BINOMIAL COEFFICIENTS

DISCRETE STRUCTURES II

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BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

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Binomial Coefficient

For a set $S, |S| = n$, the number of permutations is $n!$.

For a set $S, |S| = n$, the number of subsets of size $k, 0 \leq k \leq n$ is given by the notation:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

"n choose k" – binomial coefficient

Since these are sets, order does not matter

We will start by looking at how to choose all **sequences** of size k from a set of size n.

Binomial Coefficient

Example: $S = \{a, b, c\}$

All **sequences** of size 2 $= (a, b), (b, a), (a, c), (c, a), (b, c), (c, b)$. There are 6 such **sequences.**

All **subsets** of size 2 $= \{a, b\}, \{a, c\}, \{b, c\}$. There are 3 such **subsets**

Another way to find all **sequences.** Find all permutations of all subsets.

For each **subset** of size **2**, there are **2!=2** ways to arrange them into sequences.

Informally we can see that **#sequences = #subsets** $*$ **2!.**

Binomial Coefficient

Example: $S = \{a_1, a_2, \dots, a_n\}$. Find all sequences using the Product Rule.

Task 1: Choose a subset S_k of size k from a set $S, |S| = n$.

Task 2: Choose a permutation of S_k.

All **subsets** of size $k = \binom{n}{k}$.

For each **subset** of size k, there are $k!$ ways to arrange them into sequences.

Thus all **sequences = #subsets** $*$ **k!**

$= \binom{n}{k} k!$

Binomial Coefficient

Example: $S = \{a_1, a_2, \dots, a_n\}$

Let the **number of sequences of size $\boldsymbol{k}$** $= m$

Then the **number of subsets of size $\boldsymbol{k}$** $= m/k!$

So if we can count the number of **sequences** of size k, we can find the number of **subsets** of size k.

We will find the number of sequences in a different way.

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Procedure: write the k elements from left to right

Task 1: Choose one of n elements for position 1
Task 2: Choose one of remaining $n - 1$ elements for position 2
Task 3: Choose one of remaining $n - 2$ elements for position 3
…
Task $k - 1$: Choose one of remaining $n - (k - 2) = n - k + 2$ elements for position $k - 1$
Task k : Choose one of remaining $n - (k - 1) = n - k + 1$ elements for position k

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, a_2, a_3, a_4, a_5\}$ into a sequence.

Task 1: (, , , ,)

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, a_2, a_3, \quad , a_5\}$ into a sequence.

Task 1: $(a_4, , , ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \ , a_3, \ , a_5\}$ into a sequence.

Task 2: $(a_4, a_2, , ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{a_1, \ , \ , \ , a_5\}$ into a sequence.

Task 3: $(a_4, a_2, a_3, ,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{, \ , \ , \ , a_5\}$ into a sequence.

Task 4: $(a_4, a_2, a_3, a_1,)$

Binomial Coefficient

Use the **Product Rule** to write a **sequence** of size k elements from a set $S, |S| = n$.

Example: $S = \{,\ ,\ ,\ ,\ \}$ into a sequence.

Task 5: $(a_4, a_2, a_3, a_1, a_5)$

Binomial Coefficient

Product rule: Number of ways to write a **sequence** of size k elements from a set $S, |S| = n$, is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+2) \cdot (n-k+1)$$

which is "sort of" a factorial:

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \cdot (n-k) \cdot \ldots \cdot 2 \cdot 1}{(n-k) \cdot \ldots \cdot 2 \cdot 1}$$

$$= \frac{n!}{(n-k)!}$$

The number of permutations of a set of size k is $k!$

Thus the number of **sequences** of size k is the number of **subsets** of size k multiplied by $k!$

Binomial Coefficient

Number of ways to write a **sequence** of size k elements from a set $S, |S| = n$, is

$$\frac{n!}{(n-k)!}$$

Consider the earlier method we used to count all **sequences** of size k from a set $S, |S| = n$.

Task 1: Choose a subset S_k of size k from a set $S, |S| = n$. There are $\binom{n}{k}$ ways to do this task

Task 2: Choose a permutation of S_k. There are $k!$ ways to do this task

Using the Product rule, there are $\binom{n}{k}\, k!$ **sequences** of size k in a set $S, |S| = n$.

Binomial Coefficient

The number of **sequences** of size k in a set $S, |S| = n$ is $\binom{n}{k}\, k!$, where $\binom{n}{k}$ is the number of **subsets** of size k.

The number of **sequences** of size k in a set $S, |S| = n$ is also $\frac{n!}{(n-k)!}$.

Thus $\binom{n}{k}k! = \frac{n!}{(n-k)!}$.

We can now solve for the number of **subsets**:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial Coefficient

All subsets of size k from set S:

List all permutations of each subset. Note there are $k!$ such permutations

$$\binom{n}{k} \begin{cases} a = \{k \text{ elements}\} \\ b = \{k \text{ elements}\} \\ c = \{k \text{ elements}\} \\ \dots \end{cases}$$

$a_1, a_2, \dots, a_{k!}$

$b_1, b_2, \dots, b_{k!}$

$c_1, c_2, \dots, c_{k!}$

…

The total number of permutations listed is exactly the total number of permutations of size k, which is $\frac{n!}{(n-k)!}$.

Thus

$$\frac{n!}{(n-k)!} = \binom{n}{k} k!$$

or

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial Coefficient

The number of ways to choose a subset of size k from a set $S, |S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Given a deck of 52 cards, how many hands of 5 cards are there?

In this case we do not care about the order (though you might rearrange them, but the order you receive them does not matter).

Binomial Coefficient

The number of ways to choose a subset of size k from a set S, $|S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Given a deck of 52 cards, how many hands of 5 cards are there?

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = \frac{52!}{5!\,47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$$

Binomial Coefficient

The number of ways to choose a subset of size k from a set S, $|S| = n$ is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

How many bitstrings of length n with exactly k many 1’s are there?

1	2	3			n-2	n-1	n
1	0	1	1	0	0	0	1

Choose k positions out of n possible positions
-chosen positions get a 1
-other positions get a 0
= $\binom{n}{k}$ bitstrings of length n with k 1's.

Binomial Coefficient

Choose k positions out of n possible positions
-chosen positions get a 1
-other positions get a 0
= $\binom{n}{k}$ bitstrings of length n with k 1's.

This is not surprising, since we have seen that we can encode subsets as bitstrings and vice versa

1	2	3			n-2	n-1	n
1	0	1	1	0	0	0	1

Binomial Coefficient

$$(x+y)^2$$

$$= x^2 + 2xy + y^2$$

$$(x+y)^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^6 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

Binomial Coefficient

$$(x+y)^6 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

We can determine the coefficients of, for example, x^4y^2. Note that in each of the terms $(x+y)$, we choose either x or y. That is why the polynomials for each term sum to 6.

Binomial Coefficient

$$(x+y)^6 = (\boxed{x}+y)\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Binomial Coefficient

$$(x+y)^6 = (\boxed{x} + y)\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)\cdot(x + \boxed{y})\cdot(\boxed{x} + y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \quad x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Observe for each term there are two choices, x or y.

When choosing a subset there are two choices for each element, include or don’t include

Binomial Coefficient

$$(x+y)^6 = (\boxed{x}+y)\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)$$

$$(x+y)^6 = x^6 + \quad x^5y + \binom{6}{2}x^4y^2 + \quad x^3y^3 + \quad x^2y^4 + \quad xy^5 + \quad y^6$$

For the term x^4y^2 we choose 2 y 's and the rest are x's. What are the number of ways to do that?

Observe for each term there are two choices, x or y.

When choosing a subset there are two choices for each element, include or don’t include

Binomial Coefficient

$$(x+y)^6 = (\boxed{x}+y)\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)\cdot(x+\boxed{y})\cdot(\boxed{x}+y)$$

$$(x+y)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5y + \binom{6}{2}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{4}x^2y^4 + \binom{6}{5}xy^5 + \binom{6}{6}y^6$$

Notice that for the term x^4y^2 we can instead choose 4 x's and the rest are y's. What are the number of ways to do that? It should be the same and it is.

$$(x+y)^6 = \binom{6}{6}x^6 + \binom{6}{5}x^5y + \binom{6}{4}x^4y^2 + \binom{6}{3}x^3y^3 + \binom{6}{2}x^2y^4 + \binom{6}{1}xy^5 + \binom{6}{0}y^6$$

Newton's Binomial Theorem

In general:

$$(x+y)^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-2}x^2y^{n-2}$$

$$+ \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Newton's Binomial Theorem

Example:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(x+y)^{75}$, what is the coefficient of $x^{50}y^{25}$?

$$= \binom{75}{25}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(3x+2y)^{10}$, what is the coefficient of x^3y^7?

Let $x' = 3x$ and let $y' = 2y$. We can rewrite our expression as:

$$(3x+2y)^{10} = (x'+y')^{10}$$

$$(x'+y')^{10} = \sum_{k=0}^{10} \binom{10}{k} x'^{10-k} y'^{k}$$

We want the coefficient of the term corresponding to $k = 7$

The term of this series when $\mathrm{k} = 7$ is:

$$\binom{10}{7} x'^3 y'^7 = \binom{10}{7} (3x)^3 (2y)^7 = \binom{10}{7} (3x)^3 (2y)^7 = \binom{10}{7} 3^3 2^7 x^3 y^7$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

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The term of this series when $\mathrm{k} = 7$ is:

$$\binom{10}{7} x'^3 y'^7 = \binom{10}{7} (3x)^3 (2y)^7 = \binom{10}{7} (3x)^3 (2y)^7 = \boxed{\binom{10}{7} 3^3 2^7} \; x^3 y^7$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$(7x - 13y)^{75}$, what is the coefficient of $x^{50}y^{25}$?

$$(7x - 13y)^{75} = [(7x) + (-13y)]^{75}$$

$$[(7x) + (-13y)]^{75} = \sum_{k=0}^{75} \binom{75}{k} (7x)^{75-k} (-13y)^k$$

We want the coefficient of the term corresponding to $k = 25$

$$\binom{75}{25} (7x)^{50} (-13y)^k = \binom{75}{25} 7^{50} x^{50} (-1)^{25} 13^{25} y^{25}$$

$$= -\binom{75}{25} 7^{50} \cdot 13^{25} x^{50} y^{25}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

If we take Newton's Binomial Theorem and set $x = 1$ and $y = 1$ we get

$$(1+1)^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 1^k = \binom{n}{0} 1^n \cdot 1^0 + \binom{n}{1} 1^{n-1} \cdot 1^1 + \binom{n}{2} 1^{n-2} \cdot 1^2 + \cdots + \binom{n}{n} 1^0 \cdot 1^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

If we take Newton's Binomial Theorem and set $x = 1$ and $y = 1$ then we get the expression above. Thus

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$$

Is there another way to argue this?

Newton's Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$$

We can try induction, but let's use counting. Can we "map" it to another problem?

The number of subsets of a set of size $n = 2^n$. So let us argue that the left hand side of the above expression also represents the number of subsets of a set of size n.

How many subsets of size 0? $\binom{n}{0}$
How many subsets of size 1? $\binom{n}{1}$
How many subsets of size 2? $\binom{n}{2}$
…
How many subsets of size n? $\binom{n}{n}$

Combinatorial Proof: Show that we know what one side counts, and show the other side counts the same thing

Example

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$0 \le k \le n, \binom{n}{k} = \binom{n}{n-k}$$

We can use the formula of course.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,\big(n-(n-k)\big)!} = \frac{n!}{(n-k)!\,k!}$$

Example

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$0 \le k \le n, \binom{n}{k} = \binom{n}{n-k}$$

Or we can use a combinatorial proof:

$$\binom{5}{3} = \binom{5}{2}$$

If we choose 3 elements, we are leaving 2 elements behind. So when we choose 2 elements we are also choosing 3 elements.

Likewise if we choose 2 elements, we are leaving 3 elements behind.

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$n+1$ elements

We can use the formula again. Can we use a combinatorial proof? What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \leq k \leq n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- **1 special element**

n non-special elements

We can use the formula again. Can we use a combinatorial proof?
What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

Let us take one element, the special element.
How many subsets of size k include the special element?

How many subsets of size k do not include the special element?

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

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We can use the formula again. Can we use a combinatorial proof?
What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

Let us take one element, the special element.
How many subsets of size k include the special element?

Of the remaining n elements, take all subsets of size $k-1$

How many subsets of size k do not include the special element?

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- **1 special element**

n non-special elements

We can use the formula again. Can we use a combinatorial proof?
What are we counting?

In $\binom{n+1}{k}$ we have a set of size $n+1$ and we are counting the number of subsets of size k.

Let us take one element, the special element.
How many subsets of size k include the special element?

Of the remaining n elements, take all subsets of size $k-1$

How many subsets of size k do not include the special element?

Of the remaining n elements, take all subsets of size k

Example: Pascal's Triangle

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$1 \le k \le n, \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- **1 special element**

n non-special elements

We are counting two sets of subsets. In one set all the subsets contain the special element.

In the other set, all the subsets do not contain the special element.

Are they disjoint?

Combinatorial proofs are nice because you can understand why something is true.