HW 7

Sicong Zhao (sz163)

W1

M/G/1 queue is a queue model where:

- M stands for Markovian input modulated by a rate λ Poisson process
- G stands for general service time. We assume ith customer requires an amount of service s_i, which are independent and have a distribution G with mean $1/\mu$.
- 1 stands for there is one server.

W2

In probability theory, a transition rate matrix is an array of numbers describing the rate a continuous time Markov chain moves between states.

In a transition rate matrix Q, element q_{ij} for $i \neq j$, denotes the rate departing from state i and arriving in state j. Diagonal elements q_{ii} are defined such that:

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

and therefore the rows of the matrix sum to 0.

W3

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k$$

Where $(tA)^0$ is defined to the the identity matrix I with the same dimensions as A.

P1

Part (a)

The fire causing major loss within a year occurs at time of Poisson process with rate $\lambda p(r) = 5r$.

Therefore,

$EX = E[5r] = \frac{50}{3}$

$var(X) = E[5r] = \frac{50}{3}$

Part (b)

Let S_i denotes the casualties for ith fire. Let Z denotes the number of fire outbreaks within a year.

$$\begin{aligned} EY &= EZES_i \\ &= 50Er \\ &= \frac{500}{3} \end{aligned}$$

Part (c)

The intensity for $\{\tilde{N}_t\}$ is $\tilde{\lambda} = \frac{1}{4} * (20 + 50) = 17.5$

Part (d)

Firstly, consider fire within 3 km, the happening time occurs at a poisson process with rate λ_1, and let C_1 denotes the corresponding expected cost, X_1 denotes the number of fires within this region, z_{1i} denote the cost for each fire event:

$$\begin{aligned} \lambda_1 &= \lambda p(\text{fire happen within 3km}) \\ &= 50 * \frac{3^2}{5^2} \\ &= 18 \end{aligned}$$

$$\begin{aligned} C_1 &= EX_1 E z_{1i} \\ &= 18 * 0.5 \\ &= 9 \end{aligned}$$

Secondly, consider fire within 3-5 km, the happening time occurs at a poisson process with rate λ_2, and let C_2 denotes the corresponding expected cost, X_2 denotes the number of fires within this region, z_{2i} denote the cost for each fire event:

$$
\begin{aligned}
\lambda_2 &= \lambda p(\text{fire happen within 3-5 km}) \\
&= 50 * (1 - \frac{3^2}{5^2}) \\
&= 32
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= EX_2Ez_{2i} \\
&= 32 * 2 \\
&= 64
\end{aligned}
$$

Lastly, consider fire within 5-6 km, the happening time occurs at a poisson process with rate λ_3, and let C_3 denotes the corresponding expected cost, X_3 denotes the number of fires within this region, z_{3i} denote the cost for each fire event:

```
$$
 \begin{align}
 \lambda_3 &= 20 \\
 \\
 C_2 &= EX_2Ez_{3i} \\
 &= 20 * 3.5 \\
 &= 70

 \end{align}
$$
```

$$
\begin{aligned}
\lambda_3 &= 20 \\
\\
C_2 &= EX_2Ez_{3i} \\
&= 20 * 3.5 \\
&= 70
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 &= 20 \\
\\
C_2 &= EX_2Ez_{3i} \\
&= 20 * 3.5 \\
&= 70
\end{aligned}
$$

Therefore, the expected total cost is: $(C_1 + C_2 + C_3) * 1000 = 143000$

P2

Part (a)

(i) When $t = 0, \tilde{N}_0 = 0$

(ii) Prove increments are Poisson processes

- When $s < t < t_0$, $\tilde{N}_t - \tilde{N}_s = 0 \sim Poiss(0)$
- When $s < t_0 < t$, $\tilde{N}_t - \tilde{N}_s \sim Poiss(\lambda' t), \lambda' = \lambda \frac{t-t_0}{t}$
- When $t_0 < s < t$, $\tilde{N}_t - \tilde{N}_s \sim Poiss(\lambda'' t), \lambda'' = \lambda \frac{t-s}{t}$

Therefore, we have proved increments of $\tilde{N}_t$ are Poisson processes.

(iii) Prove increments are independent

- When $t_1 < t_2 < t_3 < t_0$, $\tilde{N}_{t_3} - \tilde{N}_{t_2} = \tilde{N}_{t_2} - \tilde{N}_{t_1} = 0$, increment independent satisfied.

- When $t_0 < t_1 < t_2 < t_3$, $\tilde{N}_{t_3} - \tilde{N}_{t_2} \sim Poiss(\lambda(t_3 - t_2))$, $\tilde{N}_{t_2} - \tilde{N}_{t_1} \sim Poiss(\lambda(t_2 - t_1))$. increment independent satisfied.

- When $t_1 < t_0 < t_2 < t_3$, $\tilde{N}_{t_3} - \tilde{N}_{t_2} \sim Poiss(\lambda(t_3 - t_2))$, $\tilde{N}_{t_2} - \tilde{N}_{t_1} = \tilde{N}_{t_2} \sim Poiss(\lambda(t_2 - t_0))$. increment independent satisfied.

- When $t_1 < t_2 < t_0 < t_3$, $\tilde{N}_{t_3} - \tilde{N}_{t_2} = \tilde{N}_{t_3} \sim Poiss(\lambda(t_3 - t_0))$, $\tilde{N}_{t_2} - \tilde{N}_{t_1} = 0$. increment independent satisfied.

To sum up, when $t > t_0$, $\tilde{N}_t$ is a Poisson process with rate $\lambda^* = \lambda \frac{t-t_0}{t}$

Part (b)

Let $N_t^1, N_t^2, N_t^3, N_t^4$ denote the number of customers will be served after 5,10,15,20 minutes, which are poisson process with rate $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

According to Theorem 2.11:

$$\begin{aligned}\lambda_1 &= \lambda P(\text{wait time} = 5) = 25 \\ \lambda_2 &= \lambda P(\text{wait time} = 10) = 25 \\ \lambda_3 &= \lambda P(\text{wait time} = 15) = 25 \\ \lambda_4 &= \lambda P(\text{wait time} = 20) = 25\end{aligned}$$

Let $\tilde{N_t^1}, \tilde{N_t^2}, \tilde{N_t^3}, \tilde{N_t^4}$ denote the number of customers being served, after waiting for 5,10,15,20 minutes respectively. According to Lemma 2.5, we have

$$\tilde{N}_t^1 = N_{t-\frac{1}{12}}^1 \sim \begin{cases} Poiss(\lambda_1 \frac{t-\frac{1}{12}}{t}) & \text{for } t > \frac{1}{12} \\ 0 & \text{for } t \leq \frac{1}{12} \end{cases}$$

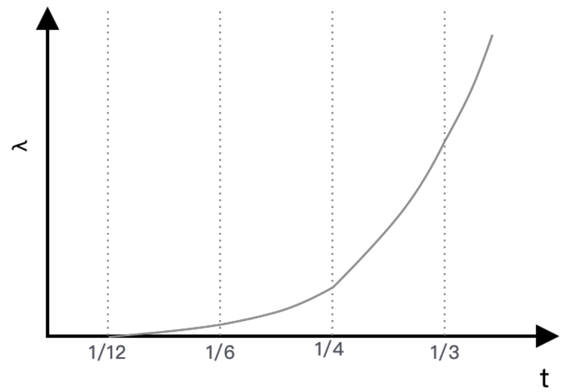
$$\tilde{N}_t^2 = N_{t-\frac{1}{6}}^2 \sim \begin{cases} Poiss(\lambda_2 \frac{t-\frac{1}{6}}{t}) & \text{for } t > \frac{1}{6} \\ 0 & \text{for } t \leq \frac{1}{6} \end{cases}$$

$$\tilde{N}_t^3 = N_{t-\frac{1}{4}}^3 \sim \begin{cases} Poiss(\lambda_3 \frac{t-\frac{1}{4}}{t}) & \text{for } t > \frac{1}{4} \\ 0 & \text{for } t \leq \frac{1}{4} \end{cases}$$

$$\tilde{N}_t^4 = N_{t-\frac{1}{3}}^4 \sim \begin{cases} Poiss(\lambda_4 \frac{t-\frac{1}{3}}{t}) & \text{for } t > \frac{1}{3} \\ 0 & \text{for } t \leq \frac{1}{3} \end{cases}$$

Combine these 4 poisson processes, we have $\tilde{N}_t = \tilde{N}_t^1 + \tilde{N}_t^2 + \tilde{N}_t^3 + \tilde{N}_t^4$.

According to Theorem 2.13, we have $\tilde{N}_t$ is a Poisson process with intensity $\tilde{\lambda} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$, its graph is as follow:



P3

The number of enthusiastic worker is a Poisson distribution with $\lambda_1 = 60 * \frac{2}{3} = 40$. For this group, the mean of cans collected is $ES_1 = 40 * 10 = 400$, the corresponding variance is $var(S_1) = 40 * 5^2 + 40 * 10^2 = 5000$.

The number of lazy worker is a Poisson distribution with $\lambda_2 = 60 * \frac{1}{3} = 20$. For this group, the mean of cans collected is $ES_2 = 20 * 3 = 60$, the corresponding variance is $var(S_2) = 20 * 2^2 + 20 * 3^2 = 260$.

To sum up, the total cans is expected to be $ES = ES_1 + ES_2 = 460$, the corresponding standard deviation is expected to be $sd(S) = \sqrt{var(S_1) + var(S_2)} = \sqrt{5260}$.

P4

$$\begin{aligned} cov(T, N_T) &= E[(T - E[T])(N_T - E[N_T])] \\ &= E[(T-\mu)(N_T - \lambda)] \\ &= E[T \cdot N_T] - \mu E[N_T] - \lambda E[T] + \mu\lambda \\ &= E[T \cdot N_T] - \mu E[N_T] - \mu\lambda + \mu\lambda \\ &= E[T \cdot N_T] - \mu E[N_T] \end{aligned}$$

$$\begin{aligned} E[T * N_T] &= \int_{t=0}^{\infty} t \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} f_T(t) dt \\ &= \int_{t=0}^{\infty} \lambda t^2 f_T(t) dt \\ &= \lambda E[T^2] \\ &= \lambda(\mu^2 + \sigma^2) \end{aligned}$$

$$\begin{aligned} E[N_T] &= \int_{t=0}^{\infty} \sum_{n=0}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!} f_T(t) dt \\ &= \int_{t=0}^{\infty} \lambda t f_T(t) dt \\ &= \lambda E[T] \\ &= \lambda\mu \end{aligned}$$

$$\begin{aligned} &\text{Therefore,} \\ &cov(T, N_T) = \lambda\sigma^2 \end{aligned}$$

P5

Part (a)

According to Theorem 2.11, $N_1(t) \sim Poiss(p\lambda)$, $N_2(t) \sim Poiss((1-p)\lambda)$.

Therefore, the joint distribution of $N_1(t), N_2(t)$ is:

$$\begin{aligned} P(N_1(t) = n_1, N_2(t) = n_2) &= e^{-p\lambda t} \frac{(p\lambda t)^{n_1}}{n_1!} e^{-(1-p)\lambda t} \frac{((1-p)\lambda t)^{n_2}}{n_2!} \\ &= \frac{e^{-\lambda t} (\lambda t)^{n_1+n_2} p^{n_1} (1-p)^{n_2}}{n_1! n_2!} \end{aligned}$$

Part (b)

$Pr(L = k) = (1 - p)^k p$

L is a geometric distribution with probability equals to $1 - p$.

P6

Part (a)

The mean of total revenue is the sum of the expectation of total revenue of speeding tickets and the expectation of total revenue of DWI tickets.

The expectation of total revenue of speeding tickets: $6 * 2/3 * 100 = 400$.

The expectation of total revenue of DWI tickets: $6 * 1/3 * 400 = 800$.

Therefore, **the mean of total revenue is** $400 + 800 = 1200$.

The variance of total revenue is the sum of the variance of total revenue of speeding tickets and the variance of total revenue of DWI tickets.

The variance of total revenue of speeding tickets: $6 * 2/3 * 100^2 = 40000$

The variance of total revenue of DWI tickets: $6 * 1/3 * 400^2 = 320000$

Therefore, **the standard deviation of total revenue is** $\sqrt{40000 + 320000} = 600$

Part (b)

Let N_1, N_2 denote the number of speeding tickets and DWI tickets. According to Theorem 2.11, N_1, N_2 are independent Poisson process with $\lambda_1 = 4, \lambda_2 = 2$.

$$\begin{aligned} P(N_1 = 5, N_2 = 1) &= P(N_1 = 5)P(N_2 = 1) \\ &= (e^{-4} * 4^5/5!) * (e^{-2} * 2^1/1!) \\ &= 0.04230404 \end{aligned}$$

Part (c)

$$P(A) = e^{-6*0.5} = 0.04978707$$

$$\begin{aligned} P(A|N=5) &= \frac{P(N=5|A)P(A)}{P(N=5)} \\ &= \frac{e^{-6*0.5}(6*0.5)^5/5!}{e^{-6}(6)^5/5!}P(A) \\ &= 0.03125 \end{aligned}$$

Therefore, $P(A|N=5) < P(A)$

P7

Part (a)

The number of truck go to Bojangle is a Poisson process with rate $\lambda_1 = 40 * 1/8 = 5$. Therefore, the probability that exactly 6 trucks arrive at Bojangle's between noon and 1PM is:

$$P(n=6) = \frac{\lambda_1^6 e^{-\lambda_1}}{6!} = 0.1462228$$

Part (b)

According to Theorem 2.15:

$$\begin{aligned} P(\text{2 trucks arrive between 12:20 to 12:40} \mid \text{6 trucks arrive between 12:00 to 13:00}) &= \binom{6}{2}(\frac{1}{3})^2(\frac{2}{3})^4 \\ &= 0.3292181 \end{aligned}$$

Part (c)

The expectation of customers arrived by truck is: $40 * 1/8 * 1 = 5$

The variance of customers arrived by truck is: $40 * 1/8 * 1^2 = 5$

The expectation of customers arrived by car is: $100 * 1/10 * 0.3 * 1 + 100 * 1/10 * 0.5 * 2 + 100 * 1/10 * 0.2 * 4 = 21$

The variance of car passenger distribution is $0.3 * (1-2.1)^2 + 0.5 * (2-2.1)^2 + 0.2 * (4-2.1)^2 = 1.09$.

The variance of customers arrived by truck is: $10 * 1.09 + 10 * (2.1)^2 = 55$

Therefore, the total expectation (mean) is $5 + 21 = 26$, the total variance is $55 + 5 = 60$, the standard deviation is $\sqrt{60} = 7.745967$