MAT273 **Unit #1 Review**

Show all your work, justify your answers and interpret any applicable solutions using complete sentences. You may use a scientific calculator like the Ti-36x Pro, but not a graphing calculator. (These problems are intended to help you prepare for the test. Do not assume that these are the only problems you need to do. You should also review your notes, homework, quizzes, and other exercises from the text.)

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1. Find the equation of the line passing through $(-2, 1, 0)$ and $(1, 3, 5)$.

2. If $\mathbf{r} = \langle 2z, 4y, 6x \rangle$ and $\mathbf{s} = \langle x, -4, 3z \rangle$, find $\mathbf{r} \cdot \mathbf{s}$ and $\mathbf{r} \times \mathbf{s}$.

3. What is the angle between the vectors $\vec{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\vec{u} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$?

4. Find two unit vectors perpendicular to the vectors $\vec{a} = \langle 3, 1, -6 \rangle$ and $\vec{b} = \langle 1, 8, 5 \rangle$.

5. Find the work done by the force $\vec{F} = 5\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ N over the distance from $(0, 0, 0)$ to $(10, 25, 30)$ m.

6. A 3-pound weight hangs from a rope at the end of a 5-foot rod held at a 45 degree angle above the horizontal. Find the torque on the end of the rod opposite the hanging weight.

7. Consider the curves given by $\mathbf{r}_1(t) = \langle 2t, t^2, 5 \rangle$ and $\mathbf{r}_2(t) = \langle 2, t^2, 5t \rangle$. Determine the following.
   a) Find a vector orthogonal to both $\mathbf{r}_1$ and $\mathbf{r}_2$ at $t = 1$.

   b) Find the angle of intersection between $\mathbf{r}_1$ and $\mathbf{r}_2$ at $t = 1$.

8. Given the space curve $\vec{r}(t) = \langle t, t^3, 3t \rangle$, find the acceleration and velocity vectors.

9. Find the unit tangent and unit normal vectors for $\vec{r}(t) = \langle 3t, 2t^2 \rangle$.

10. Find the tangential and normal components of acceleration for $\vec{r}(t) = \langle 3t, -t, t^2 \rangle$.

11. Find the arc length of the curve given by $\vec{r}(t) = \langle t, \frac{4}{3}t^{\frac{3}{2}}, \frac{1}{2}t^2 \rangle$ from $(0, 0, 0)$ to $\left(2, \frac{8}{3}\sqrt{2}, 2\right)$.

12. Find the curvature of the curve given by $\vec{r}(t) = \langle 2t, t^2, -\frac{1}{3}t^3 \rangle$. Also find the tangential and normal components of acceleration.

13. Given the function $\vec{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\mathbf{k}$, find the following.
    a) $\vec{r}'(t)$

    b) $\vec{r}''(t)$

    c) $\vec{r}'(t) \cdot \vec{r}''(t)$

    d) $\vec{r}'(t) \times \vec{r}''(t)$

14. Find the equation of the plane passing through $(2, 1, 1)$, $(0, 4, 1)$ and $(-2, 1, 4)$.

15. Find the angle between the planes given by $x - 2y + z = 0$ and $2x + 3y - 2z = 0$.

16. Classify the following surfaces and state the center and/or vertices.

a) $x^2 - y = 0$

b) $16x^2 - y^2 + 16z^2 = 4$

c) $x^2 - y^2 + z = 0$

d) $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$

17. Find two unit vectors perpendicular to the plane containing these three points $(2, 5, 1)$, $(-1, 4, 7)$, and $(1, -3, -4)$.

18. What is the angle between the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and the plane $3x + y - 3z = 0$?

19. Find the line of intersection between the two planes $2x + 5y - 2z = 6$ and $z = 2x - 3y + 12$.

Answers:

1. The direction vector will be $\langle 1-(-2), 3-1, 5-0\rangle = \langle 3,2,5\rangle$, so the line is $\vec{r}(t) = \langle -2,1,0\rangle + \langle 3,2,5\rangle t$.

2. If $\mathbf{r} = \langle 2z, 4y, 6x\rangle$ and $\mathbf{s} = \langle x, -4, 3z\rangle$, then $\mathbf{r}\cdot\mathbf{s} = 2xz - 16y + 18xz = 20xz - 16y$ and $\mathbf{r}\times\mathbf{s} = (12yz + 24x)\mathbf{i} - (6z^2 - 6x^2)\mathbf{j} + (-8z - 4xy)\mathbf{k} = \langle 12yz + 24x, -6z2 + 6x2, -8z - 4xy\rangle$.

3. If $\vec{v} = \langle 2,-1,3\rangle$ and $\vec{u} = \langle 3,1,-3\rangle$, then $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-4}{\sqrt{19}\sqrt{14}}$ which results in $\theta = 104.20°$.

4. The cross product $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, so $\vec{a}\times\vec{b} = \langle 53,-21,23\rangle$ and $|\vec{a}\times\vec{b}| = \sqrt{3779}$. Then two unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ would $\frac{1}{\sqrt{3779}}\langle 53,-21,23\rangle$ and $-\frac{1}{\sqrt{3779}}\langle 53,-21,23\rangle$.

5. Work is Force times distance, so $W = \vec{F}\cdot\vec{d} = \langle 5,5,1\rangle\, N \cdot \langle 10,25,30\rangle\, m = 205\, N\cdot m$.

6. We know $\tau = \vec{r}\times\vec{F}$, but cross products are only defined in $\mathbf{R}^3$. So think of $\vec{r}$ and $\vec{F}$ in the xy-plane with z-component of 0. Then $\tau = \vec{r}\times\vec{F} = \langle 5\cos(45°), 5\sin(45°), 0\rangle\, ft \times \langle 0,-3,0\rangle\, lbs = \langle 0,0,-10.6\rangle\, ft\cdot lbs$.

7.
   a) At $t = 1$, the intersection is $\mathbf{r}_1(t) = \mathbf{r}_2(1) = \langle 2,1,5\rangle$. To find an orthogonal vector, we need to find the directional (tangent) vector at $t = 1$. These are $\mathbf{r}_1'(t) = \langle 2, 2t, 0\rangle$ and $\mathbf{r}_2'(t) = \langle 0, 2t, 5\rangle$, so $\mathbf{r}_1'(1) = \langle 2,2,0\rangle$ and $\mathbf{r}_2'(1) = \langle 0,2,5\rangle$.
   Then $\mathbf{r}_1'(1)\times\mathbf{r}_2'(1) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & 0\\ 0 & 2 & 5\end{vmatrix} = \langle 10,-10,4\rangle$.
   b) We can use either the dot product or cross product to find the angle of intersection, so let's use $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$. This means $\theta = \sin^{-1}\left(\frac{|\mathbf{r}_1'(1)\times\mathbf{r}_2'(1)|}{|\mathbf{r}_1'(1)||\mathbf{r}_2'(1)|}\right) = \sin^{-1}\left(\frac{\sqrt{10^2+(-10)^2+4^2}}{\sqrt{2^2+2^2+0^2}\sqrt{0^2+2^2+5^2}}\right) = \sin^{-1}\left(\frac{\sqrt{216}}{\sqrt{8}\sqrt{29}}\right) = 1.3051$.

8. For $\vec{r}(t) = \langle t, t^3, 3t\rangle$, $\vec{v}(t) = \vec{r}'(t) = \langle 1, 3t^2, 3\rangle$ and $\vec{a}(t) = \vec{r}''(t) = \langle 0, 6t, 0\rangle$.

9. For $\vec{r}(t) = \langle 3t, 2t^2 \rangle$, we get $\vec{v}(t) = \langle 3, 4t \rangle$ and $\vec{a}(t) = \langle 0, 4 \rangle$. Then $\mathbf{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 3,4t \rangle}{\sqrt{9+16t^2}} = \frac{1}{\sqrt{9+16t^2}} \langle 3, 4t \rangle$ and so $\mathbf{T}' = \frac{-16t}{(9+16t^2)^{3/2}} \langle 3, 4t \rangle + \frac{1}{\sqrt{9+16t^2}} \langle 0, 4 \rangle = \frac{-16t}{(9+16t^2)^{3/2}} \langle 3, 4t \rangle + \frac{9+16t^2}{(9+16t^2)^{3/2}} \langle 0, 4 \rangle = \frac{\langle -48t, 36 \rangle}{(9+16t^2)^{3/2}} = \frac{12}{(9+16t^2)^{3/2}} \langle -4t, 3 \rangle$, which gives $\mathbf{N}(t) = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{\frac{12}{(9+16t^2)^{3/2}} \langle -4t, 3 \rangle}{\frac{12}{(9+16t^2)^{3/2}} \sqrt{16t^2+9}} = \frac{\langle -4t, 3 \rangle}{\sqrt{16t^2+9}} = \frac{1}{\sqrt{16t^2+9}} \langle -4t, 3 \rangle$.

10. For $\vec{r}(t) = \langle 3t, -t, t^2 \rangle$, we get $\vec{v}(t) = \langle 3, -1, 2t \rangle$ and $\vec{a}(t) = \langle 0, 0, 2 \rangle$. Then $a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{4t}{\sqrt{10+4t^2}}$ and $a_N = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|} = \frac{|\langle -2, -6, 0 \rangle|}{\sqrt{10+4t^2}} = \frac{2\sqrt{10}}{\sqrt{10+4t^2}}$.

11. For $\vec{r}(t) = \langle t, \frac{4}{3} t^{3/2}, \frac{1}{2} t^2 \rangle$, the point $(0, 0, 0)$ corresponds to $t = 0$ and the point $\left(2, \frac{8}{3}\sqrt{2}, 2\right)$ corresponds to $t = 2$. The derivative of $\vec{r}(t)$ is $\vec{v}(t) = \langle 1, 2t^{1/2}, t \rangle$. Then $s = \int_a^b |\vec{v}'| dt = \int_0^2 \sqrt{1 + 4t + t^2} dt \approx 4.816$.

12. For $\vec{r}(t) = \langle 2t, t^2, -\frac{1}{3} t^3 \rangle$, we get $\vec{v}(t) = \langle 2, 2t, -t^2 \rangle$ and $\vec{a}(t) = \langle 0, 2, -2t \rangle$. Then $\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|\langle -2t^2, 4t, 4 \rangle|}{\left(\sqrt{4+4t^2+t^4}\right)^3} = \frac{\sqrt{4t^4+16t^2+16}}{(t^2+2)^3} = \frac{2}{(t^2+2)^2}$. Also, $a_T = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|} = \frac{4t+2t^3}{t^2+2} = 2t$ and $a_N = \kappa |\vec{v}|^2 = \frac{2}{(t^2+2)^2} (t^2 + 2)^2 = 2$.

13.
   a) $\vec{r}'(t) = -\sin t \, \mathbf{i} + \cos t \, \mathbf{j} + 2\mathbf{k}$
   b) $\vec{r}''(t) = -\cos t \, \mathbf{i} - \sin t \, \mathbf{j}$
   c) $\vec{r}'(t) \cdot \vec{r}''(t) = \sin t \cos t - \sin t \cos t = 0$
   d) $\vec{r}'(t) \times \vec{r}''(t) = 2 \sin t \, \mathbf{i} - 2 \cos t \, \mathbf{j} + \mathbf{k}$

14. Use the points to form the vectors $\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1 \rangle = \langle -2, 3, 0 \rangle$ and $\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1 \rangle = \langle -4, 0, 3 \rangle$. Then the normal vector is $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} = \langle 9 - 0, -(-6 - 0), 0 - (-12) \rangle = \langle 9, 6, 12 \rangle$. Then the equation of the plane is $9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \rightarrow 9x + 6y + 12z = 36 \rightarrow 3x + 2y + 4z = 12$.

15. The normal vectors of the planes are $\mathbf{n}_1 = \langle 1, -2, 1 \rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2 \rangle$. The angle between the planes is the angle between the vectors, so $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-6}{\sqrt{6}\sqrt{17}}\right) = 126.45°$, or an acute angle of $53.55°$.

16.

a) Write as $y = x^2$. Since z is unspecified, this is a parabolic cylinder going up and down along the z direction. It opens along the $+y$-axis and its "vertex" is the z-axis

b) The equation $16x^2 - y^2 + 16z^2 = 4$ has a standard form of $4x^2 - \frac{y^2}{4} + 4z^2 = 1$. It is a hyperboloid of one sheet opening along the y-axis. The center is $(0,0,0)$ and the vertices are at $\left(\pm\frac{1}{2}, 0, 0\right)$ and $\left(0, 0, \pm\frac{1}{2}\right)$.

c) The equation $x^2 - y^2 + z = 0$ should be written as $z = y^2 - x^2$. This is a hyperbolic paraboloid (saddle) opening in the z-direction. The center is $(0,0,0)$.

d) Use completing-the-square to rewrite $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$ as $(x-1)^2 + \frac{9(y-2)^2}{16} + z^2 = 1$. This is an ellipsoid with center at $(1,2,0)$ and vertices at $(0,2,0)$, $(2,2,0)$, $\left(1, \frac{2}{3}, 0\right)$, $\left(1, \frac{10}{3}, 0\right)$, $(1,2,-1)$, and $(1,2,1)$.

17. Use the points to form the vectors $\mathbf{u} = \langle -1-2, 4-5, 7-1\rangle = \langle -3, -1, 6\rangle$ and $\mathbf{v} = \langle 1-2, -3-5, -4-1\rangle = \langle -1, -8, -5\rangle$. Both $\mathbf{u}$ and $\mathbf{v}$ are contained in the plane, so $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -1 & 6 \\ -1 & -8 & -5 \end{vmatrix} = \langle 5-(-48), -\big(15-(-6)\big), 24-1\rangle = \langle 53, -21, 23\rangle$ will be perpendicular to the plane. Then the perpendicular unit vectors are $\pm\frac{1}{\sqrt{3779}}\langle 53, -21, 23\rangle$.

18. The normal vector for the plane is $\mathbf{n} = \langle 3, 1, -3\rangle$. So the angle between the line and the normal vector is $\theta = \cos^{-1}\left(\frac{\langle 3,1,-3\rangle\cdot\langle 2,-1,3\rangle}{\sqrt{19}\sqrt{14}}\right) = \cos^{-1}\left(\frac{-4}{\sqrt{19}\sqrt{14}}\right) = 104.2°$, or an acute angle of 75.8°. That leaves 14.2^0 between the line and the plane.

19. The line of intersection must be contained in both planes and is therefore perpendicular to both normal vectors. Then $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & -2 \\ -2 & 3 & 1 \end{vmatrix} = \langle 5-(-6), -(2-4), 6-(-10)\rangle = \langle 11, 2, 16\rangle$. So the line is of the form $L = \langle x_0, y_0, z_0\rangle + \langle 11, 2, 16\rangle t$, for some point (x_0, y_0, z_0). To find a point on the line, let $y = 0$ and then the plane equations become a system of equations $\begin{cases} 2x - 2z = 6 \\ -2x + z - 12 \end{cases}$ whose solution is $z = -18$ and $x = -15$. Then the equation of the line is $L = \langle -15, 0, -18\rangle + \langle 11, 2, 16\rangle t$.