COMP 2804 — Assignment 3 — Solutions

Note: In these solutions, you will see numerical values for several probabilities. Most of these are rational numbers with huge numerators and huge denominators. The numerical values were are all obtained using Wolfram Alpha. In case you are not aware of Wolfram Alpha, you should familiarize yourself with it.

Question 1:

- Write your name and student number.

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Question 2: A *scrabble hand* is a set of 7 tiles, each having one of the english uppercase letters on them, drawn uniformly at random from a bag of 98 tiles. The number of tiles of each letter are as follows:

E×12, A×9, I×9, O×8, N×6, R×6, T×6, L×4, S×4, U×4, D×4, G×3, B×2, C×2, M×2, P×2, F×2, H×2, V×2, W×2, Y×2, K×1, J×1, X×1, Q×1, Z×1

1. What is the probability that a scrabble hand contains the word OCTAGON?

   First we define the sample space S by numbering the tiles $T_1, \ldots, T_{98}$ so that, even if two tiles have the same letter on them, we treat them like different tiles. Then a scrabble hand is a 7-element subset of $\{T_1, \ldots, T_{98}\}$. There are $\binom{98}{7}$ such subsets and each one is equally likely, so $\Pr(\omega) = 1/\binom{98}{7}$ for each $\omega \in S$.

   Let E be the event "Our hand contains the word OCTAGON". To compute $\Pr(E)$ we need to compute $|E|$, which we do using the Product Rule with the following procedure:

   (a) Pick 2 of the 8 O-tiles. There are $\binom{8}{2}$ ways to do this.
   (b) Pick 1 of the 2 C-tiles. There are 2 ways to do this.
   (c) Pick 1 of the 6 T-tiles. There are 6 ways to do this.
   (d) Pick 1 of the 9 A-tiles. There are 9 ways to do this.
   (e) Pick 1 of the 3 G-tiles. There are 3 ways to do this.
   (f) Pick 1 of the 6 N-tiles. There are 6 ways to do this.

   So, by the Product Rule
   $$|E| = \binom{8}{2} \cdot 2 \cdot 6 \cdot 9 \cdot 3 \cdot 6$$
   and
   $$\Pr(E) = \sum_{\omega \in E} \Pr(\omega) = \sum_{\omega \in E} 1/\binom{98}{7} = \frac{|E|}{\binom{98}{7}} = \frac{\binom{8}{2} \cdot 2 \cdot 6 \cdot 9 \cdot 3 \cdot 6}{\binom{98}{7}} \approx 3.9 \times 10^{-6} = 0.0000039 \ .$$

2. What is the probability that a scrabble hand contains the word DOODLES?

We proceed exactly as in the previous question, considering the letters D($\times 2$), O($\times 2$), L, E, S in order to get

$$\Pr(E) = \frac{\binom{4}{2} \cdot \binom{8}{2} \cdot 4 \cdot 12 \cdot 4}{\binom{98}{7}} \approx 2.3 \cdot 10^{-6} = 0.0000023.$$

3. What is the probability that a scrabble hand contains the word SMOKO?

   This question is considerably trickier than the previous two and we have to take considerable care to avoid double-counting. We will use the Sum Rule.

   (a) We could have a hand that contains SMOKOxy where x and y are two tiles, neither of which is S, M, O, or K. Note that there are $\binom{98-4-2-8-1}{2} = \binom{83}{2}$ choices for the 2-tile set $\{x, y\}$. Using the Product Rule as above and considering S, M, O, K, $\{x, y\}$ in order, we find that there are

   $$4 \cdot 2 \cdot \binom{8}{2} \cdot 1 \cdot \binom{83}{2} = 762,272$$

   such hands.

   (b) We could have a hand that contains SMOKOxy where $x \in \{S, M, O\}$ and $y \notin \{S, M, O, K\}$. (Since there is only one K tile we can rule out $x =$ K.) We use the Sum Rule over the three options (S, M, O) for x and we get

   $$\begin{aligned} &\binom{4}{2} \cdot 2 \cdot \binom{8}{2} \cdot 1 \cdot 83 && \text{(SMOKOS}y\text{)} \\ &+ 4 \cdot \binom{2}{2} \cdot \binom{8}{2} \cdot 1 \cdot 83 && \text{(SMOKOM}y\text{)} \\ &+ 4 \cdot 2 \cdot \binom{8}{3} \cdot 1 \cdot 83 && \text{(SMOKOO}y\text{)} \\ &= 27,888 + 9,296 + 37,184 = 74,368. \end{aligned}$$

   (c) We could have a hand that contains SMOKOxx where tile $x \in \{S, O\}$. (We can rule out $x = K$ and $x = M$.) We use the Sum Rule over the two options (S,O) for x and we get

   $$\begin{aligned} &\binom{4}{3} \cdot 2 \cdot \binom{8}{2} \cdot 1 && \text{(SMOKOSS)} \\ &+ 4 \cdot 2 \cdot \binom{8}{4} \cdot 1 && \text{(SMOKOOO)} \\ &= 224 + 560 = 784. \end{aligned}$$

   (d) We could have a hand that contains SMOKOxy where $x, y \in \{S, M, O\}$ and $x \neq y$.

Now we use the Sum Rule over the $\binom{3}{2} = 3$ options (SM, SO, MO) for $\{x, y\}$.

$$\begin{aligned}
&\binom{4}{2} \cdot \binom{2}{2} \cdot \binom{8}{2} \cdot 1 && \text{(SMOKOSM)} \\
+ &\binom{4}{2} \cdot 2 \cdot \binom{8}{3} \cdot 1 && \text{(SMOKOSO)} \\
+ &4 \cdot \binom{2}{2} \cdot \binom{8}{3} \cdot 1 && \text{(SMOKOMO)} \\
= &168 + 672 + 224 = 1064.
\end{aligned}$$

Adding this all up we get

$$\Pr(\text{SMOKO}) = \frac{762272 + 74368 + 784 + 1064}{\binom{98}{7}} = \frac{21}{346484} \approx 0.000061.$$

Phew! That was exhausting. If anyone needs me, tell them I'm on smoko, leave me alone.

Question 3: Two villainous professors decide to implement a randomized multiple-choice exam. Each student receives an exam containing 17 questions each having 4 multiple-choice options.

Behind the scenes is a question bank B containing 200 questions. Each question q in the question bank B has a set A_q of 10 possible answers, exactly one of which is correct.

For each student X, the 17 questions on X's exam are a uniformly random 17-element *ordered* subset of B. For each these questions q, the student is presented with a uniformly random 4-element *ordered* subset of A_q that contains the correct answer.

1. If two students X and Y write this exam, what is the probability that X writes exactly the same exam as Y? (We consider two exams to be exactly the same if they contain the same questions in the same order with the same multiple-choice options for each question, also in the same order.)

   Before beginning, remember that the number of ordered k-element subsets of an n-element set is $\binom{n}{k}k! = \frac{n!}{(n-k)!}$. (We used this when proving that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$; it's also the number of one-to-one functions from a set of size k to a set of size n.)

   Consider the set S of all possible exams. We can compute $|S|$ using the Product Rule with the following procedure:

   (a) From the 200 problems in the question bank, choose an ordered subset of size 17. There are $200!/(200 - 17)! = 200!/183!$ ways to do this.
   (b) For each $i \in \{1, \ldots, 17\}$:
      i. Choose a location to put the correct answer. There are 4 ways to do this.
      ii. Choose an ordered 3-element subset from among the 9 incorrect answers. There are $9!/6!$ ways to do this.

So by the Product Rule, we get

$$|S| = \frac{200!}{183!} \cdot \left(4 \cdot \frac{9!}{6!}\right)^{17}$$

The question says that the exam a student receives is uniformly random.

When students X and Y write an exam we get sample space $S \times S = \{(x, y) : x, y \in S\}$ where each element (x, y) consists of two exams, exam x for student X and exam y for student Y. The size of $S \times S$ is $|S|^2$ and this is a uniform probability space. Let E denote the event "X and Y get the same exam". We can write E as

$$E = \{(x, x) : x \in S\}$$

which clearly has size $|E| = |S|$. So,

$$\Pr(E) = \frac{|E|}{|S \times S|} = \frac{|S|}{|S|^2} = \frac{1}{|S|} = \frac{183!}{200!} \cdot \left(\frac{6!}{4 \cdot 9!}\right)^{17} \approx 1.023 \cdot 10^{-95}.$$

2. If two students X and Y write this exam, what is the probability that both exams contain exactly the same 17 questions (possibly in a different order and possibly with different multiple-choice options for each question)?

   This question is asking if the (unordered) 17-element subset of questions X and Y get is the same. In this case, the number of possible exams is $\binom{200}{17}$. Then using the same reasoning as above the probability that two students X and Y get exams with the same set of questions is $1/\binom{200}{17} \approx 5.46 \cdot 10^{-25}$.

3. If two students X and Y write this exam, what is the probability that at least one question on X's exam also appears on Y's exam (possibly with different answers). Give an exact answer as well as a decimal approximation.

   This is a Birthday Paradox type question, and these are always easier to answer using the complement rule. If E is the event "the two exams have at least one question in common", then the complementary event $\overline{E}$ is the event "the two exams have no questions in common". This means that the exam y given to Y has a 17-element subset of the $200 - 17 = 183$ questions that do not appear on the exam x given to X. For any exam x, the number of such y is $\binom{183}{17}$, so

$$\Pr(\overline{E}) = \frac{\binom{183}{17}}{\binom{200}{17}}$$

   and by the Complement Rule

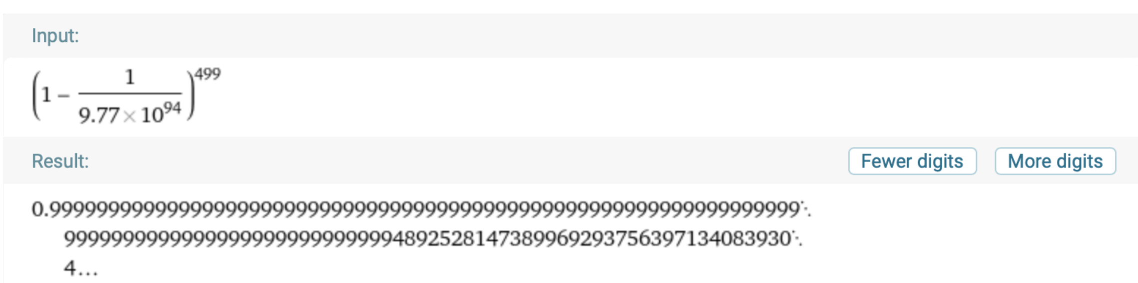
$$\Pr(E) = 1 - \Pr(\overline{E}) = 1 - \frac{\binom{183}{17}}{\binom{200}{17}} \approx 0.793 \ .$$

4. 500 students write the exam and student X decides to post the exam on Stack Overflow to ask for help during the exam. Student X is not dumb so they post it under a pseudonym $\tilde{X}$ so that they can't be identified. Of course, the professors find the Stack Overflow post. What is the probability that the professors can uniquely identify the student from the posted exam? In other words what is the probability that none of the 499 other students received exactly the same exam as X? Is this probability close to 1 or close to 0?

   For this question we have students $X_1, \ldots, X_{500}$. The sample space is $S^{500} = \{(x_1, \ldots, x_{500}) : x_i \in S,\ i \in \{1, \ldots, 500\}\}$. Without loss of generality, let $X = X_1$. We are interested in the event $E =$"student X's exam is different from everyone else's". We can easily compute $|E|$ using the Product Rule with the procedure:

   (a) Pick an exam x_1. There are $|S|$ options for this.
   (b) For each $i \in \{2, \ldots, 500\}$, pick an exam x_i different from x_1. There are $(|S|-1)^{499}$ ways to do this step.

   Therefore, by the Product Rule,
$$|E| = |S| \cdot (|S| - 1)^{499}.$$
   So we get
$$\Pr(E) = \frac{|E|}{|S|^{500}} = \left(\frac{|S|-1}{|S|}\right)^{499} = \left(1 - \frac{1}{|S|}\right)^{499}.$$
   We have seen above that
$$|S| = \frac{200!}{183!} \cdot \left(4 \cdot \frac{9!}{6!}\right)^{17} \approx 9.77 \cdot 10^{94}.$$
   We conclude that
$$\Pr(E) \approx \left(1 - \frac{1}{9.77 \cdot 10^{94}}\right)^{499},$$
   whose value is in this figure:


Input:
$$\left(1 - \frac{1}{9.77 \times 10^{94}}\right)^{499}$$
Result: Fewer digits More digits

0.9999999999999999999999999999999999999999999999999999999999999999
9999999999999999999999999999489252814738996929375639713408393 0
4...

   By the way, if x is a real number that is very close to zero, then $1 - x$ is very close to e^{-x}. Using this, we get
$$\Pr(E) \approx \left(e^{-1/(9.77 \cdot 10^{94})}\right)^{499} = e^{-499/(9.77 \cdot 10^{94})} \approx 1 - 499/(9.77 \cdot 10^{94}).$$

Question 4: Sad that Calypso Water Park is closed, Professor M opens his own backyard water park with only one slide:

Professor M invites 100 Carleton University students to his park; 10 of the students are wearing red bathing suits and the other 90 are wearing black bathing suits. The students line up in a uniformly random order and slide down the slide one after the other.

Before he gets bored, Professor M watches the first two students emerge from the slide and notes the colours c_1 and c_2 of their bathing suits, respectively.

1. Describe the sample space S for this experiment.

   There's more than one way to do this. Here's one: Call the students $x_1, \dots, x_{100}$ where $x_1, \dots, x_{10}$ wear red bathing suits and $x_{11}, \dots, x_{100}$ wear black bathing suits. Use the sample space S that consists of all permutation of $x_1, \dots, x_{100}$.

2. For each $\omega \in S$, determine $\Pr(\omega)$.

   This is easy, the question says they line up uniformly at random, so $\Pr(\omega) = 1/|S| = 1/100!$ for each $\omega \in S$.

   Let A be the event "c_1 is black" let and B be the event "c_2 is red"

3. What is $\Pr(A \cap B)$?

   An easy application of the Product Rule tells us that $|A \cap B| = 90 \cdot 10 \cdot 98! = 900 \cdot 98!$, so
   $$\Pr(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{900 \cdot 98!}{100!} = \frac{900}{100 \cdot 99} = \frac{9}{99} = \frac{3}{33}.$$

4. What is $\Pr(A \cup B)$?

   Using the Product Rule we get $|A| = 90 \cdot 99!$ and $|B| = 10 \cdot 99!$. By the Principle of Inclusion-Exclusion we get
   $$|A \cup B| = |A| + |B| - |A \cap B| = 90 \cdot 99! + 10 \cdot 99! - 900 \cdot 98! = 100 \cdot 99! - 900 \cdot 98! = 100! - 900 \cdot 98!$$
   so
   $$\Pr(A \cup B) = \frac{100! - 900 \cdot 98!}{100!} = 1 - \frac{3}{33} \ .$$
   No, $\Pr(A \cup B) = 1 - \Pr(A \cap B)$ is *not* an identity. It just happens, in this particular example, that $\overline{A \cup B}$="c_1 is red and c_2 is black" which (in this case) has the same probability as $A \cap B$="c_1 is black and c_2 is red".

5. Are the events A and B independent? In other words, is $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$?

   Let's check:

$$\Pr(A) \cdot \Pr(B) = \frac{90 \cdot 99!}{100!} \cdot \frac{10 \cdot 99!}{100!} = \frac{90 \cdot 10}{100^2} = \frac{9}{100} \neq \frac{9}{99} = \Pr(A \cap B) \ ,$$

   so A and B are *not* independent since $\Pr(A) \cdot \Pr(B) \neq \Pr(A \cap B)$.

Question 5: For this problem you have an 8-sided die. Consider the following two games:

Game A. you roll the die 8 times and you win if you roll 8 at least once.

Game B. you roll the die 24 times and you win if you roll 8 at least three times.

1. What is the probability of winning Game A?

   The sample space here is $S = \{1, \ldots, 8\}^8$ and has size $|S| = 8^8$ and the probability space is uniform, so $\Pr(\omega) = 1/8^8$ for each $\omega \in S$.

   We use the Complement Rule. If E is the event "I win Game A" then $\overline{E}$ is the event "I lose Game A" which is the same as saying "I never roll an 8". Then $\overline{E} = \{1, \ldots, 7\}^8$. Then

$$\Pr(\overline{E}) = \frac{|\overline{E}|}{|S|} = \left(\frac{7}{8}\right)^8 \quad .$$

   By the Complement Rule

$$\Pr(E) = 1 - \Pr(\overline{E}) = 1 - (7/8)^8 \approx 0.6564.$$

2. What is the probability of winning Game B?

   Here the sample space is $S = \{1, \ldots, 8\}^{24}$. Let E be the event "I win Game B" and $\overline{E}$ be the complementary event "I lose Game B". For each $i \in \{0, 1, 2\}$, let $\overline{E}_i$ be the event "I play Game B and I roll 8 exactly i times. The events, $\overline{E}_0$, $\overline{E}_1$ and $\overline{E}_2$ are pairwise disjoint and $\overline{E} = \overline{E}_0 \cup \overline{E}_1 \cup \overline{E}_2$ so, by the Sum Rule,

$$|\overline{E}| = |\overline{E}_0| + |\overline{E}_1| + |\overline{E}_2|$$

   (a) By the most straightforward application of the Product Rule, we get $|\overline{E}_0| = 7^{24}$.
   (b) We can count $|\overline{E}_1|$ using the Product Rule with the procedure "Pick a location for 8 and put something from $\{1, \ldots, 7\}$ in the 23 other locations." This gives $|\overline{E}_1| = 24 \cdot 7^{23}$.
   (c) We can count $|\overline{E}_2|$ using the Product Rule with the procedure "Pick 2 locations for 8 and put something from $\{1, \ldots, 7\}$ in the other 22 locations." This gives $|\overline{E}_2| = \binom{24}{2} \cdot 7^{22}$.

We finish up with

$$\Pr(\overline{E}) = \frac{|\overline{E}|}{8^{24}} = \frac{|\overline{E}_0| + |\overline{E}_1| + |\overline{E}_2|}{8^{24}} = \frac{7^{24} + 24 \cdot 7^{23} + \binom{24}{2} \cdot 7^{22}}{8^{24}} \approx 0.4082,$$

so

$$\Pr(E) = 1 - \Pr(\overline{E}) \approx 0.5918.$$

Question 6: You pick n numbers $x_1, \ldots, x_n$. Each x_i is chosen uniformly at random from the set $\{1, \ldots, n\}$, so $(x_1, \ldots, x_n)$ is a uniformly random n-vector from $\{1, \ldots, n\}^n$.

1. What is $\Pr(x_1 = n)$?

   The question says we're working in the uniform probability space over the sample space $S = \{1, \ldots, n\}^n$. If E is the event "$x_1 = n$" then $E = \{(n, x_2, \ldots, x_n) : (x_2, \ldots, x_n) \in \{1, \ldots, n\}^{n-1}\}$ and $|E| = n^{n-1}$, so $\Pr(E) = n^{n-1}/n^n = 1/n$.

2. What is $\Pr(\max\{x_1, \ldots, x_n\} = n)$?

   Here we use the Complement Rule. If E is the event "$\max\{x_1, \ldots, x_n\} = n$", then $\overline{E} = \{1, \ldots, n-1\}^n$ is the event "each of $x_1, \ldots, x_n$ is less than n." Now, $|\overline{E}| = (n-1)^n$ so, by the Complement Rule, $|E| = n^n - (n-1)^n$ and

   $$\Pr(E) = \frac{|E|}{|S|} = \frac{n^n - (n-1)^n}{n^n} = 1 - (1 - 1/n)^n.$$

   Notice that this is a generalization of Question 5.1.

   As a sanity check, when $n = 8$,

   $$\frac{8^8 - 7^8}{8^8} = 1 - \frac{7}{8}^8,$$

   which agrees with our solution to Question 5.1!

3. What is $\Pr(x_1 = \max\{x_1, \ldots, x_n\})$?

   Let E be the event "$x_1 = \max\{x_1, \ldots, x_n\}$" and, for each $i \in \{1, \ldots, n\}$, let E_i be the event "$x_1 = i = \max\{x_1, \ldots, x_n\}$". Then, by the Sum Rule

   $$|E| = \sum_{i=1}^{n} |E_i|.$$

   By the Product Rule, $|E_i| = i^{n-1}$ since we must choose $x_1 = i$ and each of $x_2, \ldots, x_n \in \{1, \ldots, i\}$. Therefore,

   $$|E| = \sum_{i=1}^{n} |E_i| = \sum_{i=1}^{n} i^{n-1} \quad .$$

So

$$\Pr(E) = \frac{|E|}{|S|} = \sum_{i=1}^{n} \frac{i^{n-1}}{n^n} = \frac{1}{n} \cdot \sum_{i=1}^{n} (i/n)^{n-1}.$$

This doesn't really simplify any further. In fact the sum on the right hand side has the form of a "generalized harmonic number". When a sum has a name that usually happens because someone was unable to simplify it so they gave it a name instead.

4. For $i \in \{1, \ldots, n\}$, what is $\Pr(\max\{x_1, \ldots, x_n\} = i)$? (Your formula will depend on i)

   Let E be the event "$\max\{x_1, \ldots, x_n\} = i$". We are going to determine $\Pr(E)$.

   We start with a complicated solution. For each $k \in \{1, \ldots, n\}$, let E_k be the event "$\max\{x_1, \ldots, x_n\}) = i$ and i appears exactly k times among $x_1, \ldots, x_n$". Then $|E_k| = \binom{n}{k} \cdot (i-1)^{n-k}$. So we get

$$\begin{aligned}
|E| &= \sum_{k=1}^{n} \binom{n}{k} \cdot (i-1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot (i-1)^{n-k} - \binom{n}{0} \cdot (i-1)^{n-0} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot (i-1)^{n-k} - (i-1)^{n} \\
&= \sum_{k=0}^{n} \binom{n}{k} \cdot (i-1)^{n-k} \cdot 1^k - (i-1)^{n} \\
&= ((i-1)+1)^n - (i-1)^n \qquad \text{(From Newton)} \\
&= i^n - (i-1)^n.
\end{aligned}$$

   Finish with

$$\Pr(E) = \frac{i^n - (i-1)^n}{n^n}.$$

   Here is an easier way to determine $\Pr(E)$. Let A be the event "$x_1, \ldots, x_n \le i$" and let B the event "$x_1, \ldots, x_n \le i-1$". Since $B \subseteq A$ and $E = A \setminus B$, we have

$$|E| = |A| - |B| = i^n - (i-1)^n$$

   and, thus,

$$\Pr(E) = \frac{|E|}{n^n} = \frac{i^n - (i-1)^n}{n^n}.$$

   As a sanity check, verify that this answer agrees with Part 2 when $i = n$. It does!

   As another sanity check, try $i = 1$. In this case $\max\{x_1, \ldots, x_n\} = 1$ if and only $(x_1, \ldots, x_n) = (1, \ldots, 1)$. So $\Pr(\max\{x_1, \ldots, x_n\} = 1) = 1/n^n = (1^n - 0^n)/n^n$.

Question 7: Let p be a real number with $0 < p < 1$ and consider its infinite binary representation

$$p = 0.p_1 p_2 p_3 \cdots .$$

Note that

$$\sum_{k=1}^{\infty} \frac{p_k}{2^k} = p.$$

For example, if $p = 1/\pi$, then

$$p = 0.010100010111110011 \cdots .$$

We assume that we know, for any $k \geq 1$, the bit p_k.

You would like to generate a *biased* random bit: With probability p, this bit is 1, and with probability $1-p$, it is 0. You find a *fair* coin in your pocket: This coin comes up heads with probability 1/2 and tails with probability 1/2. In this question, you will show that this coin can be used to generate a biased random bit.

Consider the following algorithm GetBiasedBit, which does not take any input:

Algorithm GetBiasedBit**:**

// all coin flips made are mutually independent
repeatedly flip the coin until it comes up heads for the first time;
let k be the number of coin flips made (including the one resulting in heads);
let $b = p_k$;
return b

- Prove that algorithm GetBiasedBit returns 1 with probability p.

  Consider the event

  $$A = \text{"algorithm GetBiasedBit returns 1"}.$$

  We have to prove that $\Pr(A) = p$.

  For each $k \geq 1$, consider the event

  $$B_k = \text{"the coin comes up heads for the first time at the } k\text{-th flip"}.$$

  This event occurs if and only if the sequence of coin flips consists of $k-1$ tails followed by one heads. Since the coin is fair and the flips are mutually independent, $\Pr(B_k) = 1/2^k$.

  The event A occurs if and only if there exists a $k \geq 1$ such that the event B_k occurs and $p_k = 1$. That is,

  $$\begin{aligned} A &\iff \bigvee_{k=1}^{\infty} (B_k \wedge p_k = 1) \\ &\iff \bigvee_{k: p_k = 1} B_k; \end{aligned}$$

in the last line, we take the logical OR over all $k \geq 1$ for which $p_k = 1$.

We observe that the events B_k, for $k \geq 1$, are pairwise disjoint. It follows that

$$
\begin{aligned}
\Pr(A) &= \Pr\left(\bigvee_{k:p_k=1} B_k \right) \\
&= \sum_{k:p_k=1} \Pr(B_k) \\
&= \sum_{k:p_k=1} 1/2^k \\
&= \sum_{k:p_k=1} p_k \cdot (1/2^k) \\
&= \sum_{k:p_k=1} p_k \cdot (1/2^k) + \sum_{k:p_k=0} p_k \cdot (1/2^k) \\
&= \sum_{k=1}^{\infty} p_k \cdot (1/2^k) \\
&= p.
\end{aligned}
$$

Question 8: Let $n \geq 0$ be an integer, let x be a real number with $0 < x < 1$, and let

$$G_n(x) = \left(\sum_{k=0}^{\infty} x^k \right)^{n+1},$$

i.e.,

$$G_n(x) = \underbrace{\left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \cdots \left(1 + x + x^2 + x^3 + \cdots\right)}_{n+1 \text{ times}}.$$

1. Let $m \geq 0$ be an integer. Determine the coefficient of x^m in the (infinite) expansion of $G_n(x)$. *Hint:* You may use a certain result that we have seen in the chapter on counting.

   The product $G_n(x)$ has $n+1$ factors $F_0, \ldots, F_n$ where each factor $F_i = (x^0 + x^1 + x^2 + \cdots)$. If we do this multiplication the (infinitely) long way we get an x^m whenever we pick x^{k_i} from F_i for each $i \in \{0, \ldots, n\}$ and $\sum_{i=0}^{n} k_i = m$. Therefore, the coefficient of x^m in $G_n(x)$ is equal to the number of solutions of

   $$k_0 + k_1 + \cdots + k_n = m$$

   where $k_0, \ldots, k_n$ are each non-negative integers. We solved this problem in class already. The number is

   $$\binom{m+n}{n}.$$

(Don't get confused by the fact that the problem we solved had variables indexed starting at 1 and this one has variables indexed starting at 0.) So,

$$G_n(x) = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m$$

2. Explain, in a few sentences, why

$$\frac{1}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m.$$

From the previous question, we see that the right hand side is $G_n(x)$. The left hand side is more recognizable if we write it like this:

$$\left(\frac{1}{1-x}\right)^{n+1}$$

Now that looks suspiciously like the definition of $G_n(x)$. Indeed, we've seen in the lecture on infinite probability spaces that, for $0 < x < 1$,

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

so the left hand side is also equal to $G_n(x)$.

3. Explain, in a few sentences, why

$$\sum_{k=n}^{\infty} \binom{k}{n} x^k = \frac{x^n}{(1-x)^{n+1}}. \tag{1}$$

From Part 2 we already learned that

$$\frac{1}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m.$$

We introduce a new summation variable $k = m + n$. Since m runs through all integers that are at least 0, the new variable k runs through all integers that are at least n. This gives

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} x^{k-n}.$$

Now multiply both sides by x^n to get

$$\frac{x^n}{(1-x)^{n+1}} = \sum_{k=n}^{\infty} \binom{k}{n} x^k.$$