HW3

Sicong Zhao (sz163)

Note: For questions not labelled as can be calculated by computer, they were calculated by hand and the steps were included in this document.

W1

π is said to satisfy the detailed balance condition if: $\pi(x)p(x, y) = \pi(y)p(y, x)$

W2

Part (a)

Let $p_{i,j}$ be an entry of P, which means the probability of transfering from state i to state j. By definition, a transition probability matrix has rows that sum to 1, i.e., $\sum_j p(i, j) = 1$.

From $P = P^T$ we can infer $p_{i,j} = p_{j,i}$. Therefore, $\sum_j p(i, j) = \sum_j p(j, i) = 1$. From which we can infer each column of P also sum up to 1.

Therefore, P is a doubly stochastic matrix.

Part (b)

According to **Therorem 1.24**, the stationary distribution of P is a uniform distribution, with $\pi(x) = 1/N$ for all x. For state i, j, we have $\pi(i) = \pi(j)$ and $p(i, j) = p(j, i)$. Then, $\pi(i)p(i, j) = \pi(j)p(j, i)$.

Such that we can infer P admits a stationary distribution π that satisfies the detailed balanced condition. And π is a uniform distribution, with $\pi(x) = 1/N$ for all x.

W3

The period of a state is of P the largest numbber that will divide all the $n \geq 1$ for which $p^n(x, x) > 0$.

W4

From $Pv = \lambda v$, we can infer $P^n v = P^{n-1} Pv = P^{n-1} \lambda v$. In this way, we can have $P^n v = \lambda^n v = |\lambda|^n v$.

Therefore, we can infer:

- If $|\lambda| < 1$, as $n \to \infty$, $P^n v \to 0$
- If $|\lambda| > 1$, as $n \to \infty$, $P^n v \to \infty$

P1

Part (a)

$\mathbb{E}^v[f(X_n)] = \sum_{n=1}^N p^v(X_n) * f(X_n)$

Let v^n denote the distribution probability after n steps with the initial distribution $v^0 = v$. Therefore, $v^n = v^{n-1}P = \cdots = v^0 P^n = vP^n$.

Because $a = (f(1), \ldots, f(N))^T$, by vectorizing we have:
$\sum_{n=1}^N p^v(X_n) * f(X_n) = v^n a = vP^n a$

Therefore, we have proved $\mathbb{E}^v[f(X_n)] = vP^n a$

Part (b)

Since $A_n := \frac{1}{n} \sum_{k=1}^n f(X_k)$, according to Cesàro Mean theorem, proofing $\lim_{n\to\infty} \mathbb{E}^v[A_n] = \pi a$ is equal to proofing $\lim_{n\to\infty} \mathbb{E}^v[f(X_n)] = \pi a$.

By definition of expectation, we can infer
$\lim_{n\to\infty} \mathbb{E}^v[f(X_n)] = \sum_{k=1}^n p^n(X_k) f(X_k)$. Besides, π is the stationary distribution when $n \to \infty$ and $a = (f(1), \ldots, f(N))^T$. Therefore, by vectorizing $\sum_{k=1}^n p^n(X_k) f(X_k)$ we can infer: $\lim_{n\to\infty} \mathbb{E}^v[f(X_n)] = \pi a$

Therefore, we have proved $\lim_{n\to\infty} \mathbb{E}^v[A_n] = \pi a$

Part (c)

Yes. The occurrence of the event "the total reward is more than C" can be determined by looking at the rewards up to step n.

Part (d)

Let's first prove X_T is in state 1.

To prove that, we can assume X_T is in state 2. According to the definition of T_C,

$T_C = min\{n \geq 1 : \sum_{k=1}^{n} f(X_k) \geq 100\}$

If X_T is in state 2, then $f(X_T) = -1$, such that $\sum_{k=1}^{n-1} f(X_k) > \sum_{k=1}^{n} f(X_k) \geq 100$. Here we know T_C is not the first time when total reward up to step n is more than 100, which goes against its definition. So X_T should be in state 1.

So, we can infer:

$\mathbb{P}(f(X_{T+1})|T < \infty) = \mathbb{P}(f(X_{T+1})|X_T = 1)$

$\mathbb{E}(f(X_{T+1})|T < \infty) = \mathbb{E}(f(X_{T+1})|X_T = 1)$

Such that:

$\mathbb{P}(f(X_{T+1}) = 1|X_T = 1) = \mathbb{P}(X_{T+1} = 1|X_T = 1) = 0.5$

$\mathbb{P}(f(X_{T+1}) = -1|X_T = 1) = \mathbb{P}(X_{T+1} = 2|X_T = 1) = 0.5$

$\mathbb{E}(f(X_{T+1})|X_T = 1) = \mathbb{P}(X_{T+1} = 1|X_T = 1)f(1) + \mathbb{P}(X_{T+1} = 2|X_T = 1)f(2) = 0$

In conclusion, we have inferred:

$\mathbb{P}(f(X_{T+1}) = 1|T < \infty) = 0.5$

$\mathbb{P}(f(X_{T+1}) = -1|T < \infty) = 0.5$

$\mathbb{E}(f(X_{T+1})|T < \infty) = 0$

P2

Part (a)

$r_{i,j}^1$ represents Player i tossed coin which shows a 'smailing face', and passed the coin to Player j. And then, j tosses the coin and detonated the bomb.

Therefore, $r_{i,j}^{1}=(1-q)qp_{i,j}$

Given $n=1$, $r_{i,j}^{1}$ represents Player i tossed coin which shows a 'smailing face', and passed the coin to Player j. And then, j tosses the coin and detonated the bomb.

Therefore, $r_{i,j}^{1}=(1-q)qp_{i,j}$

Given $n=2$, $r_{i,j}^{2}$ represents Player i tossed coin which shows a 'smailing face', and the coin will be passed to Player j in 2 steps, during the intermedia step the coin will also show a 'smailing face'. And then, j tosses the coin and detonated the bomb.

Therefore, $r_{i,j}^{2}=(1-q)q^{2}p_{i,j}^{2}$

In this way, we can generalize $r_{i,j}^{n}=(1-q)q^{n}p_{i,j}^{n}$

So, we can infer:

$r'_{i,j}=\sum_{n=1}^{\infty}r_{i,j}^{n}=\sum_{n=1}^{\infty}(1-q)q^{n}p_{i,j}^{n}=(1-q)\sum_{n=1}^{\infty}q^{n}p_{i,j}^{n}$

By definition, $p_{i,j}^{0}$ means the 0-step transition probability of $i\rightarrow j$. We can infer $p_{i,j}^{0}=0$ if $i\neq j$ and $p_{i,j}^{0}=1$ if $i=j$.

So, $r_{i,j}^{0}=(1-q)q^{0}p_{i,j}^{0}=(1-q)$, represents the bomb detonated when Player i holds it at step 0.

Therefore, we can include this term into the formula of $r'_{i,j}$, such that:

$r_{i,j}=(1-q)\sum_{n=0}^{\infty}q^{n}p_{i,j}^{n}$

Part (b)

Based on results of Part (a), we can vectorize the calculation to get the formula of R:

$(1)\ R=IR=(1-q)\sum_{n=0}^{\infty}q^{n}P^{n}$

Multiply by qP on both sides we have:

$(2)\ qPR=(1-q)\sum_{n=1}^{\infty}q^{n}P^{n}$

By $(1)-(2)$ we have:

$(I-qP)R=(1-q)q^{0}P^{0}=(1-q)I$

Therefore, $R=(1-q)(I-qR)^{-1}$

If $q = 1$, the right hand side is not well defined. Because in this case, $I - qR$ will become a singular matrix, which does not have a matrix inverse. To further explain that, if $q = 1$, there is no chance that bomb will be detonated, which means the coin will be passed on forever. So, in this case, R should be a matrix of all 0s. But each row of R is a probability distribution, which should sum up to 1. As a conclusion, R wil not be well-defined if $q = 1$.

Part (c)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

Calculating with Part (b)'s result, we have:

$$R = \begin{bmatrix} 0.40383907 & 0.15156634 & 0.18918919 & 0.10383907 & 0.15156634 \\ 0.15156634 & 0.40383907 & 0.18918919 & 0.15156634 & 0.10383907 \\ 0.14189189 & 0.14189189 & 0.43243243 & 0.14189189 & 0.14189189 \\ 0.10383907 & 0.15156634 & 0.18918919 & 0.40383907 & 0.15156634 \\ 0.15156634 & 0.10383907 & 0.18918919 & 0.15156634 & 0.40383907 \end{bmatrix}$$

Therefore, the probability distribution of the event where Player j is the loser for any $1 \leq j \leq 5$ conditioned on Player 1 being the initial bomb holder is $[0.40383907, 0.15156634, 0.18918919, 0.10383907, 0.15156634]$

Part (d)

The stationary distribution for MC above is: [0.1875, 0.1875, 0.25 , 0.1875, 0.1875]

Part (e)

Let $q = 0.999999$

$$R_q = \begin{bmatrix} 0.18749999 & 0.18749999 & 0.24999999 & 0.18749999 & 0.18749999 \\ 0.18749999 & 0.18749999 & 0.24999999 & 0.18749999 & 0.18749999 \\ 0.18749999 & 0.18749999 & 0.24999999 & 0.18749999 & 0.18749999 \\ 0.18749999 & 0.18749999 & 0.24999999 & 0.18749999 & 0.18749999 \\ 0.18749999 & 0.18749999 & 0.24999999 & 0.18749999 & 0.18749999 \end{bmatrix}$$

From above results, we can observe that each row of the matrix R_q converges to a stationary distribution of the MC.

P3

Part (a)

$$p^2 = \begin{bmatrix} 0.44 & 0.56 & 0 & 0 \\ 0.64 & 0.36 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

Part (b)

(1) The stationary distribution of p

$$\det(P^T - I) = 0$$

$$P^T - I = \begin{bmatrix} -1 & 0 & 0.8 & 0.4 \\ 0 & -1 & 0.2 & 0.6 \\ 0.1 & 0.6 & -1 & 0 \\ 0.9 & 0.4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -10 & 0 \\ 9 & 4 & 0 & -10 \\ -10 & 0 & 8 & 4 \\ 0 & -10 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -10 & 0 \\ 0 & -50 & 90 & -10 \\ 0 & 60 & -92 & 4 \\ 0 & -10 & 2 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 6 & -10 & 0 \\ 0 & -10 & 2 & 6 \\ 0 & 0 & 80 & -40 \\ 0 & 0 & -80 & 40 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -10 & 0 \\ 0 & -10 & 2 & 6 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Say $v = (v_1, v_2, v_3, v_4)$ is an eigenvector for P

Then,

$2v_3 - v_4 = 0 \to v_3 = 0.5v_4$

$-10v_2 + 2v_3 + 6v_4 = 0 \to v_2 = 0.7v_4$

$v_1 + 6v_2 - 10v_3 = 0 \to v_1 = 0.8v_4$

After normalization, $v = (\frac{4}{15}, \frac{7}{30}, \frac{1}{6}, \frac{1}{3})$

(2) The stationary distribution of p^2

$$(P^2)^T - I = \begin{bmatrix} 0.44-1 & 0.64 & 0 & 0 \\ 0.56 & 0.36-1 & 0 & 0 \\ 0 & 0 & 0.2-1 & 0.4 \\ 0 & 0 & 0.8 & 0.6-1 \end{bmatrix} = \begin{bmatrix} -0.56 & 0.64 & 0 & 0 \\ 0.56 & -0.64 & 0 & 0 \\ 0 & 0 & -0.8 & 0.4 \\ 0 & 0 & 0.8 & -0.4 \end{bmatrix}$$
$$= \begin{bmatrix} -0.56 & 0.64 & 0 & 0 \\ 0 & 0 & 0.8 & -0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -8 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Say $v' = (v'_1, v'_2, v'_3, v'_4)$ is an eigenvector of P^2

Then,

$7v'_1 - 8v'_2 = 0$

$2v'_3 - v'_4 = 0$

So, there are at least 2 stationary distributions of P^2, they are:

$v' = (\frac{8}{15}, \frac{7}{15}, 0, 0)$

$v'' = (0, 0, \frac{1}{3}, \frac{2}{3})$

The combination of v' and v'' could also be the stationary distributions of P^2.

Therefore, the stationary distributions of P^2 could be expressed in following formula:

$av' + bv'' = (\frac{8}{15}a, \frac{7}{15}a, \frac{1}{3}b, \frac{2}{3}b)$ for $(a, b) \sim 0 \leq a, b \leq 1 \text{ and } a + b = 1$

Part (c)

As $n \to \infty$, p^{2n} is a matrix with each row converge to the stationary distribution.

Such that,

$$p^{2n} = \begin{bmatrix} \frac{8}{15} & \frac{7}{15} & 0 & 0 \\ \frac{8}{15} & \frac{7}{15} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Therefore,

$p^{2n}(1, 1) = \frac{8}{15}$

$p^{2n}(2, 2) = \frac{7}{15}$

$p^{2n}(3, 3) = \frac{1}{3}$

$p^{2n}(4, 4) = \frac{2}{3}$

P4

$$p^n(i, j) = \begin{bmatrix} 0 & 0 & 0.18333333 & 0.09166667 & 0.34117647 & 0.21323529 & 0.17058824 \\ 0 & 0 & 0.26666667 & 0.13333333 & 0.28235294 & 0.17647059 & 0.14117647 \\ 0 & 0 & 0.66666667 & 0.33333333 & 0 & 0 & 0 \\ 0 & 0 & 0.66666667 & 0.33333333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.47058824 & 0.29411765 & 0.23529412 \\ 0 & 0 & 0 & 0 & 0.47058824 & 0.29411765 & 0.23529412 \\ 0 & 0 & 0 & 0 & 0.47058824 & 0.29411765 & 0.23529412 \end{bmatrix}$$

P5

Part (a)

Looking for the stationary distribution of $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$

$$P^T - I = \begin{bmatrix} 0.7-1 & 0.2 \\ 0.3 & 0.8-1 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} = \begin{bmatrix} 0.3 & -0.2 \\ 0 & 0 \end{bmatrix}$$

Say $v = (v_1, v_2)$ is the stationary distribution for P

Then,

$0.3v_1 - 0.2v_2 = 0$

After normalization, $v = (\frac{2}{5}, \frac{3}{5})$

Therefore, the limiting market share of Brand A, B is $(\frac{2}{5}, \frac{3}{5})$

Part (b)

Because a consumer who changes brands would pick another brand at random, the new transition matrix could be represented as:

$$P' = \begin{bmatrix} 0.7 & a & a \\ b & 0.8 & b \\ c & c & 0.9 \end{bmatrix}$$

As a transition matrix, each row of P should sum up to 1. Therefore, we can infer: $a = 0.15, \quad b = 0.1, \quad c = 0.05.$

$$P' = \begin{bmatrix} 0.7 & 0.15 & 0.15 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.9 \end{bmatrix}$$

The new limiting market share could be obtained in the following approach:

$$(P')T - I = \begin{bmatrix} -0.3 & 0.1 & 0.05 \\ 0.15 & -0.2 & 0.05 \\ 0.15 & 0.1 & -0.1 \end{bmatrix} = \begin{bmatrix} 0.15 & -0.2 & 0.05 \\ 0 & -0.3 & 0.15 \\ 0 & 0.3 & -0.15 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Say $v' = (v_1', v_2', v_3')$ is the stationary distribution for P'

Then,

$2v'_2 - v'_3 = 0 \rightarrow v'_2 = \frac{1}{2}v'_3$

$3v'_1 - 4v'_2 + v'_3 = 0 \rightarrow v'_1 = \frac{1}{3}v'_3$

After normalization, $v' = (\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$

Therefore, the limiting market share of Brand A, B, C is $(\frac{2}{11}, \frac{3}{11}, \frac{6}{11})$

P6

Part (a)

When $X_n = 0$, no umbrella can be taken away, so the new location should have 3 umbrellas.Therefore $P(X_{n+1} = 3 \mid X_n = 0) = 1$

When $X_n = 1$, if it's raining 1 umbrella will be taken away, so the new location should have 3 umbrellas. Therefore, $P(X_{n+1} = 3 \mid X_n = 1) = 0.2$. If it's not raining, the new location should have 2 umbrellas. Therefore, $P(X_{n+1} = 2 \mid X_n = 1) = 0.8$.

When $X_n = 2$, if it's raining 1 umbrella will be taken away, so the new location should have 2 umbrellas. Therefore, $P(X_{n+1} = 2 \mid X_n = 2) = 0.2$. If it's not raining, the new location should have 1 umbrellas. Therefore, $P(X_{n+1} = 1 \mid X_n = 2) = 0.8$.

When $X_n = 3$, if it's raining 3 umbrella will be taken away, so the new location should have 1 umbrellas. Therefore, $P(X_{n+1} = 1 \mid X_n = 3) = 0.2$. If it's not raining, the new location should have 0 umbrellas. Therefore, $P(X_{n+1} = 0 \mid X_n = 3) = 0.8$.

So, we can infer the transition matrix to be:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.8 & 0.2 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{bmatrix}$$

Part (b)

The stationary distribution of P could be calculated in following steps:

$$P^T - I = \begin{bmatrix} -1 & 0 & 0 & 0.8 \\ 0 & -1 & 0.8 & 0.2 \\ 0 & 0.8 & -0.8 & 0 \\ 1 & 0.2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 & 0 & -1 \\ 0 & 0.2 & 0 & -0.2 \\ 0 & -1 & 0.8 & 0.2 \\ 0 & 0.8 & -0.8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.2 & 0 & -1 \\ 0 & 0.2 & 0 & -0.2 \\ 0 & 0 & 0.8 & -0.8 \\ 0 & 0 & -0.8 & 0.8 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0.2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Say $v = (v_0, v_1, v_2, v_3)$ is the stationary distribution for P

Then,

$v_2 - v_3 = 0 \rightarrow v_2 = v_3$

$v_1 - v_3 = 0 \rightarrow v_1 = v_3$

$v_0 + 0.2v_1 - v_3 = 0 \rightarrow v_0 = 0.8v_3$

After normalization, $v = (\frac{4}{19}, \frac{5}{19}, \frac{5}{19}, \frac{5}{19})$

The probability for the individual getting wet is:

$$P(rain, X_n = 0) = P(rain)P(X_n = 0) = \frac{4}{95}$$

P7

Part (a)

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Part (b)

$$P^T - I = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -1 & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Say $v = (v_0, v_1, v_2, v_3)$ is the stationary distribution for P

Then,

$2v_2 - v_3 = 0 \rightarrow v_2 = \frac{1}{2}v_3$

$2v_1 - v_2 = 0 \rightarrow v_1 = \frac{1}{2}v_2 = \frac{1}{4}v_3$

$2v_0 - v_1 = 0 \rightarrow v_0 = \frac{1}{2}v_1 = \frac{1}{8}v_3$

After normalization, $v = (\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15})$

The limiting amount of time the chain spends at each site is $(\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15})$

P8

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{24} & \frac{7}{8} & \frac{1}{12} & 0 \\ \frac{1}{36} & 0 & \frac{8}{9} & \frac{1}{12} \\ \frac{1}{8} & 0 & 0 & \frac{7}{8} \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{24} & \frac{1}{36} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{1}{8} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{12} & \frac{1}{36} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{1}{8} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{12} & \frac{1}{8} \\ \frac{1}{2} & -\frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{1}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{12} & -\frac{1}{9} & 0 \\ 0 & 0 & \frac{1}{12} & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Say $v = (v_0, v_1, v_2, v_3)$ is the stationary distribution for P

Then,

$2v_2 - 3v_3 = 0 \rightarrow v_2 = \frac{3}{2}v_3$

$3v_1 - 4v_2 = 0 \rightarrow v_1 = \frac{4}{3}v_2 = 2v_3$

$4v_0 - v_1 = 0 \rightarrow v_0 = \frac{1}{4}v_1 = \frac{1}{2}v_3$

After normalization, $v = (\frac{1}{10}, \frac{2}{5}, \frac{3}{10}, \frac{1}{5})$

Therefore, the limiting fraction of land in each of the states is $(\frac{1}{10}, \frac{2}{5}, \frac{3}{10}, \frac{1}{5})$