INDEPENDENT EVENTS

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

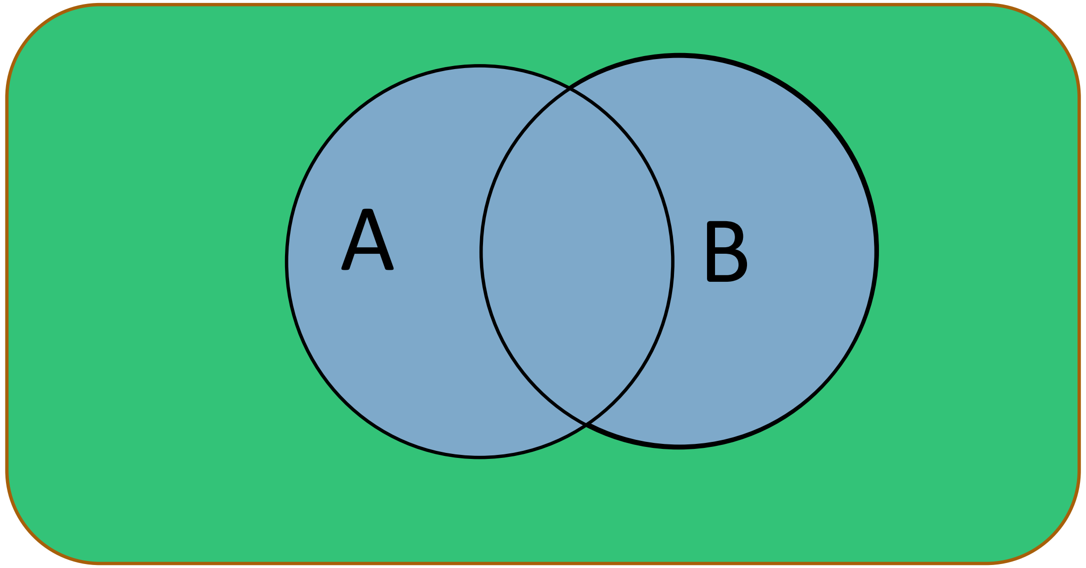
DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

BY MICHIEL SMID

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)}$$



Events A, B are independent if:

An event A has no influence on the probability of event B, and

An event B has no influence on the probability of event A

If A and B independent, then

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B independent, then
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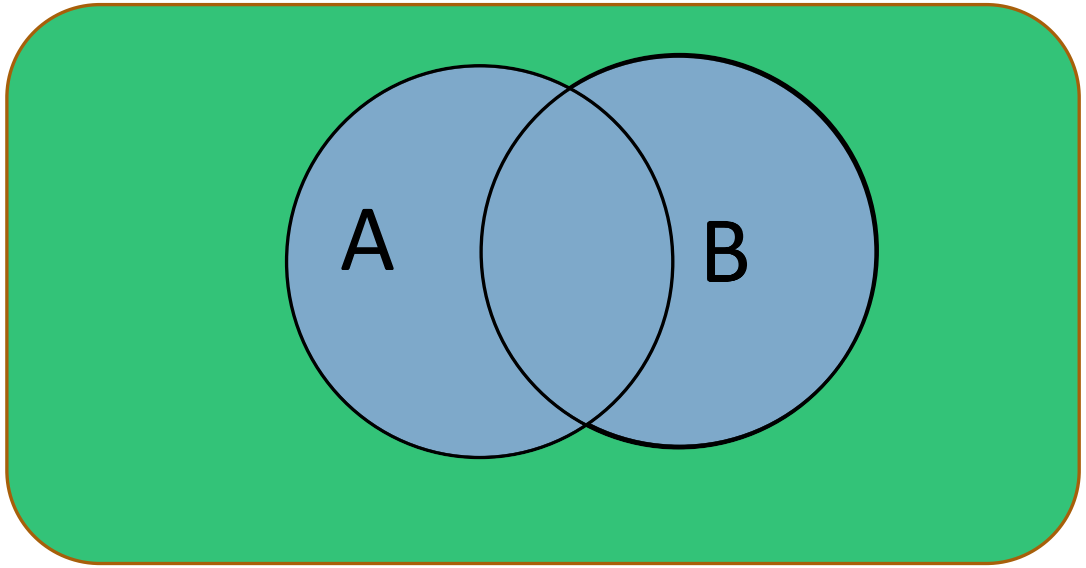

Claim:

If A and B independent, then

If $\Pr(B) > 0$ then

$$\Pr(A|B) = \Pr(A)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A) \cdot \Pr(B)}{\Pr(B)}$$

$$= \Pr(A)$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent, then
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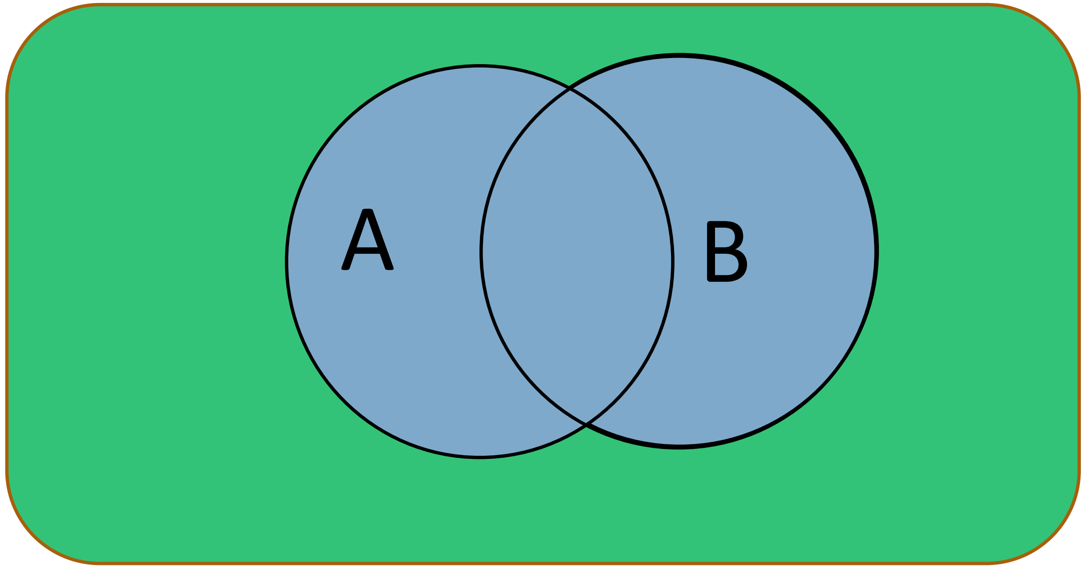

Claim:

If A and B independent, then

If $\Pr(A) > 0$ then

$$\Pr(B|A) = \Pr(B)$$

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{\Pr(A) \cdot \Pr(B)}{\Pr(A)}$$

$$= \Pr(B)$$

Independent Events

Events $A, B, \Pr(B) > 0$

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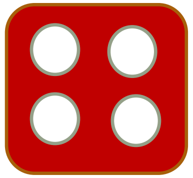

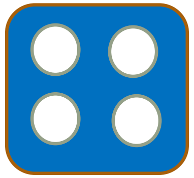

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

Event $A =$ "red + blue = 7“

Event $B =$ "red = 4“

To verify that events A and B are independent, we must verify the expression:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

That means we need to determine $\Pr(A \cap B)$, $\Pr(A)$, and $\Pr(B)$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

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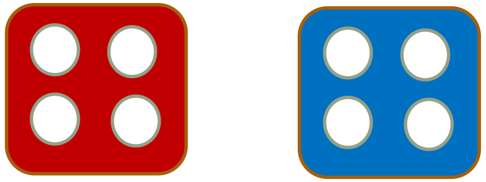

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

Event $A =$ "red + blue = 7“

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$\Pr(A) = \frac{|A|}{|S|}$$

$$= \frac{6}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

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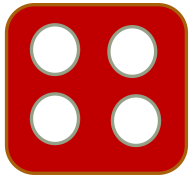

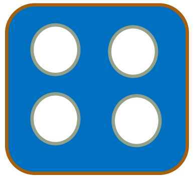

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

Event $B =$ "red = 4“

$$B = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$$

$$\Pr(B) = \frac{|B|}{|S|}$$

$$= \frac{6}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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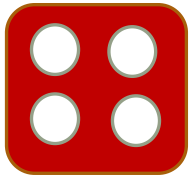

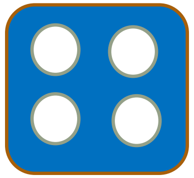

Roll fair red die, fair blue die

$$S = \{(i,j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

Event $A \cap B =$ " red + blue = 7 and red = 4“

$$A \cap B \ = \{(4,3)\}$$

$$\Pr(A \cap B) = \frac{|A \cap B|}{|S|}$$

$$= \frac{1}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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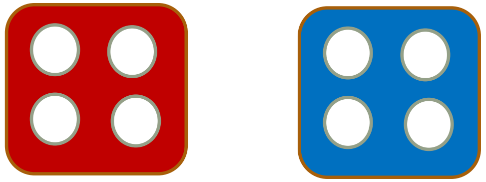

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

$$\Pr(A \cap \mathrm{B}) = \frac{|A \cap B|}{|S|} = \frac{1}{36}$$

$$\Pr(A) = \frac{1}{6}, \qquad \Pr(B) = \frac{1}{6}$$

$$\Pr(A) \cdot \Pr(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

That means $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$ and thus A and B are independent events.

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B independent,

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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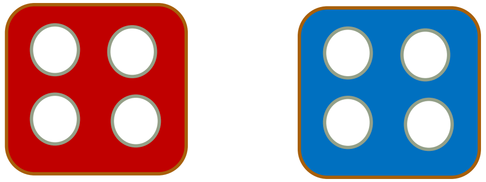

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

Observe that if A happens, then any of

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

can be the outcome.

But if we know B happens, then the only outcome of A that can happen $(4,3)$.

So B can affect what element of A can be selected, but does not affect the probability of A.

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B independent,

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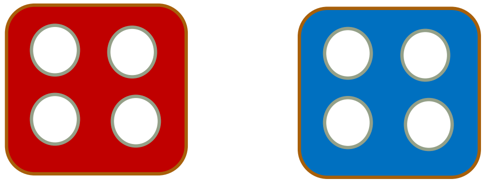

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

i = red, j = blue

Event A = "red + blue = 11“

Event B = "red = 5“

To verify that events A and B are independent, we must verify the expression:

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$.

That means we need to determine $\Pr(A \cap B)$, $\Pr(A)$, and $\Pr(B)$

Independent Events

Events $A, B, \Pr(B) > 0$

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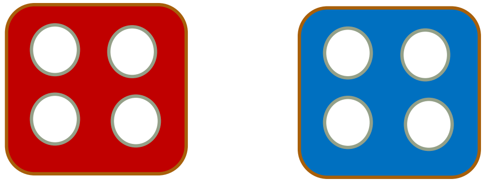

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

Event $A =$ "red + blue = 11"

$$A = \{(5,6), (6,5)\}$$

$$\Pr(A) = \frac{|A|}{|S|}$$

$$= \frac{2}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

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If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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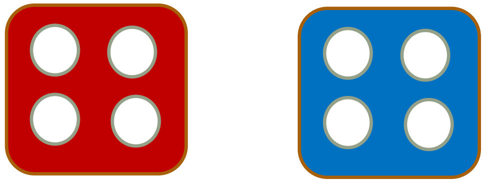

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

Event $B =$ "red = 5“

$$B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

$$\Pr(B) = \frac{|B|}{|S|}$$
$$= \frac{6}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B independent,

$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

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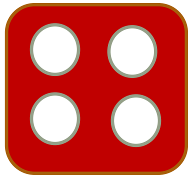

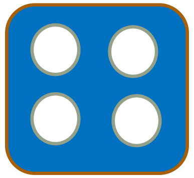

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

Event $A \cap B =$ " red + blue = 11 and red = 5"

$A \cap B = \{(5,6)\}$

$$\Pr(A \cap B) = \frac{|A \cap B|}{|S|}$$

$$= \frac{1}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

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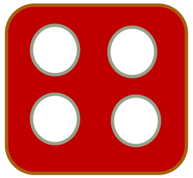

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

$$\Pr(A \cap \mathrm{B}) = \frac{|A \cap B|}{|S|} = \frac{1}{36}$$

$$\Pr(A) = \frac{1}{18}, \qquad \Pr(B) = \frac{1}{6}$$

$$\Pr(A) \cdot \Pr(B) = \frac{1}{18} \cdot \frac{1}{6} = \frac{1}{108}$$

That means $\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B)$ and thus A and B are NOT independent events.

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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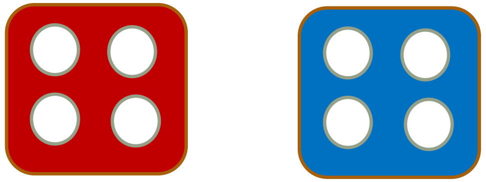

The process to determine independence is to verify the equation

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

We individually find the values for $\Pr(A \cap B)$, $\Pr(A)$, and $\Pr(B)$ then plug them into the equation.

This is the only way to verify independence.

Set based “intuitions” often do not hold up.

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

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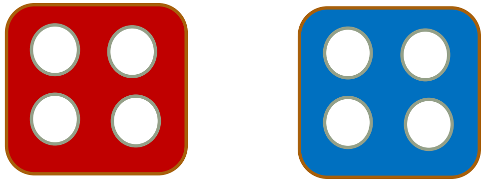

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

Event $A =$ "red + blue = 4"

Event $B =$ "red = 4"

$$A = \{(1,3), (2,2), (3,1)\}$$

$$\Pr(A) = \frac{|A|}{|S|}$$

$$= \frac{3}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

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If A and B independent,

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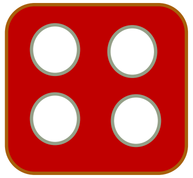

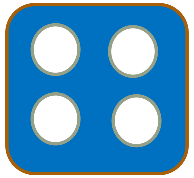

Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

$i =$ red, $j =$ blue

Event $A =$ "red + blue = 4"

Event $B =$ "red = 4"

$$B = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$$

$$\Pr(A) = \frac{|B|}{|S|}$$

$$= \frac{6}{36}$$

Independent Events

Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

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If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
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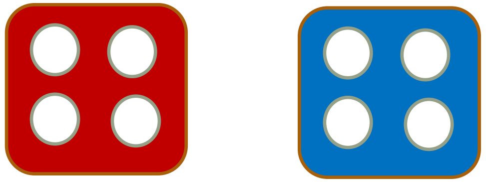


Roll fair red die, fair blue die

$$S = \{(i, j): 1 \le i \le 6, 1 \le j \le 6\}$$

i = red, j = blue

Event A = "red + blue = 4“
Event B = "red = 4“

$$A = \{(1,3), (2,2), (3,1)\}$$
$$B = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$$

$$A \cap B = \{\ \}$$

$$\Pr(A \cap B) = \frac{|A \cap B|}{|S|}$$

$$= 0$$

Independent Events

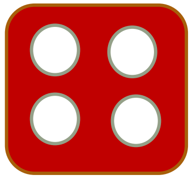
Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
If $\Pr(A) > 0$ then $\Pr(B|A) = \Pr(B)$



Roll fair red die, fair blue die

$$S = \{(i, j): 1 \leq i \leq 6, 1 \leq j \leq 6\}$$

$i =$ red, $j =$ blue

$$\Pr(A \cap \mathrm{B}) = \frac{|A \cap B|}{|S|} = 0$$

$$\Pr(A) = \frac{1}{12}, \qquad \Pr(B) = \frac{1}{6}$$

$$\Pr(A) \cdot \Pr(B) = \frac{1}{12} \cdot \frac{1}{6} = \frac{1}{72}$$

That means $\Pr(A \cap B) \neq \Pr(A) \cdot \Pr(B)$ and thus A and B are NOT independent events.

Independent Events

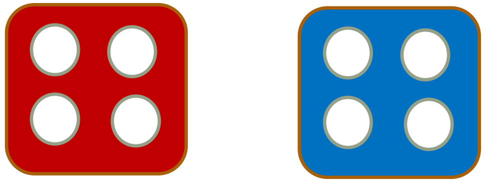
Events $A, B, \Pr(B) > 0$

$$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)}$$

If A and B independent,

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
If $\Pr(A) > 0$ then $\Pr(B|A) = \Pr(B)$



Intuitively if A and B are independent, then one event happening does not affect the probability of the other event happening.

Event A $=$ "red + blue = 4“
Event B $=$ "red = 4“

We can see that if “red = 4”, then it is impossible for "red + blue = 4”. Thus B definitely will impact $\Pr(A)$.

If they were independent,

$$\Pr(A|B) = \Pr(A)$$

But $\Pr(A|B) = 0$ and $\Pr(A) \neq 0$.

Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(\mathrm{A} \cap B)}{\Pr(B)}$

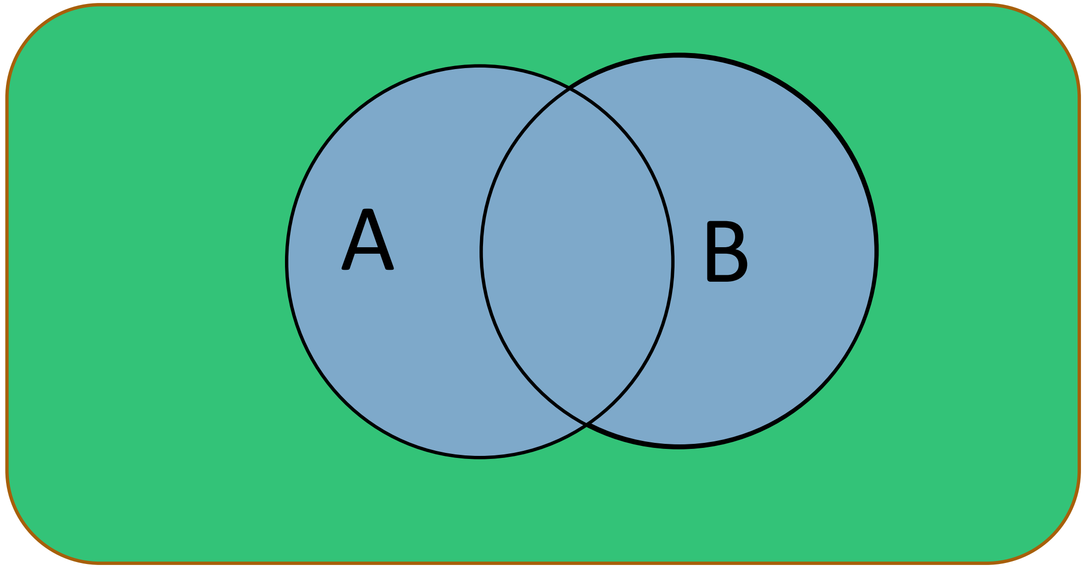
If A and B independent,
$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
If $\Pr(A) > 0$ then $\Pr(B|A) = \Pr(B)$

If A and B are independent, are A and $\bar{B}$ independent?

If A and B are independent, then whether or not B occurs has no impact on the probability of A.

Observe that "not B occurring" is equivalent to $\bar{B}$. That implies that $\bar{B}$ has no affect on $\Pr(A)$ thus A and $\bar{B}$ are independent.



Independent Events

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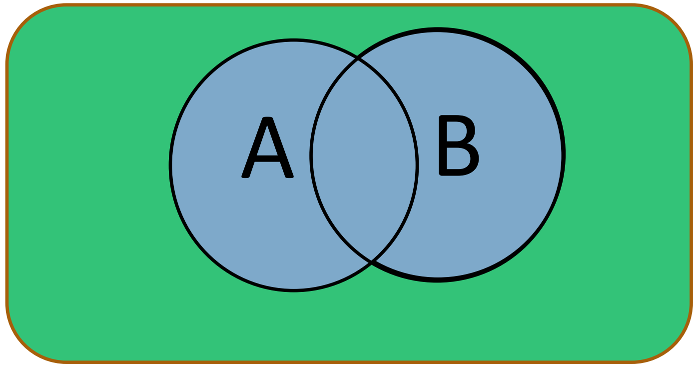
If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
If $\Pr(A) > 0$ then $\Pr(B|A) = \Pr(B)$

If A and B are independent, are A and $\bar{B}$ independent?
We will verify this using the rules of probability.

Given: A and B are independent, thus
$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

To show: A and $\bar{B}$ are independent, thus
$\Pr(A \cap \bar{B}) = \Pr(A) \cdot \Pr(\bar{B})$

$\Pr(A \cap \bar{B}) = \Pr(A) \cdot (1 - \Pr(B))$
$\Pr(A \cap \bar{B}) = \Pr(A) - \Pr(A) \cdot \Pr(B)$
$\Pr(A \cap \bar{B}) + \Pr(A) \cdot \Pr(B) = \Pr(A)$
$\Pr(A \cap \bar{B}) + \Pr(A \cap B) = \Pr(A)$



Independent Events

Events $A, B, \Pr(B) > 0$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

If A and B independent,
$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

If $\Pr(B) > 0$ then $\Pr(A|B) = \Pr(A)$
If $\Pr(A) > 0$ then $\Pr(B|A) = \Pr(B)$

If A and B are independent, are A and $\bar{B}$ independent?
We will verify this using the rules of probability.

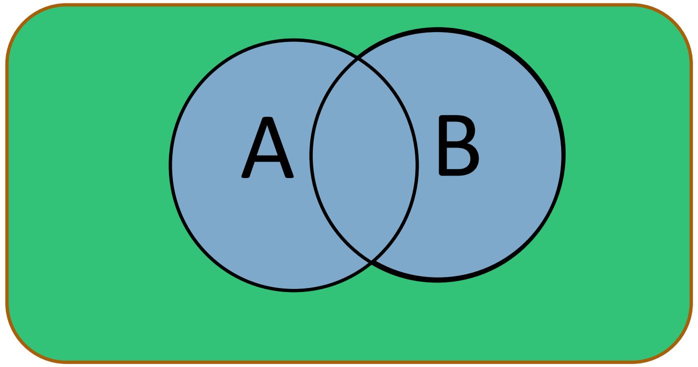
Given: A and B are independent, thus
$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

To show: A and $\bar{B}$ are independent, thus
$\Pr(A \cap \bar{B}) = \Pr(A) \cdot \Pr(\bar{B})$

$$\Pr(A \cap \bar{B}) + \Pr(A \cap B) = \Pr(A)$$

$$\frac{|A \cap \bar{B}|}{|S|} + \frac{|A \cap \mathrm{B}|}{|S|} = \frac{|A|}{|S|}$$

$$|A \cap \bar{B}| + |A \cap B| = |A|$$



Independent Events

Events $A_1, A_2, \dots, A_n$ $n \geq 2$

pairwise independent events: $\forall i \neq j$,
$\Pr(A_i \cap A_j) = \Pr(A_i) \cdot \Pr(A_j)$

mutually independent:
$\forall k, 2 \leq k \leq n, \forall i_1 < i_2 \dots < i_k$:

$$\Pr(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) \\ = \Pr(A_{i_1}) \cdot \Pr(A_{i_2}) \cdot \dots \cdot \Pr(A_{i_k})$$

We can see that mutually independent → pairwise independent, since pairwise independent is mutually independent for $k = 2$.

If we want to verify pairwise independence of all equations for a set of n events, how many equations must we verify?

$$\binom{n}{2}$$

Since there are n events and we must examine all pairs.

This behaves like n^2.

Independent Events

Events $A_1, A_2, \ldots, A_n$ $n \geq 2$

pairwise independent events: $\forall i \neq j$,
$\Pr(A_i \cap A_j) = \Pr(A_i) \cdot \Pr(A_j)$

mutually independent:
$\forall k, 2 \leq k \leq n, \forall i_1 < i_2 \ldots < i_k$:

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \Pr(A_{i_1}) \cdot \Pr(A_{i_2}) \cdot \ldots \cdot \Pr(A_{i_k})$$

We can see that mutually independent → pairwise independent, since pairwise independent is mutually independent for $k = 2$.

If we want to verify *mutual independence* of all equations for a set of n events, how many equations must we verify?

The claim is:

$$2^n - 1 - n$$

We need to examine all sets of k, $2 \leq k \leq n$.

This is all subsets 2^n

Subtract all subsets of size 1, there are n

Subtract the empty set, there is 1.

Independent Events

Events $A_1, A_2, \dots, A_n$ $n \geq 2$

pairwise independent events: $\forall i \neq j$,

$$\Pr(A_i \cap A_j) = \Pr(A_i) \cdot \Pr(A_j) \qquad \boxed{\binom{n}{2}}$$

mutually independent:

$\forall k, 2 \leq k \leq n, \forall i_1 < i_2 \dots < i_k$:

$$\Pr(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) \qquad \boxed{2^n - 1 - n}$$

$$= \Pr(A_{i_1}) \cdot \Pr(A_{i_2}) \cdot \dots \cdot \Pr(A_{i_k})$$

mutually independent → pairwise independent

Events A, B, C

To verify pairwise independent:

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$$
$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$$

To verify mutually independent:

All equations of size 2, plus all equations of size 3. So the equations above, plus:

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$$

$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

We will define the events:

$$A = \text{1st flip equals 2nd flip}$$
$$B = \text{2nd flip equals 3rd flip}$$
$$C = \text{1st flip} = \text{3rd flip}$$

We will write this as:

$$A = "f_1 = f_2"$$
$$B = "f_2 = f_3"$$
$$C = "f_1 = f_3"$$

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$

$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$$A = "f_1 = f_2" \\ B = "f_2 = f_3" \\ C = "f_1 = f_3"$$

$$A = \{HHH, HHT, TTH, TTT\} \\ B = \{HHH, THH, HTT, TTT\} \\ C = \{HHH, HTH, THT, TTT\}$$

$$|A| = |B| = |C| = 4$$

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{4}{|S|} = \frac{4}{8} = \frac{1}{2}$$

We check pairwise independence by verifying 3 equations.

Independent Events

Events A, B, C

pairwise: $\boxed{\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)}$

$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$A = "f_1 = f_2" = \{HHH, HHT, TTH, TTT\}$
$B = "f_2 = f_3" = \{HHH, THH, HTT, TTT\}$
$C = "f_1 = f_3" = \{HHH, HTH, THT, TTT\}$

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$$

Verify 1: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$$\Pr(A) \cdot \Pr(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$A \cap B = \{HHH, TTT\}$$

$$\Pr(A \cap B) = \frac{|A \cap \mathrm{B}|}{|S|} = \frac{2}{8} = \frac{1}{4},$$

$\therefore$ first equation is verified

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$$\boxed{\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)}$$

$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$A = "f_1 = f_2" = \{HHH, HHT, TTH, TTT\}$
$B = "f_2 = f_3" = \{HHH, THH, HTT, TTT\}$
$C = "f_1 = f_3" = \{HHH, HTH, THT, TTT\}$

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$$

Verify 2: $\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$

$$\Pr(A) \cdot \Pr(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$A \cap C = \{HHH, TTT\}$$

$$\Pr(A \cap C) = \frac{|A \cap C|}{|S|} = \frac{2}{8} = \frac{1}{4},$$

$\therefore$ second equation is verified

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$

$\boxed{\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)}$

mutually: above, plus

$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$A = "f_1 = f_2" = \{HHH, HHT, TTH, TTT\}$

$B = "f_2 = f_3" = \{HHH, THH, HTT, TTT\}$

$C = "f_1 = f_3" = \{HHH, HTH, THT, TTT\}$

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$$

Verify 3: $\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$

$$\Pr(B) \cdot \Pr(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$B \cap C = \{HHH, TTT\}$$

$$\Pr(B \cap C) = \frac{|B \cap C|}{|S|} = \frac{2}{8} = \frac{1}{4},$$

$\therefore$ third equation is verified

Independent Events

Events A, B, C

pairwise: $\boxed{\begin{aligned} \Pr(A \cap B) &= \Pr(A) \cdot \Pr(B) \\ \Pr(A \cap C) &= \Pr(A) \cdot \Pr(C) \\ \Pr(B \cap C) &= \Pr(B) \cdot \Pr(C) \end{aligned}}$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$A = "f_1 = f_2" = \{HHH, HHT, TTH, TTT\}$
$B = "f_2 = f_3" = \{HHH, THH, HTT, TTT\}$
$C = "f_1 = f_3" = \{HHH, HTH, THT, TTT\}$

Since all three equations hold, events $A, B,$ and C are **pairwise independent**.

To show additionally that they are mutually independent, we must show:

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$

$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$

mutually: above, plus

$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$A = "f_1 = f_2" = \{HHH, HHT, TTH, TTT\}$

$B = "f_2 = f_3" = \{HHH, THH, HTT, TTT\}$

$C = "f_1 = f_3" = \{HHH, HTH, THT, TTT\}$

To show:

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

$$\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$$

$$\Pr(A) \cdot \Pr(B) \cdot \Pr(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$A \cap B \cap C = \{HHH, TTT\}$$

$$\Pr(A \cap B \cap C) = \frac{|A \cap B \cap C|}{|S|} = \frac{2}{8} = \frac{1}{4}$$

$\therefore$ Not mutually independent

Independent Events

Events A, B, C

pairwise: $\Pr(A \cap B) = \Pr(A) \cdot \Pr(B)$

$$\Pr(A \cap C) = \Pr(A) \cdot \Pr(C)$$
$$\Pr(B \cap C) = \Pr(B) \cdot \Pr(C)$$

mutually: above, plus

$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Example: Flip a fair coin 3 times.

The sample space is:

$$S = \{HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT\}$$

$|S| = 8$, uniform probability

$$A = "f_1 = f_2"$$
$$B = "f_2 = f_3"$$
$$C = "f_1 = f_3"$$

Not mutually independent:

Assume and A and B happen. If "$f_1 = f_2$" and "$f_1 = f_3$" then it must be that "$f_1 = 3$".

So if we know A and B happen, then $\Pr(C) = 1$.

$$\Pr(C | A \cap B) = 1$$
$$\Pr(C) = \frac{1}{2}$$

Independent Events

Exercise 5.81:

Three students are writing an exam.

Annie: passes the exam with probability 0.9
Boris: passes the exam with probability 0.9
Charlie: passes the exam with probability 0.6

Assume that they are in a team setting, so as long as 2 of them pass, then all 3 pass. But if 2 or more fail, then all 3 fail.

How can they optimize their chances?

We will look at 2 different scenarios.

Independent Events

A: Annie: passes with probability 0.9
B: Boris: passes with probability 0.9
C: Charlie: passes with probability 0.6

Scenario 1: no cheating

If no one is cheating, then the results of the exams are mutually independent, i.e., none of the outcomes of A, B or C has any influence on the probability of the others, either individually or in combination.

What is the $\Pr(\geq 2 \text{ students pass})$?

$\geq 2 \text{ pass} \leftrightarrow A \cap B \cap C$ or $A \cap B \cap \overline{C}$
or $A \cap \overline{B} \cap C$ or $\overline{A} \cap B \cap C$

Since these sets are pairwise disjoint, and "or" corresponds to union → apply the sum rule:

$$\Pr(\geq 2\, pass)$$

$$= \Pr\left(A \cap B \cap C\right) + \Pr\left(A \cap B \cap \overline{C}\right)$$
$$+ \Pr(A \cap \overline{B} \cap C) + \Pr(\overline{A} \cap B \cap C)$$

Mutually independent:
$$\Pr(A \cap B \cap C) = \Pr(A) \cdot \Pr(B) \cdot \Pr(C)$$

Independent Events

A: Annie: passes with probability 0.9
B: Boris: passes with probability 0.9
C: Charlie: passes with probability 0.6

Scenario 1: no cheating

If no one is cheating, then the results of the exams are mutually independent, i.e., none of the outcomes of A, B or C has any influence on the probability of the others, either individually or in combination.

What is the $\Pr(\geq 2 \text{ students pass})$?

$\geq 2 \text{ pass} \leftrightarrow A \cap B \cap C$ or $A \cap B \cap \overline{C}$
or $A \cap \overline{B} \cap C$ or $\overline{A} \cap B \cap C$

Since these sets are pairwise disjoint, and "or" corresponds to union → apply the sum rule:

$$\Pr(\geq 2\, pass)$$

$$\begin{gathered} = 0.9 \cdot 0.9 \cdot 0.6 \\ +0.9 \cdot 0.9 \cdot 0.4 \\ +0.9 \cdot 0.1 \cdot 0.6 \\ +0.1 \cdot 0.9 \cdot 0.6 \\ = 0.918 \end{gathered}$$

Independent Events

A: Annie: passes with probability 0.9
B: Boris: passes with probability 0.9
C: Charlie: passes with probability 0.6

Scenario 2: Charlie copies from Annie

What is the $\Pr(\geq 2 \text{ students pass})$?

Before we had:

$\geq 2 \text{ pass} \leftrightarrow ABC$ or $AB\overline{C}$ or $A\overline{B}C$ or $\overline{A}BC$

Now we cannot have $AB\overline{C}$ or $\overline{A}BC$, since Annie and Charlie get the same mark.

That means now we have:

$\geq 2 \text{ pass} \leftrightarrow ABC$ or $A\overline{B}C$

$\geq 2 \text{ pass} \leftrightarrow AB$ or $A\overline{B}$

$\geq 2 \text{ pass} \leftrightarrow A$

$$\Pr(\geq 2\, pass) = \Pr(A) = 0.9$$

Whereas with no cheating:

$$\Pr(\geq 2\, pass) = 0.918$$

If students cheat, the average goes down