Math 541/Stat 621 Assignment

Rules

(i) The assignments are assigned on weekly basis (except during the week of midterms).

(ii) The assignments are usually assigned before Tuesday, and due next Tuesday (unless specified) in class, before the class starts. Late homework will not be accepted under *usual condition*.

(iii) Assignment with the lowest score will be dropped.

(iv) Answer without proper justification may receive less, or even zero credit.

(v) Collaboration is encouraged. However, students must write their own homework. Any identical homework will be treated as violation of *the Duke Community Standard*, and will not be given any credit.

Grading Scheme

(i) Each assignment has a total of 50 points, and will be graded with the following scheme:

Part I: Warm-up	10
Part II: Regular Problems	30
Overall completeness, clarity	10
Total	50

(ii) All problems from Part I, and problems marked with "$\star$", along with additionally selected problems (which in total of 2–4 problems) from Part II will be graded based on correctness.

(iii) Unstapled homework: -10 points from completeness/clarity.

Assignment 1

Due: Sept 3, 2019 (Tuesday)

Warm-up

W1. Complete the following statement about *Bayes' formula*: let $A, B_1, B_2, \cdots, B_N$ be a collection of events from a sample space Ω, where B_k's form a partition of Ω (meaning that B_k's are mutually disjoint, and $B_1 \cup \cdots \cup B_N = \Omega$). Then

$$\mathbb{P}(A) = ...$$

W2. (a) If $P = (p_{ij})_{1 \le i,j \le N}$ is a $(N \times N)$-transition matrix of a Markov chain (with N states), what are the conditions that the entries p_{ij} must satisfy?

(b) Let P be the transition matrix of a Markov chain $\{X_n\}_{n \ge 0}$ and $k \ge 1$. What does each entry of the matrix P^k represent? ($P^k = P \cdots P$ denotes the k-th power of the matrix P.)

Problems

$\star$P1. (Random knight tour) Consider the following 3×3 chessboard, with 8 boundary cells labeled as 1–8 (8 = ☠). A knight is randomly placed on a boundary cell. Its (random) movement is described as follows:

1. At any turn, the knight will make a legal move (the L-movement) at random. Each possible move has equal probability to be taken. (E.g., if the current position is 2, then the knight moves to 5, 7 with probability $\frac{1}{2}$ each).
2. If the knight reaches 8 (labeled as ☠), it stays forever afterward.

Note that the movement of the knight is a Markov chain.

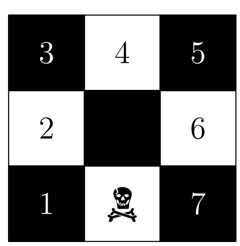


Figure 1: 3×3-chessboard

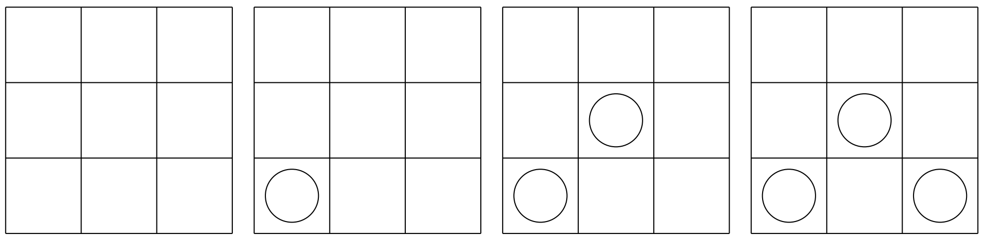
(a) Sketch the network representation of the Markov chain. (Rearrange the nodes appropriately so that the edges do not cross.)

(b) Write down the correspondent transition matrix.

⋆P2. (Lonely tic-tac-toe) Consider the following "tic-tac-toe" with single player.

- The game starts at Turn 0, with a 3-by-3 blank grid (see Figure 2(a) below).
- For every turn, the player randomly fills in an unfilled cell with a circle. Each unfilled cell has equal probability to be filled.
- The player "wins" the game when a line (horizontal, vertical, or diagonal) is formed. The game continues until all cells are filled.

The following is an example of the evolution of the game.



(a) Turn 0 (b) Turn 1 (c) Turn 2 (d) Turn 3

Figure 2: A possible evolution of game

(a) How many game states are there? (A *game state* is the picture of the grid (as Figure 2) at any fixed turn.)

(b) How many possible histories of game are there? (A *history of game* is a possible sequence of game states, until the end of the game.)

(c) What is the minimal number of steps to guarantee a win?

(d) Let u_k denote the probability of winning the game *within* k turns. Find u_k for $k = 2, 3, 4$ and 6.

(e) For $0 \leq k \leq 9$, denote X_k the random game state of the game. For any $k \geq 10$, let $X_k = X_9$ (hence, X_k freezes after 9 turns). Is the sequence $\{X_k\}_{k\geq 0}$ a Markov chain?

(f) Let (a)-(d) be the game states from Figure 2. Find the (conditional) probabilities of the following events:

i. $X_3 =$(d), conditioned on $X_2 =$(c);

ii. $X_3 =$(d), conditioned on $X_2 =$(c), X_1=(b);

iii. $X_1 = $ (b), $X_2 = $ (c), $X_3 = $ (d);

iv. $X_3 =$(d);

v. winning the game in the next step, conditioned on $X_3 =$(d).

Assignment 2

Due: Sept 10, 2019 (Tuesday)

Warm-up

W1. What is a stationary distribution of a transition matrix P?

W2. Show that if u, v are two stationary distributions of a transition matrix P, then so is the interpolation $\alpha u + (1-\alpha)v$ for any $\alpha \in [0,1]$. Therefore, if a transition matrix admits two different stationary distributions, then it admits infinitely many.

W3. Show that the following transition matrix has two distinct stationary distribution:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

W4. What is a doubly stochastic matrix? What can you say about its stationary measure?

Problems

P1. (Who kill the pub-goer?) Bob from the WPTG example was found dead on Day 4002. He gave each pub owner a fair coin, and the past history of visited pubs was fully recorded (here: MC-Murderer.xls). His wife, Mary, overheard his last phone conversation, and it goes as follows:

> *"...... Yes... What! What did you just say? You change the coin? No you didn't do that! @#$@*... I can't believe this! How dare you to change the coin! You pathetic liar #!@*%@$!... I gonna #!@* you now..."*

After hanging the call, Bob left his house, and went missing until his body was found next day. Use the data given above to find the murder.

P2. Let P be a transition probability of dimension $N \times N$.

(a) Let $\mathbb{1} = (1, 1, \cdots, 1)^T$ denote the N-dimensional *column* vector consisting of all entries with value 1. Show that $\mathbb{1}$ is a (right) eigenvector of P. What is the correspondent eigenvalue?

(b) Let $v = (v_1, \cdots, v_N)$ be a probability (row) vector. Show that $w = vP$ is also a probability vector. (*Hint:* let $\mathbb{1}$ be from (a). Then $\sum_{i=1}^N v_i = v\mathbb{1}$).

P3. Recall the brand preference example from the class (see Lecture slide 541-MC1.pdf). Let P be the 3×3-transition matrix

(a) Use the Markov simulator (MC-sim.xls)) to find the long run occupation percentage for each state.

(b) With the help of calculator, find the matrix P^3, P^{30}, P^{300}. Comparing these matrices to the result from (a), what do you observe?

P4. Exercise 1.1, pg 62 from textbook.

P5. Exercise 1.3, pg 63 from textbook.

P6. Exercise 1.6, pg 63 from textbook.

P7. Exercise 1.7, pg 63 from textbook.

P8. Exercise 1.8, pg 63 from textbook.

Assignment 3

Due: Sept 17, 2019 (Tuesday)

Warm-up

W1. Consider a N-states Markov chain with transition matrix $P = (p_{ij})_{1 \le i,j \le N}$, and let π (a probability row vector) be its stationary distribution. What does it mean to say π *satisfies the detailed balance condition*?

W2. Suppose P is a *symmetric* $(N \times N)$-transition matrix (symmetry of a matrix means $P = P^T$).

(a) Show that P is a doubly stochastic matrix.

(b) Show that P admits a stationary distribution π that satisfies the detailed balanced condition. What is π?

W3. Let x be a state of a Markov chain with a transition matrix P. What is the *period* of this state?

W4. Let P be a $(N \times N)$-matrix with a (right) eigenvector $v \in \mathbb{C}^N$ with eigenvalue $\lambda \in \mathbb{C}$. Show that

(a) if $|\lambda| < 1$, then $P^n v \to 0$ as $n \to \infty$.

(b) if $|\lambda| > 1$, then $P^n v \to \infty$ as $n \to \infty$ (in the sense that the magnitude blows up).

Problems

P1. Let $\{X_n\}$ be a Markov chain with states $\{1, \cdots, N\}$, and P be the relevant transition matrix. Consider the reward function $f : \{1, \cdots, N\} \to \mathbb{R}$ on these states. Then $f(X_n)$ is the reward of the chain at step n.

(a) Let $a = (f(1), \cdots, f(N))^T$ be the (column) reward vector. Suppose the chain starts with the initial distribution v (which is a row vector). Show that the expected reward at n-step, conditioned on the initial distribution v, is given by

$$\mathbb{E}^v[f(X_n)] = vP^n a.$$

(b) Denote $A_n := \frac{1}{n}\sum_{k=1}^{n} f(X_k)$ the *average reward* up to step n. Suppose the n-step distribution converges to a stationary distribution π. Show that

$$\lim_{n\to\infty} \mathbb{E}^v[A_n] = \pi a.$$

Hint: you may use this fact without proving it: Cesàro Mean.

(c) Let $C \in \mathbb{R}$ be some number, and T be the first time when the total reward up to step n is more than C, i.e.,

$$T = T_C := \min \left\{ n \geq 1 : \sum_{k=1}^{n} f(X_n) \geq C \right\}.$$

Is T a stopping time?

(d) Consider the case when $N = 2$, and

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix}, \qquad f(i) = \begin{cases} 1 & i = 1, \\ -1 & i = 2. \end{cases}$$

Let $C = 100$, and define $T = T_C$ as before. Find $\mathbb{P}(X_{T+1} = k | T < \infty)$ for $k = 1, 2$, and $\mathbb{E}(f(X_{T+1}) | T < \infty)$.

$\star$ P2. (Time bomb) Consider a drinking game with N players, labeled as $\{1, \cdots, N\}$. The game involves a "time bomb", a coin with two sides, ☠ or ☺, with a probability $q \in (0, 1)$ showing ☺, and a random number generator (generating numbers 1–N at random). These random numbers are generated by a simulator of a Markov chain having a $(N \times N)$ transition matrix $P = (p_{ij})_{1 \leq i,j \leq N}$. The rule of this game goes as follows:

- At turn $n = 0$, Player i (for some $i \in \{1, 2, \cdots, N\}$) holds the "time bomb". He first flips the coin. If it shows a ☺, then he is safe. He next presses the random number generator, which returns a number $j \in \{1, \cdots N\}$. Player i then passes the "time bomb" to Player j, and the game proceeds to turn $n = 1$. If the coin shows up ☠, then the "bomb" will be "detonated", in which case he is the "loser" of this round.
- At turn $n \geq 1$, the "bomb holder" repeat the procedure above. The game proceeds until the "bomb" is "detonated."

(a) Denote r_{ij} the probability of the event where Player j is the loser, conditioned on that Player i holds the "time bomb" initially. Show that

$$r_{ij} = (1 - q) \sum_{n=0}^{\infty} q^n p_{ij}^n,$$

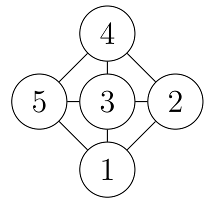
where p_{ij}^n denotes the n-step transition probability of $i \to j$.

(b) Let $R = (r_{ij})_{1 \leq i,j \leq N}$ be the matrix with entries r_{ij}. Show that

$$R = (1 - q)(I - qP)^{-1}.$$

Is the expression on the right hand side above still well-defined if $q = 1$?

(c) Let $N = 5$, $q = 0.7$, and the Markov chain generator is given by the following network:



At each state, the chain randomly choose any state connected in the graph (excluding the current state), each with equal probability. Use calculator to find the probability of the event where Player j is the loser for any $1 \leq j \leq 5$, conditioned on Player 1 being the initial bomb holder.

(d) Find the stationary distribution(s) of the MC above. (You may use calculator).

(e) From (c), by repeating the computation with $q \approx 1$, show numerically that each row of the matrix R_q from (b) converges to a stationary distribution of the MC.

P3. Exercise 1.13, pg 64 from textbook.

P4. Exercise 1.16, pg 65 from textbook. (You may use calculator.)

P5. Exercise 1.27, pg 66 from textbook.

P6. Exercise 1.37, pg 68 from textbook.

P7. Exercise 1.41, pg 69 from textbook.

P8. Exercise 1.44, pg 70 from textbook.

Remark: From P5 onwards, when the textbook problems ask "limiting fraction," you may use the convergence result (Theorem 1.19 from 541-MC2). Mostly that means "find the stationary distribution." Of course, you have to at least verify the Markov chain is irreducible. You may ignore the condition "aperiodicity" for now.

Assignment 4

Due: September 24, 2019 (Tuesday)

1 Warm-Up

W1. Imagine an absorbing random walk on $\{1, 2, \cdots, 11\}$. On every state i, the chain has $\frac{1}{2}$ probability to move to left $i-1$, and $\frac{1}{2}$ to right $i+1$, except that the boundaries $1, 11$ are absorbing (once the chain reaches there, it stays forever). For any state $i \in \{1, \cdots, 11\}$, denote r_i^1 the probability of the chain started at i eventually staying at 1 (or we say the chain exits through 1), and r_i^{11} the probability of staying at 11. Note that $r_i^1 + r_i^{11} = 1$.

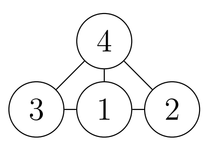
(a) What are the values of r_1^1 and r_1^{11}?

(b) Find the unique number i_0 so that the chain has equal probability to exit through 1 and 11. That is, find the unique number i_0 such that $r_{i_0}^1 = r_{i_0}^{11} = \frac{1}{2}$.

(c) If we start the chain at a state $i < i_0$, which probability of r_i^1, r_i^{11} is larger? What about if we start at $i > i_0$?

(You don't need any rigorous justification for this problem.)

W2. Consider a Markov chain with state space defined in the following graph:



At each node, the chain choose a state (directly) linked with the current state at random (each with equal probability) and move to there. For instance, if the current state is 3, then it moves to 1 or 4 each with probability $\frac{1}{2}$.

(a) Write down the transition matrix of the chain, and find its stationary distribution (you may use calculator).

(b) The degree d_i of a state/node i is the number of edges connected to it. Show that the stationary distribution $\pi = (\pi_i)_i$ is proportional to the degree vector $(d_i)_i$.

W3. (No response is required.) Complete the mid-semester survey (link here: `https://forms.gle/y8Juq2jELmDF16Bt7`) by September 24, 2019. Your score of this problem depends on the "collective response" of the survey. Mathematically, let $N \geq 1$ be the number of students in Math 541/Stat 621, and for $1 \leq j \leq N$, let $\xi_j = \mathbb{1}\{j\text{-th student submitted the survey}\}$. Then your score is

$$2 - (2 - 2i)\mathbb{1}\left\{\frac{1}{N}\sum_{j=1}^{N} \xi_j \leq 0.9\right\}, \qquad i = \sqrt{-1}.$$

Problems

P1. A Markov chain with finite state has transition matrix with strictly positive entries, except that the diagonal entries are zero. Show that

(a) P is irreducible;

(b) P is aperiodic if the number of states is 3 or more.

$\star$ P2. Consider a Markov chain $\{X_k\}_{k\geq 0}$ with 5 states labeled as $1, 2, \cdots, 5$, which has the transition matrix as below (any unfilled entry is zero):

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0.2 & 0.8 & & \\ & & & 0.2 & 0.8 \\ & & & 0.5 & 0.5 \\ 1 & & & & \\ 1 & & & & \end{bmatrix} \end{array}$$

(a) Decide if the transition matrix is irreducible.

(b) Find the period of each state.

(c) Does the transition matrix admit a unique stationary distribution?

(d) Show that P^{3k} converges. Does the matrix P^k converge?

(e) Consider the reward function $f(m) = m$. Namely, the state i as reward i. Given a state $i \in \{1, 2, \cdots, 5\}$, does the expected reward $\mathbb{E}^i[f(X_n)]$ conditioned on initial state i converge as $n \to \infty$? If so, what is the limit?

(f) Does the average reward up to time n

$$\frac{1}{n} \sum_{k=1}^{n} f(X_k)$$

converge almost surely as $n \to \infty$? If so, what is the limit?

P3. A group of 7 friends, with 3 males, 4 females, go to karaoke. After finishing a song, to decide who is the next "mic-holder," the current singer choose a friend with opposite sex at randomly, each with the same probability. Let X_n denote the "mic-holder" of the n-th song (the count of song starts at zero).

(a) Is the Markov chain $\{X_n\}$ aperiodic?

(b) Show that the Markov chain satisfies the detailed balance condition.

(c) What is the long run fraction of times for each singer holding the mic?

(d) For a female singer, on average how many songs she have to wait so that she can sing again? That is, what is the expected number of songs in between the two (consecutive) rounds (including the first song she sang) that she is a mic holder? What about for a male singer?

$\star$ P4. Consider the WPTG example with transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}.$$

Each pub has an entrance fee, with Pub 0 \$2, and Pub 1 \$4.

(a) On average, how much does Bob pay per day for entrance fee in long run?

(b) Suppose Bob want to control its long run average entrance fee (per day) at the price of \$2.5. He has the (only) option to give a new coin to Pub 0 owner. To achieve his goal, what is the parameter of the coin he should give to the owner?

P5. For each transition matrix from Problems 1.10(a), (b), (c), and 1.12(a) (from textbook), find the period of each state. For each of these cases, decide if the matrix P^k converges as $k \to \infty$. If so, write down its limit.

Assignment 5

Due: Oct 1, 2019 (Tuesday)

Warm-up

Warm-up exercises will be deferred to the next week.

Problems

$\star$P1. Consider the board game *Snakes and Ladders* with 100 cells. The player currently is at Cell 95. The rule of the game goes as follows.

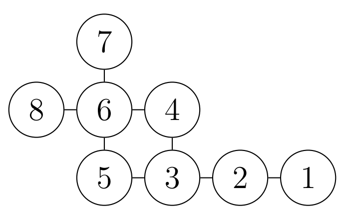
- At every turn, the player rolls a dice, and moves forward with the step of number. Cell 100 is "refelcting." For instance, if the player currently is at Cell 98, and he rolls a 5, then he goes to 97.
- When the player reaches Cell 100, he wins.
- There is a snake at Cell 98. When the player reaches this cell, it immediately sends the player to Cell 1. In this case, the player lose the game.
- There is a ladder at Cell 96. If the player reaches this cell, it immediately sends the player to Cell 99.

(a) Find the probability that the player wins the game.

(b) Find the expected time to end the game.

(Through it is tricky, you are encouraged to do this problem without the help of calculator.)

$\star$P2. Consider the frog on the lilipads arranged folows.



At each step, the frog jumps to a lilipad that is linked with the current one at random (each has the same probability).

(a) Show that the associated Markov chain satisfies the detailed balance condition. Find its stationary distribution.

(b) It the associated transition matrix a doubly stochastic matrix?

(c) From now on, assume that 1, 7, 8 are absorbing, that is, once the frog reaches there, it stays there. Find the exit distribution of each state $1, 7, 8$. Namely for each transient state $i \in \{2, 3, 4, 5, 6\}$ and absorbing state $j \in \{1, 7, 8\}$, find $q_{ij} = \mathbb{P}^i(X_\infty = j)$.

(d) If the frog reaches Pad 1, 7, 8, then it is rewarded with 5, 9, 3 bugs respectively. The frog is initially placed on Lilipads 2–6, each with probability $\frac{1}{5}$. Find the expected reward the frog gets.

(e) Find the expected time of the frog visiting any of Pad 1, 7, or 8.

P3. Consider a random number generator which generates a sequence of numbers between 1 to 4. The generator is in fact a Markov chain, with transition matrix

$$P = \begin{bmatrix} 0.25 & 0.75 & & \\ 0.6 & & 0.1 & 0.3 \\ 0.3 & 0.3 & 0.2 & 0.2 \\ & & 0.5 & 0.5 \end{bmatrix}.$$

For instance, if the current number is 2, then with probability 0.6, 0.1, 0.3, the next number will be 1, 3, 4. Suppose the first number generated by the generator is 1.

(a) Find the expected time to obtain the first 3 or 4 in the random sequence.

(b) What is the probability of obtaining the number 3 before the number 4 in the sequence.

(c) Find the expected interval between two consecutive 1.

(You are encouraged to do this problem without the help of calculator.)

P4. Exercise 48, pg 71 from textbook. (*Remark: Do not over-complicate Part (a). For (b), you may actually compute by hand, see Supplementary Notes, Example 1.39. You may also use calculator.*)

P5. Exercise 49, pg 71 from textbook.

P6. Exercise 58, pg 72 from textbook.

P7. Exercise 67, pg 74 from textbook.

Assignment 5, Cont'd

Due: October 10, 2019

Warm-Up

W1. What is a Poisson random variable? Specifically, write down its probability mass function.

W2. What is an exponential random variable? Specifically, write down its probability density function.

W3. What is the memoryless property for exponential random variables?

Assignment 6

Due: Oct 22, 2019 (Tuesday)

Warm-Up

W1. Write down *Wald's identity for random sum.* Be sure to write down all necessary conditions.

W2. What is *the superposition* $\{N_t\}$ *of two Poisson processes* $\{N_t^1\}, \{N_t^2\}$. What does the main result of the superposition of Poisson processes say about the process $\{N_t\}$?

Problems

P1. Let $T_k \sim \exp(\lambda_k)$ for $1 \leq k \leq n$ be independent exponential random variables.

(a) Find the probability of

$$\mathbb{P}(T_1 \leq T_2 \leq T_3).$$

Your answer should depends on $\lambda_1, \lambda_2, \lambda_3$.

(b) Show that

$$\mathbb{P}(T_1 \leq T_2 \leq \cdots \leq T_n) = \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} \frac{\lambda_2}{\lambda_2 + \cdots + \lambda_n} \cdots \frac{\lambda_{n-1}}{\lambda_{n-1} + \lambda_n} \frac{\lambda_n}{\lambda_n}.$$

P2. Let $T \sim \exp(1)$, and $\lambda : [0, \infty) \to \mathbb{R}^+$. be a continuous function, and $\Lambda(t) = \int_0^t \lambda(r) dr$ be the antiderivative. Assume $\Lambda(t) \to \infty$ as $t \to \infty$. Define $S = \min\{t \geq 0 : T \leq \Lambda(t)\}$, that is the first time $t \geq 0$ when T is smaller than the integral of $\lambda(s)$ on $[0, t]$.

(a) Determine the survival function $\rho(t) := \mathbb{P}(S > t)$ and density f of S.

(b) Show that if $\lambda(s) = \lambda_0 \in \mathbb{R}^+$ is a constant function, then $S \sim \exp(\lambda_0)$.

(c) Show that S satisfies the following variation of "memoryless property:"

$$\mathbb{P}(S > t + s | S > s) = e^{-\int_s^{t+s} \lambda(r) dr}.$$

P3. Exercise 2.7, pg 93 from textbook.

P4. Exercise 2.13, pg 93 from textbook.

P5. Exercise 2.22, pg 94 from textbook.

P6. Exercise 2.27, pg 95 from textbook.

P7. Exercise 2.28, pg 95 from textbook.

Assignment 7

Due: Oct 29, 2019 (Tuesday)

Warm-Up

W1. What is a M/G/1 queue? Specifically, what does the "M", "G", and "1" mean?

W2. What is *a transition rate matrix* $Q = (q_{ij})_{1 \le i,j \le N}$ with finite dimensions? What conditions do the entries q_{ij} satisfy? (*Hint: Wikipedia.*)

W3. Let A be a square matrix with finite dimensions, and $t \ge 0$. What is the correspondent matrix exponential e^{tA}? Specifically, write down the *power series* definition of e^{tA}.

Problems

P1. A fire department A is in charge of any fire outbreak that takes place within 5 kilometers. (Namely, the fire department is located at the center of a disk with radius 5.) The fire occurs at time of Poisson process with rate $\lambda = 50$ per year, uniform across the region. (The latter means whenever a fire happens, the location is uniform on the disk.)

(a) For each fire outbreak, there is a probability of $p(r) = 0.1r$ causing major loss, where r is the distance from the fire to the department. Let X be the number of fire outbreaks with major loss within a year. Find its expected value and variance. (Hint: the formula for the volume of cone $V = \frac{1}{3}\pi r^2 h$ may be useful.)

(b) The number of casualties of each fire is a binomial distribution $\text{bin}(\tilde{p}(r), 10)$, with $\tilde{p}(r) = 0.1r$, where r is the distance from the fire to the department. Let Y be the number of total casualties caused by fires within a year. Find its expected value.

(c) Because of the closure of a fire department nearby, the department A is now in charge of any fire within 6 kilometers. On the annulus $\{5 < r \le 6\}$, the fire occurs at time of Poisson with rate $\lambda = 20$, uniform across the region. Let $\{\tilde{N}_t\}$ be the total number of fires that happen on the northeast side within the region of in-charge (i.e., the first quadrant), which is a Poisson process. Find the intensity of $\{\tilde{N}_t\}$.

(d) The operational cost for every fire truck sent depends on the location of fire. If the fire happens within 3 km, the cost is a uniform distribution on $[0, 1]$ (1 unit = \$1000); if it is within 3-5km, then it is a uniform distribution on $[1, 3]$; if within 5-6km, then it is uniform on $[3, 4]$. Find the expected total cost for a year.

$\star$P2. (a) Let $\{N_t\}_{t \ge 0}$ be a Poisson process with intensity $\lambda > 0$. Show that the for any $t_0 > 0$, the "delayed" process defined by $\tilde{N}_t := N_{t-t_0}$ if $t \ge t_0$, and $\tilde{N}_t = 0$ otherwise, is a Poisson process. Find its intensity (may depend on time).

(b) Customers arrive to a restaurant at time of Poisson process with rate $\lambda = 100$ per hour. For each customer, upon his arrival, the time it takes to serve his order is a uniform distribution on $5, 10, 15, 20$ minutes. Let $\hat{N}_t$ be the process that counts the events of the order served. Show that $\hat{N}_t$ is also a Poisson process. Find its intensity. Specifically, sketch its graph.

P3. Exercise 2.37, pg 96 from textbook.

P4. Exercise 2.40, pg 96 from textbook.

P5. Exercise 2.45, pg 97 from textbook.

P6. Exercise 2.46, pg 97 from textbook.

P7. Exercise 2.47, pg 97 from textbook.

Assignment 8

Due: Nov 5, 2019 (Tuesday)

Warm-Up

W1. Let Q, R, P be square matrices with same dimensions, and P invertible. We say Q is *similar to R via P* if it holds $Q = P^{-1}RP$.

(a) Show that if Q is similar to R via P, then Q^n is similar to R^n via P for any integers $n \geq 0$.

(b) Show that if Q is similar to R via P, then e^{tQ} is similar to e^{tR} via P for any $t \geq 0$.

W2. Let $\{P_t\}_{t\geq 0}$ be a transition probability matrix of some continuous time Markov chain, and Q be its transition rate matrix. What is the relation between $\{P_t\}$ and Q? Namely, what differential equation does the matrix $\{P_t\}$ satisfy (for all $t \geq 0$)?

W3. Let $\{P_t\}, Q$ as in W2. Write down the two equivalent definitions of *stationary distributions* π in terms of $\{P_t\}$ and Q.

Problems

P1. Suppose customers arrive to a restaurant at time of Poisson process with uniform intensity $\lambda > 0$.

(a) Over the first hour of business time (8 – 9am), there are 3 customers visited the restaurant. What is the probability of the event that the second customer arrives before 8:30am?

(b) Suppose over the 8 hours of business time (8am–4pm), there are 100 customers visited the restaurant (ignore the previous setting). Assume the spend of each customer is independent. During the lunch hour (11am–1pm), the spends are identically distributed, with average and variance are \$20, 50; during other time (8–11am, 1–4pm), the spends are also identically distributed, with average and variance are \$10, 20. Find the expected value and variance of the total spend of customers over the 8 hours. (*Hint: When applying* $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$*, make sure to check independence (or zero correlation) of X and Y.*)

P2. Signals are transmitted between a generator and receiver (with no time delay). The generator sends signals at a Poisson time with periodic intensity of period 2 hours. Over a period, the intensity is given by

$$\lambda(s) = \begin{cases} 120 & s \in [0,1], \\ 60 & s \in [1,2]. \end{cases}$$

At any given time, the receiver successfully translates the signal with probability $p(s)$, which is again a periodic function with period 2. Over a period, the probability is given by

$$p(s) = \begin{cases} 2s & s \in [0, 0.5], \\ 1 & s \in [0.5, 2]. \end{cases}$$

Let $\{N_t\}$, $\{\tilde{N}_t\}$ be the counting processes which counts the transmitted and successfully translated signals up to time t.

(a) The size of each *successfully translated* signal is uniformly distributed from 1 to 2 MB (as a continuous random variable). Find the expected value and variance of the total sizes of translated signals for a day.

(b) Find the asymptotic rate of of translated signals $\lim_{t\to\infty} t^{-1}\tilde{N}_t$. (Hint: use independent increments.)

(c) Find the asymptotic portion of successfully translated signals:

$$\rho := \lim_{t\to\infty} \frac{\tilde{N}_t}{N_t}.$$

(d) Suppose over the first 2 hours, there is only one signal successfully translated by the receiver. What is the probability that it was received in the first hour?

(e) We may also operate the receiver with a *phase shift*. Namely, we delay the operation of the receiver at a time of $\tau \in [0, 2]$, so that the probability of each signal being successfully translated is $p(s - \tau)$. Let ρ_τ be the asymptotic portion of successfully translated signals defined similarly as before. Find the largest possible value of ρ_τ.

$\star$P3. (*A third world problem.*) Three kids labeled as 1, 2, 3 share a toy. Three possible sharing plans are considered.

- Plan A. The kids take turns to hold the toy, with pattern $1 \to 2 \to 3 \to 1 \to 2 \cdots$. Before a cycle $1 \to 2 \to 3$ starts, they toss two fair dice with 6 sides, and let m, n be the largest and smallest of the outcomes. Kid 1, 2, 3 then hold the toy over a period of $m, n, \frac{m+n}{2}$ unit respectively.
- Plan B. At every cycle, only a kid holds the toy. Before the cycle starts, the kids flip a fair dice with 3 sides. If it shows up k, then Kid k holds the toy over this cycle. The time of possession is determined similarly as before.
- Plan C. At every cycle, two kids take turns to hold the toy. Before the cycle, the kids flip a fair dice with 3 sides. If it shows up k, then Kid k do not possess the toy at this cycle. The other kid will take turn to keep the toy over a period of time. The time of possession for each of them is determined similarly as before.

(a) Find the asymptotic fraction of possession time for each kid for Plan A.

(b) Find the asymptotic fraction of possession time for each kid for Plan B.

(c) Find the asymptotic fraction of possession time for each kid for Plan C.

(d) For Plan B, we may replace the fair dice with an unfair one. Find the parameter of the unfair dice so that the sharing plan is fair, namely, every kid has the same asymptotic fraction of possession time.

P4. Exercise 3.5, pg 114 from textbook. (Hint: use the memoryless property.)

P5. Exercise 3.9, pg 114 from textbook.

P6. Exercise 3.13, pg 114 from textbook.

Assignment 9

Due: Nov 12, 2019 (Tuesday)

Warm-Up

W1. (a) What is the analogy of "doubly stochastic matrix" for continuous time Markov chains?

(b) Show that if the transition rate matrix Q of some continuous time Markov chain with finite states is "doubly stochastic," then $\frac{1}{N}(1, 1, \cdots, 1)$ is a stationary distribution.

W2. Write down the definition of *detailed balance* for a Markov chain with rate matrix Q.

W3. Consider a "forked" Poisson process described as follows. The chain starts at $i = 0$. It jumps from $0 \to 1 \to \cdots \to k-1$ as a Poisson process with uniform intensity λ. When the chain is at $k-1$, it jumps to the (absorbing) states k, k', k'', each with rate $\alpha, \beta, \gamma > 0$ respectively. (Therefore, the chain has $k+3$ states.)

(a) Sketch the network representation of the chain.

(b) Find the mean exit time T, where T is the first time when the chain hits any of the states k, k', k''.

(c) Find the exit probability distribution, namely, the probability $\mathbb{P}(X_T = k)$, $\mathbb{P}(X_T = k')$ and $\mathbb{P}(X_T = k'')$. (*Hint:* argue through the exponential race.)

Problems

P1. A broken traffic light switches between red, yellow, green (labeled as $1, 2, 3$) at random. The transition of these three colors is a continuous time Markov chain with transition rate matrix

$$Q = \begin{bmatrix} -3 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

(a) Sketch the network representation of the chain.

(b) Compute the transition probability matrix P_t for a given $t \geq 0$. You may leave your answer of the form UR_tU^{-1} for some invertible matrix U.

(c) Use the expression above to find the limit $\lim_{t\to\infty} P_t$.

(d) Suppose the chain has initial distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Find the distribution $v(t)$ of the chain at any given time $t \geq 0$.

(e) Find stationary distribution(s) of the Markov chain. Does it agree with the limit from (c)?

(f) When the light is green, per unit time there are 20 cars passing by; when it is yellow, there are 5 cars; when it is red, no car is passing. Find the asymptotic amount ot cars passing by the light per unit time.

P2. Let $\{Y_n\}_{n\geq 0}$ be a discrete Markov chain with transition probability P (the diagonal entires may not be zero), and $\{N_t\}_{t\geq 0}$ be a (independent) homogeneous Poisson process with intensity $\lambda > 0$. Consider $X_t = Y_{N_t}$, which is a continuous time Markov chain.

(a) Find the correspondent transition rate matrix Q (in terms of the entries p_ij of P and $\lambda > 0$).

(b) Show that π is a stationary distribution of $\{Y_n\}$ if and only if it is that of $\{X_t\}$.

P3. Exercise 3.16, pg 115 from textbook.

P4. Exercise 4.1, pg 150 from textbook.

P5. Exercise 4.2, pg 150 from textbook.

P6. Exercise 4.3, pg 150 from textbook.

Assignment 10

Due: Nov 19, 2019 (Tuesday)

No warm-up for this week. The 10 points will be relocated to regular problems.

Problems

P1. Seven friends, three males (Andy, Benny, Conan) and four females (Denise, Emily, Felicia, Grace) go for karaoke. They take turns to sing songs. The duration of the song sung by a male singer is an exponential random variable with rate $\frac{1}{3}$ per minute; the duration for a female singer is an exponential with rate $\frac{1}{4}$ per minute. After a song, the singer randomly passes the mic to an opposite gender, each with the same probability.

(a) Write down the transition rate matrix of the correspondent Markov chain. Decide if the chain satisfies the detailed balance condition.

(b) Find the long run fraction of "mic-holding" time for each singer.

(c) Suppose Grace and Andy just experienced an emotional break-up. If any of them has the mic, he/she will hold it forever. Assume the chain starts with Conan. Compute the expected value of T, where T is the first time when Andy or Grace having the mic.

(d) Who (Andy or Grace) has a higher chance to become the mic-holder eventually?

P2. The performance of the stock market can be categorized as three states: bull, bear, and badger (labeled as 1, 2, 3). The transition among these states is a continuous time Markov chain with rate matrix given as follows:

$$Q = \begin{bmatrix} -2 & 1 & 1 \\ 3 & -3 & 0 \\ 1 & 3 & -4 \end{bmatrix}.$$

The time unit being used here is a year. Consider two investment portfolios:

- Portfolio 1: during the bull, bear, badger period, it continuously earns \$10, -\$6, \$0 per unit time.
- Portfolio 2: during the bull, bear, badger period, it continuously earns \$8, -\$3, \$3 per unit time.
- The price of each portfolio is "cumulative." That is, the investor pays a constant price $\$r$ continuously per unit time.

(a) Ignoring the price, which investment plan is better off?

(b) The national bond earns at a constant rate \$2 per year. What price should we set for each portfolio so that after paying the price it is still a better investment option comparing to the bond.

P3. Consider the random knight tour on the chessboard of size 8×8. The knight stays at each cell for exponential time. The average time on white cells is triple of that of black cells. Once the time is up, the knight makes a legal move randomly, each with the same probability. Let $\{X_t\}$ be the location of the knight.

(a) Find the long run distribution of each cell. Also, find the long run fraction of time of which the knight staying at black cells.

(b) ~~Given $t \geq 0$, let Y_t be the color (black or white) of the cell X_t. Show that $\{Y_t\}$ is a CTMC. Find its rate matrix, and its stationary distribution. Compare the answer with (a).~~ *Part (b) is crossed out.*

(c) Repeat Problem (a) if the "knight" is replaced" by the "queen."

P4. Consider a clock with marks $1, 2, \cdots, 6$. The movement of minute and hour hand are independent. The minute hand moves one step clockwise with rate α, one step counterclockwise with rate β. The hour hand moves one step clockwise with rate γ, and counterclockwise with rate δ.

(a) Find the asymptotic portion of time of which the two hands overlapped. *(Hint: consider the difference of marks, mod 6.)*

(b) Set $(\alpha, \beta, \gamma, \delta) = (1, 2, 3, 4)$. Suppose at time $t = 0$ the minute and hour hand are at Mark $3, 6$. Let T be the first time when the two hands are overlapped. Find $\mathbb{E}[T]$.

P5. Exercise 4.12, pg 152 from textbook.

P6. Exercise 4.22, pg 153 from textbook.

P7. Exercise 4.28, pg 154 from textbook.

P8. Exercise 4.29, pg 154 from textbook. (You may skip Part (d).)

P9. Exercise 4.31, pg 154 from textbook.