HW 9

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W1

Part (a)

The CTMC version of "doubly stochastic matrix" is that the transition rate matrix Q whose row sum and column sum are all 0.

Part (b)

$$\frac{1}{N}(1,\ldots,1)\times Q=\frac{1}{N}(\sum_{i=1}^{p}q_{i1},\ldots,\sum_{i=1}^{p}q_{in})=(0,\ldots,0)$$

Therefore, $\frac{1}{N}(1,\ldots,1)$ is a stationary distribution.

W2

$$\pi(k)q(k,j)=\pi(j)q(j,k)\text{ for all }j\neq k$$

W3

Part (a)

Part (b)

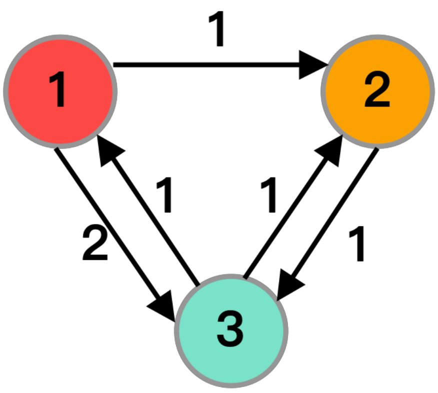
$$T=\lambda(k-1)+\frac{\alpha+\beta+\gamma}{3}$$

Part (c)

$$
\begin{aligned}
P(X_T = k) &= \frac{\alpha}{\alpha+\beta+\gamma} \\
P(X_T = k') &= \frac{\beta}{\alpha+\beta+\gamma} \\
P(X_T = k'') &= \frac{\gamma}{\alpha+\beta+\gamma}
\end{aligned}
$$

P1

Part (a)



Part (b)

$$
\begin{aligned}
\Lambda &= \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
U &= \begin{bmatrix} -5 & 1 & 1 \\ -1 & -1 & 1 \\ -3 & 1 & 1 \end{bmatrix} \\
P_t &= U \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1} =
\end{aligned}
$$

Part (c)

Passing $t \to \infty$, we have:

$$P_{t\to\infty} = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{bmatrix}$$

Part (d)

Because $v(t) = v(0)e^{tQ}$, we have

$$v(t) = v(0)P_t = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]U \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{-1}$$

Part (e)

$$Q^T = \begin{bmatrix} -3 & 0 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 4 \\ 0 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 4 \\ 0 & 0 & -0 \end{bmatrix}$$

So, set $x_3 = 1$. we have $x_2 = \frac{4}{3}, x_1 = \frac{1}{3}$. After normalization, $\pi = [\frac{1}{8}, \frac{1}{2}, \frac{3}{8}]$. This is the same result with (c).

Part (f)

Let $f(i)$ denote the number of cars passing by in state i.

$$\frac{1}{t}\int_0^t f(X_s)ds = \sum_{k=1}^{3} f(i)\pi_i = 10$$

The asymptotic amount of cars passing by the light per unit time is 10.

P2

Part (a)

$$
\begin{aligned}
p_t(i,j) &= P(X_t = j | X_0 = i) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} p_{ij}^n
\end{aligned}
$$

$$
\begin{aligned}
Q_{ij} &= \lim_{t \to 0} \frac{p_t(i,j)}{t} \\
&\text{When } t \to 0, p(n > 1) \to 0 \\
&\approx \lim_{t \to 0} \frac{p_t(i,j | n \in \{0,1\})}{t} \\
&= \lim_{t \to 0} \frac{e^{-\lambda t} p_{i,j}^0 + \lambda t e^{-\lambda t} p_{i,j}}{t} = (*)
\end{aligned}
$$

For $i \neq j$, we have $p_{i,j}^0 = 0$, therefore:

$$
(*) = \lim_{t \to 0} \lambda e^{-\lambda t} p_{i,j} \to \lambda p_{ij}
$$

For $i = j$, we have $p_{i,j}^0 = 1$, therefore

$$
(*) = \lim_{t \to 0} \frac{e^{-\lambda t} + \lambda t e^{-\lambda t} p_{i,j}}{t} \to -\lambda(1 - p_{ij})
$$

Part (b)

Therefore, if π is a stationary distribution of $\{Y_n\}$, we have:

$$
\begin{aligned}
\pi Q = 0 \to \sum_i \pi_i Q_{ij} &= 0 \\
\sum_{i \neq j} \pi_i Q_{ij} &= \lambda \sum_{i \neq j} \pi_i p_{ij} = (**) \\
\sum_{i = j} \pi_i Q_{ij} &= -\lambda \pi + \lambda \pi_j p_{jj} = (***)
\end{aligned}
$$

From $(**) + (***) = 0$ we have $\pi P = \pi$.

Similarly, we can prove if π is not the stationary distribution of $\{Y_n\}$, we would have $\pi Q \neq 0 \to \pi P \neq \pi$.

Therefore, we have proved that π is a stationary distribution of $\{Y_n\}$ if and only if it is that of $\{X_t\}$.

P3

The goal is to minimize the unit time cost.

Firstly, the expected cost for each interval is:

$$\begin{aligned} 1200 * \int_0^c f_T(t)dt + 300\int_c^{30} f_T(t)dt &= \frac{4t^2}{3}\Big|_0^c + \frac{t^2}{3}\Big|_c^{30} \\ &= c^2 + 300 = (1) \end{aligned}$$

Secondly, the expected number of months for each interval is:

$$\begin{aligned} \int_0^c tf_T(t)dt + \int_c^{30} cf_T(t)dt &= \frac{2c^3}{2700} + c(1 - \frac{c^2}{900}) \\ &= c - \frac{c^3}{2700} = (2) \end{aligned}$$

Therefore, we hope to miniimze $\frac{(1)}{(2)}$:

$$\frac{(1)}{(2)} = \frac{c^2+300}{c - \frac{c^3}{2700}} = g(c)$$
$$\text{set } \frac{d}{dc} g'(c) = 0 \Rightarrow c = 14.57605$$

P4

Part (a)

Let v denote the limiting fraction of time in each city, we have $Q^T v = 0$.

$$Q^T = \begin{bmatrix} -4 & 3 & 5 \\ 2 & -4 & 0 \\ 2 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, set $x_3 = 1$. we have $x_2 = 1, x_1 = 2$. After normalization, $\pi = [\frac{1}{2}, \frac{1}{4}, \frac{1}{4}]$.

Part (b)

About $\frac{1}{4} * \frac{3}{5+3} = \frac{3}{32}$ of the trip is from Boston to Atlanta.

P5

Part (a)

$$Q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 2 & -3 & 0 & 1 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

Let v denote the stationary distribution, we have $Q^T v = 0$.

$$Q^T = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & -3 & 2 & 0 \\ 1 & 0 & -2 & 2 \\ 0 & 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_4 = 1$. we have $x_3 = 3, x_2 = 2, x_1 = 4$. After normalization, $\pi = [\frac{4}{10}, \frac{2}{10}, \frac{3}{10}, \frac{1}{10}]$.

Part (b)

Let $f(i)$ denote the rate of sales in state i. $f(0) = 0, f(1) = f(2) = f(3) = 2$

$$\frac{1}{t} \int_0^t f(X_s) ds = \sum_{k=1}^{4} f(i) \pi_i = 1.2$$

P6

Part (a)

$\{X_t\}$ is a CTMC with states $\{0, 1, 2, 12, 21\}$. The state definition is:

$$\begin{aligned} 0 &: \text{No broken machine} \\ 1 &: \text{Only machine 1 broken} \\ 2 &: \text{Only machine 2 broken} \\ 12 &: \text{Machine 1\&2 are broken, machine 1 broken first} \\ 21 &: \text{Machine 1\&2 are broken, machine 2 broken first} \end{aligned}$$

Part (b)

The transition rate matrix is:

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 12 \\ 21 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 12 & 21 \end{array} \\ \begin{bmatrix} -4 & 1 & 3 & 0 & 0 \\ 2 & -5 & 0 & 3 & 0 \\ 4 & 0 & -5 & 0 & 1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 4 & 0 & 0 & -4 \end{bmatrix} \end{array}$$

Let v denote the stationary distribution, we have $Q^T v = 0$.

$$Q^T = \begin{bmatrix} -4 & 2 & 4 & 0 & 0 \\ 1 & -5 & 0 & 0 & 4 \\ 3 & 0 & -5 & 2 & 0 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ -4 & 2 & 4 & 0 & 0 \\ 3 & 0 & -5 & 2 & 0 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ 0 & -18 & 4 & 0 & 16 \\ 0 & 15 & -5 & 2 & -12 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & -18 & 4 & 0 & 16 \\ 0 & 15 & -5 & 2 & -12 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 4 & -12 & 16 \\ 0 & 0 & -5 & 12 & -12 \\ 0 & 0 & 1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & -12 & 32 \\ 0 & 0 & 0 & 12 & -32 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -5 & 0 & 0 & 4 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_5 = 1$. we have $x_4 = \frac{8}{3}, x_3 = 4, x_2 = \frac{16}{9}, x_1 = \frac{44}{9}$. After normalization, $\pi = [\frac{44}{129}, \frac{16}{129}, \frac{36}{129}, \frac{24}{129}, \frac{9}{129}]$.