PIGEONHOLE PRINCIPLE

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Pigeonhole Principle

Assume we have 366 people. Each has a birthday. What can we conclude?

There are at least 2 people who share a birthday. Why?

There are 365 days in a year (not in leap year).

Jan 1	Jan 2	…	Sep 27	Sep 28	Sep 29	Sep 30	…	Dec 30	Dec 31

Pigeonhole Principle

Country where everyone's last name is 1 Upper Case letter and 1 lower case letter.

Xe, Gt, Po, etc.

In a country with ≥ 677 people, at least two must have the same last name. Why?

How many last names are there in total? Can we determine using the Product Rule?

Task 1: Choose an Upper Case letter – 26 ways to choose

Task 2: Choose a lower case letter – 26 ways to choose

$$26 \cdot 26 = 676$$

Therefore there are 676 possible last names.

Aa	Ab	Ac	…	Zx	Zy	Zz

Pigeonhole Principle

k holes ("boxes")
$\geq k+1$ pigeons ("objects")

Then $\exists$ hole with ≥ 2 pigeons.

Task 1: Place pigeon 1 in an empty box

Task 2: Place pigeon 2 in an empty box

…

Task $k+1$: There are no empty boxes. Place pigeon $k+1$ in a box with another pigeon

Box 1	Box 2	Box 3	…	Box $k-2$	Box $k-1$	Box k

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a - b = 1$$

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

(Try and construct a subset where the claim is not true)

$$S = \{1, 3, 5, 7, \qquad \}$$

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a - b = 1$$

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

(Try and construct a subset where the claim is not true)

$$S = \{1, 3, 5, 7, \quad \}$$

That is the intuition.

Exercise 3.84

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$a - b = 1$

How can we prove this using the Pigeonhole principle?

We can make our own boxes…

Then start putting elements of S into their boxes… Since $|S| = n + 1$, there must be a box with 2 elements

n Boxes:

1, 2

3, 4

5, 6

$\vdots$

$\mathbf{2n - 1, 2n}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2, 3\}$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

Try and construct an example again…

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

$S = \{1, 2, 3, 4\}$

At this point we cannot add anything else.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
$$a + b = 2n + 1$$

We make n boxes such that the elements in each box sums to $2n + 1$. Then we can prove the claim using the pigeonhole principle.

n Boxes:

$\mathbf{1, 2n}$

$\mathbf{2, 2n - 2}$

$\mathbf{3, 2n - 3}$

$\vdots$

$\mathbf{n - 1, n + 2}$

$\mathbf{n, n + 1}$

We have n boxes and $n + 1$ things to place in these boxes.

By the pigeonhole principle, one box must have 2 or more items.

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5, 7\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

$n = 4$
$\{1, 2, 3, 4, 5, 6, 7, 8\}$

$|S| = 5$

Cannot add 1 because everything is a multiple of 1

$S = \{3, 5, 7, 4\}$

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true…

$$S = \{a_1, a_2, a_3, \dots, a_{n+1}\}$$

for $i = 1, \dots, n + 1$, we express each term a_i as follows:
$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$$48 = 2^4 \cdot 3$$
$$7 = 2^0 \cdot 7$$
$$1 = 2^0 \cdot 1$$

Exercise 3.85

$S \subseteq \{1, 2, \ldots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true…

$$S = \{a_1, a_2, a_3, \ldots, a_{n+1}\}$$

for $i = 1, \ldots, n + 1$, we express each term a_i as follows:
$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$O = \{q_1, q_2, \ldots, q_{n+1}\}$ are odd integers that belong to $\{1, 3, 5, \ldots, 2n - 1\}$ (a set of size n).

Thus in the set O there are two elements that are equal.

Exercise 3.85

$S \subseteq \{1, 2, \dots, 2n\}$
$|S| = n + 1$

Claim: $\exists a, b \in S$:
a is a multiple of b

Try and make a counter-example to help us understand the problem.

Seems to be true…

$$S = \{a_1, a_2, a_3, \dots, a_{n+1}\}$$

$a_i = 2^{k_i} \cdot q_i$ where $k_i \geq 0$ and q_i is an odd number.

$$O = \{q_1, q_2, \dots, q_{n+1}\}$$

$\exists\ i \neq j: q_i = q_j$

Without loss of generality, assume $2^{k_i} \geq 2^{k_j}$. Then:

$\frac{a_i}{a_j} = \frac{2^{k_i} \cdot q_i}{2^{k_j} \cdot q_j} = 2^{k_i - k_j}$ which is an integer, therefore a_i is a multiple of a_j.

TA: Amber Lager

During September, TA drinks 45 bottles of beer and $\geq$ 1 bottle per day.

Claim: $\exists$ consecutive days during which TA drinks exactly 14 bottles.

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

TA: Amber Lager

During September, TA drinks 45 bottles of beer and $\geq$ 1 bottle per day.

Claim: $\exists$ consecutive days during which TA drinks exactly 14 bottles.

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September ith, $\mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum that $= 14$.

$a_i = b_1 + b_2 + \cdots + b_i$ = total bottles drank from Sept 1st to Sept i^{th}

$a_1, a_2, a_3, \ldots, a_{30}$ all distinct.
Now add 14 to each:

$$a_1 + 14, a_2 + 14, a_3 + 14, \ldots, a_{30} + 14$$
$$a_1, a_2, a_3, \ldots, a_{30}$$

60 numbers belonging to set:

$$\{1, 2, \ldots, a_{30} + 14\}$$
$$= \{1, 2, \ldots, 45 + 14\}$$
$$= \{1, 2, \ldots, 59\}$$

Must be 2 numbers that are equal

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September $i\text{th}, \mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum that $= 14$.

$a_1 + 14, a_2 + 14, a_3 + 14, \ldots, \ a_{30} + 14$
$a_1, a_2, a_3, \ldots, a_{30}$

are 60 numbers belonging to set:

$$= \{1, 2, \ldots, 59\}$$

Must be 2 numbers that are equal. Can they both be from the same sequence?

All numbers within each sequence are distinct.

So the matching number must occur once in each sequence

TA: Amber Lager

During September, TA drinks 45 bottles of beer and ≥ 1 bottle per day.

Claim: ∃ consecutive days during which TA drinks exactly 14 bottles.

For $i = 1, \ldots, 30, b_i =$ number of bottles drank on September $i\mathrm{th}, \mathrm{b_i} \geq 1$

$$b_1 + b_2 + b_3 + \ \ldots + b_{30} = 45$$

We want to find a subsum that $= 14$.

$a_1, a_2, a_3, \ldots, a_{30}$
$a_1 + 14, a_2 + 14, a_3 + 14, \ldots ,\ a_{30} + 14$

are 60 numbers belonging to set:

$$= \{1, 2, \ldots, 59\}$$

$\exists i, j: a_i = a_j + 14$
$14 = a_i - a_j$
$= b_{j+1} + b_{j+1} + \cdots + b_i$

Therefore from Sept j+1 to Sept i the TA drank 14 bottles.