COMP 2804 — Assignment 1

Due: Sunday September 26, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as a single .pdf file. Typesetting (using Latex, Word, Google docs, etc) is recommended but not required. Marks will be deducted for illegible or messy solutions. This includes but is not limited to excessive scribbling, shadows, blurry photos, messy handwriting, etc.
- **No late assignments will be accepted.**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams (which is where most of the marks are).
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: You decide to blow off computer science to pursue your dream of being a rock star. You have a guitar and you can play 7 different chords $\{A, B, C, D, E, F, G\}$, and you are trying to figure out how long of a career you will have based on how many songs you can come up with.

A song uses 3 chords for a verse and 4 chords for the chorus, and the order you play the chords matters (that is, $(A, B, C) \neq (B, C, A)$). So a song can be expressed as $(Verse = (A, B, C), Chorus = (A, D, F, G))$. We may also express the same song as a single sequence with the chords of the verse followed by the chords of the chorus, which we call the *chord sequence* of the song. The chord sequence of the above song is (A, B, C, A, D, F, G). Given a set S of songs, two songs s_1 and s_2 in S are considered *unoriginal* if every chord in their chord sequence is the same. We say that a song $s \in S$ is *original* if the song differs from every other song $s' \in S$ in at least one chord. In the following problem, S is the set of songs that you will write throughout your career.

(a) How many original songs can you make?

Solution: Simple application of the product rule - task 1, pick the first chord, task 2, pick the second chord, etc. Since there are 7 choices for each chord and 7 possible locations, the number of original songs is $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^7$.

(b) To make sure there is enough melody in your songs, you want to never play the same chord twice in a row in your verse or in your chorus. Thus $((A, B, C), (C, D, C, D))$ is valid, but $((A, B, B), (C, D, A, A))$ is not. How many original songs is it possible to make given this restriction?

Solution: Again the procedure is to pick the chords one by one. We will start with the verse. For the first chord there are 7 choices. For the next chord we cannot repeat the first chord, so there are 6 choices. For the third chord we cannot repeat the second chord so there are 6 choices. Thus the number of ways to pick chords for the Verse are $7 \cdot 6^2$. For the chorus we do the same procedure, but now with 4 chords total, which gives us $7 \cdot 6^3$ ways to choose a chorus. Thus the total number of original songs you can make is $7 \cdot 6^2 \cdot 7 \cdot 6^3 = 7^2 \cdot 6^5$.

(c) You decide to experiment by adding a *bridge* to some of your songs. A bridge is a short interlude of anywhere from 0 chords (no bridge) up to 3 chords, and every chord in the bridge must be unique. Example: $Bridge = (), Bridge = (G, C), Bridge = (A, B, C)$ are all allowed, however $Bridge = (A, B, A)$ is not allowed, since there are duplicate chords. Now the chord sequence of a song is the original chord sequence with the chords of the bridge appended (alternatively, the triple $(Verse, Chorus, Bridge)$). Adjust your answer from (b) to determine how many original songs you can make with an optional bridge.

Solution: Using the answer from (b) and multiply by the number of bridges. The number of bridges we can make are bridges of length 0 + bridges of length 1 + bridges of length 2 + bridges of length 3. For an example we will look at the number of ways to choose a bridge of length 3. We first choose the first chord, which is 7 choices. We cannot repeat this chord, so for the next chord we have 6 choices. For the third chord we cannot repeat either of the two previous chords so there are 5 choices. The total number of bridges we can make are

Number of chords	**Number of choices**
Bridges of length 0	1
Bridges of length 1	7
Bridges of length 2	$7 \cdot 6$
Bridges of length 3	$7 \cdot 6 \cdot 5$

Thus the total number of original songs is now $7^2 \cdot 6^5(1 + 7 + 7 \cdot 6 + 7 \cdot 6 \cdot 5)$.

(d) You are not as good at guitar as you thought, so you decide to take out the bridge, but you find you are very good at the chord progression (F, A, D). You decide to include

(F, A, D) in all your songs in either the verse or chorus or both. If you ignore all the restrictions of parts (b) and (c), how many original songs can you come up with that include (F, A, D) in either the verse or the chorus or both?

Solution: Let X be the number of songs with (F, A, D) in the verse and let Y be the number of songs with (F, A, D) in the chorus. We want to know $|X \cup Y|$. Using the principle of inclusion / exclusion we have the formula $|X \cup Y| = |X| + |Y| - |X \cap Y|$. Each verse is 3 chords, that means there is 1 way to choose a verse with (F, A, D) in it. There are 7^4 ways to choose a chorus of 4 chords, thus the number of songs with (F, A, D) in the verse is $|X| = 1 \cdot 7^4$.

Next we determine $|Y|$. There are 7^3 ways to choose an arbitrary verse. For the chorus we first choose a location for the chords (F, A, D). They can either be the first 3 chords or the last 3 chords, so there are 2 choices. There is one chord left and 7 ways to choose it, so the total number of songs with (F, A, D) in the chorus is $7^3 \cdot 2 \cdot 7$.

Finally we determine $|X \cap Y|$. There is 1 way to choose a verse with (F, A, D) and $2 \cdot 7$ ways to choose a chorus with (F, A, D), so the number of songs with (F, A, D) in the verse and the chorus is $1 \cdot 2 \cdot 7$. Thus the total number of songs with (F, A, D) in either the verse or the chorus is

$$\begin{aligned} |X \cup Y| &= |X| + |Y| - |X \cap Y| \\ &= 7^4 + 7^3 \cdot 2 \cdot 7 - 2 \cdot 7 \\ &= 2401 + 4802 - 14 \\ &= 7189 \end{aligned}$$

(e) You have gotten better at guitar, so you no longer no longer need to include (F, A, D) in every song in order. However, you still like how the F, A, and D chords sound in a song together. Find how many original songs have at least one F chord, at least one A chord, **and** at least one D chord (anywhere in the song in any order). You may ignore the constraints of parts (b), (c), and of course (d).

Solution: Let X be all the songs with at least one F chord, let Y be all the songs with at least one A chord, let Z be all the songs with at least one D chord,and let S be the set of all original songs. We want to find $|X \cap Y \cap Z|$. However, it is easier to find the complement of this set. Thus we will calculate $|X \cap Y \cap Z| = |S| - |\overline{X \cap Y \cap Z}|$.

$$\begin{aligned} |X \cap Y \cap Z| &= |S| - |\overline{X \cap Y \cap Z}| \\ &= |S| - |\overline{X} \cup \overline{Y} \cup \overline{Z}| \\ &= |S| - \left(|\overline{X}| + |\overline{Y}| + |\overline{Z}| - |\overline{X} \cap \overline{Y}| - |\overline{X} \cap \overline{Z}| - |\overline{Y} \cap \overline{Z}| + |\overline{X} \cap \overline{Y} \cap \overline{Z}|\right). \end{aligned}$$

To determine the number of songs without any F chords $|\overline{X}|$ we note that we now have 6 choices for each chord and 7 chords, so $|\overline{X}| = 6^7$. The same reasoning applies to songs without A and without D, so $|\overline{X}| = |\overline{Y}| = |\overline{Z}| = 6^7$. To determine the number of songs without F and without A we observe that now for each chord we are given 5 choices. Thus $|\overline{X} \cap \overline{Y}| = 5^7$. The same reasoning applies to $|\overline{X} \cap \overline{Z}|$ and $|\overline{Y} \cap \overline{Z}|$. Finally for songs with F,A, or D we have 4 choices for each chord, thus $|\overline{X} \cap \overline{Y} \cap \overline{Z}| = 4^7$. Thus

$$
\begin{aligned}
|X \cap Y \cap Z| &= |S| - |\overline{X \cap Y \cap Z}| \\
&= |S| - |\overline{X} \cup \overline{Y} \cup \overline{Z}| \\
&= |S| - \left(|\overline{X}| + |\overline{Y}| + |\overline{Z}| - |\overline{X} \cap \overline{Y}| - |\overline{X} \cap \overline{Z}| - |\overline{Y} \cap \overline{Z}| + |\overline{X} \cap \overline{Y} \cap \overline{Z}|\right). \\
&= 7^7 - 3 \cdot 6^7 + 3 \cdot 5^7 - 4^7 \\
&= 201726
\end{aligned}
$$

(f) You no longer are constrained to include F, A or D. However, you find that some of your choruses are repetitive. You want a more originality in your chorus, so you will require that your choruses use at least 3 different chords in the 4 chord sequence. So a valid chorus is $(G, C, C, D), (A, B, A, F)$ or (F, A, D, E) but not (B, B, B, B) since it only uses 1 chord, and not (A, B, A, A) since it only uses 2 chords. Given this restriction, how many original **CHORUSES** can you make? You may ignore the constraints from all the previous parts.

Solution: We can find the total number of choruses and subtract all songs that have only 1 or 2 chords. There are 7^4 choruses total. For a chorus that uses 1 chord, there are 7 ways to choose that chord. To find all the choruses that use 2 chords, method 1: Choose 2 chords from 7, $\binom{7}{2}$. The number of choruses you can make with up to 2 chords is 2^4. This includes 2 single chord choruses, so we subtract them. Total with 2 chords is $\binom{7}{2} \cdot (2^4 - 2)$. Answer:

$$7^4 - 7 - \binom{7}{2} \cdot (2^4 - 2) = 2100$$

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Method 2 for finding songs with 2 chords: choose 2 chords in $\binom{7}{2}$ ways. We either have 3 of one chord and 1 of the other chord, or else there are 2 and 2. If we have 3 and 1, there are 2 ways to choose which chord is which. Then there are $4!/3!$ ways to arrange them. If we have 2 and 2 there is 1 way to choose which chord has 2 occurrences. Then there are $\binom{4!/2!2!=4}{2}$ ways to arrange them. Answer:

$$7^4 - 7 - \binom{7}{2} \cdot \left(2 \cdot 4!/3! + \binom{4}{2}\right) = 2100$$

Alternatively: Find the number of choruses with 3 and 4 chords directly. The number of ways to choose 4 chords from 7 is $\binom{7}{4}$. The number of ways to arrange these 4 chords is 4!. The number of ways to choose 3 chords is $\binom{7}{3}$. Exactly one of these chords is duplicate, thus there are 3 ways to choose this chord. The number of ways to arrange the 4 chords is still 4!, however, since exactly 1 chord is duplicated, we divide by the $2! = 2$ which is the number of permutations of those two chords identical chords. Then the total is

$$\binom{7}{4} \cdot 4! + \binom{7}{3} \cdot 3 \cdot 4!/2! = 2100.$$

Question 3: After a long quarantine you bike to the bar. You plan on staying long enough to have 6 drinks. Each time you order a drink you choose between 2 different beer on tap (Corona or Corona Delta Variant), or 3 different sodas (Croak™, Pepci™ or Kanata Dry Gingerale™).

(a) How many different combinations (where order matters) of drinks can you have this evening?

Solution: For each round you choose between 5 different drinks, so there are 5^6 combinations.

(b) Assume that you had i beer this evening, for an integer $0 \leq i \leq 6$. How many ways could you have had i beer over the 6 rounds?

Solution: Let i be the number of beer you had. Then, out of the 6 rounds, choose i of them to have beer in. There are $\binom{6}{i}$ ways to do this. For each of the i rounds in which you had beer there are 2 beer to choose from. So there are 2^i ways to choose the beer. For each of the $6 - i$ rounds you had soda, there are 3 soda to choose from, which means there are 3^{6-i} ways to choose soda. Thus for any given i, the number of distinct combinations of drinks is $\binom{6}{i} \cdot 2^i \cdot 3^{6-i}$.

(c) If you drink 3 or more beer this evening you will leave your bike at the bar and Uber home. Count all combinations of drinks that require you to take an Uber home, and count all combinations of drinks where you can still bike home. (Observe that since these are non-overlapping sets, the sum of these two should equal your answer from (a).)

Solution: You can get the solution by summing up all the possible values of i from (b). Observe that this is simply the binomial theorem. Let $b = 2$ be the number of beer choices and $c = 3$ be the number of soda choices. Then over 6 rounds the number of combinations is:

$$
\begin{aligned}
(b+c)^6 &= \sum_{i=0}^{6} \binom{6}{i} b^i \cdot c^{6-i} \\
&= \sum_{i=0}^{6} \binom{6}{i} 2^i \cdot 3^{6-i} \\
&= \binom{6}{0} 2^0 \cdot 3^6 + \binom{6}{1} 2^1 \cdot 3^5 + \binom{6}{2} 2^2 \cdot 3^4 + \binom{6}{3} 2^3 \cdot 3^3 + \binom{6}{4} 2^4 \cdot 3^2 \\
&\quad + \binom{6}{5} 2^5 \cdot 3^1 + \binom{6}{6} 2^6 \cdot 3^0.
\end{aligned}
$$

If we sum up all the combinations with less than 3 beer we have:

$$= \binom{6}{0} 2^0 \cdot 3^6 + \binom{6}{1} 2^1 \cdot 3^5 + \binom{6}{2} 2^2 \cdot 3^4$$

choices. All the combinations with 3 or more beer is

$$= \binom{6}{3} 2^3 \cdot 3^3 + \binom{6}{4} 2^4 \cdot 3^2 + \binom{6}{5} 2^5 \cdot 3^1 + \binom{6}{6} 2^6 \cdot 3^0.$$

Question 4: You are playing laser tag. The players are a mix of 8 adults and 5 children. The staff tell you to line yourselves up against the wall in order to pick teams.

(a) How many ways can you arrange everyone in a line if all children must be separated, that is, no two children may be beside one another?

(b) Assume that we allow up to 2 children to line up together, but never 3 or more together. How many ways can everyone be lined up now?

Hint: It is easier to place the parents first then arrange the children among them.

Solution: The first task is to arrange the 8 adults. There are 8! ways to do this. Now among these adults are 9 possible locations to place 1 or 2 children. The number of solutions depend on how we group the children. For instance, if we have 1 group of two and 3 singles, then the number of ways to group the children is the number of permutations of $2, 1, 1, 1$ which is $\binom{5}{1}$. This does not rearrange the children, but rather, once we have an arrangement of children, decides whether to group child 1 and 2 together, or group child 2 and 3 together, or group child 3 and 4 together, etc. We label each way of grouping the children as A,B or C.

Label	**Groups**	**# Ways to Group**
A	$1,1,1,1,1$	1
B	$2,1,1,1$	$\binom{4}{1} = 4$
C	$2,2,1$	$\binom{3}{2} = 3$

We calculate each scenario separately and use the sum rule (since they are non-overlapping sets).
A: we choose 5 positions out of 9. There are $\binom{9}{5}$ ways to do this. There are 5! ways to arrange the children on a line and 1 way to group them.
B: we choose 4 positions out of 9. There are $\binom{9}{4}$ ways to do this. There are 5! ways to arrange the children on a line and 4 ways to group them.
C: we choose 3 positions out of 9. There are $\binom{9}{3}$ ways to do this. There are 5! ways to arrange the children on a line and 3 ways to group them.

The total number of ways to arrange everyone on a line is the number of ways to do A plus the number of ways to do B plus the number of ways to do C. That is:

$$A + B + C = \binom{9}{5} \cdot 5! + \binom{9}{4} \cdot 5! \cdot 4 + \binom{9}{3} \cdot 5! \cdot 3.$$

We multiply this by the number of arrangements of adults to get:

$$8! \cdot 5! \cdot \left(\binom{9}{5} + \binom{9}{4} \cdot 4 + \binom{9}{3} \cdot 3 \right)$$

Question 5: The Ottawa Senators played 80 games this season and won 51 of them, and lost or tied the other 29. You watched some of the games and observed that they had a one losing streak of 5 games (that is, they lost 5 games in a row) and another losing streak of 3 games. You also heard mentioned that they did not have any 3 or 4 game winning streaks. Prove that they must have had at least one winning streak of 5 or more games.

Solution: This is a generalization of the pigeonhole principle. We can find the maximum number of "bins" (slots between losing streaks) to put the "balls". They had a losing streak of 5 games and one of 3 games. All other non-winning streaks could consist of a single loss or tie. That gives us ≤ 23 non-winning streaks. Since the Sens could start or end the season with wins, that gives us 24 "boxes" in which to put wins. We can start by putting 2 wins into each of the 24 boxes. That leaves us with $51 - 48 = 3$ wins left to place. Since there are no 3 or 4 game winning streaks, we must place all of these wins into a single "box", giving us at least one 5 game winning streak.

Question 6: You have won a prize package at your favourite store. This store has n products for sale, with unlimited stock, and you are allowed to choose k items total for your prize package (k may be larger than n). You may take k of the same product, or $\min\{k, n\}$ different products, or any combination in between. Two prize packages are the same if they consist of all the same products in the same quantities, otherwise they are considered different. For the questions below, be sure to explain your answer, i.e., what are the sequence of tasks you performed to obtain your answer? If you only put the answer you will get 0.

(a) How many different prize packages are there?

Solution: You want the number of ways to select k items from n possible items. This is the same as finding solutions to $x_1 + x_2 +, ..., +x_n = k$, where each x_i represents a product you can choose. Which is equivalent to lining up $n + k - 1$ 0's and choosing $n - 1$ of them to be 1's. So the answer is $\binom{n+k-1}{n-1}$.

(b) For an integer j where $1 \leq j \leq \min\{k, n\}$, how many different prize packages of there of exactly j different products (where the total products still sum to k)?

Solution: There are $\binom{n}{j}$ ways to choose exactly j products. There are k products total that you must partition among j products. Now we are again finding solutions to $x_1 + x_2 +, ..., +x_j = k$, except now each $x_i \geq 1$ rather than $x_i \geq 0$. We substitute variables $x'_1 = x_1 - 1, x'_2 = x_2 - 1,$, etc. Then we must find solutions to $x'_1 + x'_2 + ... + x'_j = k - j$. There are $\binom{k-1}{j-1}$ such solutions.

(c) Use the answers from above to explain why

$$\sum_{i=1}^{\min\{k,n\}} \binom{n}{i} \cdot \binom{k-1}{i-1} = \binom{n+k-1}{n-1}.$$

Solution: We are counting the same thing two different ways. $\binom{n+k-1}{n-1}$ is the total number of prize packages available, while $\sum_{i=1}^{\min\{k,n\}} \binom{n}{i} \cdot \binom{k-1}{i-1}$ counts the number of ways to choose precisely i different items and sums all the possibilities.

Question 7: You do not have enough money to eat out or get groceries until payday 4 days from now. Each day consists of three meals: breakfast, lunch, and supper. So payday is 12 meals away. Fortunately you have enough food for exactly 12 meals. You have enough eggs for 3 meals, hamburgers for 2 meals, pasta for 5 meals, tacos for 1 meal and cereal for 1 meal.

(a) How many different meal plans can you make?

Solution: There are 12 different meal slots. First place the eggs. There are $\binom{12}{3}$ ways to do so. Then choose the hamburgers. There are $\binom{9}{2}$ ways to choose hamburger meals. Pasta: $\binom{7}{5}$, tacos: $\binom{2}{1}$, cereal: $\binom{1}{1}$. Note: it does not matter what order you choose the meals. We could have started with pasta, or tacos, or whatever. The total is then:

$$\binom{12}{3} \cdot \binom{9}{2} \cdot \binom{7}{5} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{12!}{3!5!2!1!1!}$$

different meal plans.

(b) It turns out you will get a covid relief cheque tomorrow morning. So you now have only 4 meals to fill (you will eat breakfast before you get groceries). How many distinct meal combinations are there now?

Solution: There are 5 different types of meals. We will consider all possible combinations of meals and count the number of distinct ways they can be arranged.

(a) 4 different meal types are used. There are $\binom{5}{4}$ ways to choose them. There are 4! ways to arrange them. The total is $\binom{5}{4} \cdot 4!$ ways.

(b) 2 alike, 2 different meal types are used (note this is 3 different types total). That means one type of meal will be eaten twice. Note, this cannot be tacos or cereal, since there is only one of each. The two alike are chosen from eggs, hamburgers or pasta, and there are $\binom{3}{1}$ ways to do this. There are 4 types of food left to choose the two singles, so there are $\binom{4}{2}$ ways to do that. Now to arrange them, we choose 2 locations of the 4 for the two alike meals, which is $\binom{4}{2}$. There are 2 ways to then arrange the single meals. The total is then $\binom{3}{1} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 2$.

(c) 2 alike and 2 alike. There are 3 different meals types with food for 2 or more meals, so there are $\binom{3}{2}$ ways to choose them. There are $\binom{4}{2}$ ways to arrange them distinctly. The total is then $\binom{3}{2} \cdot \binom{4}{2}$.

(d) 3 alike, 1 different. For 3 alike you must choose from eggs or pasta, and there are $\binom{2}{1}$ ways to do this. There are 4 types left to choose 1 different from, thus $\binom{4}{1}$ ways to do this. There are $\binom{4}{1}$ ways to arrange them. The total is then $\binom{2}{1} \cdot \binom{4}{1} \cdot \binom{4}{1}$.

(e) 4 alike. We can only do this with pasta. There is 1 way to choose pasta and 1 way to eat pasta for 4 meals in a row. Yum! The total is 1.

The total number of meal plans is the sum of these. Thus the total number of possible meal plans is:

$$\binom{5}{4} \cdot 4! + \binom{3}{1} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 2 + \binom{3}{2} \cdot \binom{4}{2} + \binom{2}{1} \cdot \binom{4}{1} \cdot \binom{4}{1} + 1.$$