Chapter 2
Revised 6/25/2025

MTH 201

Worksheet Answers

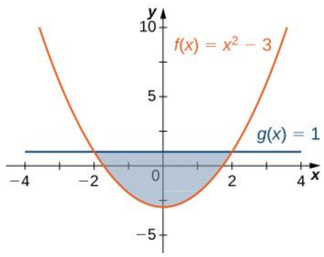
Problems found in OpenStax Calculus Volume II

Sections:

- 2.1 - Bounded Area
- 2.2 - Volume by Slicing
- 2.3 - Shells
- 2.4 - Arc Length and Surface Area
- 2.5 - Physical Applications
- 2.9 - Hyperbolic Trig Functions

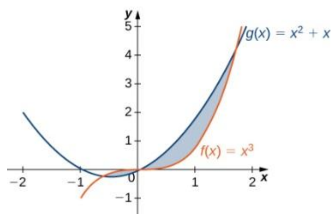
1. (2.1) Determine the area of the shaded region shown.

a.



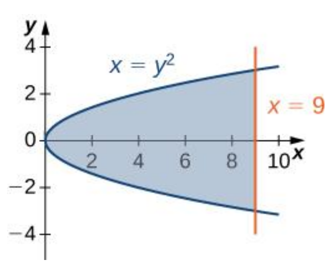
$$\int_{-2}^{2}\left(1-(x^2-3)\right)dx=\int_{-2}^{2}(4-x^2)dx=\left[4x-\frac{1}{3}x^3\right]_{-2}^{2}$$
$$=\frac{16}{3}-\frac{-16}{3}=\frac{32}{3}\ units^2$$

b.



Find where the two curves intersect:
$$x^2+x=x^3$$
$$x^3-x^2-x=0$$
$$x(x^2-x-1)=0$$
$$x=0, x=\frac{1-\sqrt{5}}{2}, x=\frac{\sqrt{5}+1}{2}$$
Set up the integral and evaluate:
$$A=\int_{-\frac{\sqrt{5}+1}{2}}^{0}\left(x^3-(x^2+x)\right)dx+\int_{0}^{\frac{\sqrt{5}+1}{2}}\left((x^2+x)-x^3\right)dx$$
$$=\int_{\frac{1-\sqrt{5}}{2}}^{0}(x^3-x^2-x)dx+\int_{0}^{\frac{\sqrt{5}+1}{2}}(-x^3+x^2+x)dx$$
$$=\left[\frac{1}{4}x^4-\frac{1}{3}x^3-\frac{1}{2}x^2\right]_{\frac{-\sqrt{5}+1}{2}}^{0}+\left[-\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2\right]_{0}^{\frac{\sqrt{5}+1}{2}}$$
$$-\left(\frac{5\sqrt{5}-13}{24}\right)+\left(\frac{5\sqrt{5}+13}{24}\right)$$
$$=\frac{13}{12}\ units^2$$

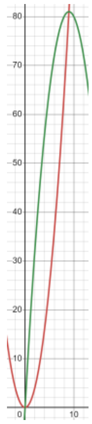
c.



If integrated over the y-axis, you only need one integral.
Find where the curve intersects the line:
$$y^2=9\Rightarrow y=\pm 3$$
Set up and evaluate the integral:
$$A=\int_{-3}^{3}(9-y^2)dy=\left[9y-\frac{1}{3}y^3\right]_{-3}^{3}=18-(-18)$$
$$=36\ units^2$$

2. (2.1) Graph the equations and determine the area between the two curves.

a. $y = x^2$ and $y = -x^2 + 18x$



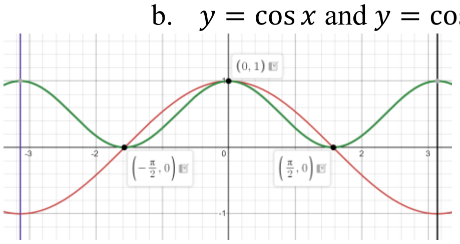
Find where the two curves intersect:

$$x^2 = -x^2 + 18x$$
$$2x^2 = 18x$$
$$2x^2 - 18x = 0$$
$$x(2x - 18) = 0$$
$$x = 0, x = 9$$

Set up and evaluate the integral:

$$A = \int_0^9 (-x^2 + 18x - x^2)dx = \int_0^9 (-2x^2 + 18x)dx$$
$$= \left[-\frac{2}{3}x^3 + 9x^2\right]_0^9$$
$$= 243\ units^2$$

b. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$



It looks like we need four separate integrals for this, but since the functions are symmetrical, we can take the area of the right side and multiply it by 2.

$$A = 2\left(\int_0^{\frac{\pi}{2}} (\cos x - \cos^2 x)dx + \int_{\frac{\pi}{2}}^{\pi} (\cos^2 x - \cos x)dx\right)$$

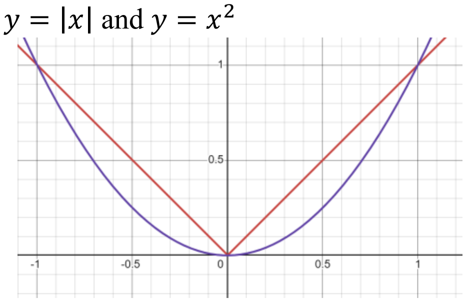
Before chapter 3 you cannot integrate $\cos^2 x$ as it is. Use the double angle identity.

$$A = 2\left([\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1 + \cos(2x)}{2}dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1 + \cos(2x)}{2}dx - [\sin x]_{\frac{\pi}{2}}^{\pi}\right)$$

I'm skipping some of the integration steps, but you just have to split the fraction from the double angle identity, integrate, then get a common denominator.

$$= 2\left(2 - \left[\frac{2x + \sin(2x)}{4}\right]_0^{\frac{\pi}{2}} + \left[\frac{2x + \sin(2x)}{4}\right]_{\frac{\pi}{2}}^{\pi}\right)$$
$$= 2\left(2 - \frac{\pi}{4} + \frac{\pi}{4}\right)$$
$$= 4\ units^2$$

c. $y = |x|$ and $y = x^2$



These areas are once again symmetric like in (b). I omit the absolute value sign because on the right side, $|x| = x$.

$$A = 2\int_0^1 (x - x^2)dx = 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$
$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\ units^2$$

d. $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$

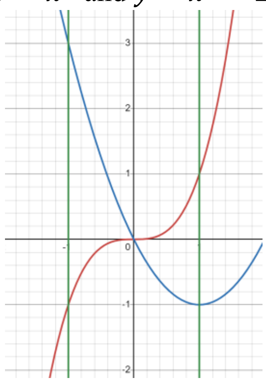
If we integrate over the y-axis we only need one integral.
See where the functions intersect:

$$12 - x = \sqrt{x}$$
$$x = 9 \Rightarrow y = 3$$

The functions can be rewritten as $x = y^2$ and $x = 12 - y$.

$$A = \int_1^3 (12 - y - y^2)dy$$
$$= \left[12y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_1^3 = \frac{45}{2} - \frac{67}{6} = \frac{34}{3}\ units^2$$

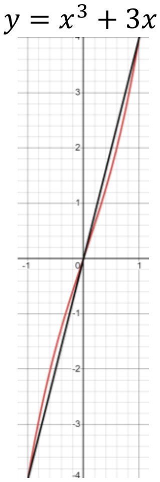
e. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1,1]$



The integral needs to be split again.

$$A = \int_{-1}^{0} (x^2 - 2x - x^3)dx + \int_0^1 (x^3 - x^2 + 2x)\,dx$$
$$= \left[\frac{1}{3}x^3 - x^2 - \frac{1}{4}x^4\right]_{-1}^{0} + \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + x^2\right]_0^1$$
$$= \frac{19}{12} + \frac{11}{12} = \frac{5}{2}\ units^2$$

f. $y = x^3 + 3x$ and $y = 4x$



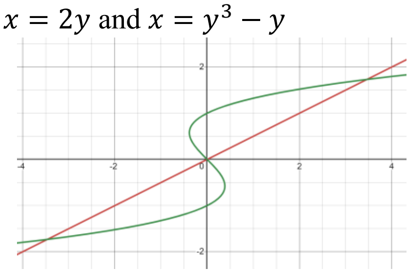
Find where the functions intersect:

$$x^3 + 3x = 4x$$
$$x^3 = x$$
$$x = 0, x = 1$$

The functions are symmetric, so double one of the areas.

$$A = 2\int_0^1 (4x - x^3 - 3x)dx = 2\left[2x^2 - \frac{1}{4}x^4 - \frac{3}{2}x^2\right]_0^1 = \frac{1}{2}\ units^2$$

g. $x = 2y$ and $x = y^3 - y$



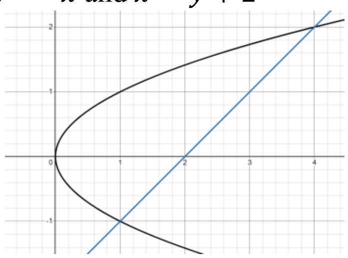
Find the y-values where the functions intersect:

$$2y = y^3 - y$$
$$2 = y^2 - 1$$
$$y^2 = 3$$
$$y = \pm\sqrt{3}$$

Integrate over the y-axis. These curves are also symmetric.

$$A = 2\int_0^{\sqrt{3}} \left(2y - (y^3 - y)\right)dy = 2\int_0^{\sqrt{3}} (3y - y^3)dy$$
$$= 2\left[\frac{3}{2}y^2 - \frac{1}{4}y^4\right]_0^{\sqrt{3}}$$
$$= \frac{9}{2}\ units^2$$

h. $y^2 = x$ and $x = y + 2$



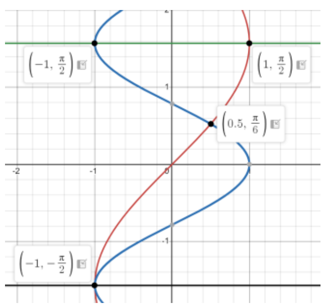
Find the y-values where the functions intersect:

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$y = -1, y = 2$$

This again will be integrated over the y-axis.

$$A = \int_{-1}^{2} (y + 2 - y^2)dy = \left[\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right]_{-1}^{2}$$
$$= \frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{9}{2}\ units^2$$

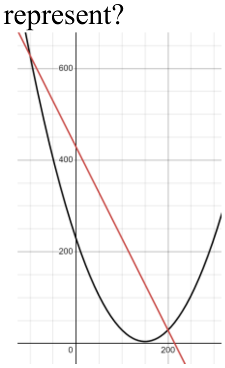
i. $x = \sin y$, $x = \cos(2y)$, $y = \frac{\pi}{2}$, and $y = -\frac{\pi}{2}$



The blue curve is $\cos(2y)$ because of the shorter wavelength.
These areas must be integrated separately across the y-axis.

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos(2y) - \sin y)dy + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin y - \cos(2y))dy$$
$$= \left[\frac{1}{2}\sin(2y) + \cos y\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}} + \left[-\cos y - \frac{1}{2}\sin(2y)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \left(\frac{3\sqrt{3}}{4} - 0\right) + \left(0 - \frac{-3\sqrt{3}}{4}\right)$$
$$= \frac{3\sqrt{3}}{2}\ units^2$$

3. (2.1) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?



Find where the two curves intersect:

$$0.01x^2 - 3x + 229 = 429 - 2x$$
$$0.01x^2 - x - 200 = 0$$
$$\cancel{x = -100}, x = 200$$

The area between the two curves is a basic integral over the x-axis.

$$A = \int_0^{200} \left(429 - 2x - (0.01x^2 - 3x + 229)\right)dx$$

$$= \int_0^{200} (-0.01x^2 + x + 200)dx = \left[-\frac{1}{300}x^3 + \frac{1}{2}x^2 + 200x\right]_0^{200}$$

$$= \frac{100000}{3} - 0 \approx \$33333.33$$

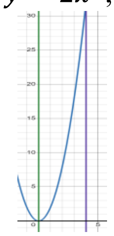
How do we know the units are dollars and what the area represents? Marginal cost and revenue have rate units (like dollars *per* phone). If you multiply the units of the x and y axes to get the units of area:

$$\frac{\$}{phone} \times phone = \$$$

The units are in dollars. The area integral is summing the infinitesimal differences between revenue and cost. Revenue minus cost equals profit, so the area represents the profit of selling 200 cell phones.

4. (2.2) Graph the region bounded by the two curves and find the volume when the region is revolved about the x-axis.

a. $y = 2x^2$, $x = 0$, $x = 4$, and $y = 0$

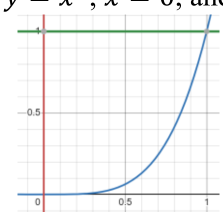


$$V = \pi \int_0^4 (2x^2)^2 dx = \pi \int_0^4 4x^4 dx = \pi \left[\frac{4}{5}x^5\right]_0^4$$

$$= \pi \cdot \frac{4}{5} \cdot 4^5$$

$$= \frac{4096\pi}{5}\ units^3$$

b. $y = x^4$, $x = 0$, and $y = 1$ for $x \geq 0$



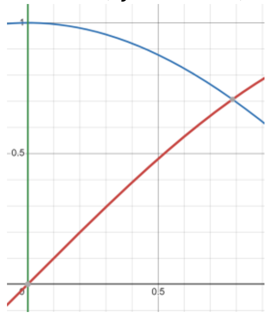
Use the washer method:

$$V = \pi \int_0^1 (1^2 - (x^4)^2)dx = \pi \int_0^1 (1 - x^8)dx$$

$$= \pi \left[x - \frac{1}{9}x^9\right]_0^1 = \pi\left(1 - \frac{1}{9}\right)$$

$$= \frac{8\pi}{9}\ units^3$$

c. $y = \sin x$, $y = \cos x$, and $x = 0$



You should already know that $\sin x$ and $\cos x$ intersect at $x = \frac{\pi}{4}$.

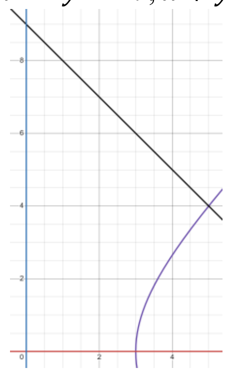
Use the washer method:

$$V = \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)dx$$

Use the double angle identity to rewrite the integrand.

$$V = \pi \int_0^{\frac{\pi}{4}} \cos(2x)\, dx = \frac{\pi}{2}\sin(2x)\, \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{2}\sin\left(\frac{2\pi}{4}\right) = \frac{\pi}{2}\ units^3$$

d. $x^2 - y^2 = 9$, $x + y = 9$, $y = 0$, and $x = 0$



Find the intersection between $x^2 - y^2 = 9$ and $y = 0$:

$$x^2 - 0 = 9$$
$$\cancel{x = -3}, x = 3$$

Finding the intersection between $x^2 - y^2 = 9$ and $x + y = 9$ requires solving the system of equations. There are many ways to do it but I will isolate x from the first equation.

$$x^2 - y^2 = 9 \Rightarrow x^2 = 9 + y^2 \Rightarrow x = \sqrt{y^2 + 9}$$

Then plug it into the linear equation:

$$y + \sqrt{y^2 + 9} = 9$$
$$y^2 + 9 = (9 - y)^2$$
$$y^2 + 9 = 81 - 18y + y^2$$
$$9 = 81 - 18y$$
$$18y - 72 = 0$$
$$y = 4$$
$$x + 4 = 9 \Rightarrow x = 5$$

This volume calculation requires two separate integrals. The first one uses the disk method and the second uses the washer method. Since we are rotating about the x-axis we will make everything in terms of x.

$$V = \pi \int_0^3 (9 - x)^2 dx + \pi \int_3^5 \left((9 - x)^2 - \left(\sqrt{x^2 - 9}\right)^2\right) dx$$

For the first integral:

$$\pi \int_0^3 (9 - x)^2 dx$$
$$u = 9 - x \Rightarrow du = -dx$$
$$-\pi \int_9^6 u^2 du = -\left[\frac{1}{3}u^3\right]_9^6 = -\pi(72 - 243) = 171\pi$$

The second integral can be split into two integrals:

$$\pi \int_3^5 \left((9 - x)^2 - \left(\sqrt{x^2 - 9}\right)^2\right) dx$$
$$= \pi \int_3^5 (9 - x)^2 dx - \pi \int_3^5 (x^2 - 9) dx$$

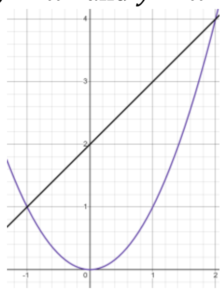
We already integrated the first part and the second part is easy.

$$-\pi\left[\frac{1}{3}u^3\right]_6^4 - \pi\left[\frac{1}{3}x^3 - 9x\right]_3^5 = -\pi\left(\frac{64}{3} - 72\right) - \pi\left(-\frac{10}{3} - (-18)\right)$$
$$= 36\pi$$

So the overall integral is:

$$171\pi + 36\pi = 207\pi\ units^3$$

e. $y = x^2$ and $y = x + 2$



Find where the functions intersect:

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$x = -1, x = 2$$

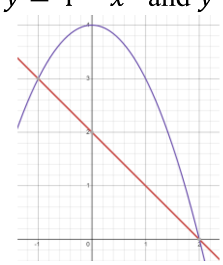
Now set up the integral:

$$V = \pi \int_{-1}^{2} ((x+2)^2 - (x^2)^2)dx$$

You can split the integral but I will combine everything into one big polynomial.

$$\pi \int_{-1}^{2} (-x^4 + x^2 + 4x + 4)dx = \pi \left[-\frac{1}{5}x^5 + \frac{1}{3}x^3 + 2x^2 + 4x\right]_{-1}^{2}$$
$$= \pi\left(\frac{184}{15} - \frac{-32}{5}\right) = \frac{72\pi}{5}\ units^3$$

f. $y = 4 - x^2$ and $y = 2 - x$



Find where the functions intersect:

$$4 - x^2 = 2 - x$$
$$x^2 - x - 2 = 0$$
$$x = -1, x = 2$$

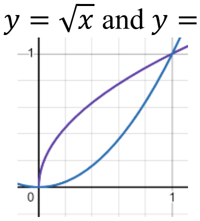
Now set up the interval:

$$V = \pi \int_{-1}^{2} ((4 - x^2)^2 - (2 - x)^2)dx$$

Once again, I'm combining everything into one polynomial:

$$\pi \int_{-1}^{2} \left(x^4 - 8x^2 + 16 - (x^2 - 4x + 4)\right)dx$$
$$= \pi \int_{-1}^{2} (x^4 - 9x^2 + 4x + 12)dx = \pi \left[\frac{1}{5}x^5 - 3x^3 + 2x^2 + 12x\right]_{-1}^{2}$$
$$= \pi\left(\frac{72}{5} - \frac{-36}{5}\right) = \frac{108\pi}{5}\ units^3$$

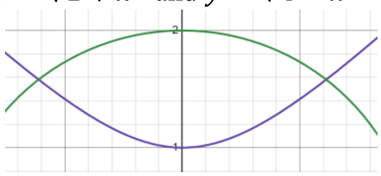
g. $y = \sqrt{x}$ and $y = x^2$



We should not have to do math to figure out that the functions intersect at $x = 0$ and $x = 1$.

$$V = \pi \int_{0}^{1} \left(\left(\sqrt{x}\right)^2 - (x^2)^2\right)dx = \pi \int_{0}^{1} (-x^4 + x)dx$$
$$= \pi \left[-\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_{0}^{1} = \pi\left(-\frac{1}{5} + \frac{1}{2}\right) = \frac{3\pi}{10}\ units^3$$

h. $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$



Find where the functions intersect:

$$\sqrt{1 + x^2} = \sqrt{4 - x^2}$$
$$1 + x^2 = 4 - x^2$$
$$2x^2 = 3$$
$$x = \pm\sqrt{\frac{3}{2}}$$

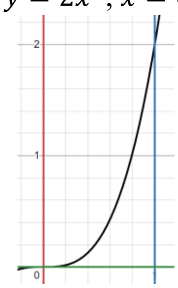
$$V = \pi \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} \left(\left(\sqrt{4 - x^2}\right)^2 - \left(\sqrt{1 + x^2}\right)^2\right) dx$$
$$= \pi \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} (|4 - x^2| - |1 + x^2|) dx$$

We can ignore the absolute value signs because both functions are positive on the interval.

$$\pi \int_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}} (-2x^2 + 3) dx = \pi \left[-\frac{2}{3}x^3 + 3x\right]_{-\sqrt{\frac{3}{2}}}^{\sqrt{\frac{3}{2}}}$$
$$= \pi\left(\sqrt{6} - (-\sqrt{6})\right) = 2\sqrt{6}\pi\ units^3$$

5. (2.2) Graph the region bounded by the two curves and find the volume when the region is revolved about the y-axis.

a. $y = 2x^3$, $x = 0$, $x = 1$, and $y = 0$



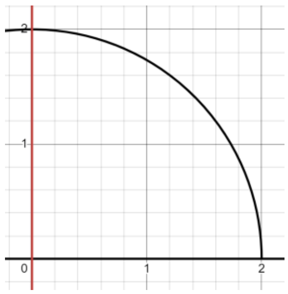
Rewrite the function in terms of y:

$$x = \sqrt[3]{\frac{y}{2}}$$

Set up and evaluate the integral:

$$V = \pi \int_0^2 \left((1)^2 - \left(\sqrt[3]{\frac{y}{2}}\right)^2\right) dy = \pi \int_0^2 \left(1 - \left(\frac{y}{2}\right)^{\frac{2}{3}}\right) dy$$
$$= \pi \int_0^2 \left(1 - 2^{-\frac{2}{3}} y^{\frac{2}{3}}\right) dy = \pi \left[y - \frac{3}{5 \cdot 2^{\frac{2}{3}}} y^{\frac{5}{3}}\right]_0^2 = \pi\left(\frac{4}{5} - 0\right)$$
$$= \frac{4\pi}{5}\ units^3$$

b. $y = \sqrt{4 - x^2}$, $y = 0$, and $x = 0$



If you recognize this volume as half the volume of a sphere of radius 2, then you can use the equation for the volume of a sphere and divide it by 2.

$$V = \frac{1}{2} \cdot \frac{4}{3} \pi (2)^2 = \frac{16\pi}{3}\ units^3$$

To solve by integration, start by solving for x:

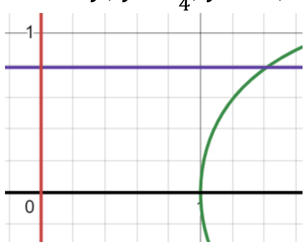
$$x = \sqrt{4 - y^2}$$

This represents the entire right side of the circle but integrating over $0 \le y \le 2$ gives us just the upper part. We can also just use the disk method.

$$V = \pi \int_0^2 \left(\sqrt{4 - y^2}\right)^2 dy = \pi \int_0^2 (4 - y^2) dy$$

$$= \pi \left[4y - \frac{1}{3} y^3\right]_0^2 = \pi \left(8 - \frac{8}{3}\right) = \frac{16\pi}{3}\ units^3$$

c. $x = \sec y$, $y = \frac{\pi}{4}$, $y = 0$, and $x = 0$

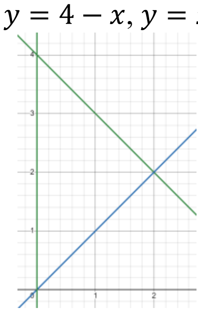


We can also use the disk method for this problem.

$$V = \pi \int_0^{\frac{\pi}{4}} \sec^2 y\, dy = \pi \cdot \tan y \,|_0^{\frac{\pi}{4}} = \pi(1 - 0)$$

$$= \pi\ units^3$$

d. $y = 4 - x$, $y = x$, and $x = 0$

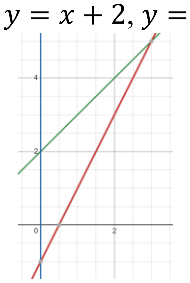


The easy way to do this is to use the symmetry argument and just evaluate the volume from 0 to 2 and multiply it by 2. The hard way is to use two separate integrals, one over $0 \le y \le 2$ and another over $2 \le y \le 4$. I will show the easy way, because the integrals are the same difficulty, just longer.

$$V = 2\pi \int_0^2 y^2 dy = 2\pi \cdot \frac{1}{3} y^3 |_0^2 = \frac{2\pi}{3} \cdot 8$$

$$= \frac{16\pi}{3}\ units^3$$

e. $y = x + 2$, $y = 2x - 1$, and $x = 0$



Solve for x in both functions:

$$x = y - 2$$

$$x = \frac{y}{2} + \frac{1}{2}$$

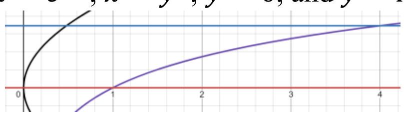
This requires two integrals, one of which uses the washer method.

$$V = \pi \int_{-1}^{2} \left(\frac{1}{2} y + \frac{1}{2}\right)^2 dy + \pi \int_2^5 \left(\left(\frac{1}{2} y + \frac{1}{2}\right)^2 - (y - 2)^2\right) dy$$

$$= \pi \int_{-1}^{2} \left(\frac{1}{4} y^2 + \frac{1}{2} y + \frac{1}{4}\right) dy + \pi \int_2^5 \left(\frac{1}{4} y^2 + \frac{1}{2} y + \frac{1}{4} - (y^2 - 4y + 4)\right)$$

$$= \pi\left[\frac{1}{12}y^3 + \frac{1}{4}y^2 + \frac{1}{4}y\right]_0^2 + \pi\int_2^5 \left(-\frac{3}{4}y^2 + \frac{9}{2}y - \frac{15}{4}\right)dy$$
$$= \pi\left(\frac{13}{6} - \frac{-1}{12}\right) + \pi\left[-\frac{1}{4}y^3 + \frac{9}{4}y^2 - \frac{15}{4}y\right]_2^5$$
$$= \frac{9\pi}{4} + \frac{27\pi}{4} = 9\pi\ units^3$$

f. $x = e^{2y}$, $x = y^2$, $y = 0$, and $y = \ln 2$



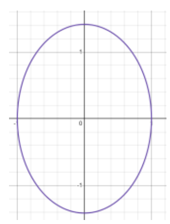
This is a regular disk method integral:

$$V = \pi\int_0^{\ln 2} \left((e^{2y})^2 - (y^2)^2\right)dy$$
$$= \pi\int_0^{\ln 2} (e^{4y} - y^4)dy = \pi\left[\frac{1}{4}e^{4y} - \frac{1}{5}y^5\right]_0^{\ln 2}$$
$$= \pi\left(\frac{1}{4}e^{4\ln 2} - \frac{1}{5}(\ln 2)^5 - \left(\frac{1}{4}\right)\right)$$
$$= \frac{\pi}{20}(75 - 4\ln^5 2)\ units^3$$

You probably don't have to simplify that much.

6. (2.2) Find the volume of a generic ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolved around the x-axis.

Here is one example of an ellipse:



All we need to do is revolve the top part around the x-axis. Start by solving for y:

$$y = \pm\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)} = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

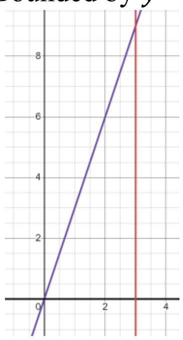
Then set up the integral. We only need to integrate with the positive case because it's the top part. The width of the ellipse is $2a$.

$$V = \pi\int_{-a}^{a} \left(b\sqrt{1 - \frac{x^2}{a^2}}\right)^2 dx = b^2\pi\int_{-a}^{a}\left(1 - \frac{x^2}{a^2}\right)dx = b^2\pi\left[x - \frac{1}{3a^2}x^3\right]_{-a}^{a}$$
$$= b^2\pi\left(\left(a - \frac{1}{3a^2}a^3\right) - \left(-a - \frac{1}{3a^2}(-a^3)\right)\right)$$
$$= b^2\pi\left(\left(a - \frac{a}{3}\right) - \left(\frac{a}{3} - a\right)\right) = b^2\pi\left(\frac{2a}{3}\right)$$

$$\frac{4ab^2\pi}{3}\ units^3$$

7. (2.3) Find the volume generated when the region between the two curves is rotated about the given axis using the shell method.

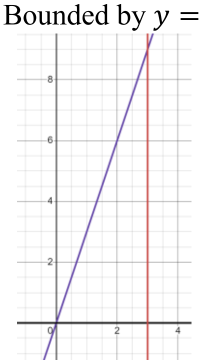
a. Bounded by $y = 3x$, $y = 0$, and $x = 3$ rotated about the y-axis.



The radius of the shells is x and the heights of the shells are $3x$.

$$V = 2\pi\int_0^3 x(3x)dx = 6\pi\int_0^3 x^2dx = 6\pi\left[\frac{1}{3}x^3\right]_0^3 = 2\pi(27)$$
$$= 54\pi\ units^3$$

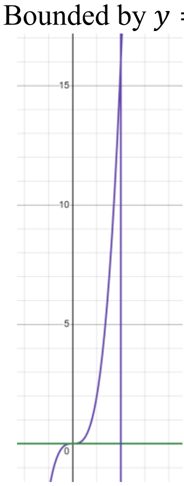
b. Bounded by $y = 3x$, $y = 0$, and $x = 3$ rotated about the x-axis.



The radius of the shells is y and the heights are $3 - \frac{y}{3}$. Don't worry about 3 being an x value, we only care about the length measurement.

$$V = 2\pi\int_0^9 y\left(3 - \frac{y}{3}\right)dy = 2\pi\int_0^9 \left(3y - \frac{y^2}{3}\right)dy$$
$$= 2\pi\left[\frac{3}{2}y^2 - \frac{1}{9}y^3\right]_0^9 = 2\pi\left(\frac{81}{2}\right)$$
$$= 81\pi\ units^3$$

c. Bounded by $y = 2x^3$, $y = 0$, and $x = 2$ rotated about the x-axis.

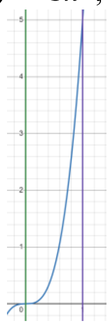


The radius of the shells is y and the heights are $2 - \sqrt[3]{\frac{y}{2}}$.

$$v = 2\pi\int_0^{16} y\left(2 - \left(\frac{y}{2}\right)^{\frac{1}{3}}\right)dy = 2\pi\int_0^{16}\left(2y - \frac{1}{2^{\frac{1}{3}}}y^{\frac{4}{3}}\right)dy$$
$$= 2\pi\left[y^2 - \frac{3}{7\sqrt[3]{2}}y^{\frac{7}{3}}\right]_0^{16} = 2\pi\left(\frac{256}{7}\right)$$
$$= \frac{512\pi}{7}\ units^3$$

8. (2.3) Find the volume when rotating the region between the given curve and the x-axis around the y-axis using the shell method.

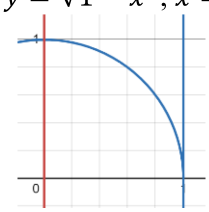
a. $y = 5x^3$, $x = 0$, and $x = 1$



The radius of the shells is x and the heights are $5x^3$.

$$V = 2\pi \int_0^1 x(5x^3)dx = 2\pi \int_0^1 5x^4 dx = 2\pi[x^5]_0^1$$
$$= 2\pi(1)$$
$$= 2\pi\ units^3$$

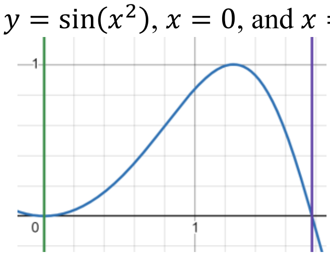
b. $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$



The radius of the shells is x and the heights are $\sqrt{1 - x^2}$.

$$V = 2\pi \int_0^1 x\sqrt{1 - x^2}dx$$
$$u = 1 - x^2 \Rightarrow du = -2xdx$$
$$V = -\pi \int_1^0 \sqrt{u}du = \pi \int_0^1 \sqrt{u}du$$
$$= \pi \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$
$$= \frac{2\pi}{3}\ units^3$$

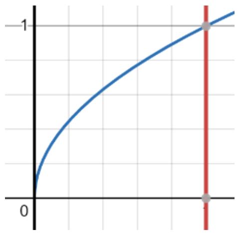
c. $y = \sin(x^2)$, $x = 0$, and $x = \sqrt{\pi}$



The radius of the shells is x and the heights are $\sin(x^2)$.

$$V = 2\pi \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$$
$$u = x^2 \Rightarrow du = 2x \cdot dx$$
$$V = \pi \int_0^{\pi} \sin u\, du = \pi[-\cos u]_0^{\pi}$$
$$= \pi\big(1 - (-1)\big) = 2\pi\ units^3$$

d. $y = \sqrt{x}$, $x = 0$, and $x = 1$

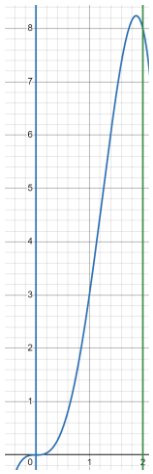


The radius of the shells is x and the heights are $\sqrt{x}$.

$$V = 2\pi \int_0^1 x\sqrt{x}dx = 2\pi \int_0^1 x^{\frac{3}{2}}dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$

$$= \frac{4\pi}{5}\ units^3$$

e. $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$



The radius of the shells is x and the heights are $5x^3 - 2x^4$.

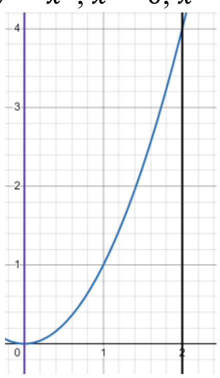
$$V = 2\pi \int_0^2 x(5x^3 - 2x^4)\,dx = 2\pi \int_0^2 (5x^4 - 2x^5)dx$$

$$= 2\pi \left[x^5 - \frac{1}{3}x^6\right]_0^2 = 2\pi\left(\frac{32}{3}\right)$$

$$= \frac{64}{3}\pi\ units^3$$

9. (2.3) Find the volume when rotating the region between the given curve and $y = 0$ around the x-axis using the shell method.

a. $y = x^2$, $x = 0$, $x = 2$, and the x-axis



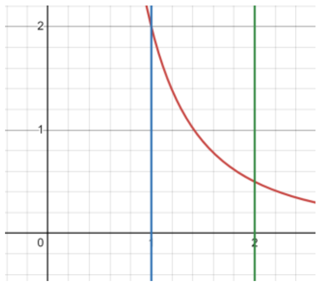
The radius of the shells is y and the heights are $2 - \sqrt{y}$.

$$V = 2\pi \int_0^4 y(2 - \sqrt{y})dy = 2\pi \int_0^4 (2y - y\sqrt{y})dy$$

$$= 2\pi \int_0^4 \left(2y - y^{\frac{3}{2}}\right)dy = 2\pi \left[y^2 - \frac{2}{5}y^{\frac{5}{2}}\right]_0^4 = 2\pi\left(\frac{16}{5}\right)$$

$$= \frac{32\pi}{5}\ units^3$$

b. $y = \frac{2}{x^2}$, $x = 1$, $x = 2$, and the x-axis



You need two separate integrals, one for the top and one for the bottom part.
Top:
The radius of the shells is y and the heights are $\sqrt{\frac{2}{y}} - 1$.
$y = \frac{2}{x^2}$ intersects $x = 2$ at $y = \frac{1}{2}$.

$$V = 2\pi \int_{\frac{1}{2}}^{2} y\left(\sqrt{\frac{2}{y}} - 1\right) dy = 2\pi \int_{\frac{1}{2}}^{2} \left(\frac{\sqrt{2}y}{y^{\frac{1}{2}}} - y\right) dy$$

$$= 2\pi \int_{\frac{1}{2}}^{2} \left(\sqrt{2y} - y\right) dy = 2\pi \left[\frac{2\sqrt{2}}{3} y^{\frac{3}{2}} - \frac{1}{2} y^2\right]_{\frac{1}{2}}^{2}$$

$$= 2\pi \left(\frac{2}{3} - \frac{5}{24}\right) = \frac{11\pi}{12}\ units^3$$

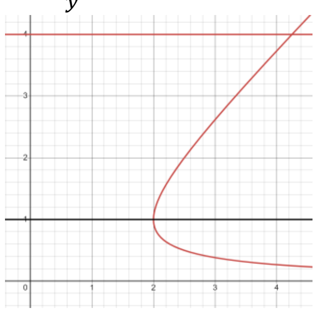
Bottom:
The radius of the shells is y and the heights are 1.

$$V = 2\pi \int_0^{\frac{1}{2}} y\, dy = 2\pi \left[\frac{1}{2} y^2\right]_0^{\frac{1}{2}} = \frac{\pi}{4}\ units^3$$

Total volume:

$$V = \frac{11\pi}{12} + \frac{\pi}{4} = \frac{7\pi}{6}\ units^3$$

c. $x = \frac{1+y^2}{y}$, $y = 1$, $y = 4$, and the y-axis



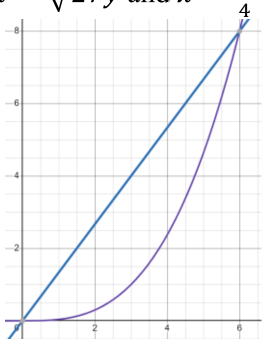
The radius of the shells is y and the heights are $\frac{1+y^2}{y}$.

$$V = 2\pi \int_1^4 y\left(\frac{1+y^2}{y}\right) dy = 2\pi \int_1^4 (1+y^2) dy$$

$$2\pi \left[y + \frac{1}{3} y^3\right]_1^4 = 2\pi \left(\frac{76}{3} - \frac{4}{3}\right)$$

$$= 48\pi\ units^3$$

d. $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$



The radius of the shells is y and the heights are $\sqrt[3]{27y} - \frac{3y}{4}$.

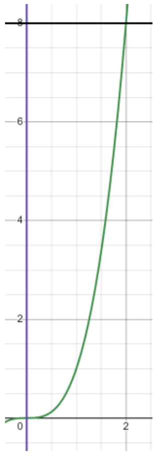
$$V = 2\pi \int_0^8 y\left(\sqrt[3]{27y} - \frac{3y}{4}\right) dy = 2\pi \int_0^8 y\left(3\sqrt[3]{y} - \frac{3y}{4}\right) dy$$

$$= 6\pi \int_0^8 \left(y^{\frac{4}{3}} - \frac{y^2}{4}\right) dy = 6\pi \left[\frac{3}{7} y^{\frac{7}{3}} - \frac{1}{12} y^3\right]_0^8 = 6\pi \left(\frac{256}{21}\right)$$

$$= \frac{512\pi}{7}\ units^3$$

10. (2.3) Find the volume when the region between the curves is rotated around the given axis using the shell method.

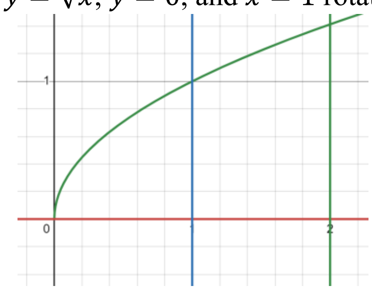
a. $y = x^3$, $x = 0$, and $y = 8$ rotated around the y-axis



The radius of the shells is x and the heights are $8 - x^3$.

$$V = 2\pi \int_0^2 x(8 - x^3)dx = 2\pi \int_0^2 (8x - x^4)dx$$
$$= 2\pi \left[4x^2 - \frac{1}{5}x^5\right]_0^2 = 2\pi\left(\frac{48}{5}\right)$$
$$= \frac{96\pi}{5}\ units^3$$

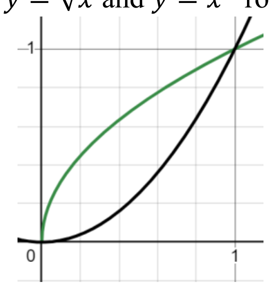
b. $y = \sqrt{x}$, $y = 0$, and $x = 1$ rotated around $x = 2$



The radius of the shells is $2 - x$ and the heights are $\sqrt{x}$.

$$V = 2\pi \int_0^1 (2 - x)\sqrt{x}dx = 2\pi \int_0^1 \left(2\sqrt{x} - x^{\frac{3}{2}}\right)dx$$
$$= 2\pi \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = 2\pi\left(\frac{4}{3} - \frac{2}{5}\right)$$
$$= \frac{28\pi}{5}\ units^3$$

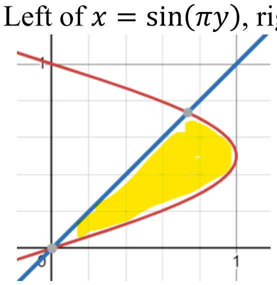
c. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis



The radius of the shells is x and the heights are $\sqrt{x} - x^2$.

$$V = 2\pi \int_0^1 x(\sqrt{x} - x^2)dx = 2\pi \int_0^1 \left(x^{\frac{3}{2}} - x^3\right)dx$$
$$2\pi \left[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4\right]_0^1 = 2\pi\left(\frac{2}{5} - \frac{1}{4}\right)$$
$$= \frac{3\pi}{10}\ units^3$$

d. Left of $x = \sin(\pi y)$, right of $y = x$ rotated around the y-axis (use a calculator)



Use a graphing calculator to find that the functions intersect at ≈ 0.736.
Solve for y in terms of x:

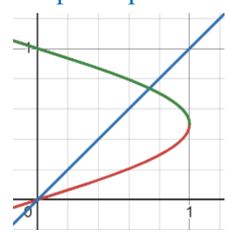
$$\pi y = \arcsin x$$
$$y = \frac{1}{\pi}\arcsin x$$

This volume calculation requires two integrals, for left and right.
Left:
The radius of the shells is x and the heights are $x - \frac{1}{\pi}\arcsin x$.

$$V = 2\pi \int_0^{0.736} x\left(x - \frac{1}{\pi}\arcsin x\right) dx \approx 0.551\, units^3$$

Right:

This is a strange integral because $\frac{1}{\pi}\arcsin x$ only represents the bottom part of the trig function. The top is represented by $1 - \frac{1}{\pi}\arcsin x$.



The radius of the shells is still x and the heights are $1 - \frac{1}{\pi}\arcsin x - \frac{1}{\pi}\arcsin x = 1 - \frac{2}{\pi}\arcsin x$.

$$V = 2\pi \int_{0.736}^{1} x\left(1 - \frac{2}{\pi}\arcsin x\right) dx \approx 0.463\, units^3$$

$$0.551\, units^3 + 0.463\, units^3 \approx 0.987\, units^3$$

11. (2.4) Find the length of the functions over the given interval.

a. $f(x) = 5x$ from $x = 0$ to $x = 2$.

$$f'(x) = 5$$

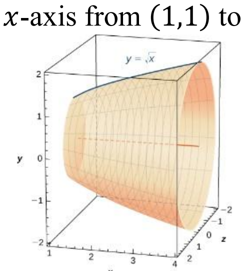
$$s = \int_0^2 \sqrt{1 + 5^2}dx = \sqrt{26}[x]_0^2 = 2\sqrt{26}\, units$$

b. $x = 4y$ from $y = -1$ to $y = 1$.

$$\frac{dx}{dy} = 4$$

$$s = \int_{-1}^{1} \sqrt{1 + 4^2} = \sqrt{17}[y]_{-1}^{1} = 2\sqrt{17}\, units$$

12. (2.4) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolved around the x-axis from $(1,1)$ to $(4,2)$ as shown.



$$S = 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = 2\pi \int_1^4 \sqrt{x}\sqrt{1 + \frac{1}{4x}}dx$$

$$= 2\pi \int_1^4 \sqrt{x}\sqrt{\frac{4x+1}{4x}}dx = \pi \int_1^4 \sqrt{x}\sqrt{\frac{4x+1}{x}}dx = \pi \int_1^4 \sqrt{4x+1}dx$$

$$u = 4x + 1 \Rightarrow du = 4dx$$

$$S = \frac{\pi}{4}\int_5^{17} \sqrt{u}du = \frac{\pi}{4}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^{17} = \frac{\pi}{4}\left[\frac{34\sqrt{17}}{3} - \frac{10\sqrt{5}}{3}\right]$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right) units^2$$

13. (2.4) Find the lengths of the functions over the given interval. Use a calculator if the integral cannot be evaluated directly.

a. $y = x^{\frac{3}{2}}$ from (0,0) to (1,1)

$$y' = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

$$s = \int_0^1 \sqrt{1 + \left(\frac{3}{2}\sqrt{x}\right)^2}\, dx = \int_0^1 \sqrt{1 + \frac{9}{4}x dx}$$

$$u = 1 + \frac{9}{4}x \Rightarrow du = \frac{9}{4}dx$$

$$s = \frac{4}{9}\int_1^{\frac{13}{4}} \sqrt{u} \cdot du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{\frac{13}{4}} = \frac{4}{9}\left(\frac{13\sqrt{13}}{12} - \frac{2}{3}\right)$$

$$= \frac{13\sqrt{13} - 8}{27}\ units$$

b. $y = \frac{1}{3}(x^2+2)^{\frac{3}{2}}$ from $x = 0$ to $x = 1$

$$\frac{dy}{dx} = x\sqrt{x^2+2}$$

$$s = \int_0^1 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2}\, dx = \int_0^1 \sqrt{1 + x^2(x^2+2)}dx = \int_0^1 \sqrt{1 + x^4 + 2x^2}dx$$

$$= \int_0^1 \sqrt{(x^2+1)^2}dx = \int_0^1 (x^2+1)dx = \left[\frac{1}{3}x^3 + x\right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}\ units$$

c. $y = e^x$ on $x = 0$ to $x = 1$

$$s = \int_0^1 \sqrt{1 + (e^x)^2}dx = \int_0^1 \sqrt{1 + e^{2x}}dx \approx 2.0035\ units$$

d. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

$$y' = x^3 - \frac{1}{4x^3}$$

$$s = \int_1^2 \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{4x^6 - 1}{4x^3}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{16x^{12} - 8x^6 + 1}{16x^6}}dx$$

$$= \int_1^2 \sqrt{\frac{16x^6}{16x^6} + \frac{16x^{12} - 8x^6 + 1}{16x^6}}dx = \int_1^2 \sqrt{\frac{16x^{12} + 8x^6 + 1}{16x^6}}dx$$

$$= \int_1^2 \sqrt{\frac{(4x^6+1)^2}{16x^6}}dx = \int_1^2 \frac{4x^6+1}{4x^3}dx = \int_1^2 \left(\frac{4x^6}{4x^3} + \frac{1}{4x^3}\right)dx = \int_1^2 \left(x^3 + \frac{1}{4}x^{-3}\right)dx$$

$$= \left[\frac{1}{4}x^4 + \frac{-1}{4 \cdot 2}x^{-2}\right]_1^2 = \left[\frac{1}{4}x^4 - \frac{1}{8x^2}\right]_1^2 = \frac{127}{32} - \frac{1}{8} = \frac{123}{32}\ units$$

e. $y = \frac{1}{27}(9x^2+6)^{\frac{3}{2}}$ from $x = 0$ to $x = 2$

$$y' = \frac{1}{18}(9x^2+6)^{\frac{1}{2}} \cdot 18x = x\sqrt{9x^2+6}$$

$$s = \int_0^2 \sqrt{1 + \left(x\sqrt{9x^2+6}\right)^2}\, dx = \int_0^2 \sqrt{1 + x^2(9x^2+6)}dx = \int_0^2 \sqrt{1 + 9x^4 + 6x^2}dx$$

$$= \int_0^2 \sqrt{(3x^2+1)^2}dx = \int_0^2 (3x^2+1)dx = [x^3 + x]_0^2 = 2^3 + 2 = 10\ units$$

f. $y = \frac{5-3x}{4}$ from $y = 0$ to $y = 4$

Get x by itself and find the derivative:

$$4y = 5 - 3x \Rightarrow -3x = 4y - 5 \Rightarrow x = \frac{-4y+5}{3}$$

$$x' = \frac{1}{3}(-4) = -\frac{4}{3}$$

$$s = \int_0^4 \sqrt{1 + \left(-\frac{4}{3}\right)^2}\, dy = \int_0^4 \sqrt{1 + \frac{16}{9}}dy = \int_0^4 \sqrt{\frac{25}{9}}dy = \int_0^4 \frac{5}{3}dy = \frac{5}{3}[x]_0^4 = \frac{5}{3} \cdot 4$$

$$= \frac{20}{3}\, units$$

g. $x = 5y^{\frac{3}{2}}$ from $y = 0$ to $y = 1$

$$x' = \frac{15}{2}y^{\frac{1}{2}} = \frac{15}{2}\sqrt{y}$$

$$s = \int_0^1 \sqrt{1 + \left(\frac{15}{2}\sqrt{y}\right)^2}\, dy = \int_0^1 \sqrt{1 + \frac{225y}{4}}dy = \int_0^1 \sqrt{\frac{225y+4}{4}}dy$$

$$= \frac{1}{2}\int_0^1 \sqrt{225y+4}dy$$

$$u = 225y + 4 \Rightarrow du = 225dy$$

$$s = \frac{1}{2 \cdot 225}\int_4^{229} \sqrt{u}du = \frac{1}{450}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{229} = \frac{1}{675}\left[u^{\frac{3}{2}}\right]_4^{229} = \frac{1}{675}\left(229\sqrt{229} - 8\right) units$$

h. $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

$$x' = \sec^2 y$$

$$s = \int_0^{\frac{3}{4}} \sqrt{1 + (\sec^2 y)^2}dy \approx 1.201\, units$$

i. $x = 4^y$ from $y = 0$ to $y = 2$

$$x' = \frac{4^y}{\ln 4}$$

$$s = \int_0^2 \sqrt{1 + (4^y \ln 4)^2}dy \approx 15.2341\, units$$

14. (2.4) Find the surface area of the volume generated when the given curves revolve about the x-axis.

a. $y = \sqrt{x}$ from $x = 2$ to $x = 6$

$$y' = \frac{1}{2\sqrt{x}}$$

$$S = 2\pi \int_2^6 \sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = 2\pi \int_2^6 \sqrt{x}\sqrt{1 + \frac{1}{4x}}dx = 2\pi \int_2^6 \sqrt{x\left(1 + \frac{1}{4x}\right)}dx$$

$$= 2\pi \int_2^6 \sqrt{x + \frac{1}{4}}dx$$

$$u = x + \frac{1}{4} \Rightarrow du = dx$$

$$S = 2\pi \int_{\frac{9}{4}}^{\frac{25}{4}} \sqrt{u}\, du = 2\pi \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{\frac{9}{4}}^{\frac{25}{4}} = 2\pi \left(\frac{125}{12} - \frac{9}{4}\right) = \frac{49\pi}{3}\ units^2$$

b. $y = 7x$ from $x = -1$ to $x = 1$

$$y' = 7$$

If you try to do this in one integral you will get an answer that makes no sense:

$$S = 2\pi \int_{-1}^{1} 7x\sqrt{1 + 7^2} dx = 14\pi\sqrt{50} \int_{-1}^{1} x dx = 14\pi\sqrt{50} \left[\frac{1}{2} x^2\right]_{-1}^{1} = 0$$

That's because you will get a negative surface area before $x = 0$ because the function is under the x-axis. You will have to integrate separately and change the sign of the left part, or multiply the right integral by 2 because the function is symmetric.

$$S = 7 \cdot 4\pi\sqrt{50} \int_0^1 x dx = 28\pi\sqrt{50} \left[\frac{1}{2} x^2\right]_0^1 = 14\pi\sqrt{50} = 14\pi \cdot 5\sqrt{2} = 70\pi\sqrt{2}\ units^2$$

c. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

$$y' = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$$

$$S = 2\pi \int_0^2 \sqrt{4 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2}}\right)^2} dx = 2\pi \int_0^2 \sqrt{4 - x^2} \sqrt{1 + \frac{x^2}{4 - x^2}} dx$$

$$= 2\pi \int_0^2 \sqrt{4 - x^2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}} dx = 2\pi \int_0^2 \sqrt{4 - x^2} \sqrt{\frac{4}{4 - x^2}} dx = 2\pi \int_0^2 \sqrt{4 - x^2} \frac{\sqrt{4}}{\sqrt{4 - x^2}} dx$$

$$= 2\pi \int_0^2 \sqrt{4} dx = 4\pi [x]_0^2 = 8\pi\ units^2$$

d. $y = 5x$ from $x = 1$ to $x = 5$

$$y' = 5$$

$$S = 2\pi \int_1^5 5x\sqrt{1 + 5^2} dx = 5 \cdot 2\pi\sqrt{26} \left[\frac{1}{2} x^2\right]_1^5 = 10\pi\sqrt{26} \left(\frac{25}{2} - \frac{1}{2}\right) = 120\pi\sqrt{26}\ units^2$$

15. (2.4) Find the surface area of the volume generated when the given curves revolve about the y-axis. Use a calculator if the integral cannot be evaluated directly.

a. $y = x^2$ from $x = 0$ to $x = 2$

Rewrite in terms of y and find the y bounds:

$$x = \sqrt{y}$$

$$x = 0 \Rightarrow y = 0$$

$$x = 2 \Rightarrow y = 4$$

$$x' = \frac{1}{2\sqrt{y}}$$

$$S = 2\pi \int_0^4 \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy = 2\pi \int_0^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}} dy = 2\pi \int_0^4 \sqrt{y\left(1 + \frac{1}{4y}\right)} dy$$

$$= 2\pi \int_0^4 \sqrt{y + \frac{1}{4}} dy$$

$$u = y + \frac{1}{4} \Rightarrow du = dy$$

$$S = 2\pi \int_{\frac{1}{4}}^{\frac{17}{4}} \sqrt{u}du = 2\pi \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{1}{4}}^{\frac{17}{4}} = 2\pi\left(\frac{17\sqrt{17}}{12} - \frac{1}{12}\right) = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) units^2$$

b. $y = x + 1$ from $x = 0$ to $x = 3$

Rewrite in terms of y and find the y bounds:

$$x = y - 1$$
$$x = 0 \Rightarrow y = 1$$
$$x = 3 \Rightarrow y = 4$$
$$x' = 1$$

$$S = 2\pi \int_1^4 (y-1)\sqrt{1+1^2}dy = 2\pi\sqrt{2}\int_1^4 (y-1)dy = 2\pi\sqrt{2}\left[\frac{1}{2}y^2 - y\right]_1^4$$

$$= 2\pi\sqrt{2}\left(4 - \left(-\frac{1}{2}\right)\right) = 9\pi\sqrt{2}\, units^2$$

c. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

Rewrite in terms of y and find the y bounds:

$$x = y^3$$
$$x = 1 \Rightarrow y = 1$$
$$x = 27 \Rightarrow y = 3$$
$$x' = 3y^2$$

$$S = 2\pi \int_1^3 y^3\sqrt{1 + (3y^2)^2}dy = 2\pi \int_1^3 y^3\sqrt{1+9y^4}dy$$

$$u = 1 + 9y^4 \Rightarrow du = 36y^3dy$$

$$\frac{2\pi}{36}\int_{10}^{730}\sqrt{u}du = \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{10}^{730} = \frac{\pi}{27}\left[u^{\frac{3}{2}}\right]_{10}^{730} = \frac{\pi}{27}\left(730\sqrt{730} - 10\sqrt{10}\right) units^2$$

If you want, you can simplify further:

$$S = \frac{10\sqrt{10}\pi}{27}\left(73\sqrt{73} - 1\right) units^2$$

d. $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

Rewrite in terms of y and find the y bounds:

$$x = \left(\frac{1}{y}\right)^2 = \frac{1}{y^2}$$
$$x = 1 \Rightarrow y = 1$$
$$x = 3 \Rightarrow y = \frac{1}{\sqrt{3}}$$
$$x' = -\frac{2}{y^3}$$

$$S = 2\pi \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{y^2}\sqrt{1 + \left(-\frac{2}{y^3}\right)^2}\,dy \approx 25.645\, units^2$$

16. (2.5) How much work is done when a person lifts a 50 lb object 3 ft off the ground?

You don't need an integral for this problem since the force is constant.

$$W = F \cdot d = 50\ lb \cdot 3\ ft = 150\ lb \cdot ft$$

Integrating yields the same result:

$$W = \int_{0\ ft}^{3\ ft} 50\ lb \cdot dy = 50\ lb \cdot [x]_{0\ ft}^{3\ ft} = 50\ lb(3\ ft) = 150\ lb \cdot ft$$

17. (2.5) How much work is done when you push a box along the floor 2 m when applying a constant force of $F = 100\ N$?

You don't need an integral for this problem either since the force is constant.

$$W = F \cdot d = 100\ N \cdot 2\ m = 200\ N \cdot m = 200\ J$$

18. (2.5) How much work is done when moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

The force changes with respect to distance so an integral is required.

$$W = \int F \cdot dx = \int_0^1 3x^2 dx = [x^3]_0^1 = 1\ N \cdot m = 1\ J$$

19. (2.5) Find the mass of a 3 ft long car antenna (starting at $x = 0$) that had a density function of $\rho(x) = 3x + 2$ lb/ft.

$$m = \int \rho(x) \cdot dx = \int_0^3 (3x + 2)dx = \left[\frac{3}{2}x^2 + 2x\right]_0^3 = \frac{39}{2}\ lb$$

20. (2.5) Find the mass of a 4 inch long pencil (starting at $x = 2$) that has a density function of $\rho(x) = \frac{5}{x}$ oz/in.

$$m = \int \rho(x) \cdot dx = \int_2^6 \left(\frac{5}{x}\right) dx = [5 \ln x]_2^6 = 5 \ln 6 - 5 \ln 2 = \ln 6^5 - \ln 2^5 = \ln \frac{6^5}{2^5}$$
$$= \ln 243\ \ oz$$

I didn't put the absolute value signs on x when integrating to $\ln x$ because the x-values are all positive. If you ever have to plug in negative numbers, remember the absolute value.

21. (2.5) Find the mass of the two-dimensional object that is centered at the origin.

a. An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

$$m = 2\pi \int_0^2 x(x^3 - 2x + 5)dx = 2\pi \int_0^2 (x^4 - 2x^2 + 5x)dx = 2\pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + \frac{5}{2}x^2\right]_0^2$$
$$= 2\pi \left(\frac{166}{15}\right) = \frac{332\pi}{15}$$

b. A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

$$m = 2\pi \int_0^5 x\sqrt{3x}dx = 2\sqrt{3}\pi \int_0^5 x^{\frac{3}{2}}dx = 2\sqrt{3}\pi \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^5 = 2\sqrt{3}\pi \cdot 10\sqrt{5} = 20\pi\sqrt{15}$$

22. (2.5) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

Find the spring constant:

If $F(x) = kx$, and $W = \int F \cdot dx$, then $W = \int kx \cdot dx$, so

$$W = \frac{1}{2}kx^2$$

Units have to be converted to meters because Joules are Newton-Meters.

$$2\ J = \frac{1}{2}k(0.05\ m)^2$$
$$k = 1600 \frac{N}{m}$$

That means $F(x) = 1600x$.

If the spring has a natural length of 10 cm, then 15 to 20 cm is actually 5 to 10 cm of additional stretch, which is what we are integrating.

$$W = \int F(x)dx = 1600 \int_{0.05}^{0.10} x \cdot dx = 800[x^2]_5^{10} = 6\,J$$

23. (2.5) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length L, of the spring?
See the above problem for more details on this process.
To stretch from 8 to 12 cm:

$$5\,J = \int_{8-L}^{12-L} kx \cdot dx$$

$$5 = \frac{1}{2}k[x^2]_{8-L}^{12-L} = \frac{k}{2}((12-L)^2 - (8-L)^2)$$

$$5 = \frac{k}{2}\big((144 - 24L + L^2) - (64 - 16L + L^2)\big)$$

$$5 = \frac{k}{2}(80 - 8L)$$

$$k = \frac{10}{80 - 8L}$$

Do the same thing for stretching 12 to 14 cm:

$$4 = \int_{12-L}^{14-L} kx \cdot dx$$

$$4 = \frac{k}{2}[x^2]_{12-L}^{14-L}$$

$$8 = k((14-L)^2 - (12-L)^2)$$

$$8 = k\big((196 - 28L + L^2) - (144 - 24L + L^2)\big) = k(52 - 4L)$$

$$k = \frac{8}{52 - 4L}$$

Now set the k values equal to each other and solve for L.

$$\frac{10}{80 - 8L} = \frac{8}{52 - 4L}$$

$$L = 5\,cm$$

Don't worry about changing all the units to meters, you will get the same result $(0.05\,m)$. The spring constant is not given, so we can have whatever units we want, in this case $\frac{J}{cm}$. We don't have to do any calculations for k, we are just setting two terms with the same units equal to each other.

24. (2.5) A rectangular dam is 40 ft high and 60 ft wide. Assume the weight density of water is $62.5\frac{lbs}{ft^3}$. Using a calculator, compute the total force F on the dam when

a. The surface of the water is at the top of the dam and

$$F = \int_a^b \rho w(x)s(x)dx = \int_0^{40} 62.5 \cdot 60 \cdot x \cdot dx = 3{,}000{,}000\,lb$$

b. The surface of the water is halfway down the dam.

$$F = \int_a^b \rho w(x)s(x)dx = \int_0^{20} 62.5 \cdot 60 \cdot x \cdot dx = 750{,}000\,lb$$

25. (2.5) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Computer the work to pump all the water to the top.

This is a symbolic application of the formula in your notes:

$$W = \int_a^b \rho \cdot x \cdot (CSA)dx$$

The water travels from $x = 0$ to $x = H$, and the cross-sectional area is A.

$$W = \int_0^H \rho \cdot x \cdot A \cdot dx$$

Factor out the constants:

$$W = \rho A \int_0^H x \cdot dx = \frac{\rho A}{2}[x^2]_0^H = \frac{\rho A H^2}{2}$$

26. (2.9) Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$.

$$\cosh x + \sinh x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} = \frac{2e^{-x}}{2} = e^{-x}$$

27. (2.9) Show that $\cosh x$ and $\sinh x$ satisfy $y'' = y$.

$y'' = y$ means the function is equal to its second derivative.

$$y = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x \Rightarrow \frac{d}{dx}[\sinh x] = \cosh x = y'' = y$$

$$y = \sinh x$$

$$\frac{d}{dx}[\sinh x] = \cosh x \Rightarrow \frac{d}{dx}[\cosh x] = \sinh x = y'' = y$$

28. (2.9) Derive $\cosh^2 x + \sinh^2 x = \cosh(2x)$ from the definition.

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x})}{4} + \frac{(e^x - e^{-x})(e^x - e^{-x})}{4}$$

$$= \frac{e^{2x} + 2\frac{e^x}{e^x} + \frac{1}{e^{2x}}}{4} + \frac{e^{2x} - 2\frac{e^x}{e^x} + \frac{1}{e^{2x}}}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x)$$

29. (2.9) Find the derivatives of the following functions.

a. $\cosh(3x + 1)$

Just use chain rule:

$$3\sinh(3x + 1)$$

b. $\frac{1}{\cosh x}$

Rewrite the function:

$$\frac{1}{\cosh x} = \operatorname{sech} x$$

$$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

c. $\cosh^2 x + \sinh^2 x$

This involves the chain rule, just like with regular trig functions.

$$\frac{d}{dx}[\cosh^2 x + \sinh^2 x] = 2\cosh x \sinh x + 2\sinh x \cosh x = 4\sinh x \cosh x$$

d. $\tanh\left(\sqrt{x^2+1}\right)$

This involves the chain rule, but there are three layers.

$$\frac{d}{dx}\left[\tanh\left(\sqrt{x^2+1}\right)\right] = 2x \cdot \frac{1}{2\sqrt{x^2+1}} \cdot \operatorname{sech}^2\left(\sqrt{x^2+1}\right) = \frac{x\operatorname{sech}^2\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

e. $\sinh^6 x$

This is chain rule again.

$$\frac{d}{dx}[\sinh^6 x] = 6\sinh^5 x \cosh x$$

30. (2.9) Find the antiderivatives for the following functions.

a. $\cosh(2x+1)$

$$\int \cosh(2x+1)dx$$
$$u = 2x+1 \Rightarrow du = 2dx$$
$$= \frac{1}{2}\int \cosh u\, du = \frac{1}{2}\sinh u + C = \frac{1}{2}\sinh(2x+1) + C$$

b. $x\cosh(x^2)$

$$\int x\cosh(x^2)\,dx$$
$$u = x^2 \Rightarrow du = 2xdx$$
$$\frac{1}{2}\int \cosh u\, du = \frac{1}{2}\sinh u + C = \frac{1}{2}\sinh(x^2) + C$$

For some reason the textbook says the answer is $\frac{1}{2}\sinh^2(x^2) + C$. I think it's wrong but I don't know.

c. $\cosh^2(x)\sinh(x)$

$$\int \cosh^2(x)\sinh(x)\,dx$$
$$u = \cosh x \Rightarrow du = \sinh x\,dx$$
$$\int u^2\sinh(x)\frac{du}{\sinh x} = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\cosh^3 x + C$$

d. $\frac{\sinh x}{1+\cosh x}$

$$\int \frac{\sinh x}{1+\cosh x}dx$$
$$u = 1+\cosh x \Rightarrow du = \sinh x\,dx$$
$$= \int \frac{1}{u}du = \ln|u| + C = \ln|1+\cosh x| + C$$

e. $\cosh x + \sinh x$

$$\int (\cosh x + \sinh x)dx = \sinh x + \cosh x + C$$

31. (2.9) Find the derivatives of the following functions.

a. $\operatorname{arctanh}(4x)$

$$\frac{d}{dx}[\operatorname{arctanh}(4x)] = 4\cdot\frac{1}{1-(4x)^2} = \frac{4}{1-16x^2}$$

b. $\operatorname{arcsinh}(\cosh x)$

$$\frac{d}{dx}[\operatorname{arcsinh}(\cosh x)] = \frac{1}{\sqrt{1+(\cosh x)^2}}\cdot\sinh x = \frac{\sinh x}{\sqrt{1+\cosh^2 x}}$$

c. $\operatorname{arctanh}(\cos x)$

$$\frac{d}{dx}[\operatorname{arctanh}(\cos x)] = \frac{1}{1-\cos^2 x} \cdot -\sin x = -\frac{\sin x}{\sin^2 x} = -\frac{1}{\sin x} = -\csc x$$

d. $\ln(\operatorname{arctanh} x)$

$$\frac{d}{dx}[\ln(\operatorname{arctanh} x)] = \frac{1}{\operatorname{arctanh} x} \cdot \frac{1}{1-x^2} = \frac{\operatorname{arccoth} x}{1-x^2}$$

MTH201 Chapter 3
Answers

Harper College*

June 25, 2025

Contents

*Problems found in Openstax Calculus Volume II
`https://openstax.org/details/books/calculus-volume-2`

1 3.1 - Integration by Parts

1. (3.1) Integrate using the simplest method. Not all problems require integration by parts.

   (a) $\int \ln x \; dx$

$$\int \ln x \; dx = \int 1 \cdot \ln x \; dx$$
$$u = \ln x, \;\; dv = 1 \cdot dx \Rightarrow du = \frac{1}{x}dx, \;\; v = x$$
$$\int \ln x \; dx = x \ln x - \int x \cdot \frac{1}{x}dx = x \ln x - \int 1 \cdot dx$$
$$= \boldsymbol{x \ln x - x + C}$$

   (b) $\int \arctan(x) \; dx$

$$\int \arctan x \; dx = \int 1 \cdot \arctan x \; dx$$
$$u = \arctan x, \;\; dv = dx \Rightarrow du = \frac{1}{1+x^2}dx, \;\; v = x$$
$$\int \arctan x \; dx = x \arctan x - \int \frac{x}{1+x^2}dx$$
$$\text{Use } u\text{-substitution to integrate}$$
$$u = 1 + x^2 \Rightarrow du = 2x \cdot dx, \;\; dx = \frac{du}{2x}$$
$$x \arctan x - \int \frac{x}{u} \cdot \frac{du}{2x}$$
$$= x \arctan x - \frac{1}{2}\int \frac{1}{u} \cdot du$$
$$= x \arctan x - \frac{1}{2} \ln |u| + C$$
$$= \boldsymbol{x \arctan x - \frac{1}{2} \ln |1 + x^2| + C}$$

   (c) $\int x \sin(2x) dx$

$$u = x, \;\; dv = \sin(2x)dx \Rightarrow du = dx, \;\; v = -\frac{1}{2}\cos(2x)$$
$$\int x \sin(2x)dx = -\frac{1}{2}x\cos(2x) - \int -\frac{1}{2}\cos(2x)dx$$
$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)dx$$
$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right)$$
$$= \boldsymbol{-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C}$$

(d) $\int xe^{-x}dx$

$$u = x,\ dv = e^{-x}dx \Rightarrow du = dx,\ v = -e^{-x}$$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} + \int e^{-x}dx$$

$$= \boldsymbol{-xe^{-x} - e^{-x} + C}$$

You can also factor out the e^{-x}

$$= \boldsymbol{e^{-x}(-x-1) + C}$$

(e) $\int x^2 \cos x \ dx$

$$u = x^2, dv = \cos x \ dx \Rightarrow du = 2x \cdot dx,\ v = \sin x$$

$$\int x^2 \cos x \ dx = x^2 \sin x - \int 2x \sin x \ dx$$

$$= x^2 \sin x - 2\int x \sin x \ dx$$

You will need to use integration by parts a second time here.

$$u = x,\ dv = \sin x \ dx \Rightarrow du = dx,\ v = -\cos x$$

$$= x^2 \sin x - 2\left(-x\cos x - \int -\cos x dx\right)$$

$$= x^2 \sin x - 2\left(-x\cos x + \int \cos x dx\right)$$

$$= \boldsymbol{x^2 \sin x - 2\left(-x\cos x + \sin x\right) + C}$$

(f) $\int \ln(2x+1)dx$

$$u = \ln(2x+1),\ dv = dx \Rightarrow du = \frac{2}{2x+1}dx,\ v = x$$

$$\int \ln(2x+1)dx = x\ln(2x+1) - \int \frac{2x}{2x+1}dx$$

Use u-substitution to evaluate the integral, then split the fraction

$$u = 2x+1 \Rightarrow du = 2 \cdot dx,\ 2x = u-1$$

$$= x\ln(2x+1) - \int \frac{u-1}{u}\frac{du}{2} = x\ln(2x+1) - \frac{1}{2}\int\left(1-\frac{1}{u}\right)du$$

$$= \boldsymbol{x\ln(2x+1) - \frac{1}{2}\left(2x+1-\ln(2x+1)\right) + C}$$

The solution in the textbook is $\boldsymbol{\frac{1}{2}(1+2x)\left(-1+\ln(1+2x)\right)+C}$.

I think they are the same but I'm not certain.

(g) $\int e^x \sin x \; dx$

$$u = \sin x,\;\; dv = e^x dx \Rightarrow du = \cos x \; dx,\;\; v = e^x$$

$$\int e^x \sin x \; dx = e^x \sin x - \int e^x \cos x \; dx$$

Repeat the IBP for the next integral.

$$u = \cos x,\;\; dv = e^x dx \Rightarrow du = -\sin x \; dx,\;\; v = e^x$$

$$= e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \; dx \right)$$

Here's the whole equation:

$$\int e^x \sin x \; dx = e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \; dx \right)$$

Distribute the signs:

$$\int e^x \sin x \; dx = e^x \sin x - e^x \cos x - \int e^x \sin x \; dx$$

Now add the integral to the left side:

$$2 \int e^x \sin x \; dx = e^x \sin x - e^x \cos x + C$$

Now divide by 2 to get the integral we want:

$$\int e^x \sin x \; dx = \boldsymbol{\frac{e^x \sin x - e^x \cos x}{2} + C}$$

(h) $\int x e^{-x^2} dx$

According to LIATE, dv should be e^{-x^2} because exponentials are usually easier to integrate.

This won't work here because e^{-x^2} cannot be integrated.

$$u = e^{-x^2},\;\; dv = x \cdot dx \Rightarrow du = -2xe^{-x^2} dx,\;\; v = \frac{1}{2}x^2$$

$$\int x e^{-x^2} dx = \frac{1}{2}x^2 e^{-x^2} - \int -2xe^{-x^2} \frac{1}{2} x^2 dx$$

$$= \frac{1}{2}x^2 e^{-x^2} + \int e^{-x^2} x^3 dx$$

Use IBP again here:

$$u = x^2,\;\; dv = xe^{-x^2} dx \Rightarrow du = 2x dx,\;\; v = -\frac{1}{2}e^{-x^2}$$

$$= \frac{1}{2}x^2 e^{-x^2} + \left(-\frac{1}{2}x^2 e^{-x^2} - \int 2x \left(-\frac{1}{2} e^{-x^2} \right) dx \right)$$

$$= 0 + \int x e^{-x^2} dx$$

$$= \boldsymbol{-\frac{1}{2} e^{-x^2} + C}$$

(i) $\int \sin(\ln(2x))dx$

I'm using substitution with the letter a so IBP does not get confusing later.

$$a = \ln(2x) \Rightarrow da = \frac{2}{2x}dx,\ dx = x \cdot da$$

$$\int \sin(\ln(2x))dx = \int \sin a \cdot x \cdot da$$

Now rewrite that x in terms of a.

$$x = \frac{1}{2}e^a$$

$$\int \sin(\ln(2x))dx = \int \sin a \frac{1}{2}e^a da = \frac{1}{2}\int \sin a \cdot e^a da$$

Now we have an integral we can do IBP on.

$$u = \sin a,\ dv = e^a da \Rightarrow du = \cos a\ da,\ v = e^a$$

$$\frac{1}{2}\int \sin a \cdot e^a da = \frac{1}{2}\left(e^a \sin a - \int e^a \cos a\ da\right)$$

Do IBP a second time.

$$u = \cos a,\ dv = e^a da \Rightarrow du = -\sin a\ da,\ v = e^a$$

$$\frac{1}{2}\int \sin a \cdot e^a da = \frac{1}{2}e^a \sin a - \frac{1}{2}\left(e^a \cos a + \int e^a \sin a\ da\right)$$

$$\frac{1}{2}\int \sin a \cdot e^a da = \frac{1}{2}e^a \sin a - \frac{1}{2}e^a \cos a - \frac{1}{2}\int e^a \sin a\ da$$

$$\int \sin a \cdot e^a da = \frac{1}{2}e^a \sin a - \frac{1}{2}e^a \cos a = \frac{1}{2}e^a\left(\sin a - \cos a\right) + C$$

$$\frac{1}{2}\int \sin a \cdot e^a da = \frac{1}{4}e^a\left(\sin a - \cos a\right) + C$$

Now plug in the original value for a.

$$\frac{1}{2}\int \sin a \cdot e^a da = \boldsymbol{\frac{1}{4}e^{\ln(2x)}\left(\sin(\ln(2x)) - \cos(\ln(2x))\right) + C}$$

(j) $\int (\ln x)^2 dx$

This is similar to the previous problem where substitution is used to create an easier integral.

Again I'm using the variable a instead of u.

$$a = \ln x \Rightarrow da = \frac{dx}{x},\ dx = a \cdot da,\ x = e^a$$

$$\int (\ln x)^2 dx = \int a^2 x\ da = \int a^2 e^a da$$

Now use IBP.

$$\int a^2 e^a da = a^2 e^a - \int 2ae^a da = a^2 e^a - 2\int ae^a da$$

$$u = a,\ dv = e^a da \Rightarrow du = da,\ v = e^a$$

$$\int a^2 e^a da = a^2 e^a - 2\left(ae^a - \int e^a da\right)$$

$$= a^2 e^a - 2ae^a + 2e^a + C$$

$$= e^a\left(a^2 - 2a + 2\right) + C$$

$$\boldsymbol{= x\left(\ln^2 x - 2\ln x + 2\right) + C}$$

(k) $\int x^2 \ln x \; dx$

$$u = \ln x,\; dv = x^2 dx \Rightarrow du = \frac{dx}{x},\; v = \frac{1}{3}x^3$$
$$\int x^2 \ln x \; dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{dx}{x}$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C$$
$$= \boldsymbol{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

(l) $\int \arccos(2x) dx$

$$u = \arccos(2x),\; dv = dx \Rightarrow du = \frac{-2\; dx}{\sqrt{1-4x^2}},\; v = x$$
$$\int \arccos(2x) dx = x \cdot \arccos(2x) - \int \frac{-2x\; dx}{\sqrt{1-4x^2}} = x \cdot \arccos(2x) + 2\int \frac{x\; dx}{\sqrt{1-4x^2}}$$

Use a regular u-substitution for the integral.

$$u = 1 - 4x^2 \Rightarrow du = -8x dx,\; dx = \frac{du}{-8x}$$
$$= x \cdot \arccos(2x) + 2\int \frac{x}{\sqrt{u}} \frac{du}{-8x}$$
$$= x \cdot \arccos(2x) - \frac{2}{8}\int \frac{1}{\sqrt{u}} du$$
$$= x \cdot \arccos(2x) - \frac{1}{4}\left(2\sqrt{u}\right) + C$$
$$= x \cdot \arccos(2x) - \frac{1}{4}\left(2\sqrt{1-4x^2}\right) + C$$
$$= \boldsymbol{x \cdot \arccos(2x) - \frac{1}{2}\sqrt{1-4x^2} + C}$$

(m) $\int x^2 \sin x \; dx$

$$u = x^2,\; dv = \sin x \; dx \Rightarrow du = 2x \; dx,\; v = -\cos x$$
$$\int x^2 \sin x \; dx = -x^2 \cos x - \int -2x \cos x \; dx = -x^2 \cos x + 2\int x \cos x \; dx$$

Use IBP a second time.

$$u = x,\; dv = \cos x \; dx \Rightarrow du = dx,\; v = \sin x$$
$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x \; dx\right)$$
$$= -x^2 \cos x + 2\left(x \sin x + \cos x\right) + C$$
$$= \boldsymbol{-x^2 \cos x + 2x \sin x + 2\cos x + C}$$

(n) $\int x^3 \sin x \; dx$

If you noticed a pattern from some of the previous problems, you can correctly infer that you have to do IBP 3 times.

$$u = x^3,\;\; dv = \sin x dx \Rightarrow du = 3x^2 dx,\;\; v = -\cos x$$

$$\int x^3 \sin x \; dx = -x^3 \cos x - \int -3x^2 \cos x \; dx$$

$$= -x^3 \cos x + 3 \int x^2 \cos x \; dx$$

Use IBP a second time.

$$u = x^2,\;\; dv = \cos x \; dx \Rightarrow du = 2x dx,\;\; v = \sin x$$

$$= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \int x \sin x dx \right)$$

Use IBP a third time.

$$u = x,\;\; dv = \sin x \; dx \Rightarrow du = dx,\;\; v = -\cos x$$

$$= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \left(-x \cos x - \int -\cos x \; dx \right) \right)$$

$$= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \left(-x \cos x + \int \cos x \; dx \right) \right)$$

$$= -x^3 \cos x + 3 \left(x^2 \sin x - 2 \left(-x \cos x + \sin x \right) \right) + C$$

$$\boldsymbol{= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C}$$

(o) $\int x \cosh x \; dx$

$$u = x,\;\; dv = \cosh x \; dx \Rightarrow du = dx,\;\; v = \sinh x$$

$$\int x \cosh x \; dx = x \sinh x - \int \sinh x \; dx$$

$$\boldsymbol{= x \sinh x - \cosh x + C}$$

2. (3.1) Compute the definite integrals.

(a) $\int_0^1 x e^{-2x} dx$

$$u = x,\;\; dv = e^{-2x} dx \Rightarrow du = dx,\;\; v = -\frac{1}{2} e^{-2x}$$

$$\int_0^1 x e^{-2x} dx = -\frac{1}{2} x e^{-2x} \Big|_0^1 - \int_0^1 -\frac{1}{2} e^{-2x} dx$$

$$= -\frac{1}{2} x e^{-2x} \Big|_0^1 + \frac{1}{2} \int_0^1 e^{-2x} dx$$

$$= -\frac{1}{2} x e^{-2x} \Big|_0^1 + \frac{1}{2} \left[-\frac{1}{2} e^{-2x} \right]_0^1$$

$$= \left(\frac{-e^{-2}}{2} - 0 \right) + \frac{1}{2} \left(\frac{-e^{-2}}{2} - \frac{-1}{2} \right) = \boldsymbol{\frac{1}{4} - \frac{3}{4e^2}}$$

(b) $\int_1^e \ln(x^2)dx$

$$u = \ln\left(x^2\right),\ dv = dx \Rightarrow du = \frac{2x}{x^2}dx,\ v = x$$

$$\int_1^e \ln(x^2)dx = x\ln\left(x^2\right)\Big|_1^e - \int_1^e 2dx$$

$$= x\ln\left(x^2\right)\Big|_1^e - (2e-2)$$

$$= 2e(1) - 2e + 2$$

$$= \mathbf{2}$$

(c) $\int_{-\pi}^{\pi} x\sin x\ dx$

$$u = x,\ dv = \sin x\ dx \Rightarrow du = dx,\ v = -\cos x$$

$$\int_{-\pi}^{\pi} x\sin x\ dx = -x\cos x\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\cos x dx = -x\cos x\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x dx$$

$$= -x\cos x\Big|_{-\pi}^{\pi} + \sin x\Big|_{-\pi}^{\pi}$$

$$= (\pi - (-\pi)) + 0 = \boldsymbol{2\pi}$$

(d) $\int_0^{\pi/2} x^2\sin x\ dx$

The indefinite version of this integral was solved previously.
I will being the solution from that problem and plug in the limits.

$$\int_0^{\pi/2} x^2\sin x\ dx = \left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi/2}$$

$$= \boldsymbol{\pi - 2}$$

3. (3.1) Evaluate $\int \cos x\ln(\sin x)dx$

$$u = \ln(\sin x),\ dv = \cos x\ dx \Rightarrow du = \frac{\cos x}{\sin x}dx, v = \sin x$$

$$\int \cos x\ln(\sin x)dx = \sin x\ln(\sin x) - \int \sin x\cdot\frac{\cos x}{\sin x}dx$$

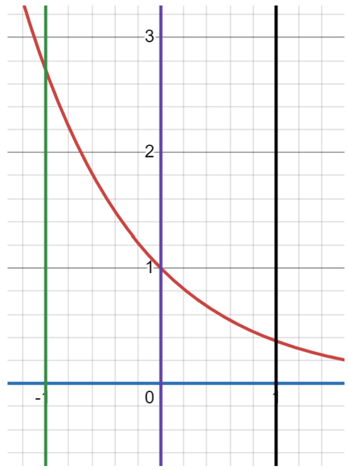
$$= \sin x\ln(\sin x) - \int \cos x\ dx$$

$$= \boldsymbol{\sin x\ln(\sin x) - \sin x + C}$$

4. (3.1) Find the volume generated by rotating the region bounded by the given curves about the line $x = 1$. Express the answer in exact form.

$$y = e^{-x}, \; y = 0, \; x = -1, \; x = 0$$

Here's what the graph looks like. The region on the left is revolved around $x = 1$ on the right.



To set up the integral using disks, get x by itself first.

$$x = -\ln y$$

The integral needs to be split at $y = 1$ because the shells start to have a different width.

The radius of the bottom disks is $1 + 1$ (because of $x = 1$).

Then subtract the volume of the center part.

$$V_{bottom} = \pi \int_0^1 \left[2^2 - 1^2\right] dy = \pi \int_0^1 3 \; dy = 3\pi \; units^3$$

For the top, the radius of the shells is $-1 - (-\ln y) + 1 = \ln(y)$.

Then add the radius of 1 for the volume of the center part.

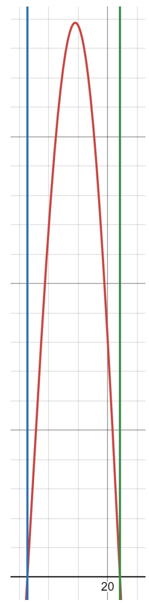
$$V = \pi \int_1^e \left[\ln^2 y - 1^2\right] dy$$

$$V = \pi \int_1^e \ln^2 y \; dy - \pi \int_1^e 1 \; dy = \pi \int_1^e \ln^2 y \; dy - \pi(e - 1)$$

We already integrated $\ln^2 x$ in a previous problem so I will just bring the answer.

$$= \pi \left[y\left(\ln^2 y - 2\ln y + 2\right)\right]_1^e - \pi(e-1) = \pi(e-2) - \pi(e-1) = -\pi \; units^3$$

5. (3.1) Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis for $\frac{11\pi}{2} \le x \le \frac{13\pi}{2}$. Express the answer in exact form.



$$A = \int_{\frac{11\pi}{2}}^{\frac{13\pi}{2}} x\cos x\ dx$$

$$u = x,\ dv = \cos x\ dx \Rightarrow du = dx,\ v = \sin x$$

$$A = x\sin x\Big|_{\frac{11\pi}{2}}^{\frac{13\pi}{2}} - \int_{\frac{11\pi}{2}}^{\frac{13\pi}{2}} \sin x\ dx$$

$$= x\sin x\Big|_{\frac{11\pi}{2}}^{\frac{13\pi}{2}} + [\cos x]_{\frac{11\pi}{2}}^{\frac{13\pi}{2}}$$

$$= \left(\left(\frac{13\pi}{2}\cdot 1\right) - \left(\frac{11\pi}{2}\cdot -1\right)\right) + (0-0)$$

$$= \boldsymbol{12\pi\ units^2}$$

2 3.2 - Trig Integration

1. (3.2) Evaluate the integrals using u-substitution
   The trig function with the power becomes u (not including the exponent).

   (a) $\int \sin^3 x\cos x\ dx$

$$u = \sin x,\ du = \cos x\ dx \Rightarrow dx = \frac{du}{\cos x}$$

$$\int \sin^3 x\cos x\ dx = \int u^3\cos x\frac{du}{\cos x} = \int u^3 du = \frac{1}{4}u^4 + C$$

$$= \boldsymbol{\frac{1}{4}\sin^4 x + C}$$

(b) $\int \tan^5(2x)\sec^2(2x)dx$

Choose $u = \tan(2x)$ because its derivative is in the integral.

$$u = \tan(2x),\ \ du = 2\sec^2(2x)\ dx \Rightarrow dx = \frac{du}{2\sec^2(2x)}$$

$$\int \tan^5(2x)\sec^2(2x)dx = \int u^5\sec^2(2x)\frac{du}{2\sec^2(2x)} = \frac{1}{2}\int u^5 du$$

$$= \frac{1}{12}u^6 + C = \boldsymbol{\frac{1}{12}\tan^6(2x) + C}$$

(c) $\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

Choose $u = \tan\left(\frac{x}{2}\right)$ because its derivative is in the integral.

$$u = \tan\left(\frac{x}{2}\right),\ \ du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \Rightarrow dx = \frac{2}{\sec^2\left(\frac{x}{2}\right)}$$

$$\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx = 2\int u\ du = u^2 + C$$

$$= \boldsymbol{\tan^2\left(\frac{x}{2}\right) + C}$$

2. (3.2) Evaluate the integrals using the guidelines for integrating powers of trig functions.

(a) $\int \sin^3 x\ dx$

Use the reduction formula for $\sin x$.

$$\int \sin^n x\ dx = \frac{-\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\ dx$$

$$\int \sin^3 x\ dx = \frac{-\cos x\sin^2 x}{3} + \frac{2}{3}\int \sin x\ dx$$

$$= \frac{-\cos x\sin^2 x}{3} - \frac{2}{3}\cos x + C$$

$$= \boldsymbol{\frac{-\cos x\sin^2 x - 2\cos x}{3} + C}$$

(b) $\int \sin x\cos x\ dx$

Use u-substitution.

$$u = \sin x,\ \ du = \cos x\ dx$$

$$\int \sin x\cos x\ dx = \int u\ du = \frac{1}{2}u^2 + C$$

$$= \boldsymbol{\frac{1}{2}\sin^2 x + C}$$

(c) $\int \sin^5 x \cos^2 x \; dx$

Split off a $\sin x$ because its power is odd. Then replace $\sin^4 x$ with the Pythagorean identity squared.

$$\int \sin^5 x \cos^2 x \; dx = \int \sin^4 x \cos^2 x \sin x \; dx = \int \left(1 - \cos^2 x\right)^2 \cos^2 x \sin x dx$$

Now do u-substitution.

$$u = \cos x, \;\; du = -\sin x dx$$

$$\int \left(1 - \cos^2 x\right)^2 \cos^2 x \sin x dx = -\int \left(1 - u^2\right)^2 u^2 du = -\int u^2 \left(1 - 2u^2 + u^4\right) du$$

$$= -\int \left(u^2 - 2u^4 + u^6\right) du = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \boldsymbol{-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C}$$

(d) $\int \sqrt{\sin x} \cos x \; dx$

Use u-substitution.

$$u = \sin x, \;\; du = \cos x \; dx$$

$$\int \sqrt{\sin x} \cos x \; dx = \int \sqrt{u} du = \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \boldsymbol{\frac{2}{3} \sin^{\frac{3}{2}} x + C}$$

(e) $\int \sec x \tan x \; dx$

You should recognize the integrand as the derivative of $\sec x$.

$$\int \sec x \tan x \; dx = \boldsymbol{\sec x + C}$$

(f) $\int \tan^2 x \sec x \; dx$

Rewrite $\tan^2 x$ using the Pythagorean identity.

After distributing, you will be able to use reduction formulas.

$$\int \tan^2 x \sec x \; dx = \int \left(\sec^2 x - 1\right) \sec x \; dx = \int \left(\sec^3 x - \sec x\right) dx = \int \sec^3 x \; dx - \int \sec x \; dx$$

$$= \left[\frac{1}{2} \sec(x) \tan(x) + \frac{1}{2} \int \sec x \; dx\right] - \ln\left|\sec x + \tan x\right|$$

$$= \frac{1}{2} \sec(x) \tan(x) + \frac{1}{2} \ln\left|\sec x + \tan x\right| - \ln\left|\sec x + \tan x\right| + C$$

$$= \boldsymbol{\frac{1}{2} \sec(x) \tan(x) - \frac{1}{2} \ln\left|\sec x + \tan x\right| + C}$$

(g) $\int \sec^4 x \; dx$

You will need to use the reduction formula.

$$\int \sec^4 x \; dx = \frac{1}{3} \sec^2(x) \tan(x) + \frac{2}{3} \int \sec^2(x) \; dx$$

We don't need another reduction formula because $\sec^2 x$ is the derivative of $\tan x$

$$= \boldsymbol{\frac{1}{3} \sec^2(x) \tan(x) + \frac{2}{3} \tan(x) + C}$$

3. (3.2) Find a general formula for the integral:

$$\int \sin^2(ax)\cos(ax)dx$$

$$\sin(ax) \text{ has the power, so make } u = \sin(ax).$$
$$u = \sin(ax),\ \ du = a\cos(ax)dx$$
$$\int \sin^2(ax)\cos(ax)dx = \frac{1}{a}\int u^2 du = \frac{1}{a}\cdot\frac{1}{3}u^3 + C$$
$$= \boldsymbol{\frac{1}{3a}\sin^3(ax) + C}$$

4. (3.2) Evaluate the integrals using double-angle formulas.

(a) $\int_0^\pi \sin^2 x\ dx$

$$\text{From the double-angle formula } \cos(2x) = 1 - 2\sin^2(x),\ \sin^2 x = \frac{1}{2}(1-\cos(2x))$$
$$\text{Now we don't need a reduction formula to integrate.}$$
$$\int_0^\pi \sin^2 x\ dx = \int_0^\pi \frac{1}{2}(1-\cos(2x))\,dx = \frac{1}{2}\int_0^\pi (1-\cos(2x))\,dx = \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^\pi$$
$$= \frac{1}{2}\left(\pi - \frac{1}{2}\sin(2\pi)\right) = \frac{1}{2}(\pi - 0)$$
$$= \boldsymbol{\frac{\pi}{2}}$$

(b) $\int \cos^2(3x)\ dx$

$$\text{From the double-angle formulas, } \cos^2 x = \frac{1+\cos(2x)}{2}.$$
$$\text{Apply the formula to } \cos^2(6x).$$
$$\int \cos^2(3x)\ dx = \int \left(\frac{1+\cos(6x)}{2}\right) dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos(6x)\right) dx$$
$$= \boldsymbol{\frac{1}{2}x + \frac{1}{12}\sin(6x) + C}$$

(c) $\int \sin^2 x\ dx + \int \cos^2 x\ dx$

$$\text{From the double-angle formulas, } \sin^2 x = \frac{\cos(2x)-1}{-2} \text{ and } \cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\int \sin^2 x\ dx + \int \cos^2 x\ dx = \int \frac{\cos(2x)-1}{-2}dx + \int \frac{1+\cos(2x)}{2}dx$$
$$= \int \left(-\frac{1}{2}\cos(2x) + \frac{1}{2}\right) dx + \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx$$

5. (3.2) Evaluate the definite integrals

(a) $\int_0^{2\pi} \cos x \sin(2x)dx$

Replace $\sin 2x$ with the double angle identity.

$$= \int_0^{2\pi} \cos x\,(2\sin x\cos x)\,dx$$

$$= \int_0^{2\pi} 2\cos^2 x \sin x\ dx$$

Use u-substitution.

$$u = \cos x \Rightarrow du = -\sin x\ dx$$

$$\int_0^{2\pi} 2\cos^2 x\sin x\ dx = -2\int_1^1 u^2 du = \mathbf{0}$$

The integral evaluates to 0 because the bounds are the same.

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}}dx$ (Round to three decimal places)

Generally, u-substitution works good with composite functions (functions inside of functions).

We could make $u = \sin x$ and see what happens.

$$u = \sin x,\ du = \cos x dx$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}}dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\cos^2 x}{\sqrt{u}}du$$

Use the Pythagorean identity to rewrite $\cos^2 x$ in terms of $\sin x = u$.

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1-u^2}{\sqrt{u}}du$$

Split the fraction into two separate ones.

This is a good technique every time you have a fraction with more than one term in the numerator, and one in the denominator.

$$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{u}} - u^{\frac{3}{2}}\right) du$$

$$= \left[2\sqrt{u} - \frac{2}{5}u^{\frac{5}{2}}\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

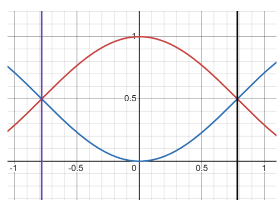
$$\approx \mathbf{0.239}$$

(c) $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}dx$
Manipulate the double angle identity for cosine:

$$\cos(2x) = 1 - 2\sin^2 x$$
$$1 - \cos(2x) = 1 - \left(1 - 2\sin^2 x\right) = 2\sin^2 x$$
$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}dx$$
$$= \sqrt{2}\int_0^{\frac{\pi}{2}} \sin x \; dx$$
$$= -\sqrt{2}\cos x\Big|_0^{\frac{\pi}{2}}$$
$$= -\sqrt{2}\,(0-1) = \boldsymbol{\sqrt{2}}$$

6. (3.2) Find the area of the region bounded by the graphs of the functions:

$$y = \cos^2 x,\;\; y = \sin^2 x,\;\; x = -\frac{\pi}{4},\;\; x = \frac{\pi}{4}$$



Set up the integral, knowing that $\cos^2 x$ is the top function.

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\cos^2 x - \sin^2 x\right) dx$$

This is another double angle identity problem. You can also use reduction formulas but I think it's too much work.

$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(2x)dx = \frac{1}{2}\sin(2x)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{2} - \frac{-1}{2}\right)$$
$$= \mathbf{1}\ \boldsymbol{unit^2}$$

7. Find the average value of the function $f(x) = \sin^2 x\cos^3 x$ over the interval $[-\pi, \pi]$
Recall the average value formula for a function $f(x)$ over $[a, b]$:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Set up and evaluate the integral:

$$f_{ave} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\cos^3 x \; dx$$

Split off a $\cos x$, then apply the Pythagorean identity to $\cos^2 x$.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^3 x \; dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x \cos^2 x \sin x \; dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1-\cos^2 x\right)\cos^2 x \sin x \; dx$$

Use u-substitution to get rid of $\sin x$.

$$u = \cos x \Rightarrow du = \sin x \; dx$$
$$f_{ave} = \frac{1}{2\pi}\int_{-1}^{-1} \left(1-u^2\right) u^2 du = \mathbf{0}$$

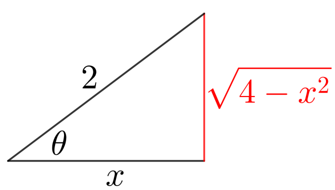
You should stop right when you see that the bounds are the same and the integral will be zero. If this was a problem with different bounds, you would just distrubute the u^2, do anti-power-rule, and plug $\cos x$ back in for u.

3 3.3 - Trigonometric Substitution

Note: Your answer and problem-solving methodology may differ from mine and still be correct. To verify, use a graphing calculator. The functions should be of the same form, only separated by a constant. This works because depending on the result of the integral, the constant of integration C will be different. It does not matter because according to the Fundamental Theorem, only the difference of the antiderivative function at the bounds matters, which will be the same no matter the value of C.

1. (3.3) Integrate using the method of trigonometric substitution.

   (a) $\int \frac{dx}{\sqrt{4-x^2}}$
   Draw the triangle and solve for $\sqrt{4-x^2}$.



$$\sin\theta = \frac{\sqrt{4-x^2}}{2}$$
$$\sqrt{4-x^2} = 2\sin\theta$$

Now find x, θ, and dx:

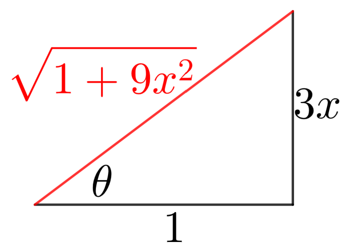
$$\cos\theta = \frac{x}{2}$$
$$x = 2\cos\theta$$
$$dx = -2\sin\theta d\theta$$
$$\theta = \arccos\frac{x}{2}$$

Set up and evaluate the integral:

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{-2\sin\theta}{2\sin\theta} d\theta = \int -d\theta$$
$$= -\theta + C$$
$$= \mathbf{-\arccos\frac{x}{2} + C}$$

Another possible answer is $\mathbf{\arcsin\frac{x}{2} + C}$.

(b) $\int \frac{dx}{\sqrt{1+9x^2}}$
Draw the triangle:



Since one of the side lengths is 1, you can solve for the entire expresssion $\frac{1}{\sqrt{1+9x^2}}$.

$$\cos\theta = \frac{1}{\sqrt{1+9x^2}}$$

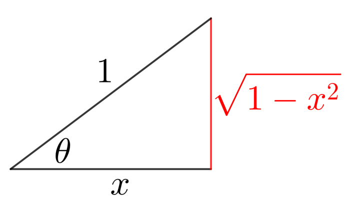
Now solve for x and thus dx.

$$x = \frac{1}{3}\tan\theta$$
$$dx = \frac{1}{3}\sec^2\theta \; d\theta$$

Set up and evaluate the integral:

$$\int \frac{dx}{\sqrt{1+9x^2}} = \frac{1}{3}\int \cos\theta\sec^2\theta \; d\theta$$
$$= \frac{1}{3}\int \sec\theta \; d\theta = \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$
$$= \mathbf{\frac{1}{3}\ln\left|\sqrt{1+9x^2} + 3x\right| + C}$$

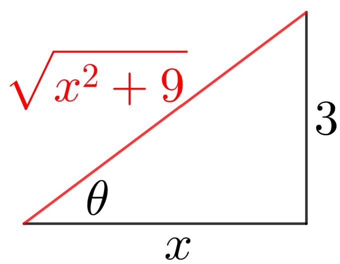
(c) $\int \frac{dx}{x^2\sqrt{1-x^2}}$
Draw the triangle:



Solve for $\frac{1}{\sqrt{1-x^2}}$, $\frac{1}{x^2}$, x, and dx.

$$\frac{1}{\sqrt{1-x^2}} = \csc\theta$$
$$\frac{1}{x^2} = \sec^2\theta$$
$$x = \cos\theta \Rightarrow dx = -\sin\theta\ d\theta$$
$$\int \frac{1}{x^2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot dx = -\int \csc\theta\sec^2\theta\sin^2\theta\ d\theta$$
$$= -\int \sec^2\theta\ d\theta = -\tan\theta + C$$
$$= \boldsymbol{-\frac{\sqrt{1-x^2}}{x} + C}$$

(d) $\int \sqrt{x^2+9}\ dx$
Draw the triangle:



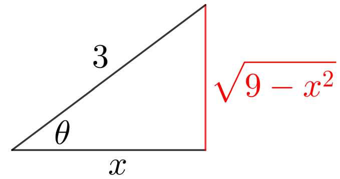
Solve for $\sqrt{x^2+9}$, x, and dx.

$$\csc\theta = \frac{\sqrt{x^2+9}}{3} \Rightarrow \sqrt{x^2+9} = 3\csc\theta$$
$$\cot\theta = \frac{x}{3} \Rightarrow x = 3\cot\theta \Rightarrow dx = -3\csc^2\theta\ d\theta$$
$$\int \sqrt{x^2+9}\ dx = -\int 3\csc\theta\cdot 3\csc^2\theta\ d\theta = -9\int \csc^3\theta\ d\theta$$

Use the reduction formula for cosecant.

$$= -9\left[-\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\int \csc\theta\ d\theta\right]$$
$$= \frac{9}{2}\csc\theta\cot\theta - \frac{9}{2}\left(-\ln|\csc\theta+\cot\theta|\right) + C$$
$$= \frac{9}{2}\csc\theta\cot\theta + \frac{9}{2}\ln|\csc\theta+\cot\theta| + C$$
$$= \frac{9}{2}\cdot\frac{\sqrt{x^2+9}}{3}\cdot\frac{x}{3} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{x} + \frac{x}{3}\right|$$
$$= \boldsymbol{\frac{x\sqrt{x^2+9}}{2} + \frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right| + C}$$

(e) $\int \frac{x^3 dx}{\sqrt{9-x^2}}$
Draw the triangle:



Solve for $\sqrt{9-x^2}$, x, and dx.

$$\sin\theta = \frac{\sqrt{9-x^2}}{3} \Rightarrow \sqrt{9-x^2} = 3\sin\theta$$
$$\cos\theta = \frac{x}{3} \Rightarrow x = 3\cos\theta \Rightarrow dx = -3\sin\theta\ d\theta$$
$$x^3 = 27\cos^3\theta$$

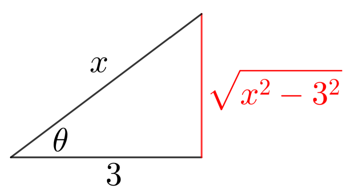
Set up and evaluate the integral using reduction formulas.

$$\int x^3 \cdot \frac{1}{\sqrt{9-x^2}} \cdot dx = \int 27\cos^3\theta \cdot \frac{1}{3}\csc\theta \cdot -3\sin\theta\ d\theta$$
$$= -27\int \cos^3\theta\ d\theta = -27\left[\frac{\sin\theta\cos^2\theta}{3} + \frac{2}{3}\int \cos\theta\ d\theta\right]$$
$$= -9\sin\theta\cos^2\theta - 18\sin\theta + C$$
$$= -9 \cdot \frac{\sqrt{9-x^2}}{3} \cdot \frac{x^2}{4} - 18 \cdot \frac{\sqrt{9-x^2}}{3} + C$$
$$= \boldsymbol{\frac{-x^2\sqrt{9-x^2}}{3} - 6\sqrt{9-x^2} + C}$$

(f) $\int \frac{dx}{(x^2-9)^{\frac{3}{2}}}$

This integral is equivalent to $\int \frac{dx}{\left(\sqrt{x^2-3^2}\right)^3} dx$.

Draw the triangle:



Solve for $\sqrt{x^2-3^2}$, x, and dx.

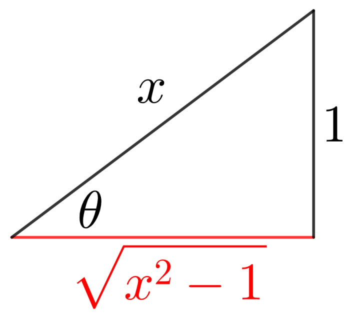
$$\tan\theta = \frac{\sqrt{x^2-3^2}}{3} \Rightarrow \sqrt{x^2-3^2} = 3\tan\theta$$
$$\sec\theta = \frac{x}{3} \Rightarrow x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\ d\theta$$

Set up and evaluate the integral:

$$\int \frac{dx}{(\sqrt{x^2-3^2})^3} dx = \int \frac{1}{(3\tan\theta)^3} \cdot 3\sec\theta\tan\theta\ d\theta$$
$$= \frac{1}{9}\int \frac{\sec\theta\tan\theta}{\tan^3\theta}\ d\theta = \frac{1}{9}\int \csc\theta\cot\theta\ d\theta$$
$$= -\frac{1}{9}\csc\theta + C = \boldsymbol{-\frac{x}{9\sqrt{x^2-9}} + C}$$

(g) $\int \frac{x^2 dx}{\sqrt{x^2-1}}$

Draw the triangle:



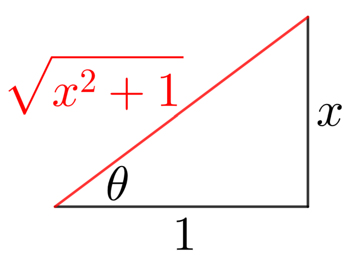
Solve for $\sqrt{x^2-1}$, x, x^2, and dx.

$$x = \csc\theta \Rightarrow dx = -\csc\theta\cot\theta\ d\theta$$
$$\sqrt{x^2-1} = \cot\theta$$
$$x^2 = \csc^2\theta$$

Set up and evaluate the integral.

$$\int \frac{x^2 dx}{\sqrt{x^2-1}} = \int \frac{-\csc^2\theta}{\cos\theta}\cdot\csc\theta\cot\theta\ d\theta = -\int \csc^3\theta\ d\theta$$
$$= -\left[-\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\int \csc\theta\ d\theta\right] = \frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\ln|\csc\theta+\cot\theta| + C$$
$$= \boldsymbol{\frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C}$$

(h) $\int \frac{dx}{x^2\sqrt{x^2+1}}$
Draw the triangle:



Solve for x, dx, x^2, and $\sqrt{x^2+1}$.

$$x = \tan\theta \Rightarrow dx = \sec^2\theta\ d\theta$$
$$x^2 = \tan^2\theta,\ \sqrt{x^2+1} = \sec\theta$$

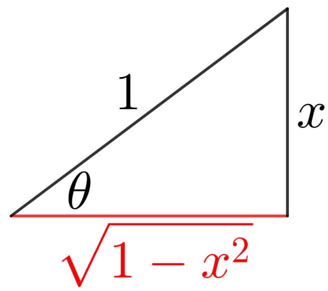
Set up and evaluate the integral.

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{1}{\tan^2\theta\sec^2\theta}\cdot\sec^2\theta\ d\theta = \int \frac{\sec\theta}{\tan^2\theta}d\theta$$
$$= \int \csc\theta\cot\theta\ d\theta$$
$$= -\csc\theta + C = \boldsymbol{-\frac{\sqrt{x^2+1}}{x} + C}$$

(i) $\int \left(1-x^2\right)^{\frac{3}{2}} dx$
Rewrite the integral:

$$\int \left(1-x^2\right)^{\frac{3}{2}} dx = \int \left(1-x^2\right)^3 d\theta$$

Draw the triangle.



Set up and evaluate the integral. You will need to use reduction formulas twice.

$$\begin{aligned}
\int \left(1-x^2\right)^{\frac{3}{2}}\,dx &= \int \left(\sqrt{1-x^2}\right)^3 dx = \int \cos^3\theta\cos\theta\; d\theta = \int \cos^4\theta\; d\theta \\
&= \frac{\sin\theta\cos^3\theta}{4} + \frac{3}{4}\int \cos^2\theta\; d\theta \\
&= \frac{\sin\theta\cos^3\theta}{4} + \frac{3}{4}\left[\frac{\sin\theta\cos\theta}{2} + \frac{1}{2}\int d\theta\right] \\
&= \frac{\sin\theta\cos^3\theta}{4} + \frac{3\sin\theta\cos\theta}{8} + \frac{3}{8}\theta + C \\
&= \boldsymbol{\frac{x\left(1-x^2\right)^{\frac{3}{2}}}{4} + \frac{3x\sqrt{1-x^2}}{8} + \frac{3}{8}\arcsin x + C}
\end{aligned}$$

2. (3.3) Use the technique of completing the square to evaluate the following integrals.

(a) $\int \frac{1}{x^2+2x+1}dx$
The integrand is already a perfect square.

$$\int \frac{1}{x^2+2x+1}dx = \int \frac{1}{(x+1)^2}dx$$

Usually for terms with the form u^2+a^2 we use the substitution $u=\tan\theta$. In this problem, $u=x+1=\tan\theta$ and $a=0$.
Set up the integral with the substitutions and evaluate.

$$x+1=\tan\theta \Rightarrow x=\tan\theta-1 \Rightarrow dx=\sec^2\theta\; d\theta$$

$$\begin{aligned}
\int \frac{1}{(x+1)^2}dx &= \int \frac{1}{\tan^2\theta}\sec^2\theta\; d\theta \\
= \int \frac{1}{\sin^2\theta}\; d\theta &= \int \csc^2\theta\; d\theta = -\cot\theta + C \\
&= \boldsymbol{-\frac{1}{x+1} + C}
\end{aligned}$$

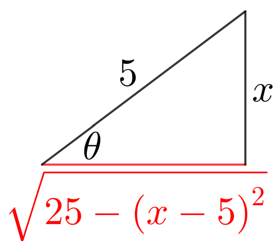
(b) $\int \frac{1}{\sqrt{-x^2+10x}}dx$
To complete the square inside the square root you will need to take out a -1 first.

$$\int \frac{1}{\sqrt{-x^2+10x}}dx = \int \frac{1}{\sqrt{-\left(x^2-10x\right)}}dx = \int \frac{1}{\sqrt{-\left(x^2-10x+25\right)+25}}dx$$

Notice that I put $+25$ on the outside instead of negative because the -1 I pulled out earlier would make the inner part -25, so the $+25$ will balance it.

$$= \int \frac{1}{\sqrt{25-(x-5)^2}}dx$$

Draw the triangle.



Solve for $\sqrt{25-(x-5)^2}$, x, and dx.

$$\cos\theta = \frac{\sqrt{25-(x-5)^2}}{5} \Rightarrow \sqrt{25-(x-5)^2} = 5\cos\theta$$
$$\sin\theta = \frac{x-5}{5} \Rightarrow x = 5\sin\theta + 5 \Rightarrow dx = 5\cos\theta\ d\theta$$

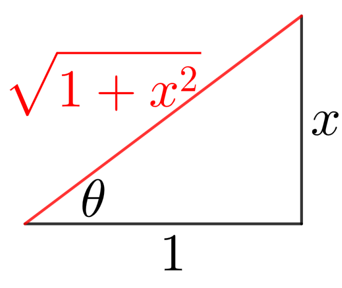
Set up and evaluate the integral.

$$\int \frac{1}{\sqrt{25-(x-5)^2}}dx = \int \frac{1}{5\cos\theta}\cdot 5\cos\theta\ d\theta = \int d\theta$$
$$= \theta + C = \boldsymbol{\arcsin\left(\frac{x-5}{5}\right) + C}$$

3. (3.3) Find the volume of the solid formed when the region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the x-axis between $x=0$ and $x=1$ is revolved about the x-axis.
This is the disk method.

$$V = \pi\int_0^1 \left(\frac{1}{1+x^2}\right)^2\ dx = \pi\int_0^1 \frac{1}{(1+x^2)^2}dx$$

Draw the triangle.



Solve for x, dx, and $1+x^2$.

$$x = \tan\theta \Rightarrow dx = \sec^2\theta\ d\theta$$
$$\sqrt{1+x^2} = \sec\theta \Rightarrow 1+x^2 = \sec^2\theta$$

Set up and evaluate the integral. Since this is a definite integral you will need to change the bounds according to $\theta = \arctan x$.

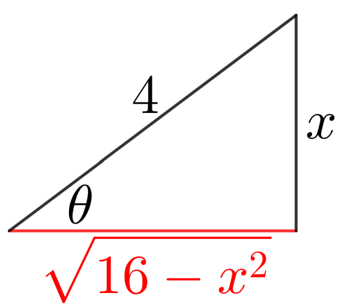
$$\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2\theta)^2} \sec^2\theta \, d\theta = \pi \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta \\
&= \pi \left[\frac{\sin\theta\cos\theta}{2} \bigg|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta \right] \\
&= \pi \left[\frac{\sin\theta\cos\theta}{2} \bigg|_0^{\frac{\pi}{4}} + \frac{1}{2}\theta \bigg|_0^{\frac{\pi}{4}} \right] \\
&= \pi \left[\frac{1}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right) \right] = \boldsymbol{\frac{\pi^2}{8} + \frac{\pi}{4}} \textbf{\textit{ units}}^{\mathbf{3}}
\end{aligned}$$

4. (3.3) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.
Find $\frac{dy}{dx}$, set up, and simplify the integral. You will need to get a common denominator inside the square root.

$$y = \left(16 - x^2\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(16 - x^2\right)^{\frac{-1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{16 - x^2}}$$

$$\begin{aligned}
S &= \int_0^2 \sqrt{1 + \left(-\frac{x}{\sqrt{16 - x^2}}\right)^2} \, dx = \int_0^2 \sqrt{1 + \frac{x^2}{16 - x^2}} = \int_0^2 \sqrt{\frac{16 - x^2 + x^2}{16 - x^2}} \, dx = \int_0^2 \sqrt{\frac{16}{16 - x^2}} \\
&= \int_0^2 \frac{4}{\sqrt{16 - x^2}} \, dx
\end{aligned}$$

Draw the triangle.



Solve for x, dx, and $\sqrt{16 - x^2}$, and evaluate the integral.

$$\sin\theta = \frac{x}{4}, \;\; x = 4\sin\theta \Rightarrow dx = 4\cos\theta \, d\theta$$

$$\cos\theta = \frac{\sqrt{16 - x^2}}{4} \Rightarrow \sqrt{16 - x^2} = 4\cos\theta$$

$$\begin{aligned}
\int_0^2 \frac{4}{\sqrt{16 - x^2}} \, dx &= 4\int_0^{\frac{\pi}{6}} \frac{1}{4\cos\theta} \cdot 4\cos\theta \, d\theta = 4\int_0^{\frac{\pi}{6}} d\theta = 4\theta \bigg|_0^{\frac{\pi}{6}} \\
&= \frac{4\pi}{6} = \boldsymbol{\frac{2\pi}{3}}
\end{aligned}$$

4 3.4 - Partial Fraction Decomposition

Note: I will be using the method of equating the coeficients to create the system of equations.

1. (3.4) Decompose the fractions as a sum or difference of simpler rational expressions.

   (a) $\frac{x^2+1}{x(x+1)(x+2)}$

$$\begin{aligned} \frac{x^2+1}{x(x+1)(x+2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \\ x^2+1 &= A(x+1)(x+2) + B(x)(x+2) + C(x)(x+1) \\ &= Ax^2 + Bx^2 + Cx^2 + 3Ax + 2Bx + Cx + 2A \end{aligned}$$

   Set up the system:

$$\begin{aligned} A+B+C &= 1 \\ 3A+2B+C &= 0 \\ 2A &= 1 \end{aligned}$$

$$\Rightarrow A = \frac{1}{2}$$

   Subtracting the bottom equation to cancel C:

$$\begin{aligned} B+C &= \frac{1}{2} \\ -(2B+C) &= -\left(-\frac{3}{2}\right) \end{aligned}$$

$$\Rightarrow -B = 2 \Rightarrow B = -2$$

   Plug in the coefficients we found:

$$\frac{x^2+1}{x(x+1)(x+2)} = \mathbf{\frac{1}{2x} - \frac{2}{x+1} + \frac{5}{2(x+2)}}$$

   (b) $\frac{3x+1}{x^2}$

$$\begin{aligned} \frac{3x+1}{x^2} &= \frac{A}{x} + \frac{B}{x^2} \\ 3x+1 &= Ax+B \\ &\Rightarrow A=3,\ B=1 \end{aligned}$$

$$\frac{3x+1}{x^2} = \boldsymbol{\frac{3}{x} + \frac{1}{x^2}}$$

   (c) $\frac{2x^4}{x^2-2x}$
   You can factor out an x and cancel it:

$$\frac{2x^4}{x^2-2x} = \frac{2x^3}{x-2}$$

   Since the deonominator is linear and has one solution, you can use synthetic division if you want, but I will show long division.

$$\begin{array}{r} 2x^2+4x+8 \\ x-2 \overline{)2x^3} \\ -\left(2x^3-4x^2\right) \\ --- \\ 4x^2 \\ -\left(4x^2-8x\right) \\ --- \\ 8x \\ -(8x-16) \\ --- \\ 16 \end{array}$$

$$\frac{2x^4}{x^2-2x} = \boldsymbol{2x^2+4x+8+\frac{16}{x-2}}$$

(d) $\frac{1}{x^2(x-1)}$

$$\begin{aligned} \frac{1}{x^2(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\ 1 &= \frac{A}{x}\left(x^2\right)(x-1) + \frac{B}{x^2}\left(x^2\right)(x-1) + \frac{C}{x-1}\left(x^2\right)(x-1) \\ &= A(x)(x-1) + B(x-1) + Cx^2 \\ &= A\left(x^2-x\right) + Bx - B + Cx^2 \\ &= Ax^2 - Ax + Bx - B + Cx^2 \end{aligned}$$

Set up and solve the system:

$$\begin{aligned} A + C &= 0 \\ B - A &= 0 \\ -B &= -1 \end{aligned}$$

$$\overline{\quad B=-1,\ A=-1,\ C=1}$$

$$\frac{1}{x^2(x-1)} = \boldsymbol{-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}}$$

(e) $\frac{1}{x(x-1)(x-2)(x-3)}$

$$\frac{1}{x(x-1)(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} + \frac{D}{x-3}$$

$$\begin{aligned} 1 &= A(x-1)(x-2)(x-2) + B(x)(x-2)(x-3) + C(x)(x-1)(x-3) + D(x)(x-1)(x-2) \\ &= A(x-1)\left(x^2-5x+6\right) + B(x)\left(x^2-5x+6\right) + C(x)\left(x^2-4x+3\right) + D(x)\left(x^2-3x+2\right) \\ &= A\left(x^3-6x^2+11x-6\right) + B\left(x^3-5x^2+6x\right) + C\left(x^3-4x^2+3x\right) + D\left(x^2-3x^2+2x\right) \\ &= Ax^3 - 6Ax^2 + 11Ax - 6A + Bx^3 - 5Bx^2 + 6Bx + Cx^3 - 4Cx^2 + 3Cx + Dx^3 - 3Dx^2 + 2Dx \end{aligned}$$

Set up and solve the system:

$$\begin{aligned} A+B+C+D&=0\\ -6A-5B-4C-3D&=0\\ 11A+6B+3C+2D&=0\\ -6A&=1 \end{aligned}$$

$$\Rightarrow A=\frac{-1}{6},\ B=\frac{1}{2},\ C=-\frac{1}{2},\ D=\frac{1}{6}$$

$$\frac{1}{x(x-1)(x-2)(x-3)}=-\mathbf{\frac{1}{6x}+\frac{1}{2(x-1)}-\frac{1}{2(x-2)}+\frac{1}{6(x-3)}}$$

(f) $\frac{3x^2}{(x-1)(x^2+x+1)}$

$$\begin{aligned} \frac{3x^2}{(x-1)(x^2+x+1)}&=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\\ 3x^2&=A\left(x^2+x+1\right)+(Bx+C)(x-1)\\ &=Ax^2+Ax+A+Bx^2-Bx+Cx-C \end{aligned}$$

Set up and solve the system:

$$\begin{aligned} A+B&=3\\ A-B+C&=0\\ A-C&=0 \end{aligned}$$

$$\Rightarrow A=1,\ B=2,\ C=1$$

$$\frac{3x^2}{(x-1)(x^2+x+1)}=\mathbf{\frac{1}{x-1}+\frac{2x+1}{x^2+x+1}}$$

2. (3.4) Evaluate the integrals using partial fraction decomposition.

(a) $\int \frac{3x}{x^2+2x-8}dx$

Rewrite the fraction in the integrand:

$$\begin{aligned} \frac{3x}{x^2+2x-8}&=\frac{A}{x-2}+\frac{B}{x+4}\\ 3x&=A(x+4)+B(x-2)\\ &=Ax+4A+Bx-2B \end{aligned}$$

Set up the system and solve:

$$\begin{aligned} A+B&=3\\ 4A-2B&=0 \end{aligned}$$

$$\Rightarrow A=1,\ B=2$$

$$\frac{3x}{x^2+2x-8}=\frac{1}{x-2}+\frac{2}{x+4}$$

$$\int\frac{3x}{x^2+2x-8}dx=\int\left(\frac{1}{x-2}+\frac{2}{x+4}\right)dx=\mathbf{\ln|x-2|+2\ln|x+4|+C}$$

(b) $\int \frac{x}{x^2-4}dx$
Rewrite the fraction in the integrand

$$\begin{aligned}\frac{x}{x^2-4} &= \frac{x}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \\ x &= A(x-2) + B(x+2) \\ &= Ax - 2A + Bx - 2B\end{aligned}$$

Set up the system and solve.

$$\begin{aligned}A + B &= 1 \\ -2A + 2B &= 0 \\ \hline \Rightarrow A &= \frac{1}{2},\ B = \frac{1}{2}\end{aligned}$$

$$\int \frac{x}{x^2-4}dx = \int \frac{1}{2(x+2)} + \frac{1}{2(x-2)}dx = \boldsymbol{\frac{1}{2}\ln|x+2| + \frac{1}{2}\ln|x-2| + C}$$

(c) $\int \frac{2x^2+4x+22}{x^2+2x+10}dx$
Perform long division on the integrand.

$$\begin{array}{r} 2 \\ x^2+2x+10\overline{)2x^2+4x+22} \\ -\left(2x^2+4x+20\right) \\ --- \\ 2 \end{array}$$

$$\Rightarrow \frac{2x^2+4x+22}{x^2+2x+10} = 2 + \frac{2}{x^2+2x+10}$$

We do not have to decompose this further since the denominator of the second term is irreducible. We will complete the square instead.

$$2 + \frac{2}{x^2+2x+10} = 2 + \frac{2}{x^2+2x+1+10-1}$$

$$\int \frac{2x^2+4x+22}{x^2+2x+10}dx = \int \left(2 + \frac{2}{(x+1)^2+3^2}\right)dx = 2x + 2\int \frac{1}{(x+1)^2+3^2}dx$$

This is a common inverse trig integral: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

$$2x + 2\int \frac{1}{(x+1)^2+3^2}dx = \boldsymbol{2x + \frac{2}{3}\arctan\left(\frac{x+1}{3}\right) + C}$$

(d) $\int \frac{2-x}{x^2+x}dx$
Rewrite the fraction in the integrand

$$\begin{aligned}\frac{2-x}{x^2+x}dx &= \frac{A}{x} + \frac{B}{x+1} \\ 2-x &= A(x+1) + B(x) \\ &= Ax + A + Bx\end{aligned}$$

Set up the system and solve.

$$\begin{aligned} A + B &= -1 \\ A &= 2 \\ \hline &\Rightarrow A = 2,\ B = -3 \end{aligned}$$

$$\int \frac{2-x}{x^2+x}dx = \int \left(\frac{2}{x} - \frac{3}{x+1}\right) dx = \mathbf{2\ln|x| - 3\ln|x+1| + C}$$

(e) $\int \frac{dx}{x^3-2x^2-4x+8}$
To factor a cubic polynomial, find one of the zeros $(x-a)$ terms, and divide the cubic by that zero to get the remaining quadratic. That zero term multiplied by the qudratic is your cubic. I will use synthetic division for this.

$$\begin{array}{r|rrrr} \mathbf{-2} & 1 & -2 & -4 & 8 \\ & & -2 & 8 & -8 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

The quotient is $x^2-4x+4 = (x-2)^2$. So the cubic is equivalent to $(x+2)(x-2)^2$.
Rewrite the fraction in the integrand.

$$\begin{aligned} \frac{1}{x^3-2x^2-4x+8} = \frac{1}{(x+2)(x-2)^2} &= \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \\ 1 &= A(x-2)^2 + B(x-2)(x+2) + C(x+2) \\ &= A\left(x^2-4x+4\right) + B\left(x^2-4\right) + C(x+2) \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 4B + Cx + 2C \end{aligned}$$

Set up and evaluate the system.

$$\begin{aligned} A + B &= 0 \\ -4A + C &= 0 \\ 4A - 4B + 2C &= 1 \\ \hline &\Rightarrow A = \frac{1}{16},\ B = -\frac{1}{16},\ C = \frac{1}{4} \end{aligned}$$

$$\int \left(\frac{1}{16(x+2)} - \frac{1}{16(x-2)} + \frac{1}{4(x-2)^2}\right) dx$$

$$= \mathbf{\frac{1}{16}\ln|x+2| - \frac{1}{16}\ln|x-2| - \frac{1}{4(x-2)} + C}$$

3. (3.4) Evaluate the integrals with irreducible quadratic factors.

(a) $\int \frac{2}{(x-4)(x^2+2x+6)}dx$ Hint: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

Rewrite the fraction in the integrand.

$$\frac{2}{(x-4)(x^2+2x+6)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+2x+6}$$
$$2 = A\left(x^2+2x+6\right) + (Bx+C)(x-4)$$
$$= Ax^2 + 2Ax + 6A + Bx^2 - 4Bx + Cx - 4C$$

Set up the system and solve.

$$A + B = 0$$
$$2A - 4B + C = 0$$
$$6A - 4C = 2$$
$$\Rightarrow A = \frac{1}{15},\ B = -\frac{1}{15},\ C = -\frac{2}{5}$$

$$\int \frac{2}{(x-4)(x^2+2x+6)}dx = \int \left(\frac{1}{15(x-4)} + \frac{\frac{-x}{15} - \frac{2}{5}}{x^2+2x+6}\right)dx$$

The first term of the integrand is a natural log integral. For the second term, get a common denominator in the numerator, then split the fraction so you have two seperate intgegrals.

$$\int \left(\frac{1}{15(x-4)} + \frac{\frac{-x}{15} - \frac{2}{5}}{x^2+2x+6}\right)dx = \frac{1}{15}\ln|x-4| + \int \frac{\frac{-x}{15} - \frac{6}{15}}{(x^2+2x+6)}dx$$
$$= \frac{1}{15}\ln|x-4| + \int \frac{-x-6}{15(x^2+2x+6)}dx$$

Factor out $-\frac{1}{15}$.

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{x^2+2x+6}dx$$

Here's where we split the integral.

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x}{x^2+2x+6}dx - \frac{1}{15}\int \frac{6}{x^2+2x+6}dx$$

Complete the square in each denominator.

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x}{(x+1)^2+5}dx - \frac{6}{15}\int \frac{1}{(x+1)^2+5}dx$$

Use u-substitution in the first integral

and use the trig integral from the hint in the second integral.

$u = x+1,\ x = u-1,\ du = dx$

$$= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{u-1}{u^2+5}dx - \frac{2}{5}\cdot\frac{1}{\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)$$
$$= \frac{1}{15}\ln|x-4| - \frac{2}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right) - \frac{1}{15}\int \left(\frac{u}{u^2+5} - \frac{1}{u^2+5}\right)du$$

Use substitution with v since u is taken.

$v = u^2+5,\ dv = 2u\ du$

The second integral is the trig integral again.

$$= \frac{1}{15}\ln|x-4| - \frac{2}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right) - \frac{1}{15}\left[\frac{1}{2}\int \frac{1}{v}dv - \frac{1}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right)\right]$$
$$= \frac{1}{15}\ln|x-4| - \frac{2}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right) - \frac{1}{15}\left[\frac{1}{2}\ln|v| - \frac{1}{\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)\right] + C$$

$$= \mathbf{\frac{1}{15}\ln|x-4| - \frac{2}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right) - \frac{1}{15}\left[\frac{1}{2}\ln\left|(x+1)^2+5\right| - \frac{1}{\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)\right] + C}$$

(b) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$
Do long division on the integrand.

$$\begin{array}{r} 1 \\ x^3+2x^2 \overline{\smash{)}x^3+6x^2+3x+6} \\ -\left(x^3+2x^2\right) \\ --- \\ 4x^2+3x+6 \end{array}$$

Decompose the fraction of the remainder.

$$\begin{aligned} 1 + \frac{4x^2+3x+6}{x^3+2x^2} &= 1 + \frac{4x^2+3x+6}{x^2(x+2)} \\ \frac{4x^2+3x+6}{x^2(x+2)} &= \frac{A}{x+2} + \frac{B}{x} + \frac{C}{x^2} \\ 4x^2+3x+6 &= Ax^2 + Bx(x+2) + C(x+2) \\ &= Ax^2 + Bx^2 + 2B + Cx + 2C \end{aligned}$$

Set up the system and solve.

$$\begin{aligned} A + B &= 4 \\ C &= 3 \\ 2B + C &= 6 \\ \hline &\Rightarrow A = 4,\ B = 0,\ C = 3 \end{aligned}$$

Evaluate the integral.

$$\begin{aligned} \int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx &= \int \left(1 + \frac{4x^2+3x+6}{x^2(x+2)}\right) \\ &= x + \int \left(\frac{4}{x+2} + \frac{3}{x^2}\right)dx \\ &= x + 4\ln|x+2| + \int 3x^-2dx \\ &= x + 4\ln|x+2| + \frac{3}{-1}x^-1 + C \\ &= \mathbf{x + 4\ln|x+2| - \frac{3}{x} + C} \end{aligned}$$

4. (3.4) Evaluate the integral $\int \frac{3x+4}{(x^2+4)(3-x)}dx$ using partial fraction decomposition.
Rewrite the fraction in the integrand.

$$\begin{aligned} \frac{3x+4}{(x^2+4)(3-x)} &= \frac{A}{3-x} + \frac{Bx+C}{x^2+4} \\ 3x+4 &= A\left(x^2+4\right) + (Bx+C)(3-x) \\ &= Ax^2 + 4A + 3Bx - Bx^2 + 3c - Cx \end{aligned}$$

Set up the system and solve.

$$\begin{aligned} A - B &= 0 \\ 3B - C &= 3 \\ 4A + 3C &= 4 \\ \hline &\Rightarrow A = 1,\ B = 1,\ C = 0 \end{aligned}$$

Be careful with the signs when integrating. The x term in $\frac{1}{3-x}$ is negative, so the natural log will be negative.

$$\begin{aligned} \int \frac{3x+4}{(x^2+4)(3-x)} dx &= \int \left(\frac{1}{3-x} + \frac{x}{x^2+4} \right) dx \\ &= -\ln|3-x| + \int \frac{x}{x^2+4} dx \end{aligned}$$

$$u = x^2 + 4 \Rightarrow du = 2x\ dx,\ x\ dx = \frac{1}{2} du$$

$$\begin{aligned} &= -\ln|3-x| + \frac{1}{2} \int \frac{1}{u} du \\ &= -\ln|3-x| + \frac{1}{2} \ln|u| + C \\ &= \boldsymbol{-\ln|3-x| + \frac{1}{2} \ln\left|x^2+4\right| + C} \end{aligned}$$

5. (3.4) Use substitution to convert the integrals to integrals of rational functions. Then evaluate using partial fraction decomposition.

   (a) $\int \frac{e^x}{e^{2x}-e^x} dx$
   Make the substitution $u = e^x$.

$$u = e^x,\ du = e^x dx$$

$$\begin{aligned} \int \frac{1}{u^2-u} du &= \int \frac{1}{u(u-1)} du \\ \frac{1}{u(u-1)} &= \frac{A}{u} + \frac{B}{u-1} \\ 1 &= A(u-1) + B(u) \\ \int \frac{1}{u(u-1)} du &= \int \left(-\frac{1}{u} + \frac{1}{u-1} \right) du \\ &= -\ln|u| + \ln|u-1| + C \\ &= \boldsymbol{-\ln|e^x| + \ln|e^x - 1| + C} \end{aligned}$$

   The $e^x - 1$ can also be rewritten as $1 - e^x$.

   (b) $\int \frac{\cos x}{\sin x(1-\sin x)} dx$
   Make the substitution $u = \sin x$.

$$u = \sin x,\ du = \cos x\ dx$$

$$\int \frac{1}{(1-u)} du = -\int \frac{1}{u(u-1)} du$$

   This is now the same integral as the previous problem.

$$\begin{aligned} &= [-\ln|u| + \ln|u-1| + C] \\ &= \ln|u| - \ln|u-1| + C \\ &= \boldsymbol{\ln|\sin x| - \ln|\sin x - 1| + C} \end{aligned}$$

   The $\sin x - 1$ can be replaced with $1 - \sin x$.

5 3.5 - Integration Tables

The tables are attached to the end of this document.

1. (3.5) Evaluate the integrals using the table of integrals.

   (a) $\int \frac{dy}{\sqrt{4-y^2}}$
   Use formula #77:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$$
$$u = y, \ a = 2$$
$$\int \frac{dy}{\sqrt{4-y^2}} = \boldsymbol{\arcsin\left(\frac{y}{2}\right) + C}$$

   (b) $\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right)dx$
   Use formula #28 and don't forget to change the bounds:

$$\int \tan^2 u \ du = \tan u - u + C$$
$$u = \frac{x}{2}, \ du = \frac{1}{2}dx, \ dx = 2 \ du$$
$$x = 0 \Rightarrow u = 0, \ x = \frac{\pi}{2} \Rightarrow u = \frac{\pi}{4}$$
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right)dx &= 2\int_0^{\frac{\pi}{4}} \tan^2 u \ du \\ &= 2\left[\tan u - u\right]_0^{\frac{\pi}{4}} \\ &= 2\left[\left(1-\frac{\pi}{4}\right) - (0-0)\right] \\ &= \boldsymbol{2 - \frac{\pi}{2}}\end{aligned}$$

   (c) $\int \tan^5(3x)dx$
   Use formula #31, which is basically a reduction formula.

$$u = 3x, \ du = 3 \ dx$$
$$\int \tan^5(3x) \ dx = \frac{1}{3}\int \tan^5 u \ du$$
$$\int \tan^n u \ du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u \ du$$
$$\begin{aligned}\frac{1}{3}\int \tan^5 u \ du &= \frac{1}{3}\left[\frac{1}{4}\tan^4 u - \int \tan^3 u \ du\right] \\ &= \frac{1}{3}\left[\frac{1}{4}\tan^4 u - \left[\frac{1}{2}\tan^2 u - \int \tan u \ du\right]\right] \\ &= \frac{1}{3}\left[\frac{1}{4}\tan^4 u - \frac{1}{2}\tan^2 u + \ln|\sec u|\right] + C \\ &= \boldsymbol{\frac{1}{3}\left[\frac{1}{4}\tan^4 3x - \frac{1}{2}\tan^2 3x + \ln|\sec 3x|\right] + C}\end{aligned}$$

2. (3.5) Evaluate the integrals using the table of integrals. You may need to complete the square or make substitutions.

(a) $\int \frac{dx}{x^2+2x+10}$
Complete the square in the denominator. Then use formula #68.

$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{x^2+2x+1+10-1} = \int \frac{dx}{(x+1)^2+9}$$

$$u = x+1,\ du = dx,\ a = 3$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

$$\int \frac{dx}{(x+1)^2+9} = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C$$

$$= \boldsymbol{\frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C}$$

(b) $\int \frac{e^x}{\sqrt{e^{2x}-4}}dx$
Make substitutions and use formula #76.

$$u = e^x,\ du = e^x\ dx,\ a = 2$$

$$\int \frac{e^x}{\sqrt{e^{2x}-4}}dx = \int \frac{1}{\sqrt{u^2-2^2}}du$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u + \sqrt{u^2-a^2}\right| + C$$

$$\int \frac{1}{\sqrt{u^2-2^2}}du = \boldsymbol{\ln\left|e^x + \sqrt{e^{2x}-4}\right| + C}$$

6 3.7 - Improper Integrals

Evaluate the following integrals, or indicate that it's divergent.

1. $\int_2^4 \frac{dx}{(x-3)^2}$
Split the integral to account for the discontinuity at $x = 3$.

$$\int_2^4 \frac{dx}{(x-3)^2} = \int_2^3 \frac{dx}{(x-3)^2} + \int_3^4 \frac{dx}{(x-3)^2}$$

Start by integrating $\int_2^3 \frac{dx}{(x-3)^2}$.

$$\int_2^3 \frac{dx}{(x-3)^2} = \int_0^3 (x-3)^{-2}\,dx - \int_0^2 (x-3)^{-2}\,dx$$

I will start by integrating the part with the discontinuity, because it's the only integral that can diverge. If it diverges, the whole thing diverges and we would save time.

$$\begin{aligned}\int_0^3 (x-3)^{-2}\,dx &= \lim_{t\to 3}\int_0^t (x-3)^{-2}\,dx \\ &= \lim_{t\to 3} -\frac{1}{x-3}\bigg|_0^t \\ &= \lim_{t\to 3}\left[-\frac{1}{t-3} - \frac{1}{3}\right] \\ &= \text{DNE (limits from the left and right don't match.)}\end{aligned}$$

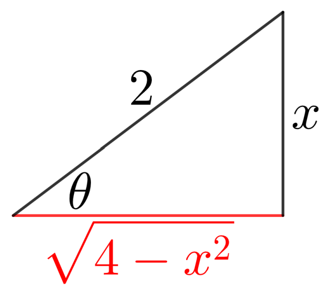
The integral diverges.

It doesn't matter if the function approaches ∞ or $-\infty$. Actually this function does both, depending on what side of $t = 3$ you're on. When integrating, we only care about area under the curve. No matter how the function diverges, it's still an infinite area. You might want to explain how the limit does not exist, stating that the inegral diverges because the limit from the left and right are not equal, and that there is a vertical asymptote at $t = 3$.

2. $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$
By inspection, this function is discontinuous at $x = 2$ since it would make the denominator 0.

$$\int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{t \to 2} \int_0^t \frac{1}{\sqrt{4-x^2}} dx$$

Make a trigangle for trig-sub.



$$\begin{aligned} \cos\theta &= \frac{\sqrt{4-x^2}}{2}, \ \sqrt{4-x^2} = 2\cos\theta \\ \sin\theta &= \frac{x}{2}, \ x = 2\sin\theta, \ dx = 2\cos\theta \ d\theta \\ \lim_{t \to 2} \int_0^t \frac{1}{\sqrt{4-x^2}} dx &= \lim_{t \to 2} \int_0^t \frac{1}{2\cos\theta} \cdot 2\cos\theta \ d\theta \\ &= \lim_{t \to 2} \int_0^t \theta \ d\theta = \lim_{t \to 2} \theta \Big|_0^t \\ &= \lim_{t \to 2} \left[\arcsin\frac{x}{2}\right]_0^t \\ &= \lim_{t \to 2} \arcsin\left(\frac{t}{2}\right) = \arcsin\frac{2}{2} \\ &= \arcsin 1 = \boldsymbol{\frac{\pi}{2}} \end{aligned}$$

3. $\int_1^\infty xe^{-x}dx$
This function is continuous everywhere, so take the limit to ∞. Use IBP.

$$\int_1^\infty xe^{-x}dx = \lim_{t\to\infty}\int_1^t xe^{-x}dx$$

$$u = x,\ du = dx,\ dv = e^{-x}dx,\ v = -e^{-x}$$

$$\begin{aligned}\lim_{t\to\infty}\int_1^t xe^{-x}dx &= \lim_{t\to\infty} -xe^{-x}\Big|_1^t + \int_1^t e^{-x}dx \\ &= \lim_{t\to\infty}\left[\left(-te^{-t} - \left(-e^{-1}\right)\right) + -e^{-x}\Big|_1^t\right] \\ &= \lim_{t\to\infty}\left(-te^{-t} + \frac{1}{e} - e^{-t} + \frac{1}{e}\right) \\ &= \lim_{t\to\infty}\left(e^{-t}\left(-t-1\right) + \frac{2}{e}\right) \\ &= 0 + \frac{2}{e} \\ &= \boldsymbol{\frac{2}{e}}\end{aligned}$$

4. $\int_0^1 \frac{\ln x}{\sqrt{x}}dx$
This function is discontinuous at the lower bound. I can take out a -1 and swap the bounds so the discontinuity is at the top. It's not necesary.

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x}}dx &= \lim_{t\to 0} -\int_1^t \frac{\ln x}{\sqrt{x}}dx \\ &= \lim_{t\to 0} -\int_1^t \ln x\ x^{-1/2}dx\end{aligned}$$

$$u = \ln x,\ du = x^{-1}dx,\ dv = x^{-1/2}dx,\ v = 2x^{1/2}$$

$$\begin{aligned}\lim_{t\to 0} -\int_1^t \ln x\ x^{-1/2}dx &= \lim_{t\to 0} -\left[2\sqrt{x}\ln x\Big|_1^t - \int_1^t 2\frac{x^{1/2}}{x}dx\right] \\ &= \lim_{t\to 0} -\left[\left(2\sqrt{t}\ln t - 0\right) - 2\int_1^t \frac{1}{\sqrt{x}}dx\right] \\ &= \lim_{t\to 0} -\left(2\sqrt{t}\ln t - 2\int_1^t x^{-1/2}dx\right) \\ &= \lim_{t\to 0} -\left(2\sqrt{t}\ln t - 2\cdot 2\sqrt{x}\Big|_1^t\right) \\ &= \lim_{t\to 0} -\left(2\sqrt{t}\ln t - 4\sqrt{t} + 4\right) \\ &= \lim_{t\to 0} -\left(\sqrt{t}\left(2\ln t - 4\right) + 4\right) \\ &= -\left(0\left(2\ln t - 4\right) + 4\right) \\ &= \boldsymbol{-4}\end{aligned}$$

5. $\int_{-\infty}^\infty \frac{1}{x^2+1}dx$
This function is continuous everywhere but we have to split the integral at 0 to take the limit at $-\infty$ and ∞ seperately.

$$\int_{-\infty}^\infty \frac{1}{x^2+1}dx = \int_{-\infty}^0 \frac{1}{x^2+1}dx + \int_0^\infty \frac{1}{x^2+1}dx$$

Since these functions are symetrical at $x = 0$, we only need to do one integral, then the other half will be the same, so multiply it by 2 at the end.

$$\begin{aligned}
\int_0^{\infty} \frac{1}{x^2+1} dx &= \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1} dx \\
&= \lim_{t\to\infty} \arctan t \Big|_0^t \\
&= \lim_{t\to\infty} (\arctan t - \arctan 0) \\
&= \lim_{t\to\infty} \arctan t \\
&= \arctan \infty = \frac{\pi}{2} \\
2\left(\frac{\pi}{2}\right) &= \boldsymbol{\pi}
\end{aligned}$$

6. $\int_{-2}^{2} \frac{1}{(1+x)^2} dx$

Split the integral at -1 since there is a discontinuity there. This function is symmetric but there's a greater distance betwen -1 and 2 than -1 to -2 so the integrals are not equal.

$$\int_{-2}^{2} \frac{1}{(1+x)^2} dx = \int_{-2}^{-1} \frac{1}{(1+x)^2} dx + \int_{-1}^{2} \frac{1}{(1+x)^2} dx$$

I'm choosing to integrate the first integral.

$$\begin{aligned}
\int_{-2}^{-1} \frac{1}{(1+x)^2} dx &= \lim_{t\to -1} \int_{-2}^{t} \frac{1}{(1+x)^2} dx \\
&= \lim_{t\to -1} -\frac{1}{1+x}\Big|_{-2}^{t} \\
&= \lim_{t\to -1} \left(-\frac{1}{1+t} + \frac{1}{1-2}\right) \\
&= \lim_{t\to -1} \left(-\frac{1}{1+t} - 1\right) \text{DNE}
\end{aligned}$$

The integral diverges.

The limit dose not exist for the same reason as problem 1. The left and right limits do not match. One side goes to ∞ and the other goes to $-\infty$. This results in an overall infinite area under the curve.

7. $\int_0^{\infty} \sin x \; dx$

Technically you should know this diverges by looking at the graph. Threre's no way the area under $\sin x$ or $\cos x$ can converge to anything if they keep oscilating symetrically. I will show the integral anyway.

$$\begin{aligned}
\int_0^{\infty} \sin x \; dx &= \lim_{t\to\infty} \int_0^t \sin x \; dx \\
&= \lim_{t\to\infty} -\cos x \Big|_0^t \\
&= \lim_{t\to\infty} (-\cos t + \cos 0) \\
&= \lim_{t\to\infty} (-\cos t + 1) \text{ DNE}
\end{aligned}$$

The integral diverges.

8. $\int_0^1 \frac{dx}{\sqrt[3]{x}} dx$

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt[3]{x}}dx &= \lim_{t\to 0}\int_t^1 x^{-1/3}dt \\ &= \lim_{t\to 0}\frac{3}{2}x^{2/3}\Big|_t^1 \\ &= \lim_{t\to 0}\left(\frac{3}{2}-\frac{3}{2}t^{2/3}\right) \\ &= \frac{3}{2}-0 \\ &= \mathbf{\frac{3}{2}}\end{aligned}$$

9. $\int_{-1}^{2}\frac{dx}{x^3}$
Split the integral at 0 where the discontinuity is. I'm choosing to start with the integral from 0 to 2 to avoid negative numbers, even though you technically won't get any because when you integrate you get a square, and squares are positive...

$$\begin{aligned}\int_{-1}^{2}\frac{dx}{x^3} &= \int_{-1}^{0}\frac{dx}{x^3}+\int_0^2\frac{d}{x^3} \\ \int_0^2 x^{-3}dx &= \lim_{t\to 0}\int_t^2 x^{-3}dx \\ &= \lim_{t\to 0}\frac{-1}{2x^2}\Big|_t^2 \\ &= \lim_{t\to 0}\left(-\frac{1}{8}+\frac{1}{2t^2}\right) \\ &= -\frac{1}{8}+\infty=\infty\end{aligned}$$

The integral diverges.

10. $\int_0^3 \frac{1}{x-1}dx$
Split the integral at -1 where the discontinuity is. I will start by integrating from 0 to 1 and see what happens.

$$\begin{aligned}\int_0^3 \frac{1}{x-1}dx &= \int_0^1\frac{1}{x-1}dx+\int_1^3\frac{1}{x-1}dx \\ \int_0^1\frac{1}{x-1}dx &= \lim_{t\to 1}\int_0^t\frac{1}{x-1}dx \\ &= \lim_{t\to 1}\ln|x-1|\big|_0^t \\ &= \lim_{t\to 1}\ln|1-1|-\ln|-1| \\ &= \ln|0|-0=-\infty\end{aligned}$$

The integral diverges.

11. $\int_0^1 \frac{dx}{x^\pi}$
Don't be afraid of the π, it's just a constant. But if you're integrating this below zero then you should be very afraid because non-rational negative exponents do not give real number results. You will probably have to take

a complex analysis class to figure it out...

$$\begin{aligned}
\int_0^1 \frac{dx}{x^\pi} &= \lim_{t\to 0}\int_t^1 x^{-\pi}dx \\
&= \lim_{t\to 0} \frac{x^{-\pi+1}}{-\pi+1}\bigg|_t^1 \\
&= \lim_{t\to 0}\left(\frac{1}{-\pi+1} - \frac{t^{-\pi+1}}{-\pi+1}\right) \\
&= \frac{1}{-\pi+1} - \frac{0^{-\pi+1}}{-\pi+1} \\
&= \frac{1}{-\pi+1} - \frac{1}{(-\pi^1)(0^{\pi-1})} \\
&= \frac{1}{-\pi+1} + \infty = \infty
\end{aligned}$$

The integral diverges.

12. $\int_0^e \ln(x)dx$

This integral will require IBP.

$$\int_0^e \ln x\ dx = \lim_{t\to 0}\int_t^e \ln x\ dx$$

$$u = \ln x,\ du = \frac{x}{dx},\ dv = dx,\ v = x$$

$$\begin{aligned}
\lim_{t\to 0}\int_t^e \ln x\ dx &= \lim_{t\to 0}\left(x\ln x - \int_t^e \frac{x}{x}dx\right) \\
&= \lim_{t\to 0}\left[x\ln x - x\right]_t^e \\
&= \lim_{t\to 0}\left((e\ln e - e) - (t\ln t - t)\right) \\
&= -0\ln 0 - 0 \\
&= \mathbf{0}
\end{aligned}$$

13. $\int_0^9 \frac{dx}{\sqrt{9-x}}$

$$\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{t\to 9}\int_0^t \frac{dx}{\sqrt{9-x}}$$

$$u = 9-x,\ du = -dx$$

$$\begin{aligned}
\lim_{t\to 9}\int_0^t \frac{dx}{\sqrt{9-x}} &= \lim_{t\to 9}\int_0^t -u^{-1/2}du \\
&= \lim_{t\to 9} -2\sqrt{u}\bigg|_0^t \\
&= \lim_{t\to 9} -2\sqrt{9-x}\bigg|_0^t \\
&= \lim_{t\to 9}\left(-2\sqrt{9-t}+2\sqrt{9}\right) \\
&= \lim_{t\to 9}\left(-2\sqrt{9-t}+6\right) \\
&= 0+6 = \mathbf{6}
\end{aligned}$$

14. $\int_0^4 x\ln(4x)dx$
Integrate this with IBP.

$$\int_0^4 x\ln(4x)dx = \lim_{t\to 0}\int_t^4 x\ln(4x)dx$$

$$u=\ln(4x),\ du=\frac{4}{4x}dx=\frac{1}{x}dx,\ dv=x\ dx,\ v=\frac{1}{2}x^2$$

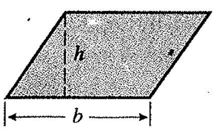
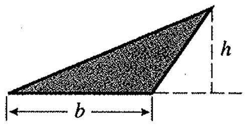
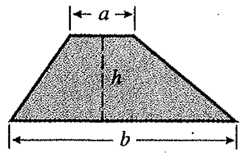
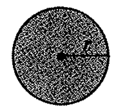
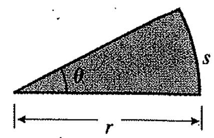
$$\begin{aligned}
\lim_{t\to 0}\int_t^4 x\ln(4x)dx &= \lim_{t\to 0}\left(\frac{1}{2}x^2\ln(4x)\Big|_t^4-\int_t^4\frac{1}{2}x\ dx\right)\\
&=\lim_{t\to 0}\left(\left(8\ln 16-\frac{1}{2}t^2\ln(4t)\right)-\frac{1}{4}x^2\Big|_t^4\right)\\
&=\lim_{t\to 0}\left(8\ln 16-\frac{1}{2}t^2\ln(4t)-\left(4-\frac{1}{4}t^2\right)\right)\\
&=\lim_{t\to 0}\left(8\ln 16-\frac{1}{2}t^2\ln(4t)-4+\frac{1}{4}t^2\right)\\
&=8\ln 16-\frac{1}{2}0^2\ln 0-4+\frac{1}{4}0^2\\
&=8\ln 16-\frac{1}{2}0^2(-\infty)-4+\frac{1}{4}0^2\\
&=\mathbf{8\ln 16-4}
\end{aligned}$$

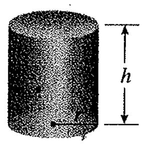
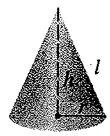
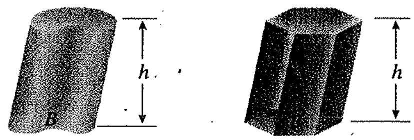
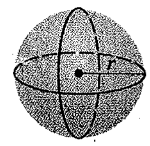
15. Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the x-axis, and on the left by x=1.
Graph the function in a graphig calculator to see what it's doing so you can set up the integral.

$$\begin{aligned}
A=\int_0^\infty\frac{7}{x^2}dx &= \lim_{t\to\infty}\int_1^t 7x^{-2}dx\\
&=\lim_{t\to\infty}-\frac{7}{x}\Big|_1^t\\
&=\lim_{t\to\infty}\left(-\frac{7}{t}+\frac{7}{1}\right)\\
&=-\frac{7}{\infty}+7=0+7\\
&=\mathbf{7\ units^2}
\end{aligned}$$

GEOMETRY FORMULAS

A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram	Triangle	Trapezoid	Circle	Sector
				
$A = bh$	$A = \frac{1}{2}bh$	$A = \frac{1}{2}(a+b)h$	$A = \pi r^2,\ C = 2\pi r$	$A = \frac{1}{2}r^2\theta,\ s = r\theta$ (θ in radians)

Right Circular Cylinder	Right Circular Cone	Any Cylinder or Prism with Parallel Bases	Sphere
			
$V = \pi r^2 h,\ S = 2\pi rh$	$V = \frac{1}{3}\pi r^2 h,\ S = \pi rl$	$V = Bh$	$V = \frac{4}{3}\pi r^3,\ S = 4\pi r^2$

ALGEBRA FORMULAS

THE QUADRATIC FORMULA	THE BINOMIAL FORMULA
The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$(x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$ $(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$
2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int \sin u\,du = -\cos u + C$
5. $\displaystyle\int \cos u\,du = \sin u + C$
6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$
8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$
9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$
10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
11. $\displaystyle\int \ln u\,du = u\ln u - u + C$
12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$
14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln\left|\tan \tfrac{1}{2}u\right| + C$
15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$
16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln\left|u + \sqrt{u^2-1}\right| + C$
17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln\left|u + \sqrt{u^2-1}\right| + C$

RECIPROCALS OF BASIC FUNCTIONS

18. $\displaystyle\int \frac{1}{1 \pm \sin u}\,du = \tan u \mp \sec u + C$

19. $\displaystyle\int \frac{1}{1 \pm \cos u}\,du = -\cot u \pm \csc u + C$

20. $\displaystyle\int \frac{1}{1 \pm \tan u}\,du = \tfrac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

21. $\displaystyle\int \frac{1}{\sin u \cos u}\,du = \ln|\tan u| + C$

22. $\displaystyle\int \frac{1}{1 \pm \cot u}\,du = \tfrac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

23. $\displaystyle\int \frac{1}{1 \pm \sec u}\,du = u + \cot u \mp \csc u + C$

24. $\displaystyle\int \frac{1}{1 \pm \csc u}\,du = u - \tan u \pm \sec u + C$

25. $\displaystyle\int \frac{1}{1 \pm e^u}\,du = u - \ln(1 \pm e^u) + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

26. $\displaystyle\int \sin^2 u\,du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

27. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

28. $\displaystyle\int \tan^2 u\,du = \tan u - u + C$

29. $\displaystyle\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

30. $\displaystyle\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

31. $\displaystyle\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

32. $\displaystyle\int \cot^2 u\,du = -\cot u - u + C$

33. $\displaystyle\int \sec^2 u\,du = \tan u + C$

34. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

35. $\displaystyle\int \cot^n u\,du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

36. $\displaystyle\int \sec^n u\,du = \frac{1}{n-1}\sec^{n-2} u\tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

37. $\displaystyle\int \csc^n u\,du = -\frac{1}{n-1}\csc^{n-2} u\cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS

38. $\displaystyle\int \sin mu \sin nu\,du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

39. $\displaystyle\int \cos mu \cos nu\,du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

40. $\displaystyle\int \sin mu \cos nu\,du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$

41. $$\int \sin^m u \cos^n u\,du = -\frac{\sin^{m-1} u\cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u\cos^n u\,du$$
$$= \frac{\sin^{m+1} u\cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u\cos^{n-2} u\,du$$

PRODUCTS OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

42. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

43. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

POWERS OF u MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. $\displaystyle\int u\sin u\,du = \sin u - u\cos u + C$

45. $\displaystyle\int u\cos u\,du = \cos u + u\sin u + C$

46. $\displaystyle\int u^2\sin u\,du = 2u\sin u + (2-u^2)\cos u + C$

47. $\displaystyle\int u^2\cos u\,du = 2u\cos u + (u^2-2)\sin u + C$

48. $\displaystyle\int u^n\sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

49. $\displaystyle\int u^n\cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

50. $\displaystyle\int u^n\ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

51. $\displaystyle\int ue^u\,du = e^u(u-1) + C$

52. $\displaystyle\int u^n e^u\,du = u^n e^u - n\int u^{n-1}e^u\,du$

53. $\displaystyle\int u^n a^u\,du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a}\int u^{n-1}a^u\,du + C$

54. $\displaystyle\int \frac{e^u\,du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1}\int \frac{e^u\,du}{u^{n-1}}$

55. $\displaystyle\int \frac{a^u\,du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1}\int \frac{a^u\,du}{u^{n-1}}$

56. $\displaystyle\int \frac{du}{u\ln u} = \ln|\ln u| + C$

POLYNOMIALS MULTIPLYING BASIC FUNCTIONS

57. $\displaystyle\int p(u)e^{au}\,du = \frac{1}{a}p(u)e^{au} - \frac{1}{a^2}p'(u)e^{au} + \frac{1}{a^3}p''(u)e^{au} - \cdots$ [signs alternate: $+ - + - \cdots$]

58. $\displaystyle\int p(u)\sin au\,du = -\frac{1}{a}p(u)\cos au + \frac{1}{a^2}p'(u)\sin au + \frac{1}{a^3}p''(u)\cos au - \cdots$ [signs alternate in pairs after first term: $+ + - - + + - - \cdots$]

59. $\displaystyle\int p(u)\cos au\,du = \frac{1}{a}p(u)\sin au + \frac{1}{a^2}p'(u)\cos au - \frac{1}{a^3}p''(u)\sin au - \cdots$ [signs alternate in pairs: $+ + - - + + - - \cdots$]

RATIONAL FUNCTIONS CONTAINING POWERS OF $a + bu$ IN THE DENOMINATOR

60. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}[bu - a\ln|a+bu|] + C$

61. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{b^3}\left[\frac{1}{2}(a+bu)^2 - 2a(a+bu) + a^2\ln|a+bu|\right] + C$

62. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$

63. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right] + C$

64. $\displaystyle\int \frac{u\,du}{(a+bu)^3} = \frac{1}{b^2}\left[\frac{a}{2(a+bu)^2} - \frac{1}{a+bu}\right] + C$

65. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

66. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

67. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} + \frac{1}{a^2}\ln\left|\frac{u}{a+bu}\right| + C$

RATIONAL FUNCTIONS CONTAINING $a^2 \pm u^2$ IN THE DENOMINATOR $(a > 0)$

68. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

69. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

70. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

71. $\displaystyle\int \frac{bu+c}{a^2+u^2}\,du = \frac{b}{2}\ln(a^2+u^2) + \frac{c}{a}\tan^{-1}\frac{u}{a} + C$

INTEGRALS OF $\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$ AND THEIR RECIPROCALS $(a > 0)$

72. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

73. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

74. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

75. $\displaystyle\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) + C$

76. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u+\sqrt{u^2-a^2}| + C$

77. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a^2-u^2}$ OR ITS RECIPROCAL

78. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

79. $\displaystyle\int \frac{\sqrt{a^2-u^2}\,du}{u} = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

80. $\displaystyle\int \frac{\sqrt{a^2-u^2}\,du}{u^2} = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$

81. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

82. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

83. $\displaystyle\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{u^2 \pm a^2}$ OR THEIR RECIPROCALS

84. $\displaystyle\int u\sqrt{u^2+a^2}\,du = \frac{1}{3}(u^2+a^2)^{3/2} + C$

85. $\displaystyle\int u\sqrt{u^2-a^2}\,du = \frac{1}{3}(u^2-a^2)^{3/2} + C$

86. $\displaystyle\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

87. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

88. $\displaystyle\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\left|\frac{u}{a}\right| + C$

89. $\displaystyle\int \frac{\sqrt{u^2+a^2}\,du}{u} = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

90. $\displaystyle\int \frac{du}{u^2\sqrt{u^2\pm a^2}} = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u} + C$

91. $\displaystyle\int u^2\sqrt{u^2+a^2}\,du = \frac{u}{8}(2u^2+a^2)\sqrt{u^2+a^2} - \frac{a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

92. $\displaystyle\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

93. $\displaystyle\int \frac{\sqrt{u^2+a^2}}{u^2}\,du = -\frac{\sqrt{u^2+a^2}}{u} + \ln(u+\sqrt{u^2+a^2}) + C$

94. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln|u+\sqrt{u^2-a^2}| + C$

95. $\displaystyle\int \frac{u^2}{\sqrt{u^2+a^2}}\,du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

96. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

INTEGRALS CONTAINING $(a^2+u^2)^{3/2}$, $(a^2-u^2)^{3/2}$, $(u^2-a^2)^{3/2}$ $(a > 0)$

97. $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

98. $\displaystyle\int \frac{du}{(u^2\pm a^2)^{3/2}} = \pm\frac{u}{a^2\sqrt{u^2\pm a^2}} + C$

99. $\displaystyle\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

100. $\displaystyle\int (u^2+a^2)^{3/2}\,du = \frac{u}{8}(2u^2+5a^2)\sqrt{u^2+a^2} + \frac{3a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

101. $\displaystyle\int (u^2-a^2)^{3/2}\,du = \frac{u}{8}(2u^2-5a^2)\sqrt{u^2-a^2} + \frac{3a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a+bu}$ OR ITS RECIPROCAL

102. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

103. $\displaystyle\int u^2\sqrt{a+bu}\,du = \frac{2}{105b^3}(15b^2u^2-12abu+8a^2)(a+bu)^{3/2}+C$

104. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2u^n(a+bu)^{3/2}}{b(2n+3)} - \frac{2an}{b(2n+3)}\int u^{n-1}\sqrt{a+bu}\,du$

105. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

106. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2-4abu+8a^2)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+C & (a>0) \\ \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}}+C & (a<0). \end{cases}$

109. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu}+a\int \frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bu}\,du}{u^{n-1}}$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{2au-u^2}$ OR ITS RECIPROCAL

112. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

113. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2}+\frac{a^3}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

114. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

115. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u}-\sin^{-1}\left(\frac{u-a}{a}\right)+C$

116. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\left(\frac{u-a}{a}\right)+C$

117. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au}+C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2}+\frac{3a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

INTEGRALS CONTAINING $(2au-u^2)^{3/2}$

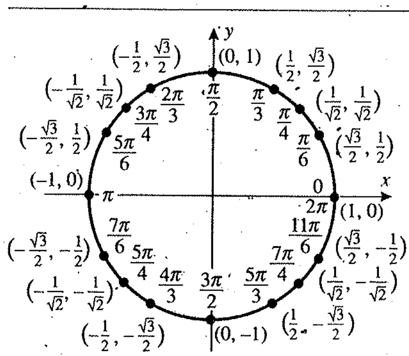
120. $\displaystyle\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}+C$

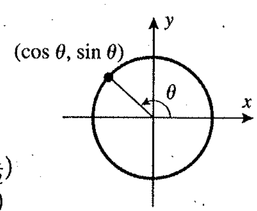
121. $\displaystyle\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}+C$

THE WALLIS FORMULA

122. $\displaystyle\int_0^{\pi/2}\sin^n u\,du = \int_0^{\pi/2}\cos^n u\,du = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot n}\cdot\frac{\pi}{2}\ \left(\begin{array}{l} n \text{ an even} \\ \text{integer and} \\ n\ge 2 \end{array}\right)$ or $\displaystyle\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(n-1)}{3\cdot 5\cdot 7\cdot\cdots\cdot n}\ \left(\begin{array}{l} n \text{ an odd} \\ \text{integer and} \\ n\ge 3 \end{array}\right)$

TRIGONOMETRY REVIEW





PYTHAGOREAN IDENTITIES

$\sin^2\theta+\cos^2\theta=1 \qquad \tan^2\theta+1=\sec^2\theta \qquad 1+\cot^2\theta=\csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta)=-\sin\theta \qquad \cos(-\theta)=\cos\theta \qquad \tan(-\theta)=-\tan\theta$

$\csc(-\theta)=-\csc\theta \qquad \sec(-\theta)=\sec\theta \qquad \cot(-\theta)=-\cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi-\theta)=\sin\theta \qquad \cos(\pi-\theta)=-\cos\theta \qquad \tan(\pi-\theta)=-\tan\theta$

$\csc(\pi-\theta)=\csc\theta \qquad \sec(\pi-\theta)=-\sec\theta \qquad \cot(\pi-\theta)=-\cot\theta$

$\sin(\pi+\theta)=-\sin\theta \qquad \cos(\pi+\theta)=-\cos\theta \qquad \tan(\pi+\theta)=\tan\theta$

$\csc(\pi+\theta)=-\csc\theta \qquad \sec(\pi+\theta)=-\sec\theta \qquad \cot(\pi+\theta)=\cot\theta$

ADDITION FORMULAS

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\displaystyle\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\displaystyle\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha=2\sin\alpha\cos\alpha \qquad \cos 2\alpha=2\cos^2\alpha-1$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha \qquad \cos 2\alpha=1-2\sin^2\alpha$

HALF-ANGLE FORMULAS

$\displaystyle\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$

MTH201 Chapter 5
Answers

Harper College*

June 25, 2025

Contents

*Problems found in Openstax Calculus Volume II and the Anton textbook.
`https://openstax.org/details/books/calculus-volume-2`

1 5.1 - Sequences

1. Write the general formula for a_n, where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.
   Write out the first few terms of the sequence. I will start at $n = 1$, not $n = 2$ since the sequence at $n \geq 2$ is a subset of the sequence at $n \geq 1$.

$$\begin{aligned}
a_1 &= 1 && \text{(n=1)} \\
a_2 &= 3 && \text{(n=2)} \\
a_3 &= 6 && \text{(n=3)} \\
a_4 &= 10 && \text{(n=4)} \\
a_5 &= 15 && \text{(n=5)} \\
&\vdots
\end{aligned}$$

   One thing we could try is to take out n and see what else is multiplied.

$$\begin{aligned}
a_1 &= 1 \cdot 1 && \text{(n=1)} \\
a_2 &= 2 \cdot \frac{3}{2} && \text{(n=2)} \\
a_3 &= 3 \cdot 2 && \text{(n=3)} \\
a_4 &= 4 \cdot \frac{10}{4} && \text{(n=4)} \\
a_5 &= 5 \cdot \frac{6}{2} && \text{(n=5)} \\
&\vdots
\end{aligned}$$

   All these terms can have a 2 in the denominator, so write them in that form:

$$\begin{aligned}
a_1 &= 1 \cdot \frac{2}{2} && \text{(n=1)} \\
a_2 &= 2 \cdot \frac{3}{2} && \text{(n=2)} \\
a_3 &= 3 \cdot \frac{4}{2} && \text{(n=3)} \\
a_4 &= 4 \cdot \frac{5}{2} && \text{(n=4)} \\
a_5 &= 5 \cdot \frac{6}{2} && \text{(n=5)} \\
&\vdots
\end{aligned}$$

   Now notice that all these numerators are equal to $n + 1$. Now we found the pattern.

$$\boldsymbol{a_n = \frac{n\,(n+1)}{2}}$$

2. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.
This is an arithmetic sequence because the difference between each consecutive term is the same $(a_{n-1} - a_n = 4)$.

$$\begin{aligned}
a_1 &= -3 && \text{(n=1)}\\
a_2 &= 1 && \text{(n=2)}\\
a_3 &= 5 && \text{(n=3)}\\
a_4 &= 9 && \text{(n=4)}\\
a_5 &= 13 && \text{(n=5)}\\
&\vdots
\end{aligned}$$

Try taking out a $4n$.

$$\begin{aligned}
a_1 &= 4 - 7 && \text{(n=1)}\\
a_2 &= 8 - 7 && \text{(n=2)}\\
a_3 &= 12 - 7 && \text{(n=3)}\\
a_4 &= 16 - 7 && \text{(n=4)}\\
a_5 &= 20 - 7 && \text{(n=5)}\\
&\vdots
\end{aligned}$$

$$\boldsymbol{a_n = 4n - 7}$$

3. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = \frac{1}{10}$ for $n \geq 1$.
Write out the terms.

$$\begin{aligned}
a_1 &= 3 && \text{(n=1)}\\
a_2 &= 0.3 && \text{(n=2)}\\
a_3 &= 0.03 && \text{(n=3)}\\
a_4 &= 0.003 && \text{(n=4)}\\
a_5 &= 0.0003 && \text{(n=5)}\\
&\vdots
\end{aligned}$$

Since the sequence is geometric, there's probably an exponent involved. We can see that each term is just 3 divided by 10 raised to some power, or multiplied by 10 raised to a negative power. Our a_n looks something like $m \times 10^{-n}$. However, if we want n to start at 1 and have $a_1 = 3$, our constant m would be 30, not 3.

$$a_n = 30 \times 10^{-n}$$

4. Find the general term for the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.
Write out the terms.

$$\begin{aligned} a_1 &= 0 && \text{(n=1)} \\ a_2 &= 1 && \text{(n=2)} \\ a_3 &= 3 && \text{(n=3)} \\ a_4 &= 7 && \text{(n=4)} \\ a_5 &= 15 && \text{(n=5)} \\ a_6 &= 31 && \text{(n=5)} \\ &\vdots \end{aligned}$$

The sequence is geometric because the difference between each consecutive term is changing.

$$\begin{aligned} a_2 - a_1 &= 1 \\ a_3 - a_2 &= 2 \\ a_4 - a_3 &= 4 \\ a_5 - a_4 &= 8 \\ &\vdots \end{aligned}$$

Those differences are powers of 2, so take out a 2^n and see what happens. Note that we are starting at $n = 2$.

$$\begin{aligned} a_2 &= 4 - 3 && \text{(n=2)} \\ a_3 &= 8 - 5 && \text{(n=3)} \\ a_4 &= 16 - 9 && \text{(n=4)} \\ a_5 &= 32 - 17 && \text{(n=5)} \\ &\vdots \end{aligned}$$

Notice that each number that is being subtracted is one more than half of the term it is subtracted from.

$$\begin{aligned} a_2 &= 2 - 1 && \text{(n=2)} \\ a_3 &= 4 - 1 && \text{(n=3)} \\ a_4 &= 8 - 1 && \text{(n=4)} \\ a_5 &= 16 - 1 && \text{(n=5)} \\ &\vdots \end{aligned}$$

And there's the pattern.

$$\boldsymbol{a_n = \frac{2^n}{2} - 1}$$

5. Find the formula for the general term of the sequence.

$$1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$$

The first thing to notice is that the terms oscillate between negative and positive numbers. Also, the first number is positive, not negative. This means the general term will have $(-1)^{n-1}$ in it. Next, notice that the denominators are the odd numbers starting at 1. Odd numbers are usually represented by $2n+1$, but since we are starting at 1, we must have $2n-1$ in the denominator.

$$\boldsymbol{a_n = \frac{(-1)^{n-1}}{2n-1}}$$

6. Find the limit of the following sequences. You may need to use L'Hôpital's Rule.

   (a) $\frac{n^2}{2^n}$
   Direct substituation yields $\frac{\infty}{\infty}$. Use L'Hôpital's Rule twice.

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2}{2^n} &= \lim_{n\to\infty}\frac{2n}{2^n\cdot\ln 2}\\ &= \lim_{n\to\infty}\frac{2}{2^n\cdot\ln 2\cdot\ln 2}\\ &= \frac{2}{\infty}\\ &= \mathbf{0}\end{aligned}$$

   (b) $\frac{\sqrt{n}}{\sqrt{n+1}}$
   Direct substituation yields $\frac{\infty}{\infty}$. If you use L'Hôpital's Rule you will get the original function back. Instead, divide the numerator and denominator by $\sqrt{n}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} &= \lim_{n\to\infty}\frac{1}{\frac{\sqrt{n+1}}{\sqrt{n}}}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}}\\ &\text{Split the fraction.}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}\\ &= \frac{1}{\sqrt{1+\frac{1}{\infty}}}\\ &= \frac{1}{\sqrt{1+0}}\\ &= \mathbf{1}\end{aligned}$$

7. Is the sequence bounded? Is it monotone increasing, or decreasing?

(a) $\frac{n}{2^n}$, $n \geq 2$

We can figure out if it's bounded by checking the limit as $n \to \infty$. Use L'Hôpital's Rule once.

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{2^n} &= \lim_{n\to\infty} \frac{1}{2^n \cdot \ln 2} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

So the sequence is **bounded** because it won't go below 0.

To test the monotonicity, the ratio test is best, because n is an exponent.

$$\begin{aligned}\frac{a^{n+1}}{a_n} &= \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} \\ &= \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} \\ &= \frac{n+1}{n \cdot 2}\end{aligned}$$

If you plug in integers, you will notice that the ratio $\frac{a^{n+1}}{a_n} \leq 1$ starting at $n = 2$. Therefore, the sequence is **monotone decreasing**.

Note: The restriction $n \geq 2$ is very important for stating the monotonicity because there is a part of the sequence before $n = 2$ where it increases.

(b) $\sin n$

We know that the range of $\sin n$ is $(-1, 1)$. Therefore, **the sequence is bounded**.

Since $\sin n$ oscillates, it is **not monotone** because it always switches between increasing and decreasing on its domain.

(c) $n^{\frac{1}{n}},\ n \geq 3$

To check if the sequence is bounded, evaluate the limit.

$$\begin{aligned}\lim_{n\to\infty} n^{\frac{1}{n}} &= \infty^{\frac{1}{\infty}} \\ &= \infty^0\end{aligned}$$

Remember from Calculus I that this is a special case of limits. We have to rewrite this in a way that allows us to use L'Hôpital's Rule or something. To do this, give the sequence a name, like y, take the natural log of both sides and then evaluate the limit.
After evaluating the limit, raise the limit as a power of e to undo the natural log.

$$\begin{aligned}y(n) &= n^{\frac{1}{n}} \\ \ln y &= \ln n^{\frac{1}{n}} \\ &= \frac{1}{n}\ln n \\ &= \frac{\ln n}{n} \\ \lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{lnn}{n} = \frac{\infty}{\infty}\end{aligned}$$

Use L'Hôpital's Rule.

$$\begin{aligned}\lim_{n\to\infty} \frac{\ln n}{n} &= \lim_{n\to\infty} \frac{\frac{1}{n}}{n} \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= \frac{1}{\infty} \\ &= 0 \\ \lim_{n\to\infty} y &= e^0 \\ &= 1\end{aligned}$$

Since the limit exists, **the sequence is bounded**.
Now, for monotonicity, I don't think you will have much luck with the ratio or difference tests; you will have to use the derivative test.
You cannot take the derivative in its current form but you can do a similar thing to when we

took the limit.

$$
\begin{aligned}
& \text{Make a name for the sequence term} \\
y(n) &= n^{\frac{1}{n}} \\
& \text{Take the natural log of both sides.} \\
\ln y &= \ln n^{\frac{1}{n}} \\
& \text{Simplify.} \\
&= \frac{1}{n} \ln n \\
&= \frac{\ln n}{n} \\
& \text{Take the derivative of both sides (implicit differentiation).} \\
\frac{d}{dn}[lny] &= \frac{d}{dn}\left[\frac{\ln n}{n}\right] \\
\frac{1}{y} \cdot y' &= \frac{n \cdot \frac{1}{n} - \ln n}{n^2} \\
&= \frac{1 - \ln n}{n^2} \\
y' &= y \cdot \frac{1 - \ln n}{n^2} \\
&= n^{\frac{1}{n}} \cdot \frac{1 - \ln n}{n^2} \\
& n^{\frac{1}{n}-2} \cdot (1 - \ln n) \\
& < 0
\end{aligned}
$$

We know that the derivative is negative because $n^{\frac{1}{n}-2}$ is positive and $1 - \ln n$ is negative since $\ln n > 1$, (but only for $n \geq 3$ since $\ln 2 < 1$ but $\ln 3 > 1$). Since the derivative is negative, the sequence is **monotone decreasing**.

(d) $\tan n$

You can figure this out by looking at the general shape of the graph. The sequence is **not bounded** because the range of $\tan n$ is $(-\infty, \infty)$.

Also, for any two consecutive positive integer inputs, $\tan n$ can go from a lower term to a higher term or from a higher term to a lower term.

It is probably not a sufficient enough proof to just say "look at the graph". Technically, you would have to provide an example of two pairs of consecutive terms, one where the second term is larger than the first and one where the second is smaller.

Proof. Let $a_n = \tan n$, where $n \in \mathbb{Z}$.

$$
\begin{aligned}
a_1 &= \tan 1 \approx 1.56 \\
a_2 &= \tan 2 \approx -2.19 \\
a_2 &< a_1 \\
a_3 &= \tan 3 \approx -0.14 \\
a_3 &> a_2
\end{aligned}
$$

Since there exists a subset of a_n where a_n both decreases and increases, $a_n = \tan n$ is not monotone on its entire domain. $\square$

8. Find the limit of the sequence using the double angle identity.

$$a_n = \frac{\cos(1/n) - 1}{1/n}$$

We will use the identity $\cos 2n = 1 - 2\sin^2 n$. Then manipulate it to find an expression for $\cos(1/n) - 1$.

$$\begin{aligned}
\cos 2n &= 1 - 2\sin^2 n \\
\cos 2n - 1 &= -2\sin^2 n \\
2n &= \frac{1}{n} \Rightarrow n = \frac{1}{2n} \\
\cos\frac{1}{n} - 1 &= -\sin^2\frac{1}{2n} \\
\frac{\cos(1/n) - 1}{1/n} &= \frac{-\sin^2\frac{1}{2n}}{\frac{1}{n}} \lim_{n\to\infty} \frac{-\sin^2\frac{1}{2n}}{\frac{1}{n}} = \frac{0}{0} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{-4\sin\frac{1}{2n} \cdot \frac{-1}{2n^2} \cdot \cos\frac{1}{2n}}{\frac{-1}{n^2}} \\
&= -2\sin 0 \cdot \frac{-1}{\infty} \cdot \cos 0 \\
&= 0
\end{aligned}$$

The sequence **converges to 0**.

9. Use the Squeeze Theorem to find the limit of the sequence.

$$a_n = \sin n \sin\frac{1}{n}$$

We are only squeezing the sequence with $\sin n$ because $\sin\frac{1}{n}$ has predictable end behavior, because it approaches 0 as $\frac{1}{n}$ approaches 0.

$$\begin{gathered}
-1 \le \sin n \le 1 \\
-\sin\frac{1}{n} \le \sin n \sin\frac{1}{n} \le -\sin\frac{1}{n} \\
\text{Evaluate the end limits.} \\
\lim_{n\to\infty} \sin\frac{1}{n} = \sin\frac{1}{\infty} = \sin 0 = 0 \\
\lim_{n\to\infty} -\sin\frac{1}{n} = 0 \\
0 \le \lim_{n\to\infty} \sin n \sin\frac{1}{n} \le 0 \\
\Rightarrow \lim_{n\to\infty} \sin n \sin\frac{1}{n} = 0
\end{gathered}$$

The sequence **converges to 0.**

10. Determine the limit of the sequence. Does the sequence converge or diverge?

(a) $a_n = (2n)^{\frac{1}{n}} - n^{\frac{1}{n}}$

We are doing something similar to previous problems with fractions of n as exponents, but now there are two terms.

Set a name, like y, for a_n, and names for the individual terms like y_1 and y_2. Take the natural log of both sides, and evaluate the limits individually. Then put them together to find the limit of the entire sequence.

$$
\begin{aligned}
y &= (2n)^{\frac{1}{n}} - n^{\frac{1}{n}} \\
&= y_1 - y_2 \\
y_1 &= (2n)^{\frac{1}{n}} \\
y_2 &= n^{\frac{1}{n}} \\
\ln y_1 &= \ln\left((2n)^{\frac{1}{n}}\right) = \frac{1}{n}\ln 2n \\
\ln y_2 &= \ln\left(n^{\frac{1}{n}}\right) = \frac{1}{n}\ln n \\
\lim_{n\to\infty} \ln y_1 &= \lim_{n\to\infty} \frac{\ln 2n}{n} = \frac{\infty}{\infty} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0 \\
\lim_{n\to\infty} \ln y_2 &= \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0 \\
&\text{Raise the limits to the base } e \text{ to undo the natural logs.} \\
\lim_{n\to\infty} y_1 = \lim_{n\to\infty} y_2 = e^0 = 1 & \\
&\text{Combine } y_1 \text{ and } y_2 \text{ into } y = a_n. \\
\lim_{n\to\infty} y = \lim_{n\to\infty} 1 - 1 = 0 &
\end{aligned}
$$

The sequence **converges to 0**.

(b) $a_n = \left(1 - \frac{2}{n}\right)^n$

Apply the natural log to both sides of a_n to bring the exponent out. Then manipulate the resulting sequence to somthing that you can use L'Hôpital's Rule on.

$$\begin{aligned}
\ln a_n &= \ln\left(1 - \frac{2}{n}\right)^n \\
&= n \cdot \ln\left(1 - \frac{2}{n}\right) \\
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \cdot \ln\left(1 - \frac{2}{n}\right) \\
&= \infty \cdot \ln\left(1 - \frac{2}{\infty}\right) = \infty \cdot \ln 1 = \infty \cdot 0
\end{aligned}$$

Instead of multiplying by n, divide by the reciprocal.

$$= \lim_{n\to\infty} \frac{\ln\left(1 - \frac{2}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

Use L'Hôpital's Rule.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{\frac{2}{n^2}}{1-\frac{2}{n}}}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} \left(\frac{2}{n^2} \cdot \frac{1}{1 - \frac{2}{n}} \cdot \frac{-n^2}{1}\right) \\
&= \lim_{n\to\infty} \frac{-2}{1 - \frac{2}{n}} \\
&= \frac{-2}{1} \\
&= -2 \\
&\Rightarrow \lim_{n\to\infty} a_n = e^{-2}
\end{aligned}$$

The sequence converges to e^{-2}.

(c) $a_n = \frac{2^n + 3^n}{4^n}$

Direct substitution of the limit will yield $\frac{\infty}{\infty}$. First we will rewrite the a_n as something of which we can take the limit easily.

$$\begin{aligned}
a_n = \frac{2^n + 3^n}{4^n} &= \frac{2^n}{4^n} + \frac{3^n}{4^n} \\
&= \frac{2^n}{2^n \cdot 2^n} + \frac{3^n}{3^n \cdot \left(\frac{4}{3}\right)^n} \\
&= \frac{1}{2^n} + \left(\frac{3}{4}\right)^n \\
\lim_{n\to\infty}\left(\frac{1}{2^n} + \left(\frac{3}{4}\right)^n\right) &= \frac{1}{\infty} + 0 \\
&= 0
\end{aligned}$$

The limit **converges to 0**.

We know $\left(\frac{3}{4}\right)^n$ goes to 0 because bases between 0 and 1 (exclusive) decrease to 0 and bases greater than 1 go to ∞.

(d) $a_n = \frac{(n!)^2}{(2n)!}$
The best choice for functions with factorials is the ratio test. First we will find what $\frac{a_{n+1}}{a_n}$ is.

$$\frac{a_{n+1}}{a_n} = \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}}$$

Division of a fraction becomes multiplication by the reciprocal.

$$= \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}$$
$$= \frac{((n+1)(n!))^2}{(2n+1)(2n+1)((2n)!)} \cdot \frac{(2n)!}{(n!)^2}$$
$$= \frac{(n+1)(n!)(n+1)(n!)}{(2n+2)(2n+1)((2n)!)} \cdot \frac{(2n)!}{(n!)(n!)}$$

Cancel some terms.

$$= \frac{(n+1)^2}{(2n+2)(2n+1)}$$

Expand everything into polynomial form.

$$= \frac{n^2+2n+1}{4n^2+6n+2}$$

Now we need to take the limit of $\left|\frac{a_{n+1}}{a_n}\right|$. The absolute value is important because you can have a ratio less than or equal to -1 and it would diverge, just to $-\infty$, not ∞. If you think absolute values are scary you can also say that the ratio has to be between -1 and 1, exclusive.
I'm going to get rid of the absolute value bars early on in this problem because both the numerator and denominator are positive after $n = 0$, so the whole thing is positive.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2}$$
$$= \frac{1}{4} \leq 1$$

$$\Rightarrow \lim_{n\to\infty}\frac{(n!)^2}{(2n)!} = 0$$

So the sequence **converges to 0**.
Note: We found the limit to be $\frac{1}{4}$ because since the degree of the numerator is equal to the degree of the denominator, the limit is equal to the ratio of the leading coefficients.

2 5.2 - Series

1. Write the following expressions as infinite series.

   (a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$
   This is the harmonic series. We are starting at 1, so the series goes from $n = 1$ to ∞. The denominator is just n.
   $$\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n}}$$

   (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
   This is the same as the previous series except it's alternating. The first term is positive so we need a $(-1)^{n-1}$
   $$\boldsymbol{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

2. Use the sequence of partial sums to determine whether the series converges or diverges.

   (a)
   $$\sum_{n=1}^{\infty} \frac{n}{n+2}$$
   Write out the first few partial sums.
   $$\frac{1}{3} = 0.\bar{3} \qquad \text{(n=1)}$$
   $$\frac{1}{3} + \frac{2}{4} = \frac{5}{6} = 0.8\bar{3} \qquad \text{(n=2)}$$
   $$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} = \frac{43}{30} = 1.4\bar{3} \qquad \text{(n=3)}$$
   $$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} = \frac{21}{10} = 2.1 \qquad \text{(n=4)}$$
   $$\vdots$$
   Also notice that the limit of $\frac{n}{n+2}$ is 1. This means that numbers are being added that get closer and closer to 1, so **the series diverges**.
   This is called the divergence test, which is from a later section. It basically means that the series can only converge if the terms being summed approach 0.

(b) Hint: Use Partial Fraction Decomposition

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\begin{aligned} \frac{1}{(n+1)(n+2)} &= \frac{A}{n+1} + \frac{B}{n+2} \\ A(n+2) + B(n+1) &= 1 \\ An + 2A + Bn + B &= 1 \end{aligned}$$

Solve the system of equations.

$$\begin{aligned} A + B &= 0 \\ 2A + B &= 1 \end{aligned}$$

$$A = 1,\ B = -1$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

This is a telescoping series.

$$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2} - \frac{1}{n+2}$$

Since the entire series equals $\frac{1}{2} - \frac{1}{n+2}$, just plug in ∞.

$$\frac{1}{2} - \frac{1}{\infty} = \mathbf{\frac{1}{2}}$$

3. Does the series converge or diverge? Explain.

(a)

$$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

Try rewriting the sum with decimals.

$$\begin{aligned} &1 + 0.1 + 0.01 + 0.001 + 0.0001 + \cdots \\ &= 1.1 + 0.01 + 0.001 + 0.0001 + \cdots \\ &= 1.11 + 0.001 + 0.0001 + \cdots \\ &= 1.111 + 0.0001 + \cdots \end{aligned}$$

It looks like the series forms the repeating decimal $1.\bar{1}$. The decimal part, $0.\bar{1} = \frac{1}{9}$, so $1.\bar{1} = 1 + \frac{1}{9}$.
So the series **converges** to $\frac{10}{9}$.
You can also say it's a geometric series with $r = \frac{1}{10} < 1$ by writing the sum.

$$\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$$

(b)

$$1 + \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$$

The powers of π are increasing by 1 but the powers of e are increasing by 2. Write the sum and rewrite as a geometric series.

$$\sum_{n=0}^{\infty} \left(\frac{\pi^n}{e^{2n}} \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{\pi}{e^2} \right)^n$$

Change the index to make the n into $n-1$.

$$= \sum_{n=1}^{\infty} \left(\frac{\pi}{e^2} \right)^{n-1}$$

Now we have the correct indices and have $n-1$ in the exponent. We see that $r = \frac{\pi}{e^2} \approx 0.43 < 1$, so **the series converges**.

4. Evaluate the telescoping series or state if it diverges.

(a)

$$\sum_{n=1}^{\infty} \left(2^{1/n} - 2^{1/(n+1)} \right)$$

Start writing out the sum.

$$\sum_{n=1}^{k} \left(2^{1/n} - 2^{1/(n+1)} \right) = \left(2 - 2^{\frac{1}{2}} \right) + \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}} \right) + \cdots + \left(2^{1/k} - 2^{1/(k+1)} \right)$$

$$= 2 - 2^{1/(k+1)}$$

$$\sum_{n=1}^{\infty} \left(2^{1/n} - 2^{1/(n+1)} \right) = 2 - 2^{\frac{1}{\infty}}$$

$$= 2 - 2^0 = 2 - 1$$

$$= \mathbf{1}$$

(b)

$$\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n+1} \right)$$

Start writing out the sum.

$$\sum_{n=1}^{k} \left(\sqrt{n} - \sqrt{n+1} \right) = \left(\sqrt{1} - \sqrt{2} \right) + \left(\sqrt{2} - \sqrt{3} \right) + \cdots + \left(\sqrt{k} - \sqrt{k+1} \right)$$

$$= \sqrt{1} - \sqrt{k+1}$$

$$= 1 - \sqrt{k+1}$$

$$\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n+1} \right) = 1 - \sqrt{\infty}$$

$$= -\infty$$

$\Rightarrow$ **The series diverges.**

5. Express the series as a telescoping sum and evaluate its kth partial sum.

(a)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

Start writing out the sum.

$$\begin{aligned}
\sum_{n=1}^{k} \ln\left(\frac{n}{n+1}\right) &= \sum_{n=1}^{k} (\ln k - \ln (k+1)) \\
&= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln k - \ln (k+1)) \\
&= \ln 1 - \ln (k+1) \\
&= 0 - \ln (k+1) \\
&= \boldsymbol{-\ln (k+1)}
\end{aligned}$$

(b)

$$\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \ln (n+1)}$$

We need to split this fraction somehow if we want to see how the series is telescoping. It would be nice if we could use log properties to get subtraction in the numerator. We can do this by getting a common denominator in $\ln\left(1+\frac{1}{n}\right)$.

$$\begin{aligned}
\frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \ln (n+1)} &= \frac{\ln\left(\frac{n+1}{n}\right)}{\ln n \ln (n+1)} \\
&= \frac{\ln (n+1) - \ln n}{\ln n \ln (n+1)} \\
&= \frac{\ln (n+1)}{\ln n \ln (n+1)} - \frac{\ln n}{\ln n \ln (n+1)} \\
&= \frac{1}{\ln n} - \frac{1}{\ln n + 1} \\
\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \ln (n+1)} &= \sum_{n=2}^{\infty} \left(\frac{1}{\ln n} - \frac{1}{\ln n + 1}\right) \\
S_k &= \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 3} - \frac{1}{\ln 4}\right) + \cdots + \left(\frac{1}{\ln k} - \frac{1}{\ln (k+1)}\right) \\
&= \frac{1}{\ln 2 - \frac{1}{\ln (k+1)}} \\
S_\infty &= \frac{1}{\ln 2 - \frac{1}{\infty}} \\
&= \boldsymbol{\frac{1}{\ln 2}}
\end{aligned}$$

6. Evaluate

$$\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$$

This can be factored into three terms. Then use Partial Fracption Decomposition.

$$\begin{aligned}
\frac{2}{n^3 - n} &= \frac{2}{n\left(n^2 - 1\right)} \\
&= \frac{2}{n\left(n-1\right)\left(n+1\right)} \\
&= \frac{A}{n} + \frac{B}{n-1} + \frac{C}{n+1} \\
A\left(n-1\right)\left(n+1\right) + B\left(n\right)\left(n+1\right) + C\left(n\right)\left(n-1\right) &= 2 \\
&= A\left(n^2 - 1\right) + B\left(n^2 + n\right) + C\left(n^2 - n\right) \\
&= An^2 - A + Bn^2 + Bn + Cn^2 - Cn
\end{aligned}$$

Set up and solve the system of equations.

$$\begin{aligned}
A + B + C &= 0 \\
B - C &= 0 \\
-A &= 2 \\
\hline
\Rightarrow A &= -2,\ B = 1,\ C = 1
\end{aligned}$$

Put the decomposed fraction back into the sum.

$$\sum_{n=2}^{\infty} = \left(-\frac{2}{n} + \frac{1}{n-1} + \frac{1}{n+1}\right)$$

This looks like a telescoping series because it contains terms like $n-1$, n, and $n+1$, but there are 3 terms. I'm going to split the $-\frac{2}{n}$ into two $-\frac{1}{n}$ terms to make two telescoping series.

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(-\frac{2}{n} + \frac{1}{n-1} + \frac{1}{n+1}\right) &= \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n} - \frac{1}{n} - \frac{1}{n+1}\right) \\
&= \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) - \sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)
\end{aligned}$$

Now both of these are telescoping series and can be evaluated individually, then combined.

$$S_1(k) = \sum_{n=2}^{k} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right)$$
$$= 1 - \frac{1}{k}$$
$$S_2(k) = \sum_{n=2}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$
$$= \frac{1}{2} - \frac{1}{k+1}$$

Plug in ∞ and subtract to get whole sum.

$$S_1(\infty) - S_2(\infty) = 1 - \frac{1}{\infty} - \frac{1}{2} + \frac{1}{\infty + 1}$$
$$= \frac{1}{2} - \frac{1}{\infty} + \frac{1}{\infty}$$
$$= \mathbf{\frac{1}{2}}$$

7. Does the series converge? If so, find the sum.

(a)

$$\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$$

This is a geometric series. It **converges** because $-1 < -\frac{3}{4} < 1$, or $\left|\frac{3}{4}\right| < 1$.

$$a = 1,\ r = -\frac{3}{4}$$
$$\frac{a}{1-r} = \frac{1}{1+\frac{3}{4}}$$
$$= \mathbf{\frac{4}{7}}$$

(b)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{7}{6^{n-1}}$$

This is a geometric series but we have to rewrite it to find a and r. The (-1) and 6 both have the correct power of $n-1$, so combine them. The 7 will become r.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{7}{6^{n-1}} = \sum_{n=1}^{\infty} 7 \cdot \frac{(-1)^{n-1}}{6^{n-1}}$$
$$= \sum_{n=1}^{\infty} 7 \left(-\frac{1}{6}\right)^{n-1}$$
$$a = 7,\ r = -\frac{1}{6}$$
$$\sum_{n=1}^{\infty} 7 \left(-\frac{1}{6}\right)^{n-1} = \frac{7}{1+\frac{1}{6}}$$
$$= \mathbf{6}$$

8. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is $\frac{3}{4}$ of the preceding height. Find the total distance the ball will travel if it assumed to bounce infinetely.
The ball first has to drop 10 m. I'm not going to include that in the sum; it will just be added on. When the ball bounces, it goes up and then travels the same distance back to the ground again. So each time the ball bounces, it travels its distance twice, so the heights will be multiplied by 2 in the sum. Then, every bounce is mutiplied by $\frac{3}{4}$ of the last one. Write out the terms to find a pattern.

$$\begin{aligned}
&10 + 2\left(\frac{3\cdot 10}{4}\right) + 2\left(\frac{3}{4}\left(\frac{3\cdot 10}{4}\right)\right) + \cdots \\
&\qquad = 10 + \sum_{n=1}^{\infty} 10\left(\frac{3}{4}\right)^{n}\cdot 2 \\
&\qquad = 10 + \sum_{n=1}^{\infty} 20\left(\frac{3}{4}\right)^{n} \\
&\qquad = 10 + \sum_{n=1}^{\infty} 20\left(\frac{3}{4}\right)^{n-1}\left(\frac{3}{4}\right)^{1} \\
&\qquad = 10 + \sum_{n=1}^{\infty} 15\left(\frac{3}{4}\right)^{n-1} \\
&\qquad = 10 + \frac{15}{1-\frac{3}{4}} \\
&\qquad = \mathbf{70}
\end{aligned}$$

3 5.3 - Divergence and Integral Tests

1. For each of the series, either find the limit or explain why the divergence test does not apply.

   (a) $a_n = \frac{n}{5n^2-3}$
   The **divergence test does not apply** because $\lim_{n\to\infty} \frac{n}{5n^2-3} = 0$. The test does not give us any information.

   (b) $a_n = \frac{(2n+1)(n-1)}{(n+1)^2}$
   Expand all terms in the fraction, then take the limit.

   $$\frac{(2n+1)(n-1)}{(n+1)^2} = \frac{2n^2-n-1}{n^2+2n+1}$$
   $$\lim_{n\to\infty} \frac{2n^2-n-1}{n^2+2n+1} = \mathbf{2}$$

   The divergence test applies because the limit is $2 \neq 0$, so the series diverges.

   (c) $a_n = \frac{2^n}{3^{n/2}}$
   To evaluate the limit of the general term we will have to rewrite it as some constant to the power of n. Based on the constant we can see if it approaches ∞ or not.

   $$\begin{aligned}\frac{2^n}{3^{n/2}} &= \frac{2^n}{\left(\sqrt{3}\right)^n} \\ &= \left(\frac{2}{\sqrt{3}}\right)^n\end{aligned}$$
   $$\lim_{n\to\infty} \left(\frac{2}{\sqrt{3}}\right)^n = \boldsymbol{\infty}$$

   Since $\frac{2}{\sqrt{3}} > 1$, the term approaches ∞. If it were less than 1, the denominator would be growing faster than the numerator and the limit would be 0.
   But the divergence test does apply because the limit is $\infty \neq 0$.

   (d) $a_n = e^{-2/n}$
   You can evaluate the limit using direct substitution.

   $$\begin{aligned}\lim_{n\to\infty} e^{-2/n} &= e^{-2/\infty} \\ &= e^0 \\ &= \mathbf{1}\end{aligned}$$

   The divergence test applies because the limit is $1 \neq 0$.

   (e) $a_n = \tan n$
   Look at the graph of $\tan n$.
   The limit **does not exist**, so the series diverges. The divergence test applies.

(f) $a_n = \frac{(\ln n)^2}{\sqrt{n}}$

Rewrite the general term so that everything is to a power of $\frac{1}{2}$.

$$\begin{aligned} \frac{(\ln n)^2}{\sqrt{n}} &= \frac{\sqrt{\ln^4 n}}{\sqrt{n}} \\ &= \left(\frac{\ln^4 n}{n}\right)^{1/2} \end{aligned}$$

Now remember the root/exponent law of limits; that the limit of something with a constant power is equal to the limit of the base all raised to the power. So we can just take the limit of $\frac{\ln^4 n}{n}$ and square root it after.

The problem is that the natural log takes a while to differentiate away because due to the chain rule, it keeps multiplying by $\frac{1}{n}$, replacing the n in the denominator. You will need to use L'Hôpital's Rule four times. I will show the first differentiation in more detail and then just follow the pattern quickly do a form where direct substitution can be used.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(\frac{\ln^4 n}{n}\right)^{1/2} \\ &= \left(\lim_{n\to\infty}\frac{\ln^4 n}{n}\right)^{1/2} \\ &= \left(\frac{\infty}{\infty}\right)^{1/2} \\ &\text{Use L'Hôpital's Rule.} \\ &= \left(\lim_{n\to\infty}\frac{4\ln^3 n \cdot \frac{1}{n}}{1}\right)^{1/2} = \left(\lim_{n\to\infty}\frac{4\ln^3 n}{n}\right)^{1/2} \\ &\text{Use L'Hôpital's Rule three more times.} \\ &= \left(\lim_{n\to\infty}\frac{12\ln^2 n}{n}\right)^{1/2} \\ &= \left(\lim_{n\to\infty}\frac{24\ln n}{n}\right)^{1/2} \\ &= \left(\lim_{n\to\infty}\frac{24}{n}\right)^{1/2} \\ &= 0^{1/2} \\ &= 0 \end{aligned}$$

The divergence test **does not apply** since the limit is 0.

2. Does the p-series converge or diverge?

(a)

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

We want only one n term raised to a power in the denominator, so combine the powers by adding them (since the bases are multiplied).

$$\begin{aligned} \frac{1}{n\sqrt{n}} &= \frac{1}{n^1 n^{1/2}} \\ &= \frac{1}{n^{3/2}} \end{aligned}$$

$$\frac{3}{2} > 1 \Rightarrow \textbf{The series converges.}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$$

The cube root represents a power of $\frac{1}{3}$, so rewrite it and combine powers by multiplying them (since a power is raised to a power).

$$\begin{aligned} \frac{1}{\sqrt[3]{n^4}} &= \frac{1}{(n^4)^{1/3}} \\ &= \frac{1}{n^{4/3}} \end{aligned}$$

$$\frac{4}{3} > 1 \Rightarrow \textbf{The series converges.}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n^{\pi}}{n^{2e}}$$

We can make the n^{π} go in the denominator when we make the exponent neagative. Then add the powers to combne them (since the bases are multiplied).

$$\begin{aligned} \frac{n^{\pi}}{n^{2e}} &= \frac{1}{n^{2e} \cdot n^{-\pi}} \\ &= \frac{1}{n^{2e-\pi}} \end{aligned}$$

$$2e - \pi > 1 \Rightarrow \textbf{The series converges.}$$

3. Use the integral test to determine whether the series converge.

(a)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$$

We will compare the series with $\frac{1}{(x+5)^{1/3}}$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(x+5)^{1/3}} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{(x+5)^{1/3}} dx \\
&= \lim_{t\to\infty} \int_1^t (x+5)^{-1/3}\, dx \\
&= \lim_{t\to\infty} \frac{3}{2} (x+5)^{2/3} \bigg|_1^t \\
&= \lim_{t\to\infty} \left[\frac{3}{2} (t+5)^{2/3} - \frac{3}{2} (6)^{2/3}\right] \\
&= \frac{3}{2} (\infty)^{2/3} - \frac{3}{2} (6)^{2/3} \\
&= \infty \Rightarrow \textbf{The series diverges} \text{ since the integral diverges.}
\end{aligned}$$

(b)

$$\sum_{n=1}^{\infty} \frac{n}{1+n^2}$$

We will compare the series with $\frac{x}{1+x^2}$.

$$\begin{aligned}
\int_1^{\infty} \frac{x}{1+x^2} dx &= \lim_{t\to\infty} \int_1^t \frac{x}{1+x^2} dx \\
u = 1+x^2,\ du &= 2x\ dx,\ x\ dx = \frac{1}{2} du \\
&= \lim_{t\to\infty} \frac{1}{2} \int_1^t \frac{1}{u} du \\
&= \lim_{t\to\infty} \frac{1}{2} \ln |u| \bigg|_1^t \\
&= \lim_{t\to\infty} \frac{1}{2} \ln \left|1+x^2\right| \bigg|_1^t \\
&= \lim_{t\to\infty} \left[\frac{1}{2} \ln \left|1+t^2\right| - \frac{1}{2} \ln 2\right] \\
&= \frac{1}{2} \ln \infty - \frac{1}{2} \ln 2 \\
&= \infty \Rightarrow \textbf{The series diverges} \text{ since the integral diverges.}
\end{aligned}$$

(c)

$$\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$$

Compare the series with $\frac{2x}{1+x^4}$. That can be integrated by setting $u = x^2$.

$$\begin{aligned}
\int_1^{\infty} \frac{2x}{1+x^4} dx &= \lim_{t\to\infty} \int_1^t \frac{2x}{1+x^4} dx \\
u &= x^2,\ du = 2x\ dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{1+u^2} du \\
&= \lim_{t\to\infty} \arctan u \Big|_1^t \\
&= \lim_{t\to\infty} \arctan x^2 \Big|_1^t \\
&= \lim_{t\to\infty} \left[\arctan t^2 - \arctan 1\right] \\
&= \arctan \infty - \arctan 1 \\
&= \frac{\pi}{2} - \frac{\pi}{4} \neq \infty \Rightarrow \textbf{The series converges} \text{ since the integral converges.}
\end{aligned}$$

4. Write the series as a p-series and determine whether it converges.

(a) Hint: $a^{\ln b} = b^{\ln a}$

$$\sum_{n=1}^{\infty} 2^{-\ln n}$$

Put the term in the denominator to make the exponent positive. Then use the identity from the hint.

$$\begin{aligned}
2^{-\ln n} &= \frac{1}{2^{\ln n}} \\
&= \frac{1}{n^{\ln 2}}
\end{aligned}$$

$\ln 2 < 1 \Rightarrow$ **The series diverges.**

(b) Hint: $a^{\ln b} = b^{\ln a}$

$$\sum_{n=1}^{\infty} n \cdot 2^{-2\ln n}$$

Get move everything into the denominator, use the property for the hint, and combine powers.

$$\begin{aligned} n \cdot 2^{-2\ln n} &= \frac{n}{2^{2\ln n}} \\ &= \frac{n}{4^{\ln n}} \\ &\text{Use the hint.} \\ &= \frac{n}{n^{\ln 4}} \\ &= \frac{1}{n^{\ln 4} \cdot n^{-1}} \\ &= \frac{1}{n^{\ln 4 - 1}} \end{aligned}$$

$\ln(4) - 1 < 1 \Rightarrow$ **The series diverges.**

5. Does the following series converge if p is large enough? If so, for which p? Hint: Don't use just p-series.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

Use the integral test and compare with $\frac{1}{x(\ln x)^p}$. This will allow us to use u-substitution to change the integrand into something easy to see if it converges.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x(\ln x)^p} dx &= \lim_{t \to \infty} \int_1^t \frac{1}{x(\ln x)^p} dx \\ u = \ln x,&\ du = \frac{1}{x} dx \\ &= \lim_{t \to \infty} \int_1^t \frac{1}{u^p} du \end{aligned}$$

You can continue integrating but we can see now that the integrand looks exactly like a power series. Recall that the integral test and the actual series either both converge or both diverge. We can see that if that integrand was in a series, it would converge if $p > 1$, so the same is true for the integral converging.

Therefore, the series **converges for $\mathbf{p > 1}$**.

4 5.4 - Comparison Tests

1. Use the comparison test to determine whether the following series converge.

   (a)
$$\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{2}\right)}$$

   First distribute the n to the other term in the denominator. Then compare with $\frac{1}{n^2}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{2}\right)} = \sum_{n=1}^{\infty} \frac{1}{n^2+\frac{n}{2}}$$
$$\frac{1}{n^2+\frac{n}{2}} < \frac{1}{n^2}$$
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (p-series).}$$
$$\text{so } \sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{2}\right)} \textbf{ converges} \text{ by the comparison test.}$$

   We know $\frac{1}{n^2+\frac{n}{2}}$ is less than $\frac{1}{n^2}$ because the denominator is bigger for all values of $n > 1$.

   (b)
$$\sum_{n=2}^{\infty} \frac{1}{2(n-1)}$$

   We can pull out the $\frac{1}{2}$ since it is a constant. The constant will not affect whether it converges or diverges. Then we compare with $\frac{1}{n}$.

$$\sum_{n=2}^{\infty} \frac{1}{2(n-1)} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n-1}$$
$$\frac{1}{n-1} > \frac{1}{n}$$
$$\sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges (harmonic series).}$$
$$\text{so } \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{n-1} \text{ diverges.}$$
$$\text{and } \sum_{n=2}^{\infty} \frac{1}{2(n-1)} \textbf{ diverges} \text{ by the comparison test.}$$

   We know $\frac{1}{n-1} > \frac{1}{n}$ because the denominator is smaller for $n > 2$.

(c)

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$$

The denominator can be simplified to pull out a $n!$ and cancel it with the one in the numerator. Then expand (FOIL) the terms to get one polynomial in the denominator that can be compared with a p-series.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} &= \sum_{n=1}^{\infty} \frac{n!}{(n+2)(n+1)\,n!} \\
&= \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} \\
\frac{1}{n^2+3n+2} &< \frac{1}{n^2} \\
\text{and } \sum_{n=1}^{\infty} \frac{1}{n^2} &\text{ converges (p-series).} \\
\text{so } \sum_{n=1}^{\infty} \frac{n!}{(n+2)!} &\textbf{ converges} \text{ by the comparison test.}
\end{aligned}$$

(d)

$$\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$$

To do the comparison we have to look at the numerator instead of the denominator. $\sin \frac{1}{n}$ is bounded between -1 and 1. That means it cant be greater than $\frac{1}{n^2}$.

$$\begin{aligned}
\frac{\sin \frac{1}{n}}{n^2} &< \frac{1}{n^2} \\
\text{and } \sum_{n=1}^{\infty} \frac{1}{n^2} &\text{ converges (p-series).} \\
\text{so } \sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2} &\textbf{ converges} \text{ by the comparison test.}
\end{aligned}$$

(e)

$$\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{(\sqrt{n})^3}$$

This is similar to the previous problem but you have to combine the powers from the square root and the cube.

$$\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{(\sqrt{n})^3} = \sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{n^{3/2}}$$

$$\frac{\sin\frac{1}{n}}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series).

so $\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{(\sqrt{n})^3}$ **converges** by the comparison test.

2. Use the Limit Comparison Test to determine whether each of the following series converges or diverges.

(a)

$$\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$$

I will choose $b_n = \frac{1}{n^2}$ because it would take n out of the denominator in the ratio. Then it would become a limit that can be evaluated with L'Hôpital's Rule.

$$b_n = \frac{1}{n^2}$$

$$\frac{a_n}{b_n} = \frac{\ln\left(1+\frac{1}{n}\right)}{n} \cdot \frac{n^2}{1}$$

$$= n \cdot \ln\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n \cdot \ln\left(1+\frac{1}{n}\right) = \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0}$$

Use L'Hôpital's Rule.

$$= \lim_{n\to\infty} \frac{\frac{\frac{-1}{n^2}}{1+\frac{1}{n}}}{\frac{-1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{1+\infty}$$

$$= 1 \neq 0$$

The series **converges** by the Limit Comparison Test since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series).

(b) Note: You must show how the comparison series converges by geometric series, not just stating it.

$$\sum_{n=1}^{\infty} \frac{1}{4^n - 3^n}$$

I chose $b_n = \frac{1}{4^n}$ since it can cancel in the ratio. It won't cancel directly but we can manipulate it.

$$\begin{aligned}
b_n &= \frac{1}{4^n} \\
\frac{a_n}{b_n} &= \frac{\frac{1}{4^n - 3^n}}{\frac{1}{4^n}} \\
&= \frac{4^n}{4^n - 3^n} \\
&\text{Divide the numerator and denominator by } 4^n. \\
&= \frac{1}{\frac{4^n - 3^n}{4^n}} \\
&= \frac{1}{1 - \left(\frac{3}{4}\right)^n} \\
\lim_{n\to\infty} \frac{1}{1 - \left(\frac{3}{4}\right)^n} &= \frac{1}{1 - 0} \\
&= 1 \neq 0
\end{aligned}$$

The series **converges** by the Limit Comparison Test since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges (geometric series).

(c) Note: You must show how the comparison series converges by geometric series, not just stating it.

$$\sum_{n=1}^{\infty} \frac{1}{e^{1.1n} - 3^n}$$

I chose $b_n = \frac{1}{e^{1.1n}}$ because it's the first term in the denominator. You could probabably choose $\frac{1}{-3^n}$ and get the same result.

$$\begin{aligned}
b_n &= \frac{1}{e^{1.1n}} \\
\frac{a_n}{b_n} &= \frac{\frac{1}{e^{1.1n} - 3^n}}{\frac{1}{e^{1.1n}}} \\
&= \frac{e^{1.1n}}{e^{1.1n} - 3^n} \\
&\text{Divide the numerator and denominator by } e^{1.1n}. \\
&= \frac{1}{1 - \left(\frac{3}{e^{1}.1}\right)^n} \\
\frac{3}{e^{1.1}} &< 1 \\
\text{so } \lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{1}{1 - 0} \\
&= 1 \neq 0
\end{aligned}$$

The series **converges** by the Limit Comparison Test since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and $\sum_{n=1}^{\infty} \frac{1}{e^{1.1n}}$ converges (geometric series).

(d)

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

I chose $b_n = \frac{1}{n}$ because if you split $n^{1+\frac{1}{n}}$ you can pull out a n^1 and cancel is with the n from b_n.

$$\begin{aligned}
b_n &= \frac{1}{n} \\
\frac{a_n}{b_n} &= \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} \\
&= \frac{n}{n^{1+\frac{1}{n}}} \\
&= \frac{n}{n \cdot n^{\frac{1}{n}}} \\
&= \frac{1}{n^{\frac{1}{n}}} \\
&= n^{-\frac{1}{n}} \\
\lim_{n\to\infty} n^{-\frac{1}{n}} &= \infty^0
\end{aligned}$$

This is a special case limit. Take the natural log, then evaluate the limit. Finally, raise it as a power of e to get the whole answer.

$$\begin{aligned}
y &= n^{-\frac{1}{n}} \\
\ln y &= -\frac{1}{n} \ln n \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} -\frac{\ln n}{n} = -\frac{\infty}{\infty}
\end{aligned}$$

Use L'Hôpital's Rule.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{-\frac{1}{n}}{1} \\
&= -\frac{1}{\infty} \\
&= 0 \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= e^0 \\
&= 1 \neq 0
\end{aligned}$$

The series **diverges** by the Limit Comparison Test since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series).

3. Use any method to evaluate the series.

$$\sum_{n=1}^{\infty} \frac{5}{3^n+1}$$

I will use the Limit Comparison Test. I chose $b_n = \frac{1}{3^k}$ because it is a simplified version of a_n.

$$\begin{aligned}
b_n &= \frac{1}{3^k} \\
\frac{a_n}{b_n} &= \frac{\frac{5}{3^n+1}}{\frac{1}{3^n}} \\
&= \frac{5 \cdot 3^n}{3^n+1} \\
&\text{Divide the numerator and denominator by } 3^k. \\
&= \frac{5}{\frac{3^k+1}{3^k}} \\
&= \frac{5}{1+\frac{1}{3^k}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{5}{1+0} \\
&= 5 \neq 0
\end{aligned}$$

The result of this limit means that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same end behavior. I won't show that $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges but you can do it with geometric series. The series **converges** by the Limit Comparison Test since $\lim_{n\to\infty} \frac{a_n}{b_n} = 5 \neq 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (geometric series).

5 5.6 - Ratio and Root Tests

1. Use the ratio test to determine if the series converges or diverges. State if the ratio test is inconclusive.
   Note: I will skip to showing division of a fraction as multiplication of the reciprocal.

   (a)
   $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

   $$\begin{aligned}
   \frac{a_{n+1}}{a_n} &= \frac{1}{(n+1)!} \cdot \frac{n!}{1} \\
   &\text{Expand the factorial and cancel.} \\
   &= \frac{n!}{(n+1)\,(n!)} \\
   &= \frac{1}{n+1} \\
   \lim_{n\to\infty} \frac{1}{n+1} &= \frac{1}{\infty} \\
   &= 0 < 1 \Rightarrow \text{The series } \textbf{converges} \text{ by the ratio test.}
   \end{aligned}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

   $$\begin{aligned}
   \frac{a_{n+1}}{a_n} &= \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\
   &\text{Split the } 2^{n+1} \text{ and combine the fraction.} \\
   &= \frac{(n+1)^2 \cdot 2^n}{2^1 \cdot 2^n \cdot n^2} \\
   &= \frac{(n+1)^2}{2n^2} \\
   &= \frac{n^2+2n+1}{2n^2} \\
   \lim_{n\to\infty} \frac{n^2+2n+1}{2n^2} &= \frac{1}{2} < 1 \\
   &\Rightarrow \text{The series } \textbf{converges} \text{ by the ratio test.}
   \end{aligned}$$

(c)

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

Be careful multiplying $n+1$ and 3. $3(n+1) = 3n+3$.

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3}$$

Expand the factorials and combine the fractions first.

$$= \frac{((n+1)(n!))^3 \cdot (3n)!}{(3n+3)(3n+2)(3n+1)((3n)!) \cdot (n!)^3}$$

Since $n+1$ and n are being multiplied inside the cube, you can distribute the power.

$$= \frac{(n+1)^3 (n!)^3 \cdot (3n)!}{(3n+3)(3n+2)(3n+1)((3n)!) \cdot (n!)^3}$$

$$= \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)}$$

$$\lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{1}{27} < 1$$

$\Rightarrow$ The series **converges** by the ratio test.

I was able to evaluate the limit in my head because I knew that the only thing that matters is the terms with the highest degree. In this case, both the numerator and denominator have powers of 3 since we are multiplying three linear terms, so we have to look at the ratio of the leading coefficients. Since all the terms have to be multiplied by each other, I just took the maximum result. In the numerator, all the coefficients are 1, so that leading coefficient must be 1. In the denominator, the highest term is going to be the result of multiplying $3n$ by itself three times $\left((3n)^3 = 27n^3\right)$. So that coefficient is $\frac{1}{27}$.

(d)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$$

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{(2n)!} \\
&\text{Expand the factorials and exponents.} \\
&= \frac{(2n+1)\,(2n+1) \cdot (2n)! \cdot n^{2n}}{(n+1)^2\,(n+1)^{2n} \cdot (2n)!} \\
&= \frac{(2n+2)\,(2n+1)\,n^{2n}}{(n+1)^2\,(n+1)^{2n}} \\
&\text{Take out the terms with the power } 2n. \\
&= \frac{(2n+2)\,(2n+1)}{(n+1)^2} \cdot \left(\frac{n}{n+1}\right)^{2n} \\
\lim_{n\to\infty} \frac{(2n+2)\,(2n+1)}{(n+1)^2} \cdot \left(\frac{n}{n+1}\right)^{2n} &= \lim_{n\to\infty} \frac{(2n+2)\,(2n+1)}{(n+1)^2} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} \\
&= 4 \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n}
\end{aligned}$$

(continues on next page)

If we use direct substitution, we will have 1^∞. We need to evaluate the limit of the natural log of it to get the exponent out and use L'Hôpital's Rule.

$$\begin{aligned}
y &= \left(\frac{n}{n+1}\right)^{2n} \\
\ln y &= \ln\left(\left(\frac{n}{n+1}\right)^{2n}\right) \\
&= 2n\ln\left(\frac{n}{n+1}\right) \\
\lim_{n\to\infty} 2n\ln\left(\frac{n}{n+1}\right) &= 2\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} \\
&= \frac{2\ln 1}{\frac{1}{\infty}} = \frac{0}{0}
\end{aligned}$$

Use L'Hôpital's Rule.

Remember to use chain rule with quotient rule.

$$\begin{aligned}
&= 2\lim_{n\to\infty}\frac{\frac{n+1}{n}\cdot\frac{(n-1)-(n+1)}{(n+1)^2}}{-\frac{1}{n^2}} \\
&= 2\lim_{n\to\infty}\frac{1}{n(n+1)}\cdot\frac{-n^2}{1} \\
&= 2\lim_{n\to\infty}\frac{-n}{n+1} \\
&= 2\cdot -1 \\
&= -2 \\
\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n} &= e^{-2} \\
\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\left(\frac{n}{n+1}\right)^{2n} &= 4e^{-2}
\end{aligned}$$

$\frac{4}{e^2} < 1 \Rightarrow$ The series **converges** by the ratio test.

(e)

$$\sum_{n=1}^{\infty} \frac{n!}{\left(\frac{n}{e}\right)^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{\left(\frac{n+1}{e}\right)^{n+1}} \cdot \frac{\left(\frac{n}{e}\right)^n}{n!}$$

Expand the factorials and exponents.

$$= \frac{(n+1)\,(n!)\left(\frac{n}{e}\right)^n}{(n!)\left(\frac{n+1}{e}\right)^1\left(\frac{n+1}{e}\right)^n}$$

$$= \frac{n+1}{\frac{n+1}{e}} \cdot \left(\frac{\frac{n}{e}}{\frac{n+1}{e}}\right)^n$$

$$= e\left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} e\left(\frac{n}{n+1}\right)^n = e \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$

I already showed how to evaluate the limit in the previous problem so I'm skipping to the result.

$$= e \cdot e^{-1}$$

$$= 1 \Rightarrow \text{The ratio test is } \textbf{inconclusive}.$$

2. Use the root test to determine if the series converges or diverges. State if the ratio test is inconclusive. Note: $\lim_{n\to\infty} n^{1/n} = 1$. I will skip to this result since it is common, but you should know how to prove it.

(a)

$$\sum_{n=1}^{\infty} \left(\frac{2n^2-1}{n^2+3}\right)^n$$

This is already in a form where there is one exponent on the outside. We can just take the limit of the term without the exponent.

$$\lim_{n\to\infty} \frac{2n^2-1}{n^2+3} = 2 > 1$$

$\Rightarrow$ The series **diverges** by the root test.

(b)

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\lim_{n\to\infty}\left(\frac{n}{2^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{2}$$

$$= \frac{1}{2} < 1$$

$\Rightarrow$ The series **converges** by the root test.

I skipped showing $\lim_{n\to\infty} n^{1/n} = 1$ but you should be able to prove it.

(c)

$$\sum_{n=1}^{\infty} \frac{n^e}{e^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n^e}{e^n}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} \frac{n^{\frac{e}{n}}}{e} \\
&= \frac{1}{e}\left(\infty^0\right) \\
&= \frac{1}{e} \lim_{n\to\infty} n^{\frac{e}{n}} \\
y &= n^{\frac{e}{n}} \\
\ln y &= \frac{e}{n} \ln(n) \\
\lim_{n\to\infty} \ln y &= e \lim_{n\to\infty} \frac{\ln n}{n} \\
&= e \lim_{n\to\infty} \frac{1}{n} \\
&= 0 \\
\lim_{n\to\infty} y = e^0 = 1 \\
\Rightarrow \frac{1}{e} \lim_{n\to\infty} n^{\frac{e}{n}} &= \frac{1}{e} < 1 \\
&\Rightarrow \text{The series } \textbf{converges} \text{ by the root test.}
\end{aligned}$$

(d)

$$\sum_{n=1}^{\infty} \left(\frac{1}{e} + \frac{1}{n}\right)^n$$

This is another series where we can just evaluate the limit without the exponent since the n-th root cancels it.

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{1}{e} + \frac{1}{n}\right) &= \frac{1}{e} + \frac{1}{\infty} \\
&= \frac{1}{e} + 0 \\
&= \frac{1}{e} < 1 \\
&\Rightarrow \text{The series } \textbf{converges} \text{ by the root test.}
\end{aligned}$$

(e)

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^{n^2}$$

If we take the n-th root of the general term, we will recude the power of n by one.

$$\begin{aligned}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n &= \left(1-\frac{1}{\infty}\right)^\infty\\ &= (1-0)^\infty\\ &= 1^\infty\end{aligned}$$

This is another indeterminate form and we will have to evaluate the limit of the natural log.

$$\begin{aligned}y &= \left(1-\frac{1}{n}\right)^n\\ \ln y &= n\ln\left(1-\frac{1}{n}\right)\\ \lim_{n\to\infty}\ln y &= \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}}\\ &= \frac{\ln(1-0)}{\frac{1}{\infty}}\\ &= \frac{\ln 1}{0} = \frac{0}{0}\\ &\text{Use L'Hôpital's Rule.}\\ &= \lim_{n\to\infty}\frac{\frac{1}{1-\frac{1}{n}}\cdot\frac{1}{n^2}}{-\frac{1}{n^2}}\\ &= \lim_{n\to\infty}\frac{-1}{1-\frac{1}{n}}\\ &= \frac{-1}{1-0}\\ &= -\frac{1}{1}\\ &= -1\\ \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n &= e^{-1}\\ &= \frac{1}{e} < 1\\ &\Rightarrow \text{The series } \textbf{converges} \text{ by the root test.}\end{aligned}$$

6 5.5 - Alternating Series

Do the series converge absolutely, conditionally, or not at all?

1. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{\sqrt{n}+3}$$

   Start by trying the divergence test without the alternating part since the degrees of both the numerator and denominator are equal.

   $$\lim_{n\to\infty} \frac{\sqrt{n}+1}{\sqrt{n}+3} = 1 \neq 0$$

   Now we know the series diverges absulutely. We have to go to the Alternating Series Test to test for conditional convergence, but the result can be seen immediately. One of the conditions for conditional convergence is that the limit is 0, but the divergence test already told us that the limit isn't 0 no matter what we do This means that if a series diverges by the divergence test, it **divereges completey**.

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+3}}{n}$$

   I'll use the Limit Comparison Test to test the series without the alternating part, and compare with $\frac{1}{\sqrt{n}}$.

   $$\begin{aligned}
   b_n &= \frac{1}{\sqrt{n}} \\
   \frac{a_n}{b_n} &= \frac{\sqrt{n+3}}{n} \cdot \frac{\sqrt{n}}{1} \\
   &= \frac{\sqrt{n\,(n+3)}}{n} \\
   &= \frac{\sqrt{n^2+3n}}{n}
   \end{aligned}$$

   $$\lim_{n\to\infty} \frac{\sqrt{n^2+3n}}{n} \neq 0 (\text{numerator and denominator have same degrees})$$

   $$\text{and } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges } (p\text{-series})$$

   $$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n+3}}{n} \textbf{diverges} \text{ by LCT.}$$

   The series diverges absolutely so we need the Alternating Series Test.

   $$\lim_{n\to\infty} \frac{\sqrt{n+3}}{n} \stackrel{?}{=} 0$$

   yes $\rightarrow$ Test for monotone.

   The term $\frac{\sqrt{n+3}}{n}$ is decreasing because the denominator is always bigger than the numerator.
   Since both conditions are met for the Alternating Series Test, the series **converges conditionally** by the AST.

3.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$$

Use the ratio test to see if $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges.

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} &= \lim_{n\to\infty} \frac{3 \cdot 3^n \cdot n!}{(n+1)\,(n!) \cdot 3^n} \\
&= \lim_{n\to\infty} \frac{3}{n+1} \\
&= \frac{3}{\infty} \\
&= 0 < 1 \to \text{Converges by ratio test.}
\end{aligned}$$

Since it converges without the alternating part, it converges with it also.
So the series converges absolutely.

4.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n}\right)^n$$

Use the divergence test to see if $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ diverges.

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n &= 0^{\infty} \\
&\text{Rewrite before using L'Hôpital's Rule.} \\
y &= \left(\frac{n+1}{n}\right)^n \\
\ln y &= n \ln\left(\frac{n+1}{n}\right) \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}} \\
&= \frac{\ln 1}{0} \\
&= \frac{0}{0} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n+1} \cdot \frac{n-n-1}{n^2}}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{-\frac{n}{n^2(n+1)}}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{n}{n+1} \\
&= 1 \neq 0
\end{aligned}$$

So the series diverges absolutely and also **diverges completely** because if it diverges by the divergence test, it completely diverges.

5.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2\left(\frac{1}{n}\right)$$

Use the divergence test to see if $\sum_{n=1}^{\infty}\cos^2\left(\frac{1}{n}\right)$ diverges.

$$\begin{aligned}\lim_{n\to\infty}\cos^2\left(\frac{1}{n}\right) &= \cos^2\left(\frac{1}{\infty}\right)\\ &= \cos^2(0)\\ &= 1 \neq 0\end{aligned}$$

So the series diverges absolutely and also **diverges completely** because if it diverges by the divergence test, it completely diverges.

6.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{1}{n}\right)$$

We need to see if $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$ converges. I will use the Limit Comparison Test and compare it to the harmonic series. In general, the harmonic series works well with natural log terms because you can use L'Hôpital's Rule easily.

$$\begin{aligned}b_n &= \frac{1}{n}\\ \frac{a_n}{b_n} &= \frac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}}\\ \lim_{n\to\infty}\frac{\ln\left(\frac{n+1}{n}\right)}{\frac{1}{n}} &= \frac{0}{0}\\ &\text{Use L'Hôpital's Rule.}\\ &= \lim_{n\to\infty}\frac{n}{n+1}\text{(see (4))}\\ &= 1 \neq 0 \text{ and } \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges.}\end{aligned}$$

Since the harmonic series and $\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$ have the same behavior and the harmoinic series diverges, this series diverges. Now we use the Alternating Series Test to see if it converges conditionally.

First see if the limit is 0.

$$\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right) = \ln\left(1+\frac{1}{\infty}\right) = \ln 1 = 0$$

We can see that the series is monotone decreasing becuase the natural log is an increasing function and we are plugging the decreasing function $1+\frac{1}{n}$ into it, so the overall function $\ln\left(1+\frac{1}{n}\right)$ decreases. Since both of these conditions are met, the series is **conditionally convergent** by the Alternating Series Test.

7.
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$$

We need to figure out if $\sum_{n=1}^{\infty}\frac{n^e}{1+n^\pi}$ converges or diverges. We can use the Limit Comparison Test, but what will we compare it to?
Think of the main case of the LCT. If the limit of $\frac{a_n}{b_n}$ is not 0, the series either both converge or both diverge. A really nice way to get a limit thats not 0 is to get a limit of 1. How do we get a limit equal to 1? We need the numerator and denominator to have the same degree, with leading coefficients of 1.
If we compare a_n to some pwoer series, then we are just multiplying a_n by n^p to get the ratio. What p do we need to get an n^π to match the denominator? We need $p = \pi - e$.

$$\begin{aligned} b_n &= \frac{1}{n^{\pi-e}} \\ \frac{a_n}{b_n} &= \frac{n^e}{1+n^\pi}\cdot\frac{n^{\pi-e}}{1} \\ &= \frac{n^\pi}{n^\pi+1} \\ \lim_{n\to\infty}\frac{n^\pi}{n^\pi+1} &= 1 \neq 0 \end{aligned}$$

Sicne the limit of the ratio is not equal to 0 and $\sum_{n=1}^{\infty}\frac{1}{n^{\pi-e}}$ diverges (p-series with $n-e<1$), $\sum_{n=1}^{\infty}\frac{n^e}{1+n^\pi}$ diverges. So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$ diverges absolutely. Now we still need the Alternating Series Test.

$$\lim_{n\to\infty}\frac{n^e}{1+n^\pi} = 0 \text{ since } \pi > e$$

We just showed that the limit of the general term is 0 since the degree of the denominator is bigger than the degree of the numerator. Now we need to see if it's monotone decreasing.
We can see it's decreasing because $1+n^\pi > n^e$ for all $n \geq 1$.
Both conditions of the Alternating Series Test are met, so the series is **conditionally convergent** by the AST.

8.

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{\frac{1}{n}}$$

We need to see if $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$ converges or diverges. Use the divegence test.

$$\begin{aligned}
\lim_{n\to\infty} n^{\frac{1}{n}} &= \infty^0 \\
y &= n^{\frac{1}{n}} \\
\ln y &= \frac{1}{n} \ln n \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{\ln n}{n} \\
&= \frac{\ln \infty}{\infty} \\
&= \frac{\infty}{\infty} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= \frac{1}{\infty} \\
&= 0 \\
\Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} &= e^0 = 1 \neq 0
\end{aligned}$$

We used the divergence test to how that the series without the alternating part diverges. This also means that with the alternating part, the series **diverges completely** because the limit is not 0.

9.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$$

We need to see if $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges. The textbook says to get a common denominator, rationalize the numerator, and use the Limit Comparison Test with $b_n = \frac{1}{n^{\frac{3}{2}}}$.
You can try that on your own but there is an easier way.
Notice two things about the general term:

- There is subtraction.
- One summand has n and the other has $n + 1$.

This is a telescoping series, which converges. We don't have to do the work of seeing what it converges to because the question didn't ask that.
Since $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ converges (telescoping series), $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ **converges completely**.

10.

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{\frac{1}{n}}}$$

Without the alternating part, we have $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}}$. Does it converge?

No. This is because we have n raised to a power, and the power $\frac{1}{n}$ is less than 1. It's like a p-series but the p is variable. Therefore, it diverges for all $n > 1$. I'm going to show the divergence test anyway because we need it for the Alternating Series Test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} &= \lim_{n\to\infty} n^{-\frac{1}{n}} \\
y &= n^{-\frac{1}{n}} \\
\ln y &= -\frac{1}{n} \ln n \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} -\frac{\ln n}{n} \\
&= -\frac{\infty}{\infty} \\
&= \lim_{n\to\infty} -\frac{1}{n} \\
&= -\frac{1}{\infty} \\
&= 0 \\
\lim_{n\to\infty} n^{-\frac{1}{n}} &= e^0 \\
&= 1 \neq 0
\end{aligned}$$

Since it diverges by the divergence test, it **diverges completely**.

MTH201 Chapter 6
Answers

Harper College*

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Contents

*Problems found in Openstax Calculus Volume II and the Anton textbook.
`https://openstax.org/details/books/calculus-volume-2`

1 6.1 - Power Series

1. Find the radius and interval of convergence for the power series.

   (a)
$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$$

   Use the ratio test to find the limit (in terms of x). You want this limit to be less than 1 to converge, becuase of the conditions of the ratio test.
   Set the limit to be less than 1 and see for which x value this happens. That maximum x value is your radius of convergence.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2x)^{n+1}}{n+1} \cdot \frac{n}{(2x)^n} \\
&= (2x) \cdot \frac{n}{n+1} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 2x \lim_{n\to\infty} \frac{n}{n+1} \\
&= 2x\,(1) \\
&= 2x \\
2x < 1 &\to x < \frac{1}{2}
\end{aligned}$$

$$\boldsymbol{R = \frac{1}{2}}$$

$$\boldsymbol{I = \left[-\frac{1}{2}, \frac{1}{2}\right)}$$

   Why do we have different brackets in the interval?
   On the right, if you plug in $x = \frac{1}{2}$, the $2x$ becomes 1 and you have a harmonic series. That diverges, so $x = \frac{1}{2}$ is not included in the interval.
   On the left, if you plug in $-\frac{1}{2}$ you will get an alternating harmonic series. I won't show it, but you can use the Alternating Series Test to show that the series converges conditionally. Therefore we must put brackets on the left.

(b)

$$\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$$

Use the ratio test to find the limit (in terms of x). You want this limit to be less than 1 to converge, becuase of the conditions of the ratio test.
Set the limit to be less than 1 and see for which x value this happens. That maximum x value is your radius of convergence.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)\, x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{nx^n} \\
&= \frac{(n+1)\, x}{2^n \cdot n} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \frac{x}{2} \lim_{n\to\infty} \frac{n+1}{n} \\
&= \frac{x}{2} \\
\frac{x}{2} < 1 \to x < 2
\end{aligned}$$

$$\boldsymbol{R = 2}$$

To see if the series is convergent at the endpoints, plug in $x = \pm 2$.

$$\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{2^n} \text{ - Diverges by the divergence test.}$$

$$\sum_{n=1}^{\infty} \frac{n \cdot (-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{n\,(-1)^n\,(2)^n}{2^n} \text{ - Diverges by the divergence test.}$$

I didn't show the whole divergence test but you just have to cancel terms and see that the limit doesn't equal 0. Since both endpoints diverge, they need parenthesis.

$$\boldsymbol{I = (-2, 2)}$$

(c)

$$\sum_{n=1}^{\infty} \frac{\pi^n x^n}{n^\pi}$$

We're using the same procedure as (a) and (b).

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{n^{\pi+1} x^{n+1}}{(n+1)^\pi} \cdot \frac{n^\pi}{\pi^n x^n} \\
&= \frac{\pi x n^\pi}{(n+1)^\pi} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \pi x \lim_{n\to\infty} \frac{n^\pi}{(n+1)^\pi} \\
&= \pi x \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^\pi \\
&= \pi x\,(1)^\pi \\
&= \pi x \\
\pi x < 1 \to x < \frac{1}{\pi}
\end{aligned}$$

$$\boldsymbol{R = \frac{1}{\pi}}$$

To find what heppens at the endpoints of the interval of convergence, plug in $x = \pm\frac{1}{\pi}$. In general, it's best to plug in the positive case first because if it converges absolutely with the positive case, it will converge when alternating from the negative case.

$$\begin{aligned}\frac{\pi^n \left(\frac{1}{\pi}\right)^n}{n^\pi} &= \frac{1^n}{n^\pi} \\ &= \frac{1}{n^\pi}\end{aligned}$$

A series with that general term will converge (p-series). It will also converge if you plug in $-\frac{1}{\pi}$. Now we can make our interval.

$$\boldsymbol{I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]}$$

(d)

$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$$

The procedure is the same for the pervious problems.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{10^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n x^n} \\ &= \frac{10x}{(n+1)} \\ \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= 10x \lim_{n\to\infty} \frac{1}{n+1} \\ &= 10x\,(0) \\ &= 0 < 1\end{aligned}$$

Since the limit is always equal to 0 no matter what x we plug in, and $0 < 1$, it always converges. Then we have an infinite radius of convergence and an interval of convergence of all real numbers.

$$\boldsymbol{R = \infty}$$

$$\boldsymbol{I = (-\infty, \infty)}$$

2. Find the radius of convergence of each series.

(a)

$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

Use the ratio test to find the limit (in terms of x). You want this limit to be less than 1 to converge, becuase of the conditions of the ratio test.

Set the limit to be less than 1 and see for which x value this happens. That maximum x value is your radius of convergence.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{((n+1)!)^2 x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 x^n} \\
&= \frac{(n+1)(n!)(n+1)(n!)((2n)!)x^{n+1}}{(2n+2)(2n+1)((2n)!)(n!)(n!)x^n} \\
&= \frac{(n+1)^2 x}{(2n+2)(2n+1)} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= x \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \\
&= \frac{x}{4} \\
\frac{x}{4} &< 1 \to x < 4
\end{aligned}$$

$$\boldsymbol{R = 4}$$

I knew the limit was $\frac{1}{4}$ because the degrees of the numerator and denominator were the same and the denominator would have a leading coefficient of 4. The numerator's leading coefficient is 1.

(b)

$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$$

Use the same procedure as the previous problems.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{((n+1)!)^3 x^{n+1}}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3 x^n} \\
&= \frac{(n+1)^3 (n!)^3 x^{n+1}((3n)!)}{(3n+3)(3n+2)(3n+1)((3n)!)(n!)^3 x^n} \\
&= \frac{(n+1)^3 x}{(3n+3)(3n+2)(3n+1)} \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= x \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \\
&= \frac{x}{27} \\
\frac{x}{27} &< 1 \to x < 27
\end{aligned}$$

$$\boldsymbol{R = 27}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

Use the same procedure as the previous problems. This time you will not be able to evaluate the linit directly because you will get an indeterminate form as 1^∞. You will have to rewrite it to use L'Hôpital's Rule.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!x^n} \\
&= x \cdot \frac{(n+1)^1 n^n}{(n+1)^{n+1}} \\
&= x \frac{n^n}{(n+1)^n} \\
&= x \left(\frac{n}{n+1} \right)^n \\
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= x \lim_{n\to\infty} 1^n \\
&= x \cdot 1^\infty \\
y &= \left(\frac{n}{n+1} \right)^n \\
\ln y &= n \cdot \ln \frac{n}{n+1} \\
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{\ln \frac{n}{n+1}}{\frac{1}{n}} = \frac{0}{0} \\
&\text{Use L'Hôpital's Rule.} \\
&= \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{n+1-n}{(n+1)^2}}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty} -\frac{(n+1)\, n^2}{n\,(n+1)^2} \\
&= \lim_{n\to\infty} -\frac{(n+1)\, n}{(n+1)^2} \\
&= \lim_{n\to\infty} -\frac{n}{n+1} \\
&= -1 \\
x \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n &= xe^{-1} \\
&= \frac{x}{e} \\
\frac{x}{e} < 1 &\to x < e
\end{aligned}$$

$$\mathbf{R = e}$$

3. Given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with center a. Identify its interval of convergence.

(a)

$$f(x) = \frac{1}{x},\ a = 1$$

We need $\frac{1}{x}$ to look like the form $\frac{1}{1-r}$. If you equate the denominators, you get $x = 1 - r$.

$$\begin{aligned} x &= 1 - r \\ \rightarrow r &= 1 - x \\ \frac{1}{x} &= \frac{1}{1-(1-x)} \end{aligned}$$

Set up the series.

$$\frac{1}{x} = \boldsymbol{\sum_{n=0}^{\infty} (1-x)^n}$$

To find the interval of convergence, we need the absolute value of the r-value to be less than 1 so that the terms tend to 0. If we remove the absolute values, we get an inequality with two bounds.

$$|1 - x| < 1$$

$$-1 < 1 - x < 1$$

$$-2 < -x < 0$$

$$0 < x < 2$$

$$\mathbf{I} = (\mathbf{0}, \mathbf{2})$$

(b)

$$f(x) = \frac{x}{1-x^2},\ a = 0$$

We need $\frac{x}{1-x^2}$ to look like the form $\frac{a}{1-r}$. We don't have to manipulate the fraction if we see that $a = x$ and $r = x^2$.
Set up the series and start simplifying:

$$\begin{aligned} \frac{x}{1-x^2} &= \sum_{n=0}^{\infty} x\left(x^2\right)^n \\ &= \sum_{n=0}^{\infty} x^1\left(x^{2n}\right) \\ &= \boldsymbol{x^{2n+1}} \end{aligned}$$

Set up the inequality to find the interval of convergence.

$$\left|x^2\right| < 1$$

$$-1 < x^2 < 1$$

$$\mathbf{I} = (\mathbf{-1}, \mathbf{1})$$

(c)

$$f(x) = \frac{x^2}{1+x^2},\ a = 0$$

This is similar to the previous problem but there is addition in the denominator. Rewrite it as subtraction before finding a and r.

$$\frac{x^2}{1+x^2} = \frac{x^2}{1-(-x^2)}$$
$$a = x^2,\ r = -x^2$$
$$= \sum_{n=0}^{\infty} x^2 \left(-x^2\right)^n$$
$$= \sum_{n=0}^{\infty} x^2 (-1)^n \left(x^2\right)^n$$
$$= \sum_{n=0}^{\infty} (-1)^n x^2 \left(x^{2n}\right)$$
$$= \boldsymbol{\sum_{n=0}^{\infty} (-1)^n x^{2n+2}}$$

Find the interval of convergence in a similar way to the previous problem.

$$\left|-x^2\right| < 1$$
$$x^2 < 1$$
$$-2 < x^2 < 1$$
$$x < 1$$
$$\mathbf{I} = (-\mathbf{1}, \mathbf{1})$$

(d)

$$f(x) = \frac{1}{1-2x},\ a = 0$$

You can plug this straight into a series and simplify if you want.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$$
$$= \boldsymbol{\sum_{n=0}^{\infty} 2^n x^n}$$

Find the interval of convergence.

$$|2x| < 1$$
$$-1 < 2x < 1$$
$$-\frac{1}{2} < x < \frac{1}{2}$$
$$\mathbf{I} = \left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}}\right)$$

(e)

$$f(x) = \frac{x^2}{1-4x^2},\ a = 0$$

You can plug this straight into the series again, but you will have to simplify more.

$$\begin{aligned}\frac{x^2}{1-4x^2} &= \sum_{n=0}^{\infty} x^2 \left(4x^2\right)^n \\ &= \sum_{n=0}^{\infty} x^2 \cdot 4^n \cdot x^{2n} \\ &= \boldsymbol{\sum_{n=0}^{\infty} 4^n x^{2n+2}}\end{aligned}$$

Find the interval of convergence.

$$\begin{gathered}\left|4x^2\right| < 1 \\ -1 < 4x^2 < 1 \\ -\frac{1}{4} < x^2 < \frac{1}{4} \\ -\frac{1}{2} < x < \frac{1}{2} \\ \mathbf{I} = \left(-\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{2}}\right)\end{gathered}$$

2 6.2 - Properties of Power Series

1. Use partial fractions to find the power series of each function.

(a)

$$\frac{4}{(x-3)(x+1)}$$

Start with PFD.

$$\begin{aligned}\frac{4}{(x-3)(x+1)} &= \frac{A}{x-3}+\frac{B}{x+1}\\ 4 &= A(x+1)+B(x-3)\\ &= Ax+A+Bx-3B\end{aligned}$$

Equate the coefficients and solve the system of equations.

$$\begin{aligned}A+B&=0\\ A-3B&=4\end{aligned}$$

$$\Rightarrow B=-1,\ A=1$$

Plug in A and B. Then rewrite each fraction in the form $\frac{a}{1-r}$ and turn them into geometric series. This includes factoring out cosntants to include in a, and factoring out -1s to flip subtraction terms.

$$\begin{aligned}\frac{4}{(x-3)(x+1)} &= \frac{1}{x-3}-\frac{1}{x+1}\\ &= \frac{1}{3\left(\frac{x}{3}-1\right)}-\frac{1}{x+1}\\ &= \frac{1}{-3\left(1-\frac{x}{3}\right)}-\frac{1}{1-(-x)}\\ &= \frac{-\frac{1}{3}}{1-\frac{x}{3}}-\frac{1}{1-(-x)}\\ &= \sum_{n=0}^{\infty}-\frac{1}{3}\left(\frac{x}{3}\right)^n-\sum_{n=0}^{\infty}(-x)^n\end{aligned}$$

Combine the 2 series into one.

$$\begin{aligned}&= \sum_{n=0}^{\infty}\left(-\frac{1}{3}\left(\frac{x}{3}\right)^n-(-x)^n\right)\\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{3^1}\frac{x^n}{3^n}-(-x)^n\right)\\ &= \sum_{n=0}^{\infty}\left(-\frac{1}{3^{n+1}}x^n-(-1)^n(x)^n\right)\end{aligned}$$

Factor out x^n.

$$\begin{aligned}&= \sum_{n=0}^{\infty}x^n\left(-\frac{1}{3^{n+1}}-(-1)^n\right)\\ &= \sum_{n=0}^{\infty}x^n\left(-\frac{1}{3^{n+1}}-(-1)^{n+1}(-1)^{-1}\right)\\ &= \sum_{n=0}^{\infty}x^n\left(-\frac{1}{3^{n+1}}+(-1)^{n+1}\right)\\ &= \boldsymbol{\sum_{n=0}^{\infty}\left((-1)^{n+1}-\frac{1}{3^{n+1}}\right)x^n}\end{aligned}$$

(b)

$$\frac{5}{(x^2+4)(x^2-1)}$$

Start with PFD.

$$\begin{aligned}
\frac{5}{(x^2+4)(x^2-1)} &= \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2-1} \\
5 &= (Ax+B)\left(x^2-1\right) + (Cx+D)\left(x^2+4\right) \\
&= Ax^3 - Ax + Bx^2 - B + Cx^3 + 4Cx + Dx^2 + 4D
\end{aligned}$$

Equate the coefficients and solve the system of equations.

$$\begin{aligned}
A + C &= 0 \\
B + D &= 0 \\
-A + 4C &= 0 \\
-B + 4D &= 5
\end{aligned}$$

$$\Rightarrow C = 0,\ A = 0,\ D = 1,\ B = -1$$

Plug in D and B. Then rewrite each fraction in the form $\frac{a}{1-r}$ and turn them into geometric series. This includes factoring out cosntants to include in a, and factoring out -1s to flip subtraction terms.

$$\begin{aligned}
\frac{5}{(x^2+4)(x^2-1)} &= -\frac{1}{x^2+4} + \frac{1}{x^2-1} \\
&= \frac{1}{x^2-1} - \frac{1}{x^2+4} \\
&= -\frac{1}{1-x^2} - \frac{1}{4\left(1+\frac{x^2}{4}\right)} \\
&= -\frac{1}{1-x^2} - \frac{1}{4\left(1-\left(-\frac{x^2}{4}\right)\right)} \\
&= \sum_{n=0}^{\infty} -1\left(x^2\right)^n - \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{x^2}{4}\right)^n \\
&= -\sum_{n=0}^{\infty} x^{2n} - \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n \left(x^2\right)^n \\
&= -\sum_{n=0}^{\infty} x^{2n} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{4}\right)^n x^{2n}
\end{aligned}$$

Combine into one series.

$$= \sum_{n=0}^{\infty}\left(-x^{2n} - \frac{1}{4}\left(\frac{1}{4}\right)^n x^{2n}\right)$$

Factor out x^{2n}.

$$= \sum_{n=0}^{\infty} x^{2n}\left(-1 - \frac{1}{4^1}\left(\frac{1}{4^n}\right)\right)$$

$$\boldsymbol{\sum_{n=0}^{\infty} x^{2n}\left(-1 - \frac{1}{4^{n+1}}\right)}$$

2. Express each series as a rational function.

(a)

$$\sum_{n=1}^{\infty} \frac{1}{x^n}$$

In this series, n starts at 1, not 0. We will have to change the index, which also changes the n inside.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{x^n} &= \sum_{n=0}^{\infty} \frac{1}{x^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{1}{x} \cdot \frac{1}{x^n} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \frac{1}{x^n} \\
&= \frac{1}{x} \sum_{n=0}^{\infty} \left(\frac{1}{x}\right)^n \\
&= \frac{\frac{1}{x}}{1 - \frac{1}{x}} \\
&= \mathbf{\frac{1}{x-1}}
\end{aligned}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{(x-3)^{2n-1}}$$

Start by changing the index to start at 0. Be careful with the new exponent because you will have to distribute the 2.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{(x-3)^{2n-1}} &= \sum_{n=0}^{\infty} \frac{1}{(x-3)^{2(n+1)-1}} \\
&= \sum_{n=0}^{\infty} \frac{1}{(x-3)^{2n+1}}
\end{aligned}$$

We want the power of n by itself, so factor out a $\frac{1}{x-3}$. Then take out the power of n and apply the $\frac{a}{1-r}$ formula.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(x-3)^{2n+1}} &= \sum_{n=0}^{\infty} \frac{1}{x-3} \cdot \frac{1}{(x-3)^{2n}} \\
&= \sum_{n=0}^{\infty} \frac{1}{x-3} \cdot \left(\frac{1}{(x-3)^2}\right)^n \\
&= \frac{\frac{1}{x-3}}{1 - \frac{1}{(x-3)^2}} \\
&= \frac{1}{(x-3)\left(1 - \frac{1}{(x-3)^2}\right)}
\end{aligned}$$

Distriute the entire $x-3$ term into $1 - \frac{1}{(x-3)^2}$ (don't FOIL). Then multiply both the numerator and denominator by $x-3$ to put it in the form of a rational function.

$$\mathbf{\frac{x-3}{(x-3)^2 - 1}}$$

3. Given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, use differentiation or integration to find the power series for each function, centered at $x = 0$.

(a)

$$\ln\left(1 - x^2\right)$$

We don't have a power series for $\ln x$ yet, but its derivative is a simple rational function that can be converted into a power series and integrated to get the original function back.
Find the derivative, write it as a geometric series, write out the series expansion, integrate the series expansion, and write the result as a new series.

$$\begin{aligned}
\frac{d}{dx}\left[\ln\left(1-x^2\right)\right] &= -\frac{2x}{1-x^2} \\
&= \sum_{n=0}^{\infty} (-2x)\left(x^2\right)^n \\
&= \sum_{n=0}^{\infty} -2x^1 x^{2n} \\
&= \sum_{n=0}^{\infty} -2x^{2n+1} \\
&= -2x^1 - 2x^3 - 2x^5 - 2x^7 - \cdots \\
\ln\left(1-x^2\right) &= \int -\frac{2x}{1-x^2} dx \\
&= -\frac{2}{2}x^2 - \frac{2}{4}x^4 - \frac{2}{6}x^6 - \frac{2}{8}x^8 - \cdots \\
&= \sum_{n=1}^{\infty} -\frac{2}{2n}x^{2n} \\
&= \boldsymbol{\sum_{n=0}^{\infty} -\frac{1}{n}x^{2n}}
\end{aligned}$$

(b)

$$\arctan\left(x^2\right)$$

We are repeating the same procedure as the pervious problem.

$$\begin{aligned}
\frac{d}{dx}\left[\arctan\left(x^2\right)\right] &= \frac{2x}{1+x^4} \\
&= \sum_{n=0}^{\infty} 2x\left(-x^4\right)^n \\
&= \sum_{n=0}^{\infty} 2x^1 (-1)^n x^{4n} \\
&= \sum_{n=0}^{\infty} 2(-1)^n x^{4n+1} \\
&= 2x^{4\cdot 0+1} - 2x^{4\cdot 1+1} + 2x^{4\cdot 2+1} - \cdots \\
\arctan\left(x^2\right) &= \int \frac{2x}{1+x^4} dx \\
&= \frac{2}{4\cdot 0+2}x^{4\cdot 0+2} - \frac{1}{4\cdot 1+2}x^{4\cdot 1+2} + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{2}{4n+2}x^{4n+2} \\
&= \boldsymbol{\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}x^{4n+2}}
\end{aligned}$$

3 6.3 - Taylor and Maclaurin Series

1. Find the Taylor polynomials of degree 2 approximating the given function centered at the given point.

   (a) $f(x) = 1 + x + x^2$ at $a = -1$

n		$f^{(n)}(x)$	$f^{(n)}(-1)$
0	$f(x)$	$1 + x + x^2$	1
1	$f'(x)$	$1 + 2x$	-1
2	$f''(x)$	2	2

$$\frac{1}{0!}(x+1)^0 - \frac{1}{1!}(x+1)^1 + \frac{2}{2!}(x+1)^2$$

$$\mathbf{1 - (x+1) + (x+1)^2}$$

   (b) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

n		$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{2}\right)$
0	$f(x)$	$\sin(2x)$	0
1	$f'(x)$	$2\cos(2x)$	-2
2	$f''(x)$	$-2\sin(2x)$	0

$$-\frac{2}{1!}\left(x - \frac{\pi}{2}\right)$$

$$\mathbf{-2\left(x - \frac{\pi}{2}\right)}$$

   (c) $f(x) = \ln x$ at $a = 1$

n		$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f(x)$	$\ln x$	0
1	$f'(x)$	$\frac{1}{x}$	1
2	$f''(x)$	$-\frac{1}{x^2}$	-1

$$\frac{1}{1!}(x-1)^1 - \frac{1}{2!}(x-1)^2$$

$$\mathbf{(x-1) - \frac{1}{2}(x-1)^2}$$

2. Verify that the given value of n yields a remainder estimate $|R_n| < \frac{1}{1000}$. Find the value of the Taylor polynomial of degree n at the point a.

   (a) $28^{1/3}$; $a = 27$; $n = 1$

   According to the theorem we need the $(n+1)$th derivative of $f(x)$. Let $f(x) = x^{1/3}$.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$
$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$

   Plug f'', a, and n into the formula.

$$\begin{aligned} R_1(x) &= \frac{f^{(2)}(c)}{2!}(x-27)^2 \\ &= \frac{-\frac{2}{9}c^{-\frac{5}{3}}}{2}(x-27)^2 \\ &= -\frac{1}{9c^{\frac{5}{3}}}(x-27)^2 \end{aligned}$$

   We want to see what the remainder is when $x = 28$, so plug it into $R_1(x)$.

$$\begin{aligned} R_1(28) &= -\frac{1}{9c^{\frac{5}{3}}}(1)^2 \\ &= -\frac{1}{9c^{\frac{5}{3}}},\ 27 < c < 28 \end{aligned}$$

   We want to find the upper bound for the remainder, so we we need to maximize it. Minimizing the denominator will maximize the overall function. So the remainder is maximized when the value of c is lowest, which is $c = 27$. Plug in $c = 27$ to bound the remainder.

$$\begin{aligned} |R_1(28)| &< \frac{1}{9(27)^{\frac{5}{3}}} \\ &= \frac{1}{2187} \\ &< \frac{1}{1000} \end{aligned}$$

   So **the choice of n satisfies** $|\mathbf{R_n}| < \boldsymbol{\frac{1}{1000}}$. Now we need to create the Taylor polynomial p_1 centered at $a = 27$ and use it to approximate $28^{1/3}$.

$$\begin{aligned} p_1(x) &= \frac{f(27)}{0!}(x-27)^0 + \frac{f'(27)}{1!}(x-27)^1 \\ &= \sqrt[3]{27} + \frac{x-27}{3\cdot(27)^{\frac{2}{3}}} \end{aligned}$$

$$\mathbf{p_1(28) \approx 3.\overline{037}}$$

$$28^{\frac{1}{3}} \approx 2.036559$$

   The approximation is in range of our desired error, so our answer is good.

(b) $\ln 2$; $a = 1$; $n = 1000$ (Use a calculator for series) According to the theorem we need the $(n+1)$th derivative of $f(x)$. Let $f(x) = \ln x$. Let's find the 1001st derivative:

$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$
$$\vdots$$
$$f^{(n)}(x) = \frac{(n-1)!}{x^n}(-1)^{n-1}$$
$$f^{(1001)}(x) = (-1)^{1000}\frac{1000!}{x^{1001}}$$

Plug f'', a, and n into the formula.

$$\begin{aligned}
R_{1000}(x) &= \frac{f^{(1000)}(c)}{1001!}(x-1)^{1001} \\
&= \frac{1000!}{1001!c^{1000}}(x-1)^{1001} \\
&= -\frac{1}{1001c^{1000}}(x-1)^{1000},\ 1 < c < 2
\end{aligned}$$

We want to find the upper bound for the remainder, so we we need to maximize it. Minimizing the denominator will maximize the overall function. So the remainder is maximized when the value of c is lowest, which is $c = 1$. Plug in $c = 1$ to bound the remainder.

$$\begin{aligned}
|R_{1000}(2)| &< \frac{1}{1001}(2-1)^{1}001 \\
&= \frac{1}{1001} \\
&< \frac{1}{1000}
\end{aligned}$$

So **the choice of n satisfies** $|\mathbf{R_n}| < \frac{\mathbf{1}}{\mathbf{1000}}$. Now we need to create the Taylor polynomial p_{1000} centered at $a = 1$ and use it to approximate $\ln 2$.

$$p_n(x) = \frac{\ln 1}{0!}(x-1)^0 + \frac{1/1}{1!}(x-1)^1 + \cdots + \frac{(n-1)!}{1^n}(-1)^{n-1}(x-1)^n +$$
$$p_{1000}(x) = \sum_{n=1}^{1001}\frac{(x-1)^n}{n}(-1)^{n-1}$$
$$\approx \mathbf{0.693646}$$
$$\ln 2 \approx 0.693147$$

The approximation is in range of our desired error, so our answer is good.

3. Find the Taylor series of the given function centered at the given point.

(a) $f(x) = 1 + x + x^2 + x^3$ at $a = -1$

You can only get up to the 4th derivative, so this Taylor series is not infinite.

n		$f^{(n)}(x)$	$f^{(n)}(-1)$
0	$f(x)$	$1 + x + x^2 + x^3$	0
1	$f'(x)$	$1 + 2x + 3x^2$	2
2	$f''(x)$	$2 + 6x$	-4
3	$f'''(x)$	6	6

$$0 + \frac{2}{1!}(x+1) - \frac{4}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$
$$\mathbf{2(x+1) - 2(x+1)^2 + (x+1)^3}$$

(b) $f(x) = \cos x$ at $a = 2\pi$

n		$f^{(n)}(x)$	$f^{(n)}(2\pi)$
0	$f(x)$	$\cos x$	1
1	$f'(x)$	$-\sin x$	0
2	$f''(x)$	$-\cos x$	-1
3	$f'''(x)$	$\sin x$	0
4	$f^{(4)}(x)$	$\cos x$	1

Ignore the 0 terms because they don't do anything. Then apply the formula and get the following:

$$\frac{1}{0!}(x-2\pi)^0 - \frac{1}{2!}(x-2\pi)^2 + \frac{1}{4!}(x-2\pi)^4 - \cdots$$

$$\boldsymbol{\sum_{n=0}^{\infty}(-1)^n \frac{(x-2\pi)^{2n}}{(2n)!}}$$

Since ony even degrees were present, we used $2n$. The $(-1)^n$ is the result of the signs alternating.

(c) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

n		$f^{(n)}(x)$	$f^{(n)}(0)$
0	$f(x)$	$(x-1)^{-3}$	-1
1	$f'(x)$	$-3(x-1)^{-4}$	$-3 = 1 \cdot -3$
2	$f''(x)$	$12(x-1)^{-5}$	$-12 = 3 \cdot -4$
3	$f'''(x)$	$-60(x-1)^{-6}$	$-60 = 12 \cdot 5$
			$\frac{(n+2)!}{2}$

Notice in the last column that one of the multiples of $f^{(n)}(0)$ is the product of the last two products. That product is multiplied by $n+2$. Thats where $(n+2)!$ comes from.

$$-\frac{1}{0!}x^0 - \frac{3}{1!}x^1 - \frac{12}{2!}x^2 - \frac{60}{3!}x^3 - \cdots$$

$$\sum_{n=0}^{\infty} \frac{-\frac{(n+2)!}{2}}{n!} \cdot x^n$$

$$\boldsymbol{\sum_{n=0}^{\infty} -\frac{(n+2)(n+1)}{2} x^n}$$

4. Compute the Taylor series of each function around $x = 1$.

(a) $f(x) = 2 - x$

n		$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f(x)$	$2-x$	1
1	$f'(x)$	-1	-2
2	$f''(x)$	0	0
3	$f'''(x)$	0	0

$$\frac{1}{0!}(x-1)^0 - \frac{1}{1!}(x-1)^1$$

$$\mathbf{1 - (x-1)}$$

$$= 2 - x$$

When a Taylor series is finite, you should get back the exact value of the function you start with.

(b) $f(x) = (x-2)^2$

n		$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f(x)$	$(x-2)^2$	1
1	$f'(x)$	$2(x-2)^1$	-2
2	$f''(x)$	2	2
3	$f'''(x)$	0	0

$$\frac{1}{0!}(x-1)^0 - \frac{2}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2$$

$$\mathbf{4 - 2(x-1) + (x-1)^2}$$

(c) $f(x) = \frac{1}{x}$

n		$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f(x)$	x^{-1}	1
1	$f'(x)$	$-x^{-2}$	-1
2	$f''(x)$	$2x^{-3}$	2
3	$f'''(x)$	$-6x^{-4}$	-6

$$\frac{1}{0!}(x-1)^0 - \frac{1}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \cdots$$

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

$$\boldsymbol{\sum_{n=0}^{\infty} (-1)^n (x-1)^n}$$

(d) $f(x) = e^{2x}$

n		$f^{(n)}(x)$	$f^{(n)}(1)$
0	$f(x)$	$1e^{2x}$	$1e^2$
1	$f'(x)$	$2e^{2x}$	$2e^2$
2	$f''(x)$	$4e^{2x}$	$4e^2$
3	$f'''(x)$	$8e^{2x}$	$8e^2$

$$\frac{e^2}{0!}(x-1)^0 - \frac{2e^2}{1!}(x-1)^1 + \frac{4e^2}{2!}(x-1)^2 - \frac{8e^2}{3!}(x-1)^3 + \cdots$$

$$\boldsymbol{\sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(x-1)^n}$$

5. Approximate $\sqrt{e}$ to 4 decimal places of accuracy using Taylor series remainder.
Start by finding $f(x)$. Since the square root is on the outside, you may want $f(x) = \sqrt{x}$. However, we usually want f to be easily differentiable and predictable so we can bound it. $\sqrt{x}$ will be a difficult function to deal with.
Instead, let $f(x) = e^x$. This is a lot better because we know the derivatives and it has an easy Mclaurin series because $e^0 = 1$. We will also center the series at $a = 0$ because it's close to $\frac{1}{2}$.

$$|R_n(x)| < \frac{M}{(n+1)!}|x-0|^{n+1}$$
$$\left|R_n\left(\frac{1}{2}\right)\right| < 0.0001 < \frac{e^c}{(n+1)!}|x|^{n+1},\ 0 < c < \frac{1}{2}$$

There's no point in plugging the endpoint $c = \frac{1}{2}$ into e^c, because that's what we're trying to approximate. Let's choose a bigger number and see what happens.
We can say that $e \approx 4$, so $4^{1/2} = 2$. We know this is bigger than the real $\sqrt{t}$ because $4 > e$ and $\sqrt{x}$ is an increasing function.

$$\begin{aligned}\left|R_n\left(\frac{1}{2}\right)\right| &< 0.0001 < \frac{2}{(n+1)!}\left|\frac{1}{2}\right|^{n+1},\ 0 < c < \frac{1}{2}\\ &= \frac{2}{(n+1)!2^{n+1}}\\ &\rightarrow n = 5\end{aligned}$$

Now find $p_5\left(\frac{1}{2}\right)$.

$$f(0) = f'(0) = \cdots = e^0 = 1$$
$$p_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$$
$$\mathbf{p_5\left(\frac{1}{2}\right) \approx 1.64870}$$
$$\sqrt{e} \approx 1.64872$$

The approximation is in range of our desired error, so our answer is good.

MTH201 Chapter 7
Answers

Harper College*

July 22, 2025

Contents

*Problems found in Openstax Calculus Volume II.
https://openstax.org/details/books/calculus-volume-2

1 7.1 - Parametric Equations

1. Sketch the curves by eliminating the parameter t. Give the orientation of the curve.

   (a) $x = t^2 + 2t$, $y = t + 1$
   Rewrite the parametric equation into a rectangular one by solving for t in terms of y, and substituting it into $x(t)$.

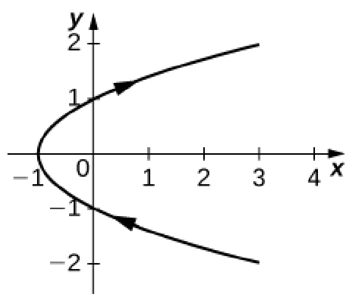
$$\begin{aligned} t &= y - 1 \\ x &= (y-1)^2 + 2(y-1) \\ &= (y-1)(y-1) + 2y - 2 \\ &= y^2 - 2y + 1 + 2y - 2 \\ &= y^2 - 1 \\ y^2 &= x + 1 \\ y &= \pm\sqrt{x+1} \end{aligned}$$

   Those are just two square root functions shifted to the left by 1 unit.
   To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \to (0, 1)$$

$$t = 1 \to (3, 2)$$

   So the orientation is bottomm to top.



   (b) $x = 2t + 4$, $y = t - 1$
   Rewrite the parametric equation into a rectangular one by solving for t in terms of y, and substituting it into $x(t)$.

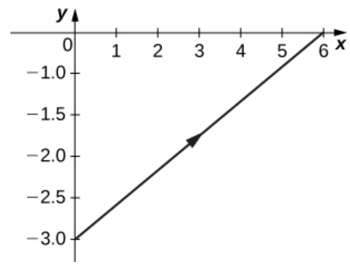
$$\begin{aligned} t &= y + 1 \\ x &= 2(y+1) + 4 \\ &= 2y + 2 + 4 \\ &= 2y + 6 \\ x &= 2y + 6 \\ \to 2y &= x - 6 \\ \to y &= \frac{1}{2}x - 3 \end{aligned}$$

   This is the equation of a line.
   To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \to (4, -1)$$

$$t = 1 \rightarrow (6, 0)$$

So the orientation is bottomm to top.



2. Eliminate the parameter and sketch the graph. $x = 2t^2$, $y = t^4 + 1$
   Don't solve for t explicitly. Solve for t^4 in terms of x, then plug it into y.

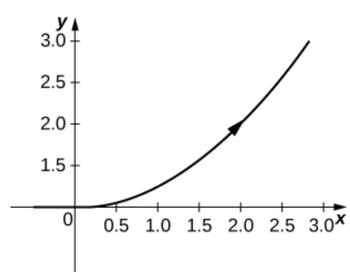
$$x = 2t^2$$
$$x^2 = 4t^4$$
$$t^4 = \frac{1}{4}x^2$$
$$\mathbf{y} = \frac{\mathbf{1}}{\mathbf{4}}\mathbf{x^2} + \mathbf{1}$$

   This is a parabola opening up, shifted 1 unit up, and stretched by a factor of 4.
   To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \rightarrow (0, 1)$$

$$t = 1 \rightarrow (2, 2)$$

   So the orientation is left to right.



3. Eliminate the parameter and sketch the graph. Indicate any asymptotes.

   (a) $x = 6\sin(2\theta)$, $y = 4\cos(2\theta)$
   We need to use the Pythagorean identity for sine and cosine:

$$\sin^2\theta + \cos^2\theta = 1$$

   Even though we have 2θ, the identity will still work.

$$\sin^2(2\theta) + \cos^2(2\theta) = 1$$

Now we just need to solve for $\sin(2\theta)$ and $\cos(2\theta)$

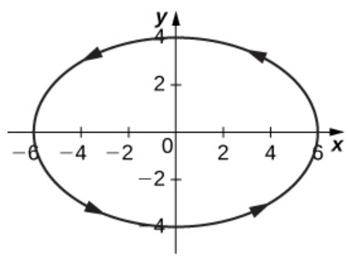
$$\sin(2\theta) = \frac{x}{6}$$
$$\cos(2\theta) = \frac{y}{4}$$
$$\begin{aligned} 1 &= \sin^2(2\theta) + \cos^2(2\theta) \\ &= \left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 \\ \mathbf{1} &= \frac{\mathbf{x^2}}{\mathbf{36}} + \frac{\mathbf{y^2}}{\mathbf{16}} \end{aligned}$$

That is the equation of an ellipse that is 12 units wide and 8 units tall.
To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \rightarrow (0, 4)$$

$$t = \frac{\pi}{2} \rightarrow (-6, 0)$$

So the orientation is counter-clockwise.



(b) $x = 3 - 2\cos\theta$, $y = -5 + 3\sin\theta$
We are using the Pythagorean Identity for sine and cosine again. Solve for $\cos\theta$ and $\sin\theta$.

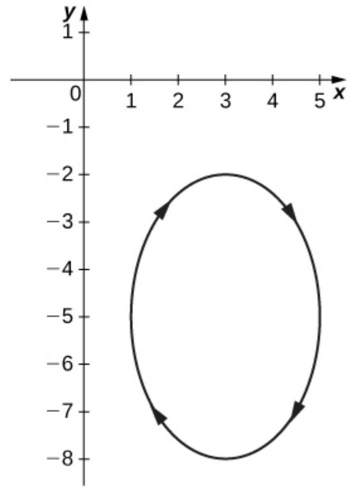
$$\sin\theta = \frac{1}{3}(y+5)$$
$$\cos\theta = 3 - x$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\frac{\mathbf{(y+5)^2}}{\mathbf{9}} + \frac{\mathbf{(-x+3)^2}}{\mathbf{1}} = \mathbf{1}$$

This is an ellipse 1 unit wide and 6 units tall centered at $(3, -5)$.
Use this link to explore ellipse transformations.
To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \rightarrow (0, 4)$$

$$t = \frac{\pi}{2} \rightarrow (-6, 0)$$

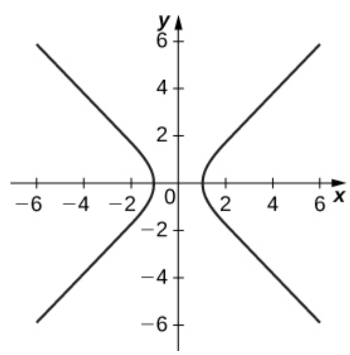
So the orientation is clockwise.



(c) $x = \sec t$, $y = \tan t$ (Do not find orientation)
Use the Pythagorean identity for secant and tangent.

$$1 = \sec^2 t - \tan^2 t$$

$$1 = x^2 - y^2$$



If you graph this you will see it's a hyperbola, and it does have asymptotes.
To find the asymptotes, look for which points make the graph undefined, or which points make the equation false.
The two asymptotes are $\mathbf{y} = \pm\mathbf{x}$, since the equation will result in $x^2 - x^2 \neq 1$.
The reason I said not to find the orientation is because it's confusing. Click this link to see how the ellipse works.

(d) $x = e^t$, $y = e^{2t}$
You can find the rectangular form of the equation by recognizing that $e^{2t} = \left(e^t\right)^2$.

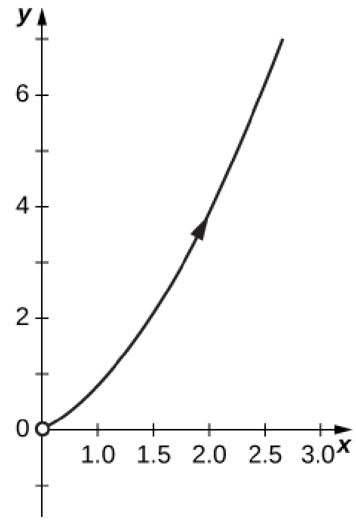
$$y = \left(e^t\right)^2 = x^2$$

That's the standard parabola, but only on Quadrant I because since $x = e^t$ is always positive, the graph does not exist on the left.
To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 0 \rightarrow (1, 1)$$

$$t = 1 \rightarrow \left(e, e^2\right)$$

So the orientation is bottom to top.



(e) $x = t^3$, $y = 3 \ln t$ (Hint: Use log properties.)
If you have a constant multiple of a log, that constant becomes the exponent of the term inside the log.

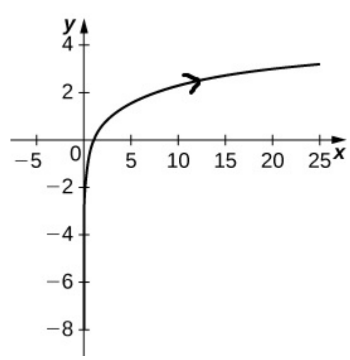
$$y = 3 \ln t = \ln \left(t^3\right) = \ln x$$

That's the standard natural log graph.
To find the orientation, pick two values for t, and see which direction the function moves.

$$t = 1 \rightarrow (1, 0)$$

$$t = e \rightarrow \left(e^3, 3\right)$$

So the orientation is bottom to top.



4. Convert the parametric equations into rectangular form. State the domain of the rectangular form. Do not sketch.
Recall that the three things that affect the domain of a function:

- Denominators equaling 0
- Negative numbers inside even roots (square root, 4th root, etc.)
- Invalid input into functions (ex: $\ln 0$)

Additionally, we also have to account for the shape of the graph (ex: width of an ellipse).

(a) $x = t^2 - 1$, $y = \frac{t}{2}$

Solve for t in terms of y and plug into x. Then solve for y.

$$\begin{aligned} y &= \frac{t}{2} \\ t &= 2y \\ \Rightarrow x &= (2y)^2 - 1 \\ x &= 4y^2 - 1 \\ y^2 &= \frac{1}{4}(x+1) \\ \Rightarrow y &= \pm\frac{\sqrt{x+1}}{\sqrt{4}} \\ &= \pm\mathbf{\frac{1}{2}\sqrt{x+1}} \end{aligned}$$

The domain is the set of x-values such that $x + 1 \geq 0$ because then we will not have negative numbers in the square root.

$$\begin{aligned} x + 1 &\geq 0 \\ \rightarrow x &\geq -1 \end{aligned}$$

You can write the domain in at least three ways:

- $\mathbf{x \geq -1}$
- $\mathbf{x \in [-1, \infty)}$
- $\mathbf{\{x |\ x \geq -1\}}$

(b) $x = 4\cos\theta$, $y = 3\sin\theta$, $\theta \in (0, 2\pi]$

Use the Pythagorean Identity for sine and cosine.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin\theta = \frac{y}{3},\ \cos\theta = \frac{x}{4}$$
$$\mathbf{\frac{y^2}{9} + \frac{x^2}{16} = 1}$$

This is an ellipse centered at the origin, that has an x-radius of 4 and a y-radius of 3.
That means the domain is from $x = -4$ to $x = 4$.
You can write the domain in at least three ways:

- $\mathbf{-4 \leq x \leq 4}$
- $\mathbf{x \in [-4, 4]}$
- $\mathbf{\{x |\ -4 \leq x \leq 4\}}$

(c) $x = 2t - 3$, $y = 6t - 7$

I would solve for t in terms of x so when you plug it into y you will immediately have the form you are looking for.

$$\begin{aligned} x &= 2t - 3 \\ t &= \frac{x+3}{2} \\ y &= 6\left(\frac{x+3}{2}\right) - 7 \\ y &= 3x + 9 - 7 \\ \mathbf{y} &\mathbf{= 3x + 2} \end{aligned}$$

This is a polynomial, so our domain is all real numbers. You can write the domain in at least three ways:

- $\boldsymbol{-\infty < x < \infty}$
- $\boldsymbol{x \in \mathbb{R}}$
- $\boldsymbol{\{x| -\infty < x < \infty\}}$

(d) $x = 1 + \cos t$, $y = 3 - \sin t$
Use the Pythagorean Identity for sine and cosine.

$$\sin^2 t + \cos^2 t = 1$$
$$\cos t = x - 1,\ \sin t = 3 - y$$
$$\mathbf{(3 - y)^2 + (x - 1)^2 = 1}$$

This is a circle of radius 1 centered at $(1, 3)$. Since the radius is 1, the domain will be one unit away from the center on each side.
You can write the domain in at least three ways:

- $\mathbf{0 \leq x \leq 2}$
- $\mathbf{x \in [0, 2]}$
- $\mathbf{\{x|\ 0 \leq x \leq 2\}}$

(e) $x = \cos(2t)$, $y = \sin t$
Use the double angle formula $\cos(2t) = 2\cos^2 t - 1$

$$x = 2\cos^2 t - 1$$
$$\cos^2 t = \frac{x+1}{2}$$
$$\frac{x+1}{2} + y^2 = 1$$
$$y^2 = 1 - \frac{1}{2}(x+1)$$
$$\mathbf{y = \pm\sqrt{1 - \frac{1}{2}(x+1)}}$$

To find the domain, start by setting the term inside the square root to be greater than or equal to zero.

$$1 - \frac{1}{2}(x+1) \geq 0$$
$$1 \geq \frac{1}{2}(x+1)$$
$$x + 1 \leq 2$$
$$x \leq 1$$

From this it looks like the domain is $(\infty, 1]$ but we have to look at the parameters. $x(t)$ and $y(t)$ are trig functions that are bounded by -1 and 1.
You can write the domain in at least three ways:

- $\mathbf{-1 \leq x \leq 1}$
- $\mathbf{x \in [-1, 1]}$
- $\mathbf{\{x|\ -1 \leq x \leq 1\}}$

(f) $x = t^2$, $y = 2\ln t$, $t \geq 1$
If you have a constant multiple of a log, that constant becomes the exponent of the term inside the log.

$$y = 2\ln t = \ln\left(t^2\right) = \mathbf{\ln x}$$

That's the standard natural log graph.
You can write the domain of the natural log function in at least three ways:

- $\mathbf{-1 \leq x < \infty}$
- $\mathbf{x \in [-1, \infty)}$
- $\mathbf{\{x| \ -1 \leq x < \infty\}}$

(g) $x = 2\sin(8t)$, $y = 2\cos(8t)$
Use the Pythagorean Identity for sine and cosine.

$$\sin^2(8t) + \cos^2(8t) = 1$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$
$$\mathbf{\frac{x^2}{4} + \frac{y^2}{4} = 1}$$

This is the graph of a circle centered at the origin with radius 2. So it spans from $x = -2$ to $x = 2$.
You can write the domain in at least three ways:

- $\mathbf{-2 \leq x \leq 2}$
- $\mathbf{x \in [-2, 2]}$
- $\mathbf{\{x| \ -2 \leq x \leq 2\}}$

2 7.2 - Calculus of Parametric Curves

1. Each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

   (a) $x = 8 + 2t$, $y = 1$

   $$\frac{dy}{dx} = \frac{y'}{x'} = \frac{0}{2} = \mathbf{0}$$

   (b) $x = -5 + 7t$, $y = 3t = -1$

   $$\frac{dy}{dx} = \frac{y'}{x'} = \frac{3}{-5} = -\frac{\mathbf{3}}{\mathbf{5}}$$

2. Find the equation of the tangent line at the given value of the parameter.

   (a) $x = \cos t$, $y = 8\sin t$, $t = \frac{\pi}{2}$
   Find $\frac{dy}{dx}$.

   $$\frac{dy}{dx} = \frac{y'}{x'} = \frac{8\cos t}{-\sin t} = -8\cot t$$
   $$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -8\cot\frac{\pi}{2} = 0$$

   Find the point of the tangent line.

   $$x\left(\frac{\pi}{2}\right) = 0$$
   $$y\left(\frac{\pi}{2}\right) = 8$$

   Then plug into point-slope form.

   $$y - 8 = 0\,(x - 0)$$
   $$\mathbf{y = 8}$$

   (b) $x = t + \frac{1}{t}$, y=$t - \frac{1}{t}$, $t = 1$
   Find $\frac{dy}{dx}$.

   $$\frac{dy}{dx} = \frac{y'}{x'} = \frac{1 + \frac{1}{t^2}}{1 - \frac{1}{t^2}}$$
   $$\left.\frac{dy}{dx}\right|_{t=1} = \frac{1+1}{1-1} = \frac{2}{0} = \text{Undefined}$$

   Since the slope is a vertical line, it only has an x-coordiante. Just plug in $t = 1$ for $x\,(t)$ only.

   $$x\,(1) = 1 + \frac{1}{1} = 2$$

   So the final equation is $\mathbf{x = 2}$.

(c) $x = e^{\sqrt{t}}$, $y = 1 - \ln\left(t^2\right)$, at $t = 1$

Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{0 - \frac{2t}{t^2}}{e^{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}}}$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{\frac{-2}{1}}{e \cdot \frac{1}{2}} = -\frac{2}{\frac{e}{2}} = -\frac{4}{e}$$

Find the point of the tangent line.

$$x(1) = e^1 = e$$
$$y(1) = 1 - \ln 1 = 1 - 0 = 1$$

Then plug into point-slope form.

$$y - 1 = -\frac{4}{e}(x - e) = -\frac{4}{e}x + 4$$
$$\mathbf{y = -\frac{4}{e}x + 5}$$

3. For $x = \sin(2t)$, $y = 2\sin t$ where $0 \le t < 2\pi$, find all values of t where the equation has a vertical tangent line.

A vertical tangent line occurs when the slope is undefined (ie. when the denominator is 0). This means we have to set $x'(t) = 0$.

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2\cos t}{2\cos(2t)} = \frac{\cos t}{\cos 2t}$$
$$\cos 2t = 0$$

You may be tempted to say $2\pi = \frac{\pi}{2}$ and $2\pi = \frac{3\pi}{2}$, then divide by 2. But turning $\cos t$ into $\cos(2t)$, we are doubling the domain.

Another way to find that is to use an inequality:

$$\cos t \rightarrow \; 0 \le t \le 2\pi$$
$$\rightarrow \; 0 \le 2t \le 4\pi$$

So now we actually have an additional 360 degrees to find solutions for.

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$
$$\boldsymbol{t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$$

4. Find the second derivative of $x = \sqrt{t}$, $y = 2t + 4$, at $t = 1$.
Start by finding the first derivative.

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{2}{\frac{1}{2\sqrt{t}}} = 4\sqrt{t}$$

Now plug into the second derivative formula.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left[4\sqrt{t}\right]}{\frac{1}{2\sqrt{t}}} \\ &= \frac{\frac{4}{2\sqrt{t}}}{\frac{1}{2\sqrt{t}}} \\ &= \mathbf{4}\end{aligned}$$

5. Find the arc length of the curve on the indicated interval of the parameter.

(a) $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \leq t \leq 1$
Find x' and y''.

$$\begin{aligned}x' &= t^2 \\ y' &= t\end{aligned}$$

Plug into the formula and evaluate the integral.

$$\begin{aligned}S &= \int_0^1 \sqrt{\left(t^2\right)^2 + (t)^2}dt \\ &= \int_0^1 \sqrt{t^4 + t^2}dt \\ &= \int_0^1 \sqrt{t^2\left(t^2 + 1\right)}dt \\ &= \int_0^1 t\sqrt{t^2 + 1}dt\end{aligned}$$

I just factored a t^2 out of the term inside the square root. Since t^2 is a perfect square, it could be further simplified by taking it out of the square root, leaving just t on the outside. Now we can do u-substitution to integrate. Don't forget to change the bounds of the integral.

$$\begin{aligned}u &= t^2 + 1 \\ du &= 2t\ dt \\ t\ dt &= \frac{du}{2}\end{aligned}$$

$$\begin{aligned}\int_0^1 t\sqrt{t^2 + 1}dt &= \int_1^2 \sqrt{u}du \\ &= \frac{1}{2} \cdot \frac{2}{3}u^{\frac{3}{2}}\Big|_1^2 \\ &= \frac{1}{3}\left(2^{\frac{3}{2}} + 1\right) \\ &= \mathbf{\frac{1}{3}\left(2\sqrt{2} - 1\right)}\end{aligned}$$

(b) $x = 1 + t^2$, $y = (1+t)^3$, $0 \le t \le 1$ Use a calculator to integrate. Round to three decimal places.
Find x' and y''.

$$x' = 2t$$
$$y' = 3(1+t)^2$$

Plug into the formula and evaluate the integral using a calculator.

$$\begin{aligned} S &= \int_0^1 \sqrt{(2t)^2 + \left((1+t)^2\right)^2} dt \\ &\approx \mathbf{7.075} \end{aligned}$$

(c) $x = a\cos^3\theta$, $y = a\sin^3\theta$, $0 \le t < 2\pi$
Find x' and y''.

$$x' = 3a\cos^2\theta\sin\theta$$
$$y' = -3a\sin^2\theta\cos\theta$$

Plug into the formula and evaluate the integral.

$$\begin{aligned} S &= \int_0^{2\pi} \sqrt{\left(3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2} d\theta \\ &= \int_0^{2\pi} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta} d\theta \\ &= \int_0^{2\pi} 3a\sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta} d\theta \\ &= 3a\int_0^{2\pi} \sqrt{\left(\sin^2\theta\cos^2\theta\right)\left(\underline{\cos^2\theta + \sin^2\theta}\right)} d\theta \\ &= 3a\int_0^{2\pi} \sin\theta\cos\theta d\theta \end{aligned}$$

If you try to integrate with these bounds you will get an answer of 0 because there is an equal amount of negative area under the curve of the integrand than positive.
We are going to need u-substitution so if we set $u = \sin x$ we need some bounds such that the sine of the lower bound does not equal the sine of the upper bound so that the integral is not 0. I chose to evaluate the integral from 0 to $\frac{\pi}{2}$ four times.

$$\begin{aligned} S &= 12a\int_0^{\frac{\pi}{2}} \sqrt{\sin\theta\cos\theta} d\theta \\ u &= \sin\theta \\ du &= \cos\theta\ d\theta \\ S &= 12a\int_0^1 u\ du \\ &= 12a \cdot \frac{1}{2}\left[u^2\right]_0^1 \\ &= \frac{12a}{2} \cdot 1 \\ &= \mathbf{6a} \end{aligned}$$

3 7.3 - Polar Coordinates

1. Convert the rectangular coordinates to two different polar coordinates. Round to three decimal places.
   If the input to the inverse tangent function is negative, your result will be an angle in the fourth quadrant. If the input is potitive, your result will be an angle in the first quadrant.

   (a) $(3, -4)$

   $$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
   $$\arctan\left(-\frac{4}{3}\right) \approx -0.927 \text{ rad}$$

   This angle gives us a result in the fourth quadrant, which is what we want.
   We can add 2π to this angle to get a positive angle, and flip the radius and only add π.

   $$(\mathbf{5}, \mathbf{5.356})$$
   $$(-\mathbf{5}, \mathbf{2.214})$$

   (b) $(-6, 8)$

   $$r = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$
   $$\arctan\left(-\frac{8}{6}\right) \approx -0.927 \text{ rad}$$

   This angle gives us a result in the fourth quadrant, but we want a point in the second quadrant, so just add π. If you flip the radius you will add 2π.

   $$(\mathbf{10}, \mathbf{2.214})$$
   $$(-\mathbf{10}, \mathbf{5.356})$$

   (c) $\left(3, -\sqrt{3}\right)$

   $$r = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3}$$
   $$\arctan\left(-\frac{\sqrt{3}}{3}\right) - \frac{\pi}{6} \text{ rad}$$

   This angle gives us a result in the fourth quadrant, which is what we want.
   We can add 2π to this angle to get a positive angle, and flip the radius and only add π.

   $$\left(\mathbf{2\sqrt{3}}, \mathbf{5.759}\right)$$
   $$\left(-\mathbf{2\sqrt{3}}, \mathbf{2.618}\right)$$

2. Convert the polar coordinates to rectangular coordinates.

   (a) $\left(-2, \frac{\pi}{6}\right)$

   $$x = -2\cos\frac{\pi}{6} = -\sqrt{3}$$
   $$y = -2\sin\frac{\pi}{6} = -1$$
   $$\left(-\sqrt{\mathbf{3}}, -\mathbf{1}\right)$$

(b) $\left(1, \frac{7\pi}{6}\right)$

$$x = \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$
$$y = \sin\frac{7\pi}{6} = -\frac{1}{2}$$
$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

(c) $\left(0, \frac{\pi}{2}\right)$
If the radius is 0, the point is the origin no matter what the angle is.

$$(\mathbf{0}, \mathbf{0})$$

4 7.4 - Area and Arc Length in Polar Coordinates

1. Set up, but do not evaluate, an integral that represents the area of the following regions.

(a) Region enclosed by $r = 3\sin\theta$
First use a calculator to find the interval of t-values that produce this graph.
If you plug in $[0, 2\pi]$, you will get a circle. If you plug in $[0, \pi]$ you will still get a circle, but it may show up lighter on your calculator. If you plug in $\left[0, \frac{\pi}{2}\right]$ you will get half of a circle. This means that that the interval for one region of that circle is $[0, \pi]$.
Now set up the integral.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\pi} (3\sin\theta)^2\, d\theta \\ &= \boldsymbol{\frac{9}{2}\int_0^{\pi} \sin^2\theta d\theta} \end{aligned}$$

(b) Region enclosed by one petal of $r = 8\sin(2\theta)$
First use a calculator to find the interval of t-values that produce one petal of the graph.
Plugging in $[0, \pi]$ gives us two petals, and $\left[0, \frac{\pi}{2}\right]$ gives us one, so we will use $\left[0, \frac{\pi}{2}\right]$.
Now set up the integral.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^2\, d\theta \\ &= \boldsymbol{32\int_0^{\frac{\pi}{2}} \sin^2(2\theta) d\theta} \end{aligned}$$

(c) Region below the polar axis and enclosed by $r = 1 - \sin\theta$
First use a calculator to find the interval of t-values that produce the bottom of the graph.
If you plug in $[0, 2\pi]$ you get the whole graph. If you plug in $[0, \pi]$ you only get the top. Try plugging in $[\pi, 2\pi]$. Then you will get the bottom.
Now set up the integral.

$$A = \boldsymbol{\frac{1}{2}\int_{\pi}^{2\pi} (1 - \sin\theta)^2\, d\theta}$$

2. Find the area of the described regions.

(a) Enclosed by $r = 6\sin\theta$

First use a calculator to find the interval of t-values that produce this graph.

If you plug in $[0, 2\pi]$, you will get a circle. If you plug in $[0, \pi]$ you will still get a circle, but it may show up lighter on your calculator. If you plug in $\left[0, \frac{\pi}{2}\right]$ you will get half of a circle. This means that that the interval for one region of that circle is $[0, \pi]$.

Now set up the integral.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} (6\sin\theta)^2\, d\theta \\
&= 18\int_0^{\pi} \sin^2\theta d\theta \\
&= 18\left[-\frac{\cos\theta\sin\theta}{2}\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} 1\; d\theta\right] \\
&= 18\left(0 + \frac{1}{2}(\pi - 0)\right) \\
&= \frac{18\pi}{2} \\
&= \mathbf{9\pi\ units^2}
\end{aligned}$$

(b) Below the polar axis and enclosed by $r = 2 - \cos\theta$

If we look at the graph from 0 to 2π we will see a whole circl-like shape. We just want the part below the axis so go from 0 to π. It will look like the top part but the graph is symmetric so the numerical value is the same.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} (2 - \cos\theta)^2\, d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(4 - 4\cos\theta + \cos^2\theta\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi} 4d\theta - \frac{1}{2}\int_0^{\pi} \cos\theta\; d\theta + \frac{1}{2}\int_0^{\pi} \cos^2\theta\; d\theta \\
&= 2\pi - \frac{1}{2}\sin\theta\Big|_0^{\pi} + \frac{1}{2}\left[\frac{\sin\theta\cos\theta}{2}\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} 1\; d\theta\right] \\
&= 2\pi + \frac{1}{2}\left[0 + \frac{\pi}{2}\right] \\
&= 2\pi + \frac{\pi}{4} \\
&= \mathbf{\frac{9\pi}{4}\ units^2}
\end{aligned}$$

(c) Enclosed by one petal of $r = 3\cos(2\theta)$

Graph the function from $0 \leq \theta \leq 2\pi$, and you will see all four petals on the graph. We can take the enitre area and divide it by 4.

$$\begin{aligned} A &= \frac{1}{4} \cdot \frac{1}{2} \int_0^{2\pi} (3\cos(2\theta))^2 \, d\theta \\ &= \frac{1}{8} \int_0^{2\pi} 9\cos^2(2\theta) d\theta \\ &= \frac{9}{8} \int_0^{2\pi} \cos^2(2\theta) d\theta \end{aligned}$$

We can use the doubble or half-angle identities to find an expression for $\cos^2(2\theta)$.

$$\begin{gathered} \cos(2x) = 2\cos^2 x = 1 \\ \cos^2 x = \frac{\cos(2x) + 1}{2} \\ \text{Let } x = 2\theta \\ \Rightarrow \cos^2(2\theta) = \frac{\cos(4\theta) + 1}{2} \end{gathered}$$

Plug that back in the integral.

$$\begin{aligned} \frac{9}{8} \int_0^{2\pi} \cos^2(2\theta) d\theta &= \frac{9}{8} \int_0^{2\pi} \frac{1}{2} (\cos(4\theta) + 1) \, \theta \\ &= \frac{9}{16} \int_0^{2\pi} (\cos(4\theta) + 1) \, d\theta \\ &= \frac{9}{16} \left[\frac{1}{4} \sin(4\theta) \Big|_0^{2\pi} + (2\pi - 0) \right] \\ &= \frac{9}{16} [0 + 2\pi] \\ &= \boldsymbol{\frac{9\pi}{8}} \textbf{ units}^{\mathbf{2}} \end{aligned}$$

(d) Enclosed by the inner loop of $r = 3 + 6\cos\theta$

You may have a hard time finding the interval of θ on your calulator. You can always figure it out algebraicly.

To get just the inner loop we need the two points where $r = 0$ because thats where the loop starts and ends.

$$\begin{aligned}
3 + 6\cos\theta &= 0 \\
6\cos\theta &= -3 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3}, \frac{4\pi}{3}
\end{aligned}$$

Use $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ for the bounds on the integral.

The hard part of this problem is accounting for your numbers because you have to use a reduction formula and it will get messy.

$$\begin{aligned}
A &= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 6\cos\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} \left(9 + 36\cos\theta + 36\cos^2\theta\right) d\theta \\
&= \frac{9}{2}\left(\frac{4\pi}{3} - \frac{2\pi}{3}\right) + \frac{36}{2}\sin\theta\Big|_{2\pi/3}^{4\pi/3} + \frac{36}{2}\left[\frac{\sin\theta\cos\theta}{2} + \frac{1}{2}\int_{2\pi/3}^{4\pi/3} 1\, dx\right] \\
&= \frac{9}{2}\left(\frac{2\pi}{3}\right) + 18\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + 18\left[\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\cdot -\frac{1}{2} - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) + \frac{1}{2}\left(\frac{4\pi}{3} - \frac{2\pi}{3}\right)\right] \\
&= 3\pi - 18\sqrt{3} + 9\left(\frac{\sqrt{3}}{2}\right) + 9\left(\frac{2\pi}{3}\right) \\
&= 9\pi - 18\sqrt{3} + \frac{9}{2}\sqrt{3} \\
&= \boldsymbol{9\pi - \frac{27\sqrt{3}}{2}}\ \textbf{units}^{\mathbf{2}}
\end{aligned}$$

3. Set up, but do not evaluate, an integral that represents the arc length of the given polar curves.

(a) $r = 1 + \sin\theta$ over $0 \le \theta \le 2\pi$

All we need to calculate is $\frac{dr}{d\theta}$.

$$\frac{dr}{d\theta} = \cos\theta$$

Now plug into arc lengh formula.

$$S = \boldsymbol{\int_0^{2\pi} \sqrt{(1 + \sin\theta)^2 + \cos^2\theta d\theta}}$$

(b) $r = e^{\theta}$ over $0 \leq \theta \leq 1$

All we need to calculate is $\frac{dr}{d\theta}$.

$$\frac{dr}{d\theta} = e^{\theta}$$

Now plug into arc lengh formula and simplify the integral.

$$\begin{aligned} S &= \int_0^1 \sqrt{(e^{\theta})^2 + (e^{\theta})^2} d\theta \\ &= \int_0^1 \sqrt{e^{2\theta} + e^{2\theta}} d\theta \\ &= \int_0^1 \sqrt{2e^{2\theta}} d\theta \\ &= \boldsymbol{\sqrt{2} \int_0^1 e^{\theta} d\theta} \end{aligned}$$

4. Find the arc length of the polar curves over the given intervals.

(a) $r = e^{3\theta}$ over $0 \leq \theta \leq 2$

Find $\frac{dr}{d\theta}$ and plug it into the arc length formula.

$$\frac{dr}{d\theta} = 3e^{3\theta}$$

$$\begin{aligned} S &= \int_0^2 \sqrt{(e^{3\theta})^2 + (3e^{3\theta})^2} d\theta \\ &= \int_0^2 \sqrt{e^{6\theta} + 9e^{6\theta}} d\theta \\ &= \int_0^2 \sqrt{10e^{6\theta}} d\theta \\ &= \int_0^2 \left(10e^{6\theta}\right)^{\frac{1}{2}} d\theta \\ &= \sqrt{10} \int_0^2 e^{3\theta} d\theta \\ &= \sqrt{10} \cdot \frac{1}{3} e^{3\theta} \Big|_0^2 \\ &= \boldsymbol{\frac{\sqrt{10}}{3} \left(e^6 - 1\right)} \end{aligned}$$

(b) $r = 8 + 8\cos\theta$ over $0 \leq \theta \leq \pi$ (Hint: Use half-angle formulas)
Find $\frac{dr}{d\theta}$ and plug it into the arc length formula.

$$\frac{dr}{d\theta} = -8\sin\theta$$

$$\begin{aligned}
S &= \int_0^{\pi} \sqrt{(8+8\cos\theta)^2 + (-8\sin\theta)^2} d\theta \\
&= \int_0^{\pi} \sqrt{64 + 128\cos\theta + 64\cos^2\theta + 64\sin^2\theta} d\theta \\
&= \int_0^{\pi} \sqrt{128 + 128\cos\theta} d\theta \\
&= \int_0^{\pi} \sqrt{64\,(2 + 2\cos\theta)} d\theta \\
&= 8\int_0^{\pi} \sqrt{2 + 2\cos\theta} d\theta \\
&= 8\int_0^{\pi} \sqrt{2\,(1 + \cos\theta)} d\theta
\end{aligned}$$

Use the half-angle formulas to find an expression for $1 + \cos\theta$.

$$\begin{aligned}
&= \int_0^{\pi} \sqrt{2\left(2\cos^2\left(\frac{x}{2}\right)\right)} d\theta \\
&= 8\int_0^{\pi} \sqrt{4\cos^2\left(\frac{x}{2}\right)} d\theta \\
&= 16\int_0^{\pi} \cos\left(\frac{x}{2}\right) d\theta \\
&= 16\sin\left(\frac{x}{2}\right)\cdot 2\Big|_0^{\pi} \\
&= 32\sin\left(\frac{x}{2}\right)\Big|_0^{\pi} \\
&= 32\,(1-0) \\
&= \mathbf{32\ units}
\end{aligned}$$

5. Find the slope of a tangent line to the polar curves.

(a) $r = 4\cos\theta,\ \left(2, \frac{\pi}{3}\right)$
Get $x'(\theta)$ and $y'(\theta)$.

$$x(\theta) = 4\cos\theta\cos\theta = 4\cos^2\theta$$
$$y(\theta) = 4\cos\theta\sin\theta$$
$$x'(\theta) = -8\cos\theta\sin\theta$$

$$\begin{aligned} y'(\theta) &= 4\left(\cos^2\theta - \sin^2\theta\right) \\ &= 4\cos^2\theta - 4\sin^2\theta \end{aligned}$$

$$\frac{dy}{dx} = \frac{4\cos^2\theta - 4\sin^2\theta}{-8\cos\theta\sin\theta}$$

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{\frac{\pi}{3}} &= -\frac{1}{2}\left(\frac{\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2}{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}\right) \\ &= -\frac{1}{2}\left(\frac{\frac{1}{4} - \frac{3}{4}}{\frac{\sqrt{3}}{4}}\right) \\ &= -\frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{4}}\right) \\ &= \boldsymbol{\frac{\sqrt{3}}{3}} \end{aligned}$$

(b) $r = 4 + \sin\theta,\ \left(3, \frac{3\pi}{2}\right)$
Get $x'(\theta)$ and $y'(\theta)$.

$$x(\theta) = (4 + \sin\theta)\cos\theta = 4\cos\theta + \sin\theta\cos\theta$$
$$y(\theta) = (4 + \sin\theta)\sin\theta = 4\sin\theta + \sin^2\theta$$

$$\begin{aligned} x'(\theta) &= -4\sin\theta + \left(\sin\theta\cdot - \sin\theta + \cos^2\theta\right) \\ &= -4\sin\theta - \sin^2\theta + \cos^2\theta \\ y'(\theta) &= 4\cos\theta + 2\sin\theta\cos\theta \end{aligned}$$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{4\cos\theta + 2\sin\theta\cos\theta}{-4\sin\theta + (\sin\theta\cdot - \sin\theta + \cos^2\theta)}$$
$$\left.\frac{dy}{dx}\right|_{\frac{3\pi}{2}} = \frac{4(0) + 2(-1)(0)}{4} = \mathbf{0}$$

Chapter 2
Revised 6/25/2025

MTH 201

Blank Worksheet

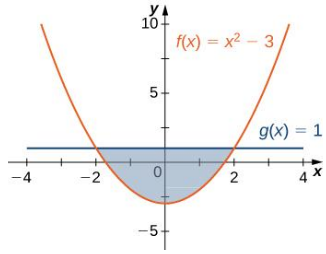
Problems found in OpenStax Calculus Volume II

Sections:

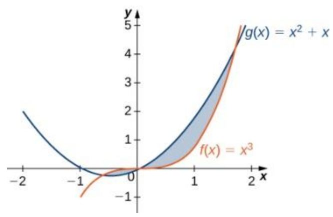
- 2.1 - Bounded Area
- 2.2 - Volume by Slicing
- 2.3 - Shells
- 2.4 - Arc Length and Surface Area
- 2.5 - Physical Applications
- 2.9 - Hyperbolic Trig Functions

Chapter 2
Revised 6/25/2025

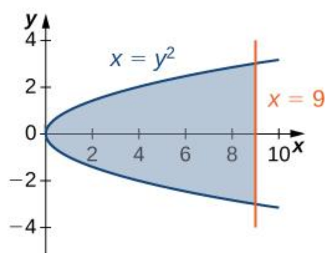
1. **1.** (2.1) Determine the area of the shaded region shown.
   a.



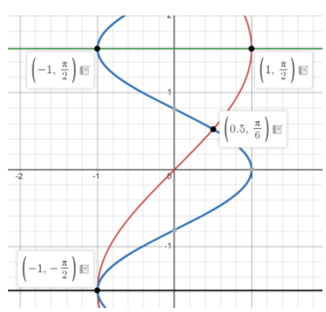
   b.



   c.



2. **2.** (2.1) Graph the equations and determine the area between the two curves.
   a. $y = x^2$ and $y = -x^2 + 18x$
   b. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$
   c. $y = |x|$ and $y = x^2$
   d. $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$
   e. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1,1]$
   f. $y = x^3 + 3x$ and $y = 4x$
   g. $x = 2y$ and $x = y^3 - y$
   h. $y^2 = x$ and $x = y + 2$
   i. $x = \sin y$, $x = \cos(2y)$, $y = \frac{\pi}{2}$, and $y = -\frac{\pi}{2}$



3. (2.1) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

4. (2.2) Graph the region bounded by the two curves and find the volume when the region is revolved about the x-axis.
a. $y = 2x^2$, $x = 0$, $x = 4$, and $y = 0$
b. $y = x^4$, $x = 0$, and $y = 1$ for $x \geq 0$
c. $y = \sin x$, $y = \cos x$, and $x = 0$
d. $x^2 - y^2 = 9$, $x + y = 9$, $y = 0$, and $x = 0$
e. $y = x^2$ and $y = x + 2$
f. $y = 4 - x^2$ and $y = 2 - x$
g. $y = \sqrt{x}$ and $y = x^2$
h. $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

5. (2.2) Graph the region bounded by the two curves and find the volume when the region is revolved about the y-axis.
a. $y = 2x^3$, $x = 0$, $x = 1$, and $y = 0$
b. $y = \sqrt{4 - x^2}$, $y = 0$, and $x = 0$
c. $x = \sec y$, $y = \frac{\pi}{4}$, $y = 0$, and $x = 0$
d. $y = 4 - x$, $y = x$, and $x = 0$
e. $y = x + 2$, $y = 2x - 1$, and $x = 0$
f. $x = e^{2y}$, $x = y^2$, $y = 0$, and $y = \ln 2$

6. (2.2) Find the volume of a generic ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolved around the x-axis.

7. (2.3) Find the volume generated when the region between the two curves is rotated about the given axis using the shell method.
a. Bounded by $y = 3x$, $y = 0$, and $x = 3$ rotated about the y-axis.
b. Bounded by $y = 3x$, $y = 0$, and $x = 3$ rotated about the x-axis.
c. Bounded by $y = 2x^3$, $y = 0$, and $x = 2$ rotated about the x-axis.

8. (2.3) Find the volume when rotating the region between the given curve and the x-axis around the y-axis using the shell method.
a. $y = 5x^3$, $x = 0$, and $x = 1$
b. $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$
c. $y = \sin(x^2)$, $x = 0$, and $x = \sqrt{\pi}$
d. $y = \sqrt{x}$, $x = 0$, and $x = 1$
e. $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

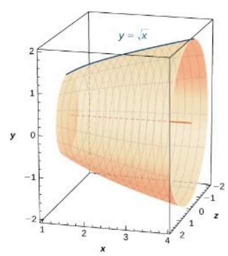
9. (2.3) Find the volume when rotating the region between the given curve and $y = 0$ around the x-axis using the shell method.
a. $y = x^2$, $x = 0$, $x = 2$, and the x-axis

b. $y = \frac{2}{x^2}$, $x = 1$, $x = 2$, and the x-axis
c. $x = \frac{1+y^2}{y}$, $y = 1$, $y = 4$, and the y-axis
d. $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$

10. (2.3) Find the volume when the region between the curves is rotated around the given axis using the shell method.
a. $y = x^3$, $x = 0$, and $y = 8$ rotated around the y-axis
b. $y = \sqrt{x}$, $y = 0$, and $x = 1$ rotated around $x = 2$
c. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis
d. Left of $x = \sin(\pi y)$, right of $y = x$ rotated around the y-axis (use a calculator)

11. (2.4) Find the length of the functions over the given interval.
a. $f(x) = 5x$ from $x = 0$ to $x = 2$.
b. $x = 4y$ from $y = -1$ to $y = 1$.

12. (2.4) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolved around the x-axis from $(1,1)$ to $(4,2)$ as shown.



13. (2.4) Find the lengths of the functions over the given interval. Use a calculator if the integral cannot be evaluated directly.
a. $y = x^{\frac{3}{2}}$ from $(0,0)$ to $(1,1)$
b. $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$ from $x = 0$ to $x = 1$
c. $y = e^x$ on $x = 0$ to $x = 1$
d. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$
e. $y = \frac{1}{27}(9x^2 + 6)^{\frac{3}{2}}$ from $x = 0$ to $x = 2$
f. $y = \frac{5-3x}{4}$ from $y = 0$ to $y = 4$
g. $x = 5y^{\frac{3}{2}}$ from $y = 0$ to $y = 1$
h. $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$
i. $x = 4^y$ from $y = 0$ to $y = 2$

14. (2.4) Find the surface area of the volume generated when the given curves revolve about the x-axis.
a. $y = \sqrt{x}$ from $x = 2$ to $x = 6$
b. $y = 7x$ from $x = -1$ to $x = 1$
c. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

d. $y = 5x$ from $x = 1$ to $x = 5$

15. (2.4) Find the surface area of the volume generated when the given curves revolve about the y-axis. Use a calculator if the integral cannot be evaluated directly.
a. $y = x^2$ from $x = 0$ to $x = 2$
b. $y = x + 1$ from $x = 0$ to $x = 3$
c. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$
d. $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

16. (2.5) How much work is done when a person lifts a 50 lb object 3 ft off the ground?

17. (2.5) How much work is done when you push a box along the floor 2 m when applying a constant force of $F = 100\ N$?

18. (2.5) How much work is done when moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

19. (2.5) Find the mass of a 3 ft long car antenna (starting at $x = 0$) that had a density function of $\rho(x) = 3x + 2$ lb/ft.

20. (2.5) Find the mass of a 4 inch long pencil (starting at $x = 2$) that has a density function of $\rho(x) = \frac{5}{x}$ oz/in.

21. (2.5) Find the mass of the two-dimensional object that is centered at the origin.
a. An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$
b. A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

22. (2.5) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

23. (2.5) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length L, of the spring?

24. (2.5) A rectangular dam is 40 ft high and 60 ft wide. Assume the weight density of water is $62.5\frac{lbs}{ft^3}$. Using a calculator, compute the total force F on the dam when
a. The surface of the water is at the top of the dam and
b. The surface of the water is halfway down the dam.

25. (2.5) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Computer the work to pump all the water to the top.

26. (2.9) Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$.

27. (2.9) Show that $\cosh x$ and $\sinh x$ satisfy $y'' = y$.

28. (2.9) Derive $\cosh^2 x + \sinh^2 x = \cosh(2x)$ from the definition.

29. (2.9) Find the derivatives of the following functions.

a. $\cosh(3x + 1)$
b. $\frac{1}{\cosh x}$
c. $\cosh^2 x + \sinh^2 x$
d. $\tanh\left(\sqrt{x^2 + 1}\right)$
e. $\sinh^6 x$

30. (2.9) Find the antiderivatives for the following functions.

a. $\cosh(2x + 1)$
b. $x \cosh(x^2)$
c. $\cosh^2(x) \sinh(x)$
d. $\frac{\sinh x}{1+\cosh x}$
e. $\cosh x + \sinh x$

31. (2.9) Find the derivatives of the following functions.

a. $\operatorname{arctanh}(4x)$
b. $\operatorname{arcsinh}(\cosh x)$
c. $\operatorname{arctanh}(\cos x)$
d. $\ln(\operatorname{arctanh} x)$

MTH201 Chapter 3

Harper College*

June 25, 2025

Contents

*Problems found in Openstax Calculus Volume II
`https://openstax.org/details/books/calculus-volume-2`

1 3.1 - Integration by Parts

1. (3.1) Integrate using the simplest method. Not all problems require integration by parts.
   (a) $\int \ln x \; dx$
   (b) $\int \arctan(x) \; dx$
   (c) $\int x \sin(2x) dx$
   (d) $\int x e^{-x} dx$
   (e) $\int x^2 \cos x \; dx$
   (f) $\int \ln(2x+1) dx$
   (g) $\int e^x \sin x \; dx$
   (h) $\int x e^{-x^2} dx$
   (i) $\int \sin(\ln(2x)) dx$
   (j) $\int (\ln x)^2 dx$
   (k) $\int x^2 \ln x \; dx$
   (l) $\int \arccos(2x) dx$
   (m) $\int x^2 \sin x \; dx$
   (n) $\int x^3 \sin x \; dx$
   (o) $\int x \cosh x \; dx$
2. (3.1) Compute the definite integrals.
   (a) $\int_0^1 x e^{-2x} dx$
   (b) $\int_1^e \ln(x^2) dx$
   (c) $\int_{-\pi}^{\pi} x \sin x \; dx$
   (d) $\int_0^{\pi/2} x^2 \sin x \; dx$
3. (3.1) Evaluate $\int \cos x \ln(\sin x) dx$
4. (3.1) Find the volume generated by rotating the region bounded by the given curves about the line $x = 1$. Express the answer in exact form.
$$y = e^{-x}, \; y = 0, \; x = -1, \; x = 0$$
5. (3.1) Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis for $\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}$. Express the answer in exact form.

2 3.2 - Trig Integration

1. (3.2) Evaluate the integrals using u-substitution
   (a) $\int \sin^3 x \cos x \ dx$
   (b) $\int \tan^5(2x)\sec^2(2x)dx$
   (c) $\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right) dx$
2. (3.2) Evaluate the integrals using the guidelines for integrating powers of trig functions.
   (a) $\int \sin^3 x \ dx$
   (b) $\int \sin x \cos x \ dx$
   (c) $\int \sin^5 x \cos^2 x \ dx$
   (d) $\int \sqrt{\sin x}\cos x \ dx$
   (e) $\int \sec x \tan x \ dx$
   (f) $\int \tan^2 x \sec x \ dx$
   (g) $\int \sec^4 x \ dx$
3. (3.2) Find a general formula for the integral:
$$\int \sin^2(ax)\cos(ax)dx$$
4. (3.2) Evaluate the integrals using double-angle formulas.
   (a) $\int_0^{\pi} \sin^2 x \ dx$
   (b) $\int \cos^2(3x) \ dx$
   (c) $\int \sin^2 x \ dx + \int \cos^2 x \ dx$
5. (3.2) Evaluate the definite integrals
   (a) $\int_0^{2\pi} \cos x \sin(2x)dx$
   (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}}dx$ (Round to three decimal places)
   (c) $\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(2x)}dx$
6. (3.2) Find the area of the region bounded by the graphs of the functions:
$$y = \cos^2 x, \ y = \sin^2 x, \ x = -\frac{\pi}{4}, \ x = \frac{\pi}{4}$$
7. Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$

3 3.3 - Trigonometric Substitution

1. (3.3) Integrate using the method of trigonometric substitution.

   (a) $\int \frac{dx}{\sqrt{4-x^2}}$

   (b) $\int \frac{dx}{\sqrt{1+9x^2}}$

   (c) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

   (d) $\int \sqrt{x^2+9}\ dx$

   (e) $\int \frac{x^3 dx}{\sqrt{9-x^2}}$

   (f) $\int \frac{dx}{(x^2-9)^{\frac{3}{2}}}$

   (g) $\int \frac{x^2 dx}{\sqrt{x^2-1}}$

   (h) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

   (i) $\int \left(1-x^2\right)^{\frac{3}{2}}\ dx$

2. (3.3) Use the technique of completing the square to evaluate the following integrals.

   (a) $\int \frac{1}{x^2+2x+1} dx$

   (b) $\int \frac{1}{\sqrt{-x^2+10x}} dx$

3. (3.3) Find the volume of the solid formed when the region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the x-axis between $x = 0$ and $x = 1$ is revolved about the x-axis.

4. (3.3) Find the length of the curve $y = \sqrt{16-x^2}$ between $x = 0$ and $x = 2$.

4 3.4 - Partial Fraction Decomposition

1. (3.4) Decompose the fractions as a sum or difference of simpler rational expressions.

   (a) $\frac{x^2+1}{x(x+1)(x+2)}$

   (b) $\frac{3x+1}{x^2}$

   (c) $\frac{2x^4}{x^2-2x}$

   (d) $\frac{1}{x^2(x-1)}$

   (e) $\frac{1}{x(x-1)(x-2)(x-3)}$

   (f) $\frac{3x^2}{(x-1)(x^2+x+1)}$

2. (3.4) Evaluate the integrals using partial fraction decomposition.

   (a) $\int \frac{3x}{x^2+2x-8}dx$

   (b) $\int \frac{x}{x^2-4}dx$

   (c) $\int \frac{2x^2+4x+22}{x^2+2x+10}dx$

   (d) $\int \frac{2-x}{x^2+x}dx$

   (e) $\int \frac{dx}{x^3-2x^2-4x+8}$

3. (3.4) Evaluate the integrals with irreducible quadratic factors.

   (a) $\int \frac{2}{(x-4)(x^2+2x+6)}dx$ Hint: $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

   (b) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$

4. (3.4) Evaluate the integral $\int \frac{3x+4}{(x^2+4)(3-x)}dx$ using partial fraction decomposition.

5. (3.4) Use substitution to convert the integrals to integrals of rational functions. Then evaluate using partial fraction decomposition.

   (a) $\int \frac{e^x}{e^{2x}-e^x}dx$

   (b) $\int \frac{\cos x}{\sin x(1-\sin x)}dx$

5 3.5 - Integration Tables

The tables are attached to the end of this document.

1. (3.5) Evaluate the integrals using the table of integrals.
   (a) $\int \frac{dy}{\sqrt{4-y^2}}$
   (b) $\int_0^{\frac{\pi}{2}} \tan^2\left(\frac{x}{2}\right)dx$
   (c) $\int \tan^5(3x)dx$
2. (3.5) Evaluate the integrals using the table of integrals. You may need to complete the square or make substitutions.
   (a) $\int \frac{dx}{x^2+2x+10}$
   (b) $\int \frac{e^x}{\sqrt{e^{2x}-4}}dx$

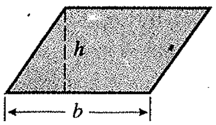
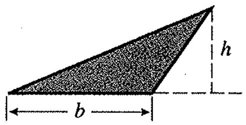
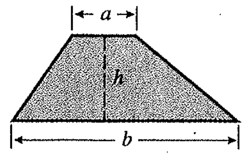
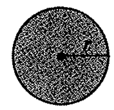
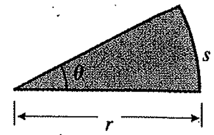
6 3.7 - Improper Integrals

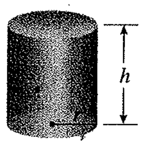
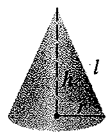
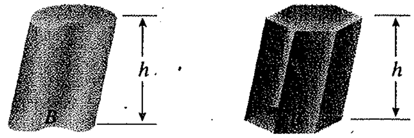
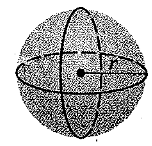
Evaluate the following integrals, or indicate that it's divergent.

1. $\int_2^4 \frac{dx}{(x-3)^2}$
2. $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$
3. $\int_1^\infty xe^{-x} dx$
4. $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$
5. $\int_{-\infty}^\infty \frac{1}{x^2+1} dx$
6. $\int_{-2}^2 \frac{1}{(1+x)^2} dx$
7. $\int_0^\infty \sin x \; dx$
8. $\int_0^1 \frac{dx}{\sqrt[3]{x}} dx$
9. $\int_{-1}^2 \frac{dx}{x^3}$
10. $\int_0^3 \frac{1}{x-1} dx$
11. $\int_0^1 \frac{dx}{x^\pi}$
12. $\int_0^e \ln(x) dx$
13. $\int_0^9 \frac{dx}{\sqrt{9-x}}$
14. $\int_0^4 x \ln(4x) dx$
15. Find the area of the region bounded by the curve $y = \frac{7}{x^2}$, the x-axis, and on the left by x=1.

GEOMETRY FORMULAS

A = area, S = lateral surface area, V = volume, h = height, B = area of base, r = radius, l = slant height, C = circumference, s = arc length

Parallelogram	Triangle	Trapezoid	Circle	Sector
				
$A = bh$	$A = \frac{1}{2}bh$	$A = \frac{1}{2}(a+b)h$	$A = \pi r^2$, $C = 2\pi r$	$A = \frac{1}{2}r^2\theta$, $s = r\theta$ (θ in radians)

Right Circular Cylinder	Right Circular Cone	Any Cylinder or Prism with Parallel Bases	Sphere
			
$V = \pi r^2 h$, $S = 2\pi rh$	$V = \frac{1}{3}\pi r^2 h$, $S = \pi rl$	$V = Bh$	$V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$

ALGEBRA FORMULAS

THE QUADRATIC FORMULA	THE BINOMIAL FORMULA
The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$(x+y)^n = x^n + nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 + \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$ $(x-y)^n = x^n - nx^{n-1}y + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}y^2 - \dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots \pm nxy^{n-1} \mp y^n$

TABLE OF INTEGRALS

BASIC FUNCTIONS

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$
2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int \sin u\,du = -\cos u + C$
5. $\displaystyle\int \cos u\,du = \sin u + C$
6. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
7. $\displaystyle\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$
8. $\displaystyle\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$
9. $\displaystyle\int \tan^{-1} u\,du = u\tan^{-1} u - \ln\sqrt{1+u^2} + C$
10. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
11. $\displaystyle\int \ln u\,du = u\ln u - u + C$
12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
13. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C = \ln\left|\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}u\right)\right| + C$
14. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C = \ln\left|\tan \tfrac{1}{2}u\right| + C$
15. $\displaystyle\int \cot^{-1} u\,du = u\cot^{-1} u + \ln\sqrt{1+u^2} + C$
16. $\displaystyle\int \sec^{-1} u\,du = u\sec^{-1} u - \ln\left|u + \sqrt{u^2-1}\right| + C$
17. $\displaystyle\int \csc^{-1} u\,du = u\csc^{-1} u + \ln\left|u + \sqrt{u^2-1}\right| + C$

RECIPROCALS OF BASIC FUNCTIONS

18. $\displaystyle\int \frac{1}{1 \pm \sin u}\, du = \tan u \mp \sec u + C$

19. $\displaystyle\int \frac{1}{1 \pm \cos u}\, du = -\cot u \pm \csc u + C$

20. $\displaystyle\int \frac{1}{1 \pm \tan u}\, du = \tfrac{1}{2}(u \pm \ln|\cos u \pm \sin u|) + C$

21. $\displaystyle\int \frac{1}{\sin u \cos u}\, du = \ln|\tan u| + C$

22. $\displaystyle\int \frac{1}{1 \pm \cot u}\, du = \tfrac{1}{2}(u \mp \ln|\sin u \pm \cos u|) + C$

23. $\displaystyle\int \frac{1}{1 \pm \sec u}\, du = u + \cot u \mp \csc u + C$

24. $\displaystyle\int \frac{1}{1 \pm \csc u}\, du = u - \tan u \pm \sec u + C$

25. $\displaystyle\int \frac{1}{1 \pm e^u}\, du = u - \ln(1 \pm e^u) + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

26. $\displaystyle\int \sin^2 u\, du = \tfrac{1}{2}u - \tfrac{1}{4}\sin 2u + C$

27. $\displaystyle\int \cos^2 u\, du = \tfrac{1}{2}u + \tfrac{1}{4}\sin 2u + C$

28. $\displaystyle\int \tan^2 u\, du = \tan u - u + C$

29. $\displaystyle\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u \cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$

30. $\displaystyle\int \cos^n u\, du = \frac{1}{n}\cos^{n-1} u \sin u + \frac{n-1}{n}\int \cos^{n-2} u\, du$

31. $\displaystyle\int \tan^n u\, du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\, du$

32. $\displaystyle\int \cot^2 u\, du = -\cot u - u + C$

33. $\displaystyle\int \sec^2 u\, du = \tan u + C$

34. $\displaystyle\int \csc^2 u\, du = -\cot u + C$

35. $\displaystyle\int \cot^n u\, du = -\frac{1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\, du$

36. $\displaystyle\int \sec^n u\, du = \frac{1}{n-1}\sec^{n-2} u \tan u + \frac{n-2}{n-1}\int \sec^{n-2} u\, du$

37. $\displaystyle\int \csc^n u\, du = -\frac{1}{n-1}\csc^{n-2} u \cot u + \frac{n-2}{n-1}\int \csc^{n-2} u\, du$

PRODUCTS OF TRIGONOMETRIC FUNCTIONS

38. $\displaystyle\int \sin mu \sin nu\, du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

39. $\displaystyle\int \cos mu \cos nu\, du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C$

40. $\displaystyle\int \sin mu \cos nu\, du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)} + C$

41. $\displaystyle\int \sin^m u \cos^n u\, du = -\frac{\sin^{m-1} u \cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u \cos^n u\, du$

$$= \frac{\sin^{m+1} u \cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u \cos^{n-2} u\, du$$

PRODUCTS OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

42. $\displaystyle\int e^{au}\sin bu\, du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

43. $\displaystyle\int e^{au}\cos bu\, du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

POWERS OF u MULTIPLYING OR DIVIDING BASIC FUNCTIONS

44. $\displaystyle\int u \sin u\, du = \sin u - u\cos u + C$

45. $\displaystyle\int u \cos u\, du = \cos u + u\sin u + C$

46. $\displaystyle\int u^2 \sin u\, du = 2u\sin u + (2-u^2)\cos u + C$

47. $\displaystyle\int u^2 \cos u\, du = 2u\cos u + (u^2-2)\sin u + C$

48. $\displaystyle\int u^n \sin u\, du = -u^n\cos u + n\int u^{n-1}\cos u\, du$

49. $\displaystyle\int u^n \cos u\, du = u^n\sin u - n\int u^{n-1}\sin u\, du$

50. $\displaystyle\int u^n \ln u\, du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

51. $\displaystyle\int u e^u\, du = e^u(u-1) + C$

52. $\displaystyle\int u^n e^u\, du = u^n e^u - n\int u^{n-1}e^u\, du$

53. $\displaystyle\int u^n a^u\, du = \frac{u^n a^u}{\ln a} - \frac{n}{\ln a}\int u^{n-1}a^u\, du + C$

54. $\displaystyle\int \frac{e^u\, du}{u^n} = -\frac{e^u}{(n-1)u^{n-1}} + \frac{1}{n-1}\int \frac{e^u\, du}{u^{n-1}}$

55. $\displaystyle\int \frac{a^u\, du}{u^n} = -\frac{a^u}{(n-1)u^{n-1}} + \frac{\ln a}{n-1}\int \frac{a^u\, du}{u^{n-1}}$

56. $\displaystyle\int \frac{du}{u\ln u} = \ln|\ln u| + C$

POLYNOMIALS MULTIPLYING BASIC FUNCTIONS

57. $\displaystyle\int p(u)e^{au}\, du = \frac{1}{a}p(u)e^{au} - \frac{1}{a^2}p'(u)e^{au} + \frac{1}{a^3}p''(u)e^{au} - \cdots$ [signs alternate: $+ - + - \cdots$]

58. $\displaystyle\int p(u)\sin au\, du = -\frac{1}{a}p(u)\cos au + \frac{1}{a^2}p'(u)\sin au + \frac{1}{a^3}p''(u)\cos au - \cdots$ [signs alternate in pairs after first term: $+ + - - + + - - \cdots$]

59. $\displaystyle\int p(u)\cos au\, du = \frac{1}{a}p(u)\sin au + \frac{1}{a^2}p'(u)\cos au - \frac{1}{a^3}p''(u)\sin au - \cdots$ [signs alternate in pairs: $+ + - - + + - - \cdots$]

RATIONAL FUNCTIONS CONTAINING POWERS OF $a + bu$ IN THE DENOMINATOR

60. $\displaystyle\int \frac{u\,du}{a+bu} = \frac{1}{b^2}[bu - a\ln|a+bu|] + C$

61. $\displaystyle\int \frac{u^2\,du}{a+bu} = \frac{1}{b^3}\left[\frac{1}{2}(a+bu)^2 - 2a(a+bu) + a^2\ln|a+bu|\right] + C$

62. $\displaystyle\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$

63. $\displaystyle\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right] + C$

64. $\displaystyle\int \frac{u\,du}{(a+bu)^3} = \frac{1}{b^2}\left[\frac{a}{2(a+bu)^2} - \frac{1}{a+bu}\right] + C$

65. $\displaystyle\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

66. $\displaystyle\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

67. $\displaystyle\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} + \frac{1}{a^2}\ln\left|\frac{u}{a+bu}\right| + C$

RATIONAL FUNCTIONS CONTAINING $a^2 \pm u^2$ IN THE DENOMINATOR ($a > 0$)

68. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

69. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

70. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

71. $\displaystyle\int \frac{bu+c}{a^2+u^2}\,du = \frac{b}{2}\ln(a^2+u^2) + \frac{c}{a}\tan^{-1}\frac{u}{a} + C$

INTEGRALS OF $\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$ AND THEIR RECIPROCALS ($a > 0$)

72. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

73. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

74. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

75. $\displaystyle\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) + C$

76. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u+\sqrt{u^2-a^2}| + C$

77. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a^2-u^2}$ OR ITS RECIPROCAL

78. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

79. $\displaystyle\int \frac{\sqrt{a^2-u^2}\,du}{u} = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

80. $\displaystyle\int \frac{\sqrt{a^2-u^2}\,du}{u^2} = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$

81. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

82. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

83. $\displaystyle\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{u^2 \pm a^2}$ OR THEIR RECIPROCALS

84. $\displaystyle\int u\sqrt{u^2+a^2}\,du = \frac{1}{3}(u^2+a^2)^{3/2} + C$

85. $\displaystyle\int u\sqrt{u^2-a^2}\,du = \frac{1}{3}(u^2-a^2)^{3/2} + C$

86. $\displaystyle\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

87. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

88. $\displaystyle\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\left|\frac{u}{a}\right| + C$

89. $\displaystyle\int \frac{\sqrt{u^2+a^2}\,du}{u} = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

90. $\displaystyle\int \frac{du}{u^2\sqrt{u^2\pm a^2}} = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u} + C$

91. $\displaystyle\int u^2\sqrt{u^2+a^2}\,du = \frac{u}{8}(2u^2+a^2)\sqrt{u^2+a^2} - \frac{a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

92. $\displaystyle\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

93. $\displaystyle\int \frac{\sqrt{u^2+a^2}}{u^2}\,du = -\frac{\sqrt{u^2+a^2}}{u} + \ln(u+\sqrt{u^2+a^2}) + C$

94. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln|u+\sqrt{u^2-a^2}| + C$

95. $\displaystyle\int \frac{u^2}{\sqrt{u^2+a^2}}\,du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

96. $\displaystyle\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

INTEGRALS CONTAINING $(a^2+u^2)^{3/2}$, $(a^2-u^2)^{3/2}$, $(u^2-a^2)^{3/2}$ ($a > 0$)

97. $\displaystyle\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

98. $\displaystyle\int \frac{du}{(u^2\pm a^2)^{3/2}} = \pm\frac{u}{a^2\sqrt{u^2\pm a^2}} + C$

99. $\displaystyle\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

100. $\displaystyle\int (u^2+a^2)^{3/2}\,du = \frac{u}{8}(2u^2+5a^2)\sqrt{u^2+a^2} + \frac{3a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

101. $\displaystyle\int (u^2-a^2)^{3/2}\,du = \frac{u}{8}(2u^2-5a^2)\sqrt{u^2-a^2} + \frac{3a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a+bu}$ OR ITS RECIPROCAL

102. $\displaystyle\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

103. $\displaystyle\int u^2\sqrt{a+bu}\,du = \frac{2}{105b^3}(15b^2u^2-12abu+8a^2)(a+bu)^{3/2}+C$

104. $\displaystyle\int u^n\sqrt{a+bu}\,du = \frac{2u^n(a+bu)^{3/2}}{b(2n+3)} - \frac{2an}{b(2n+3)}\int u^{n-1}\sqrt{a+bu}\,du$

105. $\displaystyle\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

106. $\displaystyle\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2-4abu+8a^2)\sqrt{a+bu}+C$

107. $\displaystyle\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

108. $\displaystyle\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C & (a>0) \\[2ex] \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bu}{-a}} + C & (a<0). \end{cases}$

109. $\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

110. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

111. $\displaystyle\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bu}\,du}{u^{n-1}}$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{2au-u^2}$ OR ITS RECIPROCAL

112. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

113. $\displaystyle\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

114. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2} + a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

115. $\displaystyle\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u} - \sin^{-1}\left(\frac{u-a}{a}\right)+C$

116. $\displaystyle\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\left(\frac{u-a}{a}\right)+C$

117. $\displaystyle\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au}+C$

118. $\displaystyle\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2} + a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

119. $\displaystyle\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2} + \frac{3a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

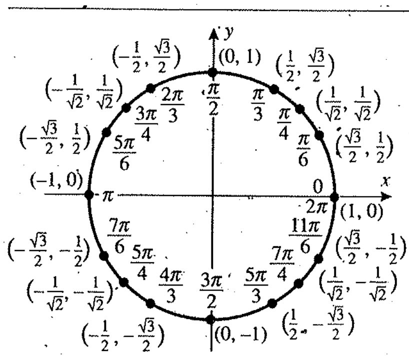
INTEGRALS CONTAINING $(2au-u^2)^{3/2}$

120. $\displaystyle\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}+C$

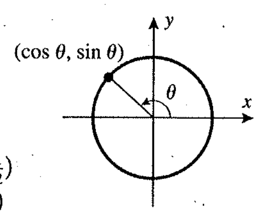
121. $\displaystyle\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}+C$

THE WALLIS FORMULA

122. $\displaystyle\int_0^{\pi/2}\sin^n u\,du = \int_0^{\pi/2}\cos^n u\,du = \frac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot6\cdot\cdots\cdot n}\cdot\frac{\pi}{2}\ \left(\begin{array}{l} n \text{ an even} \\ \text{integer and} \\ n\ge 2 \end{array}\right)$ or $\displaystyle\frac{2\cdot4\cdot6\cdot\cdots\cdot(n-1)}{3\cdot5\cdot7\cdot\cdots\cdot n}\ \left(\begin{array}{l} n \text{ an odd} \\ \text{integer and} \\ n\ge 3 \end{array}\right)$



TRIGONOMETRY REVIEW



PYTHAGOREAN IDENTITIES

$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1+\cot^2\theta = \csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi-\theta) = \sin\theta \qquad \cos(\pi-\theta) = -\cos\theta \qquad \tan(\pi-\theta) = -\tan\theta$

$\csc(\pi-\theta) = \csc\theta \qquad \sec(\pi-\theta) = -\sec\theta \qquad \cot(\pi-\theta) = -\cot\theta$

$\sin(\pi+\theta) = -\sin\theta \qquad \cos(\pi+\theta) = -\cos\theta \qquad \tan(\pi+\theta) = \tan\theta$

$\csc(\pi+\theta) = -\csc\theta \qquad \sec(\pi+\theta) = -\sec\theta \qquad \cot(\pi+\theta) = \cot\theta$

ADDITION FORMULAS

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\displaystyle\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\displaystyle\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = 2\cos^2\alpha - 1$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \cos 2\alpha = 1 - 2\sin^2\alpha$

HALF-ANGLE FORMULAS

$\displaystyle\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \qquad \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

MTH201 Chapter 5

Harper College*

June 25, 2025

Contents

*Problems found in Openstax Calculus Volume II and the Anton textbook.
https://openstax.org/details/books/calculus-volume-2

1 5.1 - Sequences

1. Write the general formula for a_n, where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

2. Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.

3. Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = \frac{1}{10}$ for $n \geq 1$.

4. Find the general term for the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

5. Find the formula for the general term of the sequence.

$$1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$$

6. Find the limit of the following sequences. You may need to use L'Hôpital's Rule.
   (a) $\frac{n^2}{2^n}$
   (b) $\frac{\sqrt{n}}{\sqrt{n+1}}$

7. Is the sequence bounded? Is it monotone increasing, or decreasing?
   (a) $\frac{n}{2^n},\ n \geq 2$
   (b) $\sin n$
   (c) $n^{\frac{1}{n}},\ n \geq 3$
   (d) $\tan n$

8. Find the limit of the sequence using the double angle identity.

$$a_n = \frac{\cos(1/n) - 1}{1/n}$$

9. Use the Squeeze Theorem to find the limit of the sequence.

$$a_n = \sin n \sin \frac{1}{n}$$

10. Determine the limit of the sequence. Does the sequence converge or diverge?
    (a) $a_n = (2n)^{\frac{1}{n}} - n^{\frac{1}{n}}$
    (b) $a_n = \left(1 - \frac{2}{n}\right)^n$
    (c) $a_n = \frac{2^n + 3^n}{4^n}$
    (d) $a_n = \frac{(n!)^2}{(2n)!}$

2 5.2 - Series

1. Write the following expressions as infinite series.

   (a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

   (b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

2. Use the sequence of partial sums to determine whether the series converges or diverges.

   (a)
   $$\sum_{n=1}^{\infty} \frac{n}{n+2}$$

   (b) Hint: Use Partial Fraction Decomposition
   $$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

3. Does the series converge or diverge? Explain.

   (a)
   $$1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$$

   (b)
   $$1 + \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$$

4. Evaluate the telescoping series or state if it diverges.

   (a)
   $$\sum_{n=1}^{\infty} \left(2^{1/n} - 2^{1/(n+1)}\right)$$

   (b)
   $$\sum_{n=1}^{\infty} \left(\sqrt{n} - \sqrt{n+1}\right)$$

5. Express the series as a telescoping sum and evaluate its kth partial sum.

   (a)
   $$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

   (b)
   $$\sum_{n=2}^{\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\ln n \ln(n+1)}$$

6. Evaluate

$$\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$$

7. Does the series converge? If so, find the sum.

   (a)

   $$\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$$

   (b)

   $$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{7}{6^{n-1}}$$

8. A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces vertically to a height that is $\frac{3}{4}$ of the preceding height. Find the total distance the ball will travel if it assumed to bounce infinetely.

3 5.3 - Divergence and Integral Tests

1. For each of the series, either find the limit or explain why the divergence test does not apply.

   (a) $a_n = \frac{n}{5n^2-3}$

   (b) $a_n = \frac{(2n+1)(n-1)}{(n+1)^2}$

   (c) $a_n = \frac{2^n}{3^{n/2}}$

   (d) $a_n = e^{-2/n}$

   (e) $a_n = \tan n$

   (f) $a_n = \frac{(\ln n)^2}{\sqrt{n}}$

2. Does the p-series converge or diverge?

   (a)
   $$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{n^{\pi}}{n^{2e}}$$

3. Use the integral test to determine whether the series converge.

   (a)
   $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n}{1+n^2}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$$

4. Write the series as a p-series and determine whether it converges.

   (a) Hint: $2^{-\ln n} = \frac{1}{n^{\ln 2}}$
   $$\sum_{n=1}^{\infty} 2^{-\ln n}$$

   (b)
   $$\sum_{n=1}^{\infty} n \cdot 2^{-2\ln n}$$

5. Does the following series converge if p is large enough? If so, for which p?
   $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

4 5.4 - Comparison Tests

1. Use the comparison test to determine whether the following series converge.

   (a)
$$\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{2}\right)}$$

   (b)
$$\sum_{n=2}^{\infty} \frac{1}{2\,(n-1)}$$

   (c)
$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$$

   (d)
$$\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{n^2}$$

   (e)
$$\sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\left(\sqrt{n}\right)^3}$$

2. Use the Limit Comparison Test to determine whether each of the following series converges or diverges.

   (a)
$$\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$$

   (b)
$$\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$$

   (c)
$$\sum_{n=1}^{\infty} \frac{1}{e^{1.1n}-3^n}$$

   (d)
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$

3. Use any method to evaluate the series.
$$\sum_{n=1}^{\infty} \frac{5}{3^n+1}$$

5 5.6 - Ratio and Root Tests

1. Use the ratio test to determine if the series converges or diverges. State if the ratio test is inconclusive.

   (a)
   $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$$

   (d)
   $$\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$$

   (e)
   $$\sum_{n=1}^{\infty} \frac{n!}{\left(\frac{n}{e}\right)^n}$$

2. Use the root test to determine if the series converges or diverges. State if the ratio test is inconclusive.

   (a)
   $$\sum_{n=1}^{\infty} \left(\frac{2n^2-1}{n^2+3}\right)^n$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{n^e}{e^n}$$

   (d)
   $$\sum_{n=1}^{\infty} \left(\frac{1}{e} + \frac{1}{n}\right)^n$$

   (e)
   $$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

6 5.5 - Alternating Series

Do the series converge absolutely, conditionally, or not at all?

1. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{\sqrt{n}+3}$$

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n+3}}{n}$$

3. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$$

4. $$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+1}{n}\right)^n$$

5. $$\sum_{n=1}^{\infty} (-1)^{n+1} \cos^2\left(\frac{1}{n}\right)$$

6. $$\sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(1+\frac{1}{n}\right)$$

7. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^e}{1+n^\pi}$$

8. $$\sum_{n=1}^{\infty} (-1)^{n+1} n^{\frac{1}{n}}$$

9. $$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

10. $$\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{\frac{1}{n}}}$$

MTH201 Chapter 6

Harper College*

June 25, 2025

Contents

*Problems found in Openstax Calculus Volume II and the Anton textbook.
https://openstax.org/details/books/calculus-volume-2

1 6.1 - Power Series

1. Find the radius and interval of convergence for the power series.

   (a)
$$\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$$

   (b)
$$\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$$

   (c)
$$\sum_{n=1}^{\infty} \frac{\pi^n x^n}{n^\pi}$$

   (d)
$$\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$$

2. Find the radius of convergence of each series.

   (a)
$$\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$$

   (b)
$$\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$$

   (c)
$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

3. Given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with center a. Identify its interval of convergence.

   (a)
$$f(x) = \frac{1}{x}, \quad a = 1$$

   (b)
$$f(x) = \frac{x}{1-x^2}, \quad a = 0$$

   (c)
$$f(x) = \frac{x^2}{1+x^2}, \quad a = 0$$

(d)

$$f\left(x\right)=\frac{1}{1-2x},\ a=0$$

(e)

$$f\left(x\right)=\frac{x^2}{1-4x^2},\ a=0$$

2 6.2 - Properties of Power Series

1. Use partial fractions to find the power series of each function.

   (a)

   $$\frac{4}{\left(x-3\right)\left(x+1\right)}$$

   (b)

   $$\frac{5}{\left(x^2+4\right)\left(x^2-1\right)}$$

2. Express each series as a rational function.

   (a)

   $$\sum_{n=1}^{\infty}\frac{1}{x^n}$$

   (b)

   $$\sum_{n=1}^{\infty}\frac{1}{\left(x-3\right)^{2n-1}}$$

3. Given that $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, use differentiation or integration to find the power series for each function, centered at $x=0$.

   (a)

   $$\ln\left(1-x^2\right)$$

   (b)

   $$\arctan\left(x^2\right)$$

3 6.3 - Taylor and Maclaurin Series

1. Find the Taylor polynomials of degree 2 approximating the given function centered at the given point.

   (a) $f(x) = 1 + x + x^2$ at $a = -1$

   (b) $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

   (c) $f(x) = \ln x$ at $a = 1$

2. Verify that the given value of n yields a remainder estimate $|R_n| < \frac{1}{1000}$. Find the value of the Taylor polynomial of degree n at the point a.

   (a) $28^{1/3}$; $a = 27$; $n = 1$

   (b) $\ln 2$; $a = 1$; $n = 1000$ (Use a calculator for series)

3. Find the Taylor series of the given function centered at the given point.

   (a) $f(x) = 1 + x + x^2 + x^3$ at $a = -1$

   (b) $f(x) = \cos x$ at $a = 2\pi$

   (c) $f(x) = \frac{1}{(x-1)^3}$ at $a = 0$

4. Compute the Taylor series of each function around $x = 1$.

   (a) $f(x) = 2 - x$

   (b) $f(x) = (x - 2)^2$

   (c) $f(x) = \frac{1}{x}$

   (d) $f(x) = e^{2x}$

5. Approximate $\sqrt{e}$ to 4 decimal places of accuracy using Taylor series remainder.

MTH201 Chapter 7

Harper College*

July 22, 2025

Contents

*Problems found in Openstax Calculus Volume II.
https://openstax.org/details/books/calculus-volume-2

1 7.1 - Parametric Equations

1. Sketch the curves by eliminating the parameter t. Give the orientation of the curve.

   (a) $x = t^2 + 2t,\ y = t + 1$

   (b) $x = 2t + 4,\ y = t - 1$

2. Eliminate the parameter and sketch the graph. $x = 2t^2,\ y = t^4 + 1$

3. Eliminate the parameter and sketch the graph. Indicate any asymptotes.

   (a) $x = 6\sin(2\theta),\ y = 4\cos(2\theta)$

   (b) $x = 3 - 2\cos\theta,\ y = -5 + 3\sin\theta$

   (c) $x = \sec t,\ y = \tan t$ (Do not find orientation)

   (d) $x = e^t,\ y = e^{2t}$

   (e) $x = t^3,\ y = 3\ln t$ (Hint: Use log properties.)

4. Convert the parametric equations into rectangular form. State the domain of the rectangular form. Do not sketch.

   (a) $x = t^2 - 1,\ y = \frac{t}{2}$

   (b) $x = 4\cos\theta,\ y = 3\sin\theta,\ \theta \in (0, 2\pi]$

   (c) $x = 2t - 3,\ y = 6t - 7$

   (d) $x = 1 + \cos t,\ y = 3 - \sin t$

   (e) $x = \cos(2t),\ y = \sin t$

   (f) $x = t^2,\ y = 2\ln t,\ t \geq 1$

   (g) $x = 2\sin(8t),\ y = 2\cos(8t)$

2 7.2 - Calculus of Parametric Curves

1. Each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.
   (a) $x = 8 + 2t$, $y = 1$
   (b) $x = -5 + 7t$, $y = 3t = -1$
2. Find the equation of the tangent line at the given value of the parameter.
   (a) $x = \cos t$, $y = 8 \sin t$, $t = \frac{\pi}{2}$
   (b) $x = t + \frac{1}{t}$, y=$t - \frac{1}{t}$, $t = 1$
   (c) $x = e^{\sqrt{t}}$, $y = 1 - \ln\left(t^2\right)$, at $t = 1$
3. For $x = \sin(2t)$, $y = 2 \sin t$ where $0 \leq t < 2\pi$, find all values of t where the equation has a vertical tangent line.
4. Find the second derivative of $x = \sqrt{t}$, $y = 2t + 4$, at $t = 1$.
5. Find the arc length of the curve on the indicated interval of the parameter.
   (a) $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \leq t \leq 1$
   (b) $x = 1 + t^2$, $y = (1+t)^3$, $0 \leq t \leq 1$ Use a calculator to integrate. Round to three decimal places.
   (c) $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq t < 2\pi$

3 7.3 - Polar Coordinates

1. Convert the rectangular coordinates to two different polar coordinates. Round to three decimal places.
   (a) $(3, -4)$
   (b) $(-6, 8)$
   (c) $\left(3, -\sqrt{3}\right)$
2. Convert the polar coordinates to rectangular coordinates.
   (a) $\left(-2, \frac{\pi}{6}\right)$
   (b) $\left(1, \frac{7\pi}{6}\right)$
   (c) $\left(0, \frac{\pi}{2}\right)$

4 7.4 - Area and Arc Length in Polar Coordinates

1. Set up, but do not evaluate, an integral that represents the area of the following regions.
   (a) Region enclosed by $r = 3\sin\theta$
   (b) Region enclosed by one petal of $r = 8\sin(2\theta)$
   (c) Region below the polar axis and enclosed by $r = 1 - \sin\theta$
2. Find the area of the described regions.
   (a) Enclosed by $r = 6\sin\theta$
   (b) Below the polar axis and enclosed by $r = 2 - \cos\theta$
   (c) Enclosed by one petal of $r = 3\cos(2\theta)$
   (d) Enclosed by the inner loop of $r = 3 + 6\cos\theta$
3. Set up, but do not evaluate, an integral that represents the arc length of the given polar curves.
   (a) $r = 1 + \sin\theta$ over $0 \leq \theta \leq 2\pi$
   (b) $r = e^{\theta}$ over $0 \leq \theta \leq 1$
4. Find the arc length of the polar curves over the given intervals.
   (a) $r = e^{3\theta}$ over $0 \leq \theta \leq 2$
   (b) $r = 8 + 8\cos\theta$ over $0 \leq \theta \leq \pi$ (Hint: Use half-angle formulas)

5. Find the slope of a tangent line to the polar curves.

(a) $r = 4\cos\theta,\ \left(2, \frac{\pi}{3}\right)$

(b) $r = 4 + \sin\theta,\ \left(3, \frac{3\pi}{2}\right)$