COMP 3804 — Assignment 1

Due: Thursday February 2, 23:59.

Assignment Policy:

- Your assignment must be submitted as one single PDF file through Brightspace.

> Use the following format to name your file:
>
> `LastName_StudentId_a1.pdf`

- **Late assignments will not be accepted. I will not reply to emails of the type "my internet connection broke down at 23:57" or "my scanner stopped working at 23:58", or "my dog ate my laptop charger".**
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you must do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

> Some useful facts:
>
> 1. for any real number $x > 0$, $x = 2^{\log x}$.
> 2. For any real number $x \neq 1$ and any integer $k \geq 1$,
>
> $$1 + x + x^2 + \cdots + x^{k-1} = \frac{x^k - 1}{x - 1}.$$
>
> 3. For any real number $0 < \alpha < 1$,
>
> $$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1 - \alpha}.$$

Question 1: Write your name and student number.

Solution: Ryan Lo (101117765)

Question 2: Consider the following recurrence, where n is a power of 6:

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ n^2 + 11 \cdot T(n/6) & \text{if } n \geq 6. \end{cases}$$

- Solve this recurrence using the *unfolding method*. Give the final answer using Big-O notation.

  $T(n) = n^2 + 11 * T(n/6)$
  $T(n) = n^2 + 11((n/6)^2 + 11 * T(n/6^2))$
  $T(n) = n^2 + 11(n/6)^2 + 11^2 * T(n/6^2)$
  $T(n) = (1 + 11/6^2)n^2 + 11^2 * T(n/6^2)$
  $T(n) = (1 + 11/6^2)n^2 + 11^2 * ((n/6^2)^2 + 11 * T(n/6^3))$
  $T(n) = (1 + 11/6^2 + 11^2/6^4)n^2 + 11^3 * T(n/6^3)$
  $T(n) = (1 + 11/6^2 + 11^2/6^4)n^2 + 11^3 * ((n/6^3)^2 + 11 * T(n/6^4))$
  $T(n) = (1 + 11/6^2 + 11^2/6^4 + 11^3/6^6)n^2 + 11^4 * T(n/6^4)$
  $. \; . \; .$
  $T(n) = (1 + 11/6^2 + 11^2/6^4 + ... + 11^{k-1}/6^{2(k-1)})n^2 + 11^k * T(n/6^k)$
  $T(n/6^k) = T(1)$
  $T(n) \leq \sum_{i=0}^{k-1} 11^i/6^{2i}n^2 + 6^k * T(1)$
  $= \frac{(11/6^2)^k - 1}{(11/6^2) - 1} 1/1 - (11/6^2)n^2 + 6^k$
  $= \frac{36}{25}n^2 + n$
  $= O(n^2)$

- Solve this recurrence using the *Master Theorem*.

  $a = 11, b = 6, d = 2$
  if $d > log_b a : T(n) = O(n^d)$
  $2 > log_6 11$
  $2 > 1.3383$
  $T(n) = O(n^2)$

Question 3: Consider the following recurrence:

$$T(n) = n + T(n/5) + T(7n/10).$$

In class, we have seen that $T(n) = O(n)$. In this question, you will prove this using the *recursion tree method.*

Recall from class: The root represents the recursion tree on an input of size n. Consider a node u in the recursion tree that represents a recursive call on an input of size m. Then we write the value m at this node u, we give u a left subtree which is a recursion tree for an input of size $m/5$, and we give u a right subtree which is a recursion tree for an input of size $7m/10$. In this way, $T(n)$ is the sum of the values stored at all nodes in the entire recursion tree.

Below, we assume that the *levels* in the recursion tree are numbered $0, 1, 2, \ldots,$, where the root is at level 0. For each $i \geq 0$, let S_i be the sum of the values of all nodes at level i.

- Determine S_0.
- Determine S_1.
- Determine S_2.
- Use induction to prove the following claim: For every $i \geq 0$,

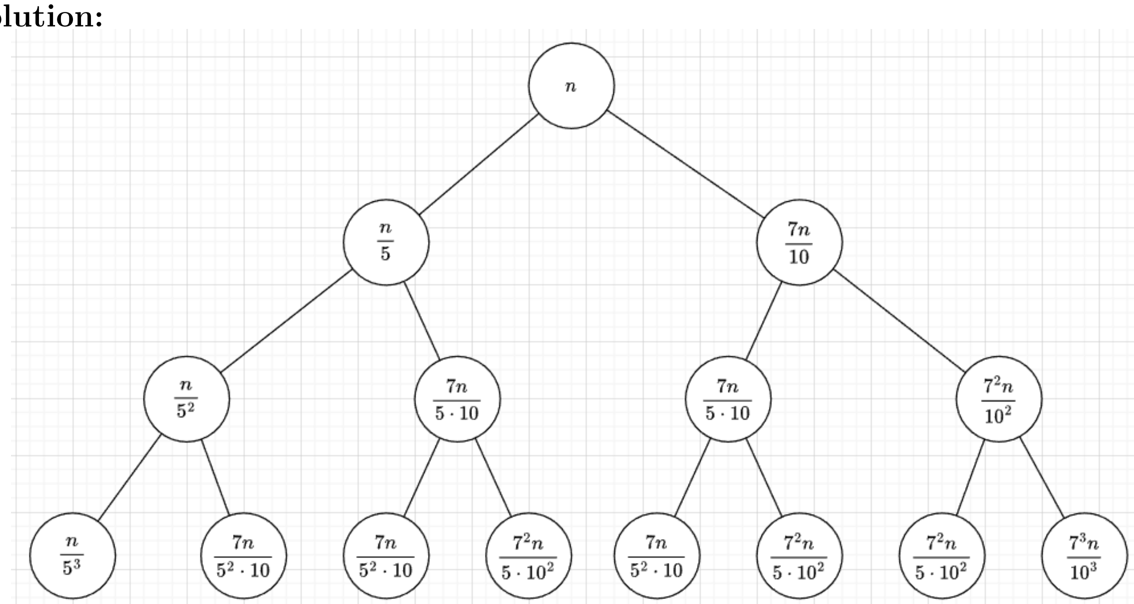
$$S_i \leq (9/10)^i \cdot n.$$

  Hint: Consider level i, let $k = 2^i$, and let the values stored at the nodes at level i be $m_1, m_2, \ldots, m_k$. What are the values stored at the nodes at level $i+1$?

- Complete the proof by showing that $T(n) = O(n)$.

$O(n)$

Solution:



The values stored at the nodes at level i + 1 are:
$(1/5^i)n$, $(7/5^{i-1} * 10)n$, $(7^{i-1}/5 * 10^{i-1})n$, $(7^i/5 * 10^i)n$
$S_0 = n$
$S_1 = (9/10)n$
$S_2 = (9/10)^2 n$
$S_3 = (9/10)^3 n$
$n/5^k = 1$
$k = log_5 n$
$n/(7/10)^k = 1$
$k_2 = log_{10/7} n$
$T(n) = n + (9/10)n + (9/10)^2 n + ... + (9/10)^k n$
$= n[1 + (9/10) + (9/10)^2 + ... + (9/10)^k]$
$T(n) = O(n)$

Question 4: Zoltan is not only your friendly TA, he is also the owner of the popular budget airline ZoltanJet that offers flights in Canada. As you all know, there are n airports in Canada. We denote these airports, in order from west to east, by $A_1, A_2, \ldots, A_n$.

William, who is the CEO of ZoltanJet, has designed a *flight plan* which is a list of ordered pairs (A_i, A_j) of airports such that there is a direct flight from A_i to A_j. This flight plan has the following two properties:

- (P.1) Every flight is going eastwards[1]. In other words, if (A_i, A_j) is in the flight plan, then $i < j$.
- (P.2) For any two indices i and j with $1 \le i < j \le n$, it is possible to fly from A_i to A_j in at most two *hops*. In other words, either (A_i, A_j) is in the flight plan, or there is an index k such that both (A_i, A_k) and (A_k, A_j) are in the flight plan. Note that, because of (P.1), $i < k < j$.

Observe that ZoltanJet can guarantee (P.1) and (P.2) by offering direct flights between all $\binom{n}{2} = \Theta(n^2)$ pairs (A_i, A_j) of airports, where $1 \le i < j \le n$.

- Prove that ZoltanJet can guarantee (P.1) and (P.2) using a flight plan having only $O(n \log n)$ pairs of airports. You may assume that n is a power of two.

  Hint: Since this is the divide-and-conquer assignment, you probably have to use ...

Solution:

From P.1 we can imagine it as a sorted array of airports. Now we can take P.2 and since it is possible to fly between A_i to A_j in at most two hops, we can randomly partition it into 3 parts. Where the flight starts in the first partition hops into the second partition, then hops towards the third partition. Dividing it into 3 parts gives us $log_3 n$ and the case of having to visit every airports n.

[1] But how do I get home? A customer service representative will tell you "that is your problem".

We get n traversals over every airport and n/3 partitions everytime so $O(nlogn)$

Question 5: Professor Justin Bieber needs a fast algorithm that searches for an arbitrary element x in a sorted array $A[1 \dots n]$ of n numbers. He remembers that there is something called "binary search", which maintains an interval $[\ell, r]$ of indices such that, if x is present in the array, then it is contained in the subarray $A[\ell \dots r]$. In one iteration, the algorithm takes the middle index, say p, in the interval $[\ell, r]$. Then the algorithm either finds x at the position p, or it recurses in the interval $[\ell, p-1]$, or it recurses in the interval $[p+1, r]$. Unfortunately, Professor Bieber does not remember the expression[2] for p in terms of ℓ and r.

Professor Bieber does remember that, instead of choosing p in the middle of the interval $[\ell, r]$, it is often enough to choose p uniformly at random in this interval. Based on this, he obtains the following algorithm: The input consists of the sorted array $A[1 \dots n]$, its size n, and a number x. If x is in the array, then the algorithm returns the index p such that $A[p] = x$. Otherwise, the algorithm returns "not present". We assume that all numbers in A are distinct.

[2] is it $\lfloor (r-\ell)/2 \rfloor$, or $\lceil (r-\ell)/2 \rceil$, or $\lfloor (r-\ell+1)/2 \rfloor$, or $\lceil (r-\ell+1)/2 \rceil$?

Algorithm BIEBERSEARCH(A, n, x):
$\ell = 1$; $r = n$;
while $\ell \leq r$
do p = uniformly random element in $\{\ell, \ell + 1, \ldots, r\}$;
  if $A[p] < x$
  then $\ell = p + 1$
  else if $A[p] > x$
    then $r = p - 1$
    else return p
    endif
  endif
endwhile;
return "not present"

Let T be the running time of this algorithm on an input array of length n. Note that T is a random variable. Prove that the expected value of T is $O(\log n)$.
Hint: Most solutions that you find on the internet are wrong.

Solution:

A is a sorted array of n numbers. x is the element that we are searching for.

We are going to show: $E(T) = O(logn)$

When we run $BieberSearch(A, n, x)$, recursive calls are generated on smaller and smaller arrays.

The expected time value of T considers all the cases where we get x.

After 1 run, 2 runs, 3 runs, ..., n runs.

Using the linearity of expectation we get:

$T(n) = T(1) + T(2) + T(3) + ... + T(n)$

$T(n) = 1 + 1/2 + 1/3 + ... + 1/n - 1$

This is actually a case of the harmonic numbers. The sum of the reciprocals becomes log n.

$T(n) = O(logn)$

Question 6: You are given a sequence S consisting of n numbers; not all of these numbers need to be distinct.

Describe an algorithm, in plain English, that decides, in $O(n)$ time, whether or not this sequence S contains a number that occurs more than $n/4$ times.

You may use any result that was proven in class. Justify the correctness of your algorithm and explain why the running time is $O(n)$.
Hint: The algorithm must be comparison-based; you are not allowed to use hashing, bucket-sort, or radix-sort.

Solution: We can start with finding the median using the selection algorithm that we learned in class. We first find the median(m) then create a 3-way partition based on that median. Where we have the first partition as $| < m|, | = m|, | > m|$. If the median(m) is the element that we are looking for then we are finished looking. Otherwise, we can call this algorithm on both sides of the partition $| < m|, | > m|$. Once we find the element we can count if the equal partition contains more than $n/4$ of that element. If we find the element as the median then it is finished otherwise the element does not exist where it occurs more than $n/4$ times because dividing it into another 3-way partition will not have enough elements for $n/4$. Finding out the median takes $O(n)$ time, running the algorithm again on the right and left sides takes $2 * O(n)$ time. In total it takes $3 * O(n)$ which is just $O(n)$.