# COMP 2804 — Solutions Assignment 1

**Question 1:**

- Write your name and student number.

**Solution:** James Bond, 007

**Question 2:** Recall that $K_n$, where $n \geq 2$ is an integer, denotes the *complete graph* with $n$ vertices. This graph has $\binom{n}{2}$ edges, i.e., one edge for each pair of distinct vertices.

Let $k \geq 1$ be an integer and let $L$ be a set of size $k$. If each edge of $K_n$ is labeled with one element of $L$, then we say that $K_n$ is *$k$-labeled.* Three pairwise distinct vertices $u$, $v$, and $w$ of $K_n$ define a *boring* triangle, if the three edges $uv$, $uw$, and $vw$ have the same label.

In this question, you will prove the following statement $S(n, k)$ for all integers $n$ and $k$ with $n \geq 3 \cdot k!$:

$$S(n, k)\text{: Every } k\text{-labeled } K_n \text{ has a boring triangle.}$$

(Note that in class, we have proved statement $S(n, 2)$ for all $n \geq 6$.) The proof will be by induction on $k$.

- Base case: Explain, in a few sentences, why statement $S(n, 1)$ is true for all integers $n \geq 3$.

  **Solution:** Let $n \geq 3$. Statement $S(n, 1)$ says that every 1-labeled $K_n$ has a boring triangle. In a 1-labeling, all edges have the same label and, thus, every triangle is boring. Since $n \geq 3$, $K_n$ has at least one triangle.

- Induction step: Let $k \geq 2$ and assume that statement $S(m, k-1)$ is true for all integers $m \geq 3 \cdot (k-1)!$. Prove that statement $S(n, k)$ is true for all integers $n \geq 3 \cdot k!$.

  *Hint:* Let $u$ be an arbitrary vertex of $K_n$. Argue that among all edges that have $u$ as a vertex, at least $\lceil (n-1)/k \rceil$ have the same label.

  **Solution:** Let $n \geq 3 \cdot k!$ and consider an arbitrary $k$-labeled $K_n$. Let $u$ be an arbitrary vertex of $K_n$. We know that there are $n - 1$ edges that have $u$ as a vertex. Each of these edges has a label from a set of size $k$.

  We first argue that among all these $n-1$ edges, at least $(n-1)/k$ have the same label. If this is not the case, then each label occurs less than $(n-1)/k$ times. But then, the number of edges having $u$ as a vertex is less than $k \cdot (n-1)/k = n - 1$. This is a contradiction.

  Consider a label, say "red", that occurs at least $(n-1)/k$ times among these $n-1$ edges. Then "red" occurs at least $\lceil (n-1)/k \rceil$ times among these edges. Observe that

$$\left\lceil \frac{n-1}{k} \right\rceil \geq \left\lceil \frac{3 \cdot k! - 1}{k} \right\rceil = \left\lceil 3 \cdot (k-1)! - \frac{1}{k} \right\rceil = 3 \cdot (k-1)!.$$

Consider the subgraph $G$ of $K_n$ whose vertices are the endpoints of all "red" edges; $u$ itself is not part of this subgraph. Note that $G$ is a complete graph on at least $3 \cdot (k-1)!$ vertices.

**Case 1:** $G$ contains a "red" edge, say $vw$.

Then $uvw$ is a boring triangle in $K_n$.

**Case 2:** All edges of $G$ are not "red".

In this case, $G$ is a complete graph on at least $3 \cdot (k-1)!$ vertices, and its edges have labels from a set of size $k-1$. Thus, by induction, $G$ contains a boring triangle. This triangle is also boring in $K_n$.

**Question 3:** Let $k \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, k\}$. For any integer $n \geq 2$, determine the number of sequences $(a_1, a_2, \ldots, a_n)$ of length $n$, where each element belongs to $S$ and no two consecutive elements are the same.

**Solution:** The procedure is "write a sequence $(a_1, a_2, \ldots, a_n)$ of length $n$, where each element belongs to $S$ and no two consecutive elements are the same". It is natural to write such a sequence from left to right. Thus, for $i = 1, 2, \ldots, n$, the $i$-th task is "write $a_i$".

1. There are $k$ ways to do the first task.
2. For the second task, we have $k-1$ choices, because $a_2$ must be different from $a_1$.
3. For the third task, we have $k-1$ choices, because $a_3$ must be different from $a_2$.
4. In general, for $i = 2, 3, \ldots, n$, there are $k-1$ choices for the $i$-th task, because $a_i$ must be different from $a_{i-1}$.

By the Product Rule, the total number of ways to do the entire procedure is equal to

$$k \cdot \underbrace{(k-1) \cdot (k-1) \cdots (k-1)}_{n-1 \text{ times}} = k \cdot (k-1)^{n-1}.$$

**Question 4:** Zoltan's House of Pizza is a popular restaurant in downtown Ottawa. The daily dinner special is Zoltan's Meal Deal: You choose between standard dough and gluten free dough; you choose one of chicken, beef, pork, and shrimp; you choose one out of five different sauces; you choose three out of seven different cheeses; you choose any subset of twenty different toppings; and, finally, you choose four out of nine different beers.

- Determine the number of Zoltan's Meal Deals, if the cheeses chosen must be distinct and the beers chosen must be distinct.

  **Solution:** The procedure is "order a Zoltan's Meal Deal". The question specifies the different tasks:

1. Task 1: Choose the dough. There are 2 ways to do this.
2. Task 2: Choose the protein. There are 4 ways to do this.
3. Task 3: Choose the sauce. There are 5 ways to do this.
4. Task 4: Choose a *subset* of three cheeses. There are $\binom{7}{3} = 35$ ways to do this.
5. Task 5: Choose the toppings. There are $2^{20} = 1,048,576$ ways to do this.
6. Task 6: Choose a *subset* of four beers. There are $\binom{9}{4} = 126$ ways to do this.

By the Product Rule, the total number of Zoltan's Meal Deals is equal to

$$2 \cdot 4 \cdot 5 \cdot 35 \cdot 1,048,576 \cdot 126 = 184,968,806,400.$$

- Determine the number of Zoltan's Meal Deals, if the cheeses chosen are not necessarily distinct and the beers chosen are not necessarily distinct.

  **Solution:** The only difference is in Tasks 4 and 6.

  Task 4: We start with a **wrong** way of counting: Choose 3 out of 7 different cheeses, where we may choose a cheese more than once. By the Product Rule, the number of ways to do this is equal to $7 \cdot 7 \cdot 7 = 343$. Why is this wrong: If we choose Feta twice and Mozarella once, then we count this option three times: We count it as (F,F,M) and (F,M,F) and (M,F,F).

  The correct way of counting is as follows:

  1. Three different types of cheese. There are $\binom{7}{3} = 35$ ways to choose these.
  2. Two different types of cheese. To count this, we choose two types of cheese. One of the two chosen types we take twice; the other type, we take once. The number of ways to do this is $\binom{7}{2} \cdot 2 = 42$.
  3. One type of cheese. There are 7 ways to do this.
  4. Overall, the number of ways to do Task 4 is equal to

  $$35 + 42 + 7 = 84.$$

  Here is a different way to get the same answer: We number the cheeses as $1, 2, \ldots, 7$. Then we are looking for the number of solutions of the equation

  $$x_1 + x_2 + \ldots + x_7 = 3,$$

  where $x_1, x_2, \ldots, x_7$ are non-negative integers. We have seen in class that the number of solutions is equal to $\binom{9}{6} = 84$.

  Task 6: We choose 4 out of 9 different beers, where we may choose a beer more than once. The number of ways to do this is equal to the number of solutions of the equation

  $$x_1 + x_2 + \ldots + x_9 = 4,$$

where $x_1, x_2, \ldots, x_9$ are non-negative integers. We have seen in class that the number of solutions is equal to $\binom{12}{8} = 495$.

Here is a different way to get the same answer:

1. All types of beer are distinct: $\binom{9}{4} = 126$.
2. Pick three types of beer, then pick one of the three to be double ordered: $\binom{9}{3} \cdot 3 = 252$.
3. Pick two types of beer, then pick how many beers of the first type you will order: $\binom{9}{2} \cdot 3 = 108$.
4. Pick one type of beer: 9.
5. This gives a total of
$$126 + 252 + 108 + 9 = 495.$$

The total number of Zoltan's Meal Deals is equal to

$$2 \cdot 4 \cdot 5 \cdot 84 \cdot 1,048,576 \cdot 495 = 1,743,991,603,200.$$

**Question 5:** This fall term, 230 students have registered for COMP 2804B. While watching video lectures, 150 students do not wear pants and 110 students drink beer.

Determine the best possible lower bound on the number of students who do not wear pants and drink beer while watching video lectures.

**Solution:** Let $S$ be the set of all 2804B-students, let $A$ be the set of all pantless students in $S$, and let $B$ be the set of all beer-drinking students in $S$. The question asks for a lower bound on the size of $A \cap B$. We are given that $|S| = 230$, $|A| = 150$, and $|B| = 110$.

Since $A \cup B \subseteq S$, we have
$$|A \cup B| \leq |S| = 230.$$
By the Principle of Inclusion-Exclusion, we have
$$|A \cup B| = |A| + |B| - |A \cap B| = 150 + 110 - |A \cap B| = 260 - |A \cap B|.$$
By combining this, we get
$$230 \geq 260 - |A \cap B|,$$
implying that
$$|A \cap B| \geq 30.$$
This is the best possible lower bound: Number the students from 1 to 230. If the first 150 students are pantless and the last 110 students drink beer, then $|A \cap B| = 30$.

**Question 6:** In class, we have seen how to determine the number of solutions of the equation
$$x_1 + x_2 + x_3 = 99, \tag{1}$$
where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.

- Determine the number of solutions of (1), where $x_1 \geq 40$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 55$ are integers.
- Determine the number of solutions of (1), where $x_1 \geq 40$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers.
- Use the Principle of Inclusion and Exclusion to determine the number of solutions of (1), where $0 \leq x_1 \leq 39$, $0 \leq x_2 \leq 49$, and $0 \leq x_3 \leq 54$ are integers.

**Solution:** We have seen the following in class: If $n \geq 0$ is an integer, then the number of solutions of the equation

$$x_1 + x_2 + x_3 = n, \tag{2}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers, is equal to $\binom{n+2}{2}$.

1. We start with (1), where $x_1 \geq 40$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers. We write the equation as

   $$(x_1 - 40) + x_2 + x_3 = 59. \tag{3}$$

   If we define $y_1 = x_1 - 40$, then this becomes

   $$y_1 + x_2 + x_3 = 59. \tag{4}$$

   Note that $x_1 \geq 40$ if and only if $y_1 \geq 0$. This implies that counting the solutions of (3), where $x_1 \geq 40$, $x_2 \geq 0$, and $x_3 \geq 0$, is the same as counting the solutions of (4), where $y_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

   Using (2), the answer to this part of the question is

   $$\binom{61}{2} = 1830.$$

2. Next we do (1), where $x_1 \geq 0$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers. We write the equation as

   $$x_1 + (x_2 - 50) + x_3 = 49. \tag{5}$$

   If we define $y_2 = x_2 - 50$, then this becomes

   $$x_1 + y_2 + x_3 = 49. \tag{6}$$

Note that $x_2 \geq 50$ if and only if $y_2 \geq 0$. This implies that counting the solutions of (5), where $x_1 \geq 0$, $x_2 \geq 50$, and $x_3 \geq 0$, is the same as counting the solutions of (6), where $x_1 \geq 0$, $y_2 \geq 0$, and $x_3 \geq 0$.

Using (2), the answer to this part of the question is

$$\binom{51}{2} = 1275.$$

3. Next we do (1), where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 55$ are integers. We write the equation as

   $$x_1 + x_2 + (x_3 - 55) = 44.$$

   Using the same reasoning as above, the answer to this part of the question is

   $$\binom{46}{2} = 1035.$$

4. Next we do (1), where $x_1 \geq 40$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers. We write the equation as

   $$(x_1 - 40) + (x_2 - 50) + x_3 = 9.$$

   Using the same reasoning as above, the answer to this part of the question is

   $$\binom{11}{2} = 55.$$

5. Finally, we do (1), where $0 \leq x_1 \leq 39$, $0 \leq x_2 \leq 49$, and $0 \leq x_3 \leq 54$ are integers.

   The parts above suggest that we should use the Complement Rule. The negation of "$0 \leq x_1 \leq 39$ and $0 \leq x_2 \leq 49$ and $0 \leq x_3 \leq 54$" is "$x_1 \geq 40$ or $x_2 \geq 50$ or $x_3 \geq 55$".

   We define the following sets:

   (a) $U$ is the set of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.

   (b) $A$ is the set of solutions of (1), where $x_1 \geq 40$, $x_2 \geq 0$, and $x_3 \geq 0$ are integers.

   (c) $B$ is the set of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 50$, and $x_3 \geq 0$ are integers.

   (d) $C$ is the set of solutions of (1), where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 55$ are integers.

   We want to determine the value of

   $$|U| - |A \cup B \cup C|.$$

   Using (2), we have $|U| = \binom{101}{2} = 5050$. By Inclusion-Exclusion, we have

   $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

We have already seen that $|A| = 1830$, $|B| = 1275$, $|C| = 1035$, and $|A \cap B| = 55$.

To determine $|A \cap C|$, we write (1) as

$$(x_1 - 40) + x_2 + (x_3 - 55) = 4.$$

From this, we get $|A \cap C| = \binom{6}{2} = 15$.

What is $|B \cap C|$? If $x_1 \geq 0$, $x_2 \geq 50$ and $x_3 \geq 55$, then $x_1 + x_2 + x_3 > 99$. Thus, $|B \cap C| = 0$. By the same reasoning, $|A \cap B \cap C| = 0$.

Thus,

$$|A \cup B \cup C| = 1830 + 1275 + 1035 - 55 - 15 - 0 + 0 = 4070.$$

The final answer to this part of the question is

$$5050 - 4070 = 980.$$

**Question 7:** Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. A permutation of $S$ can be regarded as a bijection $f : S \to S$. For any integer $i \geq 1$, define

$$f^i(x) = \underbrace{f(f(f(\cdots f}_{i \text{ times}}(x)\cdots))).$$

In words, to obtain $f^i(x)$, we start with $x$ and apply the function $i$ times.

If $x$ is an element of $S$, then the *cycle* of $x$ is the sequence

$$(x, f^1(x), f^2(x), \ldots, f^{m-1}(x)),$$

where $m$ is the smallest positive integer such that $f^m(x) = x$.

The *cycle decomposition* of $f$ is obtained as follows:

**Step 1:** Initially, all elements of $S$ are unmarked and the cycle decomposition is empty.

**Step 2:** Repeat the following, until all elements of $S$ have been marked: Take the smallest unmarked element, say $x$, in $S$. Mark all elements in the cycle of $x$, and add this cycle to the cycle decomposition.

For example, the cycle decomposition of the permutation

| $f(x)$ | 7 | 4 | 6 | 8 | 1 | 3 | 5 | 2 |
|---|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

is given by

$$(1, 7, 5), (2, 4, 8), (3, 6).$$

This cycle decompositions consists of three cycles, having lengths 3, 3, and 2, respectively.

- Let $x$ be an element of $S$. Explain why there exists a positive integer $m$ such that $f^m(x) = x$.

  **Solution:** Consider the infinite sequence

  $$x, f^1(x), f^2(x), f^3(x), \ldots$$

  Since all elements in this sequence belong to the finite set $S$, they cannot all be distinct. Thus, there are integers $k$ and $\ell$ with $\ell > k \geq 0$ such that $f^k(x) = f^\ell(x)$.

  Let $g : S \to S$ be the inverse of $f$. Note that $g$ exists, because $f$ is a bijection. Also note that $f(x) = y$ if and only if $g(y) = x$. Since

  $$g^k\left(f^k(x)\right) = x$$

  and

  $$g^k\left(f^k(x)\right) = g^k\left(f^\ell(x)\right) = f^{\ell-k}(x),$$

  we have

  $$x = g^k\left(f^k(x)\right) = f^{\ell-k}(x).$$

  Thus, if we take $m = \ell - k$, then $m > 0$ and $f^m(x) = x$.

- Prove or disprove the following statement: Let $k$ be an integer with $k > n/2$. The cycle decomposition can have more than one cycle of length $k$.

  **Solution:** Assume the statement is true. Then there are two cycles of length $k$. Since these two cycles are disjoint, together they contain $2k > n$ elements. This is a contradiction, because these elements belong to a set of size $n$. Thus, the cycle decomposition can have at most one cycle of length $k$.

- Let $k$ be an integer with $k > n/2$. Prove that the number of permutations of $S$ whose cycle decompositions contain a cycle of length $k$ is equal to

  $$\binom{n}{k} \cdot (k-1)! \cdot (n-k)!. \tag{7}$$

  **Solution:** The procedure is "specify a permutation whose cycle decompositions contain a cycle of length $k$".

  1. Task 1: Choose a subset $X$ of size $k$. There are $\binom{n}{k}$ ways to do this.
  2. Task 2: Choose a cycle consisting of the elements of $X$:
     (a) The smallest element in $X$ is the first element in the cycle.
     (b) Choose a permutation of the remaining $k-1$ elements of $X$. There are $(k-1)!$ ways to do this.
  3. Task 3: Choose a permutation of the set $S \setminus X$ and convert it to its cycle decomposition. Since $S \setminus X$ has $n-k$ elements, there are $(n-k)!$ ways to do this. (Note that each cycle in this decomposition has less than $k$ elements.)

By the Product Rule, the number of permutations of $S$ whose cycle decompositions contain a cycle of length $k$ is equal to the expression in (7).

- Use algebra to simplify the expression in (7) to $n!/k$.

  **Solution:**

$$\begin{aligned} \binom{n}{k} \cdot (k-1)! \cdot (n-k)! &= \frac{n!}{k!(n-k)!} \cdot (k-1)! \cdot (n-k)! \\ &= n! \cdot \frac{(k-1)!}{k!} \\ &= \frac{n!}{k}. \end{aligned}$$

**Question 8:** Let $n \geq 2$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. Recall that a permutation of $S$ is an ordered sequence $a_1, a_2, \ldots, a_n$, in which every element of $S$ occurs exactly once.

Let $i$ be an integer with $1 \leq i \leq n$. We say that $i$ is a *cool index*, if $a_i \neq i$. We say that $i$ is a *super cool index*, if $i$ is cool and none of the indices $i+1, i+2, \ldots, n$ is cool; in other words, $i$ is the rightmost cool index in the permutation.

For example, in the permutation below, the indices 1 and 4 are cool, whereas the index 4 is super cool.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $a_i$ | 4 | 2 | 3 | 1 | 5 | 6 | 7 | 8 |
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

- Determine the number of permutations of the set $S$ that have at least one cool index.

  **Solution:** We are going to use the Complement Rule:

  1. The total number of permutations is $n!$.
  2. There is only one permutation without any cool index, namely $1, 2, 3, \ldots, n$.
  3. Thus, the number of permutations that have at least one cool index is equal to $n! - 1$.

- Prove or disprove the following statement: There exists a permutation of $S$ in which the index 1 is super cool.

  **Solution:** The statement is false: If the index 1 is not cool, then it is not super cool.

  Assume that the index 1 is cool. Then $a_1 \neq 1$ and there is an index $i \geq 2$ such that $a_i = 1$. Thus, $i$ is also cool and, therefore, 1 is not super cool.

- Let $k$ be an integer with $1 \leq k \leq n-1$. Determine the number of permutations of $S$ in which the index $k+1$ is super cool.

  **Solution:** The permutations in this part are of the form

| $a_i$ | | | | | $\neq k+1$ | $k+2$ | $k+3$ | $\dots$ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | $\dots$ | $k$ | $k+1$ | $k+2$ | $k+3$ | $\dots$ | $n$ |

1. $a_{k+1}$ can be any element in $\{1, 2, \dots, k\}$. Thus, there are $k$ ways to choose $a_{k+1}$.
2. Once $a_{k+1}$ has been chosen, the sequence $a_1, a_2, \dots, a_k$ is a permutation of $\{1, 2, \dots, k+1\} \setminus \{a_{k+1}\}$. There are $k!$ ways to choose this permutation.
3. By the Product Rule, the number of permutations of $S$ in which the index $k+1$ is super cool is equal to $k \cdot k!$.

- Use the above results to prove that

$$\sum_{k=1}^{n-1} k \cdot k! = n! - 1.$$

**Solution:** It is clear that any permutation with at least one cool index has exactly one super cool index. We are going to divide these permutations into groups, based on their super cool index: For any $k$, let $X_k$ be the number of permutations of $S$ in which the index $k+1$ is super cool. We have seen above that only the indices $2, 3, \dots, n$ can be super cool. Thus, we only need the values $k = 1, 2, \dots, n-1$. Note that $\sum_{k=1}^{n-1} X_k$ counts the permutations with at least one cool index, which, as we have seen above, is equal to $n! - 1$. We have also seen that $X_k = k \cdot k!$. We conclude that

$$n! - 1 = \sum_{k=1}^{n-1} X_k = \sum_{k=1}^{n-1} k \cdot k!.$$

**Question 9:** Let $n \geq 1$ be an integer and consider the set $S = \{0, 1, \dots, 3n\}$. For any integer $k$, let $N_k$ be the number of subsets $X$ of $S$ for which $|X| = 2n+1$ and $\min(X) = k$.

- Determine the values of $k$ for which $N_k \neq 0$.

**Solution:** Let $k$ be such that $N_k \neq 0$. Consider a subset $X$ of $S$ for which $|X| = 2n+1$ and $\min(X) = k$.

1. Since $k = \min(X) \in S$, it is clear that $k \geq 0$.
2. Since $X \subseteq \{k, k+1, k+2, \dots, 3n\}$, we have

$$2n + 1 = |X| \leq 3n - k + 1,$$

which implies that $k \leq n$.

- Let $k$ be an integer with $0 \leq k \leq n$. Determine the value of $N_k$.

**Solution:** We are going to use the Product Rule to determine $N_k$, i.e., the number of subsets $X$ of $S$ for which $|X| = 2n+1$ and $\min(X) = k$.

The following tasks specify a set $X$:

1. Task 1: Start with $X = \{k\}$. There is one way to do this.
2. Task 2: Choose a subset $Y$ of $\{k+1, k+2, \ldots, 3n\}$ with $|Y| = 2n$; add this subset to $X$. There are $\binom{3n-k}{2n}$ ways to do this.

By the Product Rule, we have

$$N_k = \binom{3n-k}{2n}.$$

- Use the above results to prove that

$$\sum_{k=0}^{n} \binom{3n-k}{2n} = \binom{3n+1}{n}.$$

**Solution:** We first observe that

$$\binom{3n+1}{n} = \binom{3n+1}{(3n+1)-n} = \binom{3n+1}{2n+1},$$

which is equal to the number of subsets $X$ of $S$ with $|X| = 2n + 1$.

We divide all such subsets $X$ into groups, based on the value of $\min(X)$. For any $k$, the $k$-th group consists of all $X$ for which $\min(X) = k$. We have seen above that this group is non-empty if and only if $0 \leq k \leq n$. By definition, $N_k$ is equal to the size of the $k$-th group.

Observe that each subset $X$ of $S$ with $|X| = 2n+1$ is in exactly one group. Therefore, $\sum_{k=0}^{n} N_k$ counts the total number of such subsets $X$. We conclude that

$$\sum_{k=0}^{n} \binom{3n-k}{2n} = \binom{3n+1}{n}.$$