Homework Week 9

Section 10.5

(#38) (+1)

#38 $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$

$$\sum_{n=1}^{\infty} \left|\frac{n!}{(-n)^n}\right| = \sum_{n=1}^{\infty} \frac{n!}{n^n} \rightarrow \text{Ratio test}$$

$$\lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \lim_{n\to\infty} \left|\frac{n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{1}\right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^{-1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1. \text{ Thus,}$$

$\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$ converges absolutely, so it converges conditionally.

$\square$

(+1) ~~(+1)~~ ~~(+2)~~

#56 $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n+1)!(n+2)!}$ Ratio test

$$\lim_{n\to\infty} \left| \frac{(3(n+1))!}{(n+1)!(n+2)!(n+3)!} \cdot \frac{n!(n+1)!(n+2)!}{(3n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3n+3)(3n+2)(3n+1)(3n)! \cdot (n!)}{(n+3)! \quad (3n)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(3n+3)(3n+2)(3n+1)}{(n+3)(n+2)(n+1)} \right| = 27 > 1$$

Thus, the series diverges.

#58 $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$ Root test (+2)

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{(n!)^n}{n^{(n^2)}} \right|} = \lim_{n\to\infty} \left| \frac{n!}{n^n} \right| = \lim_{n\to\infty} \frac{n(n-1)\cdots(3)(2)(1)}{n \cdot n \cdot n \cdots n \cdot n}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{1}{n}\right)\cdot\left(\frac{2}{n}\right)\cdot\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right)}_{R}\left(\cancel{\frac{n}{n}}\right) = \lim_{n\to\infty} R \left(\frac{n-1}{n}\right)$$

Note $R<1$ so

$\leq \lim_{n\to\infty} \frac{1}{n}$ By the squeeze thm $L = 0 < 1$

so the series converges.

②

~~(+2)~~ (+1)

Section 10.6.

#12 $\sum_{n=1}^{\infty} (-1)^n \ln(1+\frac{1}{n})$

The solution manual says it converges by the Alternating series test. i.e.

If $f(x) = \ln(1+\frac{1}{x})$

$f'(x) = \frac{-1}{x^2} \frac{1}{1+\frac{1}{x}} < 0$ for $x \in [1, \infty)$.

Thus f is decreasing $\Rightarrow$ $a_n = \ln(1+\frac{1}{n})$ is decreasing.

Furthermore, $\lim_{n\to\infty} \ln(1+\frac{1}{n}) = \ln(1) = 0$.

By A.S.T. converges.

Note: $\sum_{n=1}^{\infty} |(-1)^n \ln(1+\frac{1}{n})|$ diverges since.

note

$S_N = \sum_{n=1}^{N} \ln(1+\frac{1}{n}) = \sum_{n=1}^{N} \ln(\frac{n+1}{n}) = \sum_{n=1}^{N} \ln(n+1) - \ln(n)$

$= [\ln(2) - \ln(1)] + [\ln(3) - \ln(2)] + \dots$

$[\ln(N+1) - \ln(N)] = \ln(N+1) \to \infty$

as $N \to \infty$.

③

~~(+5)~~ (+1)

#24 $\sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n+5^n}$

$a_n = \frac{2^n}{n+5^n}$

converges by DCT
absolutely

$\left| \frac{(-2)^{n+1}}{n+5^n} \right| = \frac{2^{n+1}}{n+5^n} \le \frac{2^{n+1}}{5^n} = 2\left(\frac{2}{5}\right)^n$

Since $2\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$ is geometric w/ $r = 2/5 < 1$ it converges

$\Rightarrow$ our series $\boxed{\text{converges absolutely by DCT.}}$

~~HW Week #~~10

Section 10.6

(+2)

#30 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n-\ln(n)}$

Let $f(x) = \dfrac{\ln x}{x-\ln x}$.

$$f'(x) = \frac{(x-\ln x)\frac{1}{x} - \ln x\left(1-\frac{1}{x}\right)}{(x-\ln x)^2}$$

$$= \frac{1 - \frac{\ln x}{x} - \ln x + \frac{\ln x}{x}}{(x-\ln x)^2}$$

$$= \frac{1-\ln x}{(x-\ln x)^2} < 0 \text{ for } x > e.$$

$a_n = \dfrac{\ln(n)}{n-\ln(n)}$ is decreasing.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n)}{n-\ln(n)}$$

$$= \lim_{n\to\infty} \frac{\frac{\ln(n)}{n}}{1-\frac{\ln(n)}{n}} = \frac{0}{1-0} = 0$$

So $\sum(-1)^n a_n$ converges conditionally.

But $n-\ln(n) < n-0 = n$

So $\dfrac{1}{n-\ln(n)} > \dfrac{1}{n} \Rightarrow$

$$\frac{\ln(n)}{n-\ln(n)} > \frac{1}{n-\ln(n)} > \frac{1}{n} \Rightarrow$$

$\displaystyle\sum_{n=1}^{\infty} \left|(-1)^n \frac{\ln(n)}{n-\ln(n)}\right|$ diverges by DCT. $\square$

Only converges conditionally!

(+1)

#42 $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$

Divergence test

$$\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{(n^2+n) - n^2}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{n\left(\sqrt{1+\frac{1}{n}} + 1\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{2} \neq 0.$$ Diverges!

#50 $s = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$ first 4 terms. (+1)

$$|s - s_4| = a_5 = \frac{1}{10^5} = .00001$$