HW5

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P1

Part (a)

Starting from Cell 95, the player could only reach Cell from 95 to 100.

We could treat Cell 98 as the losing state, because once the player reaches Cell 98, he would immediately lose the game. Besides, we can eliminate state 96 by adding its distribution to state 99.

Therefore,

The absorbing states are $\{98, 100\}$

The transient states are $\{95, 96, 97, 99\}$

Then, the transition matrix is:

$$P = \begin{array}{c} \\ 95 \\ 97 \\ 99 \\ 98 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccccc} 95 & 97 & 99 & 98 & 100 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

According to Theorem 1.27, we have: $\tilde{Q} = (I - A)^{-1} B$

$$A = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{2} \\ 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

Therefore,

$$\tilde{Q} = \begin{bmatrix} \frac{7}{13} & \frac{6}{13} \\ \frac{8}{13} & \frac{5}{13} \\ \frac{7}{13} & \frac{6}{13} \end{bmatrix}$$

As a result, the probability that the player wins the game is $\frac{6}{13}$

Part (b)

According to Theorem 1.28, $\tilde{m} = (I - A)^{-1}\mathbb{I}, \mathbb{I} = (1, \dots, 1)^T$

Therefore,

$$\tilde{m} = \begin{bmatrix} \frac{36}{13} \\ \frac{30}{13} \\ \frac{36}{13} \end{bmatrix}$$

The expected time to end the game is $\frac{36}{13}$.

P2

Part (a)

Since this is a random walk on a connected graph, according to Theorem 1.25, it satisfies the detailed balanced condition with distribution:

$$\pi = \{\frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{2}{16}, \frac{2}{16}, \frac{4}{16}, \frac{1}{16}, \frac{1}{16}\}$$

Part (b)

The transition matrix is:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Since the first column's sum is not 1, this transition matrix is not a doubly stochastic matrix.

Part (c)

According to Theorem 1.27, we have: $\tilde{Q} = (I - A)^{-1} B$

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Therefore:

$$\tilde{Q} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.714 & 0.142 & 0.142 \\ 0.428 & 0.286 & 0.286 \\ 0.286 & 0.357 & 0.357 \\ 0.286 & 0.357 & 0.357 \\ 0.143 & 0.428 & 0.428 \end{bmatrix} \end{array}$$

Part (d)

Let $f(X)$ denote the expected reward at state X. The expected reward is:

$$E(\text{reward}) = \sum_{i=2}^{6} \sum_{j \in \{1,7,8\}} P^i (X_n = j) * f(X_n = j) = 5.629$$

The expected reward is 5.629

Part (e)

According to Theorem 1.28, $\tilde{m} = (I - A)^{-1}\mathbb{I}, \mathbb{I} = (1, \ldots, 1)^T$

Therefore,

$$\tilde{m} = \begin{bmatrix} 4.428 \\ 6.857 \\ 6.571 \\ 6.571 \\ 4.286 \end{bmatrix}$$

P3

Part (a)

According to the question, we take state 3, 4 as absorbing state.

So, the modified transition matrix is:

$$P' = \begin{bmatrix} 0.25 & 0.75 & 0 & 0 \\ 0.6 & 0 & 0.1 & 0.3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.25 & 0.75 \\ 0.6 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.3 \end{bmatrix}$$

According to Theorem 1.28, $\tilde{m} = (I - A)^{-1}\mathbb{I}, \mathbb{I} = (1, \ldots, 1)^T$

Therefore,

$$\tilde{m} = \begin{bmatrix} 5.833 \\ 4.5 \end{bmatrix}$$

Starting at number 1, the expected time to obtain the first 3 or 4 is 5.833.

Part (b)

According to Theorem 1.27, we have: $\tilde{Q} = (I - A)^{-1}B$

Therefore,

$$\tilde{Q} = \begin{bmatrix} 0.25 & 0.75 \\ 0.25 & 0.75 \end{bmatrix}$$

Starting at number 1, the probability of obtaining number 3 before the number 4 is 0.25.

Part (c)

The stationary distribution for P is:

$$\pi = \{\frac{16}{54}, \frac{15}{54}, \frac{10}{54}, \frac{13}{54}\}$$

Because I, S hold, according to Theorem 1.22, $E^1T_1 = \pi_1^{-1} = \frac{27}{8}$

P4

Part (a)

The probability of stationary distribution for any state is $\frac{1}{12}$.

Because I, S hold, according to Theorem 1.22, for any state $i \in \{1, \ldots, 12\}$, the expected return time is $E^iT_i = \pi_i^{-1} = 12$

Part (b)

Because this is a symmetric system, the probability X_n will visit all the other states before returning to its starting position equals for any starting position.

So, let's assume the starting position is 1. There are two directions for X_n to visit all the other states: clockwise and counter-clockwise.

Let's start with the clockwise situation.

Step1: $X_0 = 1 \to X_1 = 2$

Step2: $X_1 = 2 \to X_2 = 3$

So, now we treat state 1 and state 2 as abosrbing state, if X_n reach state 1 before reach state 2, it will visit all the other states.

So, the modified transition matrix is:

$$P' = \begin{array}{c c} & \begin{array}{cccccccccccc} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$A = \begin{array}{c c} & \begin{array}{cccccccccc} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \end{array}$$

$$B = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix} \end{array}$$

According to Theorem 1.27, we have: $\tilde{Q} = (I - A)^{-1}B$

Therefore:

$$\tilde{Q} = \begin{bmatrix} \frac{1}{11} & \frac{10}{11} \\ \cdots & \cdots \end{bmatrix}$$

Therefore, the probability for returning at state 1 without hitting state 2 is $\frac{1}{11}$.

Besides, every time reaching state 2, there is a probability of $\frac{1}{2}$ to reach state 3 again. Therefore, the probability for going clockwise and return to 1 can be calculated by:

$$P^1(X_n = 1 \mid X_2 = 3, \text{clockwise visiting all states}) = \frac{1}{11} + \frac{1}{11} * \frac{10}{11} * \frac{1}{2} + \frac{1}{11} * (\frac{10}{11} * \frac{1}{2})^2 + \frac{1}{11} * (\frac{10}{11} * \frac{1}{2})^3 + \cdots = \frac{1}{6}$$

Similarly, $P^1(X_n = 1 \mid X_2 = 3, \textbf{counter-clockwise visiting all states}) = \frac{1}{6}$

Therefore, the probability Xn will visit all the other states before returning to its starting position is $\frac{1}{3} * (\frac{1}{6} + \frac{1}{6}) = \frac{1}{12}$

P5

Part (a)

The degrees of the vertices are given by the following table:

$$\begin{bmatrix} 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 5 & 8 & 8 & 8 & 8 & 8 & 8 & 5 \\ 3 & 5 & 5 & 5 & 5 & 5 & 5 & 3 \end{bmatrix}$$

The sum of the degree is 420, so the stationary probabilities are the degrees divided by 420.

Part (b)

Because I, S hold, according to Theorem 1.22, the expected return time for (1,1) is $E^{(1,1)} T_{(1,1)} = \pi_{(1,1)}^{-1} = 140$

P6

Part (a)

Let's denote each child use the first letter of the name. The transition matrix is:

$$P = \begin{array}{c} \\ \mathrm{D} \\ \mathrm{H} \\ \mathrm{S} \\ \mathrm{J} \\ \mathrm{T} \\ \mathrm{M} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathrm{D} & \mathrm{H} & \mathrm{S} & \mathrm{J} & \mathrm{T} & \mathrm{M} \end{array} \\ \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

State D, H, S are transient.

State J, T, M are recurrent. Specifically, state M is absorbing.

Part (b)

Since when J and T has the ball, they will only pass the ball to each other. In this perspective, they together form an 'absorbing' state. Let's using state R to denote the event that J or T have the ball.

Therefore, the modified transition matrix P' is:

$$
P = \begin{array}{c} \\ \mathrm{D} \\ \mathrm{H} \\ \mathrm{S} \\ \mathrm{R} \\ \mathrm{M} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \mathrm{D} & \mathrm{H} & \mathrm{S} & \mathrm{R} & \mathrm{M} \end{array} \\
\begin{bmatrix}
0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

In this way, we have constructed a markov chain with 3 transient states and 2 absorbing states.

According to Theorem 1.27, we have: $\tilde{Q} = (I - A)^{-1} B$

Therefore:

$$
A = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}
$$

$$
B = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}
$$

$$
\tilde{Q} = \begin{bmatrix} 0.6 & 0.4 \\ \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}
$$

Therefore, starting with Dick, the probability that Mark will end up with the ball is 0.4

P7

Part (a)

The transition matrix is:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Part (b)

According to Theorem 1.28, $\tilde{m} = (I - A)^{-1}\mathbb{I}$, $\mathbb{I} = (1, \ldots, 1)^T$

And based on the transition matrix, we know:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \end{bmatrix}$$

Therefore, $ET = 14.7$