INFINITE SAMPLE SPACES

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

ROSENCRANTZ AND GUILDENSTERN ARE DEAD

GUILDENSTERN: Heads.

He keeps flipping the coin.

Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads … Heads …

…

GUILDENSTERN: A weaker man might be moved to reexamine his faith. If for nothing else at least in the law of probability.

He flips another coin to Rosencrantz.

ROSENCRANTZ: Heads.

S: Sample space

$$\Pr(x) : x \in S \rightarrow [0,1]$$

$$\sum_{w \in S} \Pr(w) = 1$$

Up until now, all sample spaces have been finite sets.

However, there can be sample spaces that are infinite sets…

S

Take a fair coin, flip this coin until it comes up H for the first time.

Observe all coin flips are independent.

Sample space $S = \{H, TH, TTH, TTTH, \ldots\}$
$= \{T^n H : n \geq 0\}$

$$\Pr(T^n H)$$

$$= \Pr(f_1 = T \wedge f_2 = T \wedge \cdots \wedge f_n = T \wedge f_{n+1} = H)$$

T T T H

S: Sample space

$$\Pr(x) : x \in S \rightarrow [0,1]$$

$$\boxed{\sum_{w \in S} \Pr(w) = 1}$$

Fair coin, flip until it comes up H

All coin flips are independent.

$$S = \{H, TH, TTH, TTTH, \dots\}$$
$$= \{T^n H : n \geq 0\}$$

S

$$\Pr(T^n H)$$
$$= \Pr(f_1 = T \wedge f_2 = T \wedge \cdots \wedge f_n = T \wedge f_{n+1} = H)$$
$$= \Pr(f_1 = T) \cdot \Pr(f_2 = T) \cdot \dots \cdot \Pr(f_{n+1} = H)$$
$$= \left(\frac{1}{2}\right)^{n+1}$$

We know that the sample space of probabilities must sum to 1. And yet there are infinitely many outcomes.

T T T H

Infinite Series:

Given an infinite sequence $a_0, a_1, a_2, \ldots$ of real numbers, we want to define the sum of these numbers.

We know if there are finite many (say N numbers in the sequence), then we can define the sum by:

$$\sum_{n=0}^{N} a_n$$

To define it for infinity we use

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n$$

If the limit exists, then we can write it as

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

Infinite Series: $a_0, a_1, a_2, \ldots$ real numbers

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

(if the limit exists)

Consider a series where we define each term a_n as:

$$a_n = x^n, 0 < x < 1$$

Then

$$\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n = \lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x}$$

We can prove this using induction on N, however, there is another way to prove this.

$$1 + x + x^2 + x^3 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

Multiply left and right by $(1 - x)$:

<u>Infinite Series</u>: $a_0, a_1, a_2, \ldots$ real numbers

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

(if the limit exists)

Consider a series where we define each term a_n as:

$$a_n = x^n, 0 < x < 1$$

$$1 + x + x^2 + x^3 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

Multiply left and right by $(1 - x)$:

$$(1 - x)(1 + x + x^2 + x^3 + \cdots + x^N) = 1 - x^{N+1}$$

Take the LHS:

$$\text{LHS} = 1 + x + x^2 + x^3 + \cdots + x^{N-1} + x^N$$
$$-x - x^2 - x^3 - \cdots - x^{N-1} - x^N - x^{N+1}$$

<u>Infinite Series</u>: $a_0, a_1, a_2, \ldots$ real numbers

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

(if the limit exists)

Consider a series where we define each term a_n as:

$$a_n = x^n, 0 < x < 1$$

$$1 + x + x^2 + x^3 + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}$$

Multiply left and right by $(1 - x)$:

$$(1 - x)(1 + x + x^2 + x^3 + \cdots + x^N) = 1 - x^{N+1}$$

Take the LHS:

$$\begin{aligned} \text{LHS} &= 1 \cancel{+ x + x^2 + x^3 + \cdots + x^{N-1} + x^N} \\ &\quad \cancel{- x - x^2 - x^3 - \cdots - x^{N-1} - x^N} - x^{N+1} \\ &= 1 - x^{N+1} \end{aligned}$$

And

$$\text{RHS} = 1 - x^{N+1}$$

Infinite Series: $a_0, a_1, a_2, \ldots$ real numbers

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \sum_{n=0}^{\infty} a_n$$

(if the limit exists)

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n = \lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x}$$

For $0 < x < 1$:

$$\lim_{N\to\infty} x^{N+1} = 0$$

Thus:

$$\lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x} = \frac{1}{1 - x}$$

And

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

Infinite Series:

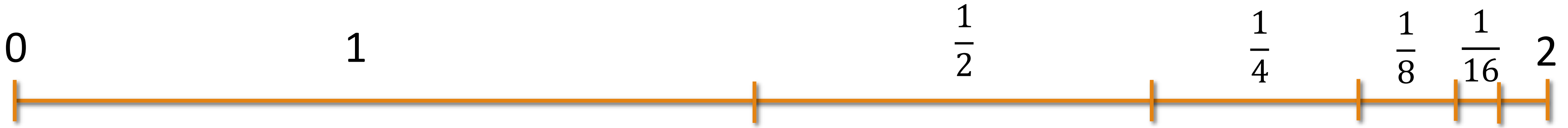
For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

We can test this out for $x = \frac{1}{2}$:

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots = \frac{1}{1-\frac{1}{2}} = 2$$

We can visually see this:



Assume this is a line of length 2.

Fair coin, flip until it comes up H

All coin flips are independent.

$$S = \{H, TH, TTH, TTTH, \ldots\}$$
$$= \{T^n H : n \geq 0\}$$

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

T T T H

$$\Pr(T^n H) = \left(\frac{1}{2}\right)^{n+1}$$

To sum all probabilities in the sample space S:

$$\sum_{n=0}^{\infty} \Pr(T^n H) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n}$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{1}{2} \cdot 2 = 1$$

Fair coin, flip until it comes up H

All coin flips are independent.

$$S = \{H, TH, TTH, TTTH, \ldots\}$$
$$= \{T^n H : n \geq 0\}$$

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

T T T H

We may also think of it like this:

$$\sum\nolimits_{x \in S} \Pr(X)$$

$$= \Pr(H) + \Pr(TH) + \Pr(TTH) + \Pr(TTTH) + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$= 1$$

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consider a game: fair coin and players P_1 and P_2 take turns flipping the coin.

The first player to flip heads wins.

$P_1 = \mathrm{T}$ $\quad$ $P_2 = \mathrm{T}$ $\quad$ $P_1 = \mathrm{T}$ $\quad$ $P_2 = \mathrm{H}$

Who do we think should win this game?

(This does not seem very fair.)

The sample space $S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H}: \mathrm{n} \geq 0\}$

(First time heads is flipped the game ends.)

Let event $A = P_1$ wins. What outcomes does A contain?

Every element of S where n is an even number.

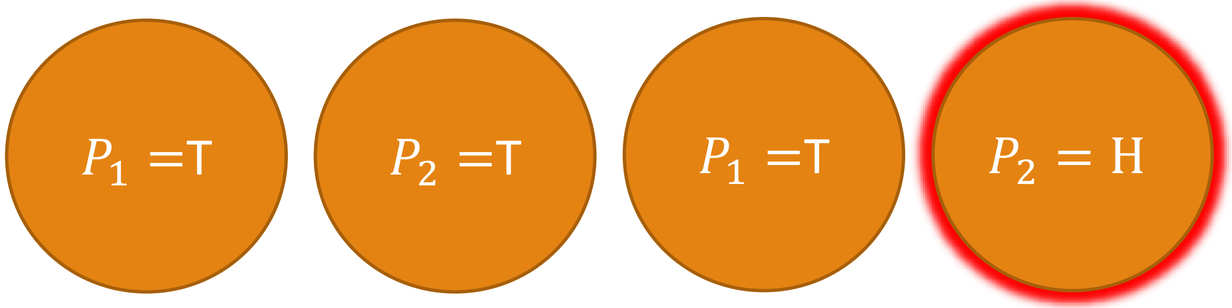
For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consider a game: fair coin and players P_1 and P_2 take turns flipping the coin.

The first player to flip heads wins.

The sample space $S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H}: \mathrm{n} \geq 0\}$



Let event $A = P_1$ wins.

$$A = \{H, TTH, TTTTH, TTTTTTH, \ldots\}$$

$$A = \{T^{2m}H: m \geq 0\}$$

Recall the $\Pr(A) = \sum_{w \in A} \Pr(w)$, thus we sum the individual probabilities of each outcome (and there are infinitely many).

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H)$$

Each coin flip is independent…

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consider a game: fair coin and players P_1 and P_2 take turns flipping the coin.

The first player to flip heads wins.

The sample space $S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H} : \mathrm{n} \geq 0\}$

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H)$$

The product of each outcome is the product of the probabilities of the coin flips.

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H) = \sum_{m=0}^{\infty} \Pr(T)^{2m} \cdot \Pr(H)$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} \cdot \frac{1}{2} \ = \ \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{2} \cdot \frac{1}{2}\right)^{m}$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consider a game: fair coin and players P_1 and P_2 take turns flipping the coin.

The first player to flip heads wins.

The sample space $S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H}: \mathrm{n} \geq 0\}$

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H)$$

The product of each outcome is the product of the probabilities of the coin flips.

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m+1}$$

$$= \frac{1}{2}\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} = \frac{1}{2}\sum_{m=0}^{\infty} \left(\frac{1}{2}\cdot\frac{1}{2}\right)^{m}$$

$$= \frac{1}{2}\sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$$

For $0 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Consider a game: fair coin and players P_1 and P_2 take turns flipping the coin.

The first player to flip heads wins.

The sample space $S = \{\mathrm{T^n H : n \geq 0}\}$

$$\sum_{m=0}^{\infty} \Pr(T^{2m}H)$$

$$\Pr(Player\ 1\ wins) = \frac{2}{3}$$

$$\Pr(Player\ 2\ wins) = \frac{1}{3}$$

$$\sum_{m=0}^{\infty} \Pr(T^{2m+1}H) = \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m+2}$$

$$\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} \left(\frac{1}{2}\right)^{2} = \frac{1}{4} \sum_{m=0}^{\infty} \left(\frac{1}{2} \cdot \frac{1}{2}\right)^{m}$$

$$= \frac{1}{4} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m} = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

Also, who has a higher chance of winning, P_1 or P_2?

What do these outcomes of S look like?

$TTTH * TTH$ or

$H * TTTH$ or

$TTTTTTH * TTTTTTH$ or

$H * H$

Etc. (* is a meaningless separator token.)

The sample space $S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

For player 1 to win, the second head must be on an even flip.

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

Also, who has a higher chance of winning, P_1 or P_2?

The sample space $S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H}\mathrm{T}^{\mathrm{m}}\mathrm{H}, : \mathrm{n} \geq 0, m \geq 0\}$

For player 1 to win, the second head must be on an odd flip.

Thus $n + m + 1$ must be an even number.

There are 2 ways for $n + m + 1$ to be even – either n is odd and m is even, or m is odd and n is even.

Let $A_1 = P_1 \text{ wins and } n \text{ is odd and } A_2 = P_1 \text{ wins and } m \text{ is odd}$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

$S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

$A_1 = P_1$ wins and n is odd

$A_2 = P_1$ wins and m is odd

Observe that A_1 and A_2 are disjoint. Thus:

$$\Pr(A) = \Pr(A_1) + \Pr(A_2)$$

As before, coin flips are independent. To determine A_1 we can rewrite $m = 2k$ since it is even and $n = 2j + 1$ since it is odd. Thus:

$$\Pr(A_1) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Pr(\mathrm{T^{2j+1} H T^{2k} H})$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2j+2k+3}$$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

$S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

$A_1 = P_1$ wins and n is odd

$A_2 = P_1$ wins and m is odd

$$\Pr(A_1) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2j+2k+3}$$

$$= \sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{2j+3}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2k}$$

$$= \left(\frac{1}{2}\right)^{3}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{2j}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{2k}$$

$$= \frac{1}{8}\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^{j}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{k}$$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

$S = \{\mathrm{T}^{\mathrm{n}}\mathrm{H}\mathrm{T}^{\mathrm{m}}\mathrm{H}, : \mathrm{n} \geq 0, m \geq 0\}$

$A_1 = P_1$ wins and n is odd

$A_2 = P_1$ wins and m is odd

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, 0 < x < 1$$

$$\Pr(A_1) = \frac{1}{8}\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^j \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$$

$$= \frac{1}{8}\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^j \cdot \frac{1}{1-\frac{1}{4}}$$

$$= \frac{1}{8} \cdot \frac{1}{1-\frac{1}{4}} \cdot \frac{1}{1-\frac{1}{4}}$$

$$= \frac{1}{8} \cdot \frac{4}{3} \cdot \frac{4}{3} = \frac{16}{72} = \frac{2}{9}$$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

$S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

$A_1 = P_1$ wins and n is odd

$A_2 = P_1$ wins and m is odd

To determine A_2 we can rewrite $m = 2k + 1$ since it is odd and $n = 2j$ since it is even. Thus:

$$\Pr(A_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Pr(\mathrm{T^{2j} H T^{2k+1} H})$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2j+2k+3}$$

$$= \frac{2}{9}$$

New game: fair coin and players P_1 and P_2 take turns flipping the coin. The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

What is $\Pr(A)$?

$S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

$A_1 = P_1$ wins and n is odd

$A_2 = P_1$ wins and m is odd

$$\Pr(A) = \Pr(A_1) + \Pr(A_2)$$

$$= \frac{2}{9} + \frac{2}{9}$$

$$= \frac{4}{9}$$

Thus the probability that P_1 wins is now $\frac{4}{9}$, and thus the probability that P_2 wins is $\frac{5}{9}$. So the odds have shifted into player 2's favour, as we suspected.

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

The sample space $S = \{\mathrm{T^n HT^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

$$T \ldots THT \ldots TH$$

However, we can think of this as 2 rounds of the first game (where the first player to flip H wins).

Round 1 ends once the first H is flipped.

Round 2 ends once the second H is flipped.

Whoever wins round 1, the other player starts round 2.

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

The sample space $S = \{\mathrm{T^n H T^m H}, : \mathrm{n} \geq 0, m \geq 0\}$

We know that whoever starts each round is more likely to win that round. But if they win the first round, they do not start the next round.

$A = P_1$wins first round and P_1 wins second round or P_2wins first round and P_1 wins second round.

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

Let $A_1 = P_1$ wins first round and P_1 wins second round

Let $A_2 = P_2$ wins first round and P_1 wins second round

$A \leftrightarrow A_1$ or A_2

These are disjoint events, thus we can use the sum rule.

$$\Pr(A) = \Pr(A_1) + \Pr(A_2)$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i \text{ starts}, P_j \text{ wins}$
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

We can rewrite A_1 and A_2 in terms of B_{ij} defined to the left.

$A_1 = P_1 \text{wins first and } P_1 \text{ wins second}$

If A_1 occurs, we know that P_1 starts the first round and P_1 wins the first round. This is event B_{11}.

If P_1 wins the first round, then P_2 must start the second round. We are assuming P_1 won the second round. This is event B_{21}. Thus:

$A_1 = B_{11}$ and B_{21}

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i \text{ starts}, P_j \text{ wins}$
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

We can rewrite A_1 and A_2 in terms of B_{ij} defined to the left.

$A_2 = P_2 \text{wins first and } P_1 \text{ wins second}$

If A_2 occurs, we know that P_1 starts the first round and P_2 wins the first round. This is event B_{12}.

If P_2 wins the first round, then P_1 must start the second round. We are assuming P_1 won the second round. This is event B_{11}. Thus:

$A_2 = B_{12}$ and B_{11}

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

We can rewrite A_1 and A_2 in terms of B_{ij} defined to the left.

$A_1 = P_1$ wins first and P_1 wins second

$A_2 = P_2$ wins first and P_1 wins second

$A_1 = B_{11}$ and B_{21}

$A_2 = B_{12}$ and B_{22}

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i \text{ starts}, P_j \text{ wins}$

$$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$$

$$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

$\mathrm{A}_1 = \mathrm{B}_{11}$ and B_{21}

Are B_{11} and B_{21} independent?

B_{21} is the event that P_1 wins *given* that P_2 started.

We know that if P_2 started, then P_1 won last round.

But since the 2 different rounds have no coin tosses in common, they are independent events.

Or consider:

$$\Pr(B_{21}|B_{11}) =$$

$$\Pr(B_{21})$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

$\mathrm{A}_1 = \mathrm{B}_{11}$ and B_{21}

B_{11} and B_{21} are independent.

$$\Pr(A_1) = \Pr(\mathrm{B}_{11} \wedge \mathrm{B}_{21})$$
$$= \Pr(B_{11}) \cdot \Pr(B_{21})$$
$$= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

$\mathrm{A}_2 = \mathrm{B}_{12}$ and B_{11}

B_{12} and B_{11} are independent.

$$\Pr(A_2) = \Pr(\mathrm{B}_{12} \wedge \mathrm{B}_{11})$$

$$= \Pr(B_{12}) \cdot \Pr(B_{11})$$

$$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:

Events: $B_{ij} = P_i \text{ starts}, P_j \text{ wins}$

$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$

$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **second** player to flip heads wins.

$$A = P_1 \text{ wins}$$

We will look at an easier way to determine $\Pr(A)$.

$$A = A_1 \cup A_2$$

$$\Pr(A) = \Pr(A_1) + \Pr(A_2)$$

$$= \frac{2}{9} + \frac{2}{9}$$

$$= \frac{4}{9}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i \text{ starts}, P_j \text{ wins}$
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **third** player to flip heads wins.

$$A = P_1 \text{ wins}$$

$$A_? = B_{1?} \wedge B_{??} \wedge B_{?1}$$

The term in the middle will determine the other two values. So the number of terms is the number of ways we can write the term in the middle.

$$A_1 = B_{12} \wedge B_{11} \wedge B_{21}$$

$$A_2 = B_{12} \wedge B_{12} \wedge B_{11}$$

$$A_3 = B_{11} \wedge B_{21} \wedge B_{21}$$

$$A_4 = B_{11} \wedge B_{22} \wedge B_{11}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2

take turns flipping the coin.

The **third** player to flip heads wins.

$$A = P_1 \text{ wins}$$

$$A_1 = B_{12} \wedge B_{11} \wedge B_{21}$$

$$\Pr(A_1) = \Pr(B_{12}) \cdot \Pr(B_{11}) \cdot \Pr(B_{21})$$

$$= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{27}$$

$$A_2 = B_{12} \wedge B_{12} \wedge B_{11}$$

$$\Pr(A_2) = \Pr(B_{12}) \cdot \Pr(B_{12}) \cdot \Pr(B_{11})$$

$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{27}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:

Events: $B_{ij} = P_i$ starts, P_j wins

$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$

$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **third** player to flip heads wins.

$$A = P_1 \text{ wins}$$

$$A_3 = B_{11} \wedge B_{21} \wedge B_{21}$$

$$\Pr(A_3) = \Pr(B_{11}) \cdot \Pr(B_{21}) \cdot \Pr(B_{21})$$

$$= \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{27}$$

$$A_4 = B_{11} \wedge B_{22} \wedge B_{11}$$

$$\Pr(A_4) = \Pr(B_{11}) \cdot \Pr(B_{22}) \cdot \Pr(B_{11})$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

Fair coin, players P_1 and P_2 take turns flipping the coin. For $i, j \in \{1,2\}$:
Events: $B_{ij} = P_i$ starts, P_j wins
$\Pr(B_{11}) = \Pr(B_{22}) = \frac{2}{3}$
$\Pr(B_{12}) = \Pr(B_{21}) = \frac{1}{3}$

Same game: fair coin and players P_1 and P_2 take turns flipping the coin.

The **third** player to flip heads wins.

$$A = P_1 \text{ wins}$$

$$\Pr(A) = \Pr(A_1 \vee A_2 \vee A_3 \vee A_4)$$

$$= \Pr(A_1) + \Pr(A_2) + \Pr(A_3) + \Pr(A_4)$$

$$= \frac{2}{27} + \frac{2}{27} + \frac{2}{27} + \frac{8}{27}$$

$$= \frac{14}{27}$$