COMP 2804 — Assignment 4

Due: Sunday December 6, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

1. Expected value - simple
2. Independent random variables
3. Expected value - harder

Question 1:

- Write your name and student number.

Question 2: Craps is a game where you roll two dice and bet on the outcome. There are many ways to play, but we will focus on one specific scenario. On your "come-out" roll you roll a 9 on two dice. Now the game is this: you must keep rolling 2 dice until you either roll 9 again, in which case you win, or you roll a 7, in which case the house wins (you lose).

1. (a) Find the probability that you win.

(b) Find the probability that the house wins.

Show all your work for both parts. Observe that the probabilities from 1a and 1b added together should total 1, but you may NOT use this fact to determine your answers.

Solution: The probability of rolling a 7 is $\frac{6}{36}$. The probability of rolling a 9 is $\frac{4}{36}$. The probability of rolling neither is $\frac{26}{36}$. The probability of rolling a 9 before a 7 is

$$\begin{aligned}
&\sum_{i=0}^{\infty}\left(\frac{26}{36}\right)^i\cdot\left(\frac{4}{36}\right)\\
=&\left(\frac{4}{36}\right)\cdot\sum_{i=0}^{\infty}\left(\frac{26}{36}\right)^i\\
=&\left(\frac{4}{36}\right)\cdot\left(\frac{1}{1-\frac{26}{36}}\right)\\
=&\left(\frac{4}{36}\right)\cdot\left(\frac{36}{10}\right)\\
=&\frac{2}{5}.
\end{aligned}$$

The probability of rolling a 7 before a 9 is

$$\begin{aligned}
&\sum_{i=0}^{\infty}\left(\frac{26}{36}\right)^i\cdot\left(\frac{6}{36}\right)\\
=&\left(\frac{6}{36}\right)\cdot\sum_{i=0}^{\infty}\left(\frac{26}{36}\right)^i\\
=&\left(\frac{6}{36}\right)\cdot\left(\frac{1}{1-\frac{26}{36}}\right)\\
=&\left(\frac{6}{36}\right)\cdot\left(\frac{36}{10}\right)\\
=&\frac{3}{5}
\end{aligned}$$

.

2. Assume you have bet \$10. You roll a 9 on your "come-out" roll, same as above. If you win the casino pays you \$10. If you lose the casino takes your \$10 bet. What is your expected winnings or losings (i.e., the expected value) of this round of craps?

   Solution: Let A be the event that you win and B the event that you lose. Let X be a random variable such that $X(A) = 10$ and $X(B) = -10$. Then

$$\begin{aligned} E(X) &= X(A) \cdot Pr(A) + X(B) \cdot Pr(B) \\ &= 10 \cdot \frac{2}{5} - 10 \cdot \frac{3}{5} \\ &= 4 - 6 \\ &= -2. \end{aligned}$$

Thus you should expect to lose \$2 per round, on average, if you roll a 9 to "come-out". Most casino games give a negative number. If you count cards in Blackjack you can sometimes inch your odds into the positive, but larger casinos have automatic shufflers that make counting cards almost useless. Plus they will ban you from the casino (or worse) if they suspect you of counting cards.

Question 3: You are playing a board game about pirates that uses special six-sided fair dice. Three sides of each die has a zero on it, while the other three sides have the numbers 1 through 3 (this represents damage done by your ship's cannons). Your ship has five cannons, so to attack you roll 5 of these six-sided dice. Let

$$\begin{aligned} A &= \text{The sum of the values of the 6 dice.} \\ B &= \text{The number of times a zero is showing.} \end{aligned}$$

1. What is $E(A)$ and E(B)?

   Solution: Let A_i be the value of die i. Then $A = \sum_{i=1}^{5} A_i$, and by linearity of expectation

   $$\begin{aligned} E(A) &= E\left(\sum_{i=1}^{5} A_i\right) \\ &= \sum_{i=1}^{5} E(A_i). \end{aligned}$$

   Each value on die i comes up with probability $\frac{1}{6}$. There are three 0's and one each of $1, 2, 3$. Then for each individual die i:

   $$\begin{aligned} E(A_i) &= 0 \cdot Pr(A_i = 0) + 1 \cdot Pr(A_i = 1) + 2 \cdot Pr(A_i = 2) + 3 \cdot Pr(A_i = 3) \\ &= 0 \cdot \frac{3}{6} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} \\ &= \frac{6}{6} = 1. \end{aligned}$$

Then

$$\begin{aligned}
E(A) &= E\left(\sum_{i=1}^{5} A_i\right) \\
&= \sum_{i=1}^{5} E(A_i) \\
&= \sum_{i=1}^{5} 1 \\
&= 5.
\end{aligned}$$

Let $B_i = 1$ if $A_i = 0$ and let $B_i = 0$ if $A_i \neq 0$. Then $B = \sum_{i=1}^{5} B_i$. By linearity of expectation we have

$$\begin{aligned}
E(B) &= E\left(\sum_{i=1}^{5} B_i\right) \\
&= \sum_{i=1}^{5} E(B_i).
\end{aligned}$$

Then for each individual die i:

$$\begin{aligned}
E(B_i) &= 1 \cdot Pr(B_i = 0) + 0 \cdot Pr(B_i \neq 0) \\
E(B_i) &= 1 \cdot \frac{3}{6} + 0 \cdot \frac{3}{6} \\
&= \frac{3}{6} = \frac{1}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
E(B) &= E\left(\sum_{i=1}^{5} B_i\right) \\
&= \sum_{i=1}^{5} E(B_i) \\
&= \sum_{i=1}^{5} \frac{1}{2} \\
&= \frac{5}{2}.
\end{aligned}$$

2. Are A and B independent random variables?

   Solution: A and B are independent if all events $A = k$ and $B = \ell$ are independent. Or they are not independent if you find one counter-example.

   Consider the events $B = 6$ (every die comes up 0) and $A = 18$ (every die comes up 3). If A and B are independent it must be that $Pr(A = 18 \wedge B = 6) = Pr(A = 18) \cdot Pr(B = 6)$. Observe that if $B = 6$ then all dice came up 0, which means that $A = 0$. Thus $Pr(A = 18 \wedge B = 6) = 0$. $Pr(B = 6) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$. $Pr(A = 18) = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$. Thus $Pr(A = 18) \cdot Pr(B = 6) = \frac{1}{32} \cdot \frac{1}{7776} \neq 0$, therefore A and B are not independent.

Question 4: When FX and his girlfriend XF have a child, this child is a boy with probability 1/2 and a girl with probability 1/2, independently of the sex of their other children. FX and XF stop having children as soon as they have three girls or two boys.

Consider the random variables

$$\begin{aligned} C &= \text{the number of children that FX and XF have,} \\ G &= \text{the number of girls that FX and XF have.} \\ B &= \text{the number of boys that FX and XF have.} \end{aligned}$$

1. Determine the expected values $E(C)$ and $E(G)$ **Solution:** The sample space is $S = \{bb, gbb, bgb, ggbb, gbgb, bggb, ggg, bggg, gbgg, ggbg\}$, with corresponding probabilites $\{\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}\}$ (you may check that they sum to 1).

$$\begin{aligned} E(C) &= \sum_{k=2}^{4} k \cdot Pr(C = k) \\ &= (2 \cdot Pr(C = 2) + 3 \cdot Pr(C = 3) + 4 \cdot Pr(C = 4)) \\ &= \left(2 \cdot \frac{1}{4} + 3 \cdot \left(3 \cdot \frac{1}{8}\right) + 4 \cdot \left(6 \cdot \frac{1}{16}\right)\right) \\ &= \left(\frac{2}{4} + \frac{9}{8} + \frac{24}{16}\right) \\ &= \frac{25}{8} \end{aligned}$$

$$
\begin{aligned}
E(G) &= \sum_{k=0}^{3} k \cdot Pr(G = k) \\
&= (0 \cdot Pr(G = 0) + 1 \cdot Pr(G = 1) + 2 \cdot Pr(G = 2) + 3 \cdot Pr(G = 3)) \\
&= \left(0 \cdot \frac{1}{4} + 1 \cdot \left(2 \cdot \frac{1}{8}\right) + 2 \cdot \left(3 \cdot \frac{1}{16}\right) + 3 \cdot \left(\frac{1}{8} + 3 \cdot \frac{1}{16}\right)\right) \\
&= \left(\frac{2}{8} + \frac{6}{16} + \frac{15}{16}\right) \\
&= \frac{25}{16}
\end{aligned}
$$

2. We are told that the last child FX and XF had was a boy. Determine the expected values $E(B)$ and $E(G)$ and $E(C)$. **Solution:** Let $A =$"The last child is a boy". Then $A = \{bb, gbb, bgb, ggbb, gbgb, bggb\}$, and $Pr(A) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{11}{16}$.

$$
\begin{aligned}
E(B) &= 2 \cdot Pr(B = 2|A) \\
&= 2 \cdot \frac{Pr(B = 2 \cap A)}{Pr(A)} \\
&= 2 \cdot \frac{Pr(A)}{Pr(A)} \\
&= 2.
\end{aligned}
$$

$$
\begin{aligned}
E(G) &= \sum_{k=0}^{2} k \cdot Pr(G = k|A) \\
&= \sum_{k=0}^{2} k \cdot \frac{Pr(G = k \cap A)}{Pr(A)} \\
&= \frac{1}{Pr(A)} \sum_{k=0}^{2} k \cdot Pr(G = k \cap A) \\
&= \frac{1}{Pr(A)} \left(0 \cdot Pr(G = 0 \cap A) + 1 \cdot Pr(G = 1 \cap A) + 2 \cdot Pr(G = 2 \cap A)\right) \\
&= \frac{16}{11} \left(0 \cdot \frac{1}{4} + 1 \cdot \left(2 \cdot \frac{1}{8}\right) + 2 \cdot \left(3 \cdot \frac{1}{16}\right)\right) \\
&= \frac{16}{11} \left(\frac{2}{8} + \frac{6}{16}\right) \\
&= \frac{16}{11} \left(\frac{5}{8}\right) \\
&= \frac{80}{88} \\
&= \frac{10}{11}
\end{aligned}
$$

$$
\begin{aligned}
E(C) &= \sum_{k=2}^{4} k \cdot Pr(C = k|A) \\
&= \sum_{k=2}^{4} k \cdot \frac{Pr(C = k \cap A)}{Pr(A)} \\
&= \frac{1}{Pr(A)} \sum_{k=2}^{4} k \cdot Pr(C = k \cap A) \\
&= \frac{1}{Pr(A)} \left(2 \cdot Pr(C = 2 \cap A) + 3 \cdot Pr(C = 3 \cap A) + 4 \cdot Pr(C = 4 \cap A)\right) \\
&= \frac{16}{11} \left(2 \cdot \frac{1}{4} + 3 \cdot \left(2 \cdot \frac{1}{8}\right) + 4 \cdot \left(3 \cdot \frac{1}{16}\right)\right) \\
&= \frac{16}{11} \left(\frac{2}{4} + \frac{6}{8} + \frac{12}{16}\right) \\
&= \frac{16}{11} \left(\frac{16}{8}\right) \\
&= \frac{16}{11} \cdot 2 \\
&= \frac{32}{11}
\end{aligned}
$$

Question 5: In the Dungeons and Dragons question from last assignment we talked about how to roll for stats (that is, we take the sum of 3 six-sided dice). It was briefly mentioned that to get higher stats, we can roll 4 six-sided dice and take the sum of the 3 highest dice. In this question we will compare the expected outcome from both of these techniques.

1. Determine the expected value of the sum of rolling 3 fair six-sided dice.

   Solution: We can use linearity of expectation. Let X be the sum of 3 dice, and let d_i be the value of the ith die. Using linearity of expectation:

$$
\begin{aligned}
E(X) &= E(d_1 + d_2 + d_3) \\
&= E(d_1) + E(d_2) + E(d_3).
\end{aligned}
$$

The expected value of a die d_i is

$$
\begin{aligned}
E(d_i) &= \sum_{j=1}^{6} j \cdot Pr(d_i = j) \\
&= \sum_{j=1}^{6} j \cdot \frac{1}{6} \\
&= \frac{21}{6} \\
&= \frac{7}{2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
E(X) &= E(d_1) + E(d_2) + E(d_3) \\
&= \frac{7}{2} + \frac{7}{2} + \frac{7}{2} \\
&= 10.5
\end{aligned}
$$

2. What is the size of the sample space S of all possible dice rolls with 4 dice?

   Solution: 4 dice with 6 possible outcomes each, $|S| = 6^4$.

Let d_1, d_2, d_3, d_4 be random variables corresponding to the values of 4 six-sided dice after being rolled. Let Y be the value of the three highest dice. That is, $Y = \left(\sum_{i=1}^{4} d_i\right) - \min\{d_1, d_2, d_3, d_4\}$. Thus our goal is to find $E(Y)$ and compare it to the answer from 1.

Let the random variable X_i be the sum of the highest 3 dice if the lowest die is equal to i, that is $\min\{d_1, d_2, d_3, d_4\} = i$. Let $X_i = 0$ if the lowest die is not equal to i. Let A_i be the event that the lowest die out of d_1, d_2, d_3, and d_4 is equal to i.

3. In a few sentences explain why

$$
\sum_{\omega \in S} Y(\omega) = \sum_{i=1}^{6} \sum_{\omega \in S} X_i(\omega) = \sum_{i=1}^{6} \sum_{\omega \in A_i} X_i(\omega)
$$

   Solution: For the second equality, X_i only takes on values when $\omega \in A_i$, and it is 0 otherwise. To show that $\sum_{\omega \in S} Y(\omega) = \sum_{i=1}^{6} \sum_{\omega \in A_i} X_i(\omega)$, observe that $Y = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$, and all of $A_i, 1 \le i \le 6$ are dijoint sets. Thus $\sum_{i=1}^{6} \sum_{\omega \in A_i}$ iterates over each $\omega \in Y$ exactly once. $X_i(\omega) = Y(\omega)$ when $\omega \in A_i$, which completes the explanation.

4. Use the above expression and linearity of expectation to express $E(Y)$ in terms of $E(X_i(\omega))$ given that $\omega \in A_i$.

   Solution: By definition of expected value we have

$$
\begin{aligned}
E\left(\sum_{\omega \in S} Y(\omega)\right) &= \sum_{\omega \in S} E(Y(\omega)) \\
&= |S| E(Y(\omega)) && \text{there are } |S| \text{ elements in S} \\
E\left(\sum_{\omega \in S} Y(\omega)\right) &= E\left(\sum_{i=1}^{6} \sum_{\omega \in A_i} X_i(\omega)\right) \\
&= \sum_{i=1}^{6} \sum_{\omega \in A_i} E\left(X_i(\omega)\right)
\end{aligned}
$$

   So we get:

$$
\begin{aligned}
E(Y) &= E(Y(\omega)) \\
&= \frac{E\left(\sum_{\omega \in S} Y(\omega)\right)}{|S|} \\
&= \frac{\sum_{i=1}^{6} \sum_{\omega \in A_i} E\left(X_i(\omega)\right)}{|S|}
\end{aligned}
$$

5. Let A_i' be the event that d_1 is the lowest die out of $\{d_1, d_2, d_3, d_4\}$ and that $d_1 = i$. Let $X_i'(\omega) = d_2 + d_3 + d_4$ if $\omega \in A_i'$ and let $X_i'(\omega) = 0$ if $\omega \notin A_i'$. Find $E(X_i'(\omega))$ given that $\omega \in A_i'$. That is, assuming that i is the lowest die roll and $d_1 = i$, find the expected value of X_i'. Recall that $\forall \omega \in A_i', X_i'(\omega) = d_2(\omega) + d_3(\omega) + d_4(\omega)$, where each of d_2, d_3, and d_4 is at least i.

   Solution:

$$
\begin{aligned}
E(X_i') &= E(d_2 + d_3 + d_4) \\
&= E(d_2) + E(d_3) + E(d_4).
\end{aligned}
$$

   Observe that $E(d_2) = E(d_3) = E(d_4)$. So we must find the value of a single die roll d

given that $d \geq i$. Observe that $Pr(d \geq i) = \frac{6-i+1}{6}$.

$$\begin{aligned} E(d) &= \sum_{j=1}^{6} j \cdot Pr(d = j | d \geq i) \\ &= \sum_{j=1}^{6} j \cdot \frac{Pr(d = j \cap d \geq i)}{Pr(d \geq i)}. \end{aligned}$$

If $j < i$, then $Pr(d = j \cap d \geq i) = 0$, and when $j \geq i, Pr(d = j \cap d \geq i) = Pr(d = j)$, thus

$$\begin{aligned} E(d) &= \sum_{j=1}^{6} j \cdot \frac{Pr(d = j \cap d \geq i)}{Pr(d \geq i)} \\ &= \sum_{j=i}^{6} j \cdot \frac{Pr(d = j \cap d \geq i)}{Pr(d \geq i)} \\ &= \frac{6}{6-i+1} \sum_{j=i}^{6} j \cdot Pr(d = j) \\ &= \frac{6}{6-i+1} \sum_{j=i}^{6} j \cdot \frac{1}{6} \\ &= \frac{1}{6-i+1} \sum_{j=i}^{6} j \\ &= \frac{1}{6-i+1} \left(\sum_{j=1}^{6} j - \sum_{k=1}^{i-1} k \right) \\ &= \frac{1}{6-i+1} \left(21 - \frac{i(i-1)}{2} \right) \\ &= \frac{42 - i^2 + i}{14 - 2i} \end{aligned}$$

Using the quadratic equation we can see that $-i^2+i+42 = -(i-7)(i+6) = (7-i)(6+i)$. Thus $\frac{42-i^2+i}{14-2i} = \frac{(7-i)(6+i)}{2 \cdot (7-i)} = 3 + i/2$. Now using linearity of expectation, we get

$$\begin{aligned} E(X_i') &= E(d_2 + d_3 + d_4) \\ &= E(d_2) + E(d_3) + E(d_4) \\ &= 3 \cdot \left(3 + \frac{i}{2} \right). \end{aligned}$$

What a mess.

I feel like there must be an easier method based on the observation that $E(d)$ given that $d \geq i$ is simply the average of the values from i to 6, which you can find by adding 6 and i and dividing by 2, giving you $3 + i/2$. Then using linearity of expectation over 3 dice gives you $3 \cdot \left(3 + \frac{i}{2}\right)$. If anyone thinks of a rigourous yet simple solution roughly following those precepts, please let me know.

6. Observe that the term $E(X_i(\omega))$ in $\sum_{\omega \in A_i} E(X_i(\omega))$ is a constant, since it is the weighted average of the values $\forall \omega \in A_i, X_i(\omega)$. Similarly the term $E(X_i'(\omega))$ in $\sum_{\omega \in A_i'} E(X_i'(\omega))$ is also a constant. We will not prove it at this time, but we will use the fact that for the expressions given above, $E(X_i'(\omega)) < E(X_i(\omega))$ to show a lower bound on $E(Y)$. Given that $E(X_i'(\omega)) < E(X_i(\omega))$, briefly explain why $\sum_{\omega \in A_i} E(X_i'(\omega)) < \sum_{\omega \in A_i} E(X_i(\omega))$. **Solution:** Free marks.

7. Recall that A_i is the event that $\min\{d_1, d_2, d_3, d_4\} = i$. Show that $|A_i| = (6 - i + 1)^4 - (6 - i)^4$.

   Solution: You should give the students the hint: Question 8 part 2 in Assignment 3 is a generalization of this question. The easy solution is to let B_i be the event that each of $\{d_1, d_2, d_3, d_4\}$ is at least i. Then $|B_i| = (6 - i + 1)^4$ ways, since each d_i can take on a value from i to 6 and thus there are $(6 - i + 1)$ possible values for each die. Observe that $B_{i+1} \subseteq B_i$, and $A_i = B_i - B_{i+1}$, since $B_i - B_{i+1}$ leaves only rolls where the least die is i. Thus $|A_i| = |B_i| - |B_{i+1}| = (6 - i + 1)^4 - (6 - i)^4$.

8. Explain why $\sum_{\omega \in A_i} E(X_i'(\omega)) = ((6 - i + 1)^4 - (6 - i)^4) \cdot E(X_i'(\omega))$.

   Solution: We've established that $|A_i| = (6 - i + 1)^4 - (6 - i)^4$, thus $\sum_{\omega \in A_i}$ iterates $(6 - i + 1)^4 - (6 - i)^4$ times.

9. Now show that

$$E(Y) > \sum_{i=1}^{6} \left(3 \cdot \left(3 + \frac{i}{2}\right)\right) \cdot \frac{(6 - i + 1)^4 - (6 - i)^4}{6^4}$$

   which, if you plug into Wolfram alpha, is > 11.63.

   Solution:

   Observe that $\sum_{\omega \in S} E(Y) = |S| \cdot E(Y)$. Then

$$
\begin{aligned}
E(Y) &= \sum_{i=1}^{6} \sum_{\omega \in A_i} E\left(X_i(\omega)\right) \cdot \frac{1}{|S|} \\
&> \sum_{i=1}^{6} \sum_{\omega \in A_i} E\left(X_i'(\omega)\right) \cdot \frac{1}{|S|} \\
&= \sum_{i=1}^{6} \sum_{\omega \in A_i} \left(3 \cdot (3 + \frac{i}{2})\right) \cdot \frac{1}{6^4} \\
&= \sum_{i=1}^{6} \left(3 \cdot \left(3 + \frac{i}{2}\right)\right) \cdot \frac{(6-i+1)^4 - (6-i)^4}{6^4}
\end{aligned}
$$

Bonus: In part 6 we provide that $E(X_i) > E(X_i')$. That is, the average value of the highest 3 dice of all the rolls in A_i is higher than the average value of the highest 3 dice of all the rolls in A_i'. Explain the idea behind why this is the case. You do not need to prove it, so you may use examples to help articulate it. Hint: $A_i' \subseteq A_i$.

Solution: When the first die is the lowest, all the other dice have values $i \leq d_j \leq 6$. However, A_i also includes the event that d_1 is not the lowest, but the lowest die is still equal to i. For example consider the event that d_2 is the lowest and d_1 is not the lowest. Then $i < d_1 \leq 6$, while $i \leq d_3, d_4 \leq 6$. Then the average becomes slightly higher since d_1 must be strictly greater than i. Likewise, if d_3 is the lowest, then $i < d_1, d_2 \leq 6$, while $i \leq d_4 \leq 6$ and the average becomes higher still.

Question 6: Bonus: Michiel's Craft Beer Company (MCBC) sells n different brands of India Pale Ale (IPA). When you place an order, MCBC sends you one bottle of IPA, chosen uniformly at random from the n different brands, independently of previous orders.

Simon places m orders with MCBC. Define the random variable X to be the total number of distinct brands that Simon receives. Determine the expected value $E(X)$ of X.

Hint: Use indicator random variables. Note that your answer will have two variables, n and m, and thus might not simplify very much.

Solution: MCBC has n different IPA's. We will define the indicator random variables:

$$X_i = \begin{cases} 1 \text{ if beer } i \text{ is chosen at least once} \\ 0 \text{ otherwise} \end{cases}$$

Since X_i is an indicator random variable, $E(X_i) = Pr(X_i = 1)$, which is the probability that, out of m orders, beer i is selected at least once. We will use the complement rule, thus $Pr(X_i = 1) = 1 - Pr(X_i = 0)$, and $X_i = 0$ is the event that beer i is not selected out of m orders. Let a_j be the event that beer i is selected for order j. $Pr(a_j) = \frac{1}{n}$, and thus $Pr(\overline{a}_j) = \frac{n-1}{n}$. Since for each order the beer is chosen independently of the other orders, the probability of never choosing beer i over m orders is

$$\begin{aligned} Pr(X_i = 0) &= Pr(\overline{a}_1 \wedge \overline{a}_2 \wedge \ldots \wedge \overline{a}_m) \\ &= Pr(\overline{a}_1) \cdot Pr(\overline{a}_2) \cdot \ldots \cdot Pr(\overline{a}_m) \\ &= \left(\frac{n-1}{n}\right)^m . \end{aligned}$$

Thus $Pr(X_i = 1) = 1 - Pr(X_i = 0) = 1 - \left(\frac{n-1}{n}\right)^m$. Linearity of expectation tells us:

$$\begin{aligned} E(X) &= E(X_1 + X_2 + ... + X_n) \\ &= E(X_1) + E(X_2) + ... + E(X_n) \\ &= \sum_{j=1}^{n} Pr(X_i = 1) \\ &= \sum_{j=1}^{n} \left(1 - \left(\frac{n-1}{n}\right)^m\right) \\ &= n \cdot \left(1 - \left(\frac{n-1}{n}\right)^m\right) . \end{aligned}$$