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COMP 2804 — Assignment 4

Due: Sunday December 5, 11:59 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted.**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.
  - Each part of each question is worth 2 marks, except for 4.2 (4 marks) and 4.2 (6 marks)

Question 1:

- Write your name and student number.

Question 2: Craps is a game where you roll two dice and bet on the outcome. There are many ways to play, but we will focus on one specific scenario. On your "come-out" roll you roll a 9 on two dice. Now the game is this: you must keep rolling 2 dice until you either roll 9 again, in which case you win, or you roll a 7, in which case the house wins (you lose).

1. (a) Find the probability that you win.

   (b) Find the probability that the house wins.

   Show all your work for both parts. Observe that the probabilities from 1a and 1b added together should total 1, but you may NOT use this fact to determine your answers.

   Let A be the event that we roll a 9.

   Let B be the event that we roll a 9 or 7.

   Let X be the event that we win and Y be the event that the house wins.

$\sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$

a) Probability that you win = probability of you rolling a 9 before rolling a 7.

$Pr(X) = \sum_{i=0}^{\infty} Pr(Bc)^i * Pr(A)$

$Pr(A) = \frac{4}{36}$

$Pr(B) = \frac{10}{36}$

$Pr(Bc) = \frac{26}{36}$

$\sum_{i=0}^{\infty} Pr(Bc)^i * Pr(A)$

$= \frac{1}{1-\frac{26}{36}} * Pr(A)$

$= \frac{1}{1-\frac{26}{36}} * \frac{4}{36}$

$Pr(X) = \frac{2}{5}$

b) Probability that the house wins = probability of the house rolling a 7 before rolling a 9.

Let C be the event that we roll a 7.

$Pr(Y) = \sum_{i=0}^{\infty} Pr(Bc)^i * Pr(C)$

$Pr(C) = \frac{6}{36}$

$Pr(B) = \frac{10}{36}$

$Pr(Bc) = \frac{26}{36}$

$\sum_{i=0}^{\infty} Pr(Bc)^i * Pr(C)$

$= \frac{1}{1-\frac{26}{36}} * Pr(C)$

$= \frac{1}{1-\frac{26}{36}} * \frac{6}{36}$

$Pr(Y) = \frac{3}{5}$

2. Assume you have bet \$10. You roll a 9 on your "come-out" roll, same as above. If you win the casino pays you \$10. If you lose the casino takes your \$10 bet. What is your expected winnings or losings (i.e., the expected value) of this round of craps?

   Define random variable Z is the amount that you win or lose

   Z(event where you win) = 10

   Z(event where you lose) = -10

   E(X) = Z(event where you win) x Pr(X) + Z(event where you lose) x Pr(Y)

   $E(X) = 10 * \frac{2}{5} + (-10) * \frac{3}{5}$

   $E(X) = -2$

Question 3: You are playing a board game about pirates that uses special six-sided fair dice. Three sides of each die has a zero on it, while the other three sides have the numbers 1 through 3 (this represents damage done by your ship's cannons). Your ship has five cannons, so to attack you roll 5 of these six-sided dice. Let

$$\begin{aligned} A &= \text{The sum of the values of the 5 dice.} \\ B &= \text{The number of times a zero is showing.} \end{aligned}$$

1. What is $E(A)$ and E(B)?
   $E(X) = \sum_{w \in S} X(w) * Pr(w)$
   $E(A) = 5 * (0 * Pr(0) + 0 * Pr(0) + 0 * Pr(0) + 1 * Pr(1) + 2 * Pr(2) + 3 * Pr(3))$
   $E(A) = 5 * (1 * Pr(1) + 2 * Pr(2) + 3 * Pr(3))$
   $E(A) = 5 * (1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6})$
   $E(A) = 5 * \frac{1}{6}(1 + 2 + 3)$
   $E(A) = 5 * \frac{6}{6}$
   $E(A) = 5$
   $E(B) = 5 * Pr(0)$
   $E(B) = 5 * \frac{1}{2}$
   $E(B) = 2.5$

2. Are A and B independent random variables?

   Find a case where A and B are not independent random variables.
   $Pr(A = 30 \cap B = 5) = Pr(A = 30) * Pr(B = 5)$?
   $Pr(A = 30 \cap B = 5) = 0$
   $Pr(A = 30) = 1/6^5$, (6,6,6,6,6)
   $Pr(B = 5) = 1/6^5$, (0,0,0,0,0)
   $Pr(A = 30 \cap B = 5) \neq Pr(A = 30) * Pr(B = 5)$
   $\therefore$ A and B are not independent random variables.

Question 4: When FX and his girlfriend XF have a child, this child is a boy with probability 1/2 and a girl with probability 1/2, independently of the sex of their other children. FX and XF stop having children as soon as they have three girls or two boys.

Consider the random variables

$$\begin{aligned} C &= \text{the number of children that FX and XF have,} \\ G &= \text{the number of girls that FX and XF have.} \\ B &= \text{the number of boys that FX and XF have.} \end{aligned}$$

1. Determine the expected values $E(C)$ and $E(G)$

   $E(C) = \sum_{w \in S} C(w) * Pr(w)$

   Pr(2 children) = BB, $|S| = 2^2 = 4$

   Pr(3 children) = GGG, GBG, GBB, $|S| = 2^3 = 8$

   Pr(4 children) = GGBG, GBGG, BGGG, BGGB, GBGB, GGBB, $|S| = 2^4 = 16$

   $E(C) = 2 * (1/4) + 3 * (3/8) + 4 * (6/16) = 25/8$

   $E(G) = \sum_{w \in S} G(w) * Pr(w)$

   Pr(0 girls) = BB, $|S| = 2^2 = 4$

   Pr(1 girls) = GBB, BGB, $|S| = 2^3 = 8$

   Pr(2 girls) = GGBB, GBGB, BGGB, $|S| = 2^4 = 16$

   Pr(3 girls) = GGG, GGBG, GBGG, BGGG, $|S1| = 2^3 = 8$, $|S2| = 2^4 = 16$

   $E(G) = 0(1/4) + 1 * (2/8) + 2 * (3/16) + 3 * (1/8) + 3 * (3/16) = 25/16$

2. We are told that the last child FX and XF had was a boy. Determine the expected values $E(B)$ and $E(G)$ and $E(C)$.

   Last Child was a boy so it has to end with a B. Stopped having children from the amount of boys.

   Let A be the event that the last child was a boy.

   $E(B) = \sum_{w \in S} B(w) * Pr(w)$

   $Pr(B = 2|A) = Pr(B = 2 \cap A)/Pr(A) = Pr(A)/Pr(A) = 1$

   $E(B) = 2 * 1 = 2$

   A = BB, GBB, BGB, GGBB, GBGB, BGGB

   $Pr(A) = Pr(A \cap C = 2) + Pr(A \cap C = 3) + Pr(A \cap C = 4) = 1/4 + 2/8 + 3/16 = 11/16$

   $E(C) = \sum_{w \in S} C(w) * Pr(w)$

   $E(C) = \frac{1}{Pr(A)} * \sum_{k=2}^{4} Pr(C = k \cap A)$

   $E(C) = 2 * 1/4 + 3 * 2/8 + 4 * 3/16 = 32/16 = 32/11$

   $E(G) = \sum_{w \in S} G(w) * Pr(w)$

   Cannot have 3 girls cause that would mean it ends on a girl but we can only end on a boy.

   Pr(0 girls) = BB, $|S| = 2^2 = 4$

   Pr(1 girls) = GBB, BGB, $|S| = 2^3 = 8$

   Pr(2 girls) = GGBB, GBGB, BGGB, $|S| = 2^4 = 16$

   $Pr(G|A) = Pr(G \cap A)/Pr(A)$

$E(G) = \frac{1}{Pr(A)} * \sum_{k=2}^{4} Pr(G = k \cap A)$

$E(G) = 0 * 1/4 + 1 * 2/8 + 2 * 3/16 = 10/16/(P(A) = 10/16/11/16 = 10/11$

Question 5: In the Dungeons and Dragons question from last assignment we talked about how to roll for stats (that is, we take the sum of 3 six-sided dice). It was briefly mentioned that to get higher stats, we can roll 4 six-sided dice and take the sum of the 3 highest dice. In this question we will compare the expected outcome from both of these techniques.

1. Determine the expected value of the sum of rolling 3 fair six-sided dice.

   $E(X) = \sum_{w \in S} X(w) * Pr(w)$

   $E(A) = 3 * (1 * Pr(1) + 2 * Pr(2) + 3 * Pr(3) + 4 * Pr(4) + 5 * Pr(5) + 6 * Pr(6))$

   $Pr(1) = Pr(2) = Pr(3) = Pr(4) = Pr(5) = Pr(6) = \frac{1}{6}$

   $E(A) = 3 * \frac{1}{6} * (1 + 2 + 3 + 4 + 5 + 6)$

   $E(A) = 10.5$

2. What is the size of the sample space S of all possible dice rolls with 4 dice?

   $|S| = 6 * 6 * 6 * 6 = 6^4$

Let d_1, d_2, d_3, d_4 be random variables corresponding to the values of 4 six-sided dice after being rolled. Let Y be the value of the three highest dice. That is, $Y = \left(\sum_{i=1}^{4} d_i\right) - \min\{d_1, d_2, d_3, d_4\}$. Thus our goal is to find $E(Y)$ and compare it to the answer from 1.

Let the random variable X_i be the sum of the highest 3 dice if the lowest die is equal to i, that is $\min\{d_1, d_2, d_3, d_4\} = i$. Let $X_i = 0$ if the lowest die is not equal to i. Let A_i be the event that the lowest die out of d_1, d_2, d_3, and d_4 is equal to i.

3. In a few sentences explain why

$$\sum_{\omega \in S} Y(\omega) = \sum_{i=1}^{6} \sum_{\omega \in S} X_i(\omega) = \sum_{i=1}^{6} \sum_{\omega \in A_i} X_i(\omega)$$

   $Y = \left(\sum_{i=1}^{4} d_i\right) - \min\{d_1, d_2, d_3, d_4\} =$ Taking the 3 highest rolls

   S = rolling 4 dices

   $S = \{\{1, 1, 1, 1\} \dots \{6, 6, 6, 6\}\}$

   w = $\{1, 2, 3, 4\}$

   Y(w) = 1+2+3+4 - 1 = 9

   X1 = 2 + 3 + 4 , X2 = 0, X3 = 0, . . . , X6 = 0

   When we roll a die, we get event w, if min calue is i, then Ai happened, then Y(w) = Xi, Xj = 0 such that j != i Y(w) = X1

The first or left most summation above gives us EVERY conbination's sum of 3 largest die. The middle equation gets the sum of all the combos sum of 3 largest die once even though we literate through the set 6 times. The reason we don't add each combo's sum of 3 largest die more than once is because of X_i, which ensures that each combo is only added once.

4. Use the above expression and linearity of expectation to express $E(Y)$ in terms of $E(X_i(\omega))$ given that $\omega \in A_i$.

   E(Y) in terms of E(Xi(w))

   $E(Y) = E(Y(w))$

   $Y(w) = \sum_{w \in S} Y(w)/|S|$, by rearranging $Y(w) * |S| = \sum_{w \in S} Y(w)$

   $E(Y(w)) = \sum_{w \in S} Y(w)/|S|$

   $E(Y(w)) = E(\sum_{i=1}^{6} \sum_{w \in A}(Xi(w)/|S|)) = (\sum_{i=1}^{6} \sum_{w \in A} E(Xi(w)/|S|))$

   $\sum_{w \in S} E(Y(w)) = (\sum_{i=1}^{6} \sum_{w \in A} E(Xi(w)))$

5. Let A_i' be the event that d_1 is the lowest die out of $\{d_1, d_2, d_3, d_4\}$ and that $d_1 = i$. Let $X_i'(\omega) = d_2 + d_3 + d_4$ if $\omega \in A_i'$ and let $X_i'(\omega) = 0$ if $\omega \notin A_i'$. Find $E(X_i'(\omega))$ given that $\omega \in A_i'$. That is, assuming that i is the lowest die roll and $d_1 = i$, find the expected value of X_i'. Recall that $\forall \omega \in A_i', X_i'(\omega) = d_2(\omega) + d_3(\omega) + d_4(\omega)$, where each of d_2, d_3, and d_4 is at least i.

   $E(x_i) = E(d_2 + d_3 + d_4) = E(d_2) + E(d_3) + E(d_4) = \sum_{i=1}^{3} E(d_i)$

   $E(d_2) + E(d_3) + E(d_4)$, $d \geq i$

   i = the value that each particular die has to be greater than or equal together

   k is a die from die 2 to die 4

   j is the value from i = 6 (all legal values that d_2, d_3, d_4 can that on)

   $E(d_k) = \sum_{j=i}^{6} j * P(j \geq i)$

   $E(d_k) = \sum_{j=i}^{6} j * (6 - j + 1)/6$

   $E(d_k) = 1/6 \sum_{j=i}^{6} j * (6 - j + 1)$

   $E(d_k) = 3 + i/2$

   $E(d_1 + d_2 + d_3) = 3 * (3 + i/2)$

6. Observe that the term $E(X_i(\omega))$ in $\sum_{\omega \in A_i} E(X_i(\omega))$ is a constant, since it is the weighted average of the values $\forall \omega \in A_i, X_i(\omega)$. Similarly the term $E(X_i'(\omega))$ in $\sum_{\omega \in A_i'} E(X_i'(\omega))$ is also a constant. We will not prove it at this time, but we will use the fact that for the expressions given above, $E(X_i'(\omega)) < E(X_i(\omega))$ to show a lower bound on $E(Y)$. Given that $E(X_i'(\omega)) < E(X_i(\omega))$, briefly explain why $\sum_{\omega \in A_i} E(X_i'(\omega)) < \sum_{\omega \in A_i} E(X_i(\omega))$.

$E(X_i') < E(X_i)$ since X_i' is rolling 3 die so less than X_i which is rolling 4 die

Therefore X_i' is a subset of X_i.

one has constrant while the other doesn't Xi' is subset Xi doesn't matter

7. Recall that A_i is the event that $\min\{d_1, d_2, d_3, d_4\} = i$. Show that $|A_i| = (6-i+1)^4 - (6-i)^4$.

   6-i+1 is the number of different values to get a value = i

   6-i is the number of different values for the die to be greater than i

   $(6-i+1)^4$ at least one of our 4 die has to be = to i

   We have to subtract $(6-i)^4$ to ensure that at least one of our die is equal to i

   This is because not all of our die can be greater than i.

8. Explain why $\sum_{\omega \in A_i} E(X_i'(\omega)) = ((6-i+1)^4 - (6-i)^4) \cdot E(X_i'(\omega))$.
   $|A_i| = (6-i+1)^4 - (6-i)^4$
   $\sum_{w \in A} E(X_i'(w))$, we iterate through that by $(6-i+1)^4 - (6-i)^4$ many times
   $(6-i+1)^4 - (6-i)^4 = E(X_i'(w))$, given the lineraity of expectation
   $E(X_i'(w))$ is much like an average, so we can do the above and do linearity of expectation.

9. Now show that

$$E(Y) > \sum_{i=1}^{6} \left(3 \cdot \left(3 + \frac{i}{2}\right)\right) \cdot \frac{(6-i+1)^4 - (6-i)^4}{6^4}$$

   which, if you plug into Wolfram alpha, is > 11.63.
   $E(Y) = \sum_{i=1}^{6} \sum_{w \in A} E(X_i(w)/|S|)$, $E(X_i') < E(X_i)$
   $E(Y) > \sum_{i=1}^{6} \sum_{w \in A} E(X_i'(w)/|S|)$, $E(X_i') = 3(3 + i/2)$
   $E(Y) > \sum_{i=1}^{6} \sum_{w \in A} 3(3 + i/2)/|S|)$, $|S| = 6^4$
   $E(Y) > \sum_{i=1}^{6} \sum_{w \in A} 3(3 + i/2)/6^4)$, $|A_i| = (6-i+1)^4 - (6-i)^4$
   $E(Y) > \sum_{i=1}^{6} 3(3 + i/2) * (6-i+1)^4 - (6-i)^4/6^4)$

Bonus: In part 6 we provide that $E(X_i) > E(X_i')$. That is, the average value of the highest 3 dice of all the rolls in A_i is higher than the average value of the highest 3 dice of all the rolls in A_i'. Explain the idea behind why this is the case. You do not need to prove it, so you may use examples to help articulate it. Hint: $A_i' \subseteq A_i$.

Question 6: Bonus: Michiel's Craft Beer Company (MCBC) sells n different brands of India Pale Ale (IPA). When you place an order, MCBC sends you one bottle of IPA, chosen uniformly at random from the n different brands, independently of previous orders.

Simon places m orders with MCBC. Define the random variable X to be the total number of distinct brands that Simon receives. Determine the expected value $E(X)$ of X.

Hint: Use indicator random variables. Note that your answer will have two variables, n and m, and thus might not simplify very much.