1. Consider the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 - y \rangle$.
   a) Show $\mathbf{F}(x, y)$ is conservative.
   b) Determine the potential function f where $\nabla f = \mathbf{F}$.
   c) Determine $\int_C \boldsymbol{F}(x, y) \cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \langle \ln t \,, e^{t-1} \rangle$ from $(0, 1)$ to $(\ln 3 \,, \mathrm{e}^2)$.

2. Consider the conservative vector field $\mathbf{F}(x, y, z) = \langle 3x^2 z, z^2, x^3 + 2yz \rangle$.
   a) Determine the potential function f where $\nabla f = \mathbf{F}$.
   b) Determine $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ where C is the curve $\mathbf{r}(t) = \langle t^2 \sin(t+1)\,, t^3, t\cos(t+1) \rangle$ from $(1, -1, -1)$ to $(0, 0, 0)$.

Answers:

1.

a) $\mathbf{F}(x,y) = \langle 2xy, x^2 - y\rangle = \langle P, \mathrm{Q}\rangle \rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ which gives $\frac{\partial(x^2-y)}{\partial x} = 2x = \frac{\partial(2xy)}{\partial y}$

b) Given $f_x = 2xy$ and $f_y = x^2 - y$, we get $\int f_x dx = \int(2xy)dx = x^2y + G(y) + C = f(x,y)$ and $\int f_y dy = \int(x^2 - y)dy = x^2y - \frac{1}{2}y^2 + H(x) + C = f(x,y)$. Thus $f(x,y) = x^2y - \frac{1}{2}y^2 + C$.

c) From $(0,1)$ to $(\ln 3, e^2)$ is $t_1 = 1$ and $t_2 = 3$, so $\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = f(\ln 3, e^2) - f(0,1)$
$= \left((\ln 3)^2 * e^2 - \frac{1}{2} * (e^2)^2\right) - \left(0^2 * 1 - \frac{1}{2} * 1^2\right) = (\ln 3)^2 * e^2 - \frac{1}{2} * (e^2)^2 + \frac{1}{2}.$

2.

a) Given $f_x = 3x^2z$, $f_y = z^2$, and $f_z = x^3 + 2yz$, we get
$\int f_x dx = \int(3x^2z)dx = x^3z + G(y,z) + C = f(x,y,z)$,
$\int f_y dy = \int(z^2)dy = yz^2 + H(x,z) + C = f(x,y,z)$, and
$\int f_z dz = \int(x^3 + 2yz)dz = x^3z + yz^2 + I(x,y) + C = f(x,y,z)$.
Therefore $f(x,y,z) = x^3z + yz^2 + C$.

b) $\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r} = f(0,0,0) - f(1,-1,-1) = (0^3 * 0 + 0 * 0^2) - (1^3 * -1 - 1 * (-1)^2) = 1 + 1 = 2$