X_k's are i.i.d Bernouli random variables, with $\mathbb{P}(X_k = 1) = p$.

If N_t is a poisson process with rate λ, then the new processes denoted by $\{N_t^1\}, \{N_t^0\}$ is given by

$N_t^1 = \sum_{k\geq 1} X_k^1 \mathbb{I}(T_k \leq t)$ and $N_t^0 = \sum_{k\geq 1} X_k^0 \mathbb{I}(T_k \leq t)$

$\{N_t^1\}, \{N_t^0\}$ are two independent Poisson processes, with rate $p\lambda$, $(1-p)\lambda$.

P1

Part (a)

$$\begin{aligned}P(T_1 \leq T_2 \leq T_3) &= P(T_1 = \min\{T_1, T_2, T_3\})P(T_2 = \min\{T_2, T_3\}) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3}\end{aligned}$$

Part (b)

$$
\begin{aligned}
P(T_1 \le T_2 \le \cdots \le T_n) &= P(T_1 = \min\{T_1, T_2, \ldots, T_n\}) P(T_2 = \min\{T_2, \ldots, T_n\}) \cdots P(T_{n-1} = \min\{T_{n-1}, T_n\}) P(T_n = \min\{T_n\}) \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \cdot \frac{\lambda_2}{\lambda_2 + \cdots + \lambda_n} \cdots \frac{\lambda_{n-1}}{\lambda_{n-1} + \lambda_n} \frac{\lambda_n}{\lambda_n}
\end{aligned}
$$

P2

Part (a)

$$S = \min\{t \ge 0 : T \le \Lambda(t)\}$$

$$P(S > t) = P(T > \Lambda(t)) = 1 - P(T \le \Lambda(t)) = e^{-\Lambda(t)}$$

$$f_S(t) = \frac{dP(S \le t)}{dt} = \frac{d(1 - e^{-\Lambda(t)})}{dt} = \lambda(t) e^{-\Lambda(t)}$$

Part (b)

If $\lambda(s) = \lambda_0 \in \mathbb{R}^+$, we have

$$\Lambda(t) = \int_0^t \lambda_0 dr = \lambda_0 t$$

$$f_S(t) = \lambda_0 e^{-\lambda_0 t}$$

Therefore, $S \sim \exp(\lambda_0)$

Part (c)

$$
\begin{aligned}
P(S > t + s \mid S > s) &= \frac{P(S > t + s)}{P(S > s)} \\
&= \frac{e^{-\Lambda(t+s)}}{e^{-\Lambda(s)}} \\
&= e^{-\Lambda(t+s) + \Lambda(s)} \\
&= e^{-(\int_0^{t+s} \lambda(r) dr - \int_0^s \lambda(r) dr)} \\
&= e^{-\int_s^{t+s} \lambda(r) dr}
\end{aligned}
$$

P3

Part (a)

$U = \min\{S, T\}$, according to Lemma 2.1, $U \sim \exp(\lambda + \mu)$. Therefore, $EU = \frac{1}{\lambda + \mu}$

Part (b)

$V+U=S+T$, therefore, $E[V+U]=E[S+T]=ES+ET=\frac{1}{\mu}+\frac{1}{\lambda}$

$E[V-U]=E[V+U-2U]=\frac{1}{\mu}+\frac{1}{\lambda}-\frac{2}{\lambda+\mu}$

Part (c)

$$\begin{aligned} P(V<t) &= P(S<t, T<t) \\ &= P(S<t)P(T<t) \\ &= (1-e^{-\mu t})(1-e^{-\lambda t}) \\ &= 1-e^{-\mu t}-e^{-\lambda t}+e^{-(\mu+\lambda)t} \end{aligned}$$

$$\begin{aligned} f_V(t) &= \frac{dP(V<t)}{dt} \\ &= \mu e^{-\mu t}+\lambda e^{-\lambda t}-(\mu+\lambda)e^{-(\mu+\lambda)t} \end{aligned}$$

$$\begin{aligned} EV &= \int_{t=0}^{\infty} t f_V(t)dt \\ &= \int_{t=0}^{\infty} (\mu t e^{-\mu t}+\lambda t e^{-\lambda t}-(\mu+\lambda)te^{-(\mu+\lambda)t})dt \\ &= \frac{1}{\mu}+\frac{1}{\lambda}-\frac{1}{\lambda+\mu} \end{aligned}$$

Part (d)

$EV=E[S+T-U]=ES+ET-EU=\frac{1}{\mu}+\frac{1}{\lambda}-\frac{1}{\lambda+\mu}$

P4

Part (a)

Let T_1, T_2, T_3 denote the time when 3 types of shock occur.

$$\begin{aligned} P(U>s, V>t) &= P(T_1>s, T_2>t, T_3>\max\{s,t\}) \\ &= P(T_1>s)P(T_2>t)P(T_3>\max\{s,t\}) \\ &= e^{-(\lambda_1 s+\lambda_2 t+\lambda_3 \max\{s,t\})} \end{aligned}$$

Part (b)

$$
\begin{aligned}
P(U < t) &= 1 - P(T_1 > t, T_3 > t) \\
&= 1 - e^{-(\lambda_1 + \lambda_3)t}
\end{aligned}
$$

$$
\begin{aligned}
P(V < t) &= 1 - P(T_2 > t, T_3 > t) \\
&= 1 - e^{-(\lambda_2 + \lambda_3)t}
\end{aligned}
$$

Part (c)

No, U and V are not independent due to:

$$P(U > s)P(V > t) = e^{-(\lambda_1 + 2\lambda_2 + \lambda_3)t} \neq P(U > s, V > t)$$

P5

Part (a)

$$E(T_{12}) = E(\tau_1 + \cdots \tau_{12}) = 12E(\tau_1) = 4$$

Part (b)

$$
\begin{aligned}
E(T_{12} \mid N(2) = 5) &= E(T_{12} \mid T_5 = 2) \\
T_{12} - T_5 &\sim Poiss(3t) \\
E(T_{12} - T_5) &= \frac{7}{3} \\
E(T_{12}) = E(T_{12} - T_5) + E(T_5) &= \frac{13}{3}
\end{aligned}
$$

Part (c)

$$
\begin{aligned}
E(N(5) \mid N(2) = 5) &= E(N(5) - N(2)) + E(N(2)) \\
&= E(N(3)) + 5 \\
&= 14
\end{aligned}
$$

P6

The rate for cars can take the professor to town is: $\lambda = 6/3 = 2$.

The probability that no car which can take the professor to town arrive earlier than the bus' arrival is: $P(N_t = 0) = e^{-2t}$.

The probability that professor will take the bus is:

$$P(\text{professor take the bus}) = \int_{t=0}^{1} e^{-2t} dt = -\frac{1}{2}e^{-2} + \frac{1}{2}$$

P7

Part (a)

Let X_i denote the number of arrival passengers in i-th unit time.

Let N denotes the waiting time, which is a uniform distribution of [1,2].

$$EX = ENEX_1 = 1.5 * 24 = 36$$

Part (b)

$$var(X) = E[N]var(X_1) + var(N)E[X_1]^2 = 84$$