CONDITIONAL PROBABILITY

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Let's Make a Deal!

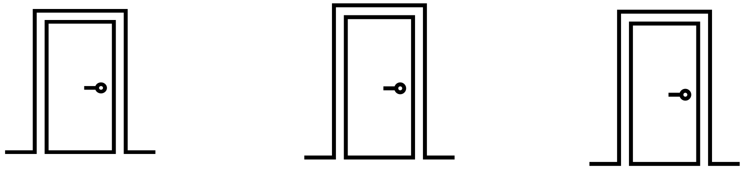
One door – Tesla Roadster
Two doors – Dogecoin

D_1 D_2 D_3

Three doors, and you do not know what is behind any of them.

The game is as follows:

1. Choose uniformly random door (but don't open it, ex. D_1)



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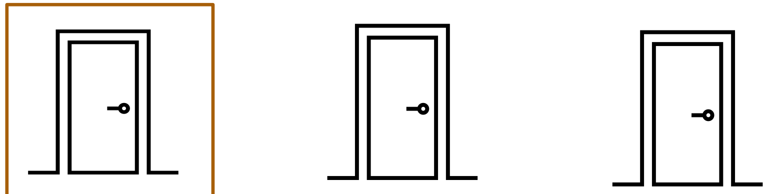
2. Out of the unselected doors (D_2 and D_3) Monty Hall opens one door with Dogecoin (ex. D_3).



D_1 D_2 D_3

You choose D_1

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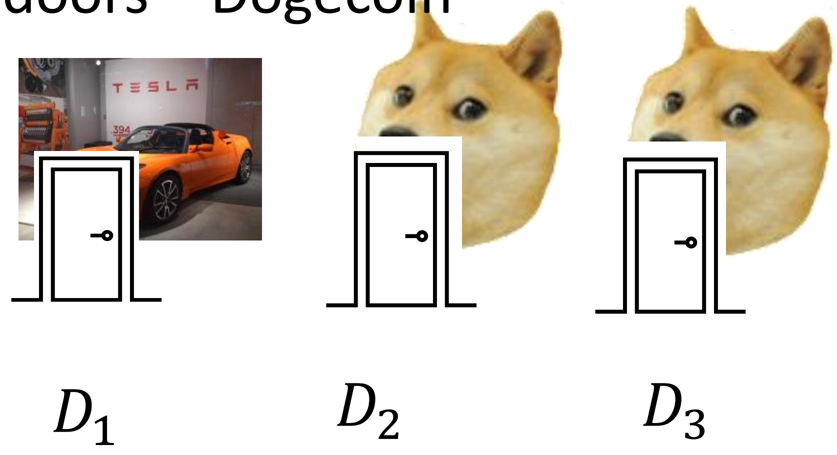

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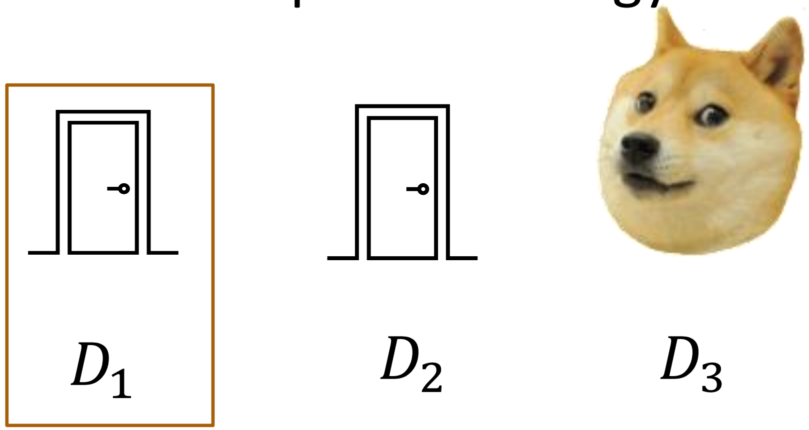
The game is as follows:

1. Choose uniformly random door (but don't open it, ex. D_1)
2. Out of the unselected doors (D_2 and D_3) Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

What do you do?
Is there a superior strategy?



Do you keep D_1 or choose D_2?

You choose D_1

Monty shows you D_3

Let's Make a Deal!

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2. Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

What sort of strategy should we use?

One thought - Monty shows you the Doge, then probability of car being behind each remaining door is ½

This is wrong – why?

What do we know?

Monty Hall knows where the Tesla is.

Monty Hall will never show you the door hiding the Tesla.

He will always show you a door hiding Doge.

Let's Make a Deal!

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1. Choose uniformly random door (but don't open it, ex. D_1)
2. Monty Hall opens one door with Dogecoin (ex. D_3).
3. Make decision – keep your door D_1 or open other door D_2?

Monty is actually giving you information because his selection is not random.

Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Let's try this strategy:

The first door we pick has something random behind it.

What happens if we select a door with Dogecoin?

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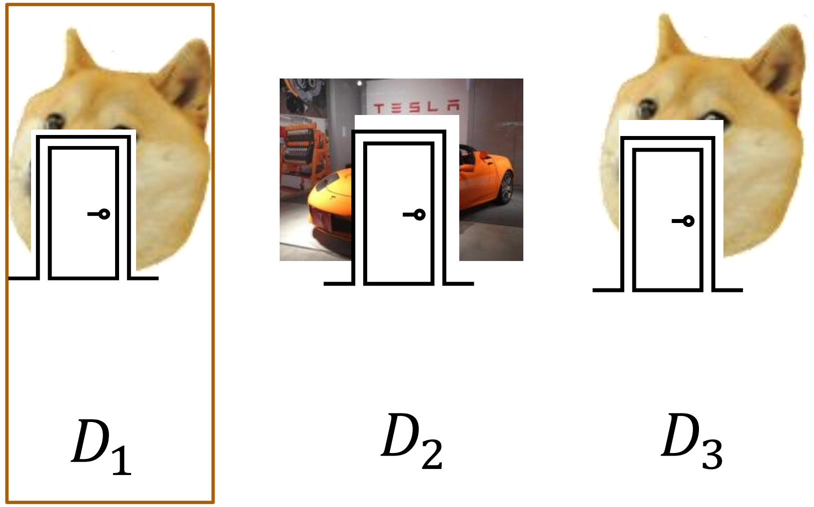
3. Make decision – keep your door D_1 or open other door D_2?

Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Dogecoin.

Monty reveals D_2 or D_3. But of course he must show D_3.



You choose D_1 Monty shows you D_3

Let's Make a Deal!

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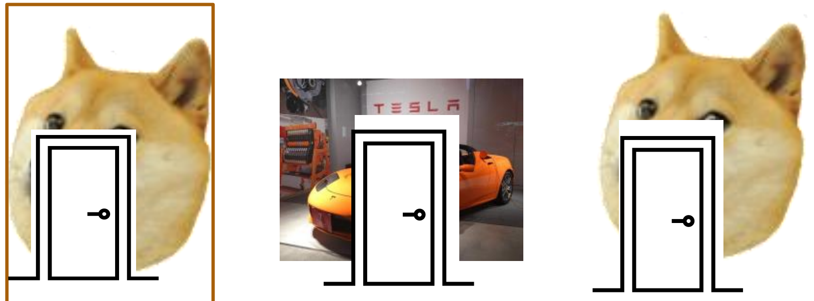
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Dogecoin.

Monty reveals D_2 or D_3. But of course he must show D_3.

Now we switch:
We always win the Tesla if we first select Dogecoin



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You choose D_1 Monty shows you D_3

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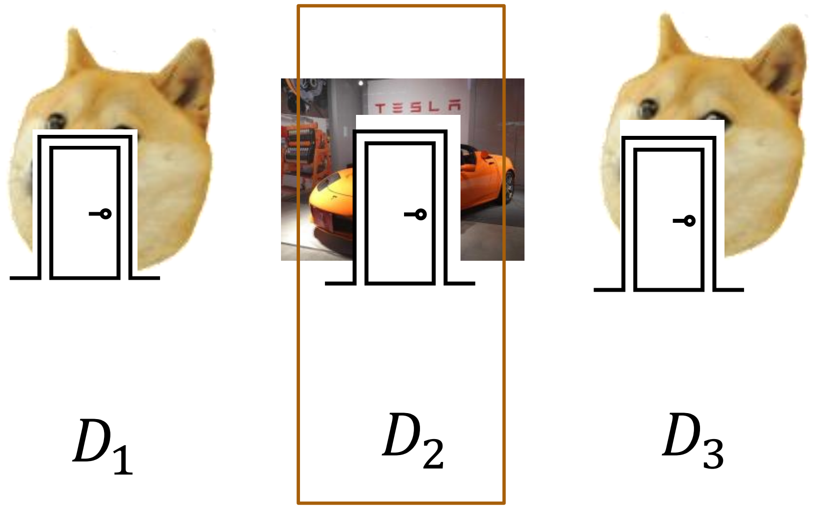
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Tesla.

Monty reveals D_1 or D_3. In this case he can choose either.



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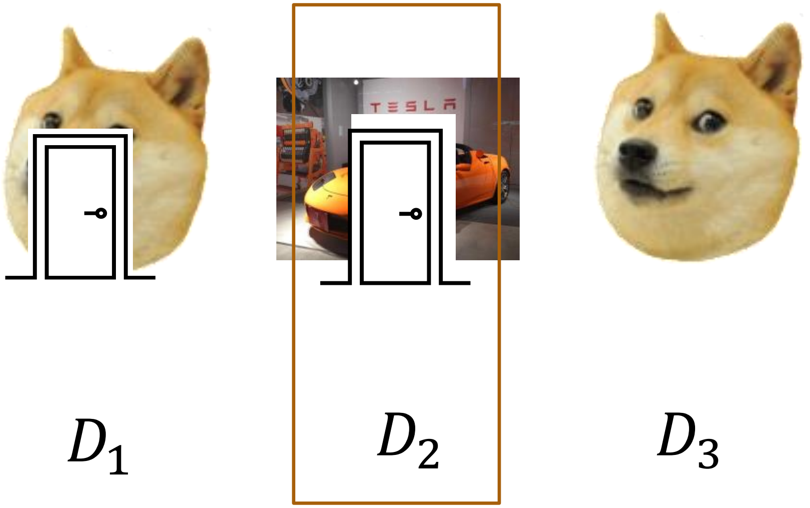
Claim: After Monty reveals Dogecoin, it is always better to switch doors.

Assume our first selection is Tesla.

Monty reveals D_1 or D_3. In this case he can choose either.

We switch.

We always find Dogecoin.



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If we always switch doors:

Win Tesla $\leftrightarrow$ door chosen in step 1 has Dogecoin.

$$\Pr(\text{first door has Doge}) = \frac{2}{3}$$

Win Dogecoin $\leftrightarrow$ door chosen in step 1 has Tesla.

$$\Pr(\text{first door has Tesla}) = \frac{1}{3}$$

What is behind the door selected in step 1 is random. So what are the probabilities?
Always switching gives us probability $\frac{2}{3}$ of winning

Let's Make a Deal!

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1. Choose uniformly random door (but don't open it, ex. D_1)
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By knowing that Monty always reveals Doge, we arrived at different probabilities than we would suspect.

$$\Pr(\text{first door has Doge}) = \frac{2}{3}$$

$$\Pr(\text{first door has Tesla}) = \frac{1}{3}$$

This is conditional probability, which we will formalize.

Anil's Kids

Anil Maheshwari has 2 kids.

We are told that at least 1 of his kids is a boy.

When each child was born:

$$\Pr(\text{child is a boy}) = \frac{1}{2}$$

$$\Pr(\text{child is a girl}) = \frac{1}{2}$$

Given that at least 1 kid is a boy, what is the $\Pr(\text{both are boys}) = ?$

We know 1 is a boy, so guess might be that the probability other is a boy is ½.

But again, we are given some (incomplete) information, and we should account for it.

Anil's Kids

Anil Maheshwari has 2 kids, at least 1 is a boy.

$\Pr(\text{child is a boy}) = ½$
$\Pr(\text{child is a girl}) = ½$

$\Pr(\text{both are boys}) = ?$

We know one is a boy, but we don't know which one.

Let's look at the sample space S: Anil has 2 kids.

All the possible combinations of 2 kids is

$S = \{bb, bg, gb, gg\}$

The first character represents the older child.

The second character represents the younger child.

Each of these outcomes has equal probability.

Our extra information – at least 1 is a boy – shrinks the sample space

Anil's Kids

Anil Maheshwari has 2 kids, at least 1 is a boy.

$\Pr(\text{child is a boy}) = ½$
$\Pr(\text{child is a girl}) = ½$

$\Pr(\text{both are boys}) = ?$

If we said the *oldest* is a boy, then $S = \{bb, bg\}$

$\Pr(\text{both are boys}) =$ ½.

$S = \{bb, bg, gb, gg\}$

What are the outcomes of S that have at least 1 boy?

$$S' = \{bb, bg, gb\}$$

We know that the outcome cannot be gg.

Now we have a sample space S' and an event $BB = \{bb\}$. What is the probability of BB?

$$\Pr(BB) = \frac{|BB|}{|S'|}$$

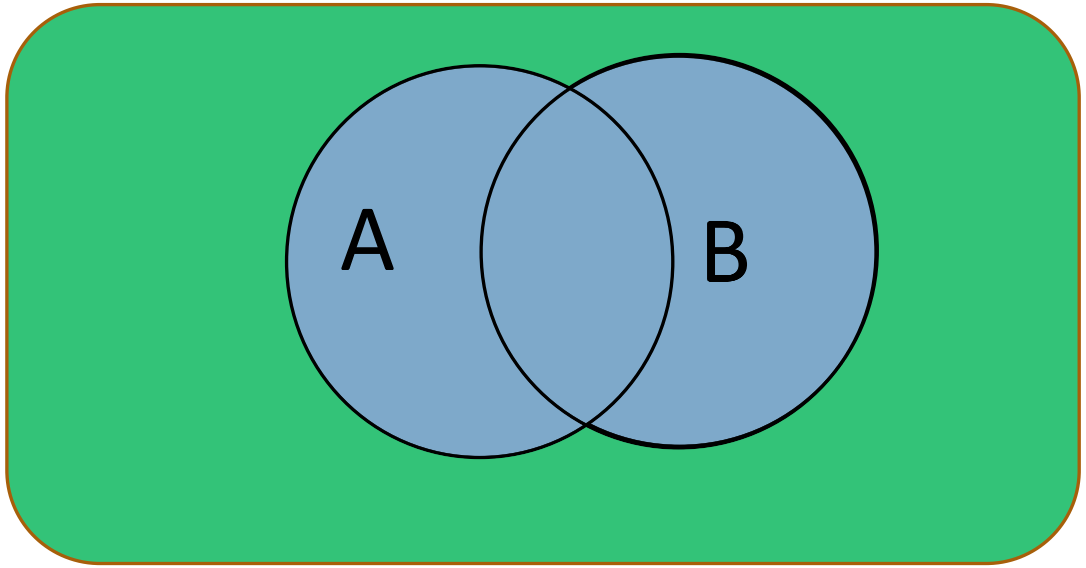
$$= \frac{1}{3}$$

Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

$$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)}$$



One way to think of it is since we are told B is true, the set B becomes the new sample space.

Then $\Pr(A|B)$ is the probability of selecting an element of A from the sample space B.

Note that if we have uniform probability, then this is the probability of event $A \cap B$ occurring in the sample space B.

$$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)} = \frac{|A \cap B|/|S|}{|B|/|S|} = \frac{|A \cap \mathrm{B}|}{|B|}$$

This does NOT generalize (but can be useful).

Conditional Probability

Events $A, B, \Pr(B) > 0$

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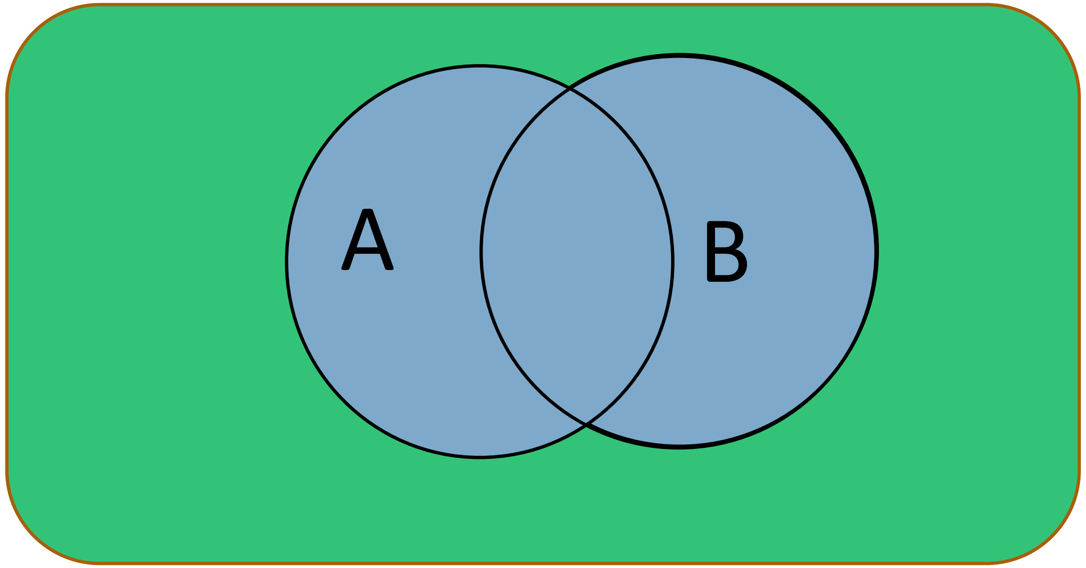


$$\Pr(A|A) = \frac{\Pr(A \cap A)}{\Pr(A)} = \frac{\Pr(A)}{\Pr(A)} = 1$$

Anil's kids:

Sample space S = two kids = $\{gg, gb, bg, bb\}$

Event B = at least one boy = $\{gb, bg, bb\}$

Event A = both are boys = $\{bb\}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{1/4}{3/4} = \frac{1}{3}$$

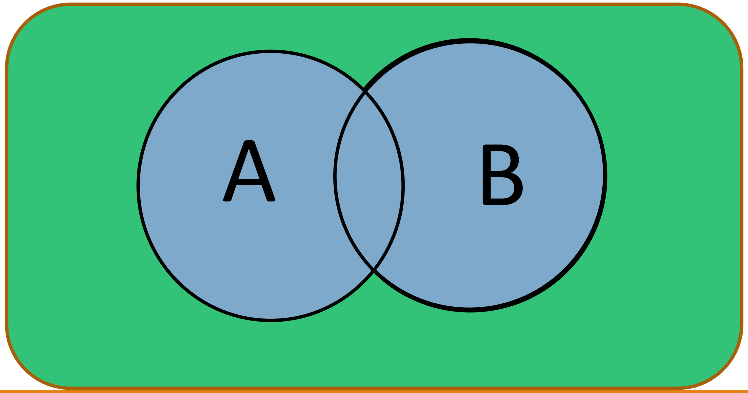
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

Is there a relationship between $\Pr(A|B)$ and $\Pr(B|A)$?



Roll fair die: $S = \{1, 2, 3, 4, 5, 6\}$

$A =$ "result is 3" $= \{3\}$
$B =$ "result is odd" $= \{1, 3, 5\}$
$C =$ "result is prime" $= \{2, 3, 5\}$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/6}{1/6} = 1$$

In general $\Pr(A|B) \neq \Pr(B|A)$

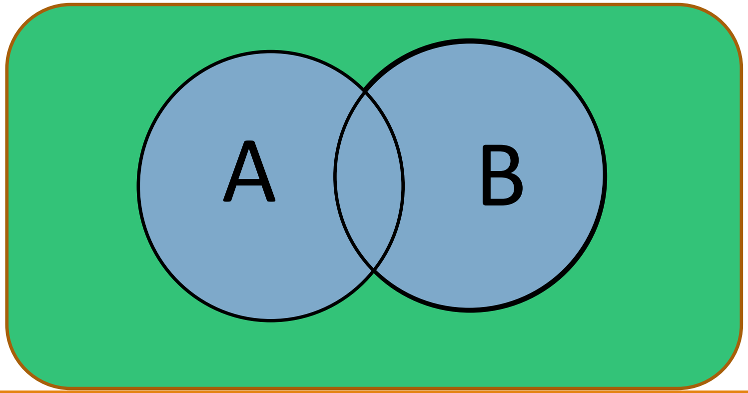
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B)$ = probability of A given B

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Is there a relationship between $\Pr(C|B)$ and $\Pr(C|\bar{B})$?



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$\bar{B}$ = "result is even" $= \{2, 4, 6\}$

$$\Pr(C|\bar{B}) = \frac{\Pr(C \cap \bar{B})}{\Pr(\bar{B})} = \frac{1/6}{3/6} = \frac{1}{3}$$

$$\Pr(C|B) = \frac{\Pr(C \cap B)}{\Pr(B)} = \frac{2/6}{3/6} = \frac{2}{3}$$

$$\Pr(C|\bar{B}) + \Pr(C|B) = \frac{1}{3} + \frac{2}{3} = 1$$

Is this always true?

Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

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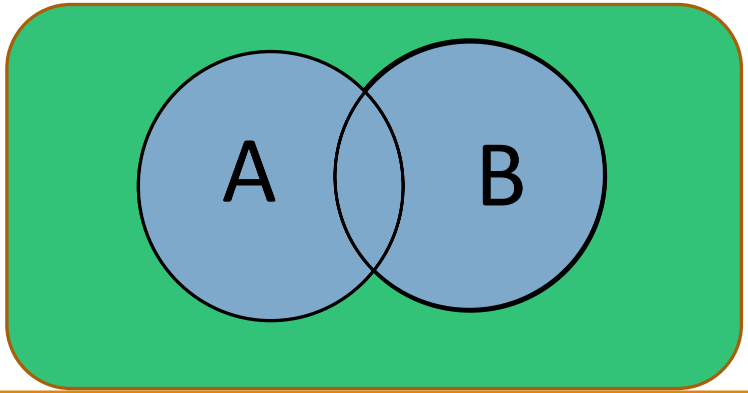

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$\bar{B} =$ "result is even" $= \{2, 4, 6\}$
$\bar{A} = \{1,2,4,5,6\}$

$$\Pr(C|A) + \Pr(C|\bar{A}) = \frac{\Pr(C \cap A)}{\Pr(A)} + \frac{\Pr(C \cap \bar{A})}{\Pr(\bar{A})}$$
$$= \frac{1/6}{1/6} + \frac{2/6}{5/6}$$
$$= \frac{1}{1} + \frac{2}{5} > 1$$

Not true in general.

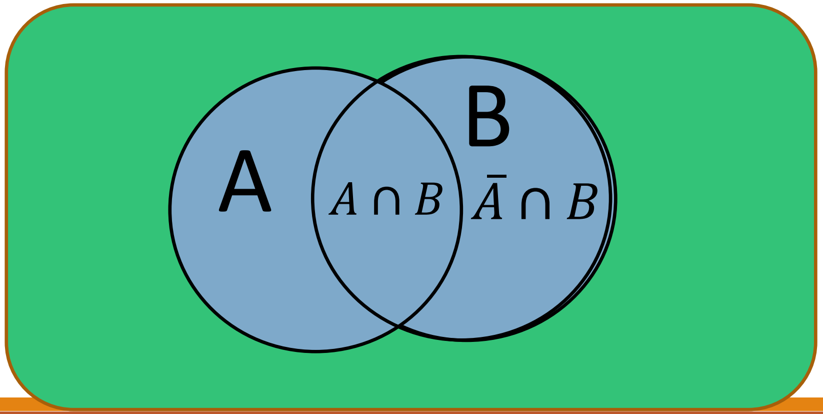
Conditional Probability

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) =$ probability of A given B

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Is there a relationship between $\Pr(A|B)$ and $\Pr(\bar{A}|B)$?



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$\bar{A} = \{1,2,4,5,6\}$

$$\Pr(A|B) + \Pr(\bar{A}|B)$$
$$= \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(\bar{A} \cap B)}{\Pr(B)}$$
$$= \frac{1/6}{3/6} + \frac{2/6}{3/6}$$

$$= \frac{1}{3} + \frac{2}{3} = 1 \text{ always}$$

Conditional Probability

Anil has 2 kids.

1 kid is a boy who was born on Sunday.

What is the probability that Anil has 2 boys?

The first thing we should do is determine the sample space.

The sample space is all combinations of 2 children. But now the children have 2 "stats".

Each child has a gender and they were born on some day of the week.

We can count S using the Product Rule, by building each individual element.

$S = \{(g_1, d_1, g_2, d_2) \mid$

for $i \in \{1,2\}$, $g_i = \text{gender of child } i$
$d_i = \text{day of the week child } i \text{ was born on}$

where

$g_i \in \{\text{girl}, \text{boy}\}$
$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$

Thus $(\text{girl}, \text{Fri}) \in S$, $(\text{boy}, \text{Sun}) \in S$, etc.

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

2 ways to choose g_1
7 ways to choose d_1
2 ways to choose g_2
7 ways to choose d_2

Thus there are $2 \cdot 7 \cdot 2 \cdot 7 = 196$ elements in S, or $|S| = 196$.

Let A be the event that Anil has 2 boys.

Let B be the event that Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(B) = \frac{|\mathrm{B}|}{|\mathrm{S}|}$$

$B = B_1 \cup B_2$ where
$B_1 = \text{1st kid is a boy born on Sunday}$
$B_2 = \text{2nd kid is a boy born on Sunday}$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\} \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$B_1 = 1^{\text{st}} \text{ kid is a boy born on Sunday}$$

$$B_1 = \{ (\text{boy}, \text{Sun}, g_2, d_2) |$$
$$g_2 \in \{\text{girl}, \text{boy}\},$$
$$d_2 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in B_1?

1 way to choose g_1
1 way to choose d_1
2 ways to choose g_2
7 ways to choose d_2

$$|B_1| = 1 \cdot 1 \cdot 2 \cdot 7 = 14$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\}, d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B_2 = 2^{\text{nd}}$ kid is a boy born on Sunday

$$B_2 = \{\ (g_1, d_1, \text{boy}, \text{Sun}) | g_1 \in \{\text{girl}, \text{boy}\}, d_1 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in B_2?

2 ways to choose g_1
7 ways to choose d_1
1 ways to choose g_2
1 ways to choose d_2

$$|B_2| = 2 \cdot 7 \cdot 1 \cdot 1 = 14$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B =$ Anil has ≥ 1 boy born on Sunday

$$B_1 = 1^{\text{st}} \text{ kid is a boy born on Sunday}$$
$$B_2 = 2^{\text{nd}} \text{ kid is a boy born on Sunday}$$

$$B = B_1 \cup B_2$$

Thus $|B| = |B_1| + |B_2| - |B_1 \cap B_2|$

$$B_1 \cap B_2 = \{ (\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

How many elements in $\mathrm{B}_1 \cap B_2$?

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
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$$|S| = 196$$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$B_1 =$ 1st $\text{kid is a boy born on Sunday}$
$B_2 =$ 2nd $\text{kid is a boy born on Sunday}$

How many elements in $B_1 \cap B_2$?

$$B_1 \cap B_2 = \{ (\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

$$|B_1 \cap B_2| = 1$$

$$B = B_1 \cup B_2$$

$$\begin{aligned} |B| &= |B_1| + |B_2| - |B_1 \cap B_2| \\ &= 14 + 14 - 1 \\ &= 27 \end{aligned}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\} \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$A \cap B = \text{has 2 boys and}$
$\geq 1 \text{ boy was born on Sunday}$

$$A \cap B = AB_1 \cup AB_2$$

Where:

$$AB_1 = \text{2 boys, first born Sun}$$
$$AB_2 = \text{2 boys, second born Sun}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_1 = 2 \text{ boys, first born Sun}$$

$$AB_1 = \{\ (\text{boy}, \text{Sun}, \text{boy}, d_2) |$$
$$d_2 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in AB_1?

1 way to choose g_1
1 way to choose d_1
1 way to choose g_2
7 ways to choose d_2

$$|AB_1| = 1 \cdot 1 \cdot 1 \cdot 7 = 7$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_2 = 2 \text{ boys, second born Sun}$$

$$AB_2 = \{\ (\text{boy}, d_1, \text{boy}, \text{Sun}) |$$
$$d_1 \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\}$$

How many elements in AB_2?

1 way to choose g_1
7 ways to choose d_1
1 way to choose g_2
1 way to choose d_2

$$|AB_2| = 1 \cdot 7 \cdot 1 \cdot 1 = 7$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
$$d_i \in \{\text{Mon}, \text{Tues}, \text{Wed}, \text{Thurs}, \text{Fri}, \text{Sat}, \text{Sun}\}\ \}$$

$|S| = 196$
Let $A =$ Anil has 2 boys.
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$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$AB_1 = 2 \text{ boys, first born Sun}$$
$$AB_2 = 2 \text{ boys, second born Sun}$$

$$AB_1 \cap AB_2 = \{\ (\text{boy}, \text{Sun}, \text{boy}, \text{Sun})\}$$

How many elements in $AB_1 \cap AB_2$?

$$|AB_1 \cap AB_2| = 1$$

$$A \cap B = AB_1 \cup AB_2$$

$$\begin{aligned} |AB_1 \cup AB_2| &= |AB_1| + |AB_2| - |AB_1 \cap AB_2| \\ &= 7 + 7\ - 1 \\ &= 13 \end{aligned}$$

Conditional Probability

Anil has 2 kids. 1 kid is a boy who was born on Sunday.
What is the probability that Anil has 2 boys?

$$S = \{(g_1, d_1, g_2, d_2) | g_i \in \{\text{girl}, \text{boy}\},$$
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$|S| = 196$

Let $A =$ Anil has 2 boys.

Let $B =$ Anil has ≥ 1 boy born on Sunday.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{|\mathrm{A} \cap \mathrm{B}|/|\mathrm{S}|}{|\mathrm{B}|/|\mathrm{S}|}$$

$$= \frac{13/196}{27/196}$$

$$= \frac{13}{27} \approx 0.48$$