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1. Determine if the following series converges or diverges. If it converges, find its sum. Clearly show all work, including any applicable tests for convergence. **(6 points)**

$$\sum_{n=2}^{\infty} 5\left(\frac{-1}{3}\right)^{n}$$

Solution. *This series converges since it is geometric with* $r = \frac{-1}{3} < 1$. *The series converges to*

$$\sum_{n=2}^{\infty} 5\left(\frac{-1}{3}\right)^{n} = 5\frac{(-1/3)^2}{1+\frac{1}{3}} = 5\left(\frac{1}{9}\right)\frac{1}{\frac{4}{3}} = \frac{5}{12} \tag{1}$$

2. Determine if the following series converge or diverge. If they converge, find their sum. Clearly show all work, including any applicable tests for convergence. **(12 points)**

(a)

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n}}$$

Solution.

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{2^{2n}} = 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n} \tag{2}$$

which is geometric with $r = 3/4 < 1$ *and converges to*

$$3\frac{1}{1-\frac{3}{4}} = 12 \tag{3}$$

(b)

$$\sum_{n=3}^{\infty} \frac{n^2}{(n+2)(n-2)}$$

Solution.

$$\lim_{n\to\infty} \frac{n^2}{(n+2)(n-2)} = \lim_{n\to\infty} \frac{n^2}{n^2-4} \tag{4}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{4}{n^2}} = 1 \tag{5}$$

Thus, the series diverges by the nth term test (divergence test).

3. Determine if the following series converge or diverge. Clearly show all work, including any applicable tests for convergence. **(16 points)**

(a)

$$\sum_{n=2}^{\infty} \frac{2n^2}{\sqrt[3]{n^7+n}}$$

Solution. *We will use a limit comparison test with* $\frac{n^2}{n^{7/3}} = \frac{1}{n^{1/3}}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{\left(\frac{2n^2}{\sqrt[3]{n^7+n}}\right)}{\frac{1}{n^{1/3}}} &= \lim_{n\to\infty} \frac{2n^{7/3}}{\sqrt[3]{n^7\left(1+\frac{1}{n^6}\right)}} \\ &= \lim_{n\to\infty} \frac{2}{\sqrt[3]{1+\frac{1}{n^6}}} = \frac{2}{1} = 2 < \infty\end{aligned}$$

Since $\sum \frac{1}{n^{1/3}}$ *diverges by the p-test (with* $p = \frac{1}{3}$*) and the LCT holds we have that the original series diverges.*

(b)

$$\sum_{n=1}^{\infty} \frac{n}{e^{-n}+n^3}$$

Solution. *We will use the limit comparison test with* $\frac{1}{n^2}$*:*

$$\begin{aligned}\lim_{n\to\infty} \frac{\left(\frac{n}{e^{-n}+n^3}\right)}{\frac{1}{n^2}} &= \lim_{n\to\infty} \frac{n^3}{e^{-n}+n^3} \\ &= \frac{1}{\frac{1}{e^n n^3}+1} = 1\end{aligned}$$

Since $\sum \frac{1}{n^2}$ *converges (by p-test,* $p = 2$*) the original series* $\sum_{n=1}^{\infty} \frac{n}{e^{-n}+n^3}$ *converges by limit comparison test.*

4. Determine if the series converges conditionally, converges absolutely, or diverges. Clearly show all work, including any applicable tests for convergence. **(10 points)**

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}$$

Solution. *We will first determine if the series converges absolutely. We proceed with the integral test, noting that* $\frac{1}{x\sqrt{\ln(x)}}$ *is positive, and decreasing since* $x\sqrt{\ln(x)}$ *is increasing:*

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}dx \qquad u = lnx \; du = 1/xdx$$

$$= \int_{ln(2)}^{\infty} u^{-1/2}du = 2u^{1/2}|_{ln2}^{\infty} = \infty$$

Thus, the series will not converge absolutely by the integral test. It can still converge conditionally, and indeed it does. We proceed with the alternating series test. Note that the sequence *is positive, decreasing, and tends to* 0 *as* $n \to \infty$*. By the alternating series test the series* $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}$ *converges conditionally.*

5. Suppose that $f(n) > 0$ for all integers n and that the series $\sum_{n=0}^{\infty} f(n)$ is known to be convergent. Is the above series absolutely convergent, conditionally convergent, divergent, or is it unable to determine? Carefully explain how you know, or what necessary information you are missing. **(4 points)**

   Solution. *We claim that the series is absolutely convergent. Note that* $|f(n)| = f(n)$ *for all* $n \in \mathbb{Z}$*.*

   Proof.

   $$\sum_{n=0}^{\infty} |f(n)| = \sum_{n=0}^{\infty} f(n) \tag{6}$$

   which converges. Thus, the series converges absolutely. □

6. Find the radius of convergence and interval of convergence for the power series

   $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n3^n} \qquad \textbf{(12 points)}$$

   Solution. *We begin with the ratio test:*

   $$\begin{aligned}\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{(x-1)^n}{n3^n} \right| &= \lim_{n\to\infty} \frac{1}{3} \left| \frac{n}{n+1} \right| |x-1| \\ &= \frac{1}{3}|x-1| = L < 1\end{aligned}$$

   for $|x-1| < 3 \equiv R$*. Thus the radius of convergence is* $R = 3$*. Then, the interval of absolute convergence is* $(-2, 4)$ *(with center* $x = 1$*). We now check the endpoints: If* $x = -2$ *we have:*

   $$\sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

   which is the alternating harmonic series and converges by the alternating series test since $\frac{1}{n}$ *is positive, decreasing, and tends to* 0*. If* $x = 4$ *we have:*

   $$\sum_{n=1}^{\infty} \frac{3^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

   which is the harmonic series and diverges by the p-test ($p = 1$*). Therefore the interval of convergence is* $[-2, 4)$*.*

7. (a) Find the Taylor Polynomial $P_3(x)$ that uses a Taylor Series centered around $x = 3$ to approximate $f(x) = \ln(x-2)$ **(5 points)**

Solution.

$$\begin{aligned}
f(x) &= \ln(x-2) \qquad f(3) && = \ln(1) = 0 \\
f'(x) &= \frac{1}{x-2} \qquad f'(3) && = 1 \\
f''(x) &= \frac{-1}{(x-2)^2} \qquad f''(3) && = -1 \\
f'''(x) &= \frac{2}{(x-2)^3} \qquad f'''(3) && = 2
\end{aligned}$$

$$\begin{aligned}
\implies P_3(x) &= \frac{f'(3)(x-3)}{1!} + \frac{f''(3)(x-3)^2}{2!} + \frac{f'''(3)(x-3)^3}{3!} \\
&= (x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} \\
&= \frac{x^3}{3} - \frac{7x^2}{2} + 13x - \frac{33}{2}
\end{aligned}$$

(b) Find the series representation of the full Taylor series from the function above.

Solution. *To do this we begin with the geometric series:*

$$\begin{aligned}
\frac{1}{x+1} &= \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x| < 1 \\
\ln(x+1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \qquad |x| < 1 \\
\ln(x-2) = \ln((x-3)+1) &= \sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^{n+1}}{n+1} \qquad |x-3| < 1
\end{aligned}$$

8. (a) Find the full Taylor Series representation for $f(x) = e^{-x/2}$ centered around $x = 1$.

Solution. *We construct this by using the existing representation for e^x:*

$$\begin{aligned}
e^{-x/2} &= e^{\frac{-(x-2)-1}{2}} = e^{-1/2} e^{\frac{-(x-1)}{2}} \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \textit{for all } x \\
e^{-x/2} &= e^{-1/2} e^{\frac{-(x-1)}{2}} = e^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n!}
\end{aligned}$$

which should converge for all x. We will show this in (b).

(b) Find the Radius of Convergence and Interval of Convergence for this Taylor series by performing an appropriate convergence test on the power series above. **(7 points)**

Solution. *We proceed as usual with the ratio test:*

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}(n+1)!} \cdot \frac{2^n n!}{(x-1)^n} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{1}{n+1} |x-1| = 0
\end{aligned}$$

Thus, the radius of convergence is $R = \infty$ and thus the interval of convergence is all of $\mathbb{R}$.

9. (a) Find a Maclaurin Series for the function $f(x) = \cos(x^2)$ (hint: the substitution method using a well-known Maclaurin series will be the quickest) **(5 points)**

Solution.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \forall x \in \mathbb{R}$$
$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} \qquad \forall x \in \mathbb{R}$$

(b) Find a Maclaurin Series representation for the indefinite integral $\int \cos(x^2)dx$ **(5 points)**

Solution.

$$\begin{aligned}\int \cos(x^2)dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \int x^{4n} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}\end{aligned}$$

(c) Estimate the integral $\int_0^1 \cos(x^2)dx$ with an error of magnitude less than 0.01 by using a Taylor polynomial of the lowest possible degree. **(5 points)**

Solution. *We need to determine what the lowest possible degree to use is- that is, we need N (the truncation point). First note that:*

$$\begin{aligned}\int_0^1 \cos(x^2)dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}\Big|_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}\end{aligned}$$

We know further that the remainder for truncating an alternating series is bounded:

$$|R_n| = |S - S_N| < a_{N+1} = \frac{1}{(2N+2)!(4N+5)}$$

We must then check possible values of N and check the lowest possible N such that $a_{N+1} < \frac{1}{100}$. Note that $a_0 = 1$ and $a_1 = \frac{1}{10} > \frac{1}{100}$. We do have, however, that $a_2 = \frac{1}{4!\cdot 9} = \frac{1}{216} < \frac{1}{100}$. Therefore we only need two terms ($n = 0, 1$) to ensure the desired accuracy. Hence,

$$\int_0^1 \cos(x^2)dx \approx 1 - \frac{1}{10} = \frac{9}{10} = 0.9$$