HW4

Sicong Zhao (sz163)

W1

Part (a)

r_1^1 refers to starting at state 1 and eventually staying at state 1. Because state 1 is absorbing, so starting at 1 will staying there. Therefore, $r_1^1 = 1$.

Because $r_1^{11} + r_1^1 = 1$, so we can infer $r_1^{11} = 0$

Part (b)

Intuitively, the state at the middle will have equal probability for reaching at both end. That is, i_0 should be 6.

Part (c)

If we start at $i < i_0$, $r_i^1 > r_i^{11}$. If we start at $i > i_0$, $r_i^1 < r_i^{11}$.

W2

Part (a)

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

It's stationary distribution is: [0.3, 0.2, 0.2, 0.3]

Part (b)

$d = [3, 2, 2, 3]$

$\pi = [0.3, 0.2, 0.2, 0.3]$

$d = 10\pi$, therefore, $d \propto \pi$

P1

Part (a)

Let S be the state space.

$p_{ij} > 0, p_{ji} > 0$ for any state $i, j \in S$ and $i \neq j$, this mean any two states belong to S

are mutually communicated. Therefore, P is irreducible.

Part (b)

If the number of states is no less than 3, we can have 3 different state i, j, k.

Since $p_{ij} > 0, p_{ji} > 0$, we can infer, $p_{ii}^2 \geq p_{ij} * p_{ji} > 0$. Therefore, $2 \in I_i$

Since $p_{ij} > 0, p_{jk} > 0, p_{ki} > 0$, we can infer, $p_{ii}^3 \geq p_{ij} * p_{jk} * p_{ki} > 0$. Therefore, $3 \in I_i$

Given $2, 3 \in I_i$, $g.c.d$ of I_i is 1.

Therefore, P is aperiodic if the number of states is 3 or more.

P2

Part (a)

The transition matrix is irreducible because any two states are mutually communicated.

Part (b)

$I_1 = [3, 6, 9, \ldots]$, so the period of state 1 = 3

$I_2 = [3, 6, 9, \ldots]$, so the period of state 2 is 3.

$I_3 = [3, 6, 9, \ldots]$, so the period of state 3 is 3.

$I_4 = [3, 6, 9, \ldots]$, so the period of state 4 is 3.

$I_5 = [3, 6, 9, \ldots]$, so the period of state 5 is 3.

Part (c)

Markov chian $\{X_n\}$ has finite states, according to Lemma 1.14, it has stationary distributions.

Markov chian $\{X_n\}$ has finite states and irreducible, according ro Theorem 1.16, it admits a unique stable distributions.

Part (d)

Let $M = P^3$, so $M_{ii} = P^3_{ii} > 0$ for $i \in \{1, 2, 3, 4, 5\}$.

$$P^3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.44 & 0.56 \\ 0 & 0 & 0 & 0.44 & 0.56 \end{bmatrix}$$

Let $M = P^3$, $M \cdot M = M$, so P^3k converges.

P does not converge. Because for any $n \geq 1$, when $k = 3n$:

$$P^k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.44 & 0.56 \\ 0 & 0 & 0 & 0.44 & 0.56 \end{bmatrix}$$

When $k = 3n + 1$:

$$P^k = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.44 & 0.56 \\ 0 & 0 & 0 & 0.44 & 0.56 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When $k = 3n + 2$:

$$P^k = \begin{bmatrix} 0 & 0 & 0 & 0.44 & 0.56 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 & 0 \end{bmatrix}$$

Therefore, P does not converge.

Part (e)

No, $E^i[f(X_n)]$ does not converge as $n \to \infty$. Because the chain is bouncing between 3 sets of states, so the expected reward would not converge.

Part (f)

Yes, since the transition matrix admits a unique stanble distributuion π, and it's irreducible, according to Ergodic theoremthe expected reward would converge to:

$$E^i[f(X_n)] \to \sum_{i=1}^{5} f(i)\pi_i$$

The stationary distribution can be calculated by following steps:

$$P^T - I = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0.2 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.8 & 0.5 & 0 & 0 \end{bmatrix} - I = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 0.2 & -1 & 0 & 0 & 0 \\ 0.8 & 0 & -1 & 0 & 0 \\ 0 & 0.2 & 0.5 & -1 & 0 \\ 0 & 0.8 & 0.5 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 1 & -5 & 0 & 0 & 0 \\ 4 & 0 & -5 & 0 & 0 \\ 0 & 2 & 5 & -10 & 0 \\ 0 & 8 & 5 & 0 & -10 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 5 & 0 & -1 & -1 \\ 0 & 0 & 5 & -4 & -4 \\ 0 & 8 & 5 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -12 & 8 \\ 0 & 0 & 0 & 14 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the stationary distribution is: $[\frac{25}{75}, \frac{5}{75}, \frac{20}{75}, \frac{11}{75}, \frac{14}{75}]$

So, we can derive:$E^i[f(X_n)] = \sum_{i=1}^{5} f(i)\pi_i = \frac{209}{75}$

P3

Part (a)

No, the period of $\{X_n\}$ is 2. Let states $1, 2, 3$ represent male and states $4, 5, 6, 7$ represent female.

Because current singer choose a friend with opposite gender randomly, so the chain is bouncing between states $1, 2, 3$ and states $4, 5, 6, 7$. Therefore, the period is 2.

Part (b)

The transition matrix is:

$$P = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \end{bmatrix}$$

The stationary distribution is: $[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}]$

We can prove $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j in $\{1, 2, 3, 4, 5, 6, 7\}$, therefore Markov chain satisfies the detailed balance condition.

Part (c)

According to the ergodic theorem, we can infer:

$$\frac{1}{n} \sum_{k=1}^{n} 1_x(X_k) \to \pi_x$$

This means the long run percentage of occupation time for each state is equal to the stationary distribution. So, the long run fraction of times for each singer holding the mic is the same as stationary distribution, $[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}]$.

Part (d)

In this chain, irreducible and sationary distribution hold. Therefore, $E^x T_x = \pi_x^{-1} = 8$.

Therefore, a female is expected to wait for 8 songs.

Similarly, the male is expected to wait for 6 songs.

P4

Part (a)

Let $f(X)$ denote the entrance fee for pub X, $X \in \{0, 1\}$

Since P has finite state, according to Lemma 1.14, it has stationary distributions.

Since P is irreducible has stationary distributions, therefore, according to ergodic theorem, the long run average entrance fee could be calculated using:

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) = f(0)\pi_0 + f(1)\pi(1)$$

Let's calculate π

$$P^T - I = \begin{bmatrix} -0.5 & 0.4 \\ 0.5 & -0.4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 0 & 0 \end{bmatrix}$$

Let $v_0 = 1$, we can infer $5v_0 - 4v_1 = 0$, $v_1 = \frac{5}{4}$. After normalizing, we can infer $\pi = [\frac{4}{9}, \frac{5}{9}]$.

Therefore, Bob is expected to pay $3.11 in the long run.

Part (b)

Let a, b be the probability that Bob goes to Pub 0 and Pub 1. We can infer:

$2a + 4b = 2.5$

$a + b = 1$

Such that, $a = \frac{3}{4}, b = \frac{1}{4}$.

So, the stationary distribution for the new transition matrix P' is $\pi' = [\frac{3}{4}, \frac{1}{4}]$.

Let $P' = \begin{bmatrix} c & d \\ 0.4 & 0.6 \end{bmatrix}$, because $\pi' P' = \pi'$, such that c, d must satisfy:

$$\frac{3}{4} = \frac{3}{4}c + \frac{1}{10}$$
$$\frac{1}{4} = \frac{3}{4}d + \frac{3}{20}$$

Therefore, we can derive: $c = \frac{13}{15}, d = \frac{2}{15}$

So, the parameter of the coin should be $[\frac{13}{15}, \frac{2}{15}]$, which means this coin will give a probability of $\frac{13}{15}$ that Bob will return to Pub 0 tomorrow if he is at Pub 0.

P5

1.10 (a)

Accoding to the transition matrix, we know that:

$1 \in I_1$

$\{2,3\} \subset I_2$

$\{2,3\} \subset I_3$

$1 \in I_4$

Therefore, the period for each state is 1.

Because I, A, S hold, P^k converges. The limit is

$$P^k = \begin{bmatrix} \frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21} \\ \frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21} \\ \frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21} \\ \frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21} \end{bmatrix}$$

1.10(b)

Accoding to the transition matrix, we know that:

$1 \in I_1$

$1 \in I_2$

$1 \in I_3$

$1 \in I_4$

Therefore, the period for each state is 1.

Because I, A, S hold, P^k converges. The limit is

$$P^k = \begin{bmatrix} \frac{4}{19} & \frac{6}{19} & \frac{6}{19} & \frac{3}{19} \\ \frac{4}{19} & \frac{6}{19} & \frac{6}{19} & \frac{3}{19} \\ \frac{4}{19} & \frac{6}{19} & \frac{6}{19} & \frac{3}{19} \\ \frac{4}{19} & \frac{6}{19} & \frac{6}{19} & \frac{3}{19} \end{bmatrix}$$

1.10(c)

Accoding to the transition matrix, we know that:

$1 \in I_1$

$1 \in I_2$

$1 \in I_3$

$1 \in I_4$

Therefore, the period for each state is 1.

Because I, A, S hold, P^k converges. The limit is

$$P^k = \begin{bmatrix} \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{3}{15} \\ \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{3}{15} \\ \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{3}{15} \\ \frac{4}{15} & \frac{4}{15} & \frac{4}{15} & \frac{3}{15} \end{bmatrix}$$

1.12(a)

Accoding to the transition matrix, we know that:

$I_1 = \{2, 4, 6 \dots\}$

$I_2 = \{2, 4, 6 \dots\}$

$I_3 = \{2, 4, 6 \dots\}$

$I_4 = \{2, 4, 6 \dots\}$

Therefore, the period for each state is 2. P is not aperiodic, and P^k will bounce between 2 different matrices. So P^k does not converge.