INDICATOR RANDOM VARIABLES

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$

Experiment -> success with prob p

-> failure prob $1 - p$

Coin: H with prob p

T with prob $1 - p$

We perform an experiment until success.

X is number of trials. Then:

$$\mathrm{E}(\mathrm{X}) = \frac{1}{p}$$

For example, if flip a coin with probability $p = \frac{1}{2}$ of flipping heads, then

$$X = \text{number of coin flips until heads comes up}$$

$$E(X) = \frac{1}{p} = \frac{1}{1/2} = 2.$$

If we play the lottery each week with a probability $p = \frac{1}{1\,000\,000}$ of winning the jackpot, then (on average)

$$X = \text{number of tickets bought until winner}$$

$$E(X) = \frac{1}{1/1\,000\,000} = 1\,000\,000$$

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$

Coin: H with prob p
T with prob $1-p$

Flip the coin n times.

$$X = \text{number of heads}$$

What is E(X)?

What would our intuition be?

If $p = \frac{1}{2}$, thus we flip a fair coin n times, then we would think $E(X) = \frac{1}{2}n$.

If $p = \frac{1}{3}$...

On average every third coin flip should be H.

Thus $E(X) = \frac{1}{3}n$
If $p = \frac{1}{4}$, $E(X) = \frac{1}{4} \cdot p$ .... we think

T H T T

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1 - p$

Flip the coin n times.

$$X = \text{number of heads}$$

$$\mathrm{E(X)} = pn?$$

We will solve this by iterating over the range of X.

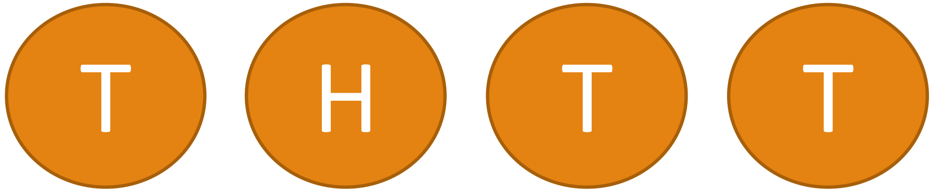
$$E(X) = \sum_k k \cdot \Pr(X = k)$$

where k takes on all possible values of X.

X can be 0, if all n flips come up tails (0 heads).

Or X can be n if all flips come up heads.

Thus X (and by extension, k) can take on all values from 0 to n (inclusive).



Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X)} = pn?$$

$$S = \{f_1 f_2 f_3 \dots f_n : f_i \in \{H, T\}\}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \Pr(X = k)$$

Each $X = k$ is an Event, thus a subset of S.
The probability of an Event is the sum of the probabilities of each Outcome in the Event.

Consider: $n = 4$ and the Event:

"$X = 2$" = "2 heads were flipped"

= $\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$

What is the probability of each of these Outcomes?

T H H T

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$

H with prob p

T with prob $1 - p$

Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn?}$$

$$S = \{f_1 f_2 f_3 \ldots f_n : f_i \in \{H, T\}\}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \Pr(X = k)$$

$n = 4$ and "$X = 2$":

$$\begin{aligned}
HHTT&: \quad p \cdot p \cdot (1-p) \cdot (1-p) &&= p^2(1-p)^2 \\
HTHT&: \quad p \cdot (1-p) \cdot p \cdot (1-p) &&= p^2(1-p)^2 \\
HTTH&: \quad p \cdot (1-p) \cdot (1-p) \cdot p &&= p^2(1-p)^2 \\
THHT&: \quad (1-p) \cdot (1-p) \cdot p \cdot p &&= p^2(1-p)^2 \\
THTH&: \quad (1-p) \cdot p \cdot (1-p) \cdot p &&= p^2(1-p)^2 \\
TTHH&: \quad (1-p) \cdot p \cdot p \cdot (1-p) &&= p^2(1-p)^2
\end{aligned}$$

We sum up all the probabilities in an Event, thus

$$\Pr(X = 2) = 6 \cdot p^2(1-p)^2 = \binom{4}{2} \cdot p^2(1-p)^2$$

T H H T

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn?}$$

$$S = \{f_1 f_2 f_3 \ldots f_n : f_i \in \{H, T\}\}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \Pr(X = k)$$

$$\Pr(X = 2) = \binom{4}{2} \cdot p^2 (1-p)^2$$

If there n coin flips, then the number of flips that have k heads showing is (choose k out of n possible):

$$\binom{n}{k}$$

Each of the k heads has probability p of being flipped, and each of the $n-k$ tails has probability $(1-p)$:

$$\Pr(X = k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

T H H T

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn?}$$

$$S = \{f_1 f_2 f_3 \ldots f_n : f_i \in \{H, T\}\}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \boxed{\Pr(X = k)}$$

$$\boxed{\Pr(X = k)} = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

And this should simplify to pn.
Again, if k was taken out, then it looks like Newton.

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

How do we introduce a k?

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn?}$$

$$S = \{f_1 f_2 f_3 \dots f_n : f_i \in \{H, T\}\}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

We can take the derivative with respect to y.

RHS:

$$\frac{d}{dy} \cdot \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^{k-1}$$

LHS:

$$\frac{d}{dy}(x+y)^n = n \cdot (x+y)^{n-1}$$

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X)} = \mathrm{pn?}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

$$n \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^{k-1}$$

Now multiply both sides by y.

$$n \cdot y \cdot (x+y)^{n-1} = y \cdot \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^{k-1}$$

$$n \cdot y \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^{k}$$

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$n \cdot y \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1-p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn}$$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

Now we substitute:

$x = (1-p)$ and
$y = p$

$$= n \cdot p \cdot \big(p + (1-p)\big)^{n-1}$$

$$= n \cdot p \cdot 1^{n-1}$$

$$= np$$

Is there an easier way? (I should hope so.)

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$n \cdot y \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
T with prob $1 - p$
Flip the coin n times.

$$X = \text{number of heads}$$
$$\mathrm{E(X) = pn}$$

Much easier way to prove $E(X) = pn$.

We will make new random variables:

$X_1 \dots X_n$:

$$X_i = \begin{cases} 1, & \text{if ith flip is heads} \\ 0, & \text{if ith flip is tails} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

Since each time $X_i = 1$ that corresponds to a time we flipped heads.

$$E(X) = E(X_1 + X_2 + \cdots + X_n)$$
$$E(X_1) + E(X_2) + \cdots + E(X_n)$$

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$n \cdot y \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$

H with prob p

T with prob $1 - p$

Flip the coin n times.

$$X = \text{number of heads}$$

$$\mathrm{E(X) = pn}$$

Much easier way to prove $E(X) = pn$.

$X_1 \ldots X_n$:

$$X_i = \begin{cases} 1, & \text{if ith flip is heads} \\ 0, & \text{if ith flip is tails} \end{cases}$$

$$E(X_i) = \sum_k \mathrm{k} \cdot \Pr(X_i = k)$$

However, $k = 1$ or $k = 0$.

$$E(X_i) = 0 \cdot \Pr(X_i = 0) + 1 \cdot \Pr(X_i = 1)$$

$$\mathrm{E(X_i)} = \Pr(X_i = 1)$$

$$\boxed{\Pr(X_i = 1) = p}$$

$$E(X_i) = p$$

Newton:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$n \cdot y \cdot (x+y)^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{n-k} y^k$$

$0 < p < 1$
H with prob p
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Flip the coin n times.

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$$\mathrm{E(X) = pn}$$

Much easier way to prove $E(X) = pn$.

$X_1 \dots X_n$:

$$X_i = \begin{cases} 1, & \text{if ith flip is heads} \\ 0, & \text{if ith flip is tails} \end{cases}$$

$$E(X_i) = p$$

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$= p + p + \cdots + p$$

$$= np$$

These are known as *Indicator Random Variables.*

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	**1**	0	**1**	**1**	**1**	**1**	0	**1**	**1**

1 2 3

"A maximal contiguous subsequence of 1's."

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

$X =$ number of blocks

For this example $X = 3$.

What is $E(X)$?

How can we compute it?

Attempt 1: iterate over the possible values.

$$E(X) = \sum_k k \cdot \Pr(X = k)$$

What are the possible values for k?

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	**1**	0	**1**	**1**	**1**	**1**	0	**1**	**1**

1 2 3

"A maximal contiguous subsequence of 1's."

Given random bitstrings of length n

Each bit -> 0 with prob ½
-> 1 with prob ½

X = number of blocks

$$E(X) = \sum_k k \cdot \Pr(X = k)$$

What are the possible values for k?

00000000000000000000000000…

If the bitstring is all 0's, then $X = 0$.

1010101010101010101010…1[0]

The largest possible number of blocks is $\lceil n/2 \rceil$

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

1 2 3

"A maximal contiguous subsequence of 1's."

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

X = number of blocks, what is $E(X)$?

Attempt 1: Iterate over the range of X:

$$E(X) = \sum_{k=0}^{\lceil n/2 \rceil} k \cdot \Pr(X = k)$$

What is $\Pr(X = k)$?

For example, $\Pr(X = 3)$?

1010100000000000
1100110011000000
1101111000001100
1100011000011100
...

Trying to count bitstrings of length n with exactly 5 blocks looks like too much work.

We will use Indicator Random Variables to make it easier

A “block” of 1’s is a consecutive sequence of 1’s with 0’s or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

$X = \#$ of blocks

$= \#$ of starting positions

What is $E(X)$?

Notice that each block has a leftmost 1.

Instead of explicitly counting blocks, we can count each position that is the leftmost 1 of a block.

Define Indicator Random Variables:

$$X_1, X_2, X_3, \dots, X_n$$

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

Then

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

For example:

$X = 0 + 1 + 0 + 1 + 0 + 0 + 0 + 0 + 0 + 1 + 0$

A “block” of 1’s is a consecutive sequence of 1’s with 0’s or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	**1**	0	**1**	**1**	**1**	**1**	0	**1**	**1**

Given random bitstrings of length n

Each bit -> 0 with prob ½
-> 1 with prob ½

$X = \#\text{ of blocks}$
$\quad = \#\text{ of starting positions}$

What is $E(X)$?

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Then

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

For example:

$$X = 0 + \boxed{1} + 0 + \boxed{1} + 0 + 0 + 0 + 0 + 0 + \boxed{1} + 0$$

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½

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$X = \#$ of blocks

$= \#$ of starting positions

What is $E(X)$?

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

$$E(X) = E(X_1 + X_2 + X_3 + \cdots + X_n)$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

If we can determine each $E(X_i)$ correctly, we can easily determine $E(X)$.

$$E(X_i) = 0 \cdot \Pr(X_i = 0) + 1 \cdot \Pr(X_i = 1)$$

$$E(X_i) = \Pr(X_i = 1)$$

A “block” of 1’s is a consecutive sequence of 1’s with 0’s or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½
-> 1 with prob ½

X = # of blocks
= # of starting positions

What is $E(X)$?

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

$$E(X) = E(X_1 + X_2 + X_3 + \cdots + X_n)$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

If we can determine each $E(X_i)$ correctly, we can easily determine $E(X)$.

$$E(X_i) = \Pr(X_i = 1)$$

$2 \leq i \leq n$: $X_i = 1 \leftrightarrow$

	i-1	i	
	0	1	

$$E(X_i) = \Pr(X_i = 1) = 1/2 \cdot 1/2 = 1/4$$

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

$X = \#$ of blocks

$= \#$ of starting positions

What is $E(X)$?

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$

$$E(X) = E(X_1 + X_2 + X_3 + \cdots + X_n)$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

If we can determine each $E(X_i)$ correctly, we can easily determine $E(X)$.

$$E(X_i) = \Pr(X_i = 1)$$

$i = 1$: $X_1 = 1 \leftrightarrow$

1	2	3	…
1			

$$E(X_1) = \Pr(X_1 = 1) = 1/2$$

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

$X = \#$ of blocks

$= \#$ of starting positions

What is $E(X)$?

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

$$X = X_1 + X_2 + X_3 + \cdots + X_n$$
$$E(X) = E(X_1 + X_2 + X_3 + \cdots + X_n)$$
$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

$$E(X) = \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} \ldots + \frac{1}{4}\right)$$

$$E(X) = \frac{1}{2} + (n-1) \cdot \left(\frac{1}{4}\right)$$

$$E(X) = \frac{2+n-1}{4} = \frac{n+1}{4}$$

A "block" of 1's is a consecutive sequence of 1's with 0's or nothing on either side.

1	2	3	4	5	6	7	8	9	10
0	1	0	1	1	1	1	0	1	1

Given random bitstrings of length n

Each bit -> 0 with prob ½

-> 1 with prob ½

$X = \#$ of blocks

$= \#$ of starting positions

What is $E(X)$?

$$X_i = \begin{cases} 1 \text{ if a block starts at } i \\ 0 \text{ otherwise} \end{cases}$$

$$E(X) = E(X_1) + E(X_2) + E(X_3) + \cdots + E(X_n)$$

$$E(X) = \frac{n+1}{4}$$

Are these indicator random variables independent? NO

If $X_2 = 1$, what do we know about X_3?

X_3 must equal 0

X_1 must also equal 0

Linearity of expectation does not need independent random variables.