1. Determine the curl and divergence of the vector field $\mathbf{F}(x, y, z) = \langle 2xyz, x^2 - yz, \cos(z) \rangle$.

2. Show that the vector field $\mathbf{F}(x, y, z) = \langle xy - \sin(z), \frac{1}{2}x^2, -x\cos(z) \rangle$ over a simply connected domain D is conservative and determine its potential function.

Answers:

1. Given $\mathbf{F}(x,y,z) = \langle 2xyz, x^2 - yz, \cos(z)\rangle = \langle f, g, h\rangle$, $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} = 2yz - z - \sin(z)$ and

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2 - yz & \cos(z) \end{vmatrix} = \langle 0 + y, -(0 - 2xy), 2x - 2xz\rangle = \langle y, 2xy, 2x - 2xz\rangle.$$

2. Given $\mathbf{F}(x,y,z) = \langle xy - \sin(z), \frac{1}{2}x^2, -x\cos(z)\rangle$, then

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy - \sin(z) & \frac{1}{2}x^2 & -x\cos(z) \end{vmatrix} = \langle 0 - 0, -\big(-\cos(z) - (-\cos(z))\big), x - x\rangle = \langle 0, 0, 0\rangle$$

and so $\mathbf{F}$ is conservative. Then for $\varphi_x = xy - \sin z$, $\varphi_y = \frac{1}{2}x^2$ and $\varphi_z = -x\cos(z)$, we get

$\int \frac{\partial \varphi}{\partial x} dx = \int (xy - \sin z) dx = \frac{1}{2}x^2 y - x\sin(z) + G(y,z) + C = \varphi(x,y,z)$,

$\int \frac{\partial \varphi}{\partial y} dy = \int \left(\frac{1}{2}x^2\right) dy = \frac{1}{2}x^2 y + H(x,z) + C = \varphi(x,y,z)$,

$\int \frac{\partial \varphi}{\partial z} dz = \int (-x\cos(z)) dz = -x\sin(z) + I(x,y) + C = \varphi(x,y,z)$.

Therefore $\varphi(x,y,z) = \frac{1}{2}x^2 y - x\sin(z) + C$ (may assume that $C = 0$).