MAT273 — Unit #4 Review

Show all your work, justify your answers and interpret any applicable solutions using complete sentences. You may use a scientific calculator like the Ti-36x Pro, but not a graphing calculator. (These problems are intended to help you prepare for the test. Do not assume that these are the only problems you need to do. You should also review your notes, homework, quizzes, and other exercises from the text.)

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1. Set up the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle x, 0\rangle$ and the curve $C: y = \sin(x)$, for $0 \le x \le \frac{\pi}{2}$.

2. Use Green's Theorem to change the line integral $\int_C xy\,dx + y^5 dy$ to an integral over an area, with C being the triangle with vertices $(0,0)$, $(2,0)$, and $(2,1)$.

3. Set up the surface integral $\iint_S yz\,dS$ where S is the part of the plane $z = y + 3$ inside the cylinder $x^2 + y^2 = 1$.

4. Set up the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = \langle x, y, z\rangle$ and the sphere $S: x^2 + y^2 + z^2 = 9$ with outward orientation.

5. Use Stoke's Theorem to change the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ into a surface integral where $\mathbf{F} = \langle xz, 2xy, 3xy\rangle$ and C is the boundary of the part of the plane $3x + y + z = 3$ in the first octant oriented counter-clockwise looking downward.

6. Use the Divergence Theorem to change the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ into a volume integral where $\mathbf{F} = \langle 3y^2z^3, 9x^2yz^2, -4xy^2\rangle$ and S is the cube with vertices (±1, ±1, ±1).

7. Set up the integral for the surface area of the part of the surface $z = xy$ that lies above the cylinder $x^2 + y^2 = 1$.

8. Consider the Helicoid surface $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$ with $-1 \le u \le 1$ and $0 \le v \le 2\pi$.
   a) Determine the surface area.

   b) Determine the tangent plane at $\mathbf{r}\left(\frac{1}{2}, \frac{\pi}{3}\right)$.

Answers:

1. Parameterizing with $x = t$ gives $\mathbf{r}(t) = \langle t, \sin t\rangle$, $\mathbf{r}'(t) = \langle 1, \cos t\rangle$, and $\mathbf{F} = \langle t, 0\rangle$ for $0 \le t \le \frac{\pi}{2}$. Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \langle t, 0\rangle \cdot \langle 1, \cos t\rangle\, dt = \int_0^{\pi/2} t\, dt$.

2. The integral $\int_C xy\, dx + y^5 dy$ gives a vector field $F = \langle f, g\rangle = \langle xy, y^5\rangle$. The triangle area within C is bounded by $0 \le x \le 2$ and $0 \le y \le \frac{1}{2}x$. So by Green's Theorem we wet $\iint_R (g_x - f_y) dA = \int_0^2 \int_0^{x/2} (-x) dy\, dx$.

3. This is a scalar function over a surface with $z = g(x, y)$, so the integral is $\iint_S f(x, y, z)\, dS = \iint_R f(x, y, g)\sqrt{z_x^2 + z_y^2 + 1} dA$. We are given $z = g(x, y) = y + 3$, so $f(x, y, g) = y(y + 3) = y^2 + 3y$ and $\sqrt{z_x^2 + z_y^2 + 1} = \sqrt{0^2 + y^1 + 1} = \sqrt{2}$. The region R is the circle $x^2 + y^2 = 1$, so the integral might be best expressed in cylindrical coordinates as $= \int_0^{2\pi} \int_0^1 (r^2 \sin^2\theta + 3r\sin\theta)\sqrt{2}\, r\, dr\, d\theta$.

4. The flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is equivalent to $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ which is equivalent to $\iiint_D \nabla \cdot \mathbf{F}\, dV$ by the Divergence theorem, since the first partial derivatives of $\mathbf{F}$ are continuous and the region D inside S is connected and simply connected. Then $\nabla \cdot \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle x, y, z\rangle = 1 + 1 + 1 = 3$. The region D can be expressed in spherical coordinates where $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, and $0 \le \rho \le 3$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \int_0^3 3\rho^2 \sin\varphi\ d\rho\, d\varphi\, d\theta$.

5. The boundary C is a triangle which is a piece-wise smooth oriented curve, and $\mathbf{F}$ has continuous first partial derivatives. Thus we can use Stoke's Theorem, so $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$. The curl cross product is $\nabla \times \mathbf{F} = \langle 3x, x - 3y, 2y\rangle$, and the unit normal vector for $C\colon z = g(x, y) = 3 - 3x - y$ is $\mathbf{n} = \langle -z_x, -z_y, 1\rangle = \langle 3, 1, 1\rangle$. Then $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (9x) + (x - 3y) + (2y) = 10x - y$, which gives the integral $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_A 10x - y\, dA$ where A is the triangle in the xy-plane with vertices at $(0, 0)$, $(1, 0)$, and $(0, 3)$, bounded by $0 \le x \le 1$ and $0 \le y \le 3 - 3x$. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_0^{3-3x} (10x - y) dy\, dx$.

6. The cube can be expressed as the rectangular bounds $-1 \le x, y, z \le 1$. The flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$, according to the Divergence Theorem. Then $\nabla \cdot \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle 3y^2z^3, 9x^2yz^2, -4xy^2\rangle = 0 + 9x^2z^2 + 0 = 9x^2z^2$. Then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 9x^2z^2\, dy\, dz\, dx$.

7. The surface integral $\iint_S f(x, y, z) dS = \iint_R f(x, y, g)\sqrt{z_x^2 + z_y^2 + 1}\, dA$, with $z = g(x, y)$, represents the surface area of S if $f = 1$. We have $z_x = y$ and $z_y = x$, and R is the unit circle. Then $\iint_S f(x, y, z) dS = \iint_R \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1}\, r\, dr\, d\theta$.

8. We need $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\cos^2 v + u\sin^2 v\rangle = \langle \sin v, -\cos v, u\rangle$, and $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2 + 1}$.

   a) The surface area is $A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA = \int_0^{2\pi} \int_{-1}^1 \sqrt{u^2 + 1}\, du\, dv = 14.42$.

   b) At the point $\mathbf{r}\left(\frac{1}{2}, \frac{\pi}{3}\right) = \langle \frac{1}{2}\cos\left(\frac{\pi}{3}\right), \frac{1}{2}\sin\left(\frac{\pi}{3}\right), \frac{\pi}{3}\rangle = \langle \frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\pi}{3}\rangle$ with normal vector is $n = \langle \sin\left(\frac{\pi}{3}\right), -\cos\left(\frac{\pi}{3}\right), \frac{1}{2}\rangle = \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$, the tangent plane is $\frac{\sqrt{3}}{2}\left(x - \frac{1}{4}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{1}{2}\left(z - \frac{\pi}{3}\right) = 0$.