HW2

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W1

A vector v, such that $vP = v$. In other words, it's the state distribution over the long run.

W2

Let $w = \alpha u + (1-\alpha)v$

$$
\begin{aligned}
&wP \\
&= (\alpha u + (1-\alpha)v)P \\
&= \alpha uP + (1-\alpha)vP
\end{aligned}
$$

Because $vP = v$ and $uP = u$, we know that

$wP = \alpha u + (1-\alpha)v = w$

Since $\alpha \in [0, 1]$, so we have infinite many w, which is the stationary distribution of transition matrix P.

W3

Let $v = [0.5, 0.5, 0, 0]$, $u = [0, 0, 0.5, 0.5]$.

We have $vP = v$ and $uP = u$.

So, P has 2 distinct stationary distribution.

W4

A doubly stochastic matrix is a square (m x m) matrix P, which has following properties:

- $\forall p_{i,j} \geq 0$ for $1 \leq i, j \leq m$
- $\sum_{j=1}^{m} p_{i,j} = 1$ for $1 \leq i \leq m$
- $\sum_{i=1}^{m} p_{i,j} = 1$for $1 \leq j \leq m$

If p is a doubly stochastic transition probability for a Markov chain with N states, then the uniform distribution, $\pi(x) = 1/N$ for all x, is a stationary distribution.

P1

According to the data, the coin in Pub 1 is not fair.

$P(1|Pub1) \approx 0.6$ and $P(1|Pub1) \approx 0.4$, while $P(1|Pub0) \approx 0.5$ and $P(0|Pub0) \approx 0.5$.

Such that, the transition matrix is:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$$

So, Pub1 seems to be the murder.

P2

Part (a)

Use I denote the N-dimentional column vector consisting of all entries with value 1.

$$PI = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N} \\ \cdots & & & \\ p_{N,1} & p_{N,2} & \cdots & p_{N,N} \end{bmatrix} I = \begin{bmatrix} \sum_{j=1}^{N} p_{1,j} \\ \sum_{j=1}^{N} p_{2,j} \\ \cdots \\ \sum_{j=1}^{N} p_{N,j} \end{bmatrix}$$

Due to the property for Markov transition matrix, $\sum_{j=1}^{N} p_{i,j} = 1$, suth that:

$$\begin{bmatrix} \sum_{j=1}^{N} p_{1,j} \\ \sum_{j=1}^{N} p_{2,j} \\ \dots \\ \sum_{j=1}^{N} p_{N,j} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix} = I$$

Part (b)

$$wI = vPI = vI = \sum_{i=1}^{N} v_i = 1$$

So, w os also a probability vector.

P3

Part (a)

The probability vector is $(P(AP), P(SS), P(HW)) = (0.22, 0.59, 0.19)$

Part (b)

$$P^3 = \begin{bmatrix} 0.3 & 0.525 & 0.175 \\ 0.175 & 0.65 & 0.175 \\ 0.175 & 0.525 & 0.3 \end{bmatrix}$$
$$P^{30} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$
$$P^{300} = \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$$

With the increase of the exponent, each row is converging to long-run distribution.

P4

If $X_n = 1$ and $X_{n-1} = 0$, we can infer $Y_n = 1$ and $Y_{n-1} = 0$.

Therefore:

- $P(X_{n+1} = 0 | X_n = 1, X_{n-1} = 0) = 0$

- $P(X_{n+1} = 1 | X_n = 1, X_{n-1} = 0) = 0.5$
- $P(X_{n+1} = 2 | X_n = 1, X_{n-1} = 0) = 0.5$

If we only consider $X_n = 1$, there could be 2 situations with equal probabilities:

1. $Y_n = 0, Y_{n-1} = 1$
   - Similarly, we can infer: $P(X_{n+1} = 1 | Y_n = 0, Y_{n-1} = 1) = 0.5$, and $P(X_{n+1} = 0 | Y_n = 0, Y_{n-1} = 1) = 0.5$
2. $Y_n = 1, Y_{n-1} = 0$
   - Similarly, we can infer: $P(X_{n+1} = 1 | Y_n = 1, Y_{n-1} = 0) = 0.5$, and $P(X_{n+1} = 2 | Y_n = 1, Y_{n-1} = 0) = 0.5$

Such that, we have:

- $P(X_{n+1} = 0 | X_n = 1) = 0.25$
- $P(X_{n+1} = 1 | X_n = 1) = 0.5$
- $P(X_{n+1} = 2 | X_n = 1) = 0.25$

$P(X_{n+1} | X_n = 1, X_{n-1} = 0) \neq P(X_{n+1} | X_n = 1)$, so historical state matters with respect to future states' probability distribution.

Therefore, X_n is not a Markov Chain.

P5

The state space for Y is 2,3,4,5,6,7,8. Its corresponding probability vector is $(\frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{3}{16}, \frac{2}{16}, \frac{1}{16})$

Since $X_n = S_n$ (mod 6), we can apply mod 6 calculation to Y thus mutate its sample space to 0,1,2,3,4,5, with corresponding probability vector to be: $(\frac{3}{16}, \frac{2}{16}, \frac{2}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16})$

Based on the probability vector, we can calculate the transition matrix to be:

$$P = \begin{bmatrix} \frac{3}{16} & \frac{2}{16} & \frac{2}{16} & \frac{2}{16} & \frac{3}{16} & \frac{4}{16} \\ \frac{4}{16} & \frac{3}{16} & \frac{2}{16} & \frac{2}{16} & \frac{2}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{4}{16} & \frac{3}{16} & \frac{2}{16} & \frac{2}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{3}{16} & \frac{4}{16} & \frac{3}{16} & \frac{2}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{2}{16} & \frac{3}{16} & \frac{4}{16} & \frac{3}{16} & \frac{2}{16} \\ \frac{2}{16} & \frac{2}{16} & \frac{2}{16} & \frac{3}{16} & \frac{4}{16} & \frac{3}{16} \end{bmatrix}$$

W6

Part (a)

The transition matrix is:

$$P = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0 & 0.25 \\ 0.75 & 0.25 & 0 \end{bmatrix} \end{array}$$

Part (b)

Let X_n denote the location of the driver in time n.

The two-step transition matrix is:

$$P^2 = PP = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{bmatrix} 0.75 & 0.125 & 0.125 \\ 0.1875 & 0.4375 & 0.375 \\ 0.1875 & 0.375 & 0.4375 \end{bmatrix} \end{array}$$

So, we can infer:

$P(X_2 = \mathrm{A}|X_0 = A) = 0.75$

$P(X_2 = \mathrm{B}|X_0 = A) = 0.125$

$P(X_2 = \mathrm{C}|X_0 = A) = 0.125$

$$P^3 = \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{bmatrix} 0.1875 & 0.40625 & 0.40625 \\ 0.609375 & 0.1875 & 0.203125 \\ 0.609375 & 0.203125 & 0.1875 \end{bmatrix} \end{array}$$

So, we can infer:

$P(X_3 = \mathrm{B} | X_0 = A) = 0.40625$

W7

Part (a)

$$P = \begin{array}{c} \\ \mathrm{RR} \\ \mathrm{RS} \\ \mathrm{SR} \\ \mathrm{SS} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{RR} & \mathrm{RS} & \mathrm{SR} & \mathrm{SS} \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

Part (b)

$$P^2 = \begin{array}{c} \\ \mathrm{RR} \\ \mathrm{RS} \\ \mathrm{SR} \\ \mathrm{SS} \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{RR} & \mathrm{RS} & \mathrm{SR} & \mathrm{SS} \end{array} \\ \begin{bmatrix} 0.36 & 0.24 & 0.24 & 0.16 \\ 0.36 & 0.24 & 0.12 & 0.28 \\ 0.36 & 0.24 & 0.24 & 0.16 \\ 0.18 & 0.12 & 0.21 & 0.49 \end{bmatrix} \end{array}$$

Part (c)

$$P(\text{rain on Wed.} \mid \text{not rain on Sun. and Mon.}) = P^2(\mathrm{SR} \mid \mathrm{SS}) + P^2(\mathrm{RR} \mid \mathrm{SS}) = 0.39$$

As above, the probability that it will rain on Wednesday given that it did not rain on Sunday or Monday is 0.39.

W8

Part (a)

Transient State: 1, 3, 5

Recurrent State: 2, 4

Irreducible closed sets: {2, 4}

Part (b)

Transient State: 2, 3,

Recurrent State: 1, 4, 5, 6

Irreducible closed sets: {1, 4, 5, 6}

Part (c)

Transient State: 3

Recurrent State: 1, 2, 4, 5

Irreducible closed sets: {1, 5} {2, 4}

Part (d)

Transient State: 3, 6

Recurrent State: 1, 2, 4, 5

Irreducible closed sets: {1, 4} {2, 5}