THE PROBABILISTIC METHOD II

DISCRETE STRUCTURES II

DARRYL HILL

BASED ON THE TEXTBOOK:

DISCRETE STRUCTURES FOR COMPUTER SCIENCE: COUNTING, RECURSION, AND PROBABILITY

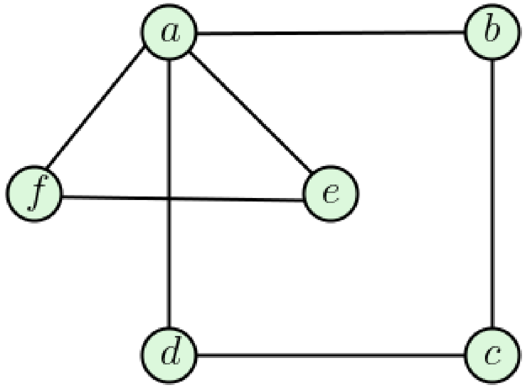
BY MICHIEL SMID

Graph $G = (V, E)$
V is a set of vertices
E is a set of edges (pairs of vertices)

For the graph below,

$$V = \{a, b, c, d, e, f\}$$

$$E = \{\{a, b\}, \{a, d\}, \{a, e\}, \{a, f\}, \\ \{b, c\}, \{c, d\}, \{e, f\}\}$$



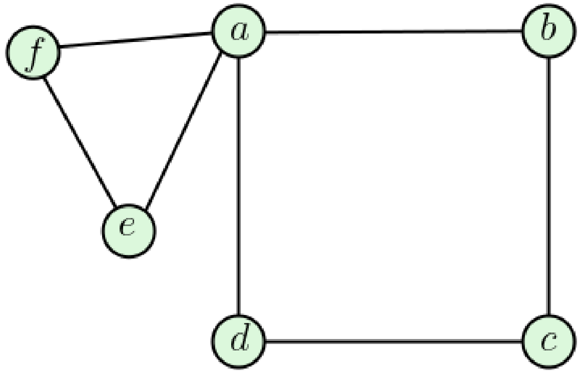
We often draw graphs:

drawing: vertex → point
edge → line segment

Notice in our drawing, $\{e, f\}$ and $\{a, d\}$ cross (and there is no vertex at the intersection).

We could draw this graph another way, and now there are no crossing edges:

This is the same set of vertices and the same set of edges, drawn differently.

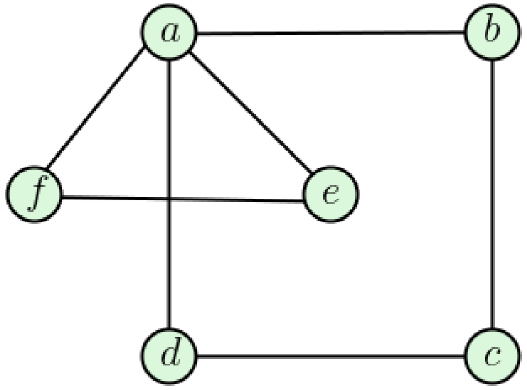


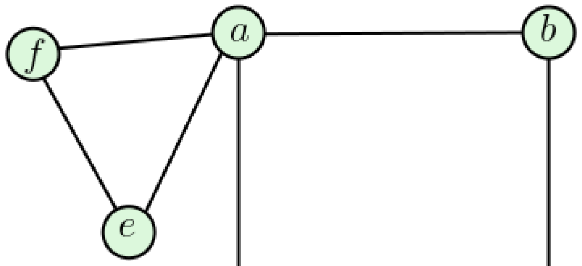
Graph $G = (V, E)$
V is a set of vertices
E is a set of edges (pairs of vertices)

A graph $G = (V, E)$ is called *planar* if $\exists$ drawing without crossings.

Planar graphs are important for all kinds of networks, road, computer, microchips.

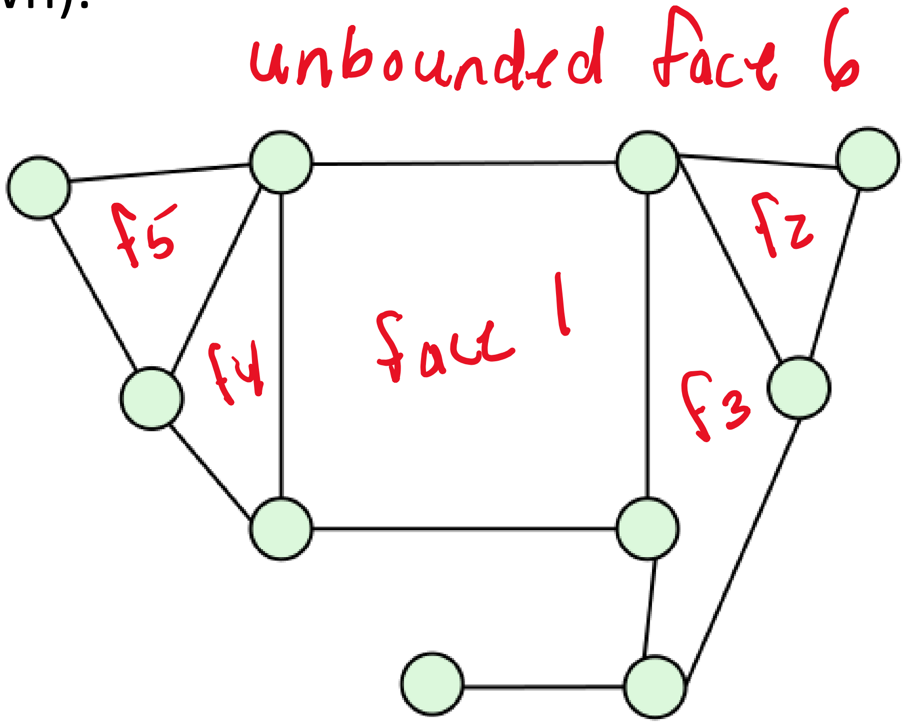
Crossings require extra infrastructure.





Planar graphs that are drawn without crossings have another property called *faces.*

The faces of a graph represent partitions of the plane (the area where the graph is drawn).



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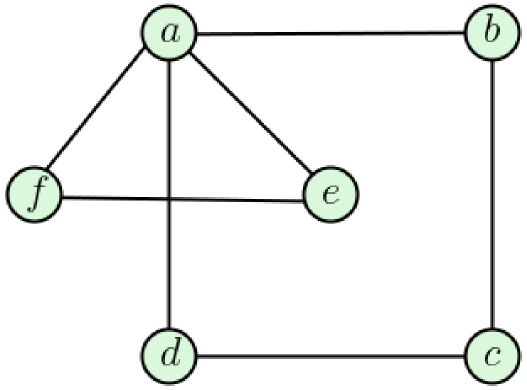
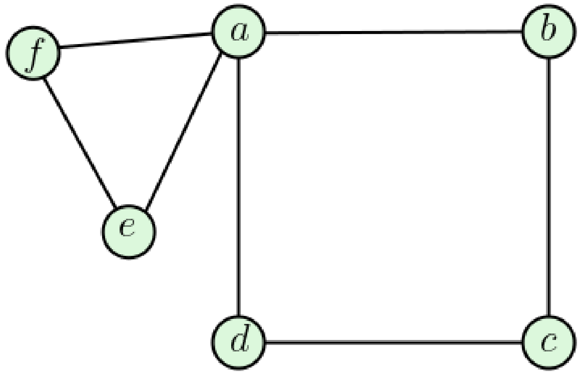




Given the (planar) graph below:

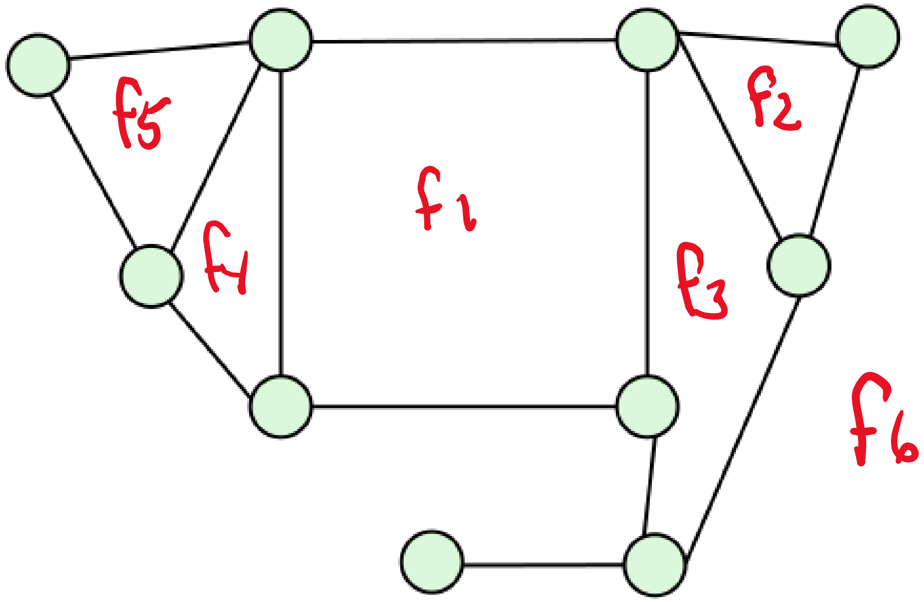
#vertices $= v = 10$

#edges $= e = 14$

#faces $= f = 6$

$$v - e + f$$
$$= 10 - 14 + 6$$
$$= 2$$

True for every connected planar graph



Graph $G = (V, E)$

Euler proved that for connected planar graphs:

$$v - e + f = 2$$

vertices = $v = 10$

edges = $e = 14$

faces = $f = 6$

$v - e + f = 2$

$10 - 14 + 6 = 2$

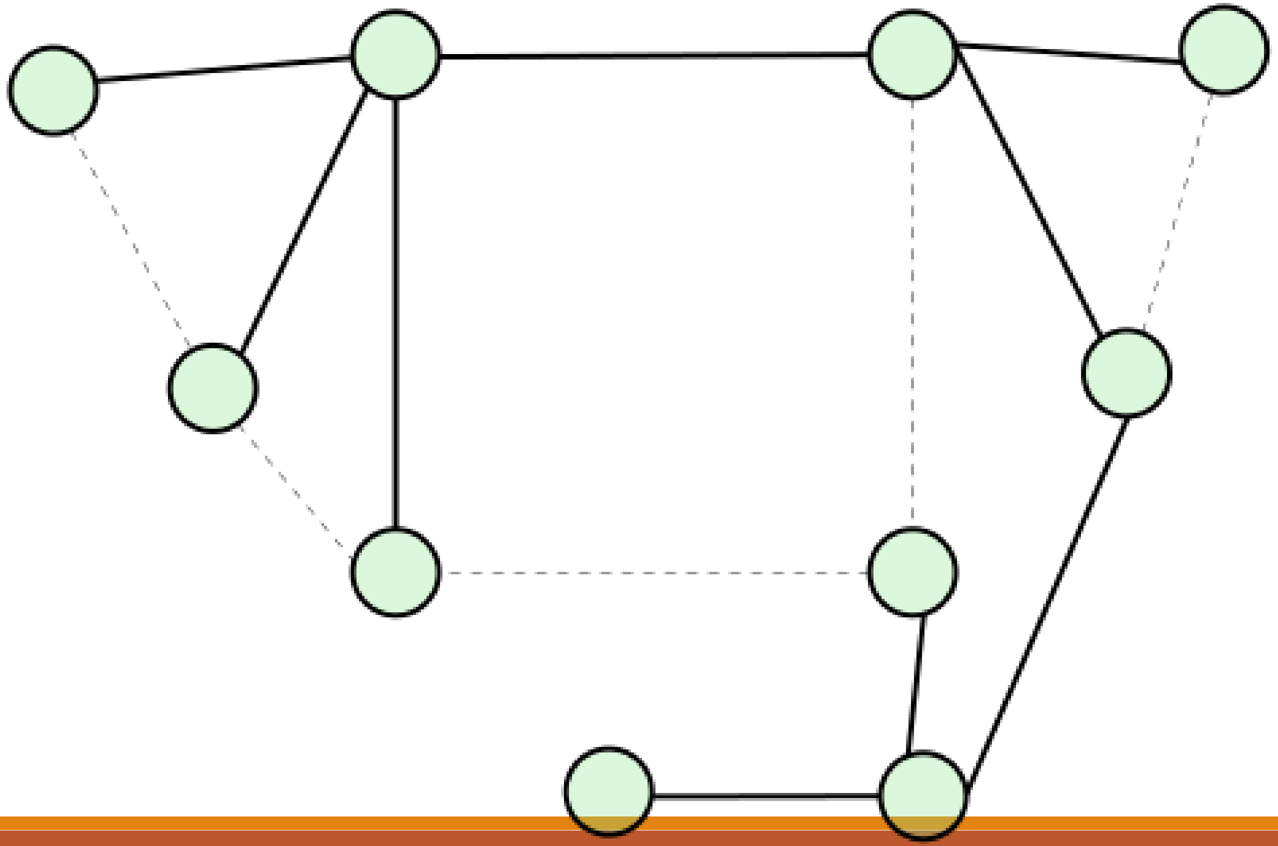
Proof:

while ∃ cycle:

  remove an edge from that cycle

This is simply a spanning tree algorithm, there are others.

We will remove edges from the graph until it is a (spanning) tree.



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

We know that for a tree:

$$e = v - 1$$

Also note that if there are no cycles, then

$$f = 1$$

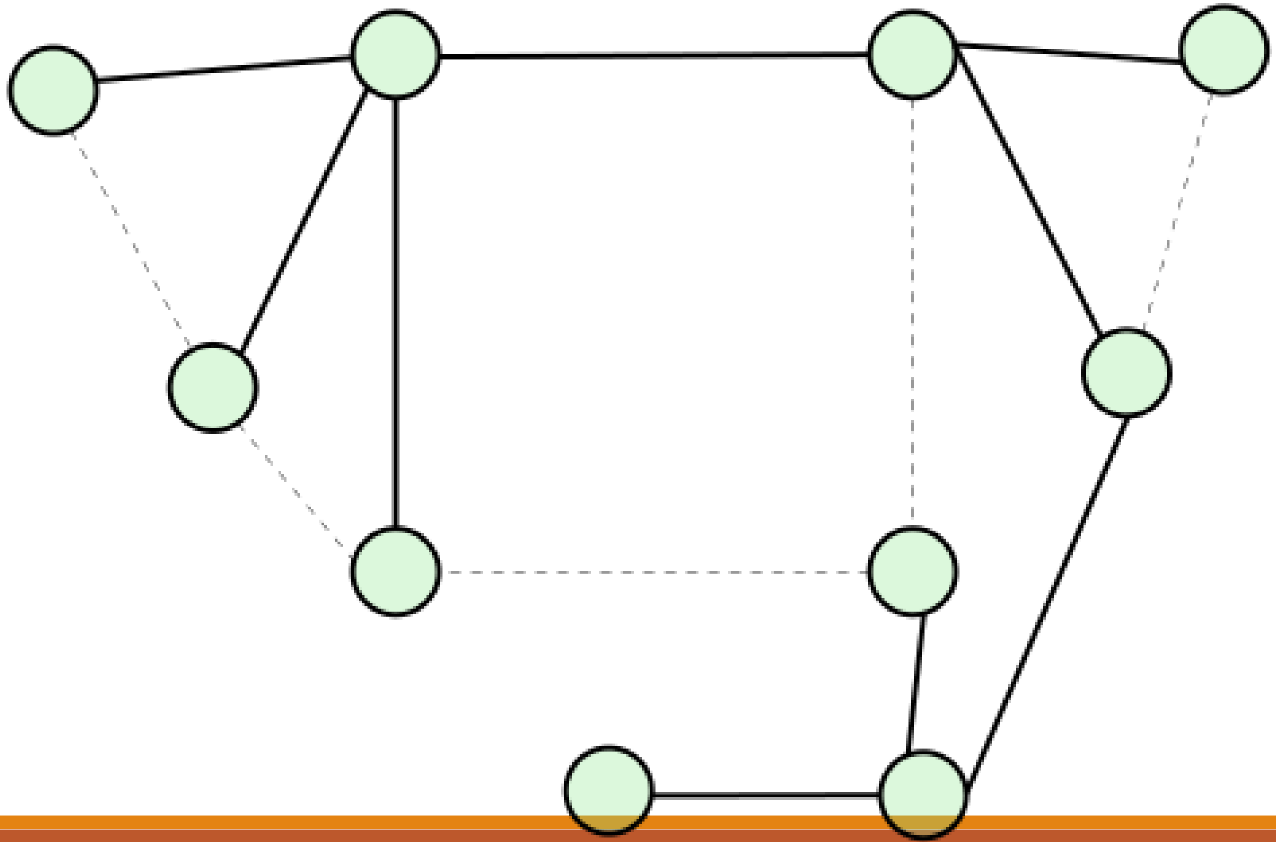
If we plug these values into Euler's formula:

$$v - e + f = v - (v - 1) + 1 = 1 + 1 = 2$$

(This is our base case).

We will remove edges from the graph until it is a (spanning) tree.

We start our proof with a spanning tree of G.



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

We will remove edges from the graph until it is a (spanning) tree.

We start our proof with a spanning tree of G.

In the spanning tree of this example, we have:

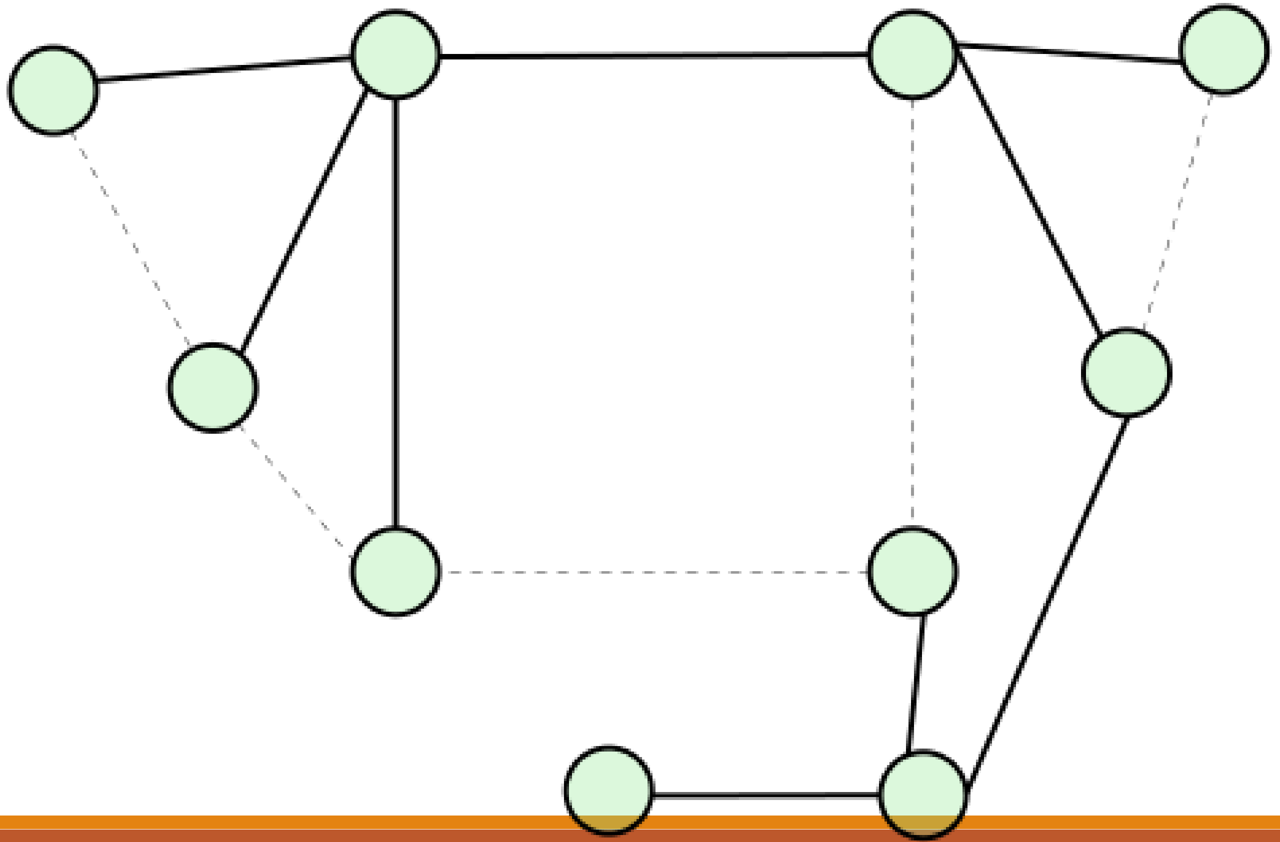
vertices = $v = 10$

edges = $e = 9$

faces = $f = 1$

$v - e + f = 2$

$10 - 9 + 1 = 2$



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

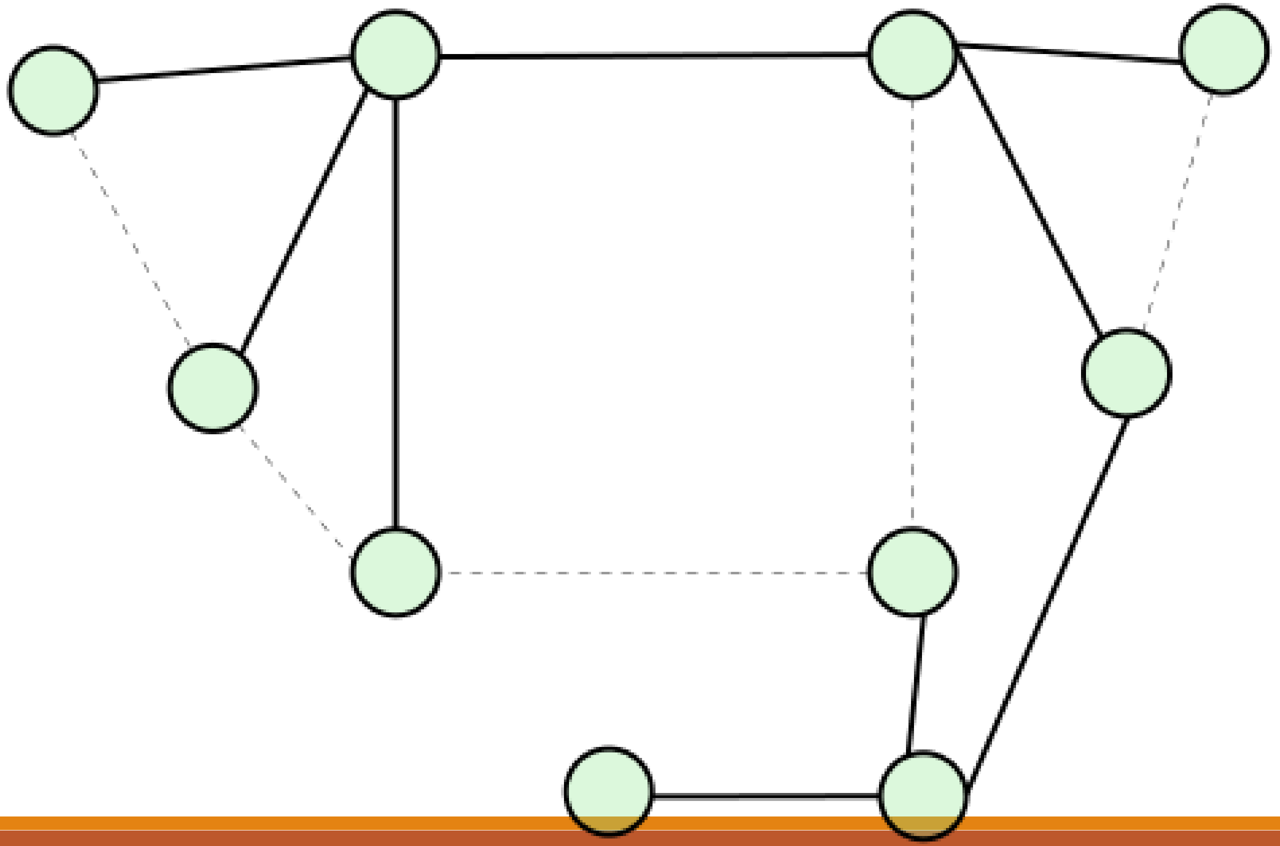
$$v - e + f = 2$$

We start adding the edges back one by one.

Notice that every edge we add creates a new cycle (and thus a new face).

vertices = $v = 10$
edges = $e = 9$
faces = $f = 1$

$v - e + f = 2$
$10 - 9 + 1 = 2$



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

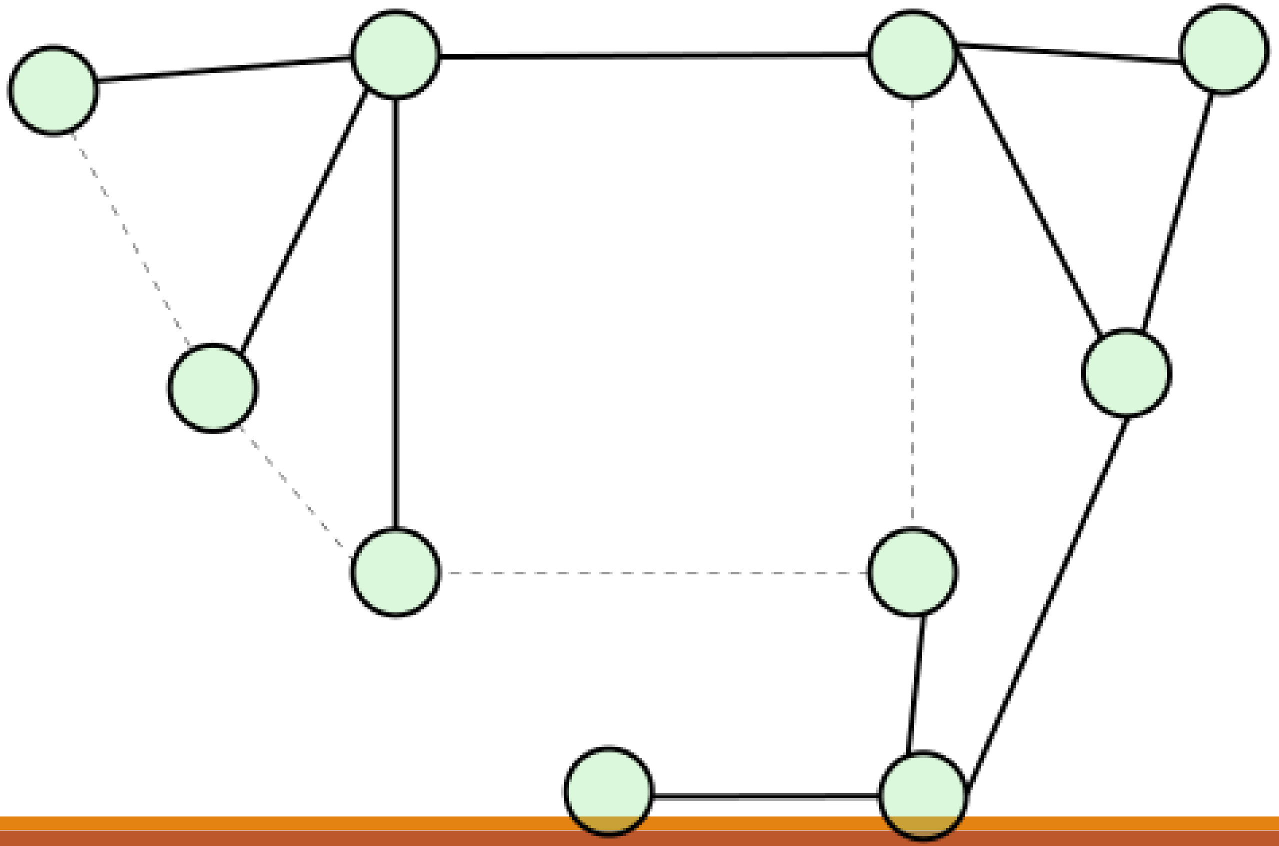
We start adding the edges back one by one.

Notice that every edge we add creates a new cycle (and thus a new face).

vertices $= \; v = 10$
edges $= e = 10$
faces $= f = 2$

$v - e + f = 2$
$10 - 10 + 2 = 2$

That means we add 1 and subtract 1 from the LHS, so it still has the same value (2).



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

We start adding the edges back one by one.

Notice that every edge we add creates a new cycle (and thus a new face).

vertices = $v = 10$

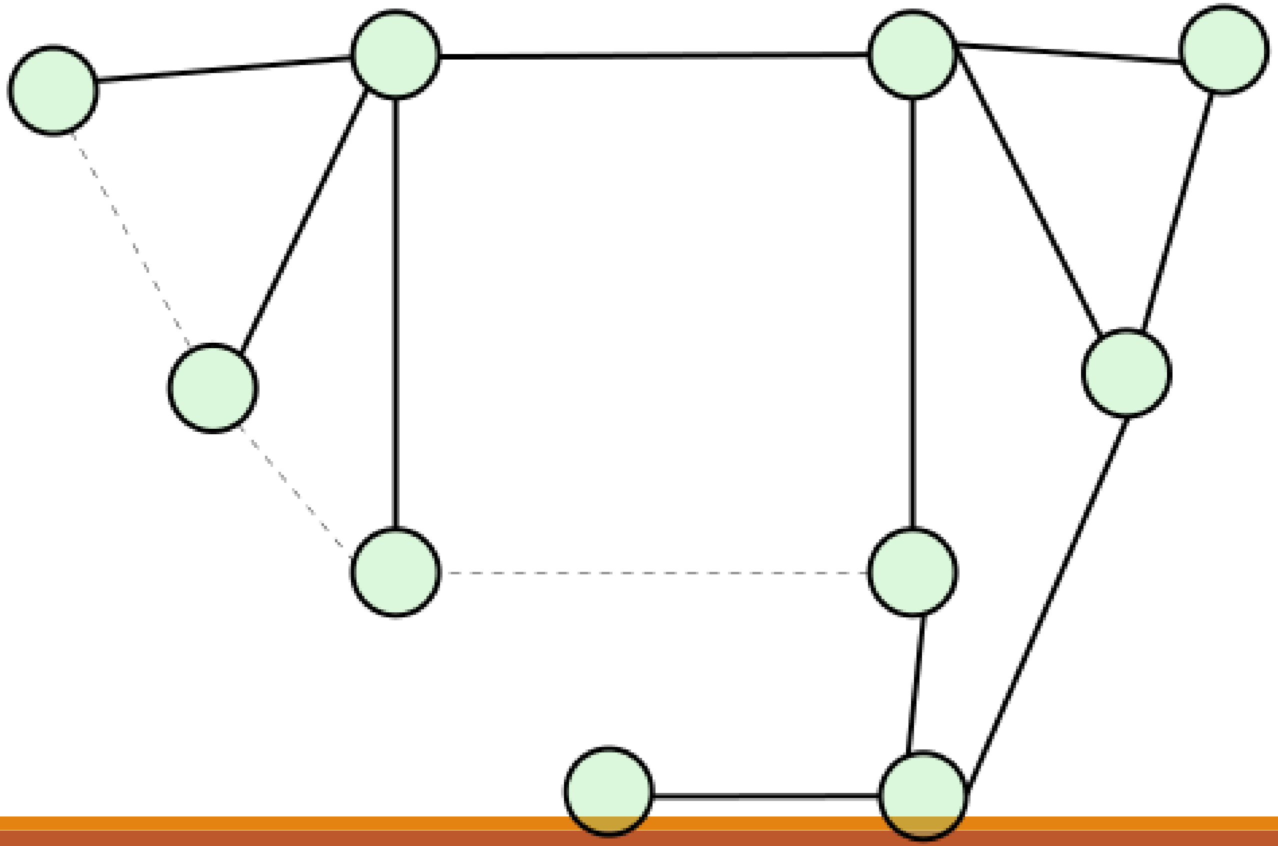
edges = $e = 11$

faces = $f = 3$

$v - e + f = 2$

$10 - 11 + 3 = 2$

That means we add 1 and subtract 1 from the LHS, so it still has the same value (2).



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

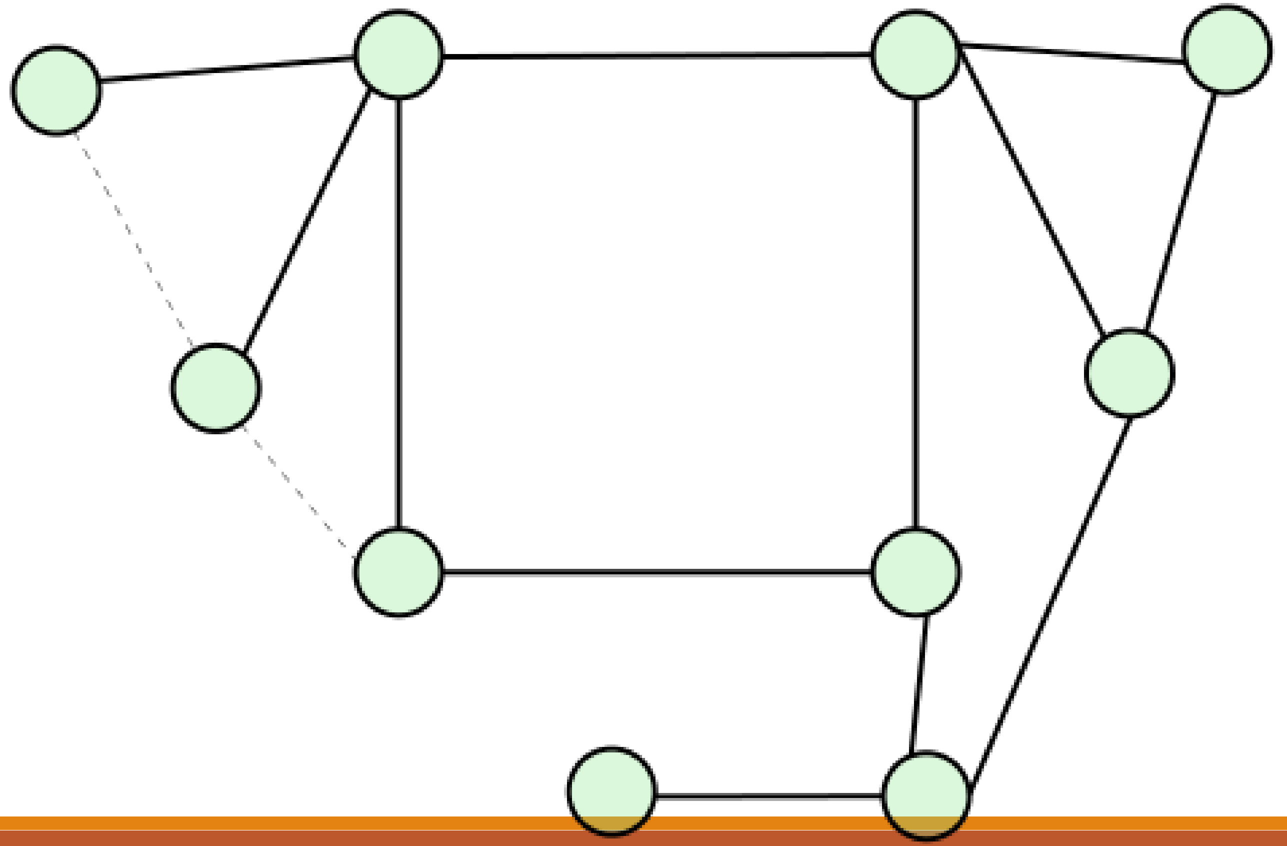
We start adding the edges back one by one.

Notice that every edge we add creates a new cycle (and thus a new face).

vertices = $v = 10$
edges = $e = 12$
faces = $f = 4$

$v - e + f = 2$
$10 - 12 + 4 = 2$

That means we add 1 and subtract 1 from the LHS, so it still has the same value (2).



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

That means if we know two of v, e, f we can determine the 3rd number.

vertices $=\ v = 10$

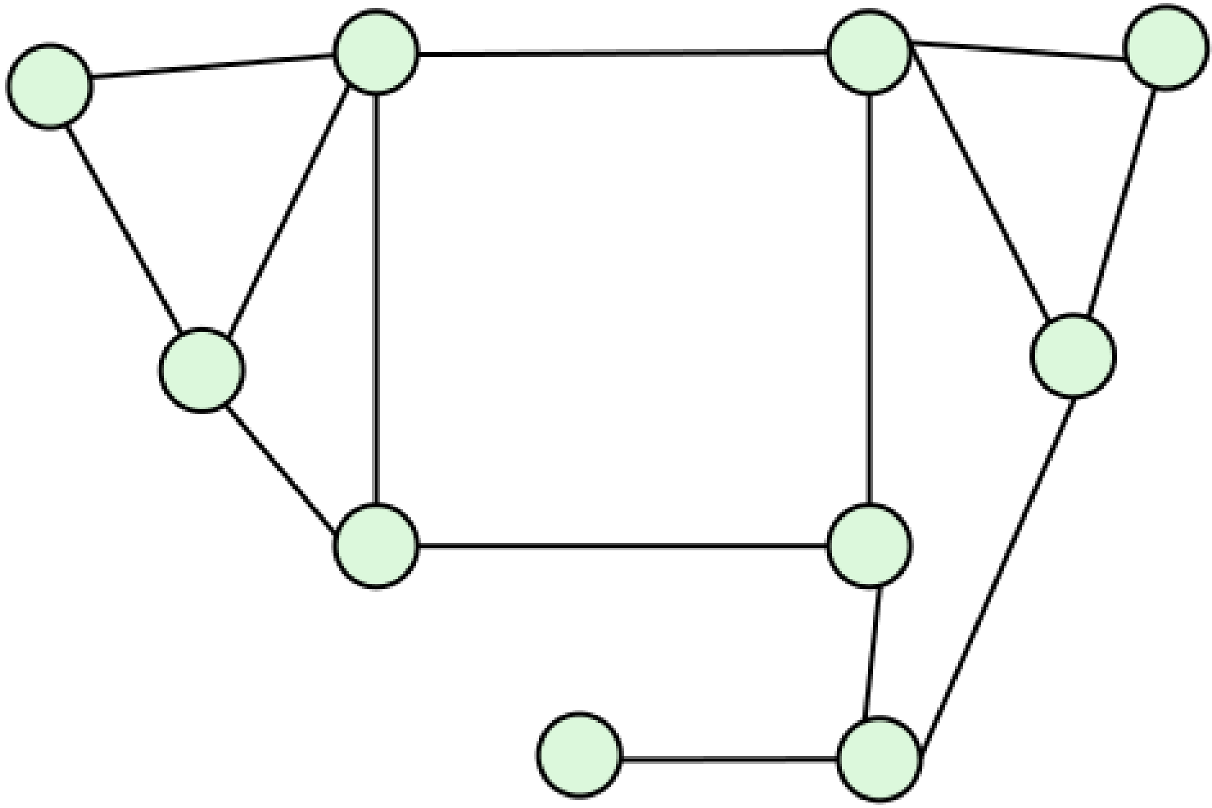
edges $= e = 14$

faces $= f = 6$

$v - e + f = 2$

$10 - 14 + 6 = 2$

Eventually we are back to the original graph, and Euler's formula still holds.



Graph $G = (V, E)$
Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

The maximum number of edges in a simple graph is

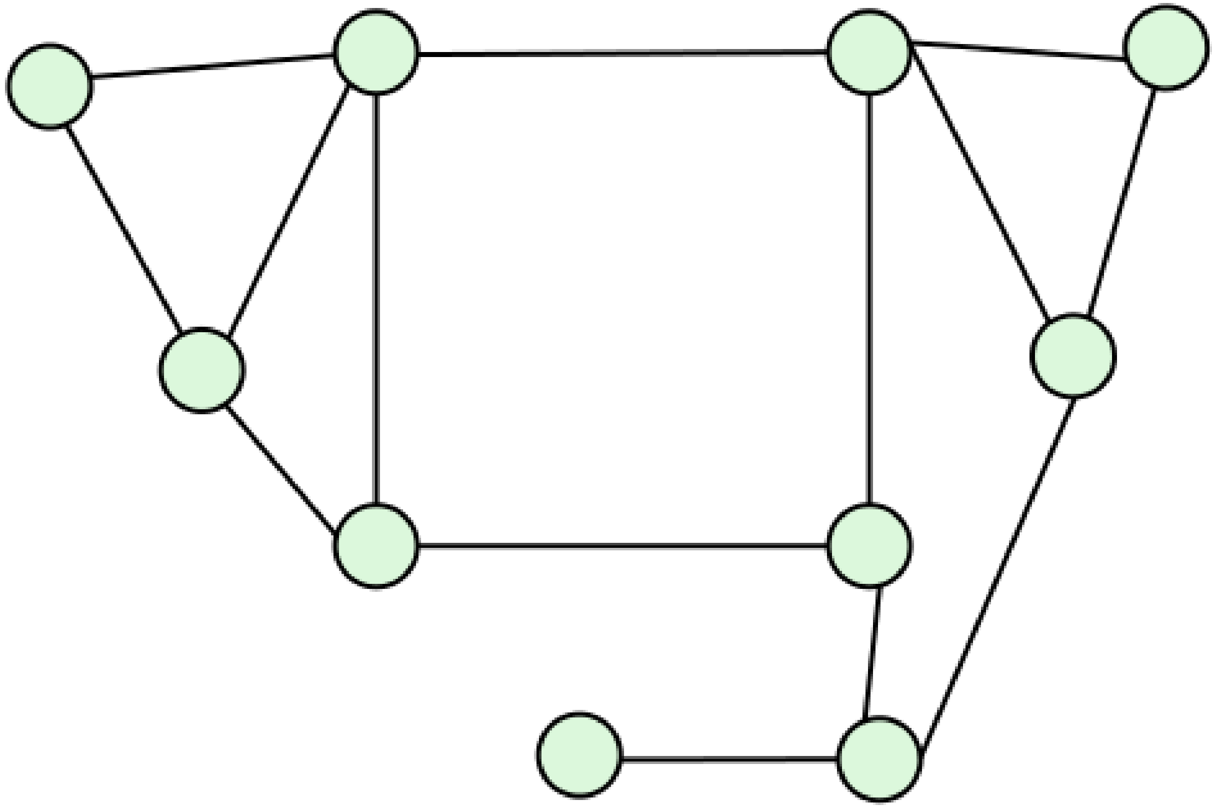
$$e \leq \binom{v}{2} = \frac{v(v-1)}{2} = O(v^2)$$

Since that accounts for all pairs in V.

However, we want to see if we can get a different upper bound on a planar graph using Euler's formula.

Planar connected graph, then we claim:

$$e \leq 3v - 6$$
$$f \leq 2v - 4$$



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

$e \leq 3v - 6$

$f \leq 2v - 4$

Number the faces: $1, 2, \dots, f$

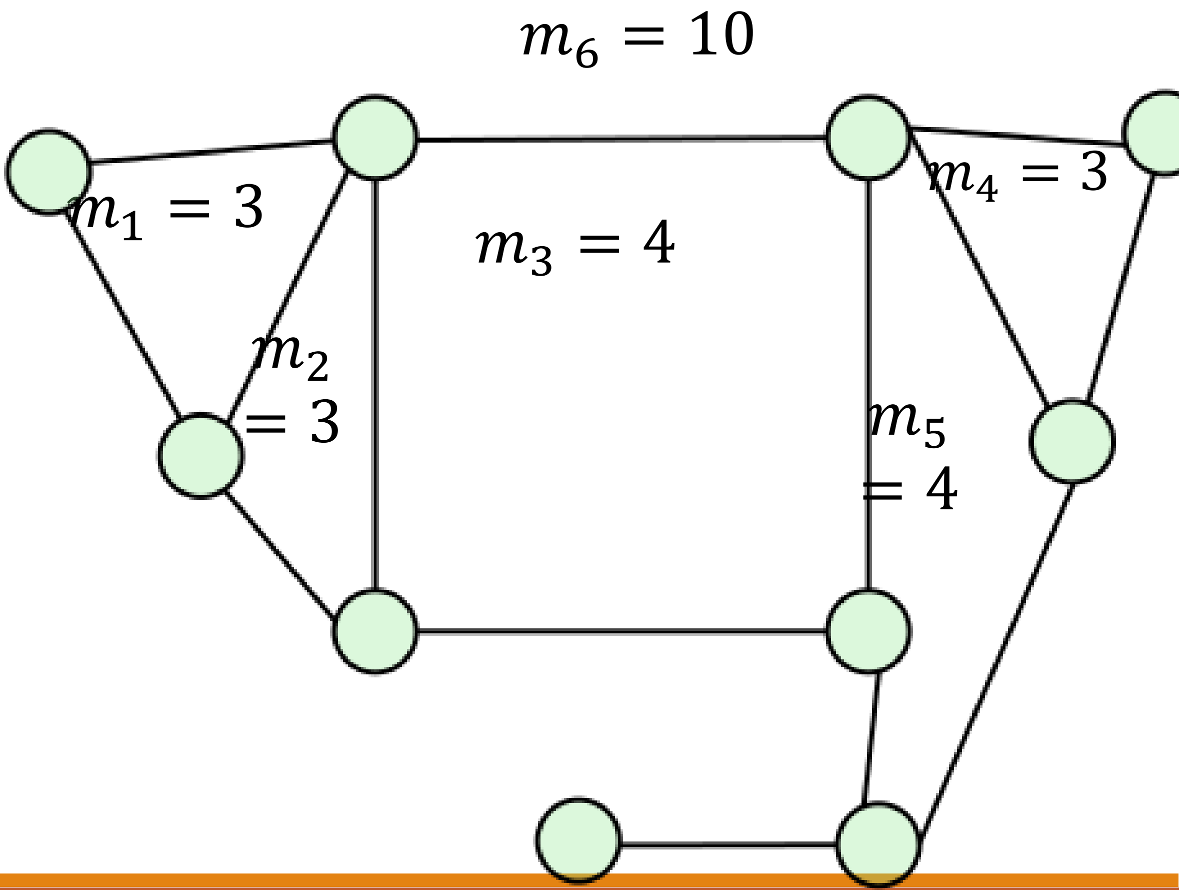
m_i= # of edges on face i

Each edge bounds either 2 different faces, or 1 face, thus:

$$\sum_{i=1}^{f} m_i \leq 2e$$

That give us an upper bound. Lower bound?

$$\sum_{i=1}^{f} m_i \geq 3f$$



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

$$3f \leq \sum_{i=1}^{f} m_i \leq 2e, \qquad 3f \leq 2e$$

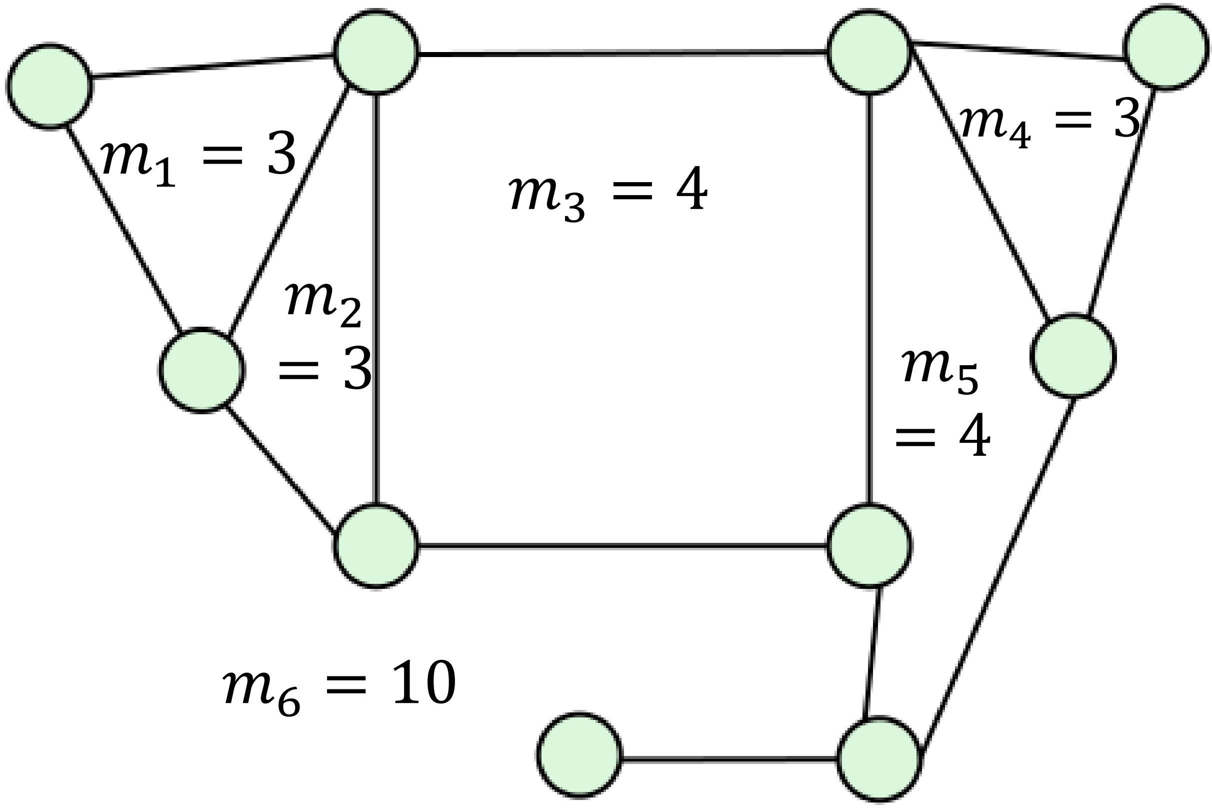
$$e \leq 3v - 6$$
$$f \leq 2v - 4$$

$$f \leq \frac{2e}{3}$$

Number the faces: $1,2, \ldots, f$

m_i= # of edges on face i

$$\sum_{i=1}^{f} m_i \leq 2e \quad \text{and} \quad \sum_{i=1}^{f} m_i \geq 3f$$



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

$e \leq 3v - 6$ ✓

$f \leq 2v - 4$

Number the faces: $1,2, \ldots, f$

m_i= # of edges on face i

$$f \leq \frac{2e}{3}$$

$$v - e + f = 2$$

$$e = v + f - 2$$

$$\leq v + \frac{2e}{3} - 2$$

$$\frac{e}{3} \leq v - 2$$

$$e \leq 3v - 6$$

Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

$e \leq 3v - 6$ ✓

$f \leq 2v - 4$ ✓

Number the faces: $1, 2, \ldots, f$

m_i = # of edges on face i

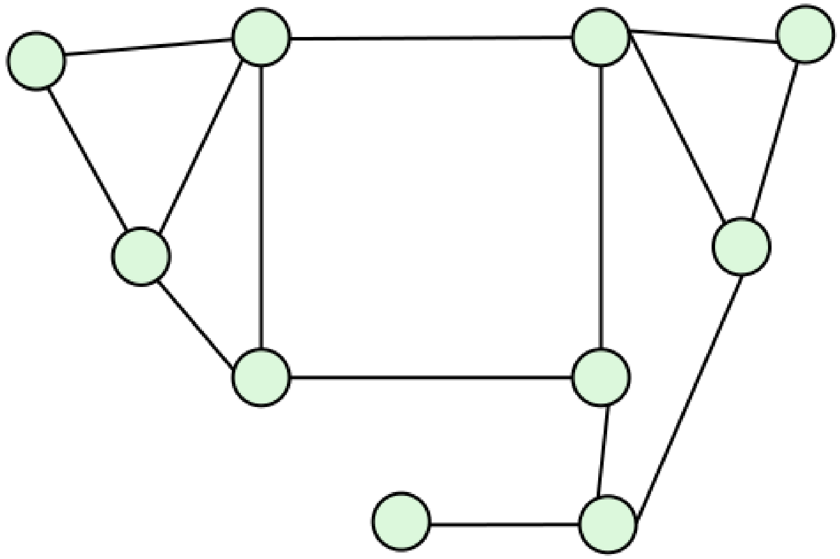
$$f \leq \frac{2e}{3}$$

$$v - e + f = 2$$

$$f \leq \frac{2e}{3}$$

$$\leq \frac{2(3v - 6)}{3}$$

$$= 2v - 4$$



Graph $G = (V, E)$

Euler's theorem for connected planar graphs:

$$v - e + f = 2$$

$e \leq 3v - 6$

$f \leq 2v - 4$

Number the faces: $1,2, \dots, f$

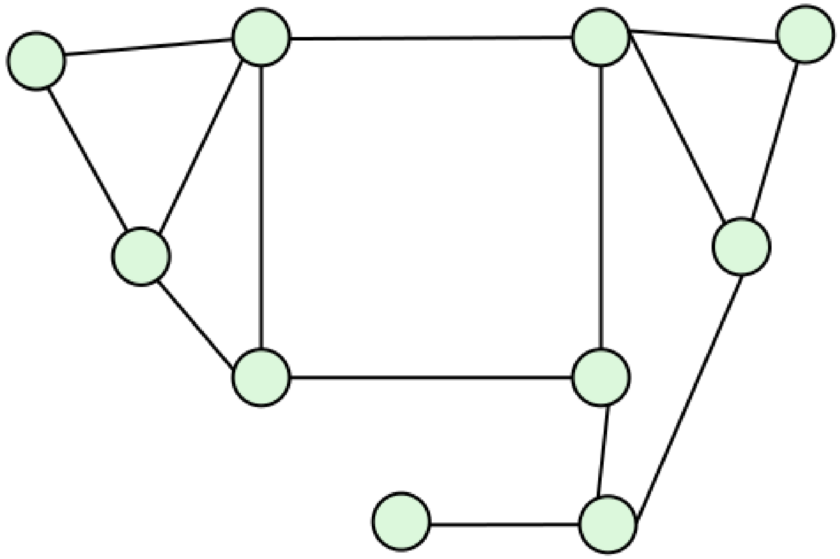
m_i= # of edges on face i

$$f \leq \frac{2e}{3}$$

$$v - e + f = 2$$

Upper bound on edges should not be surprising, since intuitively, once we add too many edges, some must cross.

An upper bound on edges implies an upper bound on faces (since they are bound by Euler’s formula)



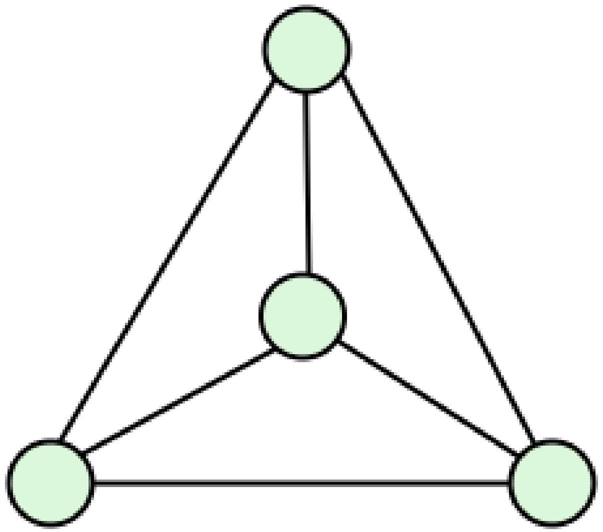
Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$

K_n is the complete graph on n vertices.

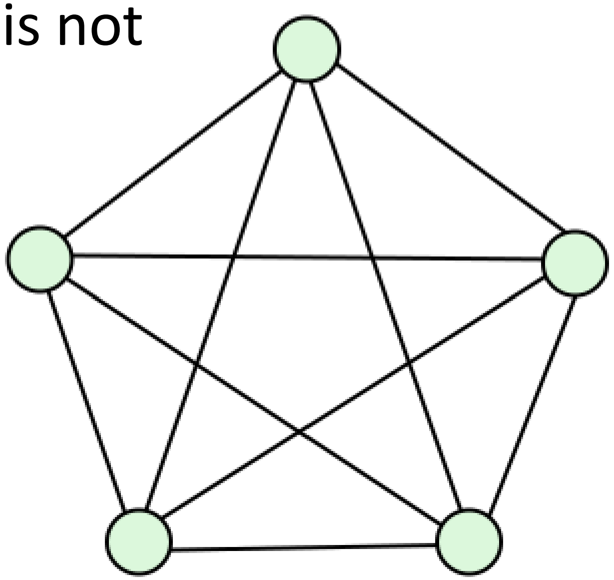
Complete means an edge between every pair of vertices in the graph. That means, for K_n:

$$e = \binom{v}{2} = \frac{v(v-1)}{2}$$

K_4 we can see is planar



K_5 we claim is not planar



Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$

K_5 has 5 vertices, and thus

$$e = \binom{5}{2} = \frac{5(4)}{2} = 10$$

edges.

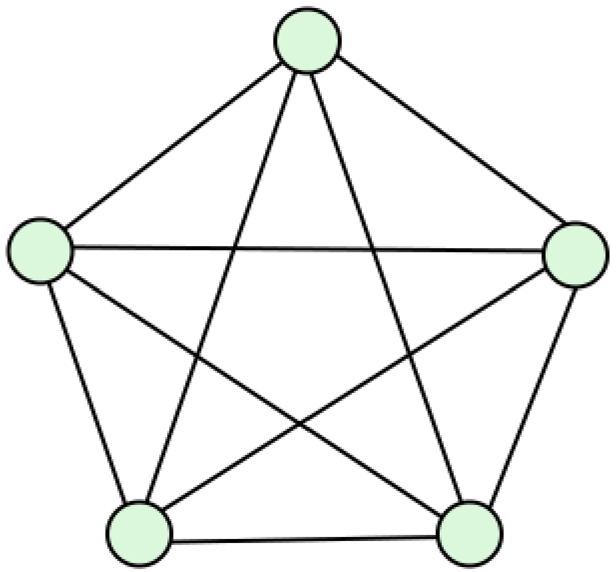
Euler's formula tells us:

$$e \leq 3v - 6$$

$$10 \leq 3 \cdot 5 - 6$$

$$10 \leq 9$$

This is not true, thus K_5 is not planar.



Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$

Once we have a certain number of edges in a graph, some of them must cross.

The next natural question we ask is, if the graph is not planar, what is the minimum number of crossings that it can be drawn with?

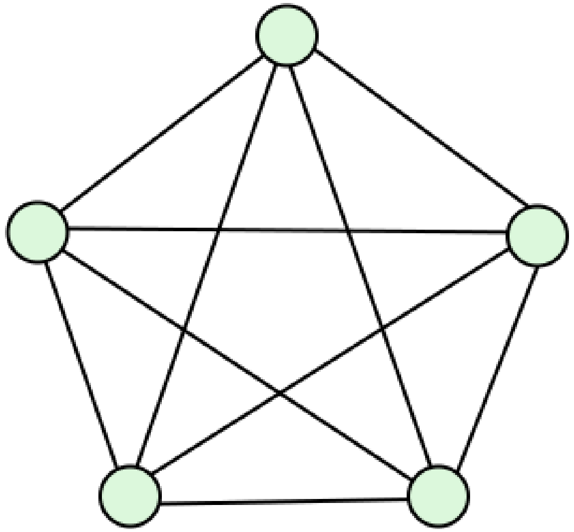
This is called the *crossing number* of a graph.

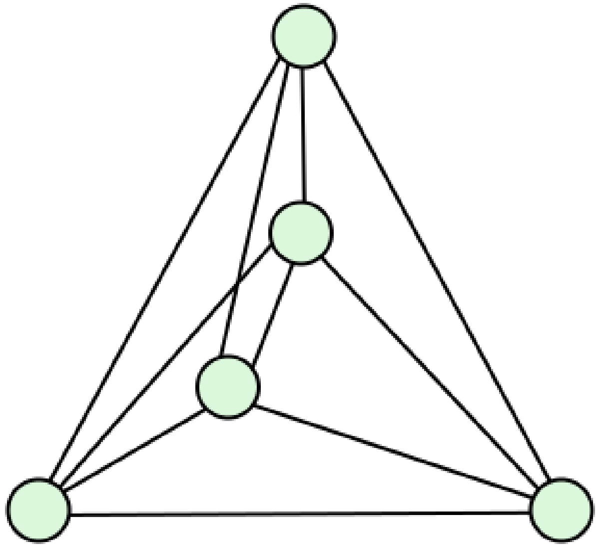
Formally, for a graph G, $Cr(G)$ is the minimum number of crossings.

For a large graph, this is an NP-complete problem.

There are 5 crossings in this drawing of K_5.

Can we draw K_5 with fewer crossings?





Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

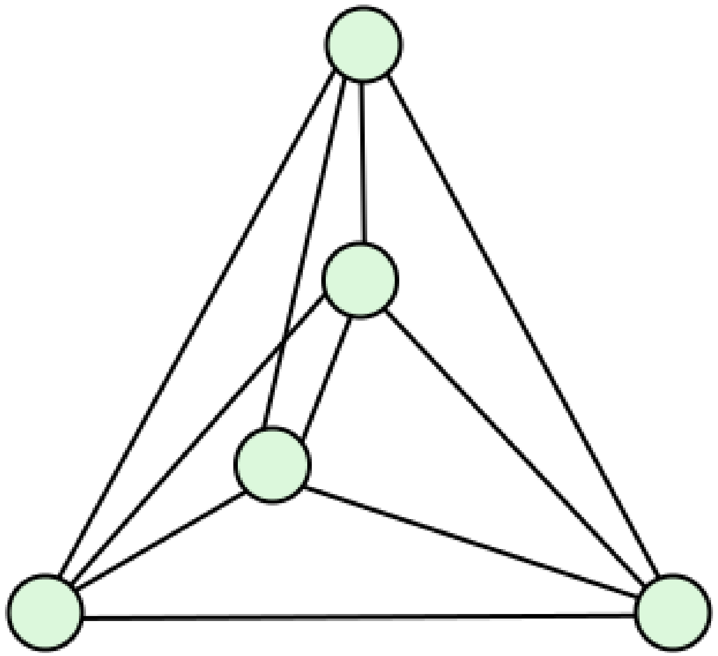
K_5 is not planar, therefore it must have at least 1 crossing.

We can draw K_5 with 1 crossing.

Thus $Cr(K_5) = 1$.

What about K_6?

We will start with K_5 with the minimum number of crossings, and add a vertex.



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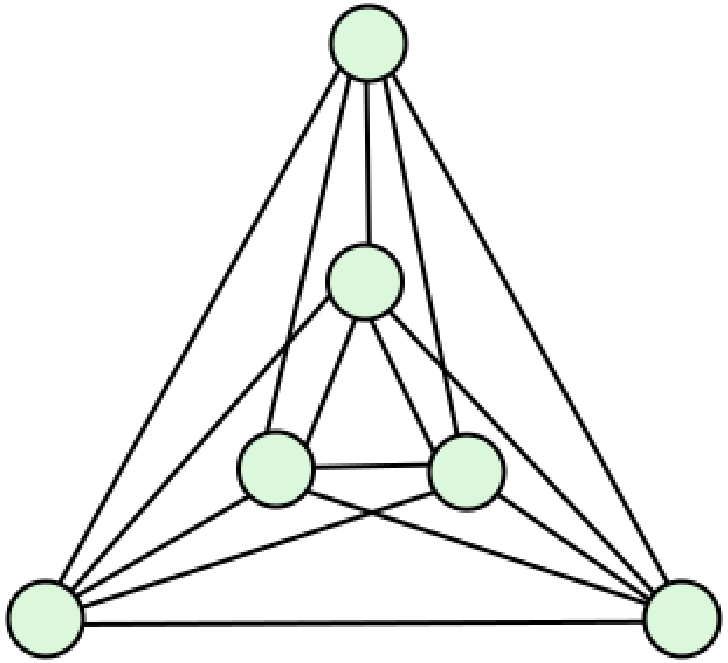
What about K_6?

We will start with K_5 with the minimum number of crossings, and add a vertex.

$Cr(K_6) \leq 3$, since we can draw it with 3.



Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

Saying a graph G is planar is the same as saying $Cr(G) = 0$.

We want to put a lower bound on the crossing number of a graph based on v and e.

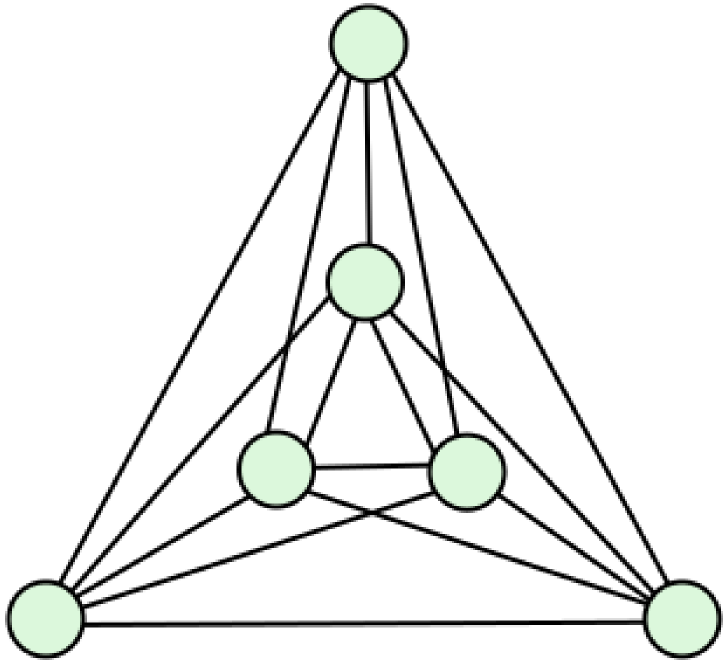
$$Cr(G) \geq \text{some function of } v \text{ and } e$$

Lower bound is least crossings necessary.

Take a drawing of G with $Cr(G)$ crossings.

(Assume we know how to draw this.)

We can make it planar by adding new vertices at each crossing.



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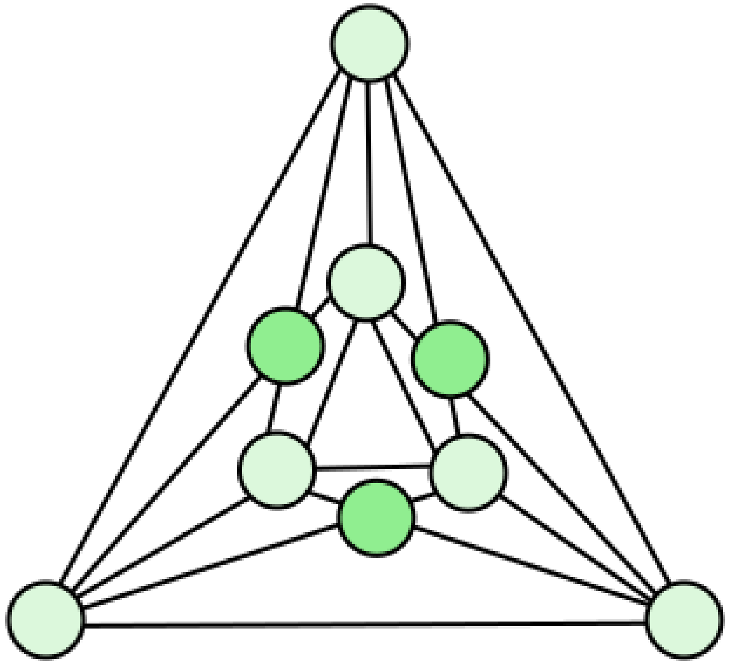


Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

Note that each crossing "splits" each edge into 2 edges.

vertices = $v + Cr(G)$

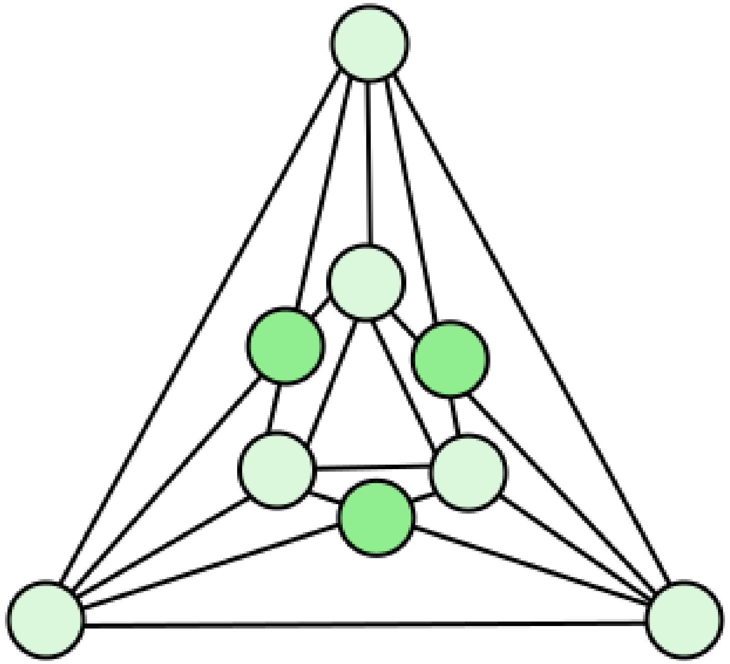
Every time we insert a vertex, we split 2 edges.

edges = $e + 2 \cdot Cr(G)$

This "new" graph is planar, thus we can apply Euler's formula.

$$e \leq 3v - 6$$
$$e + 2 \cdot Cr(G) \leq 3\big(v + Cr(G)\big) - 6$$

$$Cr(G) \geq e - 3v + 6$$



Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

$$Cr(G) \geq e - 3v + 6$$

Let's look at K_6:

$$v = 6, e = \binom{6}{2}$$

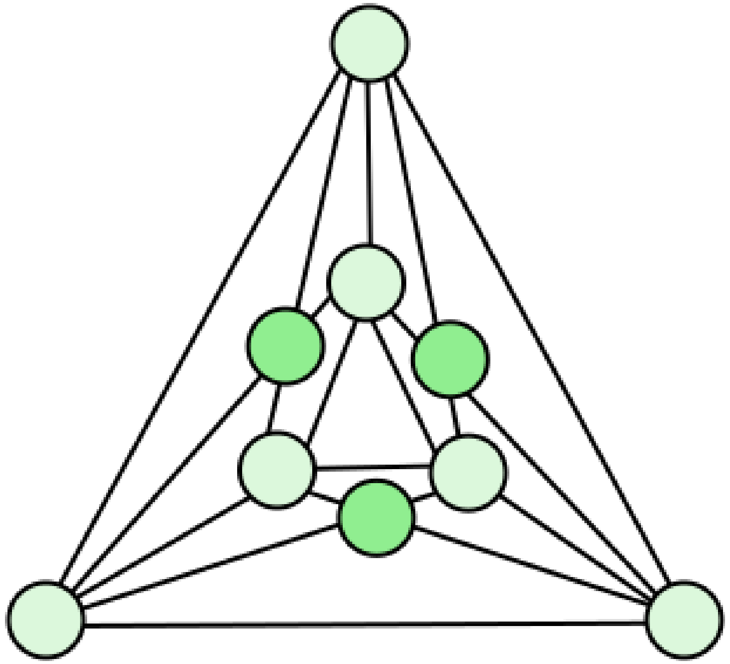
$$Cr(K_6) \geq e - 3v + 6$$

$$\geq \binom{6}{2} - 18 + 6 = 3$$

Therefore any drawing of K_6 must have ≥ 3 crossings.

We have found a drawing with 3 crossings.

Thus $Cr(K_6) = 3$.



Euler's theorem:

$v - e + f = 2$

$e \le 3v - 6$

$f \le 2v - 4$

For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

$$Cr(G) \ge e - 3v + 6$$

Let's look at K_2:

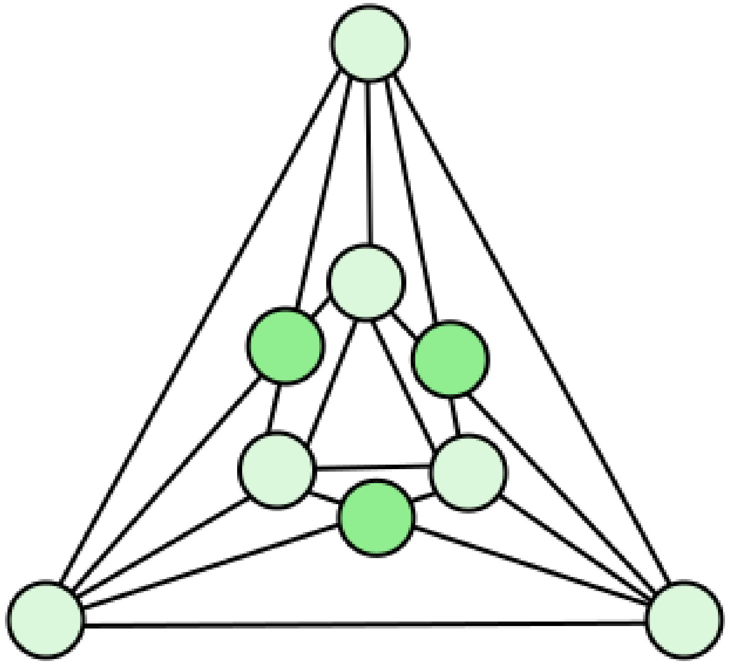
$$v = n, e = \binom{n}{2}$$

$$Cr(K_n) \ge e - 3v + 6$$

$$\ge \binom{n}{2} - 3n + 6$$

$$= \Omega(n^2)$$

This is a lower bound that follows from Euler's formula.



Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.
$$Cr(G) \geq e - 3v + 6$$

$\mathrm{Cr}(K_n) = \Omega(n^2)$. Is $\mathrm{Cr}(K_n) = O(n^2)$?

What is a simple upper bound on $\mathrm{Cr}(K_n)$?

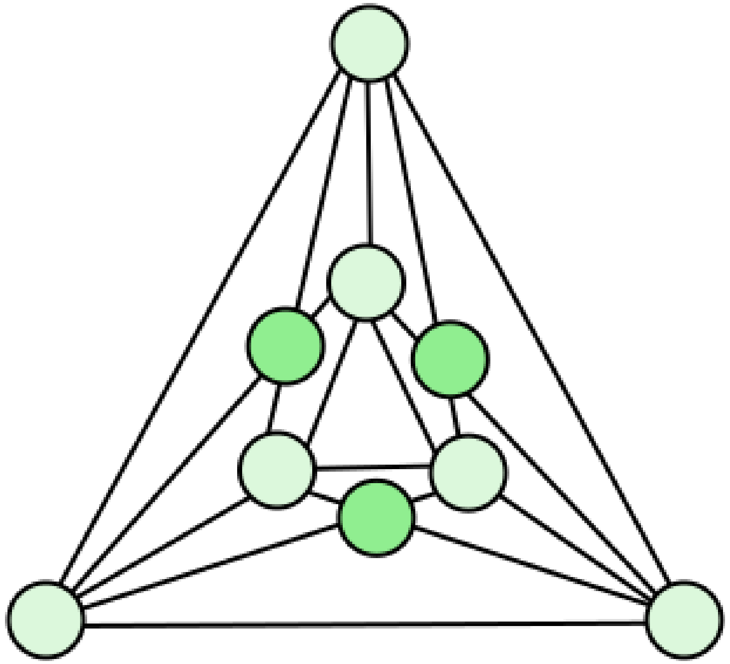
Every pair of edges crosses at most once.

$$\mathrm{Cr}(K_n) \leq \binom{e}{2}$$

$$\mathrm{Cr}(K_n) \leq \binom{\binom{n}{2}}{2} = O(n^4)$$

Which of these is correct?

It turns out $\mathrm{Cr}(K_n) = \Omega(n^4)$.



We will use probability and random variables to show that indeed $Cr(K_n) = \Omega(n^4)$

Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.
$$Cr(G) \geq e - 3v + 6$$

$\mathrm{Cr}(K_n) = \Omega(n^2)$. $\mathrm{Cr}(K_n) = O(n^2)$

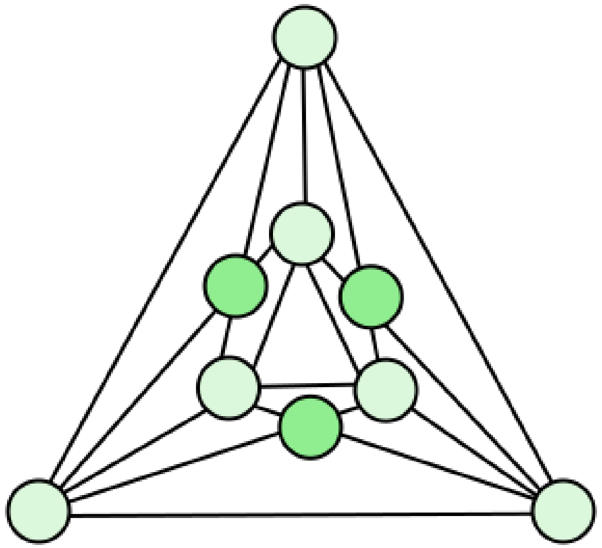
Take an arbitrary graph G, and a probability $0 < p < 1$.

Random graph G_p:
Each vertex of G is in G_p with probability p.

Each edge $\{a, b\}$ appears in G_p if both a and b are in G_p.

Random variables:

v_p = # vertices in G_p
e_p = # edges in G_p
X_p = # of crossings in best drawing of G_p



We can apply the lower bound

$Cr(G_p)$
$\geq e - 3v + 6$

Euler's theorem:
$v - e + f = 2$
$e \leq 3v - 6$
$f \leq 2v - 4$
For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

$$Cr(G) \geq e - 3v + 6$$

$$X_p \geq e_p - 3 \cdot v_p + 6$$
$$X_p - e_p + 3 \cdot v_p \geq 6$$

$$X_p - e_p + 3 \cdot v_p \geq 0$$
$$E(X_p - e_p + 3 \cdot v_p) \geq 0$$

$$E(X_p) - E(e_p) + 3 \cdot E(v_p) \geq 0$$

Each vertex of G in G_p with probability p.
$\{a, b\}$ appears in G_p if a and b are in G_p.

V_p = # vertices in G_p
E_p = # edges in G_p
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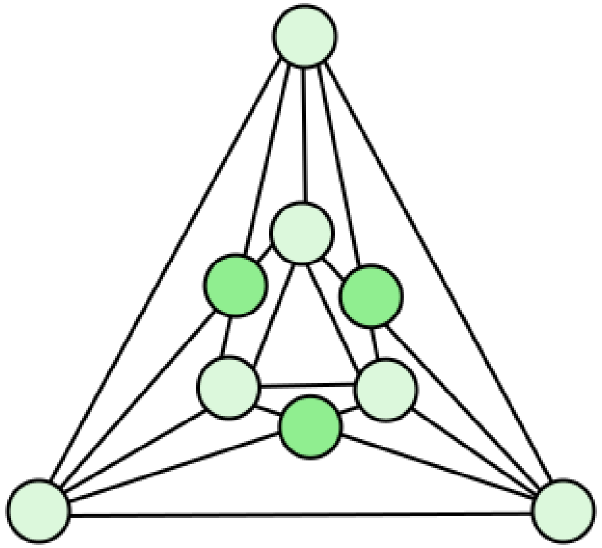
$$Cr(G_p) \geq e - 3v + 6$$



We must determine these expected values.

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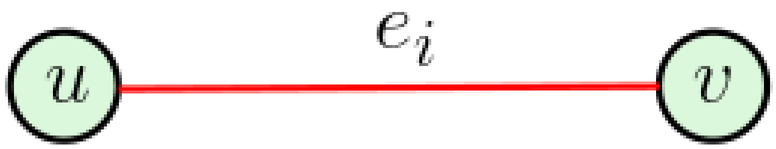
$$E(X_p) - E(e_p) + 3 \cdot E(v_p) \geq 0$$

$$E(v_p) = pv$$

An edge $\{u, v\}$ is in G_p iff both u and v were selected to be in G_p.

$$\Pr(u \text{ } and \text{ } v \text{ } in \text{ } G_p) = p \cdot p = p^2$$

Thus $E(e_p) = p^2 \cdot e$ (can use i.r.v. as well).



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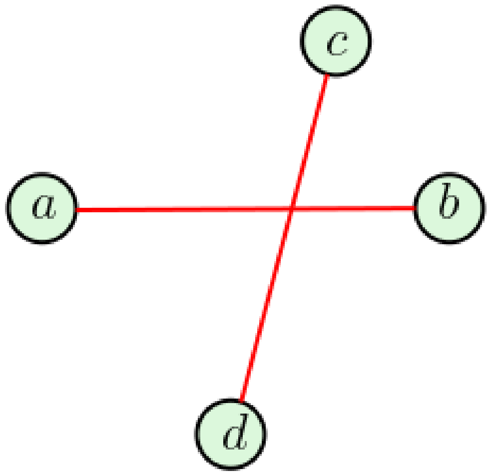
$$E(v_p) = pv$$
$$E(e_p) = p^2 \cdot e$$

Crossing occurs in G_p $\leftrightarrow$ all of a, b, c, d are in G_p.

$$E(X_p) = p^4 \cdot Cr(G)$$



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$$E(X_p) - E(e_p) + 3 \cdot E(v_p) \geq 0$$

$$p^4 \cdot Cr(G) - p^2 \cdot e + 3 \cdot pv \geq 0$$

$$Cr(G) \geq \frac{p^2 e - 3pv}{p^4}$$

This is true for every value p. So we can choose any p.

Let $p = \frac{4v}{e}$. If $e > 4v$:

$$Cr(G) \geq \frac{1}{64} \cdot \frac{e^3}{v^2}$$

New lower bound.

Euler's theorem:

$v - e + f = 2$

$e \leq 3v - 6$

$f \leq 2v - 4$

For a graph G, the crossing number $Cr(G)$ is the min number of crossings.

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$$Cr(G_p) \geq e - 3v + 6$$

New lower bound:

$$Cr(G) \geq \frac{1}{64} \cdot \frac{e^3}{v^2}$$

Apply it to K_n:

$$Cr(K_n) = \frac{1}{64} \cdot \frac{\binom{n}{2}^3}{n^2}$$

$$\approx \frac{1}{64} \cdot \frac{n^6}{n^2}$$

$$= \Omega(n^4)$$

For large values of n, the crossing number of $K_n = \Theta(n^4)$.