1. Let $f(x, y) = 2x^3 - 3y$. Determine the line integral $\int_C f(x, y)\, ds$ where $C$ is the curve $y = x^2 + 4$ in the second quadrant from $(-2, 8)$ to $(0, 4)$.

2. Let $f(x, y, z) = z + e^{xy}$. Determine the line integral $\int_C f(x, y, z)\, ds$ where $C$ is the line segment from $(1, 8, 2)$ to $(3, 4, 3)$.

3. Let $\mathbf{F}(x, y, z) = \langle z, -y, x \rangle$. Determine the line integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ where $C$ is the twisted cubic $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ from $(1, 1, 1)$ to $(2, 4, 8)$.

Answers:

1. The curve $C$ is a scalar path. Parameterizing the curve gives $\mathbf{r}(t) = \langle t, t^2 + 4 \rangle$ where $-2 \leq t \leq 0$. Then $\mathbf{r}'(t) = \langle 1, 2t \rangle$, so $|\mathbf{r}'(t)| = \sqrt{1 + 4t^2}$. Restricting $f$ to $C$ yields the function $f(C) = f\big(\mathbf{r}(t)\big) = 2t^3 - 3(t^2 + 4) = 2t^3 - 3t^2 - 12$. Then the line integral is $\int_C f\, ds = \int_a^b f\left(\mathbf{r}(t)\right) |\mathbf{r}'(t)|\, dt = \int_{-2}^{0} (2t^3 - 3t^2 - 12)\sqrt{1 + 4t^2} dt = -108.06$.
2. The curve $C$ is a scalar path. Parameterizing the curve gives $\mathbf{r}(t) = \langle 1, 8, 2 \rangle + \langle 2, -4, 1 \rangle t = \langle 1 + 2t, 8 - 4t, 2 + t \rangle$ where $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle 2, -4, 1 \rangle$, so $|\mathbf{r}'(t)| = \sqrt{2^2 + (-4)^2 + 1^2} = \sqrt{21}$. Restricting $f$ to $C$ yields the function $f(C) = f\big(\mathbf{r}(t)\big) = (2 + t) + e^{(1+2t)(8-4t)} = 2 + t + e^{-8t^2+12t+8}$. Then the line integral is $\int_C f\, ds = \int_a^b f\left(\mathbf{r}(t)\right) |\mathbf{r}'(t)|\, dt = \int_0^1 \left(2 + t + e^{-8t^2+12t+8}\right)\sqrt{21}\, dt = 647{,}298.98$.
3. The curve $C$ is a vector path. Parameterizing the curve gives $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ where $1 \leq t \leq 2$. Then $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. Restricting $\mathbf{F}$ to $C$ yields the function $\mathbf{F}(C) = \mathbf{F}\big(\mathbf{r}(t)\big) = \langle t^3, -t^2, t \rangle$. Then $\mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}'(t) = \langle t^3, -t^2, t \rangle \cdot \langle 1, 2t, 3t^2 \rangle = t^3 - 2t^3 + 3t^3 = 2t^3$. Finally, the line integral is $\int_C f\, ds = \int_C \mathbf{F}\big(\mathbf{r}(t)\big) \cdot d\mathbf{r} = \int_1^2 (2t^3)\, dt = \frac{15}{2} = 7.5$.