THE PROBABILISTIC METHOD

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

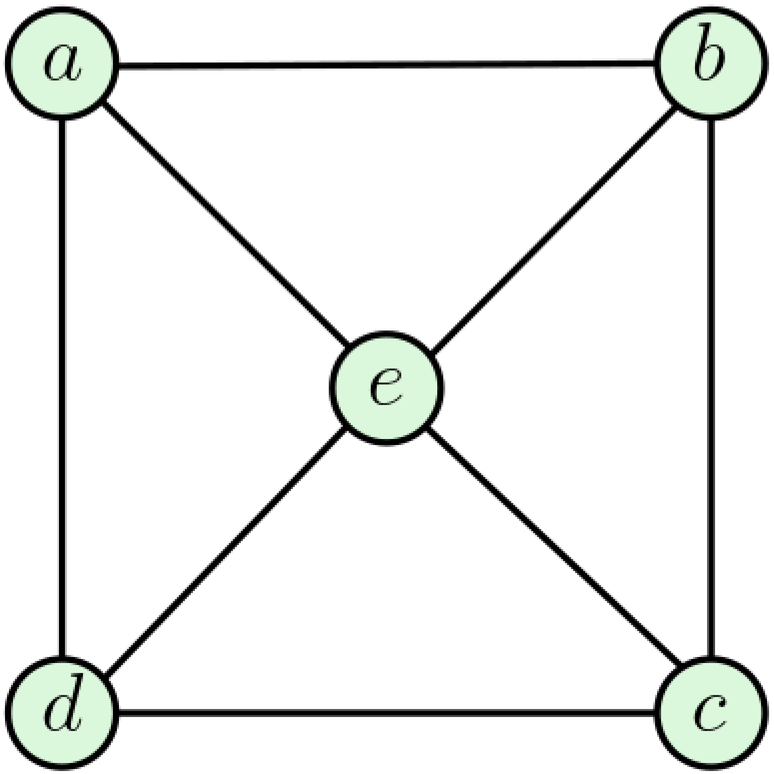
How to solve problems using probability where probability is not part of the problem.

We can use probability to prove the following:

Graph $G = (V, E), m = |E|$
partition of $V = A \cup B, A \cap B = \emptyset$

edge $\{u, v\}$ is between A and B if
$u \in A \wedge v \in B$ or
$v \in A \wedge u \in B$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$



$m = 8$
$A = \{a, d\}$
$B = \{b, c, e\}$

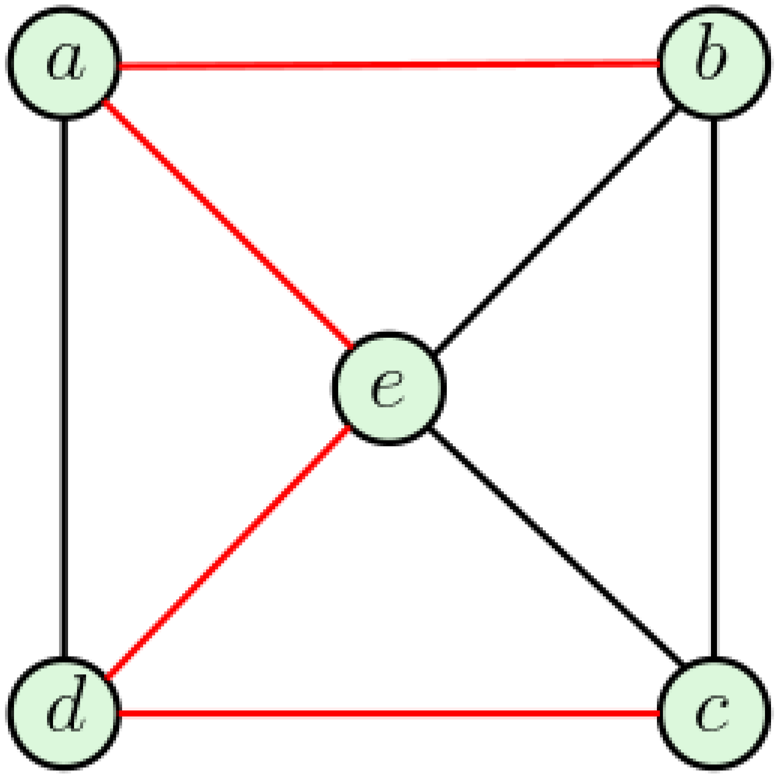
How to solve problems using probability where probability is not part of the problem.

We can use probability to prove the following:

Graph $G = (V, E), m = |E|$
partition of $V = A \cup B, A \cap B = \emptyset$

edge $\{u, v\}$ is between A and B if
$u \in A \wedge v \in B$ or
$v \in A \wedge u \in B$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$



$m = 8$
$A = \{a, d\}$
$B = \{b, c, e\}$

Is this always true?

Notice there is no randomness here

Edges between A and B are $4 \geq \frac{8}{2}$

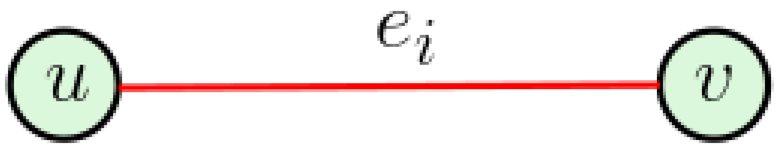
Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

We start by randomly partitioning the vertices into A and B.

Start with: $A = \emptyset, B = \emptyset$

$\forall\, u \in V$:
- flip fair coin
- if coin = H: add u to A
- if coin = T: add u to B



Once we are done the algorithm, we have partitioned the vertices, that is:

$$V = A \cup B, A \cap B = \emptyset$$

Define a Random Variable
$X =$ # edges between A and B

We will determine $E(X)$, which will allow us to prove the claim.

We will use indicator random variables to count the edges between A and B.

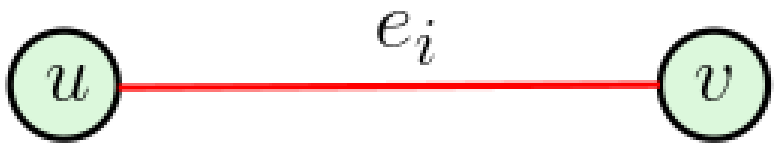
Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
    flip fair coin
    if coin = H: add u to A
    if coin = T: add u to B

X = # edges between A and B

$E(X) = ?$



$E = \{e_1, \ldots, e_m\}$

$$X_i = \begin{cases} 1 \text{ if } e_i \text{ between } A \text{ and } B \\ 0 \text{ otherwise} \end{cases}$$

$E(X_i) = \Pr(X_i = 1)$
$= \Pr(e_i \text{ between } A \text{ and } B)$

We can use the fact that these sets were generated randomly

What are the possibilities?
$u \in A, v \in A$
$u \in B, v \in B$
$u \in A, v \in B$
$u \in B, v \in A$

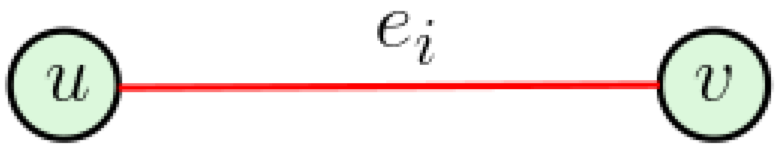
Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
flip fair coin
if coin = H: add u to A
if coin = T: add u to B

$X =$ # edges between A and B

$E(X) = ?$



$E = \{e_1, \ldots, e_m\}$

$$X_i = \begin{cases} 1 \text{ if } e_i \text{ between } A \text{ and } B \\ 0 \text{ otherwise} \end{cases}$$

$E(X_i) = \Pr(X_i = 1)$
$= \Pr(e_i \text{ between } A \text{ and } B)$

We can use the fact that these sets were generated randomly

What are the possibilities?
$u \in A, v \in A$ → $X_i = 0$ with prob ¼
$u \in B, v \in B$ → $X_i = 0$ with prob ¼
$u \in A, v \in B$ → $X_i = 1$ with prob ¼
$u \in B, v \in A$ → $X_i = 1$ with prob ¼

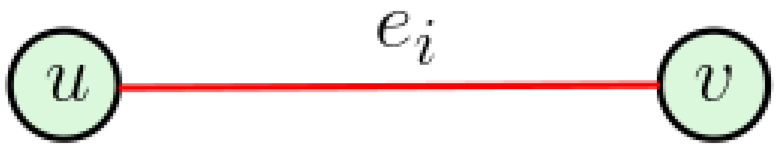
Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
flip fair coin
if coin = H: add u to A
if coin = T: add u to B

X = # edges between A and B

$E(X) = ?$



$E = \{e_1, \ldots, e_m\}$

$$X_i = \begin{cases} 1 \text{ if } e_i \text{ between } A \text{ and } B \\ 0 \text{ otherwise} \end{cases}$$

$E(X_i) = \Pr(X_i = 1)$
$= \Pr(e_i \text{ between } A \text{ and } B) = ½$

What are the possibilities?
$u \in A, v \in A \rightarrow X_i = 0$ with prob ¼
$u \in B, v \in B \rightarrow X_i = 0$ with prob ¼
$u \in A, v \in B \rightarrow X_i = 1$ with prob ¼
$u \in B, v \in A \rightarrow X_i = 1$ with prob ¼

Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
flip fair coin
if coin = H: add u to A
if coin = T: add u to B

$X =$ # edges between A and B

$E(X) = ?$

$E = \{e_1, \dots, e_m\}$

$$X_i = \begin{cases} 1 \text{ if } e_i \text{ between } A \text{ and } B \\ 0 \text{ otherwise} \end{cases}$$

$$E(X_i) = \Pr(X_i = 1)$$
$$= \Pr(e_i \text{ between } A \text{ and } B) = \frac{1}{2}$$

$$E(X) = E\left(\sum_{i=1}^{m} X_i\right)$$
$$= \sum_{i=1}^{m} E(X_i)$$
$$= \sum_{i=1}^{m} \frac{1}{2} = \frac{m}{2}$$

This implies the claim.

Why?

Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
flip fair coin
if coin = H: add u to A
if coin = T: add u to B

$X =$ # edges between A and B

$E(X) = \frac{m}{2}$

This implies the claim. Why?

Proof of the Claim:

Observe that $E(X) = \frac{m}{2}$ says that on average, the number of edges between A and B is $\frac{m}{2}$. Then there must be a partition of edges between A and $B \geq \frac{m}{2}$.

Assume this is not true. That is, $E(X) = \frac{m}{2}$ but $\forall$ partitions of V into A and B, # edges between A and $B < \frac{m}{2}$.

Then it is always the case that $X < \frac{m}{2}$.

Therefore $E(X) < \frac{m}{2}$. This is a contradiction.

Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
	flip fair coin
	if coin = H: add u to A
	if coin = T: add u to B

$X =$ # edges between A and B

$E(X) = \frac{m}{2}$

This implies the claim. Why?

Proof of the Claim:

Observe that $E(X) = \frac{m}{2}$ says that on average, the number of edges between A and B is $\frac{m}{2}$. Then there must be a partition of edges between A and $B \geq \frac{m}{2}$.

Saying the claim is not true is similar to saying there exists a set S of numbers with average x, but $\forall\, y \in S, y < x$. Then it is impossible to have an average of x.

Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

Start with: $A = \emptyset, B = \emptyset$
$\forall\, u \in V$:
    flip fair coin
    if coin = H: add u to A
    if coin = T: add u to B

X = # edges between A and B

$E(X) = \frac{m}{2}$

This implies the claim. Why?

Proof of the Claim (slightly more rigorous):

Contradiction: assume $E(X) = \frac{m}{2}$, but

$$\forall w \in S, X(w) < \frac{m}{2}$$

$$E(X) = \sum_{w \in S} X(w) \cdot \Pr(X = w)$$

Let $X' = \max\{X(w), w \in S\}$

Then $X \leq X' < \frac{m}{2}$, and $E(X) \leq E(X')$.

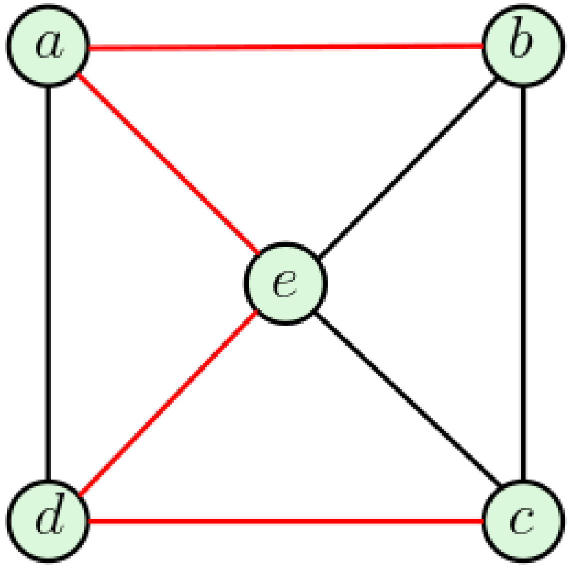
$$E(X) \leq E(X') = \sum_k k \cdot \Pr(X' = k) = X' < \frac{m}{2}$$

Graph $G = (V, E), m = |E|$
partition $V = A \cup B, A \cap B = \emptyset$

Claim:
$\exists$ partition # edges between A and $B \geq \frac{m}{2}$

True!

We have used probability to prove a property that has nothing to do with randomization.



Problem from the first lecture:

Subsets $S_1, S_2, \dots, S_m$ of $\mathrm{S} = \{1,2,\dots,n\}$ such that $\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$

Divide S into subsets such that no two subsets are subsets of each other.
What is the largest number of subsets that we can find with this property?

$$S = \{1,2,3,4,5\}$$

$\{\{1,4\},\{2,3,1\},\{2,3,5\},\{4,5\}\}$ has property

$$\{\{1,3\},\{2,3,1\},\{2,3,5\},\{4,5\}\}$$
$\{1,3\} \subseteq \{2,3,1\}$ does not have property

However, we saw experimentally that, for certain sets, $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$.

$$S = \{1,2,3,4,5\}$$
$n = 5$

If we take all subsets of size 2:

$$\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},$$
$$\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

Then
$$m = \binom{5}{2} = 10$$
In general, if the subsets are all the same size it is easier to verify.

Subsets $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$ such that

$$\forall i \neq j, S_i \not\subseteq S_j \wedge S_j \not\subseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

$m = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ is possible:

take all subsets of size $\left\lfloor \frac{n}{2} \right\rfloor$

Can we find a set of subsets with size greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$?

Let $a_1, \dots, a_n$ be a uniformly random permutation of $1, \dots, n$

Define sets:

$$A_1, \dots, A_n : A_j = \{a_1, \dots, a_j\}$$

For example, for $n = 4$, consider the set

$$3,2,4,1$$

Then the sets $A_1, \dots, A_4$ would be:

$$A_1 = \{3\}$$
$$A_2 = \{2,3\}$$
$$A_3 = \{2,3,4\}$$
$$A_4 = \{1,2,3,4\}$$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n : A_j = \{a_1, \ldots, a_j\}$$

We are given subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

$$n = 4, \qquad 3,2,4,1$$

$$A_1 = \{3\}$$
$$A_2 = \{2,3\}$$
$$A_3 = \{2,3,4\}$$
$$A_4 = \{1,2,3,4\}$$

In general, it is clear that

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Each set is completely contained in all following sets.
Since the permutation is random, for any given subset S_i of a possible solution, it is possible that $S_i = A_j$ for some value j

Subsets $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$ such that

$$\forall i \neq j, S_i \not\subseteq S_j \wedge S_j \not\subseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \dots, a_n$ be a uniformly random permutation of $1, \dots, n$

$$A_1, \dots, A_n : A_j = \{a_1, \dots, a_j\}$$

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$.

S_i occurs in $A_1, \dots, A_n$ if

$$\exists j : S_i = A_j$$

$X =$ # of values i for which S_i occurs in $A_1, \dots, A_n$

X counts how many subsets S_i of a given solution occur in the sequence $A_1, \dots, A_n$

Our claim is that $X \in \{0,1\}$. Proof by contradiction.

If there were 2 sets, S_i and S_j occurring in $A_1, \dots, A_n$, then $S_i = A_k$ and $S_j = A_\ell$, but then either $S_i \subset S_j$ or $S_j \subset S_i$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n : A_j = \{a_1, \ldots, a_j\}$$
$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

S_i occurs in $A_1, \ldots, A_n$ if $\exists j : S_i = A_j$

X = # of values i for which S_i occurs in $A_1, \ldots, A_n$

Since X $\in \{0,1\}$, $E(X) \leq 1$

for $i = 1, \ldots, m$:

$$X_i = \begin{cases} 1 \; if \; S_i \; occurs \; in \; A_1, \ldots, A_n \\ 0 \; otherwise \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_m$$

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_m)$$

Since they are indicator random variables:

$$E(X_i) = \Pr(X_i = 1)$$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n \colon A_j = \{a_1, \ldots, a_j\}$$

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

S_i occurs in $A_1, \ldots, A_n$ if $\exists j \colon S_i = A_j$

$X = \#$ of values i for which S_i occurs in $A_1, \ldots, A_n$

$$E(X) \leq 1$$

$$X_i = \begin{cases} 1 \; if \; S_i \; occurs \; in \; A_1, \ldots, A_n \\ 0 \; otherwise \end{cases}$$

$$E(X_i) = \Pr(X_i = 1)$$

S_i is not random, but $A_1, \ldots, A_n$ is based off of a random permutation.

Based on the size of S_i, there is only one possible element in $A_1, \ldots, A_n$ it can match with.

$$|S_i| = k, X_i = 1 \leftrightarrow S_i = A_k$$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n : A_j = \{a_1, \ldots, a_j\}$$
$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

S_i occurs in $A_1, \ldots, A_n$ if $\exists j : S_i = A_j$

$$E(X_i) = \Pr(X_i = 1)$$

$$|S_i| = k, X_i = 1 \leftrightarrow S_i = A_k$$

$$\underbrace{\boldsymbol{a_1 \ldots a_k}}\ \ \boldsymbol{a_{k+1} \ldots a_n}$$

Permutation of S_i

What we are asking is this: given a random permutation of $\{1,2,\ldots,n\}$ what is the probability that the first k elements are the same k elements in S_i

$$\Pr(X_i = 1) = \frac{|X_i = 1|}{\#\text{ permutations of } n \text{ numbers}}$$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n : A_j = \{a_1, \ldots, a_j\}$$

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

S_i occurs in $A_1, \ldots, A_n$ if $\exists j : S_i = A_j$

$$E(X_i) = \Pr(X_i = 1)$$

$$|S_i| = k, X_i = 1 \leftrightarrow S_i = A_k$$

$\boldsymbol{a_1 \ldots a_k}$	$\boldsymbol{a_{k+1} \ldots a_n}$

We can count $|X_i = 1|$ using the product rule.

We are given S_i, and we know it contains k elements. We place these k elements in the first k positions of our sequence:
$k!$ ways.

Place the remaining $n-k$ elements in $n-k$ positions: $(n-k)!$

Subsets $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \dots, a_n$ be a uniformly random permutation of $1, \dots, n$

$$A_1, \dots, A_n : A_j = \{a_1, \dots, a_j\}$$
$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$.

S_i occurs in $A_1, \dots, A_n$ if $\exists j: S_i = A_j$

$$E(X_i) = \Pr(X_i = 1)$$

$$|S_i| = k, X_i = 1 \leftrightarrow S_i = A_k$$

$a_1 \dots a_k$	$a_{k+1} \dots a_n$

$$\Pr(X_i = 1) = \frac{|X_i = 1|}{\#\text{ permutations of } n \text{ numbers}}$$

$$\Pr(X_i = 1) = \frac{k!\,(n-k)!}{n!}$$

$$= \frac{1}{\binom{n}{k}}$$

$$= \frac{1}{\binom{n}{|S_i|}}$$

Subsets $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \ldots, a_n$ be a uniformly random permutation of $1, \ldots, n$

$$A_1, \ldots, A_n : A_j = \{a_1, \ldots, a_j\}$$

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \ldots, S_m$ of $\{1,2,\ldots,n\}$.

S_i occurs in $A_1, \ldots, A_n$ if $\exists j: S_i = A_j$

$$E(X_i) = \Pr(X_i = 1)$$

$$|S_i| = k, X_i = 1 \leftrightarrow S_i = A_k$$

$a_1 \ldots a_k$	$a_{k+1} \ldots a_n$

$$E(X_i) = \Pr(X_i = 1) = \frac{1}{\binom{n}{|S_i|}}$$

$$E(X) = E\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} E(X_i)$$

$$= \sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}}$$

Subsets $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$ such that

$$\forall i \neq j, S_i \not\subseteq S_j \wedge S_j \not\subseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \dots, a_n$ be a uniformly random permutation of $1, \dots, n$

$$A_1, \dots, A_n : A_j = \{a_1, \dots, a_j\}$$
$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

Given $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$.

S_i occurs in $A_1, \dots, A_n$ if $\exists j : S_i = A_j$

$$E(X) = \sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}}$$

also

$E(X) \leq 1$, thus

$$1 \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}}$$

There are no random variables here anymore (but we got here with r.v. and expected value)

We want to use the above expression to bound m. We will use $\binom{n}{|S_i|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

For a fixed n, what is the largest number we can obtain using binomial coefficients?

$$\binom{n}{|S_i|} \leq \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$$

Not a proof, but note how for a particular row n, Pascal's triangle gets bigger in the middle and smaller on the edges.

$$\binom{n}{|S_i|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Thus $\frac{1}{\binom{n}{|S_i|}} \geq \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$

$$1 \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{|S_i|}} \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{m}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

$$1 \geq \frac{m}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ (QED)

Subsets $S_1, S_2, \dots, S_m$ of $\{1,2,\dots,n\}$ such that

$$\forall i \neq j, S_i \nsubseteq S_j \wedge S_j \nsubseteq S_i$$

How large can m be?

Sperner (1928): $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$

Let $a_1, \dots, a_n$ be a uniformly random permutation of $1, \dots, n$

$$A_1, \dots, A_n : A_j = \{a_1, \dots, a_j\}$$

$$A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots A_n$$

S_i occurs in $A_1, \dots, A_n$ if $\exists j : S_i = A_j$

We have shown that

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

using random variables and expected value.

However, this is a condition that has nothing to do with randomness or expectation.