HW 10

Sicong Zhao (sz163)

P1

Part (a)

$$Q = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{bmatrix} -\frac{1}{3} & 0 & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & -\frac{1}{3} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & -\frac{1}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & -\frac{1}{4} & 0 \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix} \end{array}$$

According to theorem 4.17, the stationary distribution is $\pi = \frac{1}{7}(1,1,1,1,1,1,1)$.

Since $diag(\pi)Q$ is symmetric, therefore $\pi_i q_{i,j} = \pi_j q_{ji}$ holds for all i, j. So, it satisfied detailed balance condition.

Part (b)

Since the transition rate matrix satisfies the detailed balance condition, $\pi = \frac{1}{7}(1,1,1,1,1,1,1)$ could be the stationary distribution because $\pi Q = 0$.

Additionally, due to the CTMC has finite states and it's irreducible, the stationary distribution should be unique.

Therefore, we can derive the long-run 'mic-holding' time for each singer is $\pi = \frac{1}{7}(1,1,1,1,1,1,1)$.

Part (c)

Relabel the transition rate matrix as $Q = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, we have

$$A = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & -\frac{1}{3} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 & -\frac{1}{4} & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{1}{12} \\ 0 & \frac{1}{12} \\ \frac{1}{12} & 0 \\ \frac{1}{12} & 0 \\ \frac{1}{12} & 0 \end{bmatrix}$$

The main exit time $m = -A^{-1}1 = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})^T$. Therefore, $E[T] = \frac{1}{12}$

Part (d)

$$R = -A^{-1}B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Eventually, Andy has higher chance to hold the mic.

P2

Part (a)

The stationary distribution for the states of stock market can be derived using $\pi Q = 0$

$$Q^T = \begin{bmatrix} -2 & 3 & 1 \\ 1 & -3 & 3 \\ 1 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

So, set $x_3 = 1$. we have $x_2 = \frac{7}{3}, x_1 = 4$. After normalization, $\pi = [\frac{12}{22}, \frac{7}{22}, \frac{3}{22}]$.

Therefore, the expected return per year for portfolio 1 is:

$$\sum_{i \in 1,2,3} f_1(i)\pi(i) = \frac{39}{11}$$

The expected return per year for portfolio 2 is:

$$\sum_{i \in 1,2,3} f_2(i)\pi(i) = \frac{42}{11}$$

$\frac{42}{11} > \frac{39}{11}$ Therefore, portfolio 2 is better.

Part (b)

For portfolio 1, r could be less than $\frac{17}{11}$.

For portfolio 2, r could be less than $\frac{20}{11}$

P3

Part (a)

Let $\lambda_{white} = 1, \lambda_{black} = 3$

The degree for each state can be represented by following matrix:

$$\begin{bmatrix} 2 & 3 & 4 & 4 & 4 & 4 & 3 & 2 \\ 3 & 4 & 6 & 6 & 6 & 6 & 4 & 3 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 4 & 6 & 8 & 8 & 8 & 8 & 6 & 4 \\ 3 & 4 & 6 & 6 & 6 & 6 & 4 & 3 \\ 2 & 3 & 4 & 4 & 4 & 4 & 3 & 2 \end{bmatrix}$$

According to theorem 4.17, the stationary distribution can be derived as:

$\sum_j \lambda_j^{-1} deg(j) = 224$

$$\pi = \begin{bmatrix} \frac{2}{224} & \frac{3}{672} & \frac{4}{224} & \frac{4}{672} & \frac{4}{224} & \frac{4}{672} & \frac{3}{224} & \frac{2}{672} \\ \frac{3}{672} & \frac{4}{224} & \frac{6}{672} & \frac{6}{224} & \frac{6}{672} & \frac{6}{224} & \frac{4}{672} & \frac{3}{224} \\ \frac{4}{224} & \frac{6}{672} & \frac{8}{224} & \frac{8}{672} & \frac{8}{224} & \frac{8}{672} & \frac{6}{224} & \frac{4}{672} \\ \frac{4}{672} & \frac{6}{224} & \frac{8}{672} & \frac{8}{224} & \frac{8}{672} & \frac{8}{224} & \frac{6}{672} & \frac{4}{224} \\ \frac{4}{224} & \frac{6}{672} & \frac{8}{224} & \frac{8}{672} & \frac{8}{224} & \frac{8}{672} & \frac{6}{224} & \frac{4}{672} \\ \frac{4}{672} & \frac{6}{224} & \frac{8}{672} & \frac{8}{224} & \frac{8}{672} & \frac{8}{224} & \frac{6}{672} & \frac{4}{224} \\ \frac{3}{224} & \frac{4}{672} & \frac{6}{224} & \frac{6}{672} & \frac{6}{224} & \frac{6}{672} & \frac{4}{224} & \frac{3}{672} \\ \frac{2}{672} & \frac{3}{224} & \frac{4}{672} & \frac{4}{224} & \frac{4}{672} & \frac{4}{224} & \frac{3}{672} & \frac{2}{224} \end{bmatrix}$$

The long run fraction of time that the knight stays at black cell is $\frac{1}{4}$

Part (b)

The degree for each state can be represented by following matrix:

$$\begin{bmatrix} 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\ 21 & 23 & 23 & 23 & 23 & 23 & 23 & 21 \\ 21 & 23 & 25 & 25 & 25 & 25 & 23 & 21 \\ 21 & 23 & 25 & 27 & 27 & 25 & 23 & 21 \\ 21 & 23 & 25 & 27 & 27 & 25 & 23 & 21 \\ 21 & 23 & 25 & 25 & 25 & 25 & 23 & 21 \\ 21 & 23 & 23 & 23 & 23 & 23 & 23 & 21 \\ 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \end{bmatrix}$$

According to theorem 4.17, the stationary distribution can be derived as:

$\sum_j \lambda_j^{-1} deg(j) = \frac{2912}{3}$

$$\pi = \begin{bmatrix} \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} \\ \frac{21}{2912} & \frac{69}{2912} & \frac{23}{2912} & \frac{69}{2912} & \frac{23}{2912} & \frac{69}{2912} & \frac{23}{2912} & \frac{63}{2912} \\ \frac{63}{2912} & \frac{23}{2912} & \frac{75}{2912} & \frac{25}{2912} & \frac{75}{2912} & \frac{25}{2912} & \frac{69}{2912} & \frac{21}{2912} \\ \frac{21}{2912} & \frac{69}{2912} & \frac{25}{2912} & \frac{81}{2912} & \frac{27}{2912} & \frac{75}{2912} & \frac{23}{2912} & \frac{63}{2912} \\ \frac{63}{2912} & \frac{23}{2912} & \frac{75}{2912} & \frac{27}{2912} & \frac{81}{2912} & \frac{25}{2912} & \frac{69}{2912} & \frac{21}{2912} \\ \frac{21}{2912} & \frac{69}{2912} & \frac{25}{2912} & \frac{75}{2912} & \frac{25}{2912} & \frac{75}{2912} & \frac{23}{2912} & \frac{63}{2912} \\ \frac{63}{2912} & \frac{23}{2912} & \frac{69}{2912} & \frac{23}{2912} & \frac{69}{2912} & \frac{23}{2912} & \frac{69}{2912} & \frac{21}{2912} \\ \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} & \frac{21}{2912} & \frac{63}{2912} \end{bmatrix}$$

The long run fraction of time that the queen stays at black cell is $\frac{1}{4}$

P4

Part (a)

Let the difference be the clock-wise steps from the hour hand to the minute hand. The corresponding transition rate matrix is (Let w denote $\alpha+\beta+\gamma+\delta$):

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} -w & \alpha+\delta & 0 & 0 & 0 & \beta+\gamma \\ \beta+\gamma & -w & \alpha+\delta & 0 & 0 & 0 \\ 0 & \beta+\gamma & -w & \alpha+\delta & 0 & 0 \\ 0 & 0 & \beta+\gamma & -w & \alpha+\delta & 0 \\ 0 & 0 & 0 & \beta+\gamma & -w & \alpha+\delta \\ \alpha+\delta & 0 & 0 & 0 & \beta+\gamma & -w \end{bmatrix} \end{array}$$

Since the transition rate matrix satisfies the detailed balance condition, $\pi = \frac{1}{6}(1,1,1,1,1,1)$ could be the stationary distribution because $\pi Q = 0$.

So, the asymptotic portion of time of which the 2 hands overlapped is $\frac{1}{6}$

Part (b)

Based on our definition, the initial state is 3.

Relabel the transition rate matrix as $Q = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, we have

$$A = \begin{bmatrix} -10 & 5 & 0 & 0 & 0 \\ 5 & -10 & 5 & 0 & 0 \\ 0 & 5 & -10 & 5 & 0 \\ 0 & 0 & 5 & -10 & 5 \\ 0 & 0 & 0 & 5 & -10 \end{bmatrix}$$

The main exit time $m = -A^{-1}1 = [0.5, 0.8, 0.9, 0.8, 0.5]^T$. Therefore, $E[T] = 0.9$

P5

Part (a)

The transition rate matrix of $\{X_t\}$ is:

$$Q = \begin{bmatrix} -6 & 6 & 0 & 0 \\ 1 & -5 & 4 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

The stationary distribution can be derived using $\pi Q = 0$

$$Q^T = \begin{bmatrix} -6 & 1 & 0 & 0 \\ 6 & -5 & 2 & 0 \\ 0 & 4 & -4 & 3 \\ 0 & 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_4 = 1$. we have $x_3 = \frac{3}{2}, x_2 = \frac{3}{4}, x_1 = \frac{1}{8}$. After normalization, $\pi = [\frac{1}{27}, \frac{6}{27}, \frac{12}{27}, \frac{8}{27}]$.

Part (b)

Let state be the location, so the state is $\{S, L\}$. For each frog, the transition rate matrix is:

$$Q = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

The stationary distribution is: $[\frac{2}{3}, \frac{1}{3}]$.

For each frog, it has $\frac{2}{3}$ probability to be in the sun.

$$\begin{aligned} P(X_t = 0) &= (\frac{2}{3})^0 (\frac{1}{3})^3 = \frac{1}{27} \\ P(X_t = 1) &= \binom{3}{1} (\frac{2}{3})^1 (\frac{1}{3})^2 = \frac{6}{27} \\ P(X_t = 2) &= \binom{3}{2} (\frac{2}{3})^2 (\frac{1}{3})^1 = \frac{12}{27} \\ P(X_t = 3) &= \binom{3}{3} (\frac{2}{3})^3 (\frac{1}{3})^0 = \frac{8}{27} \end{aligned}$$

P6

Part (a)

Let $\{X_t\}$ be the number of available cabs at time t. The rate matrix is:

$$Q = \begin{bmatrix} -9 & 9 & 0 & 0 \\ 2 & -8 & 6 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

The stationary distribution can be derived using $\pi Q = 0$

$$Q^T = \begin{bmatrix} -9 & 2 & 0 & 0 \\ 9 & -8 & 2 & 0 \\ 0 & 6 & -5 & 2 \\ 0 & 0 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -2 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_4 = 1$. we have $x_3 = \frac{2}{3}, x_2 = \frac{2}{9}, x_1 = \frac{4}{81}$. After normalization, $\pi = [\frac{4}{157}, \frac{18}{157}, \frac{54}{157}, \frac{81}{157}]$.

The probability that all cabs are busy when a call comes in is $\frac{4}{157}$

Part (b)

Let $f(i)$ denote the number of customers are served in state i. We have:

$$f(X_t = 0) = 3$$
$$f(X_t = 1) = 2$$
$$f(X_t = 2) = 1$$
$$f(X_t = 3) = 0$$

Therefore, the expected number of customers served per hour is: $\sum_{i \in 0,1,2,3} f(i)\pi(i) = \frac{102}{157}$

P7

Part (a)

The stationary distribution can be derived using $\pi Q = 0$

$$Q^T = \begin{bmatrix} -4 & 4 & 2 & 0 \\ 3 & -6 & 3 & 0 \\ 1 & 2 & -6 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_4 = 1$. we have $x_3 = 2, x_2 = 3, x_1 = 4$. After normalization, $\pi = [\frac{4}{10}, \frac{3}{10}, \frac{2}{10}, \frac{1}{10}]$.

Part (b)

$diag(\pi)Q$ is symmetric, therefore $\pi_i q_{i,j} = \pi_j q_{ji}$ holds for all i, j. So, it satisfied detailed balance condition.

Part (c)

$$A = \begin{bmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{bmatrix}$$
$$m = -A^{-1} \cdot 1 = [\frac{63}{12}, \frac{62}{12}, \frac{54}{12}]^T$$

Therefore, the expected amount of time until depression sets in is $\frac{63}{12}$ months.

P8

Part (a)

$$Q = \begin{bmatrix} -1.5 & 1.5 & 0 & 0 & 0 \\ 0.5 & -2 & 1.5 & 0 & 0 \\ 0 & 1 & -2.5 & 1.5 & 0 \\ 0 & 0 & 1.5 & -3 & 1.5 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$
$$Q^T = \begin{bmatrix} -1.5 & 0.5 & 0 & 0 & 0 \\ 1.5 & -2 & 1 & 0 & 0 \\ 0 & 1.5 & -2.5 & 1.5 & 0 \\ 0 & 0 & 1.5 & -3 & 2 \\ 0 & 0 & 0 & 1.5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, set $x_5 = 1$. we have $x_4 = \frac{4}{3}, x_3 = \frac{4}{3}, x_2 = \frac{8}{9}, x_1 = \frac{8}{27}$. After normalization, $\pi = [\frac{8}{131}, \frac{24}{131}, \frac{36}{131}, \frac{36}{131}, \frac{27}{131}]$.

Part (b)

The unfulfilled rate come in at rate $\frac{27}{131} * 1.5 = \frac{81}{262}$

If only 3 cars in service, the rate would increase.

Part (c)

Let state {4} be the absorbing state. Therefore,

$$A = \begin{bmatrix} -1.5 & 1.5 & 0 & 0 \\ 0.5 & -2 & 1.5 & 0 \\ 0 & 1 & -2.5 & 1.5 \\ 0 & 0 & 1.5 & -3 \end{bmatrix}$$
$$m = -A^{-1} \cdot 1 = [4.74074074, 4.07407407, 3.18518519, 1.92592593]$$

Therefore,
$g(0) = 4.74074074, g(1) = 4.07407407, g(2) = 3.18518519, g(3) = 1.92592593$.
And by the definition of $g(i)$ we can derive $g(4) = 0$

P9

Let $1, 2, 3$ denote raking leaves, cleaning bathroom, cleaning kitchen. Let 0 denotes finished.

Let (J_t, K_t) denote Jill and Kelly's status at time t. Therefore, $\{J_t, K_t\}$ could be a CTMC with state: $(3,2), (1,2), (3,1), (1,1), (0,2), (3,0), (0,0)$

The A matrix inside rate matrix is:

$$A = \begin{bmatrix} -\frac{7}{2} & 2 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & -3 & 2 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$
$$m = -A^{-1} \cdot 1 = [1.1952381, 0.96666667, 0.83333333, 0.5, 0.66666667, 0.5]$$

Therefore, the expected time is 1.1952381 hour.