1. Evaluate the line integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ where C is from $(4, 0, 0)$ to $(0, 8, 0)$ and then to $(0, 0, 8)$ and then back to $(4, 0, 0)$ along straight line paths and $\mathbf{F}(x, y, z) = \langle -y^2, z^2, x \rangle$.

2. Verify Stokes' Theorem for line the integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$ where C is the circle $x^2 + y^2 = 16$ traversed once around counter-clockwise, $\mathbf{F}(x, y, z) = \langle z^2, 2x, y \rangle$, and the surface S is:
   a) The region contained within the circle in xy-plane.
   b) The paraboloid $z = 16 - x^2 - y^2$.

Answers:

1. Stokes' Theorem applies since this is a closed, simple, positively oriented curve along the boundary of the plane $S: 2x + y + z = 8$ in the first octant. The curl of $\mathbf{F}$ would be

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & z^2 & x \end{vmatrix} = \langle 0 - 2z, -(1-0), 0 - (-2y) \rangle = \langle -2z, -1, 2y \rangle$$

and the normal vector for S will be $z = g(x,y) = 8 - 2x - y \rightarrow \mathbf{n} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle = \langle 2, 1, 1 \rangle$. Instead of parameterizing, we can restrict the function to the surface, so

$$\nabla \times \mathbf{F} = \langle -2z, -1, 2y \rangle \rightarrow \nabla \times \mathbf{F}(g) = \langle -2(8 - 2x - y), -1, 2y \rangle$$

and

$$\nabla \times \mathbf{F}(g) \cdot \mathbf{n} = \langle -2(8 - 2x - y), -1, 2y \rangle \cdot \langle 2,1,1 \rangle = -4(8 - 2x - y) - 1 + 2y = 8x + 6y - 33.$$

The triangle region in the xy-plane below S is bounded by $0 \le x \le 4$ and $0 \le y \le 8 - 2x$. This sets up the surface integral

$$\int_0^4 \int_0^{8-2x} (8x + 6y - 33)\, dy\, dx = -\frac{304}{3} = -101.33.$$

2. Stokes' Theorem applies since this is a closed, simple, positively oriented curve. The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & 2x & y \end{vmatrix} = \langle 1 - 0, -(0 - 2z), 2 - 0 \rangle = \langle 1, 2z, 2 \rangle.$$

a) The normal vector for S is $z = g(x,y) = 0 \rightarrow \mathbf{n} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle = \langle 0, 0, 1 \rangle$. Restricting $\mathbf{F}$ to S gives $\nabla \times \mathbf{F}(g) = \langle 1, 0, 2 \rangle$ and $\nabla \times \mathbf{F}(g) \cdot \mathbf{n} = \langle 1, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2$. The circular region in the xy-plane of S can be converted to polar coordinates and the surface integral is

$$\int_0^{2\pi} \int_0^4 (2) r\, dr\, d\theta = 32\pi = 100.53.$$

b) The normal vector for S is $z = g(x,y) = 16 - x^2 - y^2 \rightarrow \mathbf{n} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle = \langle 2x, 2y, 1 \rangle$. Restricting $\mathbf{F}$ to S gives $\nabla \times \mathbf{F}(g) = \langle 1, 2(16 - x^2 - y^2), 2 \rangle$ and

$$\begin{aligned} \nabla \times \mathbf{F}(g) \cdot \mathbf{n} &= \langle 1, 2(16 - x^2 - y^2), 2 \rangle \cdot \langle 2x, 2y, 1 \rangle \\ &= 2x + 4y(16 - x^2 - y^2) + 2 \\ &= 2x + 64y - 4x^2 y - 4y^3 + 2. \end{aligned}$$

The circular region in the xy-plane below S can be converted to polar coordinates and the surface integral is

$$\begin{aligned} &\int_0^{2\pi} \int_0^4 (2x + 64y - 4x^2 y - 4y^3 + 2)\, dA \\ &= \int_0^{2\pi} \int_0^4 (2r\cos\theta + 64r\sin\theta - 4r^2\cos^2\theta\, r\sin\theta - 4r^3\sin^3\theta + 2)\, r\, dr\, d\theta = 32\pi = 100.53. \end{aligned}$$

- Notice that the integrals for both surfaces are equal, since they both can be evaluated based on the same curve C. If we parameterized C so that $\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 0 \rangle$ with $0 \le t \le 2\pi$, then $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle 0, 8\cos t, 4\sin t \rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle = 32\cos^2 t$ and the line integral is $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'^{(t)} dt = \int_0^{2\pi} (32\cos^2 t)\, dt = 32\pi = 100.53.$