VANDERMONDE IDENTITY AND PASCAL'S TRIANGLE

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Vandermonde Identity

Continue our work on combinatorial proofs.

Recall that means we proof things (like equalities) by counting them.

Consider m, n, r where $m \geq 0, n \geq 0, r \geq 0,$ and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

$$= \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \binom{m}{2}\binom{n}{r-2} + \cdots + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r}\binom{n}{0}$$

Notice that we are always choosing r things (from two different sets)

Vandermonde Identity

m men

n women

Prove by counting

$m \geq 0, n \geq 0, r \geq 0$, and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

What does this mean? We know that $\binom{m+n}{r}$ represents the number of subsets of size r in a set of size $m + n$.

Can be two sets, or one set divided into two.

$\binom{m+n}{r}$ represents the number of ways to choose a subset of r people from among these two groups.

Vandermonde Identity

We want to argue that

$$\binom{m+n}{r}$$

and

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

Count the same thing.

If we are taking a subset of r people, we can split it into cases (which would correspond to our summation). Perhaps based on number of men chosen and number of women.

m men	n women

Product Rule!

Sum Rule!

	men chosen	**women chosen**
$\binom{m}{0}\binom{n}{r}$	0	r
$\binom{m}{1}\binom{n}{r-1}$	1	r-1
$\binom{m}{2}\binom{n}{r-2}$	2	r-2
	…	…
$\binom{m}{r-1}\binom{n}{1}$	r-1	1
$\binom{m}{r}\binom{n}{0}$	r	0

Vandermonde Identity

We have shown combinatorially that for $m \geq 0, n \geq 0, r \geq 0$, and $r \leq m, r \leq n$

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Special Case: $m = n = r$

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{r-k} = \binom{n+n}{n} = \binom{2n}{n}$$

$$\sum_{k=0}^{n} \binom{n}{k}\underbrace{\binom{n}{n-k}} = \binom{2n}{n}$$

Recall:

$$\binom{n}{n-k} = \binom{n}{k}$$

Rewrite:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0}$$ Row 0

$$\binom{1}{0} \quad \binom{1}{1}$$ Row 1

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$ Row 2

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

$$\binom{0}{0} \qquad (x+y)^0$$

$$\binom{1}{0} \quad \binom{1}{1} \qquad (x+y)^1$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \qquad (x+y)^2$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's Triangle

$$\binom{0}{0} \quad \text{Row 0}$$

$$\binom{1}{0} \quad \binom{1}{1} \quad \text{Row 1}$$

$$\binom{2}{0} \quad \boxed{\binom{2}{1}} \quad \boxed{\binom{2}{2}} \quad \text{Row 2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \boxed{\binom{3}{2}} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & \boxed{2} & & \boxed{1} & & \\ & 1 & & 3 & & \boxed{3} & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pascal's Triangle

Some rules we've learned:

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 $\boxed{4}$ 6 $\boxed{4}$ 1

$\binom{4}{1}$ $\binom{4}{3}$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

row 3, $2^3 = 8 = \boxed{1 + 3 + 3 + 1}$

1 4 6 4 1

Sum of any row n is 2^n

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

				1					
			1		1				
		1		2		1			
row 3	1^2	+	3^2	+	3^2	+	1^2		
1		4		6		4		1	

Sum of the squares of any row n is $\binom{2n}{n}$

You can find as the middle element of row $2n$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

row 3 $\boxed{1^2 + 3^2 + 3^2 + 1^2}$

row 4 1 4 6 4 1

1 6 15 $\boxed{20}$ 15 6

row 6

Sum of the squares of any row n is $\binom{2n}{n}$

You can find as the middle element of row $2n$

Pascal's Triangle

Some rules we've learned:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k}$$

$$\binom{n}{0} = \binom{n}{n}$$

$n \geq 2, 1 \leq k \leq n-1$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

1

1 1

1 2 1

$$(x+y)^3 = \boxed{1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot x y^2 + 1 \cdot y^3}$$

1 4 6 4 1

You can find the coefficients of the n^{th} polynomial by looking a row n

Pascal's Triangle Example

MISSISSIPPI
→ MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

(We don't need meaningful words, simply arrangements).

Our first idea might be to take all permutations.

11 letters $= 11!$ permutations

One such permutation is to swap 3rd and 4th letters, which gives us:

MISSISSIPPI

Since we want distinct arrangements, this is no good.

Pascal's Triangle Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

M: 1
I: 4
S: 4
P: 2
11

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11
S	P	I	I	M	P	I	S	S	I	S

Task 1: Place 1 M in 11 possible locations. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's in 10 possible locations. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's in 6 possible locations. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 P's in 2 possible locations. There are $\binom{2}{2}$ ways to do this.

Pascal's Triangle Example

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 P's. There are $\binom{2}{2}$ ways to do this.

MISSISSIPPI

M: 1
I: 4
S: 4
P: 2
11

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2}$$

$$= \frac{11!}{1!10!} \cdot \frac{10!}{4!6!} \cdot \frac{6!}{4!2!} \cdot \frac{2!}{2!0!}$$

$$= \frac{11!}{4!4!2!} = 34650$$

Pascal's Triangle Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

M: 1
I: 4
S: 4
P: 2
11

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 2 S's. There are $\binom{2}{2}$ ways to do this.

What if we change the order?

Pascal's Triangle Example

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

Task 1: Place 4 S's. There are $\binom{11}{4}$ ways to do this.

Task 2: Place 2 P's. There are $\binom{7}{2}$ ways to do this.

Task 3: Place 4 I's. There are $\binom{5}{4}$ ways to do this.

Task 3: Place 1 M. There are $\binom{1}{1}$ ways to do this.

MISSISSIPPI

M: 1
I: 4
S: 4
P: 2

11

$$\binom{11}{4} \cdot \binom{7}{2} \cdot \binom{5}{4} \cdot \binom{1}{1}$$

$$= \frac{11!}{4!7!} \cdot \frac{7!}{2!5!} \cdot \frac{5!}{4!1!} \cdot \frac{1!}{1!0!}$$

$$= \frac{11!}{4!4!2!} = 34650$$

Pascal's Triangle Example

MISSISSIPPI

How many ways can we rearrange the letters of the word MISSISSIPPI?

For every letter, count how many times it can occur.

M: 1
I: 4
S: 4
P: 2

11

Then place the letters in an empty array (using Product Rule)

1	2	3	4	5	6	7	8	9	10	11

Task 1: Place 1 M. There are $\binom{11}{1}$ ways to do this.

Task 2: Place 4 I's. There are $\binom{10}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{6}{4}$ ways to do this.

Task 3: Place 4 S's. There are $\binom{2}{2}$ ways to do this.

Can you think of another way to compute this?

Pascal's Triangle Example

There are 11! ways to arrange 11 letters.

How many are duplicates?

1	2	3	4	5	6	7	8	9	10	11
M	I	S	S	I	S	S	I	P	P	I

There are 4! permutations with the same arrangement of S

There are 4! permutations with the same arrangement of I

There are 2! permutations with the same arrangement of P

There is 1! permutations with the same arrangement of M

MISSISSIPPI

M: 1
I: 4
S: 4
P: 2
11

$$11!/(4! \cdot 4! \cdot 2! \cdot 1!) = 34650$$

Binomial Coefficient Example

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

How many solutions are there to this problem?

$$x_1 + x_2 + x_3 = 7$$

$(2,1,4), (1,1,5), (5,1,1), (0,7,0)$, etc

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

We will once again map it to bitstrings.

$$(2,1,4) \to 001010000$$
$$(1,4,2) \to 010000100$$
$$(0,7,0) \to 100000001$$

Procedure: Write down nine 0's.
Choose 2 0's to flip into a 1

How many ways can we write bitstrings of this type?

Number of 0's = $x_1 + x_2 + x_3 = 7$

Number of 1's $\quad + \quad + \quad = 2$

Binomial Coefficient Example

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

How many solutions are there to this problem?

$$x_1 + x_2 + x_3 = 7$$

$(2,1,4), (1,1,5), (5,1,1), (0,7,0)$, etc

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

We will once again map it to bitstrings.

$$(2,1,4) \rightarrow 001010000$$
$$(1,4,2) \rightarrow 010000100$$
$$(0,7,0) \rightarrow 100000001$$

There are $\binom{9}{2}$ solutions. To argue this we must argue the other direction also.

How many ways can we write bitstrings of this type?

Number of 0's = $x_1 + x_2 + x_3 = 7$

Number of 1's $\quad + \quad + \quad = 2$

Binomial Coefficient Example

We've mapped linear equation to a bitstring of length 9 with exactly 2 1’s. To prove a bijection, we must argue that for every bitstring of length 9 with exactly 2 1's there is a corresponding linear equation.

$$010000100 \rightarrow (1,4,2)$$

We count leading 0’s. There is 1 leading 0, so write a 1.

We count one 1, then count 0’s until next 1.

There are 4 0’s so write a 4.

Count one 1, then count the rest of the 0’s. There are 2, so write a 2.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 = 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

There are $\binom{9}{2}$ solutions.

Binomial Coefficient Example

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

We've mapped linear equation to a bitstring of length 9 with exactly 2 1's. To prove a bijection, we must argue that for every bitstring of length 9 with exactly 2 1's there is a corresponding linear equation.

$x_1 + x_2 + x_3 = 7$

$$\underbrace{0 \dots 0}_{x_1}1\underbrace{0 \dots 0}_{x_2}1\underbrace{0 \dots 0}_{x_3}$$

$$010000100 \to (1,4,2)$$

So we can map back and forth, and they are functions (at most one solution), thus these are equivalent functions.

There are $\binom{9}{2}$ solutions.

Bitstrings and Linear Equations

We can do more general equations

$$x_1 + x_2 + \cdots + x_k = n$$

We can map this to a bitstring with $k-1$ many 1's and n 0's.

Thus it is a bitstring with length $n+k-1$, where we choose $k-1$ of the bits to be 1’s

There are $\binom{n+k-1}{k-1}$ solutions.

Bitstrings and Linear Equations

How many solutions are there to this problem?

There are the original solutions

$(2,1,4), (1,1,5), (5,1,1), (0,7,0)$, etc

And also solutions of the type

$(2,1,3), (1,1,1), (0,1,1), (0,0,0)$, etc.

$x_1 + x_2 + x_3 + x_4 = 7$

The claim is these are the same. We need to show a bijection.

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 \leq 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}$$

But now possibly less than 7 0's.

Bitstrings and Linear Equations

Here are some solutions to $x_1 + x_2 + x_3 \leq 7$. We will map these to solutions to $x_1 + x_2 + x_3 + x_4 = 7$

$(2,1,4)$, $(2,1,3)$, $(0,1,1)$, $(0,7,0)$, $(0,0,0)$

$(2,1,4,0)$, $(2,1,3,1)$, $(0,1,1,5)$, $(0,7,0,0)$, $(0,0,0,7)$

$x_1 + x_2 + x_3 \leq 7$

Let $x_4 = 7 - (x_1 + x_2 + x_3)$. Then

$x_1 + x_2 + x_3 + x_4 = 7$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

$x_1 + x_2 + x_3 \leq 7$

$0 \ldots 010 \ldots 010 \ldots 010 \ldots 0$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$x_1 + x_2 + x_3 + x_4 = 7$

Subtract x_4 from both sides: $x_1 + x_2 + x_3 \leq 7$

Bitstrings and Linear Equations

How many solutions are there to this problem? There are the original solutions

$$(\underbrace{2,1,4}_{(2,1,4)},0),\ (\underbrace{2,1,3}_{(2,1,3)},1),\ (\underbrace{0,1,1}_{(0,1,1)},5),\ (\underbrace{0,7,0}_{(0,7,0)},0),\ (\underbrace{0,0,0}_{(0,0,0)},7)$$

$$x_1 + x_2 + x_3 + x_4 = 7$$
$$x_1 + x_2 + x_3 \leq 7$$

Thus the number of solutions is $\binom{10}{3}$ for both $x_1 + x_2 + x_3 \leq 7$ $x_1 + x_2 + x_3 + x_4 = 7$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

$$x_1 + x_2 + x_3 \leq 7$$

$$\underbrace{0 \ldots 0}_{x_1}1\underbrace{0 \ldots 0}_{x_2}1\underbrace{0 \ldots 0}_{x_3}1\underbrace{0 \ldots 0}_{x_4}$$