RANDOM VARIABLES

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

S: Sample space
Outcome: Element of S
Event: Subset of S
$\Pr(x) : x \in S \rightarrow [0,1]$
$\sum_{w \in S} \Pr(w) = 1$

What is a Random Variable?

"neither random, nor variable."

It is a function that maps each outcome from a given sample space S to a value.

We have already seen random variables when we talked about rolling two dice and taking the sum.

Roll two dice:

$$S = \{(i,j):\ 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

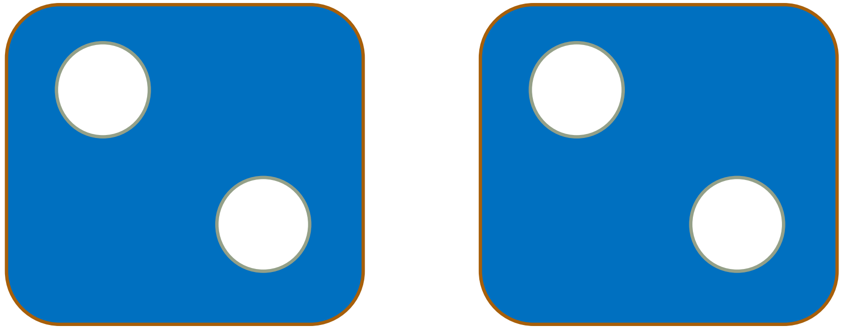
The sample space is all pairs (i,j). The function takes these outcomes as an argument and returns a value.

function $X: S \rightarrow \mathbb{R}$

$$X(i,j) = i + j$$

For example:

$$X(2,2) = 4$$



S: Sample space
Outcome: Element of S
Event: Subset of S
$\Pr(x) : x \in S \to [0,1]$
$\sum_{w \in S} \Pr(w) = 1$

In general:

Random variable is a function that takes an element of the sample space as an argument and maps it to a value.

function $X: S \to \mathbb{R}$

defines the random variable X.

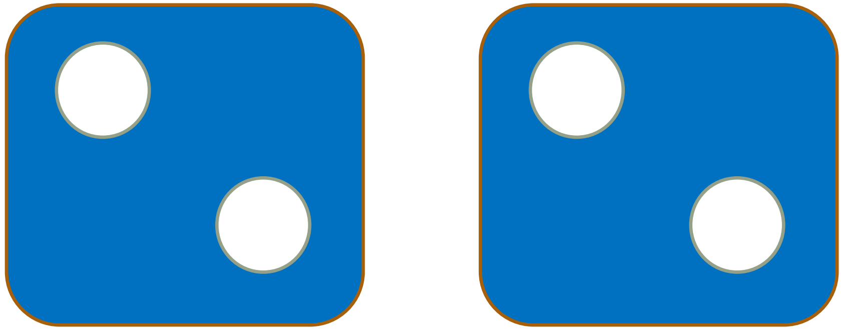
Roll two dice:

$$S = \{(i,j):\ 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

function $X: S \to \mathbb{R}$

$$X(i,j) = i + j$$
$$X(2,2) = 4$$

We often want to map outcomes to values.
We typically describe it in English.
The random variable $X =$ Sum of two rolls.
We often omit the argument.



S: Sample space

Outcome: Element of S

Event: Subset of S

$\Pr(x) : x \in S \rightarrow [0,1]$

$\sum_{w \in S} \Pr(w) = 1$

Random Variable $X: S \rightarrow \mathbb{R}$

"neither random nor variable"

Flip a fair coin 3 times:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

X = number of heads in 3 flips

$$Y = \begin{cases} 1 \text{ all three flips are equal} \\ 0 \text{ otherwise} \end{cases}$$

$$\begin{aligned} X(HHH) &= 3 \\ X(HHT) &= 2 \\ X(HTH) &= 2 \\ X(HTT) &= 1 \\ X(TTT) &= 0 \\ &etc \end{aligned}$$

$$\begin{aligned} Y(HHH) &= 1 \\ Y(HHT) &= 0 \\ Y(HTH) &= 0 \\ Y(TTT) &= 1 \\ &etc \end{aligned}$$

Much easier to simply describe them in English

T H T

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

We can now describe events using Random Variables.

An event is a subset of the sample space. Each value of a random variable also corresponds to a subset of the sample space.

Event "X=0" $\leftrightarrow$ subset $\{TTT\}$
Event "X=0“ = ”There are no heads in 3 flips”

Event "X=1" $\leftrightarrow$ $\{HTT, THT, TTH\}$
Event "X=1“ = “There was one head in 3 coin flips”

Event "X=4" $\leftrightarrow$ $\emptyset$
Event "X=4" = “There were 4 heads in 3 coin flips”

Event "Y=1" $\leftrightarrow$ $\{HHH, TTT\}$
"Y=2 " $\leftrightarrow$ $\emptyset$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

X = number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

Event "X=0" $\leftrightarrow$ subset $\{TTT\}$

"X=1" $\leftrightarrow$ $\{HTT, THT, TTH\}$

"X=4" $\leftrightarrow$ $\emptyset$

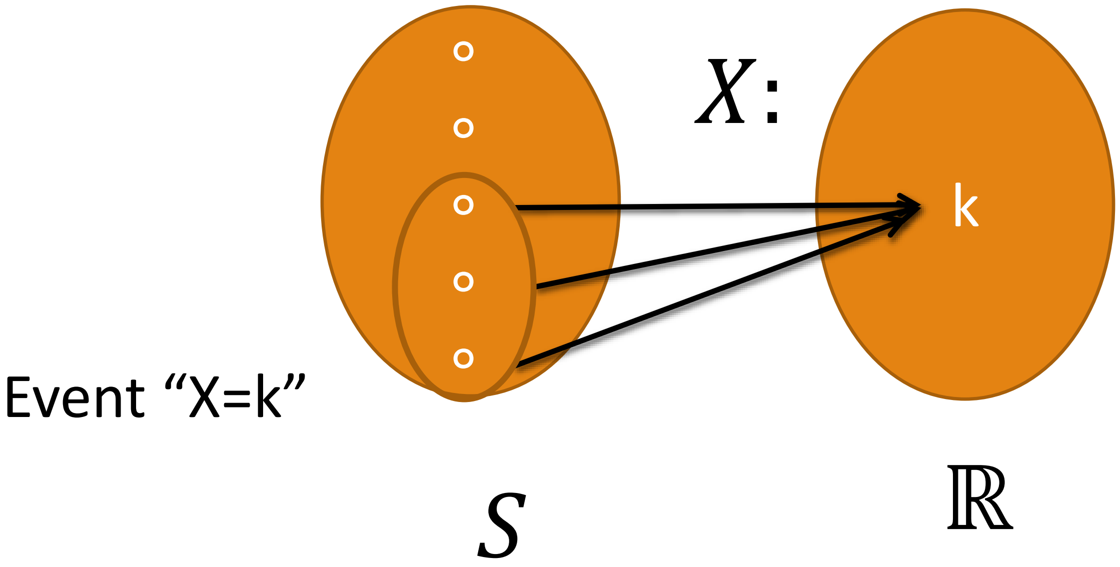
Event "Y=1" $\leftrightarrow$ $\{HHH, TTT\}$

"Y=2 " $\leftrightarrow$ $\emptyset$

In general: $X: S \rightarrow \mathbb{R}$

Event "X=k" $\leftrightarrow$ $\{w \in S: X(w) = k\}$

$$\Pr(X = k) = \sum_{w:X(w)=k} \Pr(w)$$



Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

Event "X=0" $\leftrightarrow$ subset $\{TTT\}$

"X=1" $\leftrightarrow \{HTT, THT, TTH\}$

"X=2" $\leftrightarrow \{HHT, HTH, THH\}$

"X=3" $\leftrightarrow \{HHH\}$

Event "Y=1" $\leftrightarrow \{HHH, TTT\}$

"Y=2 " $\leftrightarrow \emptyset$

$$\Pr(X = 0) = \Pr(\{TTT\}) = \frac{1}{8}$$

$$\Pr(X = 1) = \Pr(\{HTT, THT, TTH\}) = \frac{3}{8}$$

$$\Pr(X = 2) = \Pr(\{HHT, HTH, THH\}) = \frac{3}{8}$$

$$\Pr(X = 3) = \Pr(\{HHH\}) = \frac{1}{8}$$

$$\Pr(X = 4) = \Pr(\emptyset) = 0$$

Observe that the sum is 1.

In this case the random variable spans all possibilities.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

Event "X=0" ↔ subset $\{TTT\}$

"X=1" ↔ $\{HTT, THT, TTH\}$

"X=2" ↔ $\{HHT, HTH, THH\}$

"X=3" ↔ $\{HHH\}$

Event "Y=1" ↔ $\{HHH, TTT\}$

"Y=2 " ↔ ∅

$$\Pr(Y = 1) = \Pr(\{TTT, HHH\}) = \frac{2}{8} = \frac{1}{4}$$

$$\Pr(Y = 0) = \Pr(\{HHT, HTH, THH, TTH, THT, TTH\}) = \frac{6}{8} = \frac{3}{4}$$

We can also observe that the values of Y span every possibility in the sample space. Since there are only 2 possibilities, each event is the complement of the other. Thus:

$$\Pr(Y = 0) = 1 - \Pr(Y = 1) = \frac{3}{4}$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

Event "X=0" $\leftrightarrow$ subset $\{TTT\}$

"X=1" $\leftrightarrow \{HTT, THT, TTH\}$

"X=2" $\leftrightarrow \{HHT, HTH, THH\}$

"X=3" $\leftrightarrow \{HHH\}$

Event "Y=1" $\leftrightarrow \{HHH, TTT\}$

"Y=2 " $\leftrightarrow \emptyset$

We can define other events based on the values of random variables:

Event: $X \geq 2$

$\leftrightarrow \{HHH, HHT, HTH, THH\}$

$$\Pr(X \geq 2) = \Pr(\{HHH, HHT, HTH, THH\}) = \frac{4}{8} = \frac{1}{2}$$

Or we can express this event as:

$$\Pr(X \geq 2) = \Pr(X = 2\ or\ X = 3)$$

Since these are disjoint we can use the sum rule:

$$\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

Event "X=0" ↔ subset $\{TTT\}$

"X=1" ↔ $\{HTT, THT, TTH\}$

"X=2" ↔ $\{HHT, HTH, THH\}$

"X=3" ↔ $\{HHH\}$

Event "Y=1" ↔ $\{HHH, TTT\}$

"Y=2 " ↔ ∅

We can define other events based on the values of random variables:

Event: $X \geq 2$

$\leftrightarrow \{HHH, HHT, HTH, THH\}$

$$\Pr(X \geq 2) = \Pr(\{HHH, HHT, HTH, THH\}) = \frac{4}{8} = \frac{1}{2}$$

Uniform probability

$$\Pr(X \geq 2) = \Pr(X = 2\ or\ X = 3)$$

Since these are disjoint we can use the sum rule:

$$\Pr(X \geq 2) = \Pr(X = 2) + \Pr(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$$

S: Sample space
Outcome: Element of S
Event: Subset of S
$\Pr(x): x \in S \to [0,1]$
$\sum_{w \in S} \Pr(w) = 1$

Random Variable:

function $X: S \to \mathbb{R}$

"neither random nor variable"

Take two random variables
$X: S \to \mathbb{R}$ and
$Y: S \to \mathbb{R}$

Recall if events A and B then

$$\Pr(A \wedge B) = \Pr(A) \cdot \Pr(B)$$

There is also a notion of 2 random variables being independent.

“If I know the value of X, then that has no influence on the probability that Y has any particular value”

“If I know the value of Y, then that has no influence on the probability that X has any particular value”

$$\Pr(X = k | Y = \ell) = \Pr(X = k)$$

S: Sample space
Outcome: Element of S
Event: Subset of S
$\Pr(x) : x \in S \rightarrow [0,1]$
$\sum_{w \in S} \Pr(w) = 1$

Random Variable:

function $X: S \rightarrow \mathbb{R}$

"neither random nor variable"

Take two random variables
$X: S \rightarrow \mathbb{R}$ and
$Y: S \rightarrow \mathbb{R}$

Formally, random variables X and Y are independent if:

$\forall k, \forall \ell$, the events "$X = k$" and "$Y = \ell$" are independent.

Events "$X = k$" and "$Y = \ell$“ are independent if

$\Pr(X = k \wedge Y = \ell) = \Pr(X = k) \cdot \Pr(Y = \ell)$.

We must verify this for all possible combinations of values of k and ℓ.

Difficult to do since there can be many possible values k, ℓ for X and Y.

S: Sample space
Outcome: Element of S
Event: Subset of S
$\Pr(x) : x \in S \rightarrow [0,1]$
$\sum_{w \in S} \Pr(w) = 1$

Random Variable:

function $X: S \rightarrow \mathbb{R}$

"neither random nor variable"

Take two random variables
$X: S \rightarrow \mathbb{R}$ and
$Y: S \rightarrow \mathbb{R}$

An easier thing to prove is that they are NOT independent.

$\neg(\forall k, \forall \ell,$ the events "$X = k$" and "$Y = \ell$" are independent)

$\exists k\ \exists \ell,$ the events "$X = k$" and "$Y = \ell$" are NOT independent

We must find two values k and ℓ such that:

$\exists k\ \exists \ell: \Pr(X = k \wedge Y = \ell) \neq \Pr(X = k) \cdot \Pr(Y = \ell)$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

X and Y independent?

To verify, look at all possible values

$$X = k, k \in \{0,1,2,3\}$$
$$Y = \ell, \ell \in \{0,1\}$$

4 choices for k, 2 choices for ℓ, must verify 8 equations

To show that X and Y are independent, we should verify the following:

1. $\Pr(X = 0 \wedge Y = 0) = \Pr(X = 0) \cdot \Pr(Y = 0)$
2. $\Pr(X = 1 \wedge Y = 0) = \Pr(X = 1) \cdot \Pr(Y = 0)$
3. $\Pr(X = 2 \wedge Y = 0) = \Pr(X = 2) \cdot \Pr(Y = 0)$
4. $\Pr(X = 3 \wedge Y = 0) = \Pr(X = 3) \cdot \Pr(Y = 0)$
5. $\Pr(X = 0 \wedge Y = 1) = \Pr(X = 0) \cdot \Pr(Y = 1)$
6. $\Pr(X = 1 \wedge Y = 1) = \Pr(X = 1) \cdot \Pr(Y = 1)$
7. $\Pr(X = 2 \wedge Y = 1) = \Pr(X = 2) \cdot \Pr(Y = 1)$
8. $\Pr(X = 3 \wedge Y = 1) = \Pr(X = 3) \cdot \Pr(Y = 1)$

This is a lot. Perhaps we can use our intuition to find events that are not independent.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

X and Y independent?

To verify, look at all possible values

$$X = k, k \in \{0,1,2,3\}$$
$$Y = \ell, \ell \in \{0,1\}$$

To show NOT independent, find one counter-example

Does the value of one imply something about the value of the other?

If $Y = 1$ this is the event $\{HHH, TTT\}$

What values are possible for X?

We may have $X = 3$ or $X = 0$

Consider if $X = 2$ this is the event

$\{HHT, HTH, HTT, THH, THT, TTH\}$

Then $Y = 0$.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

X and Y independent?

To verify, look at all possible values

$$X = k, k \in \{0,1,2,3\}$$
$$Y = \ell, \ell \in \{0,1\}$$

To show NOT independent, find one counter-example

Does the value of one imply something about the value of the other?

If $Y = 1$ then $X = 0$ or $X = 3$
if $X = 2$ then $Y = 0$

So they clearly have an effect on each other

$\exists k\ \exists \ell$: $\Pr(X = k \land Y = \ell) \neq \Pr(X = k) \cdot \Pr(Y = \ell)$

Easiest to find values that make it impossible for both to be true, i.e., $\Pr(X = k \land Y = \ell) = 0$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

X and Y independent?

To verify, look at all possible values

$$X = k, k \in \{0,1,2,3\}$$
$$Y = \ell, \ell \in \{0,1\}$$

To show NOT independent, find one counter-example

We can try the values:

$$k = 2 \wedge \ell = 1$$

We have

$$\Pr(X = 2 \wedge Y = 1) = 0$$

while

$$\Pr(X = 2) \cdot \Pr(Y = 1)$$

$$= \frac{3}{8} \cdot \frac{1}{4}$$

$$= \frac{3}{32}$$

$$\neq 0$$

Therefore X and Y are not independent

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.
Are Y and Z independent?

Y and Z each may take on 2 values, therefore each defines 2 events, and all combinations give us 4 equations to verify:

1. $\Pr(Y = 1 \wedge Z = 1) = \Pr(Y = 1) \cdot \Pr(Z = 1)$
2. $\Pr(Y = 1 \wedge Z = 0) = \Pr(Y = 1) \cdot \Pr(Z = 0)$
3. $\Pr(Y = 0 \wedge Z = 1) = \Pr(Y = 0) \cdot \Pr(Z = 1)$
4. $\Pr(Y = 0 \wedge Z = 0) = \Pr(Y = 0) \cdot \Pr(Z = 0)$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT,\\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.

Are Y and Z independent?

Equation 1:

$$\Pr(Y = 1 \wedge Z = 1) = \Pr(Y = 1) \cdot \Pr(Z = 1)$$

Event $(Y = 1 \wedge Z = 1) = \{HHH\}$

$$\Pr(Y = 1 \wedge Z = 1) = \frac{|\{HHH\}|}{|S|} = \frac{1}{8}$$

Event $(Y = 1) = \{HHH, TTT\}$

$$\Pr(Y = 1) = \frac{|\{HHH, TTT\}|}{|S|} = \frac{1}{4}$$

Event $(Z = 1) = \{HHH, HHT, HTH, HTT\}$

$$\Pr(Z = 1) = \frac{|\{HHH, HHT, HTH, HTT\}|}{|S|} = \frac{1}{2}$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.

Are Y and Z independent?

Equation 1:

$$\Pr(Y = 1 \wedge Z = 1) = \Pr(Y = 1) \cdot \Pr(Z = 1)$$

$$\frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2}$$

$$\frac{1}{8} = \frac{1}{8}$$

Thus events $Y = 1$ and $Z = 1$ are independent.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.
Are Y and Z independent?

Equation 2:

$$\Pr(Y = 1 \wedge Z = 0) = \Pr(Y = 1) \cdot \Pr(Z = 0)$$

Event $(Y = 1 \wedge Z = 0) = \{TTT\}$

$$\Pr(Y = 1 \wedge Z = 0) = \frac{|\{TTT\}|}{|S|} = \frac{1}{8}$$

Event $(Y = 1) = \frac{1}{4}$

Event $(Z = 0) = \{THH, THT, TTH, TTT\}$

$$\Pr(Z = 1) = \frac{|\{THH, THT, TTH, TTT\}|}{|S|} = \frac{1}{2}$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

X = number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

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We know that X and Y are not independent.
Are Y and Z independent?

Equation 2:

$$\Pr(Y = 1 \wedge Z = 0) = \Pr(Y = 1) \cdot \Pr(Z = 0)$$

$$\frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2}$$

$$\frac{1}{8} = \frac{1}{8}$$

Thus events $Y = 1$ and $Z = 0$ are independent.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.
Are Y and Z independent?

Equation 3:

$$\Pr(Y = 0 \wedge Z = 1) = \Pr(Y = 0) \cdot \Pr(Z = 1)$$

Event $(Y = 0 \wedge Z = 1) = \{HHT, HTH, HTT\}$

$$\Pr(Y = 0 \wedge Z = 1) = \frac{|\{HHT, HTH, HTT\}|}{|S|}$$

$$= \frac{3}{8}$$

Event $(Y = 0) = 1 - \Pr(Y = 1) = 1 - \frac{1}{4} = \frac{3}{4}$

Event $(Z = 1) = \{HHH, HHT, HTH, HTT\}$

$$\Pr(Z = 1) = \frac{1}{2}$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.
Are Y and Z independent?

Equation 3:

$$\Pr(Y = 0 \wedge Z = 1) = \Pr(Y = 0) \cdot \Pr(Z = 1)$$

$$\frac{3}{8} = \frac{3}{4} \cdot \frac{1}{2}$$

$$\frac{3}{8} = \frac{3}{8}$$

Thus events $Y = 0$ and $Z = 1$ are independent.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.
Are Y and Z independent?

Equation 4:

$$\Pr(Y = 0 \land Z = 0) = \Pr(Y = 0) \cdot \Pr(Z = 0)$$

Event $(Y = 0 \land Z = 0) = \{TTH, THT, THH\}$

$$\Pr(Y = 0 \land Z = 0) = \frac{|\{TTH, THT, THH\}|}{|S|}$$

$$= \frac{3}{8}$$

Event $(Y = 0) = 1 - \Pr(Y = 1) = 1 - \frac{1}{4} = \frac{3}{4}$

Event $(Z = 0) = \frac{1}{2}$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

We know that X and Y are not independent.

Are Y and Z independent?

Equation 4:

$$\Pr(Y = 0 \wedge Z = 0) = \Pr(Y = 0) \cdot \Pr(Z = 0)$$

$$\frac{3}{8} = \frac{3}{4} \cdot \frac{1}{2}$$

$$\frac{3}{8} = \frac{3}{8}$$

Thus events $Y = 0$ and $Z = 0$ are independent.

We have verified all 4 equations, thus Y and Z are independent.

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

Are X and Z independent?

We would have to verify 8 equations, however…

What if $X = 0$?

Then $Z = 0$.

The value of $X = 0$ completely determines the value of Z.

Then the event $(X = 0 \land Z = 1) = \emptyset$

$X = 0$ and $Z = 1$ are candidates to prove not independent, since

$$Pr(X = 0 \land Z = 1) = 0$$

Flip a fair coin 3 times

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$X =$ number of heads in 3 flips

$$Y = \begin{cases} 1, & all\ 3\ flips\ equal \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & first\ flip\ is\ H \\ 0, & otherwise \end{cases}$$

Are X and Z independent?

We would have to verify 8 equations, however…

Verify:

$$\Pr(X = 0 \wedge Z = 1) = \Pr(X = 0) \cdot \Pr(Z = 1)$$

$$\Pr(\emptyset) = 0 \neq \frac{1}{8} \cdot \frac{1}{2}$$

Some other possible candidates:

If $Z = 0$ then $X \in \{0,1,2\}$, and $X \neq 3$
If $Z = 1$ then $X \in \{1,2,3\}$, and $X \neq 0$

Many such dependencies, but it is sufficient to find only 1 such example

Consider the sample set and random variables below:

$$S = \{2, 3, 5, 30\}$$

$$X = \begin{cases} 1, & if\ x\ is\ divisible\ by\ 2 \\ 0, & otherwise \end{cases}$$

$$Y = \begin{cases} 1, & if\ x\ is\ divisible\ by\ 3 \\ 0, & otherwise \end{cases}$$

$$Z = \begin{cases} 1, & if\ x\ is\ divisible\ by\ 5 \\ 0, & otherwise \end{cases}$$

Can we determine if they are mutually independent? What do we need to verify?

We need to show that they are pairwise independent, but also…

We need to show for all combinations of 3 that the equation holds.

For example:

$$\Pr(X = 0 \wedge Y = 0 \wedge Z = 0)$$
$$= \Pr(X = 0) \cdot (\Pr(Y = 0) \cdot \Pr(Z = 0)$$

We need to do this for all values of X, Y, and Z.