COMP 2804 — Assignment 3

Due: Sunday November 22, 11:55 pm.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through cuLearn.
- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 11:53pm" or "my scanner stopped working at 11:54pm", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1:

- Write your name and student number.

Question 2: You join a group that plays Dungeons and Dragons. Each character has 6 stats (short for statistics): Strength, Dexterity, Constitution, Intelligence, Wisdom, Charisma. The value of each stat is determined[1] by rolling three 6-sided dice and taking the sum. So each stat has a starting value from 3 to 18.

1. What is the size of the sample space for rolling an individual stat?

   Solution: The sample space S is all the ways that 3 dice can be rolled. If we label the dice d_1,d_2, and d_3 then $S = \{(d_1, d_2, d_3) \mid 1 \leq d_1, d_2, d_3 \leq 6\}$ which is 6^3.

2. What is the probability that an individual stat is 16 or higher?

   Solution: The probability that a stat is 16 or higher is the number of ways $d_1 + d_2 + d_3 \geq 16$ over the size of the sample space $|S|$.

   - We could have dice showing $4, 6, 6$ in some order. There are 3 ways to choose one of d_1, d_2, d_3 to equal 4 and one way to choose the two remaining dice to be 6 and 6.

[1] There are variations, such as rolling 4 dice and taking the highest 3.

- We could have dice showing $5, 5, 6$. There are 3 ways to choose one of d_1, d_2, d_3 to be 6 and one way for the remaining two dice to equal 5.
- We could have dice showing $5, 6, 6$. There are 3 ways to choose one of d_1, d_2, d_3 to be 5 and one way for the remaining two dice to equal 6.
- We could have dice showing $6, 6, 6$. There is one way to have all three dice equal to 6.

The total number of ways to roll 16 or higher is then $3+3+3+1 = 10$. Therefore the probability of rolling 16 or higher is $\frac{10}{6^3} = \frac{5}{108} \approx 0.046$.

3. What is the size of the sample space over all six stats?

   Solution: To determine the size of the sample space we can use the product rule. First we roll one stat: there are 6^3 ways to do that. Then we roll the next stat: there are 6^3 ways to do that, etc. Thus the size of the sample space over 6 stats is $(6^3)^6 = 6^{18}$.

4. Given that you have rolled dice for all 6 stats, what is the probability that **exactly** one out of your six stats is 16 or higher?

   Solution: This is the number of ways exactly one stat can be 16 or higher over the size of the sample space. We have previously seen there are 10 ways for a stat to be 16 or higher. Thus there are $6^3 - 10$ ways for a stat to be 15 or lower. Then the procedure is this:

   - Choose exactly 1 stat. There are 6 ways to do this.
   - Have this stat be 16 or higher. There are 10 ways to do this.
   - Have the other 5 stats be 15 or lower. There are $(6^3 - 10)^5$ ways to do this.

   Thus the probability of having exactly one stat be 16 or higher is

$$\frac{6 \cdot 10 \cdot (6^3 - 10)^5}{6^{18}} \approx 0.22.$$

5. What is the probability that at least one of your six stats is 16 or higher?

   Solution: This is the same as 1 minus the probability that all 6 stats are 15 or lower. This is

$$1 - \frac{(6^3 - 10)^6}{6^{18}} \approx 0.25$$

6. What is the probability that at least two of the six stats are 16 or higher?

   Solution: Let $A =$"At least 2 stats are 16 or higher". We will use the complement rule for this. Thus we want to count $\overline{A} =$ "at most 1 stat is 16 or higher". We can break this down into two sets:

A_0="0 stats are ≥ 16 and

A_1="1 stat is ≥ 16".

$|A_0| = (6^3 - 10)^6$ and $|A_1| = 6 \cdot 10 \cdot (6^3 - 10)^5$. We subtract this from the size of the sample space and we get $6^{18} - (6 \cdot 10 \cdot (6^3 - 10)^5 + (6^3 - 10)^6)$ ways to have at least two stats be 16 or higher. Then the probability of it happening is this total over the size of the sample space. Or we can use the formula $Pr(A) = 1 - Pr(\overline{A})$, which amounts to the same thing.

$$\begin{aligned}
&\frac{6^{18} - (6 \cdot 10 \cdot (6^3 - 10)^5 + (6^3 - 10)^6)}{6^{18}} \\
=&1 - \frac{6 \cdot 10 \cdot (6^3 - 10)^5 + (6^3 - 10)^6}{6^{18}} \\
=&1 - \frac{6 \cdot 10 \cdot (6^3 - 10)^5}{6^{18}} - \frac{(6^3 - 10)^6}{6^{18}} \\
\approx&0.0284
\end{aligned}$$

These are all equivalent.

Question 3: We learned in the Newton-Pepys lecture that there is a higher probability of getting at least one 6 on six dice than at least three 6's on eighteen dice. We will explore this idea further by looking some similar games. Compute the probability of winning each of the games listed below.

1. To win game (a): Roll 8 dice and have at least one die showing 6. To win game (b): Roll 24 dice and have at least three dice showing 6.

   Solution:

   (a) Let A be the event that no 6's are showing. Then probability at least one 6 is showing is $1 - Pr(A)$. The number of ways to roll 8 dice is 6^8. The number of ways to roll 8 dice without any 6's showing is 5^8. Therefore the probability of at least one 6 showing is

   $$\begin{aligned}
   1 - \frac{|A|}{|S|} &= 1 - \frac{5^8}{6^8} \\
   &\approx 0.767
   \end{aligned}$$

   (b) Let S be the set of all ways to roll 24 dice. $|S| = 6^{24}$. We define the following events:

   A: "at least 3 dice show a 6"
   B: "exactly 2 dice show a 6"
   C: "exactly 1 die shows a 6"
   D: "exactly 0 dice show a 6"

Then $\overline{A} = B \cup C \cup D$, and $|A| = |S| - |\overline{A}| = |S| - |B \cup C \cup D|$.

B: There are $\binom{24}{2}$ ways to choose 2 sixes. There are 5^{22} ways to roll between 1 and 5 on 22 dice.

C: There are 24 ways to choose one 6 and 5^{23} ways to roll between 1 and 5 on 23 dice.

D: There are 5^{24} ways to roll between 1 and 5 on 24 dice.

Observe that B, C and D have no dice rolls in common. Thus

$$|B \cup C \cup D| = |B| + |C| + |D| = \binom{24}{2} \cdot 5^{22} + \binom{24}{1} \cdot 5^{23} + \binom{24}{0} 5^{24}.$$

Therefore the probability of rolling at least 3 sixes on 24 dice is

$$\begin{aligned}
\frac{|A|}{|S|} &= \frac{|S| - |\overline{A}|}{|S|} \\
&= 1 - \frac{|B \cup C \cup D|}{|S|} \\
&= 1 - \frac{|B| + |C| + |D|}{|S|} \\
&= 1 - \frac{\binom{24}{2} \cdot 5^{22} + \binom{24}{1} \cdot 5^{23} + \binom{24}{0} 5^{24}}{6^{24}} \\
&\approx 0.788
\end{aligned}$$

So as we look for 6's on more dice, the probability swings in favour of Game (b).

Question 4:[Birthday paradox] You scan the rest of the questions in this assignment and decide it is time to get back to your music career. You are now in a band called *The Inclusions*. Each song is now defined by a single sequence of 7 chords from the set $\{A, B, C, D, E, F, G\}$ with no other restrictions (for example, (A, B, G, G, G, A, D) is a song). Each band consists of 4 members each with a unique instrument (we consider the singer as having an instrument). There is always exactly 1 singer and 1 drummer. The other two members must choose an instrument from the following set: {guitar, bass, banjo, saxophone, keyboard}. Each instrument in a band must be unique. Given a set S of songs, two songs s_1 and $s_2 \in S$ are unoriginal if they have the same 7 chords in the same order played with the same instruments. For the questions below we will make the unrealistic assumption that all songs are chosen uniformly at random from the set of all possible original songs.

1. What is the number of possible original songs?

   Solution: There are 7 chords and 7 locations for them, and two instruments chosen from a set of 5. The total number of possible songs is then $7^7 \cdot \binom{5}{2}$. There was a mistake in the orignal solutions, and some TAs probably encouraged you that the answer was $7^7 \cdot 5^2$, so we will accept that as well (even though it is wrong).

2. One of your songs is about to break the top 100, that is, the top 100 most popular songs in the nation for a given time period. Assuming all the songs in the top 100 are chosen uniformly at random, what is the probability that all the top 100 songs are original?

   Solution: This is essentially the complement of the birthday paradox. The number of original songs is the number of days in a year, and the number of people would be 100. The sample space consists of the number of ways to choose 100 songs. This is $|S| = (7^7 \cdot \binom{5}{2})^{100}$. The number of ways to choose 100 songs such that no two the same are selected is $7^7 \cdot \binom{5}{2} \cdot (7^7 \cdot \binom{5}{2} - 1) \cdot (7^7 \cdot \binom{5}{2} - 2) \cdot \ldots \cdot (7^7 \cdot \binom{5}{2} - 99)$. Thus the probability that all 100 songs are original is

$$\begin{aligned}
&\prod_{i=0}^{99} \frac{7^7 \cdot \binom{5}{2} - i}{7^7 \cdot \binom{5}{2}} \\
=&\frac{7^7 \cdot \binom{5}{2}!}{(7^7 \cdot \binom{5}{2})^{100}(7^7 \cdot \binom{5}{2} - 100)!} \\
\approx&0.99939.
\end{aligned}$$

3. Your song will not get into the top 100 if it is unoriginal. Assuming that each song of the top 100 is original, what is the probability that your song is original when compared to the top 100? Note: Assume that the top 100 is chosen uniformly at random but without replacement, i.e., each song is chosen randomly but each song is chosen only once.

   Solution: The probability of selecting any of 100 songs out of $7^7 \cdot \binom{5}{2}$ is

$$\frac{100}{7^7 \cdot \binom{5}{2}}.$$

   Therefore the probability that our song is original is the complement of this or

$$1 - \frac{100}{7^7 \cdot \binom{5}{2}} = \frac{823533}{823543} \approx 0.999999.$$

4. There is another band called *The Exclusions* who also have an up and coming hit song. The record label will promote both of your songs as long as they don't sound similar. Two songs are *similar* if they have all the same chords in the same quantity (ignoring the order of the chords). Both your song and The Exclusions' song use exactly 3 chords. What is the probability that the two songs use the same 3 chords in the same quantity? For example, if your band's song was (A, A, B, C, A, B, C) then it uses 3 A's, 2 B's and 2 C's. If the Exclusions song was (C, B, A, C, B, A, A) then these songs would be considered similar because each has 3 A's, 2 B's and 2 C's. If the Exclusions

song was (C, B, C, C, B, A, A) it would not be similar to your song since they use 3 C's and 2 A's.

Solution: To find the probability of this requires us to define our sample space properly. There are $\binom{7}{3}$ ways to choose 3 out of the 7 chords. Then to count the number of quantities for each chord we find the number of solutions to $x_1 + x_2 + x_3 = 7$ where each x_i must be at least 1. If we let $x'_i = x_1 - 1$, then we want the number of solutions to $x'_1 + x'_2 + x'_3 = 4$. There are $\binom{6}{2}$ such solutions. Therefore the number of dissimilar songs is $\binom{7}{3} \cdot \binom{6}{2}$. To determine the probability that the two songs are similar we look at the sample space which is the number of ways to choose 2 songs, or $(\binom{7}{3} \cdot \binom{6}{2})^2$. The number of ways to choose 2 songs the same is $\binom{7}{3} \cdot \binom{6}{2}$. Thus the probability that two songs are the same is

$$\frac{\binom{7}{3} \cdot \binom{6}{2}}{\left(\binom{7}{3} \cdot \binom{6}{2}\right)^2} = \frac{1}{\binom{7}{3} \cdot \binom{6}{2}}.$$

Alternatively since you wrote your song rather than generated it at random there is 1 way to write it. We stipulated earlier that the Exclusions' song is drawn uniformly at random. Thus the size of the sample space is $1 \cdot \binom{7}{3} \cdot \binom{6}{2}$, and the number of ways the Exclusions can choose a song similar to yours is 1. Thus the probability is

$$\frac{1}{\binom{7}{3} \cdot \binom{6}{2}}.$$

Question 5: In Blackjack you want a hand that totals as close to 21 as possible without going over. The dealer will deal you two cards, then deal themselves one card, all face up. All numbered cards have a value equal to their number. All face cards (King = K, Queen = Q, Jack=J) are worth 10. Aces=A are worth 1 or 11. Let C be the event that your first card is a black suited card worth 10. That is, C is the event that your first card is in the set $\{10\spadesuit, 10\clubsuit, J\spadesuit, J\clubsuit, Q\spadesuit, Q\clubsuit, K\spadesuit, K\clubsuit\}$. Let D be the event that your second card is a red Ace. That is, D is the event that your second card is in the set $\{A\heartsuit, A\diamondsuit\}$.

1. What is $Pr(C \cap D)$?

   Solution: The sample set S is all the possible pairs of cards that you could be dealt. $C \cap D$ is the event that your first card is a black suited card worth 10 and your second card is a red ace. Thus $Pr(C \cap D) = \frac{|C \cap D|}{|S|}$. We count all of the elements in $C \cap D$ as follows. Choose a black suited card worth 10 for the first card. There are 8 ways to do this. Then choose a red ace for the second card. There are 2 ways to do this. Thus $|C \cap D| = 8 \cdot 2 = 16$. To count all the elements in S we do as follows. Choose one of the 52 cards for the first card. There are 52 ways to do this. Choose one of the remaining 51 cards for the second card. There are 51 ways to do this. Thus $|S| = 52 \cdot 51$. Thus

$$Pr(C \cap D) = \frac{|C \cap D|}{|S|} = \frac{16}{52 \cdot 51} \approx 0.006.$$

2. What is $Pr(C \cup D)$?

Solution: $|C \cup D| = |C| + |D| - |C \cap D|$. To determine $|C|$ we choose a black suited card worth 10 for the first card. There are 8 ways to do this. Then we choose one of the remaining 51 cards for the second card. Thus $|C| = 8 * 51 = 408$. To determine $|D|$ we choose one of the red aces for the **second** card. There are 2 ways to do this. Then choose one of the remaining 51 cards for the first card. There are 51 ways to do this. Thus $|D| = 2 \cdot 51 = 102$. If you are not satisfied with this because we did them out of order, consider the following procedure to determine $|D|$. We split D into two events non-intersecting events. D_1 is we choose a red ace for the first card, and D_2 is we do not choose a red ace for the first card. Since D_1 and D_2 do not intersect, $|D| = |D_1 \cup D_2| = |D_1| + |D_2|$. To determine $|D_1|$ we first choose a red ace: there are 2 ways to do that. Then we choose the second red ace for the second card: there is 1 way to do that. Thus $|D_1| = 2 \cdot 1 = 2$. To determine $|D_2|$ we first choose any card that is not a red ace: there are 50 ways to do this. Then we choose a red ace for the second card: there are 2 ways to do this. Thus $|D_2| = 50 \cdot 2$. Therefore $|D| = |D_1 \cup D_2| = |D_1| + |D_2| = 2 + 50 \cdot 2 = 102$.

Therefore

$$Pr(C \cup D) = \frac{|C \cup D|}{|S|} = \frac{|C| + |D| - |C \cap D|}{|S|} = \frac{408 + 102 - 16}{52 \cdot 51} \approx 0.186.$$

3. Are the events C and D independent? In other words, is $Pr(C \cap D) = Pr(C) \cdot Pr(D)$?

Solution: $Pr(C \cap D) = \frac{|C \cap D|}{|S|} = \frac{16}{52 \cdot 51} = 0.0060...$ while $Pr(C) \cdot Pr(D) = \frac{408}{52 \cdot 51} \cdot \frac{102}{52 \cdot 51} = \frac{1}{169} = 0.0059...$, so they are not independent.

Question 6: You are playing 5 card poker with your friends. In this game you are initially dealt 5 cards and on each round of betting you can exchange any number of cards to try and get a better hand. You are using a standard deck of 52 cards, and any cards you exchange go into a discard pile and are no longer in play for this hand. All cards are dealt uniformly at random.

Hint: Since you know your own cards but no one else's, you may assume that your next cards are drawn uniformly at random from the 47 cards not currently in your hand (since that is, in effect, exactly what happens).

1. In your first hand you are dealt $\{5\diamondsuit, 5\spadesuit, 7\diamondsuit, 8\clubsuit, 9\diamondsuit\}$. You keep the pair of 5's and exchange the 7, 8 and 9 for three more cards. What is the probability that the three cards you receive are three of a kind (thus giving you a full house)? What is the probability that you receive at least one other 5?

Solution: The easy approach is to compute the probability directly. The sample space S is all the ways we can draw 3 cards from the 47 that are left. Thus $|S| = \binom{47}{3}$. Let

$A =$"we draw 3 of a kind from the 47 cards left". Thus $Pr(A) = \frac{|A|}{|S|}$ so we need to count A.

We will break A down into 3 of a kind chosen from all ranks with 4 cards left, and 3 of a kind chosen from all ranks with 3 cards left. The ranks with 4 cards left are $A, 2, 3, 4, 6, 10, J, Q, K$. There are 9 ways to choose one of these, and $\binom{4}{3}$ to choose 3 of them. The ranks with 3 cards left are $7, 8, 9$. There are 3 ways to choose one of these and 1 way to choose all three. These are disjoint sets, so we can use the sum rule, and $|A| = 9 \cdot \binom{4}{3} + 3$. Thus

$$\begin{aligned} Pr(A) =& \frac{|A|}{|S|} \\ =& \frac{9 \cdot \binom{4}{3} + 3}{\binom{47}{3}} \end{aligned}$$

I had originally thought this would be a conditional probability question, but this way is more straightforward. However, if they use conditional probability:

Let $A =$"we draw 3 of a kind from the 47 cards left"
Let $B =$"initially dealt $\{5\diamondsuit, 5\spadesuit, 7\diamondsuit, 8\clubsuit, 9\diamondsuit\}$".

We want to determine $Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$.

$Pr(B) = \frac{1}{\binom{52}{5}}$.
$Pr(A \cap B) = Pr(A) \cdot Pr(B)$

2. Texas Hold 'em is a variety of poker where each player is dealt 2 cards initially. Five other cards are dealt to the middle for all players to share, however for this exercise we will focus on the first 2 cards you are dealt. Assume the dealer is using a single deck of 52 cards.

   (a) What is the probability that the two cards you are dealt are a matching pair? A matching pair is two cards of the same denomination. For example $\{2\spadesuit, 2\heartsuit\}$ and $\{Q\clubsuit, Q\diamondsuit\}$ are matching pairs, but $\{5\spadesuit, 7\spadesuit\}$ and $\{Q\clubsuit, K\diamondsuit\}$ are NOT matching pairs.

   Solution: Let c_1 and c_2 be the first and second card respectively that you are dealt. Let the sample space S be the set of all possible hands of two cards (c_1, c_2). Let P be the event that you are dealt a matching pair. Then $Pr(P) = \frac{|P|}{|S|}$.
   To generate a hand of two cards *dealt in order*, we first choose 1 of the 52 cards in the deck. There are 52 ways to do this. Then choose 1 of the remaining cards. There are 51 ways to do this. Thus the number of possible hands of two cards is $|S| = 52 \cdot 51$. To find the number of matching pairs $|P|$ we are dealt the first card.

There are 52 ways to do this. Then we are dealt one of the 3 matching cards. There are 3 ways to do this. Thus

$$Pr(Pr) = \frac{|P|}{|S|} = \frac{52 \cdot 3}{52 \cdot 51} = \frac{3}{51} = \approx 0.0588.$$

We can compute it ignoring the order they are dealt. Let S be all pairs of cards from a deck of 52, and $|S| = \binom{52}{2}$. To count all pairs P we first choose a rank. There are 13 ways to do that. Then we choose 2 cards of the 4 available, and there are $\binom{4}{2}$ ways to do that. Thus $|P| = 13 \cdot \binom{4}{2}$, and

$$\begin{aligned} Pr(A) &= \frac{|A|}{|S|} \\ &= \frac{13 \cdot \binom{4}{2}}{\binom{52}{2}} \\ &= \frac{13 \cdot 6}{\frac{52 \cdot 51}{2}} \\ &= \frac{13 \cdot 6}{26 \cdot 51} \\ &= \frac{3}{51} \end{aligned}$$

(b) The dealer deals your two cards by grabbing both cards at once and tossing them at you carelessly. As a result you happen to catch a glimpse of one of your cards and see a black 7. That is, you know at least one of your cards is from the set $\{7\spadesuit, 7\clubsuit\}$ (although you did not notice which one). What is the probability that you have a matching pair now? That is, what is the probability that both of your cards are 7's given that at least one of your cards is a black 7?

Solution: Let B be the event that one of your cards is a black 7, and let P be the event that you were dealt a matching pair. We want to determine

$$Pr(P \mid B) = \frac{Pr(P \cap B)}{Pr(B)}.$$

We will start by determining $|B|$. Let C be the event your hand contains $7\clubsuit$ and let D be the event that your hand contains $7\spadesuit$. Then $B = C \cup D$, and $|B| = |C| + |D| - |C \cap D|$. The procedure to determine C is we choose a location (out of two available) for the $7\clubsuit$. There are 2 ways to do that. Then there are 51 ways to choose the other card. The same procedure applies to D. Thus $|C| = |D| = 2 \cdot 51$. To determine $|C \cap D|$ we choose a location for $7\clubsuit$. There are 2 ways to do this, then there is 1 way to choose a location for $7\spadesuit$. Thus $|C \cap D| = 2$. Therefore

$$|B| = |C| + |D| - |C \cap D| = 2 \cdot 2 \cdot 51 - 2 = 202.$$

Thus

$$Pr(B) = \frac{|B|}{|S|} = \frac{202}{52 \cdot 51}.$$

The event $P \cap B$ is all of the hands that are two 7's and at least one of the 7's is black. We can determine the size of this set using the following procedure. Choose 7♣ or 7♠. There are 2 ways to do this. Choose a location. There are 2 ways to do this. Choose one of the other 3 7's for the second location. There are 3 ways to do this. That gives us $2 \cdot 2 \cdot 3 = 12$ but we have double counted all the hands with both black 7's. There are 2 of these, therefore $|P \cap B| = 12 - 2 = 10$. Therefore

$$Pr(P \cap B) = \frac{|P \cap B|}{|S|} = \frac{10}{52 \cdot 51}.$$

Putting it all together we have

$$Pr(P \mid B) = \frac{Pr(P \cap B)}{Pr(B)} = \frac{\frac{10}{52 \cdot 51}}{\frac{202}{52 \cdot 51}} = \frac{10}{202} \approx 0.0495.$$

3. Assuming you have the same hand as above: $\{5\diamondsuit, 5♠, 7\diamondsuit, 8♣, 9\diamondsuit\}$. This time you trade in the 5♠ and the 8♣. What is the probability that the two cards you receive are both diamonds ($\diamondsuit$)? (Five cards of the same suit is called a flush, which is superior to a full house or three of a kind.)

   Solution: It has been pointed out to me that a flush is worse than full house. We will do this without conditional probability, though it is possible to use conditional probability.

   There are 47 cards left in the deck, of which 10 are diamonds. Let $A =$"both cards are diamonds". Then $|A| = \binom{10}{2}$. The sample space S is all ways to choose 2 cards from 47, thus $|S| = \binom{47}{2}$. Thus the probability of getting 2 diamonds is $Pr(A) = \frac{|A|}{|S|} = \frac{\binom{10}{2}}{\binom{47}{2}} = \frac{90}{2162} \approx 0.04$.

Question 7: We have designed k-redundant circuits. That is, we have designed a circuit C with n components $C_1, C_2, ..., C_n$ where C fails only if k or more circuits fail for $1 \leq k \leq n$. Each of the circuits fails with probability p, and all of them are mutually independent.

1. Determine the probability that exactly j circuits fail.

   Solution: Let A_i, $1 \leq i \leq n$ be the event that component C_i does not fail. Then $Pr(A_i) = (1-p)$, and $Pr(\overline{A_i}) = p$. Choose a subset of j components, and let $a_1, ..., a_j$ be the indices of the j components, and let $b_1, b_2, ...b_{n-j}$ be the indices of the remaining $n - j$ components. The probability that those j components fail and the other $n - j$ components do not fail is

$$Pr(\overline{A}_{a_1} \wedge \overline{A}_{a_2} \wedge ... \wedge \overline{A}_{a_j} \wedge A_{b_1} \wedge A_{b_2} \wedge ... \wedge A_{b_{n-j}}).$$

Since these circuits are mutually independent, this value is equal to

$$\begin{aligned}&Pr(\overline{A}_{a_1}) \cdot Pr(\overline{A}_{a_2}) \cdot \ldots \cdot Pr(\overline{A}_{a_j}) \cdot Pr(A_{b_1}) \cdot Pr(A_{b_2}) \cdot \ldots \cdot Pr(A_{b_{n-j}})\\ =&p^j \cdot (1-p)^{n-j}\end{aligned}$$

To determine the probability that exactly j circuits fail, we look at all the possible ways that j of n circuits fail. There are $\binom{n}{j}$ ways for this to happen. If the sample space S is all the possible ways that the components of C can fail, then each of these $\binom{n}{j}$ events represents a distinct element of the sample space. Thus the probability that exactly j components fail is the sum of the $\binom{n}{j}$ elements of S where precisely j components fail. Thus $Pr(A) = \binom{n}{j} \cdot p^j \cdot (1-p)^{n-j}$.

2. Determine the probability that C fails.

   Solution: To determine C we sum up all the possible events in S where k or more circuits fail. This is

$$\sum_{i=k}^{n} \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

3. Let A be the event that C fails. Prove that, for $k = 0$ (that is, C fails if 0 or more circuits fail) $Pr(A) = 1$.

   Solution: This is simply the binomial theorem.

$$\begin{aligned}&\sum_{i=0}^{n} \binom{n}{i} p^i \cdot (1-p)^{n-j}\\ =&(p + (1-p))^n\\ =&1^n\\ =&1\end{aligned}$$

Question 8:[Mutually independent] Let $d_1, d_2, ..., d_n$ be n 6-sided dice. Assuming we roll all n dice, determine the probabilities below.

1. What is the probability that d_1 is the highest roll? Note this may not be as simple as it first seems, since even if d_1 is the highest, it may not be the only highest.

   Solution: Let E be the event that d_1 is the highest roll. We want to determine $Pr(E) = \frac{|E|}{|S|}$. The sample space is all possible rolls, so $|S| = 6^n$.

   Let E_i be the event that d_1 is the highest roll and $d_1 = i$. Note that $|E| = \sum_{i=1}^{6} |E_i|$ since these are disjoint sets. We count each set E_i using the product rule. There is one way to choose $d_1 = i$ and i ways to choose the values of each of the $n-1$ other dice (since they all have a value from $1..i$). Thus by the product rule $|E_i| = i^{n-1}$. Thus

$$|E| = \sum_{i=1}^{6} i^{n-1}.$$

Then

$$\begin{aligned} Pr(E) &= \frac{|E|}{|S|} \\ &= \sum_{i=1}^{6} \frac{i^{n-1}}{6^n} \\ &= \frac{1}{6} \sum_{i=1}^{6} \left(\frac{i}{6}\right)^{n-1}. \end{aligned}$$

Either of the above forms is fine.

2. What is the probability that the highest roll is i, for $1 \leq i \leq 6$?

   Solution: Let E_i be the event that the highest roll is i.

$$Pr(E_i) = \frac{|E_i|}{|S|}$$

To determine $|E_i|$:

$$\begin{aligned} |E_i| &= \sum_{j=1}^{n} \binom{n}{j} (i-1)^{n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j} (i-1)^{n-j} - \binom{n}{0} (i-1)^{n-0} \\ &= \sum_{j=0}^{n} \binom{n}{j} (i-1)^{n-j} - (i-1)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (i-1)^{n-j} \cdot 1^j - (i-1)^n \\ &= (i-1+1)^n - (i-1)^n \qquad \text{(Binomial Theorem)} \\ &= i^n - (i-1)^n. \end{aligned}$$

The easier way to solve this is to observe that if A is the event that the highest roll is at most i and B is the event that the highest roll is at most $i-1$, then $B \subseteq A$ and $E_i = A - B$. Thus $|E_i| = |A| - |B| = i^n - (i-1)^n$.

Then $Pr(E_i) = \frac{i^n-(i-1)^n}{6^n}$.

Note that due to laziness (but also because I liked the question) this is nearly the same as the question from last year's assignment. I simply used dice instead of vectors.