MTH 200

Blank Worksheet

Problems found in OpenStax Calculus Volume II

Sections:

- 2.1 – A Preview of Calculus
- 2.2 – The Limit of a Function
- 2.3 – The Limit Laws
- 2.4 – Continuity

Chapter 2
Rev 8/25/2024

1. (2.1)

a. $P = (1,2), Q = (x, y), f(x) = x^2 + 1$

x	y	$Q(x,y)$	m_{sec} **(P to Q)**
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

b. Guess the slope of the tangent line to f at $x = 1$.

c. Find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

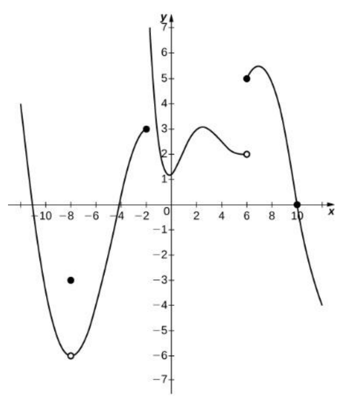
2. (2.1) Consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time *t* seconds is $h(t) = 15t - 4.9t^2$

Compute the average velocity of the stone over the given time intervals.

a. $[1, 1.05]$
b. $[1, 1.01]$
c. $[1, 1.005]$
d. $[1, 1.001]$

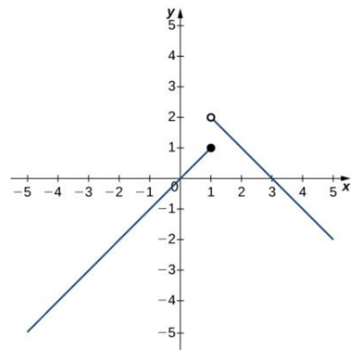
Guess the instantaneous velocity of the stone at t=1 sec.

3. (2.2) Look at the graph and classify the following statements as true or false. If false, explain and find the correct limit.



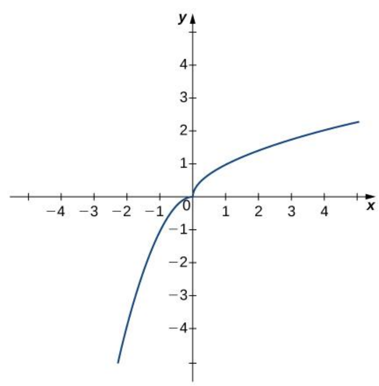
a. $\lim_{x \to 10} f(x) = 0$
b. $\lim_{x \to -2^+} f(x) = 3$
c. $\lim_{x \to -8} f(x) = f(-8)$
d. $\lim_{x \to 6} f(x) = 5$

4. (2.2) Given the graph $y = f(x)$, determine the following values.



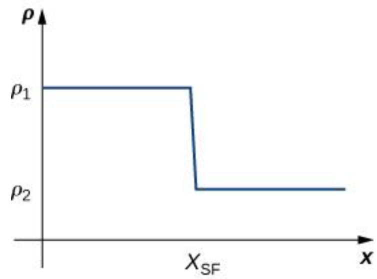
a. $\lim_{x \to 1^-} f(x)$
b. $\lim_{x \to 1^+} f(x)$
c. $\lim_{x \to 1} f(x)$
d. $\lim_{x \to 2} f(x)$
e. $f(1)$

5. (2.2) Given the graph $y = h(x)$, determine the following values.



a. $\lim_{x \to 0^-} h(x)$
b. $\lim_{x \to 0^+} h(x)$
f. $\lim_{x \to 0} h(x)$

6. (2.2) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



c. Evaluate $\lim_{x \to X_{SF}^+} \rho(x)$

d. Evaluate $\lim_{x \to X_{SF}^-} \rho(x)$

e. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

7. (2.2) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{x \to 2} x(t)$? What does it mean physically?

***t* (sec)**	***x* (m)**
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

8. (2.3) Evaluate $\lim_{x \to 0}(4x^2 - 2x + 3)$.

9. (2.3) Evaluate $\lim_{x \to -2} \sqrt{x^2 - 6x + 3}$.

10. (2.3) Evaluate $\lim_{x \to 0} \frac{1}{1+\sin x}$.

11. (2.3) Evaluate $\lim_{x \to 4} \frac{x^2-16}{x-4}$.

12. (2.3) Evaluate $\lim_{x \to 6} \frac{3x-18}{2x-12}$.

13. (2.3) Evaluate $\lim_{t \to 9} \frac{t-9}{\sqrt{t}-3}$.

14. (2.3) Evaluate $\lim_{\theta \to \pi} \frac{\sin\theta}{\tan\theta}$.

15. (2.3) Evaluate $\lim_{x \to 1^-} \frac{2x^2+7x-4}{x^2+x-2}$.

16. (2.3) If $\lim_{x \to 6} f(x) = 4$ and $\lim_{x \to 6} f(x) = 9$, Evaluate $\lim_{x \to 6} \sqrt{g(x) - f(x)}$.

17. (2.3) Evaluate $\lim_{\theta \to 0} \theta^2 \cos \frac{1}{\theta}$.

18. (2.4) When are the functions discontinuous? Classify the discontinuity.

a. $f(x) = \frac{1}{\sqrt{x}}$
b. $f(x) = \frac{x}{x^2 - x}$
c. $f(x) = \frac{5}{e^x - 2}$
d. $f(x) = \tan 2x$

19. (2.4) Is the function discontinuous? If so, classify the discontinuity.

a. $f(x) = \frac{2x^2 - 5x + 3}{x - 1}$, at $x = 1$
b. $f(x) = \begin{cases} \frac{6x^2 + x - 2}{2x - 1}, x \neq \frac{1}{2} \\ \frac{7}{2}, x = \frac{1}{2} \end{cases}$, at $x = \frac{1}{2}$
c. $f(x) = \frac{\sin(\pi x)}{\tan(\pi x)}$, at $x = 1$
d. $f(x) = \begin{cases} x^2 - e^x, x < 0 \\ x - 1, x \geq 0 \end{cases}$, at $x = 0$
e. $f(x) = \begin{cases} x \sin x, x \leq \pi \\ x \tan x, x > \pi \end{cases}$, at $x = \pi$

20. (2.4) What value of k will make the function continuous? $f(\theta) = \begin{cases} \sin\theta, 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

21. (2.4) What value of k will make the function continuous? $f(x) = \begin{cases} \frac{x^2 + 3x + 2}{x + 2}, x \neq -2 \\ k, x = -2 \end{cases}$

22. (2.4) Let $h(x) = \begin{cases} 3x^2 - 4, x \leq 2 \\ 5 + 4x, x > 2 \end{cases}$. Over the interval $[0,4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

23. (2.4) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

24. (2.4) Let $f(x) = \begin{cases} 3x, x > 1 \\ x^3, x < 1 \end{cases}$. Graph the function and determine if it is possible to find a value k such that $f(1) = k$ which makes $f(x)$ continuous for all real numbers? Explain.

25. (2.4) Are the following statements true? Explain.

a. $f(t) = \frac{2}{e^t - e^{-t}}$ is continuous everywhere.
b. If a function is not continuous at a point, then it is not defined at that point.

MTH 200

Blank Worksheet

Problems found in OpenStax Calculus Volume II

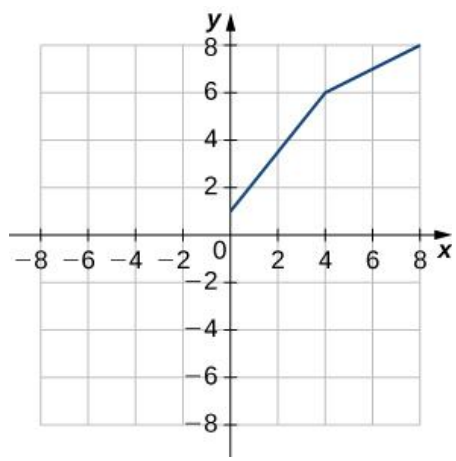
Sections:

- 3.1 – Defining the Derivative
- 3.2 – The Derivative as a Function
- 3.3 – Differentiation Rules
- 3.5 – Derivatives of Trigonometric Functions
- 3.6 – The Chain Rule
- 3.7 – Derivatives of Inverse Functions
- 3.8 – Implicit Differentiation
- 3.9 – Derivatives of Exponential and Logarithmic Functions

Chapter 3
Rev 7/23/25

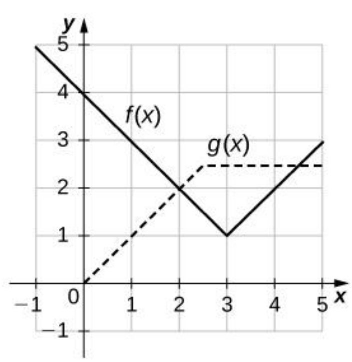
1. (3.1) Find the slope of the secant line between the values x_1 and x_2 for the following functions.
   a. $f(x) = 4x + 7; x_1 = 2, x_2 = 5$
   b. $f(x) = x^2 + 2x + 1; x_1 = 3, x_2 = 3.5$
   c. $f(x) = x^{\frac{1}{3}} + 1; x_1 = 0, x_2 = 8$

2. (3.1) Find the equation of the tangent line at the given value for the following functions using the definition of a derivative.
   a. $f(x) = 3 - 4x; a = 2$
   b. $f(x) = x^2 + x; a = 1$
   c. $f(x) = \frac{7}{x}; a = 3$
   d. $f(x) = 2 - 3x^2; a = -2$
   e. $f(x) = \frac{2}{x+3}; a = -4$
3. (3.1) Find $f'(6)$ for the function $f(x) = \sqrt{x-2}$.
4. (3.1) Find $f'(1)$ for the function $f(x) = \frac{1}{x^3}$.
5. (3.1) Evaluate $f'(1)$ and $f'(6)$.



6. (3.1) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle travelling x miles. Suppose the vehicle gets 30 mpg.
   a. Find a mathematical expression for $N(x)$.
   b. What is $N(100)$? Explain the physical meaning.
   c. What is $N'(100)$? Explain the physical meaning.
7. (3.2) Use the definition of the derivative to find $f'(x)$.
   a. $f(x) = 2 - 3x$
   b. $f(x) = 4x^2$
   c. $f(x) = \sqrt{2x}$
   d. $f(x) = \frac{9}{x}$
   e. $f(x) = \frac{1}{\sqrt{x}}$

8. (3.2) The limit represents the derivative of a function $f(x)$ at $x = a$. Find $f(x)$ and a.
   a. $\lim_{h \to 0} \frac{[3(2+h)^2+2]-14}{h}$
   b. $\lim_{h \to 0} \frac{(2+h)^4-16}{h}$
   c. $\lim_{h \to 0} \frac{e^h-1}{h}$
9. (3.2) Find the second derivative using the limit definition of derivatives.
   a. $f(x) = 2 - 3x$
   b. $f(x) = x + \frac{1}{x}$
10. (3.2) $C(x)$ denotes the average cost of college tuition x years after 2000. What do the following expressions represent.
    a. $\frac{C(x+h)-C(x)}{h}$
    b. $\lim_{h \to 0} \frac{C(10+h)-C(10)}{h}$
11. (3.3) Find $f'(x)$.
    a. $f(x) = 8x^4 + 9x^2 - 1$
    b. $f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$
    c. $f(x) = \frac{4x^3-2x+1}{x^2}$
12. (3.3) Find the equation of the tangent line to $y = \frac{2}{x^2} + 1$ at $(1, 3)$.
13. (3.3) Find the equation of the tangent line to $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$.
14. (3.3) If $f(x)$ and $g(x)$ are differentiable functions, find $h'(x)$.
    a. $h(x) = x^3 f(x)$
    b. $h(x) = \frac{3f(x)}{g(x)+2}$
15. (3.3) If $h(x) = f(x)g(x)$, use the following graph to find:
    a. $h'(1)$
    b. $h'(3)$
    c. $h'(4)$



16. (3.3) Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x-intercept of 6.
17. (3.3) According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.
    a. Suppose G, m_1, and m_2 are constants. Find the rate of change of force F with respect to distance d.
    b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.
18. (3.5) Find $\frac{dy}{dx}$ for the following functions.
    a. $y = x^2 - \sec x + 1$
    b. $y = x^2 \cot x$
    c. $y = \frac{\sec x}{x}$
    d. $y = (x + \cos x)(1 - \sin x)$
    e. $y = \frac{1 - \cot x}{1 + \cot x}$

19. (3.5) Find the equation of the tangent line to $f(x) = x^2 - \tan x$ at $x = 0$.
20. (3.5) Find $\frac{d^2y}{dx^2}$ for the following functions.
    a. $y = x \sin x - \cos x$
    b. $y = x - \frac{1}{2} \sin x$
    c. $y = 2 \csc x$
21. (3.5) Let $f(x) = \cot x$. Determine the points on the graph f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

22. (3.5) Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a \cos t + b \sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is $3 \frac{cm}{s}$ when $t = 0$, find the values of a and b.
23. (3.6) Find $\frac{dy}{dx}$ using the definition of the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
    a. $y = 6u^3, u = 7x - 4$
    b. $y = \sqrt{4u + 3}, u = x^2 - 6x$
24. (3.6) Find $\frac{dy}{dx}$.
    a. $y = (3x^2 + 1)^3$
    b. $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$
    c. $y = \csc(\pi x + 1)$

d. $y = \frac{-6}{(\sin x)^3}$
e. $y = \sin(\cos 7x)$

25. (3.6) Let $y = [f(x)]^2$ and suppose that $f(-1) = 4$ and $\frac{dy}{dx} = 10$ when $x = 1$. Find $f'(1)$.
26. (3.6) The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m. Approximate to one decimal place.
27. (3.7) Find the derivative at $x = a$, $x = f^{-1}(y)$, and $\frac{d}{dy}[f^{-1}(y)]$ at $y = f(a)$.
    a. $f(x) = 2x^3 - 3, x = 1$
    b. $f(x) = \sin x, x = 0$
28. (3.7) Find $(f^{-1})'(a)$
    a. $f(x) = x^3 + 2x + 3, a = 0$
    b. $f(x) = x - \frac{2}{x}, x < 0, a = 1$
29. (3.7) Find $\frac{dy}{dx}$.
    a. $y = \arcsin x^2$
    b. $y = \operatorname{arcsec}\frac{1}{x}$
    c. $y = (1 + \arctan x)^3$
    d. $y = \frac{1}{\arctan x}$
    e. $y = \operatorname{arccot}\sqrt{4 - x^2}$
30. (3.7) $f\left(\frac{1}{3}\right) = -8, f'\left(\frac{1}{3}\right) = 2$, Find $(f^{-1})'(-8)$.
31. (3.7) The position of a moving hockey puck after t seconds is $s(t) = \arctan t$ where s is in meters.
    a. Find the velocity function.
    b. Find the acceleration function.
    c. Find the velocity and acceleration at $t = 2, 4$, and 6 seconds.
    d. What conclusion can be drawn from the results in c?
32. (3.7) A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached. Round to four decimal places.
33. (3.8) Find $\frac{dy}{dx}$.
    a. $6x^2 + 3y^2 = 12$
    b. $3x^3 + 9xy^2 = 5x^3$

c. $y\sqrt{x+4} = xy + 8$
d. $y\sin(xy) = y^2 + 2$
e. $x^3y + xy^3 = -8$

34. (3.8) Find the equation of the tangent line to the graph at the given point.
a. $x^2y^2 + 5xy = 14, (2, 1)$
b. $xy^2 + \sin(\pi y) - 2x^2 = 10, (2, -3)$
c. $xy + \sin x = 1, \left(\frac{\pi}{2}, 0\right)$

35. (3.8) For the equation $x^2 + 2xy - 3y^2 = 0$,
a. Find the equation of the normal line to the tangent line at the point $(1, 1)$.
b. At what other point does the normal line in a. intersect the graph of the equation?

36. (3.8) For the equation $x^2 + xy + y^2 = 7$,
a. Find the x-intercept(s).
b. Find the slope of the tangent line(s) at the x-intercept(s).
c. What does (do) the value(s) in b. indicate about the tangent line(s)?

37. (3.8) The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is constant. Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16$.

38. (3.8) The surface area of a closed rectangular box with a square base with side x and height y is 78 square feet. Find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

39. (3.9) Find $\frac{dy}{dx}$.
a. $y = x^2 e^x$
b. $y = e^{x^3 \ln x}$
c. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
d. $y = 2^{4x} + 4x^2$
e. $y = x^\pi \cdot \pi^x$
f. $y = \ln\sqrt{5x - 7}$
g. $y = \log(\sec x)$
h. $y = 2^x \cdot \log_3 7^{x^2 - 4}$
i. $y = (\sin 2x)^{4x}$
j. $y = x^{\log_2 x}$
k. $y = x^{\cot x}$

40. (3.9) The position of an object is given by the function $f(x) = \log(10x^3) + x$. Find the velocity and acceleration function. Is the object speeding up or slowing down?

41. (3.9) Find the equation of the normal line to $f(x) = x \cdot 5^x$ at the point where $x = 1$.

42. (3.9) The number of cases of influenza in New York City form the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}$,

$(0 \leq t \leq 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

MTH 200

Blank Worksheet

Problems found in OpenStax Calculus Volume II and AP Calculus AB FRQ (in red).

Sections:

- 4.1 – Related Rates
- 4.2 – Linear Approximations and Differentials
- 4.3 – Maxima and Minima
- 4.4 – The Mean Value Theorem
- 4.5 – Derivatives and the Shape of a Graph
- 4.6 – Limits at Infinity and Asymptotes
- 4.7 – Applied Optimization Problems
- 4.8 – L'Hôpital's Rule
- 4.9 – Antiderivatives

1. (4.1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.
2. (4.1) Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.
3. (4.1) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of $2\frac{ft}{sec}$, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?
$$c = 10\,ft, \frac{db}{dt} = -2\frac{ft}{s}, a = 5\,ft$$
4. (4.1) Two airplanes are flying in the air at the same height: airplane *A* is flying east at $250\frac{mi}{h}$ and airplane *B* is flying north at $300\frac{mi}{h}$. If they are both heading to the same airport, located 30 miles east of airplane *A* and 40 miles north of airplane *B*, at what rate is the distance between the airplanes changing?
5. (4.1) A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of $3\frac{ft}{s}$.
   a. What is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?
   b. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?
6. (4.1) The radius of a sphere is increasing at a rate of $9\frac{cm}{s}$. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.
7. (4.1) Consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.
   a. How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10\frac{ft^3}{min}$?
   b. If the water level is decreasing at a rate of $3\frac{in}{min}$ when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.
8. (4.1) A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is $10\frac{cm}{min}$ when the height is 1 m.
9. (4.1) A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}\frac{m^3}{sec}$?
10. (4.1) You are stationary on the ground and are watching a bird fly horizontally at a rate of $10\frac{m}{sec}$. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

11. (4.1) A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?

12. (4.1) An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$. The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days.

$V = \frac{1}{3}\pi r^2 h$

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

13. (4.2) Find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$.

a. $f(x) = \frac{1}{x}, a = 2$

b. $f(x) = \sin x, a = \frac{\pi}{2}$

c. $f(x) = \sin^2 x, a = 0$

14. (4.2) Compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$.

a. $\sin 0.02$

b. $15.99^{\frac{1}{4}}$

c. $\sin 3.14$

d. $\cos 0.01$

e. 1.01^{-3}

f. $\sqrt{8.99}$

15. (4.2) Find the differential of the function.

a. $y = x \cos x$

b. $y = \frac{x^2+2}{x-1}$

16. (4.2) Find the differential and evaluate for the given x and dx.

a. $y = \frac{1}{x+1}, x = 1, dx = 0.25$

b. $y = \frac{3x^2+2}{\sqrt{x+1}}, x = 0, dx = 0.1$

c. $y = x^3 + 2x + \frac{1}{x}, x = 1, dx = 0.05$

17. (4.2) Find the change in volume dV or in surface area dA.
   a. dA if the sides of a cube change from x to $x + dx$.
   b. dV if the radius of a sphere changes from r by dr.
   c. dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9\ cm$.
   d. A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

18. (4.3) The formula for the position of the maximum or minimum of a quadratic $y = ax^2 + bx + c$ is $h = -\frac{b}{2a}$. Prove this formula using calculus.

19. (4.3) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain using graphical arguments.

20. (4.3) Is it possible to have more than one absolute maximum? Explain using graphical arguments.

21. (4.3) Consider the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

22. (4.3) Find the critical points in the domains of the following functions.
   a. $y = 4\sqrt{x} - x^2$
   b. $y = \ln(x - 2)$
   c. $y = \sqrt{4 - x^2}$
   d. $y = \frac{x^2-1}{x^2+2x-3}$
   e. $y = x + \frac{1}{x}$

23. (4.3) Find the local and/or absolute extrema for the functions over the specified domain.
   a. $y = x^2 + \frac{2}{x}$ over $[1, 4]$
   b. $y = \frac{1}{x-x^2}$ over $(0, 1)$
   c. $y = x + \sin x$ over $[0, 2\pi]$
   d. $y = \sin x + \cos x$ over $[0, 2\pi]$

24. (4.3) Find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.
   a. $y = x^2 + 4x + 5$
   b. $y = 3x^4 + 8x^3 - 18x^2$
   c. $y = \frac{x^2+x+6}{x-1}$

25. (4.3) Find the critical points, maxima, and minima for the piecewise function $y = \begin{cases} x^2 + 1, & x \le 1 \\ x^2 - 4x + 5, & x > 1 \end{cases}$

26. (4.4) Over what intervals (if any) does the MVT apply? Explain.
   a. $y = \frac{1}{x^3}$
   b. $y = \sqrt{x^2 - 4}$

27. (4.4) Use the MVT to find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$
  a. $f(x) = x^3$
  b. $f(x) = \cos 2\pi x$
  c. $f(x) = (x - 1)^{10}$

28. (4.4) Show that there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the MVT does not apply over the interval $[-1, 1]$.
  a. $f(x) = \left|x - \frac{1}{2}\right|$
  b. $f(x) = \sqrt{|x|}$

29. (4.4) Does the MVT apply for the functions over the given interval? Explain.
  a. $y = e^x$ over $[0, 1]$
  b. $f(x) = \tan(2\pi x)$ over $[0, 2]$
  c. $y = \frac{1}{|x+1|}$ over $[0, 3]$
  d. $y = \frac{x^2+3x+2}{x}$ over $[-1, 1]$
  e. $y = \ln(x + 1)$ over $[0, e - 1]$
  f. $y = 5 + |x|$ over $[-1, 1]$

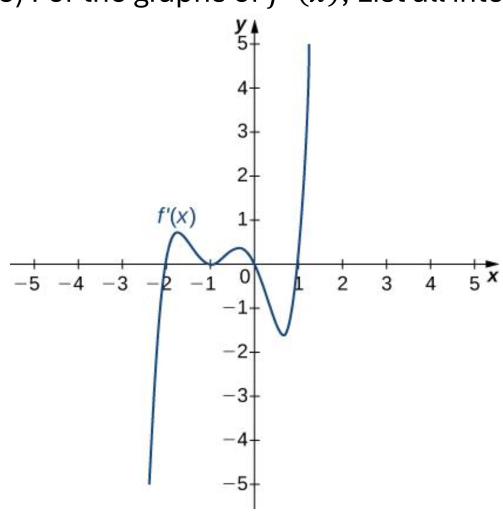
30. (4.4) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

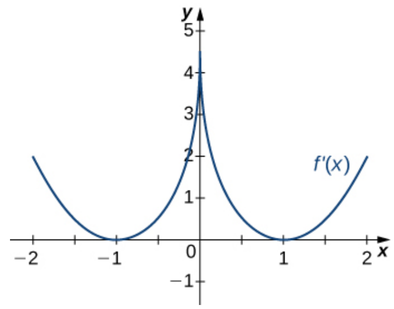
31. (4.5) For $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum? Explain.

32. (4.5) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function? Explain no.

33. (4.5) For the graphs of $f'(x)$, List all intervals where f is increasing or decreasing.

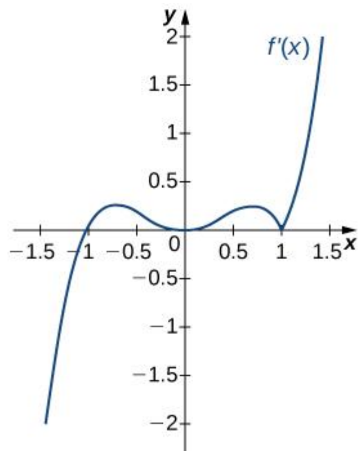
  a.



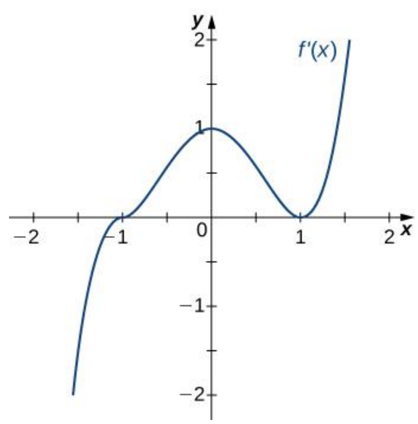


b.

34. (4.5) From the graph of f', list all intervals where f is increasing and decreasing, and where the minima and maxima are located.



35. (4.5) From the graph of f', list all inflection points and intervals where f is concave up and concave down.



36. (4.5) List intervals where f is increasing or decreasing. List the local minima and maxima of f. $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

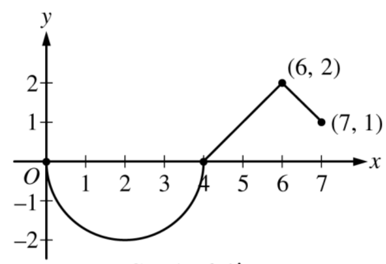
37. (4.5) Determine the intervals where f is increasing or decreasing, the local minima and maxima of f, intervals where f Is concave up and concave down, and the inflection points of f.

a. $f(x) = x^3 - 6x^2$

b. $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c. $f(x) = \frac{1}{1-x}, x \neq 1$

38. (4.5) Let f be a differentiable function. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure below.



Graph of f'

a. Find the x-coordiantes of all points of inflection on the graph of f for $0 < x < 7$. Justify your answer.

b. Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

39. (4.6) For the following functions, determine whether there is an asymptote at $x = a$. Explain.

a. $f(x) = \frac{x}{x-2}, a = 2$

b. $f(x) = (x-1)^{-\frac{1}{3}}, a = 1.$

40. (4.6) Evaluate the limit.

a. $\lim_{x \to \infty} \frac{1}{3x+6}$

b. $\lim_{x \to \infty} \frac{x^2-2x+5}{x+2}$

c. $\lim_{x \to -\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4}$

d. $\lim_{x \to -\infty} \frac{\sqrt{4x^2-1}}{x+2}$

e. $\lim_{x \to -\infty} \frac{4x}{\sqrt{x^2-1}}$

41. (4.6) Find the horizontal and vertical asymptotes.

a. $f(x) = x - \frac{9}{x}$

b. $f(x) = \frac{x^3}{4-x^2}$

c. $f(x) = \sin x \sin 2x$

d. $f(x) = \frac{x \sin x}{x^2-1}$

e. $f(x) = \frac{1}{x^3+x^2}$

f. $f(x) = \frac{x^3+1}{x^3-1}$

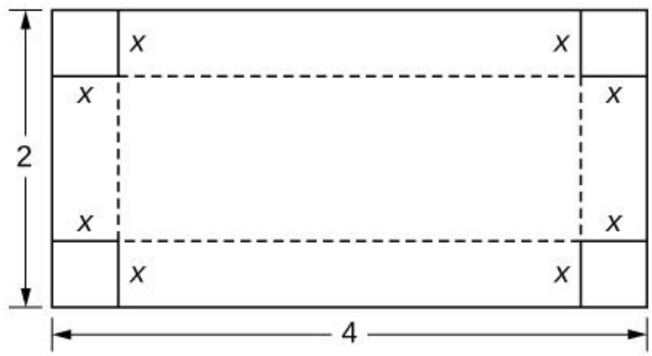
g. $f(x) = x - \sin x$

42. (4.6) For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$ must have what relation?

43. (4.7) Why do you need to check the sign of the derivative around the critical points in an optimization problem.

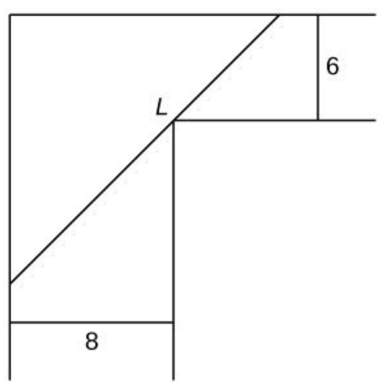
44. (4.7) True or false? For every continuous nonlinear function, you can find the value x that maximizes the function.

45. (4.7) You are constructing a cardboard box with the dimensions $2\ m$ by $4\ m$. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



46. (4.7) You need to construct a fence around an area of 1600 ft^2. What are the dimensions of the rectangular pen to minimize the amount of material needed?

47. (4.7) You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?

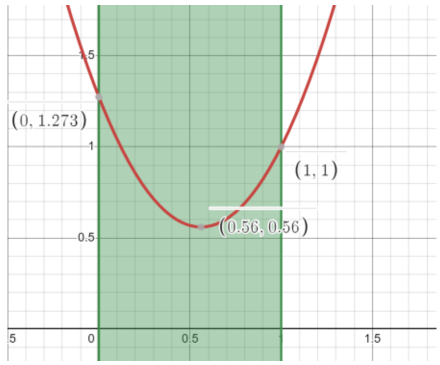


48. (4.7) A Truck uses gas as $g(v) = av + \frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. Assuming a and b are positive, at what speed is fuel consumption minimized?

49. (4.7) Consider a limousine that gets $m(v) = \frac{120-2v}{5}$ miles per gallon at speed v, the chauffeur costs $\frac{\$15}{hr}$, and gas is $\frac{\$3.5}{gal}$. Find the cheapest driving speed.

50. (4.7) A pizzeria sells pizzas for a revenue of $R(x) = 10x$ and costs $C(x) = 2x + x^2$, where x represents the number of pizzas. How many pizzas sold maximizes the profit?

51. (4.7) Consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x. Chose x to maximize the sum of their areas.



52. (4.7) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

53. (4.7) Where is the line $y = 5 - 2x$ closest to the origin? (2, 1)

54. (4.7) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/ft^2$ and the material for the sides costs $\$2/ft^2$. You need a box with volume $4ft^3$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

55. (4.7) You are the manager of an apartment complex with 50 units. When you set rent at \$800/month, all apartments are rented. As you increase rent by \$25/month, one fewer apartment is rented. Maintenance costs run \$50/month for each occupied unit. What is the rent that maximizes the total amount of profit?

56. (4.8) Evaluate the limit

a. $\lim_{x\to\infty} \frac{e^x}{x^k}$

b. $\lim_{x\to a} \frac{x-a}{x^2-a^2}$

c. $\lim_{x\to a} \frac{x-a}{x^n-a^n}$

57. (4.8) Can you apply L'Hôpital's rule directly? Why or why not? Is there a way to alter the limit to apply L'Hôpital's rule?

a. $\lim_{x\to\infty} x^{1/x}$

b. $\lim_{x\to 0} \frac{x^2}{1/x}$

58. (4.8) Evaluate the limits.

a. $\lim_{x\to 3} \frac{x^2-9}{x-3}$

b. $\lim_{x\to 0} \frac{(1+x)^{-2}-1}{x}$

c. $\lim_{x\to\pi} \frac{x-\pi}{\sin x}$
d. $\lim_{x\to 0} \frac{(1+x)^n-1}{x}$
e. $\lim_{x\to 0} \frac{\sin x-\tan x}{x^3}$
f. $\lim_{x\to 0} \frac{e^x-x-1}{x^2}$
g. $\lim_{x\to 1} \frac{x-1}{\ln x}$
h. $\lim_{x\to 1} \frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$
i. $\lim_{x\to\infty} x \sin\frac{1}{x}$
j. $\lim_{x\to 0^+} x\ln(x^4)$
k. $\lim_{x\to\infty} x^2 e^{-x}$
l. $\lim_{x\to 0} \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$
m. $\lim_{x\to\infty} xe^{1/x}$
n. $\lim_{x\to 0^+} x^{\frac{1}{x}}$
o. $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x$

59. (4.9) Write Newton's Formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.
a. $f(x) = x^3 + 2x + 1$
b. $f(x) = e^x$

60. (4.9) Solve to four decimal places using Newton's method and a calculator. Chose an x_0 that is not the exact root.
a. $x^4 - 100 = 0$
b. $x^3 - x = 0$
c. $x + \tan x = 0$. Chose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
d. $1 + x + x^2 + x^3 + x^4 = 2$
e. $x = \sin^2 x$

61. (4.9) Use Newton's method to find the following. Round to three decimals.
a. Minimum of $f(x) = 3x^3 + 2x^2 - 16$
b. Maximum of $f(x) = x + \frac{1}{x}$
c. Maximum of $f(x) = \frac{\sqrt{x}-\sqrt[3]{x}}{x}$

62. (4.9) Can Newton's method be used to solve $0 = e^x$? Why or why not?

Note: Chapter 4.10 is included in the chapter 5 sheet.

MTH 200

Blank Worksheet

Problems found in OpenStax Calculus Volume II

Sections:

- 4.10 - Antiderivatives
- 5.1 – Approximating Areas
- 5.2 – The Definite Integral
- 5.3 – The Fundamental Theorem of Calculus
- 5.5 - Substitution
- 5.6 – Integrals Involving Exponential and Logarithmic Functions
- 5.7 – Integrals Resulting in Inverse Trigonometric Functions

Chapter 5 and 4.10
Rev 12/3/24

1. (4.10) Find the antiderivative of the function.
   a. $f(x) = e^x - 3x^2 + \sin x$
   b. $f(x) = x - 1 + 4\sin 2x$
   c. $f(x) = x + 12x^2$
   d. $f(x) = \left(\sqrt{x}\right)^3$
   e. $f(x) = \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}}$
   f. $f(x) = \sec^2 x + 1$
   g. $f(x) = \sin^2 x \cos x$
   h. $f(x) = \frac{1}{2}\csc^2 x + \frac{1}{x^2}$
   i. $f(x) = 4\csc x \cot x - \sec x \tan x$
   j. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

2. (4.10) Evaluate the integral.
   a. $\int \frac{3x^2+2}{x^2} \cdot dx$
   b. $\int \left(4\sqrt{x} + \sqrt[4]{x}\right) \cdot dx$
   c. $\int \frac{14x^3+2x+1}{x^3} \cdot dx$

3. (4.10) Solve for $f(x)$, when $f'(x) = \cos x + \sec^2 x$ and $f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$.

4. (4.10) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerated at a constant rate of $10\,\frac{ft}{sec^2}$. How long before the car stops?

5. (5.1) Compute the sum $\sum_{i=5}^{10} i^2$.

6. (5.1) Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. Compute the sum $\sum_{i=1}^{100} 5a_i + 4b_i$.

7. (5.1) Use summation properties and formulas to rewrite and evaluate $\sum_{k=1}^{25}[(2k)^2 - 100k]$.

8. (5.1) Compute the left or right endpoint Riemann sum.
   a. R_4 for $g(x) = \cos \pi x$ on $[0, 1]$
   b. L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

9. (5.1) Express the endpoint sums in sigma notation but do not evaluate them.
   a. L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$
   b. R_{100} for $f(x) = \ln x$ on $[1, e]$

10. (5.1) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

11. (5.2) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n}(5(x_i^*)^2 - 3(x_i^*)^3)\Delta x$ over $[0, 2]$ as an integral.

12. (5.2) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\,\Delta x$ over $[0, 1]$ as an integral.

13. (5.2) Given R_n, express their limits as $n \to \infty$ as a definite integral.

a. $R_n = \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$

b. $R_n = \frac{3}{n}\sum_{i=1}^{n}\left(3 + 3\frac{i}{n}\right)$

c. $R_n = \frac{1}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right)\log\left(\left(1 + \frac{i}{n}\right)^2\right)$

14. (5.2) Evaluate the integral using area formulas.

a. $\int_0^6 (3 - |x - 3|)\, dx$

b. $\int_1^5 \sqrt{4 - (x - 3)^2}\, dx$

15. (5.2) Suppose $\int_0^4 f(x)dx = 5$, $\int_0^2 f(x)dx = -3$, $\int_0^4 g(x)dx = -1$, and $\int_0^2 g(x)dx = 2$. Evaluate the integrals.

a. $\int_2^4 \left(f(x) + g(x)\right)dx$

a. $\int_2^4 \left(f(x) - g(x)\right)dx$

b. $\int_2^4 \left(4f(x) - 3g(x)\right)dx$

16. (5.2) Find the net signed area between the function and the x-axis. $\int_2^4 (x - 3)^3 dx$. Hint: look at the graph.

17. (5.2) Find the average value f_{ave} of $f(x)$ between a and b and find a point c where $f(c) = f_{ave}$.

$$f(x) = (3 - |x|), a = -3, b = 3.$$

18. (5.3) Let $F(x) = \int_1^x (1 - t)dt$. Find $F'(2)$ and the average value of F' over $[1,2]$.

19. (5.3) Use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

a. $\frac{d}{dx}\int_1^x e^{\cos t} dt$

b. $\frac{d}{dx}\int_3^x \frac{ds}{\sqrt{16-s^2}}$

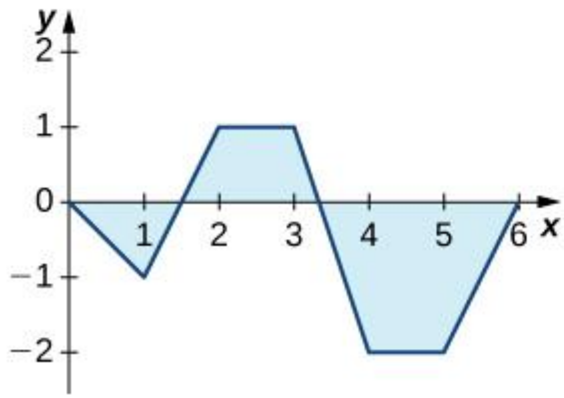
c. $\frac{d}{dx}\int_0^{\sqrt{x}} t dt$

d. $\frac{d}{dx}\int_{\cos x}^1 \sqrt{1 - t^2} dt$

e. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t} dt$

f. $\frac{d}{dx}\int_1^{e^x} \ln u^2\, du$

20. (5.3) The graph $y = \int_0^x f(t)dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
b. What are the minimum and maximum values of f?
c. What is the average value of f?

21. (5.3) Evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

a. $\int_{-2}^{3}(x^2 + 3x - 5)dx$

b. $\int_{2}^{3}(t^2 - 9)(4 - t^2)dt$

c. $\int_{0}^{1} x^{99}dx$

d. $\int_{1/4}^{4}\left(x^2 - \frac{1}{x^2}\right)dx$

e. $\int_{1}^{4}\frac{1}{2\sqrt{x}}dx$

f. $\int_{1}^{16}\frac{dt}{t^{\frac{1}{4}}}$

g. $\int_{0}^{\frac{\pi}{2}}\sin\theta\, d\theta$

h. $\int_{0}^{\frac{\pi}{4}}\sec\theta\tan\theta\, d\theta$

i. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\csc^2\theta\, d\theta$

j. $\int_{-2}^{-1}\left(\frac{1}{t^2} - \frac{1}{t^3}\right)dt$

22. (5.3) Express the integral as a function $F(x)$.

a. $\int_{1}^{x} e^t\, dt$

b. $\int_{-x}^{x}\sin t\, dt$

23. (5.3) Identify the roots of the integrand to remove absolute values. Then evaluate the definite integral.

$$\int_{-2}^{4} |t^2 - 2t - 3|dt$$

24. (5.3) Suppose the rate of gasoline consumption over the course of a year in the United States can be modeled by a sinusoidal function of the form

$$\left(11.21 - \cos\left(\frac{\pi t}{6}\right)\right) \times 10^9 \text{ gal/mo.}$$

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?
b. What is the number of gallons of gasoline consumed in the United States in a year?
c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April ($t = 3$), and the end of September ($t = 9$).

25. (5.5) Find the antiderivative using the indicated substitution.
a. $\int (x+1)^4 dx; u = x + 1$
b. $\int (2x-3)^{-7}\, dx; u = 2x - 3$
c. $\int \frac{x}{\sqrt{x^2+1}} dx; u = x^2 + 1$
d. $\int (x-1)(x^2-2x)^3 dx; u = x^2 - 2x$

26. (5.5) Determine the indefinite integral.
a. $\int x(1-x)^{99} dx$
b. $\int (11x-7)^{-3} dx$
c. $\int \cos^3\theta \sin\theta\, d\theta$
d. $\int t \sin(t^2)\cos(t^2)\, dt$
e. $\int \frac{x^2}{(x^3-3)^2} dx$

27. (5.5) Evaluate the definite integral.

$$\int_{0}^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\, d\theta$$

28. (5.6) Compute the indefinite integral.
a. $\int e^{-3x} dx$
b. $\int 3^{-x} dx$
c. $\int \frac{2}{x} dx$
d. $\int \frac{dx}{x(\ln x)^2}$
e. $\int \frac{\cos x - x\sin x}{x\cos x} dx$
f. $\int \ln(\cos x)\tan x\, dx$
g. $\int x^2 e^{-x^3} dx$

29. (5.6) Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$. Evaluate the integral.

30. (5.6) Express the trig integrals in terms of compositions with logarithms.

a. $\int \frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)} dx$

b. $\int \ln(\csc x) \cot x \, dx$

31. (5.6) Evaluate the definite integral.

$$\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$$

32. (5.6) Integrate using the indicated substitution.

a. $\int \frac{y-1}{y+1} dy \,; u = y + 1$

b. $\int \ln(x) \frac{\sqrt{1-(\ln x)^2}}{x} dx \,; u = \ln x$

33. (5.7) Evaluate the integral in terms of an inverse trig function.

a. $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$

b. $\int_{\sqrt{3}}^1 \frac{dx}{1+x^2}$

c. $\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

34. (5.7) Given the following relationship, does $\arccos t = -\arcsin t$?

$$-\arccos t + C = \int \frac{dt}{\sqrt{1 - t^2}} = \arcsin t + C$$

35. (5.7) What is wrong with this integral?

$$\int_1^2 \frac{dt}{\sqrt{1 - t^2}}$$

36. (5.7) Find the antiderivative using substitutions.

a. $\int \frac{\arcsin t}{\sqrt{1-t^2}} dt$

b. $\int \frac{\arctan(2t)}{1+4t^2} dt$

c. $\int \frac{\operatorname{arcsec}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}} dt$

d. $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

37. (5.7) Evaluate the definite integral.

a. $\int_0^{\frac{1}{2}} \frac{\tan(\arcsin t)}{\sqrt{1-t^2}} dt$

b. $\int_0^{\frac{1}{2}} \frac{\sin(\arctan t)}{1+t^2} dt$

MTH 200

Blank Worksheet

Problems found in OpenStax Calculus Volume II

Sections:

- 2.1 – A Preview of Calculus
- 2.2 – The Limit of a Function
- 2.3 – The Limit Laws
- 2.4 – Continuity

1. (2.1)

a. $P = (1,2), Q = (x, y), f(x) = x^2 + 1$

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{Q(x,y)}$	$\boldsymbol{m_{\text{sec}}}$ **($\boldsymbol{P}$ to $\boldsymbol{Q}$)**
1.1	a. 2.2100000	e. (1.1000000, 2.2100000)	i. 2.1000000
1.01	b. 2.0201000	f. (1.0100000, 2.0201000)	j. 2.0100000
1.001	c. 2.0020010	g. (1.0010000, 2.0020010)	k. 2.0010000
1.0001	d. 2.0002000	h. (1.0001000, 2.0002000)	l. 2.0001000

Formula for m_{sec}:

$$m_{sec} = \frac{Q_y - 2}{Q_x - 1}$$

b. Guess the slope of the tangent line to f at $x = 1$.

As Q gets infinitely closer to P, the slope of the secant line becomes the slope of the tangent line. From the graph, m_{sec} clearly approaches 2.

c. Find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

We know that at $x = 1$ the slope of the tangent line is 2, but we still need a y-intercept. We will let $y = 0, x = 1, m = 2$ and see what b is.

$$y = mx + b$$

$$0 = 2 \times 1 + b, b = \frac{0}{2} = 0$$

Final equation is $y = 2x$.

2. (2.1) Consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$

Compute the average velocity of the stone over the given time intervals.

a. $[1,1.05]$

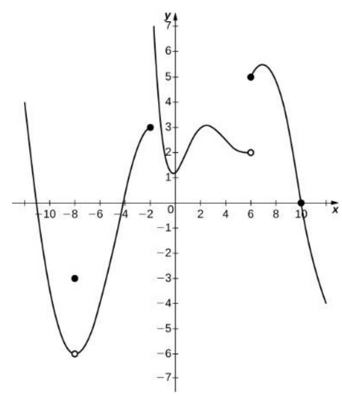
$$\frac{h(1.05) - h(1)}{1.05 - 1} = \frac{10.35 - 10.1}{0.05} = 4.995$$

b. $[1,1.01]$ 5.151
c. $[1,1.005]$ 5.176
d. $[1,1.001]$ 5.195

Guess the instantaneous velocity of the stone at t=1 sec.

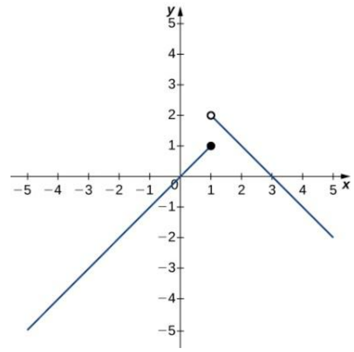
The numbers are approaching 5.2 as the interval gets smaller.

3. (2.2) Look at the graph and classify the following statements as true or false. If false, explain and find the correct limit.



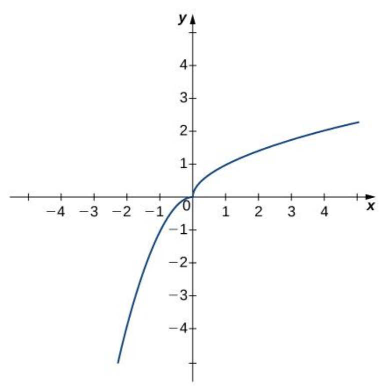
a. $\lim_{x \to 10} f(x) = 0$ – true
b. $\lim_{x \to -2^+} f(x) = 3$ – false. From the right, limit approaches ∞
c. $\lim_{x \to -8} f(x) = f(-8)$ – false. The limit means it doesn't touch the value the function approaches. The function approaches -6 but its value is different.
d. $\lim_{x \to 6} f(x) = 5$ – false. The limit does not exist because the left limit does not equal the right limit.

4. (2.2) Given the graph $y = f(x)$, determine the following values.



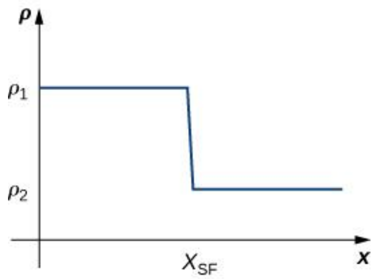
a. $\lim_{x \to 1^-} f(x)$ 1
b. $\lim_{x \to 1^+} f(x)$ 2
c. $\lim_{x \to 1} f(x)$ DNE
d. $\lim_{x \to 2} f(x)$ 1
e. $f(1)$ 1

5. (2.2) Given the graph $y = h(x)$, determine the following values.



a. $\lim_{x \to 0^-} h(x)$ 0
b. $\lim_{x \to 0^+} h(x)$ 0
f. $\lim_{x \to 0} h(x)$ 0

6. (2.2) Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x, is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



c. Evaluate $\lim_{x \to X_{SF}^+} \rho(x)$ ρ_2

d. Evaluate $\lim_{x \to X_{SF}^-} \rho(x)$ ρ_1

e. Evaluate $\lim_{x \to X_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

DNE unless $\rho_1 = \rho_2$. Approaching X_{SF} from left is the high-density area of the shock. From the right is a lower shock density.

7. (2.2) A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{x \to 2} x(t)$? What does it mean physically?

t (sec)	*x* (m)
1.75	4.5
1.95	6.1
1.99	6.42
2.01	6.58
2.05	6.9
2.25	8.5

$$\lim_{x \to 2} x(t) = 6.5$$

The runner is at the 6-meter position after 2 seconds.

8. (2.3) Evaluate $\lim_{x \to 0}(4x^2 - 2x + 3)$.

Since all polynomials exist on $(-\infty, \infty)$, the limit can be calculated by direct substitution with 0.

$$\lim_{x \to 0}(4x^2 - 2x + 3) = 0x^2 - 2(0) + 3 = 3$$

9. (2.3) Evaluate $\lim_{x \to -2} \sqrt{x^2 - 6x + 3}$.

Using the root law, $\lim_{x \to -2} \sqrt{x^2 - 6x + 3} = \sqrt{\lim_{x \to -2} x^2 - 6x + 3} = \sqrt{(-2)^2 - 6(-2) + 3} =$ $\sqrt{19}$

10. (2.3) Evaluate $\lim_{x \to 0} \frac{1}{1+\sin x}$.

This function does exist at $x = 0$, so direct substitution can be used.

$$\lim_{x \to 0} \frac{1}{1 + \sin x} = \frac{1}{1 + \sin 0} = 1$$

11. (2.3) Evaluate $\lim_{x \to 4} \frac{x^2-16}{x-4}$.

Direct substitution in this function returns indeterminate form.

$$\frac{4^2 - 16}{4 - 4} = \frac{0}{0}$$

But the numerator can be factored so that a term cancels with the denominator.

$$\frac{x^2 - 16}{x - 4} = \frac{(x+4)(x-4)}{x - 4} = x + 4$$

$$\lim_{x \to 4} (x + 4) = 8$$

12. (2.3) Evaluate $\lim_{x \to 6} \frac{3x-18}{2x-12}$.

Direct substitution in this function returns indeterminate form.

$$\frac{3(6) - 18}{2(6) - 12} = \frac{0}{0}$$

Both the numerator and denominator can be factored to cancel terms.

$$\frac{3x - 18}{2x - 12} = \frac{3(x-6)}{2(x-6)} = \frac{3}{2}$$

$$\lim_{x \to 6} \frac{3}{2} = \frac{3}{2}$$

13. (2.3) Evaluate $\lim_{t \to 9} \frac{t-9}{\sqrt{t}-3}$.

Indeterminate form. $\frac{9-9}{\sqrt{9}-3} = \frac{0}{0}$

Multiply by the conjugate to remove square root.

$$\frac{t - 9}{\sqrt{t} - 3} \cdot \frac{\sqrt{t}+3}{\sqrt{t}+3} = \frac{(t-9)(\sqrt{t}+3)}{t - 9} = \sqrt{t} + 3$$

$$\lim_{x \to 9} \sqrt{t} + 3 = \sqrt{9} + 3 = 6$$

14. (2.3) Evaluate $\lim_{\theta\to\pi}\frac{\sin\theta}{\tan\theta}$.

$\sin\pi$ and $\tan\pi$ are both 0, so it's indeterminate form. Since there are ratios of trig functions, rewrite them into a simpler form.

$$\frac{\sin\theta}{\tan\theta}=\frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}=\frac{\sin\theta}{1}\cdot\frac{\cos\theta}{\sin\theta}=\cos\theta$$

$$\lim_{\theta\to\pi}\cos\theta=\cos\pi=-1$$

15. (2.3) Evaluate $\lim_{x\to 1^-}\frac{2x^2+7x-4}{x^2+x-2}$.

We don't have indeterminate form, but the denominator is equal to 0. Start by factoring the numerator and denominator.

$$\frac{2x^2+7x-4}{x^2+x-2}=\frac{(2x-1)(x+4)}{(x-1)(x+2)}$$

Direct substitution with $x=1$ makes the denominator 1 with the term $(x-1)$, so it will be separated.

$$\lim_{x\to 1^-}\frac{1}{x-1}\cdot\frac{(2x-1)(x+4)}{x+2}$$

$$\lim_{x\to 1^-}\frac{1}{x-1}=-\infty \text{ and } \lim_{x\to 1^-}\frac{(2x-1)(x+4)}{(x+2)}=\frac{5}{3}$$

$$-\infty\cdot\frac{5}{3}=-\infty$$

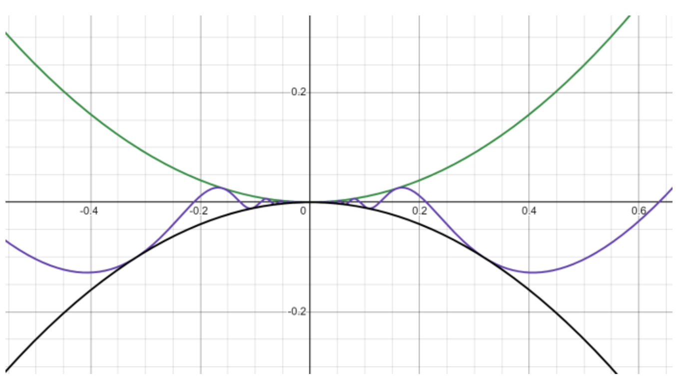
16. (2.3) If $\lim_{x\to 6}f(x)=4$ and $\lim_{x\to 6}f(x)=9$, Evaluate $\lim_{x\to 6}\sqrt{g(x)-f(x)}$.

$$\lim_{x\to 6}\sqrt{g(x)-f(x)}=\sqrt{\lim_{x\to 6}g(x)-f(x)}=\sqrt{\lim_{x\to 6}g(x)-\lim_{x\to 6}f(x)}=\sqrt{9-4}=\sqrt{5}$$

17. (2.3) Evaluate $\lim_{\theta\to 0}\theta^2\cos\frac{1}{\theta}$.

Cosine has a range of $(-1,1)$, so $\theta^2(-1)<\theta^2\cos\frac{1}{\theta}<\theta^2(1)$. If the limits at $\theta=0$ for both outer functions are the same, then the limit for the given function is the same.

$$\lim_{\theta\to 0}-\theta^2=0 \; and \; \lim_{\theta\to 0}\theta^2=0 \;\therefore\; \lim_{\theta\to 0}\theta^2\cos\frac{1}{\theta}=0$$



18. (2.4) When are the functions discontinuous? Classify the discontinuity.

a. $f(x) = \frac{1}{\sqrt{x}}$
The function is continuous on its domain $(0, \infty)$. It is not discontinuous at $x = 0$ since it is not part of the domain.

b. $f(x) = \frac{x}{x^2 - x}$
The function is discontinuous at $x = 0$ and $x = 1$. Factoring an x on the numerator and denominator makes the limit easy to calculate. $\frac{x}{x^2-x} = \frac{1}{x-1}$
At $x = 0$, $\lim_{x \to 0} \frac{1}{x-1} = \frac{1}{-1} = -1$. Since the function is discontinuous but the limit exists, there is a removable discontinuity at $x = 0$.
At $x = 1$, $\lim_{x \to 1} \frac{1}{x-1} = \infty$ since the denominator is approaching 0 and the function is getting larger. Since the limit is ∞, the function has an infinite discontinuity at $x = 1$.

c. $f(x) = \frac{5}{e^x - 2}$
The function is discontinuous at $x = \ln 2$. $\lim_{x \to \ln 2} \frac{5}{e^x - 2} = \infty$ so the function has a removable discontinuity at $x = 2$.

d. $f(x) = \tan 2x$
Since the domain of $\tan x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the domain of $\tan 2x$ is $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ to make $2x$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It would only be discontinuous for odd multiples of $\frac{\pi}{4}$ $\left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, etc.\right)$. The tangent function always has infinite discontinuities, so the function has infinite discontinuities at $\frac{(2k+1)\pi}{4}$, where k is an integer.

19. (2.4) Is the function discontinuous? If so, classify the discontinuity.

a. $f(x) = \frac{2x^2 - 5x + 3}{x - 1}$, at $x = 1$
The function is not continuous at $x = 1$. To see if the limit exists, factor the quadratic and cancel terms.

$$\frac{2x^2 - 5x + 3}{x - 1} = \frac{(2x - 3)(x - 1)}{x - 1} = 2x - 3$$
$$\lim_{x \to 1} 2x - 3 = -1$$

Since the function is discontinuous but the limit exists, there is a removable discontinuity at $x = 1$.

b. $f(x) = \begin{cases} \frac{6x^2+x-2}{2x-1}, x \neq \frac{1}{2} \\ \frac{7}{2}, x = \frac{1}{2} \end{cases}$, at $x = \frac{1}{2}$

The function exists at $x = \frac{1}{2}$, so the first condition is satisfied.

$$\lim_{x \to \frac{1}{2}} f(x) = \lim_{x \to \frac{1}{2}} \frac{6x^2 + x - 2}{2x - 1} = \lim_{x \to \frac{1}{2}} \frac{(3x + 2)(2x - 1)}{2x - 1} = \lim_{x \to \frac{1}{2}} 3x + 2 = \frac{3}{2} + 2 = \frac{7}{2}$$

The limit exists, so the second condition is satisfied.

The function equals the limit, so the third condition is satisfied, and the function is continuous.

c. $f(x) = \frac{\sin(\pi x)}{\tan(\pi x)}$, at $x = 1$

The function is discontinuous at $x = 1$ since the function does not exist because the denominator $\tan \pi = 0$. To find the limit, simplify the function.

$$\frac{\sin \pi x}{\tan \pi x} = \frac{\sin \pi x}{\frac{\sin \pi x}{\cos \pi x}} = \cos \pi x$$

$$\lim_{x \to 1} \cos \pi x = \cos \pi = -1$$

Since the limit exists and does not equal ∞ but the function value does not exist, there is a removable discontinuity at $x = 1$.

d. $f(x) = \begin{cases} x^2 - e^x, x < 0 \\ x - 1, x \geq 0 \end{cases}$, at $x = 0$

The function value $f(0) = 0 - 1 = -1$.

$$\lim_{x \to 0^-} f(x) = 0^2 - e^0 = -1$$

$$\lim_{x \to 0^+} f(x) = 0 - 1 = -1$$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = -1$, the function is continuous.

e. $f(x) = \begin{cases} x \sin x, x \leq \pi \\ x \tan x, x > \pi \end{cases}$, at $x = \pi$

$$f(\pi) = \pi \sin \pi = 0$$

$$\lim_{x \to \pi^-} f(x) = \pi \sin \pi = 0$$

$$\lim_{x \to \pi^+} f(x) = \pi \tan \pi = 0$$

Since $\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi) = 0$, the function is continuous.

20. (2.4) What value of k will make the function continuous? $f(\theta) = \begin{cases} \sin\theta, 0 \le \theta < \frac{\pi}{2} \\ \cos(\theta + k), \frac{\pi}{2} \le \theta \le \pi \end{cases}$

Sine and cosine are both continuous always. The point to test for continuity is $\frac{\pi}{2}$. The function value on the left side must equal the function value on the right side so that the limits on both sides equal each other and so do the function values.

$$\sin\theta = \cos(\theta + k)$$
$$\sin\frac{\pi}{2} = \cos\left(\frac{\pi}{2} + k\right)$$
$$\cos\left(\frac{\pi}{2} + k\right) = 1$$
$$\arccos\left(\cos\left(\frac{\pi}{2} + k\right)\right) = \arccos 1$$
$$\frac{\pi}{2} + k = 0$$
$$k = -\frac{\pi}{2}$$

21. (2.4) What value of k will make the function continuous? $f(x) = \begin{cases} \frac{x^2+3x+2}{x+2}, x \neq -2 \\ k, x = -2 \end{cases}$

The top function will look like a line with a hole at $x = -2$. The bottom function provides a value at $x = -2$. For the whole function to be continuous, the value at $x = -2$ needs to match what the top function would be at $x = -2$ without the hole. To do this, we could factor the top part and cancel an $x + 2$ to get the equation of the line.

$$\frac{x^2 + 3x + 2}{x + 2} = \frac{(x + 1)(x + 2)}{x + 2} = x + 1$$

Without the hole, the line looks like $y = x + 1$. Now we plug in $x = -2$ into this equation to get the k value.

$$k = -2 + 1 = -1$$
$$k = -1$$

22. (2.4) Let $h(x) = \begin{cases} 3x^2 - 4, x \le 2 \\ 5 + 4x, x > 2 \end{cases}$. Over the interval $[0,4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

The IVT requires continuity to be valid, but $h(x)$ is not continuous because the function is not continuous at $x = 2$.

$$3(2)^2 - 4 = 8 \neq 5 + 4(2) = 13$$

23. (2.4) Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

$$2^x = x^3$$
$$2^x - x^3 = 0$$

$2^x - x^3$ is continuous, so when applied at the endpoints of an interval, the IVT guarantees that $2^x - x^3 = 0$ when $a \le 0 \le b$, or when a and b have opposite signs.

For $[1.25, 1.375]$:

$$2^{1.25} - 1.25^3 \approx 0.425$$
$$2^{1.375} - 1.375^3 \approx -0.006$$

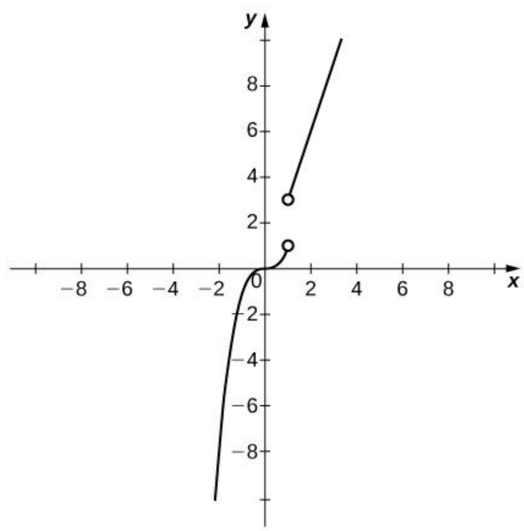
For $[1.375, 1.5]$:

$$2^{1.375} - 1.375^3 \approx -0.006$$
$$2^{1.5} - 1.5^3 \approx -0.547$$

$2^x = x^3$ has a solution in $[1.375, 1.5]$.

24. (2.4) Let $f(x) = \begin{cases} 3x, x > 1 \\ x^3, x < 1 \end{cases}$. Graph the function and determine if it is possible to find a value k such that $f(1) = k$ which makes $f(x)$ continuous for all real numbers? Explain.

No. There is a jump discontinuity, so one function value at $x = 1$ will not make the whole discontinuous part continuous.



25. (2.4) Are the following statements true? Explain.

a. $f(t) = \frac{2}{e^t - e^{-t}}$ is continuous everywhere.

False. It is not continuous at $t = 0$. $f(0) = \frac{2}{e^0 - e^{-0}} = \frac{2}{1-1} = \frac{2}{0}$

b. If a function is not continuous at a point, then it is not defined at that point.

False. A function can be defined but not continuous at a point. Some piecewise functions are an example of this.

MTH 200

Worksheet Answers

Problems found in OpenStax Calculus Volume II

Sections:

- 3.1 – Defining the Derivative
- 3.2 – The Derivative as a Function
- 3.3 – Differentiation Rules
- 3.5 – Derivatives of Trigonometric Functions
- 3.6 – The Chain Rule
- 3.7 – Derivatives of Inverse Functions
- 3.8 – Implicit Differentiation
- 3.9 – Derivatives of Exponential and Logarithmic Functions

1. (3.1) Find the slope of the secant line between the values x_1 and x_2 for the following functions.
   a. $f(x) = 4x + 7; x_1 = 2, x_2 = 5$
$$Q(x) = \frac{f(x) - f(a)}{x - a}$$
$$Q(2) = \frac{f(2) - f(5)}{2 - 5} = \frac{15 - 27}{-3} = 4$$
   b. $f(x) = x^2 + 2x + 1; x_1 = 3, x_2 = 3.5$
$$Q(x) = \frac{f(x) - f(a)}{x - a}$$
$$Q(3) = \frac{f(3) - f(3.5)}{3 - 3.5} = \frac{16 - 20.25}{-0.5} = 8.5$$
   c. $f(x) = x^{\frac{1}{3}} + 1; x_1 = 0, x_2 = 8$
$$(x) = \frac{f(x) - f(a)}{x - a}$$
$$Q(0) = \frac{f(0) - f(8)}{0 - 8} = \frac{1 - 3}{-8} = \frac{1}{4}$$

2. (3.1) Find the equation of the tangent line at the given value for the following functions using the definition of a derivative.
   a. $f(x) = 3 - 4x; a = 2$
   This function is already a line so the slope and y-intercept are given. The y-intercept is 3.
   Solving for the slope of the tangent line using the definition:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{3 - 4x + 5}{x - 2} = \lim_{x \to 2} \frac{-4x + 8}{x - 2} = \lim_{x \to 2} \frac{-4(x - 2)}{x - 2}$$
$$\lim_{x \to 2} -4 = -4$$
   Equation of tangent line at $x = 2$ is $y = -4x + 3$.
   b. $f(x) = x^2 + x; a = 1$
   Using the definition of the derivative at a point, factor the quadratic and cancel terms:
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{x - 1}$$
$$\lim_{x \to 1} x + 2 = 1 + 2 = 3$$
   To find the equation of the tangent line, use point slope form at $x = 1$.
$$y - y_1 = m(x - x_1)$$
$$y - 2 = 3(x - 1) = 3x - 3$$
$$y = 3x - 1$$

c. $f(x) = \frac{7}{x}; a = 3$

Using another definition of the derivative at a point, rewrite the resulting complex fraction and cancel the h so the limit can be evaluated.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(3) = \lim_{h\to 0} \frac{\frac{7}{3+h} - \frac{7}{3}}{h} = \lim_{h\to 0} \frac{\frac{21}{9+3h} - \frac{21+7h}{9+3h}}{h} = \lim_{h\to 0} \frac{-7h}{(9+3h)h} = \lim_{h\to 0} -\frac{7}{9+3h}$$

$$= -\frac{7}{9+3(0)} = -\frac{7}{9}$$

Equation from point-slope form and the point $\left(3, \frac{7}{3}\right)$:

$$y - \frac{7}{3} = -\frac{7}{9}(x-3)$$

$$y - \frac{7}{3} = -\frac{7}{9}x + \frac{7}{3}$$

$$y = -\frac{7}{9}x + \frac{14}{3}$$

d. $f(x) = 2 - 3x^2; a = -2$

Using the definition of the derivative at a point, simplify the polynomial and cancel an h so the limit can be evaluated.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(-2) = \lim_{h\to -0} \frac{2 - 3(h-2)^2 + 10}{h} = \lim_{h\to 0} \frac{12 - 3(h^2 - 4h + 4)}{h}$$

$$= \lim_{h\to 0} \frac{12 - 3h^2 + 12h - 12}{h} = \lim_{h\to 0} \frac{-3h^2 + 12h}{h} = \lim_{h\to 0} -3h + 12$$

$$= 12$$

Equation of tangent line using point-slope form:

$$y - (-10) = 12(x+2)$$

$$y + 10 = 12x + 24$$

$$y = 12x + 14$$

e. $f(x) = \frac{2}{x+3}; a = -4$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(-4) = \lim_{h\to 0} \frac{\frac{2}{h-4+3} + 2}{h} = \lim_{h\to 0} \frac{\frac{2}{h-1} + \frac{2}{1}}{h} = \lim_{h\to 0} \frac{\frac{2}{h-1} + \frac{2h-2}{h-1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2h}{h-1}}{h} = \lim_{h\to 0} \frac{2}{h-1} = \frac{2}{0-1} = -2$$

(problem continues on next page)

Equation of tangent line from point-slope form:

$$y - (-2) = -2(x + 4)$$
$$y + 4 = -2x - 6$$
$$y = -2x - 10$$

3. (3.1) Find $f'(6)$ for the function $f(x) = \sqrt{x - 2}$.

   Start by setting up the derivative at a point.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(6) = \lim_{x \to 6} \frac{\sqrt{x-2} - 2}{x - 6}$$

   Now multiply by the conjugate to remove the root.

$$\lim_{x \to 6} \frac{\sqrt{x-2} - 2}{x - 6} \cdot \frac{\sqrt{x-2} + 2}{\sqrt{x-2} + 2} = \lim_{x \to 6} \frac{x - 2 - 4}{(x-6)\left(\sqrt{x-2} + 2\right)} = \lim_{x \to 6} \frac{x - 6}{(x-6)\left(\sqrt{x-2} + 2\right)}$$
$$= \lim_{x \to 6} \frac{1}{\sqrt{x-2} + 2} = \frac{1}{4}$$
$$f'(6) = \frac{1}{4}$$

4. (3.1) Find $f'(1)$ for the function $f(x) = \frac{1}{x^3}$.

   Using the "h-way" while direct substituting $x = 1$. This is allowed because we are not plugging in the variable of the limit, h, but rather x, which does not change in the context of the limit.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$\lim_{h \to 0} \frac{\frac{1}{(1+h)^3} - \frac{1}{1}}{h}$$
$$\lim_{h \to 0} \left[\left(\frac{1}{h^3 + 3h^2 + 3h + 1} - \frac{1}{1}\right) \cdot \frac{1}{h}\right]$$

   Getting a common denominator, then subtracting the fractions.

$$\lim_{h \to 0} \left[\left(\frac{1}{h^3 + 3h^2 + 3h + 1} - \frac{h^3 + 3h^2 + 3h + 1}{h^3 + 3h^2 + 3h + 1}\right) \cdot \frac{1}{h}\right]$$
$$\lim_{h \to 0} \left[\frac{1 - h^3 - 3h^2 - 3h}{h^3 + 3h^2 + 3h + 1} \cdot \frac{1}{h}\right]$$

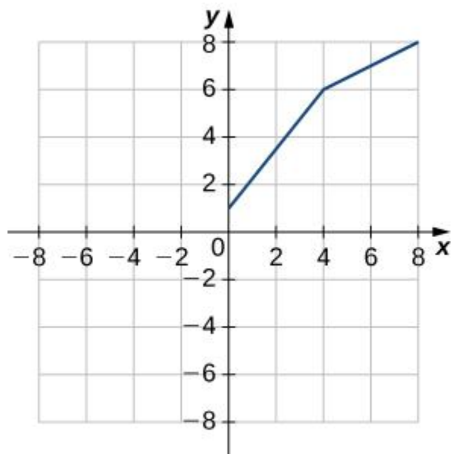
   Now every term in the numerator has an h that can be factored and canceled with the $\frac{1}{h}$.

$$\lim_{h \to 0} \left[\frac{h(-h^2 - 3h - 3)}{h^3 + 3h^2 + 3h + 1} \cdot \frac{1}{h}\right]$$

$$\lim_{h\to 0}\frac{-h^2-3h-3}{h^3+3h^2+3h+1}=-\frac{3}{1}$$
$$f'(1)=-3$$

You can verify this with the power rule.

5. (3.1) Evaluate $f'(1)$ and $f'(6)$.



$$f'(1)=1.25$$
$$f'(6)=0.5$$

6. (3.1) Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle travelling x miles. Suppose the vehicle gets 30 mpg.
   a. Find a mathematical expression for $N(x)$.
   Since the car always gets 30 mpg, use stoichiometry to convert miles into gallons. Dividing x miles by 30 miles cancels miles and results in a unit of gallons.
$$x\ miles\times\frac{1\ gal}{30\ mi}=\frac{x}{30}\ gallons$$
$$N(x)=\frac{x}{30}$$
   b. What is $N(100)$? Explain the physical meaning.
$$N(100)=\frac{100}{30}=\frac{10}{3}\approx 3.33$$
   The car uses about 3.33 gallons of gas to travel 100 miles.
   c. What is $N'(100)$? Explain the physical meaning.
$$N'(a)=\frac{N(x)-N(a)}{x-a}$$
$$N'(100)=\lim_{x\to 100}\frac{\frac{x}{30}-\frac{100}{30}}{x-100}=\lim_{x\to 100}\frac{\frac{x-100}{30}}{x-100}=\lim_{x\to 100}\frac{x-100}{30(x-100)}=\lim_{x\to 0}\frac{1}{30}$$
$$=\frac{1}{30}\frac{gal}{mi}$$
   After travelling 100 miles, the car's rate of fuel consumption is $\frac{1}{30}\frac{gal}{mi}$.

7. (3.2) Use the definition of the derivative to find $f'(x)$.

Always start with $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

a. $f(x) = 2 - 3x$

$$f'(x) = \lim_{h\to 0} \frac{\left(2 - 3(x+h)\right) - (2 - 3x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{2 - 3x - 3h - 2 + 3x}{h} = \lim_{h\to 0} \frac{-3h}{h} = -3$$

$$f'(x) = -3$$

b. $f(x) = 4x^2$

$$f'(x) = \lim_{h\to 0} \frac{(4(x+h)^2) - (4x^2)}{h} = \lim_{h\to 0} \frac{4(x^2 + 2hx + h^2) - 4x^2}{h}$$

$$\lim_{h\to 0} \frac{4x^2 + 8hx + 4h^2 - 4x^2}{h} = \lim_{h\to 0} \frac{8hx + 4h^2}{h} = \lim_{h\to 0} \frac{h(8x + 4h)}{h} = \lim_{h\to 0} 8x + 4h$$

$$f'(x) = \lim_{h\to 0} 8x + 4h = 8x$$

c. $f(x) = \sqrt{2x}$

$$f'(x) = \lim_{h\to 0} \frac{\left(\sqrt{2(x+h)}\right) - \left(\sqrt{2x}\right)}{h} = \lim_{h\to 0} \frac{\sqrt{2x+2h} - \sqrt{2x}}{h}$$

$$\lim_{h\to 0} \frac{\sqrt{2x+2h} - \sqrt{2x}}{h} \cdot \frac{\sqrt{2x+2h} + \sqrt{2x}}{\sqrt{2x+2h} + \sqrt{2x}} = \lim_{h\to 0} \frac{2x + 2h - 2x}{h\left(\sqrt{2x+2h} + \sqrt{2x}\right)}$$

$$\lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h} + \sqrt{2x}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h} + \sqrt{2x}} = \frac{2}{\sqrt{2x + 2(0)} + \sqrt{2x}} = \frac{2}{2\sqrt{2x}}$$

$$f'(x) = \frac{1}{\sqrt{2x}}$$

d. $f(x) = \frac{9}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{9}{x+h}\right) - \left(\frac{9}{x}\right)}{h} = \lim_{h\to 0} \frac{\frac{9x}{x(x+h)} - \frac{9(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{9x - 9x - 9h}{x(x+h)}}{h}$$

$$\lim_{h\to 0} \frac{\frac{-9h}{x^2 + xh}}{h} = \lim_{h\to 0} \left[\frac{-9h}{x^2 + xh} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \frac{-9}{x^2 + xh} = \frac{-9}{x^2}$$

e. $f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{\left(\frac{1}{\sqrt{x+h}}\right) - \left(\frac{1}{\sqrt{x}}\right)}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x}}{\sqrt{x}\cdot\sqrt{x+h}} - \frac{\sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}}}{h}$$

$$\lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\cdot\sqrt{x+h}} \cdot \frac{\left(\sqrt{x} + \sqrt{x+h}\right)}{\left(\sqrt{x} + \sqrt{x+h}\right)}}{h}$$

$$\lim_{h\to 0}\frac{\dfrac{x-(x+h)}{\left(\sqrt{x}\cdot\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}}{h}=-\frac{1}{x\cdot\sqrt{2x}}$$

$$f'(x)=-\frac{1}{2x^{\frac{3}{2}}}$$

8. (3.2) The limit represents the derivative of a function $f(x)$ at $x=a$. Find $f(x)$ and a.
   Look for terms of the form $a+h$. These will become x.
   a. $\lim_{h\to 0}\frac{[3(2+h)^2+2]-14}{h}$

   $$f(x)=3x^2+2$$
   $$a=2$$

   b. $\lim_{h\to 0}\frac{(2+h)^4-16}{h}$

   $$f(x)=x^4$$
   $$a=2$$

   c. $\lim_{h\to 0}\frac{e^h-1}{h}$
   This doesn't have an $a+h$ form, so a is assumed to be 0.

   $$f(x)=e^x$$
   $$a=0$$

9. (3.2) Find the second derivative using the limit definition of derivatives.
   a. $f(x)=2-3x$

   $$f'(x)=\lim_{h\to 0}\frac{2-3(x+h)-(2-3x)}{h}=\lim_{h\to 0}\frac{2-3(x+h)-2+3x}{h}$$
   $$=\lim_{h\to 0}\frac{2-3(x+h)-2+3x}{h}=\lim_{h\to 0}\frac{-3h}{h}=-3$$
   $$f''(x)=\lim_{h\to 0}\frac{3-3}{h}=0$$

   b. $f(x)=x+\frac{1}{x}$
   Set up the limit definition of a derivative.

   $$f'(x)=\lim_{h\to 0}\frac{x+h+\dfrac{1}{x+h}-x-\dfrac{1}{x}}{h}=\lim_{h\to 0}\frac{h+\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$$

   Get a common denominator between $\frac{1}{x+h}$ and $\frac{1}{x}$.

   $$=\lim_{h\to 0}\frac{h+\dfrac{x}{x(x+h)}-\dfrac{x+h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{h+\dfrac{x-x-h}{x(x+h)}}{h}$$
   $$=\lim_{h\to 0}\frac{h+\dfrac{-h}{x^2+hx}}{h}=\lim_{h\to 0}\frac{1}{h}\left(h+\frac{-h}{x^2+hx}\right)=1-\frac{1}{x^2}$$

   Now set up the limit definition of the derivative to find the second derivative.

   $$f''(x)=\lim_{h\to 0}\frac{1-\dfrac{1}{(x+h)^2}-1+\dfrac{1}{x^2}}{h}=\lim_{h\to 0}\frac{\dfrac{1}{x^2}-\dfrac{1}{(x+h)^2}}{h}$$

Get a common denominator again, and factor out h to cancel with the one in the denominator.

$$= \lim_{h\to 0} \frac{\frac{(x+h)^2}{x^2(x+h)^2} - \frac{x^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{(x+h)^2 - x^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2+2xh+h^2-x^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{h(2x+h)}{x^2(x+h)^2}\right) = \lim_{h\to 0} \frac{(2x+h)}{x^2(x+h)^2} = \frac{2x}{x^2\cdot x^2} = \frac{2}{x^3}$$

10. (3.2) $C(x)$ denotes the average cost of college tuition x years after 2000. What do the following expressions represent.
    a. $\frac{C(x+h)-C(x)}{h}$
    The average rate of change of college tuition between two different years.
    b. $\lim\limits_{h\to 0} \frac{C(10+h)-C(10)}{h}$
    The instantaneous rate of change of college tuition in 2010.
11. (3.3) Find $f'(x)$.
    a. $f(x) = 8x^4 + 9x^2 - 1$
    This is a polynomial, so we can use the power rule only.
    $$f'(x) = 32x^3 + 18x$$
    b. $f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right)$
    We need to use multiple derivative rules to solve this problem. We need the product rule because of the product of $3x$ and $18x^4 + \frac{13}{x+1}$. We need power rule for $18x^4$ and we need quotient rule for $\frac{13}{x+1}$.
    If you want to avoid power rule you can distribute the $3x$, but the quotient rule will be harder:
    $$3x\left(18x^4 + \frac{13}{x+1}\right) = 54x^5 + \frac{39x}{x+1}$$
    $$\frac{d}{dx}\left[54x^4 + \frac{39x}{x+1}\right] = 270x^4 + \frac{d}{dx}\left[\frac{39x}{x+1}\right]$$
    $$= 270x^4 + \frac{(x+1)(39) - (39x)(1)}{(x+1)^2}$$
    $$= 270x^4 + \frac{39x + 39 - 39x}{(x+1)^2}$$
    $$= 270x^4 + \frac{39}{(x+1)^2}$$

c. $f(x) = \frac{4x^3-2x+1}{x^2}$

This uses quotient rule and power rule.

$$\frac{d}{dx}\left[\frac{4x^3-2x+1}{x^2}\right] = \frac{(x^2)(12x^2-2)-(4x^3-2x+1)(2x)}{(x^2)^2}$$
$$= \frac{12x^4-2x^2-8x^4+4x^2-2x}{x^4}$$
$$= \frac{4x^4+2x^2-2x}{x^4}$$

12. (3.3) Find the equation of the tangent line to $y = \frac{2}{x^2} + 1$ at $(1, 3)$.

All we need to make a line is a point and a slope. We have a point, so we need the slope. Take the derivative. I will turn $\frac{2}{x^2}$ into $2x^{-2}$ so I can use power rule instead of quotient rule.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{2}{x^2}+1\right] = \frac{d}{dx}\left[2x^{-2}+1\right]$$
$$= -4x^{-3} + 0$$
$$= -\frac{4}{x^3}$$

To find the slope of the line at this specific point, plug the x-value into the derivative.

$$\frac{dy}{dx}|_{x=1} = -\frac{4}{1^3} = -4$$

Now plug the point and the slope into point-slope form of a line. Then convert to slope-intercept form.

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -4(x-1)$$
$$y - 3 = -4x + 4$$
$$y = -4x + 7$$

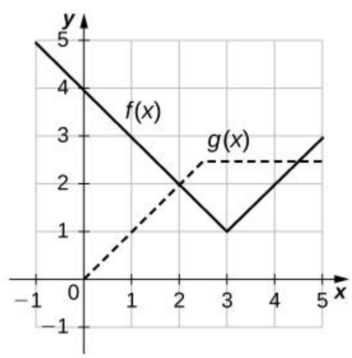
You can graph the original equation, the point, and the line equation to see if the answer makes sense.

13. (3.3) Find the equation of the tangent line to $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$.

Just like the previous problem, take the derivative, plug in the point, and make the equation of the line.

$$\frac{d}{dx}\left[\frac{2}{x} - \frac{3}{x^2}\right] = \frac{d}{dx}\left[2x^{-1} - 3x^{-2}\right]$$
$$= -\frac{2}{x^2} + \frac{6}{x^3}$$
$$\frac{dy}{dx}|_{x=1} = -2 + 6 = 4$$
$$y - y_1 = m(x - x_1)$$
$$y - (-1) = 4(x-1)$$
$$y + 1 = 4x - 4$$
$$y = 4x - 5$$

14. (3.3) If $f(x)$ and $g(x)$ are differentiable functions, find $h'(x)$.
   a. $h(x) = x^3 f(x)$
      Use the product rule to take the derivative.
      $$h'(x) = x^3 f'(x) + f(x)(3x^2)$$
   b. $h(x) = \frac{3f(x)}{g(x)+2}$
      Use the quotient rule to take the derivative.
      $$h'(x) = \frac{(g(x)+2)(3f'(x)) - (3f(x))(g'(x))}{(g(x)+2)^2}$$
15. (3.3) If $h(x) = f(x)g(x)$, use the following graph to find:
   For all three problems we need to find $h'(x)$ using the product rule:
   $$h'(x) = f(x)g'(x) + g(x)f'(x)$$
   a. $h'(1)$
   $$h'(1) = f(1)g'(1) + g(1)f'(1)$$
   $$= 3(1) + 1(-1)$$
   $$= 3 - 1 = 2$$
   b. $h'(3)$
   $$h'(3) = f(3)g'(3) + g(3)f'(3)$$
   $$= 1(0) + 2.5(DNE)$$
      $h'(3)$ does not exist because f is not differentiable at $x = 3$.
   c. $h'(4)$
   $$h'(1) = f(1)g'(1) + g(1)f'(1)$$
   $$= 3(1) + 1(-1)$$
   $$= 3 - 1 = 2$$



16. (3.3) Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x-intercept of 6.
   Since we need a point, we will use point-slope form:
   $$y - y_1 = m(x - x_1)$$
   Starting with the slope, m can be found by taking the derivative of x^3.
   $$m = f'(x) = 3x_1^2$$
   I wrote x_1 because this slope is based on what x_1 is because it is the slope at a specific point.

To account for the x-intercept, we recognize that having an x-intercept of 6 means that the point $(6,0)$ exists on the graph. This means we can plug the point in for x and y and find x_1. Do not plug them in for x_1 and y_1 because (x_1, y_1) represents a point on x^3, not the line.

$$0 - y_1 = 3x_1^2(6 - x_1)$$
$$-y_1 = 18x_1^2 - 3x_1^3$$

Substitute y_1 with x_1^3.

$$-x_1^3 = 18x_1^2 - 3x_1^3$$
$$18x_1^2 = 2x_1^3$$
$$9x_1^2 = x_1^3$$
$$x_1 = 9$$
$$y_1 = 9^3 = 729$$

The tangent line to the point $(9{,}729)$ has a x-intercept of 6.

17. (3.3) According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.
    a. Suppose G, m_1, and m_2 are constants. Find the rate of change of force F with respect to distance d.

       You can rewrite F as $Gm_1m_2 \cdot \frac{1}{d^2}$. The constants are not affected by the differentiation.

$$F = Gm_1m_2 \cdot \frac{1}{d^2}$$
$$F = Gm_1m_2 \cdot d^{-2}$$
$$\frac{dF}{dd} = Gm_1m_2 \cdot -2d^{-3}$$
$$\frac{dF}{dd} = -\frac{2Gm_1m_2}{d^3}$$

    b. Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

       Just plug everything in.

$$\frac{dF}{dd} = -\frac{2\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)(1000\ kg)(1000\ kg)}{(10\ m)^3}$$
$$\frac{dF}{dd} = 1.334 \times 10^{-7} \frac{N}{m}$$

18. (3.5) Find $\frac{dy}{dx}$ for the following functions.

a. $y = x^2 - \sec x + 1$

The derivative of $\sec x$ is $\sec x \tan x$.

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

b. $y = x^2 \cot x$

This involves the product rule. The derivative of $\cot x$ is $-\csc^2 x$.

$$\frac{dy}{dx} = x^2(-\csc^2 x) + \cot x\,(2x)$$

Simplify:

$$\frac{dy}{dx} = -x^2 \csc^2 x + 2x \cot x$$
$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

c. $y = \frac{\sec x}{x}$

This involves the quotient rule. The derivative of $\sec x$ is $\sec x \tan x$.

$$\frac{dy}{dx} = \frac{x(\sec x \tan x) - \sec x\,(1)}{x^2}$$

Simplify.

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

d. $y = (x + \cos x)(1 - \sin x)$

Use the product rule.

$$\frac{dy}{dx} = (x + \cos x)\frac{d}{dx}[1 - \sin x] + (1 - \sin x)\frac{d}{dx}[x + \cos x]$$
$$\frac{dy}{dx} = (x + \cos x)(-\cos x) + (1 - \sin x)(1 - \sin x)$$
$$\frac{dy}{dx} = (1 - \sin x)^2 - \cos x\,(x + \cos x)$$

e. $y = \frac{1 - \cot x}{1 + \cot x}$

Use the quotient rule. The derivative of $\cot x$ is $-\csc^2 x$.

$$\frac{dy}{dx} = \frac{(1 + \cot x)(\csc^2 x) - (1 - \cot x)(-\csc^2 x)}{(1 + \cot x)^2}$$

Simplify.

$$\frac{dy}{dx} = \frac{\csc^2 x\,(1 + \cot x) + \csc^2 x\,(1 - \cot x)}{(1 + \cot x)^2}$$

Factor out $\csc^2 x$.

$$\frac{dy}{dx} = \frac{\csc^2 x\,(1 + \cot x + 1 - \cot x)}{(1 + \cot x)^2}$$
$$\frac{dy}{dx} = \frac{\csc^2 x\,(2)}{(1 + \cot x)^2}$$

$$\frac{dy}{dx} = \frac{2\csc^2 x}{(1+\cot x)^2}$$

19. (3.5) Find the equation of the tangent line to $f(x) = x^2 - \tan x$ at $x = 0$.
    Find the derivative value at $x = 0$ to find the slope of the tangent line:
$$f'(x) = 2x - \sec^2 x$$
$$f'(0) = 0 - \sec^2(0)$$
$$f'(0) = 0 - 1 = -1$$
    We have the slope, and the x-coordinate of the point. To make the equation of the tangent line, we need the y-coordinate also. Plug $x = 0$ into the original function $f(x)$.
$$f(0) = 0^2 - \tan(0)$$
$$f(0) = 0 - 0 = 0$$
    Write the equation of the line in point-slope form, then convert to slope-intercept form.
$$y - y_1 = m(x - x_1)$$
$$y - 0 = -1(x - 0)$$
$$y = -x$$
20. (3.5) Find $\frac{d^2y}{dx^2}$ for the following functions.
    a. $y = x\sin x - \cos x$
       Use product rule on $x\sin x$ for the first derivative:
$$\frac{dy}{dx} = (x)(\cos x) + (\sin x)(1) + \sin x$$
$$\frac{dy}{dx} = x\cos x + 2\sin x$$
       Use it again for the second derivative:
$$\frac{d^2y}{dx^2} = (x)(-\sin x) + (\cos x)(1) + 2\sin x$$
$$\frac{d^2y}{dx^2} = -x\sin x + \cos x + 2\cos(x)$$
$$\frac{d^2y}{dx^2} = 3\cos x - x\sin x$$
    b. $y = x - \frac{1}{2}\sin x$
$$\frac{dy}{dx} = 1 - \frac{1}{2}\cos x$$
$$\frac{d^2y}{dx^2} = 0 - \frac{1}{2}(-\sin x)$$
$$\frac{d^2y}{dx^2} = \frac{1}{2}\sin x$$

c. $y = 2\csc x$

$$\frac{dy}{dx} = 2(-\csc x \cot x)$$
$$\frac{dy}{dx} = -2\csc x \cot x$$

Use product rule for the second derivative:

$$\frac{d^2y}{dx^2} = -2\big((\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x)\big)$$
$$\frac{d^2y}{dx^2} = -2(-\csc^3 x - \cot^2 x \csc x)$$

21. (3.5) Let $f(x) = \cot x$. Determine the points on the graph f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

If the tangent line is parallel to $y = -2x$, they both have a slope of -2. Find the derivative of $f(x)$ and set it equal to 2.

$$f'(x) = -\csc^2 x$$
$$-\csc^2 x = -2$$

Eliminate the negative and rewrite $\csc x$ as $\frac{1}{\sin x}$:

$$\csc^2 x = 2$$
$$\frac{1}{\sin^2 x} = 2$$

Take the reciprocal and square root both sides:

$$\sin^2 x = \frac{1}{2}$$
$$\sin x = \pm\sqrt{\frac{1}{2}}$$
$$\sin x = \pm\frac{1}{\sqrt{2}}$$

To find the angles that make this expression true, draw right triangles in every quadrant with hypotenuse length of $\sqrt{2}$ and side length either -1 or 1, depending on if it's the $+$ case or the $-$ case.

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Plug these x-values into $f(x)$ to find the y-values and list the points.

$$f\left(\frac{\pi}{4}\right) = 1$$
$$f\left(\frac{3\pi}{4}\right) = -1$$

$$f\left(\frac{5\pi}{4}\right) = 1$$

$$f\left(\frac{7\pi}{4}\right) = -1$$

$$\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{5\pi}{4}, 1\right), \left(\frac{7\pi}{4}, -1\right)$$

22. (3.5) Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a\cos t + b\sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is $0\ cm$ and the velocity is $3\frac{cm}{s}$ when $t = 0$, find the values of a and b.

Velocity is the derivative of $s(t)$. Here's what we're given:

$$s(0) = 0$$
$$s'(0) = 3$$

We have two unknowns a and b that we need to solve for. That means we need a system of two equations. Our first equation can be $s(t)$ and the second can be $s'(t)$ since we are given initial conditions from both these equations.

$$s'(t) = -a\sin t + b\cos t$$

Starting with $s(t)$, plug in $t = 0$ and we can find one of the constants.

$$s(0) = 0 = a\cos 0 + b\sin 0$$
$$a = 0$$

Now with $s'(t)$, we can plug in $a = 0$ because it's just a constant.

$$s'(0) = 3 = -0\sin 0 + b\cos 0$$
$$s'(0) = 3 = b\cos 0$$
$$b = 3$$

23. (3.6) Find $\frac{dy}{dx}$ using the definition of the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

a. $y = 6u^3, u = 7x - 4$

$$\frac{dy}{dx} = 18u^2 \cdot \frac{d}{dx}[7x - 4]$$

$$\frac{dy}{dx} = 18u^2(7)$$

$$\frac{dy}{dx} = 126(7x - 4)^2$$

b. $y = \sqrt{4u + 3}, u = x^2 - 6x$

Notice that this function has three layers, not two. The outer layer is the square root, the inner is $u = x^2 - 6x$, and the middle is $4u + 3$. Since the middle is differentiable, we need to multiply by its derivative.

$$y = (4u + 3)^{\frac{1}{2}}$$

Multiply by the derivative of $4u + 3$.

$$\frac{dy}{dx} = \frac{1}{2}(4u + 3)^{-\frac{1}{2}} \cdot \frac{d}{dx}[4u + 3]$$

Finally, multiply by the derivative of u.

$$= \frac{1}{2}(4u+3)^{-\frac{1}{2}} \cdot 4 \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}(4u+3)^{-\frac{1}{2}} \cdot 4 \cdot (2x-6)$$

Simplify and plug in u:

$$\frac{dy}{dx} = \frac{2(2x-6)}{\sqrt{4(x^2-6x)+3}}$$

$$\frac{dy}{dx} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

24. (3.6) Find $\frac{dy}{dx}$.

a. $y = (3x^2+1)^3$

This function has two differentiable parts: the inner $u = 3x^2 + 1$ and the outer $y = u^3$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 3(3x^2+1)^2 \cdot \frac{d}{dx}[3x^2+1]$$

$$\frac{dy}{dx} = 3(3x^2+1)^2(6x)$$

$$\frac{dy}{dx} = 18x(3x^2+1)^2$$

b. $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

This function has two differentiable parts: the inner $u = \frac{x}{7} + \frac{7}{x}$ and the outer $y = u^7$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \frac{d}{dx}\left[\frac{x}{7} + \frac{7}{x}\right]$$

$$\frac{dy}{dx} = 7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \left(\frac{1}{7} - \frac{7}{x^2}\right)$$

c. $y = \csc(\pi x + 1)$

This function has two differentiable parts: the inner $u = \pi x + 1$ and the outer $y = \csc u$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = -\csc(\pi x+1)\cot(\pi x+1) \cdot \frac{d}{dx}[\pi x + 1]$$

$$\frac{dy}{dx} = -\pi\csc(\pi x+1)\cot(\pi x+1)$$

d. $y = \frac{-6}{(\sin x)^3}$

This function has two differentiable parts: the inner $u = \sin x$ and the outer $y = -\frac{6}{u^3}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can start by rewriting y without a fraction so we can do power rule easily.

$$y = -6(\sin x)^{-3}$$
$$\frac{dy}{dx} = 18(\sin x)^{-4} \cdot \frac{d}{dx}[\sin x]$$
$$\frac{dy}{dx} = 18(\sin x)^{-4} \cdot \cos x$$

Simplify:

$$\frac{dy}{dx} = \frac{18 \cos x}{\sin^4 x}$$

e. $y = \sin(\cos 7x)$

This function has three differentiable parts: the inner $u = 7x$, the middle $v = \cos(u)$ and the outer $y = \sin\big(v(u)\big)$.

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$
$$\frac{dy}{dx} = \cos(\cos 7x) \cdot \frac{d}{dx}[\cos 7x]$$
$$\frac{dy}{dx} = \cos(\cos 7x) \cdot -\sin 7x \cdot \frac{d}{dx}[7x]$$
$$\frac{dy}{dx} = \cos(\cos 7x) \cdot -\sin 7x \cdot 7$$

Simplify:

$$\frac{dy}{dx} = -7\sin(7x)\cos(\cos(7x))$$

25. (3.6) Let $y = [f(x)]^2$ and suppose that $f(-1) = 4$ and $\frac{dy}{dx} = 10$ when $x = 1$. Find $f'(1)$.

Start by taking the derivative of y, remembering the chain rule with $f(x)$, since it is differentiable.

$$\frac{dy}{dx} = 2f(x)^1 \cdot f'(x)$$

We can plug in the conditions we are given at $x = -1$, and we will have enough information to find $f'(-1)$.

$$10 = 2(4) \cdot f'(-1)$$
$$f(-1) = \frac{10}{8}$$
$$f(-1) = \frac{5}{4}$$

26. (3.6) The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m. Approximate to one decimal place.
All we need to do is take the derivative of $D(t)$ and plug in $t = 6$, but this requires the chain rule since $\frac{\pi}{6}t - \frac{7\pi}{6}$ is differentiable.

$$D'(t) = 5\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) \cdot \frac{d}{dx}\left[\frac{\pi}{6}t - \frac{7\pi}{6}\right] + 0$$
$$D'(t) = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right)$$

Now plug in $t = 6$.

$$D'(6) = \frac{5\pi}{6}\cos\left(\pi - \frac{7\pi}{6}\right)$$
$$D'(6) = \frac{5\pi}{6}\cos\left(-\frac{\pi}{6}\right)$$
$$D'(6) \approx 2.3\frac{ft}{hr}$$

27. (3.7) Find the derivative at $x = a$, $x = f^{-1}(y)$, and $\frac{d}{dy}[f^{-1}(y)]$ at $y = f(a)$.

$$\frac{d}{dy}[f^{-1}(y)] = \frac{1}{f'(f^{-1}(y))}$$

When $x = a$,$y = f(a)$.

$$\frac{d}{dy}[f^{-1}(y)]|_{y=f(a)} = \frac{1}{f'\left(f^{-1}(f(a))\right)}$$

But $f^{-1}(f(a)) = a$, so

$$\frac{d}{dy}[f^{-1}(y)]|_{y=f(a)} = \frac{1}{f'(a)}$$

a. $f(x) = 2x^3 - 3, x = 1$
   Find the derivative using the power rule.
   $$f'(x) = 6x^2$$
   $$f(1) = 6$$

   Find the inverse by solving for x.
   $$y = 2x^3 - 3$$
   $$2x^3 = y + 3$$
   $$x^3 = \frac{1}{2}(y + 3)$$

$$x = f^{-1}(y) = \sqrt[3]{\frac{1}{2}(y+3)}$$

Find the derivative of the inverse function using the inverse theorem.

$$\frac{d}{dx}[f^{-1}(y)]|_{y=1} = \frac{1}{f'(1)}$$
$$\frac{d}{dx}[f^{-1}(y)] = \frac{1}{6}$$

b. $f(x) = \sin x, x = 0$

Find the derivative.

$$f'(x) = \cos x$$
$$f(0) = 1$$

Find the inverse.

$$y = \sin x$$
$$x = f^{-1}(y) = \arcsin y$$

Find the derivative of the inverse function using the inverse theorem.

$$\frac{d}{dx}[f^{-1}(y)]|_{y=0} = \frac{1}{f'(0)}$$
$$\frac{d}{dx}[f^{-1}(y)]|_{y=0} = 1$$

28. (3.7) Find $(f^{-1})'(a)$

First, we'll find the inverse of $f(x)$, say $g(x)$. Then we'll use the rule and plug in a:

$$g'(x) = \frac{1}{f'\big(g(x)\big)}$$
$$g'(a) = \frac{1}{f'\big(g(a)\big)}$$
$$g'(a) = \frac{1}{f'\big(g(a)\big)}$$

The first step is to find the derivative of f. Next find the inverse at $x = a$. Then, plug it into the derivative and the formula.

a. $f(x) = x^3 + 2x + 3, a = 0$

Find $f'(x)$ using the power rule.

$$f'(x) = 3x^2 + 2$$

If we need the inverse at $x = 0$, that means $y = 0$ on the original function. You can solve it by inspection or testing simple values.

$$x^3 + 2x + 3 = 0$$
$$x = -1$$

So $g(0) = -1$. Plug this value into the formula.

$$g'(0) = \frac{1}{f'\big(g(0)\big)}$$

$$g'(0) = \frac{1}{f'(-1)} = \frac{1}{3(-1)^2 + 2}$$

$$g'(0) = \frac{1}{5}$$

b. $f(x) = x - \frac{2}{x}, x < 0, a = 1$

Find $f'(x)$.

$$f'(x) = 1 + \frac{2}{x^2}$$

To find the inverse at $x = 1$, that means $y = 1$ on the original function.

$$x - \frac{2}{x} = 1$$

Multiply both sides by x to get rid of the x in the denominator, making the equation a polynomial. Note that we only need the negative value of x, as stated in the problem.

$$x^2 - 2 = x$$
$$x^2 - x - 2 = 0$$
$$x = -1, 2$$

In the negative case we get $x = -1$. So $g(1) = -1$. Plkug this value into the formula.

$$g'(2) = \frac{1}{f'\big(g(2)\big)} = \frac{1}{f'(-1)}$$

$$g'(2) = \frac{1}{1 + \frac{2}{(-1)^2}} = \frac{1}{3}$$

29. (3.7) Find $\frac{dy}{dx}$.

a. $y = \arcsin x^2$

Don't forget the chain rule.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^2)^2}} \cdot 2x$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{1 - x^4}}$$

b. $y = \operatorname{arcsec} \frac{1}{x}$

$$\frac{dy}{dx} = \frac{1}{\left|\frac{1}{x}\right| \sqrt{\left(\frac{1}{x}\right)^2 - 1}} \cdot -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{\left|\frac{1}{x}\right| \cdot x^2 \cdot \sqrt{\frac{1}{x^2} - 1}} = -\frac{1}{|x|\sqrt{\frac{1}{x^2} - 1}}$$

Since that $|x|$ is multiplied by the square root, you can bring it in the square root as an x^2 (no need for absolute values since squares are positive).

$$\frac{dy}{dx} = -\frac{1}{\sqrt{\frac{x^2}{x^2} - x^2}} = -\frac{1}{\sqrt{1 - x^2}}$$

c. $y = (1 + \arctan x)^3$

This time we differentiate the inverse trig function as part of the chain rule.

$$\frac{dy}{dx} = 3(1 + \arctan x)^2 \cdot \frac{1}{1 + (x)^2}$$

$$\frac{dy}{dx} = \frac{3(1 + \arctan x)^2}{1 + x^2}$$

d. $y = \frac{1}{\arctan x}$

I don't want to use the quotient rule, so I will rewrite it without a fraction.

$$y = \arctan^{-1} x$$

Now we can use the power rule to take the derivative.

$$\frac{dy}{dx} = -\arctan^{-2} x \cdot \frac{1}{1 + x^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1 + x^2)\arctan^2 x}$$

e. $y = \operatorname{arccot}\sqrt{4 - x^2}$

Here we have an inverse trig function on the outside and a square root function on the inside to do chain rule with twice.

$$\frac{dy}{dx} = -\frac{1}{1 + \left(\sqrt{4 - x^2}\right)^2} \cdot \frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot -2x$$

$$\frac{dy}{dx} = -\frac{-2x}{(1 + 4 - x^2)2\sqrt{4 - x^2}}$$

$$\frac{dy}{dx} = \frac{x}{(5 - x^2)\sqrt{4 - x^2}}$$

30. (3.7) $f\left(\frac{1}{3}\right) = -8$, $f'\left(\frac{1}{3}\right) = 2$, Find $(f^{-1})'(-8)$.

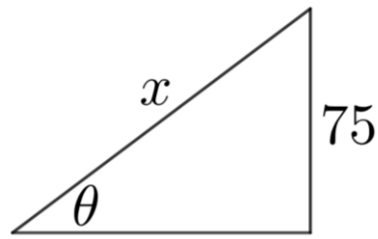
$$(f^{-1})'(-8) = \frac{1}{f'\left(f^{-1}(-8)\right)}$$

We are not given $f^{-1}(-8)$, but that $f\left(\frac{1}{3}\right) = -8$. Since x and y are flipped with inverse functions, we know $f^{-1}(-8) = \frac{1}{3}$. Now we are given $f'\left(\frac{1}{3}\right)$, so we can just plug it in.

$$(f^{-1})'(-8) = \frac{1}{f'\left(\frac{1}{3}\right)} = \frac{1}{2}$$

31. (3.7) The position of a moving hockey puck after t seconds is $s(t) = \arctan t$ where s is in meters.
    a. Find the velocity function.
    Take the derivative of $s(t)$ to get the velocity function.
    $$s'(t) = \frac{1}{1+t^2}$$
    b. Find the acceleration function.
    We can rewrite $s'(t)$ without a fraction so that we don't have to use the quotient rule when differentiating.
    $$s'(t) = (1+t^2)^{-1}$$
    $$s''(t) = -1(1+t^2)^{-2} \cdot 2t$$
    $$s''(t) = \frac{-2t}{(1+t^2)^2}$$
    c. Find the velocity and acceleration at $t = 2, 4,$ and 6 seconds.
    $$s'(2) = \frac{1}{5}$$
    $$s''(2) = -\frac{4}{25}$$
    $$s'(4) = \frac{1}{17}$$
    $$s''(4) = -\frac{8}{289}$$
    $$s'(6) = \frac{1}{37}$$
    $$s''(6) = -\frac{12}{1369}$$
    d. What conclusion can be drawn from the results in c?
    Velocity and acceleration have opposite signs for these values.
    The puck is slowing down at $2, 4,$ and 6 seconds.

    Note: you cannot say that it is slowing down for the entire interval $0 \leq t \leq 6$; we have only proved it for three specific time values.
32. (3.7) A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached. Round to four decimal places.
    Here's our triangle:



We first need an expression for θ in terms of x before we take the derivative it.

$$\sin\theta = \frac{75}{x}$$

$$\theta = \arcsin\frac{75}{x}$$

Now take the derivative and plug in $x = 90$.

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-\left(\frac{75}{x}\right)^2}} \cdot -\frac{75}{x^2}$$

$$\frac{d\theta}{dx}\Big|_{x=90} = \frac{1}{\sqrt{1-\left(\frac{75}{90}\right)^2}} \cdot -\frac{75}{90^2} \approx -0.0168\frac{rad}{ft}$$

33. (3.8) Find $\frac{dy}{dx}$.

a. $6x^2 + 3y^2 = 12$

Just apply the power rule to both sides.

$$12x + 6y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{12x}{6y}$$

$$\frac{dy}{dx} = -2\frac{x}{y}$$

b. $3x^3 + 9xy^2 = 5x^3$

This requires the product rule.

$$9x^2 + \left[(9x)\left(2y\frac{dy}{dx}\right) + (y^2)(9)\right] = 15x^2$$

$$9x^2 + 18xy\frac{dy}{dx} + 9y^2 = 15x^2$$

$$18xy\frac{dy}{dx} = 15x^2 - 9x^2 - 9y^2$$

$$\frac{dy}{dx} = \frac{6x^2 - 9y^2}{18xy}$$

Simplify and split into two fractions.

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

c. $y\sqrt{x+4} = xy + 8$

Use the product rule.

$$y \cdot \frac{1}{2\sqrt{x+4}} + \sqrt{x+4} \cdot \frac{dy}{dx} = x \cdot \frac{dy}{dx} + y + 0$$

$$\sqrt{x+4} \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} = y - \frac{y}{2\sqrt{x+4}}$$

$$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

d. $y\sin(xy) = y^2 + 2$

This requires product rule with the chain rule because xy is differentiable and inside of the sine.

$$(y)\left(\cos(xy) \cdot \frac{d}{dx}[xy]\right) + (\sin(xy))\left(\frac{dy}{dx}\right) = 2y\frac{dy}{dx} + 0$$

I didn't show the derivative of xy yet because that requires chain rule again. I will simplify first before evaluating $\frac{d}{dx}[xy]$.

$$y\cos(xy)\frac{d}{dx}[xy] + \frac{dy}{dx}\sin(xy) = 2y\frac{dy}{dx}$$

Now evaluate it.

$$y\cos(xy)\left((x)\left(\frac{dy}{dx}\right) + (y)(1)\right) + \frac{dy}{dx}\sin(xy) = 2y\frac{dy}{dx}$$

$$y\cos(xy)\left(x\frac{dy}{dx} + y\right) + \frac{dy}{dx}\sin(xy) = 2y\frac{dy}{dx}$$

$$xy\cos(xy)\frac{dy}{dx} + y^2\cos(xy) + \frac{dy}{dx}\sin(xy) = 2y\frac{dy}{dx}$$

Get the terms with $\frac{dy}{dx}$ on the same side.

$$xy\cos(xy)\frac{dy}{dx} - 2y\frac{dy}{dx} + \frac{dy}{dx}\sin(xy) = -y^2\cos(xy)$$

Factor out $\frac{dy}{dx}$.

$$\frac{dy}{dx}(xy\cos(xy) - 2y + \sin(xy)) = -y^2\cos(xy)$$

$$\frac{dy}{dx} = -\frac{y^2\cos(xy)}{xy\cos(xy) - 2y + \sin(xy)}$$

e. $x^3y + xy^3 = -8$

We need the product rule twice.

$$(x^3)\left(\frac{dy}{dx}\right) + (y)(3x^2) + (x)\left(3y^2\frac{dy}{dx}\right) + (y^3)(1) = 0$$

$$x^3\frac{dy}{dx} + 3x^2y + 3y^2x\frac{dy}{dx} + y^3 = 0$$

Get the terms with $\frac{dy}{dx}$ on the same side.

$$x^3\frac{dy}{dx}+3y^2x\frac{dy}{dx}=-3x^2y-y^3$$

Factor out $\frac{dy}{dx}$.

$$\frac{dy}{dx}(x^3+3y^2x)=-3x^2y-y^3$$
$$\frac{dy}{dx}=\frac{-3x^2y-y^3}{x^3+3y^2x}$$

34. (3.8) Find the equation of the tangent line to the graph at the given point.

We can make an equation of a line using just a point and a slope with point-slope form. We are already given the point. Just use implicit differentiation to find the slope at that point.

a. $x^2y^2+5xy=14, (2,1)$

$$(x^2)\left(2y\frac{dy}{dx}\right)+(y^2)(2x)+(5x)\left(\frac{dy}{dx}\right)+(y)(5)=0$$
$$2x^2y\frac{dy}{dx}+2xy^2+5x\frac{dy}{dx}+5y=0$$
$$2x^2y\frac{dy}{dx}+5x\frac{dy}{dx}=-2xy^2-5y$$
$$\frac{dy}{dx}(2x^2y+5x)=-2xy^2-5y$$
$$\frac{dy}{dx}=\frac{-2xy^2-5y}{2x^2y+5x}$$
$$\frac{dy}{dx}|_{(2,1)}=\frac{-2\cdot 2\cdot 1^2-5(1)}{2(2^2)\cdot 1+5(2)}=\frac{-9}{18}=-\frac{1}{2}$$
$$y-1=-\frac{1}{2}(x-2)$$
$$y-1=-\frac{1}{2}x+1$$
$$y=-\frac{1}{2}x+2$$

b. $xy^2+\sin(\pi y)-2x^2=10, (2,-3)$

$$x\left(2y\cdot\frac{dy}{dx}\right)+y^2+\cos(\pi y)\cdot\pi\cdot\frac{dy}{dx}-4x=0$$
$$2xy\cdot\frac{dy}{dx}+y^2+\pi\cdot\cos(\pi y)\cdot\frac{dy}{dx}-4x=0$$
$$2xy\cdot\frac{dy}{dx}+\pi\cdot\cos(\pi y)\cdot\frac{dy}{dx}=4x-y^2$$
$$\frac{dy}{dx}[2xy+\pi\cdot\cos(\pi y)]=4x-y^2$$
$$\frac{dy}{dx}=\frac{4x-y^2}{2xy+\pi\cdot\cos(\pi y)}$$
$$\frac{dy}{dx}|_{(2,-3)}=\frac{1}{\pi+12}$$

$$y - y_0 = m(x - x_0)$$
$$y + 3 = \frac{1}{\pi + 12}(x - 2)$$
$$y = \frac{1}{\pi + 12}x - \frac{2}{\pi + 12} - 3$$
$$y = \frac{1}{\pi + 12}x - \frac{3\pi + 38}{\pi + 12}$$

c. $xy + \sin x = 1, \left(\frac{\pi}{2}, 0\right)$

$$(x)\left(\frac{dy}{dx}\right) + (y)(1) + \cos x = 0$$
$$x\frac{dy}{dx} + y + \cos x = 0$$
$$x\frac{dy}{dx} = -y - \cos x$$
$$\frac{dy}{dx} = \frac{-y - \cos x}{x}$$
$$\frac{dy}{dx}\Big|_{\left(\frac{\pi}{2},0\right)} = \frac{-0 - \cos\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{0}{0} = 0$$
$$y = 0$$

35. (3.8) For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal line to the tangent line at the point $(1, 1)$.

The slope of the normal line is the opposite reciprocal of the slope of the tangent line. Find the slope of the tangent line with implicit differentiation, then get the opposite reciprocal.

$$2x + (2x)\left(\frac{dy}{dx}\right) + (y)(2) - 6y\frac{dy}{dx} = 0$$
$$2x\frac{dy}{dx} - 6y\frac{dy}{dx} = -2x - 2y$$
$$\frac{dy}{dx}(2x - 6y) = -2x - 2y$$
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 6y}$$
$$\frac{dy}{dx}\Big|_{(1,1)} = \frac{-2 - 2}{2 - 6} = \frac{-4}{-4} = 1$$
$$m_T = 1$$

Slope of normal line:

$$m_N = -\frac{1}{1} = -1$$

Write the equation from point-slope form to slope-intercept form.

$$y - 1 = -1(x - 1)$$
$$y - 1 = -x + 1$$
$$y = -x + 2$$

b. At what other point does the normal line in a. intersect the graph of the equation?

We can substitute $-x+2$ for y in the original equation and find the x-values. Since normal line intersects in more than one location, we will get the $x=1$ that we had before, and some other value.

$$x^2+2x(-x+2)-3(-x+2)^2=0$$
$$x^2-2x^2+4x-3(x^2-4x+4)=0$$
$$x^2-2x^2+4x-3x^2+12x-12=0$$
$$-4x^2+16x-12=0$$

This quadratic can be factored to find our x-values. If you don't want to factor, you can use synthetic division since we know one of the factors already.

$$(x-1)(x-3)=0$$

So, our other x-value is $x=3$. Plug it in the original equation to solve for y.

$$3^2+2(3)y-3y^2=0$$
$$-3y^2+6y+9=0$$

Factor this quadratic.

$$(y+1)(y-3)=0$$

To find which y-value will work, we have to plug it into the normal line equation to see if it works.

Try $y=3$:

$$3=-x+2$$
$$x=-1$$

That isn't $x=3$, which is what we need. Try $y=-1$.

$$-1=-x+2$$
$$x=3$$

This gives us the result we want, so the normal line intersects the graph at $(3,-1)$.

36. (3.8) For the equation $x^2+xy+y^2=7$,

a. Find the x-intercept(s).

The x-intercepts occur when $y=0$.

$$x^2=7$$
$$x=\pm\sqrt{7}$$

The x-intercepts are $(-7,0)$ and $(7,0)$.

b. Find the slope of the tangent line(s) at the x-intercept(s).

First differentiate implicitly, solving for $\frac{dy}{dx}$. Then plug in the x-intercept points.

$$2x+(x)\left(\frac{dy}{dx}\right)+(y)(1)+2y\frac{dy}{dx}=0$$
$$2x+x\frac{dy}{dx}+y+2y\frac{dy}{dx}=0$$
$$x\frac{dy}{dx}+2y\frac{dy}{dx}=-2x-y$$

$$\frac{dy}{dx}(x+2y) = -2x - y$$
$$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$$
$$\frac{dy}{dx}|_{(-7,0)} = \frac{-2(-7)-0}{-7+0} = \frac{14}{-7} = -2$$
$$\frac{dy}{dx}|_{(7,0)} = \frac{-2(7)-0}{7+0} = \frac{-14}{7} = -2$$

c. What does (do) the value(s) in b. indicate about the tangent line(s)?
They are parallel since the lines have the same slope.

37. (3.8) The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is constant. Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16$.
If V is constant, it's derivative is 0.

$$V = \frac{1}{3}\pi x^2 y$$
$$\frac{dV}{dx} = 0 = \frac{1}{3}\pi\left[(x^2)\left(\frac{dy}{dx}\right) + (y)(2x)\right]$$
$$0 = \frac{\pi}{3}\left(x^2\frac{dy}{dx} + 2xy\right)$$
$$x^2\frac{dy}{dx} + 2xy = 0$$
$$\frac{dy}{dx} = -\frac{2xy}{x^2} = -\frac{2y}{x}$$
$$\frac{dy}{dx}|_{(4,16)} = -\frac{2(16)}{4} = -8$$

38. (3.8) The surface area of a closed rectangular box with a square base with side x and height y is 78 square feet. Find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.
We need an equation for the surface area before taking the derivative of it.
The surface area would be the area of two square bases, plus the area of four side lengths.

$$S = 2x^2 + 4xy$$

We can also assume that S is constant, and equal to 78. It doesn't matter what the constant is because it becomes 0 with differentiation.

$$78 = 2x^2 + 4xy$$

Take the derivative implicitly.

$$0 = 4x + 4x\frac{dy}{dx} + 4y$$
$$\frac{dy}{dx} = \frac{-4x-4y}{4x} = \frac{-x-y}{x}$$

$$\frac{dy}{dx}\Big|_{(3,5)} = \frac{-3-5}{3} = -\frac{8}{3}$$

39. (3.9) Find $\frac{dy}{dx}$.

a. $y = x^2 e^x$

Use the product rule when differentiating.

$$\frac{dy}{dx} = x^2 e^x + e^x(2x)$$
$$\frac{dy}{dx} = e^x(x^2 + 2x)$$

b. $y = e^{x^3 \ln x}$

On the outside is an exponential function, which has its own derivative. Also multiply by the derivative of $x^3 \ln x$ for chain rule.

$$\frac{dy}{dx} = e^{x^3 \ln x}\left(\frac{x^3}{x} + \ln x\,(3x^2)\right)$$
$$\frac{dy}{dx} = e^{x^3 \ln x}(x^2 + 3x^2 \ln x)$$

c. $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

This problem involves the quotient rule combined with chain rule for the $-x$ exponents.

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$
$$\frac{dy}{dx} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{(e^x + e^{-x})^2}$$
$$\frac{dy}{dx} = \frac{4}{(e^x + e^{-x})^2}$$

d. $y = 2^{4x} + 4x^2$

$$\frac{dy}{dx} = 2^{4x} \cdot \ln 2 \cdot 4 + 8x$$
$$\frac{dy}{dx} = 4 \ln 2 \cdot 2^{4x} + 8x$$

e. $y = x^\pi \cdot \pi^x$

We need the product rule. Also, π is a constant, so you can use the power rule on x^π.

$$\frac{dy}{dx} = x^\pi \cdot \pi^x \ln \pi + \pi^x(\pi x^{\pi - 1})$$

f. $y = \ln\sqrt{5x - 7}$

I showed the chain rule in a bit more detail here.

$$\frac{dy}{dx} = \frac{1}{\sqrt{5x-7}} \cdot \frac{d}{dx}[\sqrt{5x-7}]$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{5x-7}} \cdot \frac{1}{2}(5x-7)^{-\frac{1}{2}} \cdot \frac{d}{dx}[5x-7]$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{5x-7}} \cdot \frac{1}{2}(5x-7)^{-\frac{1}{2}} \cdot 5$$
$$\frac{dy}{dx} = \frac{5}{2\sqrt{5x-7}\sqrt{5x-7}}$$
$$\frac{dy}{dx} = \frac{5}{2(5x-7)}$$
$$\frac{dy}{dx} = \frac{5}{10x-14}$$

g. $y = \log(\sec x)$

$$\frac{dy}{dx} = \frac{1}{\sec x \ln 10} \cdot \sec x \tan x$$
$$\frac{dy}{dx} = \frac{\tan x}{\ln 10}$$

h. $y = 2^x \cdot \log_3 7^{x^2-4}$

This is a mess, but you can do it if you keep your product rule and chain rules organized.

$$\frac{dy}{dx} = (2^x)\left(\frac{1}{(7^{x^2-4})\ln 3} \cdot 7^{x^2-4} \ln 7 \cdot 2x\right) + \left(\log_3 7^{x^2-4}\right)(2^x \ln 2)$$
$$\frac{dy}{dx} = (2^x)\left(\frac{2x\ln 7}{\ln 3}\right) + \left(\log_3 7^{x^2-4}\right)(2^x \ln 2)$$

i. $y = (\sin 2x)^{4x}$

Apply natural log to both sides.

$$\ln y = \ln((\sin 2x)^{4x})$$
$$\ln y = 4x \ln(\sin 2x)$$

Perform implicit differentiation.

$$\frac{1}{y}\frac{dy}{dx} = (4x)\left(\frac{1}{\sin 2x} \cdot 2\cos 2x\right) + (\ln(\sin 2x))(4)$$
$$\frac{1}{y}\frac{dy}{dx} = 8x \cot 2x + 4\ln(\sin 2x)$$

Multiply by y.

$$\frac{dy}{dx} = y(8x \cot 2x + 4\ln(\sin 2x))$$

Replace y with $(\sin 2x)^{4x}$.

$$\frac{dy}{dx} = (\sin 2x)^{4x}(8x \cot 2x + 4\ln(\sin 2x))$$

j. $y = x^{\log_2 x}$

We are doing logarithmic differentiation again.

Apply natural log to both sides.

$$\ln y = \ln\left(x^{\log_2 x}\right)$$
$$\ln y = \log_2 x \ln x$$

Perform implicit differentiation.

$$\frac{1}{y}\frac{dy}{dx} = (\log_2 x)\left(\frac{1}{x}\right) + (\ln x)\left(\frac{1}{x\ln 2}\right)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\log_2 x}{x} + \frac{\ln x}{x\ln 2}$$

Multiply by y.

$$\frac{dy}{dx} = y\left(\frac{\log_2 x}{x} + \frac{\ln x}{x\ln 2}\right)$$

Replace y with $x^{\log_2 x}$.

$$\frac{dy}{dx} = x^{\log_2 x}\left(\frac{\log_2 x}{x} + \frac{\ln x}{x\ln 2}\right)$$

k. $y = x^{\cot x}$

We are doing logarithmic differentiation again.
Apply natural log to both sides.

$$\ln y = \ln(x^{\cot x})$$
$$\ln y = \cot x \ln x$$

Perform implicit differentiation.

$$\frac{1}{y}\frac{dy}{dx} = (\cot x)\left(\frac{1}{x}\right) + (\ln x)(-\csc^2 x)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{\cot x}{x} - \ln x \csc^2 x$$

Multiply by y.

$$\frac{dy}{dx} = y\left(\frac{\cot x}{x} - \ln x \csc^2 x\right)$$

Replace y with $x^{\cot x}$.

$$\frac{dy}{dx} = x^{\cot x}\left(\frac{\cot x}{x} - \ln x \csc^2 x\right)$$

40. (3.9) The position of an object is given by the function $f(x) = \log(10x^3) + x$. Find the velocity and acceleration function. Is the object speeding up or slowing down?
The velocity function is the first derivative and the acceleration function is the second derivative.

$$f'(x) = \frac{30x^2}{10x^3 \ln 10} + 1$$
$$f'(x) = \frac{3}{x\ln 10} + 1$$
$$f'(x) = \frac{3}{\ln 10} \cdot \frac{1}{x} + 1$$
$$f''(x) = \frac{3}{\ln 10} \cdot -\frac{1}{x^2}$$
$$f''(x) = -\frac{3}{x^2 \ln 10}$$

An object is speeding up when the velocity and acceleration functions are either both positive, or both negative.
The velocity can be both positive or negative, depending on the value of x.
The acceleration function is always negative since x^2 is always positive.
We will assume $x > 0$ if it represents time.
When $x > 0$, the velocity is positive and the acceleration is negative. Therefore, the object is slowing down since the functions have different signs.

41. (3.9) Find the equation of the normal line to $f(x) = x \cdot 5^x$ at the point where $x = 1$.
We will find the slope of the tangent line, and its opposite reciprocal will be the slope of the normal line.

$$\mathrm{f}'(x) = (x)(5^x \ln 5) + (5^x)(1)$$
$$f'(x) = 5^x + 5^x x \ln 5$$
$$f'(1) = 5 + 5 \ln 5$$

So, the slope of the normal line is $\frac{-1}{5+5\ln 5}$.

$$f(1) = 5$$

Write the equation of the normal line in point-slope form and convert it to slope-intercept form.

$$y - y_1 = m(x - x_1)$$
$$y - 5 = \frac{-1}{5 + 5\ln 5}(x - 1)$$
$$y - 5 = -\frac{x}{5 + 5\ln 5} + \frac{1}{5 + 5\ln 5}$$
$$y = -\frac{x}{5 + 5\ln 5} + \frac{1}{5 + 5\ln 5} + 5$$

42. (3.9) The number of cases of influenza in New York City form the beginning of 1960 to the beginning of 1964 is modeled by the function $N(t) = 5.3e^{0.093t^2 - 0.87t}$, $(0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.
   a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
   Plug those values into a calculator.

$$N(0) = 5.3$$
$$N(4) \approx 0.723$$

   Between 1960 and 1964, the number of influenza cases decreased, on average, from 5,300 to 723.
   b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

$$N'(t) = 5.3e^{0.093t^2 - 0.87t}(2 \cdot 0.093t - 0.87)$$
$$N'(t) = 5.3e^{0.093t^2 - 0.87t}(0.186t - 0.87)$$

   Now plug in 0 and 4. It helps to figure out the units also.

$$N'(0) = -4.611\,\frac{cases}{year}$$

$$N'(3) \approx -0.281\,\frac{cases}{year}$$

At the beginning of 1960, the number of influenza cases was decreasing at a rate of 4611 per year. At the beginning of 1963, the number of cases was decreasing at a rate of -281 per year.

MTH 200

Worksheet Answers

Problems found in OpenStax Calculus Volume II and AP Calculus AB FRQ (in red).

Sections:

1. (4.1) Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

$$\frac{dy}{dx} = 2x \cdot \frac{dx}{dt}$$

$$\frac{dy}{dt}\Big|_{x=1} = 2(1)(4) = 8$$

2. (4.1) Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

$$2z \cdot \frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}}{2z}$$

We still need the value of z to have enough variables to solve for $\frac{dz}{dt}$.

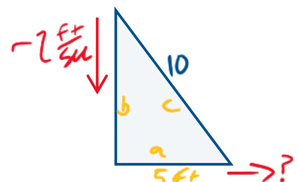
Plug $(x, y) = (1, 3)$ into the original equation to get z.

$$z^2 = x^2 + y^2 = 1^2 + 3^2 = 10$$

$$z = \pm\sqrt{10}$$

$$\frac{dz}{dt}\Big|_{(x,y)=(1,3)} = \frac{2(1)(4) + 2(3)(3)}{\pm 2\sqrt{10}} = \pm\frac{28}{2\sqrt{10}} = \pm\frac{13}{\sqrt{10}}$$

3. (4.1) A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of $2\frac{ft}{sec}$, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



$$c = 10\,ft, \frac{db}{dt} = -2\frac{ft}{s}, a = 5\,ft$$

Start by finding the height of the ladder when the bottom is 5 feet from the ground $(b = 5)$.

$$c^2 = a^2 + b^2$$

$$10^2 = 5^2 + b^2$$

$$b = \pm\sqrt{75} = \pm 5\sqrt{3}$$

Now use implicit differentiation with the Pythagorean theorem to get an equation with $\frac{da}{dt}$.

$$2c \cdot \frac{dc}{dt} = 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt}$$

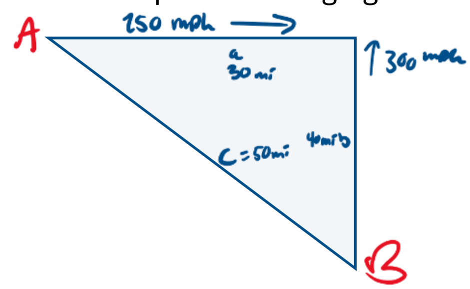
$\frac{dc}{dt} = 0$ since the length of the ladder is constant.

$$0 = 2(5) \cdot \frac{da}{dt} + 2\left(5\sqrt{3}\right)(-2)$$

$$0 = 10 \cdot \frac{da}{dt} - 20\sqrt{3}$$

$$\frac{da}{dt} = \frac{20\sqrt{3}}{10} = 2\sqrt{3}\ ft/s$$

4. (4.1) Two airplanes are flying in the air at the same height: airplane *A* is flying east at $250\,\frac{mi}{h}$ and airplane *B* is flying north at $300\,\frac{mi}{h}$. If they are both heading to the same airport, located 30 miles east of airplane *A* and 40 miles north of airplane *B*, at what rate is the distance between the airplanes changing?



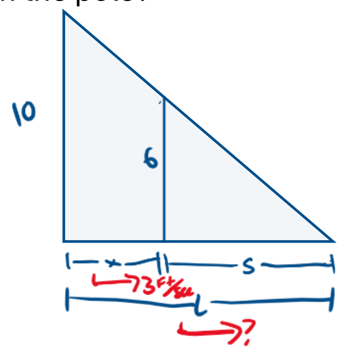
The instantaneous distance between the two planes can be determined with the special 3-4-5 triangle. $c = 50\ mi$
Now use implicit differentiation with the Pythagorean theorem to get an equation with $\frac{dc}{dt}$.

$$c^2 = a^2 + b^2$$
$$2c \cdot \frac{dc}{dt} = 2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt}$$
$$2(50) \cdot \frac{dc}{dt} = 2(30)(250) + 2(40)(300)$$
$$\frac{dc}{dt} = 390\ mph$$

5. (4.1) A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of $3\frac{ft}{s}$.

a. What is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

b. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



The distance between the person and the lamppost is $x = 10\ ft$, and the rate is:

$$x' = 3\ ft/s$$

The distance between the person and the shadow is s. The distance between the tip of the shadow and the pole is L. Question (a) asks for s' and question (b) asks for L'. A simple equation can be derived that shows the relationship between these two variables":

$$L = x + s$$

$$L' = x' + s'$$

a. s' can be found without finding L' first. Since we already have x', we need a relationship between them. Use similar triangles to find the ratio of the side lengths and simplify to get an equation. The derivative of the resulting equation includes s'.

$$\frac{10}{x+s} = \frac{6}{s}$$
$$10s = 6x + 6x$$
$$4s = 6x$$
$$4s' = 6s' = 6\left(3\frac{ft}{s}\right)$$
$$s' = \frac{9}{2}\frac{ft}{s}$$

b. L' can be found easily by plugging in s' and x' into the first equation.

$$L' = x' + s' = 3\frac{ft}{s} + \frac{9}{2}\frac{ft}{s} = \frac{15}{2}\frac{ft}{s}$$
$$L' = \frac{15}{2}\frac{ft}{s}$$

6. (4.1) The radius of a sphere is increasing at a rate of $9\frac{cm}{s}$. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

$$\frac{dr}{dt} = 9\frac{cm}{sec}$$

$$\frac{dV}{dt} = 9\frac{cm^3}{sec}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4\pi}{3}\cdot 3r^2 \cdot \frac{dr}{dt}$$

$$9 = \frac{4\pi}{3}\cdot 3 \cdot r^2 \cdot 9$$

$$1 = 4\pi r^2$$

$$r^2 = \frac{1}{4\pi}$$

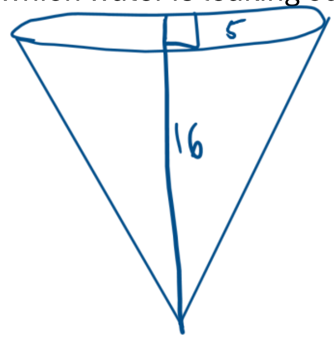
$$r = \frac{\sqrt{1}}{\sqrt{4\pi}}$$

$$r = \frac{1}{2\sqrt{\pi}}$$

7. (4.1) Consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

a. How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10\frac{ft^3}{min}$?

b. If the water level is decreasing at a rate of $3\frac{in}{min}$ when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.



a.

$$\frac{dh}{dt}\Big|_{h=10\,ft} = ?$$

$$\frac{dV}{dt} = -10\frac{ft^3}{min}$$

$$V = \frac{\pi}{3}r^2h$$

With a right cone, the ratio of height to radius is always the same (unless otherwise stated). Since we already have $\frac{dh}{dt}$, rewrite the equation in terms of h only to make it simpler.

$$\frac{r}{h} = \frac{5}{16}$$

$$r = \frac{5h}{16}$$

$$V = \frac{\pi}{3}\left(\frac{5h}{16}\right)^2 h = \frac{25\pi}{768}h^3$$

$$\frac{dV}{dt} = \frac{25\pi}{768}\cdot 3h^2\cdot\frac{dh}{dt}$$

$$-10 = \frac{25\pi}{768}\cdot 3(10)^2\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{128}{125\pi}\frac{ft}{min}$$

The height decreases at a rate of $\frac{128}{125\pi}\frac{ft}{min}$.

b.

$$\frac{dh}{dt}\Big|_{h=8\,ft} = 3\frac{in}{min} \times \frac{1\,ft}{12\,in} = \frac{1}{4}\frac{ft}{min}$$

$$\frac{dV}{dt} = ?$$

$$\frac{dV}{dt} = \frac{25\pi}{768} \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{25\pi}{768} \cdot 3(8)^2 \cdot \frac{1}{4}$$

$$\frac{dV}{dt} = \frac{25\pi}{16}\frac{ft^3}{min}$$

8. (4.1) A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is $10\frac{cm}{min}$ when the height is 1 m.

$$\frac{dV}{dt}\Big|_{h=1\,m} = ?$$

$$\frac{dh}{dt}\Big|_{h=1\,m} = 10\frac{cm}{min} \times \frac{1\,m}{100\,cm} = 0.1\frac{m}{min}$$

$$V = \pi r^2 h$$

The radius is always 2 for this cylinder, so it acts as a constant in the equation, so it can be included before differentiating.

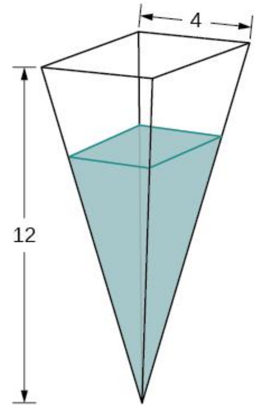
$$V = 4\pi h$$

$$\frac{dV}{dt} = 4\pi\frac{dh}{dt}$$

$$\frac{dV}{dt}\Big|_{h=1\,m} = 4\pi(0.1) = \frac{2\pi}{5}\frac{m^3}{min}$$

$$\frac{dV}{dt}\Big|_{h=1\,m} = \frac{2\pi}{5}\frac{m^3}{min}$$

9. (4.1) A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}\frac{m^3}{sec}$?



To relate the volume and height, use the equation for the volume of a pyramid.

$$V = \frac{1}{3}A_{base} \cdot h$$

Since it is a square pyramid, the area of the base is the base length squared.

$$V = \frac{1}{3}b^2 h$$

Differentiating this equation in its current form would result in having to know the instantaneous base length and rate of change of the base $\frac{db}{dt}$. This information is not given, but the base length variable b can be expressed in terms of the height h because the base and height always have the same ratio.

$$\frac{h}{b} = \frac{12}{4} = 3$$

$$b = \frac{h}{3}$$

$$V = \frac{1}{3}\left(\frac{h}{3}\right)^2 h$$

$$V = \frac{1}{27}h^3$$

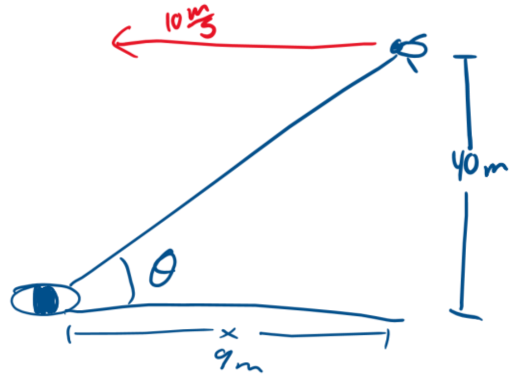
Now the derivative can be taken to get an equation with only the necessary variables.

$$\frac{dV}{dt} == \frac{1}{9}h^2 \cdot \frac{dh}{dt}$$

$$\frac{2}{3} = \frac{1}{9}(2)^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{3}{2}\frac{m}{s}$$

10. (4.1) You are stationary on the ground and are watching a bird fly horizontally at a rate of $10\,\frac{m}{sec}$. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?



$$x = 9\ m$$

$$\frac{dx}{dt} = -10\,\frac{m}{s}$$

$$\frac{d\theta}{dt} = ?$$

$$\tan\theta = \frac{40}{x}$$

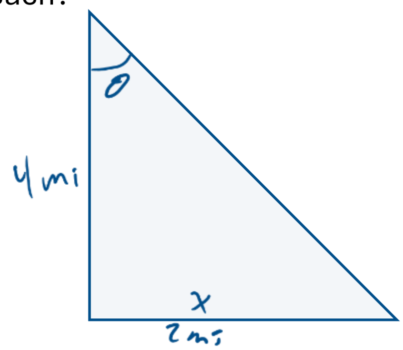
$$\theta = \arctan\frac{40}{x}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{40}{x}\right)^2}\cdot -\frac{40}{x^2}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt}\Big|_{x=9} = \frac{1}{1+\left(\frac{40}{9}\right)^2}\cdot -\frac{40}{9^2}\cdot -10 = \frac{400}{1681}$$

$$\frac{d\theta}{dt} = \frac{400}{1681}\,\frac{rad}{s}$$

11. (4.1) A lighthouse, L, is on an island 4 mi away from the closest point, P, on the beach If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



$$x = 2\ mi$$

$$\frac{d\theta}{dt} = 10\frac{rev}{min} \times \frac{2\pi\ rad}{1\ rev} = 20\pi\frac{rad}{min}$$

$$\frac{dx}{dt}\Big|_{x=2} = ?$$

$$\tan\theta = \frac{x}{4}$$

$$\theta = \arctan\frac{x}{4}$$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{4}\right)^2}\cdot\frac{1}{4}\cdot\frac{dx}{dt}$$

$$20\pi = \frac{1}{1+\left(\frac{2}{4}\right)^2}\cdot\frac{1}{4}\cdot\frac{dx}{dt}\Big|_{x=2}$$

$$\frac{dx}{dt} = 100\pi\frac{mi}{min}$$

12. (4.1) An ice sculpture melts in such a way that it can be modeled as a cone that maintains a conical shape as it decreases in size. The radius of the base of the cone is given by a twice-differentiable function r, where $r(t)$ is measured in centimeters and t is measured in days. The table gives selected values of $r'(t)$, the rate of change of the radius, over the time interval $0 \leq t \leq 12$. The height of the cone decreases at a rate of 2 centimeters per day. At time $t = 3$ days, the radius is 100 centimeters and the height is 50 centimeters. Find the rate of change of the volume of the cone with respect to time, in cubic centimeters per day, at time $t = 3$ days.

$V = \frac{1}{3}\pi r^2 h$

t (days)	0	3	7	10	12
$r'(t)$ (centimeters per day)	-6.1	-5.0	-4.4	-3.8	-3.5

$$r(3) = 100\ cm$$
$$h(3) = 50\ cm$$
$$\frac{dr}{dt}\Big|_{t=3} = -5\frac{cm}{day}$$
$$\frac{dh}{dt} = -2\frac{cm}{day}$$
$$\frac{dv}{dt}\Big|_{t=3} = ?$$

The ice sculpture maintains a conical shape, but the radius and height are not in a constant ratio. The height changes at a constant rate but the rate of change of the radius is changing with respect to time. The volume equation has to be differentiated using the product rule and the given values from the question and table.

$$V = \frac{1}{3}\pi r^2 h$$
$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dh}{dt} + \frac{2\pi}{3} r \frac{dr}{dt} h$$
$$\frac{dV}{dt}\Big|_{t=3} = \frac{\pi}{3}\left(r(3)\right)^2 h'(3) + \frac{2\pi}{3} r(3) r'(3) h(3)$$
$$\frac{dV}{dt}\Big|_{t=3} = \frac{\pi}{3}(100)^2(-2) + \frac{2\pi}{3}(100)(-5)(50)$$
$$\frac{dV}{dt}\Big|_{t=3} = -\frac{70000\pi}{3}\frac{cm^3}{day}$$

13. (4.2) Find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$.

a. $f(x) = \frac{1}{x}, a = 2$

$$f(2) = \frac{1}{2}$$
$$f'(x) = -\frac{1}{x^2}$$
$$f'(2) = -\frac{1}{4}$$
$$L(x) = -\frac{1}{4}(x-2) + \frac{1}{2}$$

b. $f(x) = \sin x, a = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = 1$$
$$f'(x) = \cos x$$
$$f'\left(\frac{\pi}{2}\right) = 0$$
$$L(x) = 0\left(x - \frac{\pi}{2}\right) + 1$$
$$L(x) = 1$$

c. $f(x) = \sin^2 x, a = 0$

$$f(0) = 0$$
$$f'(x) = 2\sin x$$
$$f'(0) = 0$$
$$L(x) = 0(x-0) + 0$$
$$L(x) = 0$$

14. (4.2) Compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$.

a. $\sin 0.02$

$$f(x) = \sin x\,, f'(x) = \cos x$$
$$x = 0$$
$$f(0) = 0, f'(0) = 1$$
$$L(x) = 1(x-0) + 0$$
$$L(x) = x$$
$$\sin 0.02 \approx L(0.02) = 0.02$$

b. $15.99^{\frac{1}{4}}$

$$f(x) = (16+x)^{\frac{1}{4}}, f'(x) = \frac{1}{4}(16+x)^{-\frac{3}{4}}$$
$$x = 0$$

$$f(0)=2, f'(0)=\frac{1}{32}$$
$$x=0$$
$$L(x)=\frac{1}{32}(x-0)+2$$
$$L(x)=\frac{x}{32}+2$$
$$(15.99)^{1/4}\approx L(-0.01)=1.9996875$$

c. $\sin 3.14$

$$f(x)=\sin x\,, f'(x)=\cos x$$
$$x=\pi$$
$$f'(\pi)=0, f'(\pi)=-1$$
$$L(x)=-1(x-\pi)+0$$
$$\sin 3.14\approx L(3.14)=0.001593$$

d. $\cos 0.01$

$$f(x)=\cos x\,, f'(x)=-\sin x$$
$$x=0$$
$$f(0)=1, f'(0)=0$$
$$L(x)=0(x-0)+1=1$$
$$\cos 0.01\approx L(0.01)=1$$

e. 1.01^{-3}

$$f(x)=(1+x)^{-3}, f'(x)=-\frac{3}{(1+x)^4}$$
$$x=0$$
$$f(0)=1, f'(0)=-3$$
$$L(x)=-3(x-0)+1$$
$$1.01^{-3}\approx L(0.01)=0.97$$

f. $\sqrt{8.99}$

$$f(x)=\sqrt{9-x}, f'(x)=\frac{1}{2\sqrt{9-x}}$$
$$x=0$$
$$f(0)=3, f'(0)=\frac{1}{6}$$
$$L(x)=\frac{1}{6}(x-0)+3$$
$$\sqrt{8.99}\approx L(-0.01)=3-\frac{1}{600}\approx 2.99833$$

15. (4.2) Find the differential of the function.

a. $y = x\cos x$

Use the product rule to find the differential.

$$dy = (-x\sin x + \cos x)dx$$

$$dy = (\cos x - x\sin x)dx$$

b. $y = \frac{x^2+2}{x-1}$

Use the quotient rule to find the differential.

$$dy = \frac{(x-1)(2x) - (x^2+2)}{(x-1)^2}dx$$

$$dy = \frac{2x^2 - 2x - x^2 - 2}{(x-1)^2}dx$$

$$dy = \frac{x^2 - 2x - 2}{(x-1)^2}dx$$

16. (4.2) Find the differential and evaluate for the given x and dx.

a. $y = \frac{1}{x+1}, x = 1, dx = 0.25$

$$y = (x+1)^{-1}, dy = -1(x+1)^{-2}dx$$

$$dy = -\frac{1}{(x+1)^2}dx$$

$$dy = -\frac{1}{(1+1)^2}\cdot 0.25$$

$$dy = -\frac{1}{16}$$

b. $y = \frac{3x^2+2}{\sqrt{x+1}}, x = 0, dx = 0.1$

Start with the quotient rule, then get a common denominator and simplify the numerator.

$$dy = \frac{\frac{6x\sqrt{x+1}}{1} - (3x^2+2)\frac{1}{2\sqrt{x+1}}}{x+1}dx$$

$$dy = \frac{12x(x+1) - 3x^2 - 2}{2\sqrt{x+1}}\cdot\frac{1}{x+1}dx = \frac{12x^2 + 12x - 3x^2 - 2}{2(x+1)^{\frac{3}{2}}}dx$$

$$dy = \frac{9x^2 + 12x - 2}{2(x+1)^{3/2}}dx$$

$$dy = \frac{9(0)^2 + 12(0) - 2}{2(0+1)^{\frac{3}{2}}}\cdot 0.1$$

$$dy = -0.1$$

c. $y = x^3 + 2x + \frac{1}{x}, x = 1, dx = 0.05$

$$dy = \left(3x^2 + 2 - \frac{1}{x^2}\right) dx$$
$$dy = \left(3(1)^2 + 2 - \frac{1}{(1)^2}\right) \cdot 0.05$$
$$dy = 0.2$$

17. (4.2) Find the change in volume dV or in surface area dA.

a. dA if the sides of a cube change from x to $x + dx$.

The formula for the surface area of a cube is $A = 6x^2$. Differentiate it to get the change in area. dx represents the infinitesimal change in the side length. When this change occurs, the resulting change in surface area is represented by dA.

$$dA = 12x \cdot dx$$

b. dV if the radius of a sphere changes from r by dr.

Similarly to the previous question, when the radius changes by dr, the chane in volume is dV, so the formula for the volume of a sphere will have to be differentiated.

$$V = \frac{4}{3}\pi r^3$$
$$dV = 4\pi r^2 \cdot dr$$

c. dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9\ cm$.

$$V = \pi r^2 h$$
$$h = 3$$
$$dr = 1.9\ cm - 2\ cm = -0.1\ cm$$
$$V = 3\pi r^2$$
$$dV = 6\pi r \cdot dr$$
$$dV = 6\pi(2\ cm) \cdot -0.1\ cm$$
$$dV = -1.2\pi\ cm^3$$

d. A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Start by making an equation for the volume of this pool.

$$V = 10 \cdot 20 \cdot h = 200h$$
$$dV = 200 \cdot dh$$
$$dh = 5.5\ ft - 6\ ft = -0.5\ ft$$
$$dV = 200 \cdot -0.5\ ft$$
$$dV = -100\ ft^3$$

18. (4.3) The formula for the position of the maximum or minimum of a quadratic $y = ax^2 + bx + c$ is $h = -\frac{b}{2a}$. Prove this formula using calculus.

The derivative at the maximum/minimum equals 0, so take the derivative and set it to 0.

$$y = ax^2 + bx + c$$
$$0 = \frac{dy}{dx} = 2ax + b$$

$$2ax = -b$$
$$x = -\frac{b}{2a}$$

19. (4.3) Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain using graphical arguments.
No. Cubic functions have opposite end behavior, so its range is $(-\infty, \infty)$. There can be no absolute maximum or minimum when the function reaches infinitely in both directions.

20. (4.3) Is it possible to have more than one absolute maximum? Explain using graphical arguments.
No. The term "absolute maximum" or "absolute minimum" refers to the y-value because it is the maximum/minimum value of the function (value means y). The function can only reach one maximum/minimum height. The function can reach that height at multiple x-values but the function will always have the same value.

21. (4.3) Consider the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?
An exponential function will go to ∞ or $-\infty$ no matter what a is, unless it is not an exponential function anymore. $a = 0$ forms the line $y = 0$, which has both an absolute maximum and minimum of $y = 0$ on $(-\infty, \infty)$.

22. (4.3) Find the critical points in the domains of the following functions.

a. $y = 4\sqrt{x} - x^2$

$$\frac{dy}{dx} = \frac{4}{2\sqrt{x}} - 2x$$
$$\frac{2}{\sqrt{x}} - 2x = 0$$
$$\frac{2}{\sqrt{x}} = 2x$$
$$\frac{1}{\sqrt{x}} = x$$
$$x^{\frac{3}{2}} = 1$$
$$x = 1$$

b. $y = \ln(x - 2)$

$$\frac{dy}{dx} = \frac{1}{x-2}$$
$$\frac{1}{x-2} = 0$$
$$1 \neq 0$$

There are no critical points for this function.

The value $x = 2$ does make the equation $\frac{1}{x-2}$ undefined, but it is not classified as a critical point because $x = 2$ is not in the domain of $\ln(x-2)$, because $\ln(2-2) = \ln 0$ does not exist.

c. $y = \sqrt{4-x^2}$

$$\frac{dy}{dx} = \frac{-2x}{\sqrt{4-x^2}}$$
$$-\frac{2x}{\sqrt{4-x^2}} = 0$$
$$-2x = 0$$
$$x = 0$$

The derivative can also be made undefined by making the denominator equal to 0.

$$\sqrt{4-x^2} = 0$$
$$4 - x^2 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

d. $y = \frac{x^2-1}{x^2+2x-3}$

$$\frac{dy}{dx} = \frac{(x^2+2x-3)(2x) - (x^2-1)(2x+2)}{(x^2+2x-3)^2}$$
$$\frac{dy}{dx} = \frac{2x^3+4x^2-6x-2x^3-2x^2+2x+2}{(x^2+2x-3)^2}$$
$$\frac{dy}{dx} = \frac{2x^2-2x+2}{(x^2+2x-3)^2} = \frac{(2x-2)(x-1)}{\left((x-1)(x+3)\right)^2}$$

$x = 1$ and $x = -3$ will make $\frac{dy}{dx}$ 0 or undefined, but neither are included in the domain of $y = \frac{x^2-1}{x^2+2x-3}$ since they make the denominator 0. The function has no critical points.

e. $y = x + \frac{1}{x}$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$
$$1 - \frac{1}{x^2} = 0$$
$$\frac{1}{x^2} = 1$$
$$x^2 = 1$$
$$x = \pm 1$$

23. (4.3) Find the local and/or absolute extrema for the functions over the specified domain.

a. $y = x^2 + \frac{2}{x}$ over $[1, 4]$

Find $\frac{dy}{dx}$ and set it equal to 0 to get critical points.

$$\frac{dy}{dx} = 2x - \frac{2}{x^2}$$
$$2x - \frac{2}{x^2} = 0$$
$$2x = \frac{2}{x^2}$$
$$x = \frac{1}{x^2}$$
$$x^3 = 1$$
$$x = 1$$

Since $x = 1$ is also an endpoint, we only need to test the endpoints for absolute extrema.

$$1^2 + \frac{2}{1} = 3$$
$$4^2 + \frac{2}{4} = \frac{33}{2}$$

So, the maximum occurs at $\left(4, \frac{33}{2}\right)$ and the minimum occurs at $(1, 3)$.

b. $y = \frac{1}{x - x^2}$ over $(0, 1)$

Checking the endpoints is not necessary because there are no endpoints on an open interval. Just check for critical points with the derivative of the function.

$$y = (x - x^2)^{-1}$$
$$\frac{dy}{dx} = -1(x - x^2)^{-2} \cdot (1 - 2x) = \frac{2x - 1}{(x - x^2)^2}$$
$$\frac{2x - 1}{(x - x^2)^2} = 0$$
$$2x - 1 = 0, x = \frac{1}{2}$$
$$(x - x^2)^2 = 0, x - x^2 = 0, x = x^2, x = 0, 1$$

Since $x = 0$ and $x = 1$ are not in the interval, they are not relevant.

$$\frac{1}{\frac{1}{2} - \left(\frac{1}{2}\right)^2} = 4$$

$\left(\frac{1}{2}, 4\right)$ is either a minimum or a maximum. Check the sign of the derivative around the critical point to see if the function changes from increasing to decreasing or decreasing to increasing. You can also look at the graph on a calculator.

$$\frac{dy}{dx}\Big|_{x=\frac{1}{4}} = -\frac{128}{9}$$

$$\frac{dy}{dx}\Big|_{x=\frac{3}{4}} = \frac{128}{9}$$

The derivative changes from negative to positive, so $\left(\frac{1}{2}, 4\right)$ is a minimum.

c. $y = x + \sin x$ over $[0, 2\pi]$

Checking for critical points from the derivative.

$$\frac{dy}{dx} = 1 + \cos x$$
$$1 + \cos x = 0$$
$$\cos x = -1$$
$$x = \pi$$

Now check the endpoints.

$$0 + \sin 0 = 0$$
$$2\pi + \sin 2\pi = 2\pi$$

0 and 2π are clearly the absolute extrema, but we need to see if $x = \pi$ is a local extremum or not. Check the derivative around the point and see if it changes signs. You can also look at the graph on a calculator.

$$\frac{dy}{dx}\Big|_{x=\frac{\pi}{2}} = 1$$
$$\frac{dy}{dx}\Big|_{x=\frac{3\pi}{2}} = 1$$

The derivative does not change signs at $x = \pi$, so it is not a local extremum.

The absolute maximum occurs at $(2\pi, 2\pi)$ and the absolute minimum occurs at $(0, 0)$.

d. $y = \sin x + \cos x$ over $[0, 2\pi]$

Take the derivative and set it equal to 0.

$$\frac{dy}{dx} = \cos x - \sin x$$
$$\cos x - \sin x = 0$$
$$\cos x = \sin x$$
$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \sqrt{2}$$
$$\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\sqrt{2}$$

Check the endpoints.

$$\sin 0 + \cos 0 = 1$$
$$2\pi + \cos 2\pi = 1$$

The endpoints are not critical points, nor are they the highest and lowest values, so they are not relevant.

$\left(\frac{\pi}{4}, \sqrt{2}\right)$ is the absolute maximum and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$ is the absolute minimum.

24. (4.3) Find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$. Note: This is an open interval, so there are no endpoints to check.

a. $y = x^2 + 4x + 5$

Set $\frac{dy}{dx}$ equal to 0, find the critical points.

$$\frac{dy}{dx} = 2x + 4$$
$$2x + 4 = 0$$
$$2x = -4$$
$$x = -2$$
$$(-2)^2 + 4(-2) + 5 = 1$$

We know that $(-2,1)$ is an absolute minimum because this function is a parabola opening up.

b. $y = 3x^4 + 8x^3 - 18x^2$

Set $\frac{dy}{dx}$ equal to 0, find the critical points.

$$\frac{dy}{dx} = 12x^3 + 24x^2 - 36x$$
$$12x(x^2 + 2x - 3) = 0$$
$$12x(x - 1)(x + 3) = 0$$
$$x = 0, 1, -3$$

Organize the critical points easier by lining them up in a table. Plugging the function in a calculator can help get the values faster.

x	y
-3	-135
0	0
1	-7

The absolute minimum is $(-3, -135)$.

$(0, 0)$ is a local maximum. It is not an absolute maximum because this function has an even degree with a positive leading coefficient, so the function reaches ∞ on both ends.

Check if the derivative changes signs at $x = 1$ to check for local extrema.

$$\frac{dy}{dx}\Big|_{x=\frac{1}{2}} = -\frac{21}{2}$$
$$\frac{dy}{dx}\Big|_{x=2} = 120$$

The derivative changes from negative to positive, so $(1, -7)$ is a local minimum. You can also look at the graph.

c. $y = \frac{x^2+x+6}{x-1}$

Set $\frac{dy}{dx}$ equal to 0, find the critical points.

$$\frac{dy}{dx} = \frac{(x-1)(2x+1)-(x^2+x+6)}{(x-1)^2} = \frac{2x^2-x-1-x^2-x-6}{(x-1)^2} = \frac{x^2-2x-7}{(x-1)^2}$$

$$\frac{x^2-2x-7}{(x-1)^2} = 0$$

$$x^2-2x-7=0$$

We are not setting the denominator equal to 0 because $x = 1$ is not in the domain of the original function.

The quadratic formula is needed to factor the quadratic.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{4+28}}{2}$$

$$x = \frac{2 \pm 4\sqrt{2}}{2}$$

$$x = 1+2\sqrt{2}, 1-2\sqrt{2}$$

By looking at the graph, the function will have no absolute extrema, so the critical points represent local extrema.

By looking at the graph:

$\left(1+2\sqrt{2}, 3+4\sqrt{2}\right)$ is a local minimum.

$\left(1-2\sqrt{2}, 3-4\sqrt{2}\right)$ is a local maximum.

25. (4.3) Find the critical points, maxima, and minima for the piecewise function $y = \begin{cases} x^2+1, & x \le 1 \\ x^2-4x+5, & x > 1 \end{cases}$

For the left side, we are checking the interval $(-\infty, 1]$. For the right side we are checking $(1, \infty)$.

Left:

$$\frac{dy}{dx} = 2x = 0, x = 0$$

$$0^2+1=1$$

Right:

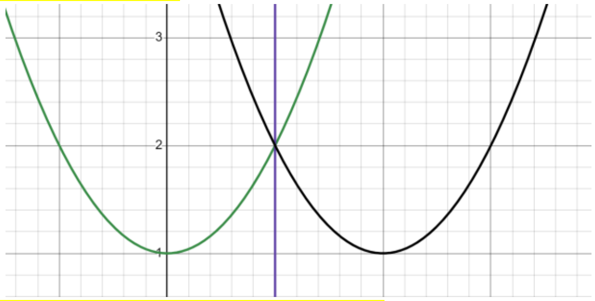
$$\frac{dy}{dx} = 2x-4=0$$

$$2x = 4$$

$$x = 2$$

$$2^2 - 4(2) + 5 = 1$$

Both parts of the piecewise function are parabolas opening up, and the two minimums are the same, so they are both the absolute minimum. For the same reason, $(1, 2)$ is a local maximum because it forms a cusp (sharp point).



The absolute minimum occurs at $(0,1)$ and $(2,1)$.

26. (4.4) Over what intervals (if any) does the MVT apply? Explain.

a. $y = \frac{1}{x^3}$

The function is continuous and differentiable everywhere except $x = 0$. Therefore, the MVT applies on any closed interval in $(-\infty, 0)$ or $(0, \infty)$.

b. $y = \sqrt{x^2 - 4}$

For this function, x cannot be between $(-2, 2)$ because the square root would be negative. Also, x cannot equal -2 and 2 because the function is not differentiable at those values:

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 4}}$$

So the MVT applies on any closed interval in $(-\infty, -2)$ or $(2, \infty)$.

27. (4.4) Use the MVT to find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$

Find what c must equal, then set the derivative equal to that.

a. $f(x) = x^3$

$$f'(c) = \frac{8}{2} = 4$$
$$f'(x) = 3x^2$$
$$3c^2 = 4$$
$$c = \sqrt{\frac{4}{3}}$$
$$c = \frac{2\sqrt{3}}{3}$$

b. $f(x) = \cos 2\pi x$

$$f'(c) = \frac{1 - 1}{2} = 0$$
$$\frac{dy}{dx} = -2\pi \sin 2\pi x = 0$$
$$\sin 2\pi x = 0$$

$$\sin 2\pi c = 0$$

$\sin x = 0$ at $x = 0, \pi, 2\pi, \frac{5\pi}{2}, \ldots$

Set the inside term of the derivative to what makes the parent function equal to 0 to see the new c values.

$$2\pi c = 0, c = 0$$
$$2\pi c = \pi, c = \frac{1}{2}$$
$$2\pi c = 2\pi, c = 1$$
$$2\pi c = \frac{5\pi}{2}, c = \frac{3}{2}$$

The applicable c values that are between 0 and 2 are $c = \frac{1}{2}, 1, \frac{3}{2}$

c. $f(x) = (x-1)^{10}$

$$f'(c) = \frac{1-1}{2} = 0$$
$$10(x-1)^9 = 0$$
$$10(c-1)^9 = 0$$
$$c - 1 = 0$$
$$c = 1$$

28. (4.4) Show that there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the MVT does not apply over the interval $[-1, 1]$.

a. $f(x) = \left|x - \frac{1}{2}\right|$

$f(x)$ is not differentiable on $[-1, 1]$ so MVT does not apply.

b. $f(x) = \sqrt{|x|}$

There is a cusp at $x = 0$, so it is not differentiable, and MVT does not apply.

29. (4.4) Does the MVT apply for the functions over the given interval? Explain.

a. $y = e^x$ over $[0, 1]$

Since e^x is continuous and differentiable over this interval, MVT applies.

b. $f(x) = \tan(2\pi x)$ over $[0, 2]$

The function is discontinuous at many points in this interval $\left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\right)$, so MVT does not apply.

c. $y = \frac{1}{|x+1|}$ over $[0, 3]$

In the given interval, y acts like $\frac{1}{x+1}$ which is continuous and differentiable on it, so MVT applies.

d. $y = \frac{x^2+3x+2}{x}$ over $[-1, 1]$

The interval contains $x = 0$, which makes the function discontinuous, so MVT does not apply.

e. $y = \ln(x+1)$ over $[0, e-1]$

The function is continuous and differentiable after $x = -1$, which includes the given interval, so MVT applies.

f. $y = 5 + |x|$ over $[-1, 1]$

$|x|$ is not differentiable at $x = 0$, so MVT does not apply.

30. (4.4) Two cars drive from one stoplight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

The MVT states that between two points in a graph, there is a point that has a tangent line equal to the secant line between the two points. On position vs time graphs for the individual cars, the secant line between the start and end is the average velocity, and the MVT states that each car must achieve that average velocity at least once. If they leave and arrive at the same time, both achieve the same average velocity.

31. (4.5) For $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum? Explain.

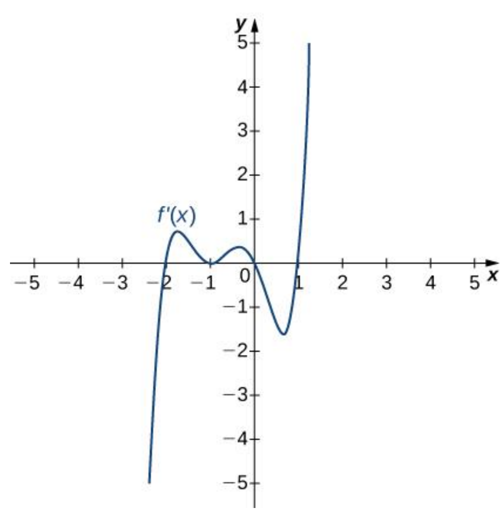
No. $x = 0$ cannot be a local extremum because $\frac{dy}{dx}$ does not change signs.

32. (4.5) Is it possible for a point c to be both an inflection point and a local extremum of a twice differentiable function? Explain no.

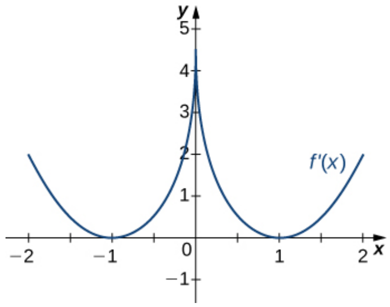
No. A twice differentiable function is a parabola. A parabola does not change concavity and has no inflection points. The second derivative stays constant.

33. (4.5) For the graphs of $f'(x)$, List all intervals where f is increasing or decreasing.

f is increasing when $f'(x) > 0$, and decreasing when $f'(x) < 0$.

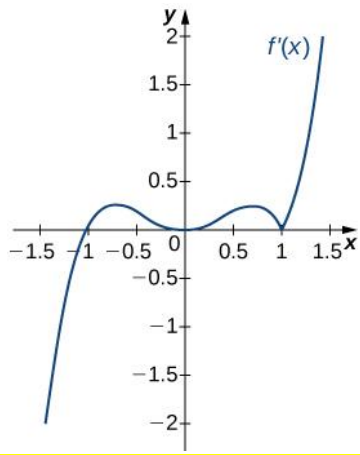
a. 

f is increasing over $-2 < x < 0$ and $x > 1$. f is decreasing over $x < -2$ and $0 < x < 1$.

b. 

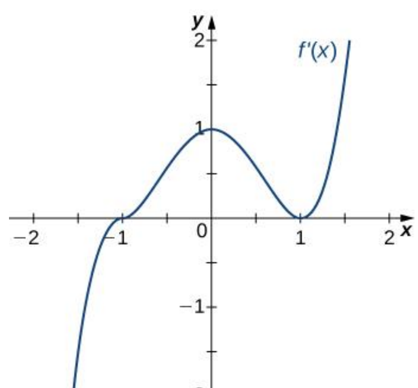
f is increasing over $(-\infty, \infty)$

34. (4.5) From the graph of f', list all intervals where f is increasing and decreasing, and where the minima and maxima are located.



f is increasing on $-1 < x < \infty$ and decreasing on $x < -1$.
A maximum occurs when the derivative changes from positive to negative. A minimum occurs when the derivative changes from negative to positive.
f has no maximum.
f has a minimum at $x = -1$.

35. (4.5) From the graph of f', list all inflection points and intervals where f is concave up and concave down.



An inflection point of f is when the slope of f' changes sign. f is concave up when f' is increasing, and concave down when f is decreasing.

f is concave up on $-\infty < x < 0$ and $x > 1$.
f is concave down on $0 < x < 1$.
f has an inflection point at $x = 0$ and $x = 1$.

36. (4.5) List intervals where f is increasing or decreasing. List the local minima and maxima of f. $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

Start by taking the derivative of f and set it equal to 0 to find the critical points.

$$f'(x) = \cos x + 3\sin^2 x \cos x = 0$$
$$\cos x\,(1 + 3\sin^2 x) = 0$$
$$\cos x = 0, x = -\frac{\pi}{2}, \frac{\pi}{2}$$
$$1 + 3\sin^2 x = 0$$
$$\sin^2 x \neq -\frac{1}{3}$$

The second part of the derivative cannot equal zero because you can't take the square root of a negative. Our only critical points are $x = -\frac{\pi}{2}, \frac{\pi}{2}$. Make the number line of f' and choose points between the critical points to check the signs. You can put the function in the calculator.

$$f'\left(-\frac{3\pi}{4}\right) = -\frac{5\sqrt{2}}{4} < 0$$
$$f'(0) = 1 > 0$$
$$f'\left(\frac{3\pi}{4}\right) = \frac{-5\sqrt{2}}{4} < 0$$

f is increasing on $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and decreasing on $x < -\frac{\pi}{2}$ and $x > \frac{\pi}{2}$

f has a local maximum at $x = \frac{\pi}{2}$ and a local minimum at $x = -\frac{\pi}{2}$

37. (4.5) Determine the intervals where f is increasing or decreasing, the local minima and maxima of f, intervals where f Is concave up and concave down, and the inflection points of f.

a. $f(x) = x^3 - 6x^2$

Set the derivative equal to 0 to find the critical points.

$$f'(x) = 3x^2 - 12x = 0$$
$$3x^2 = 12x$$
$$x^2 = 4x$$
$$x = 0, 4$$

Make the number line for $f'(x)$ to check signs. Plug in points in the calculator (see #36).

f is increasing on $x < 0$ and $x > 4$
f is decreasing on $0 < x < 4$
There is a local minimum at $x = 4$ and a local maximum at $x = 0$.
Do the same thing with the second derivative to find possible inflection points.

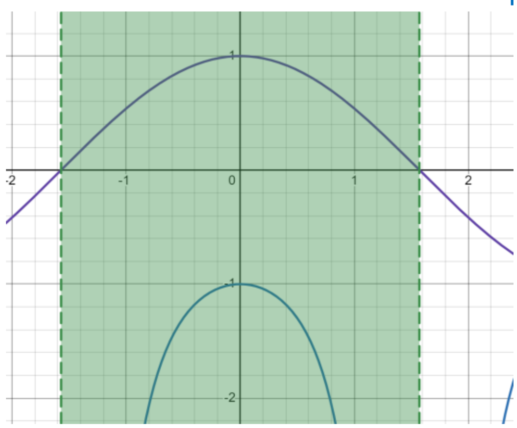
$$f''(x) = 6x - 12 = 0$$
$$6x = 12$$
$$x = 2$$

f is concave up on $x > 2$ and concave down on $x < 2$
There is an inflection point at $x = 2$.

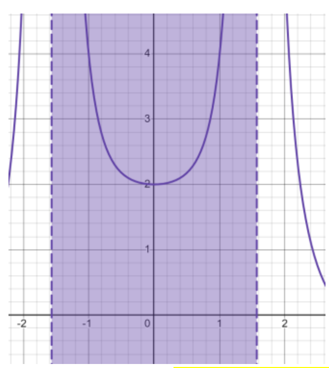
b. $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
Set the derivative equal to 0 to find critical points.

$$f'(x) = \cos x + \sec^2 x = 0$$
$$\cos x = -\sec^2 x$$

I would graph this on a calculator and find the intersection points.



$\cos x$ and $-\sec^2 x$ do not intersect at this interval so the derivative does not equal zero. You could have also graphed the whole derivative and seen if it hits the x-axis.



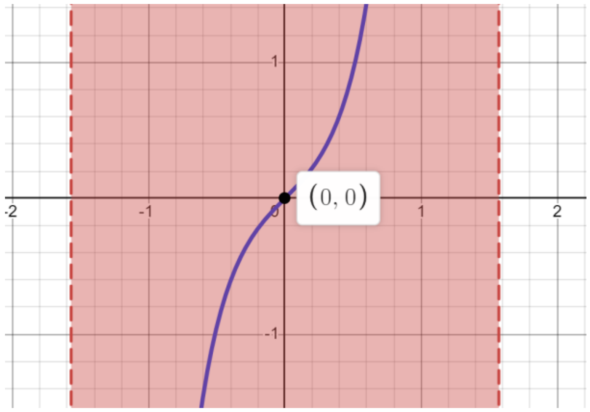
Since the derivative never changes signs there are no local extrema. We don't need to make a number line like #36 because there are no critical points, and we can see from the graph that the derivative is always positive. We could also plug in a point like $x = 0$ and see what sign it has.

f is increasing on $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Set the second derivative equal to 0.

$$f''(x) = -\sin x + 2\sec x \sec x \tan x = 2\sec^2 x \tan x - \sin x = 0$$

Graph this in the calculator to make it easier to see the intersection.



The graph of the second derivative equals 0 at $x = 0$ and it changes signs at that point, so $x = 0$ is an inflection point.

Since $f'' < 0$ on the left, f is concave down on $x < 0$.

Since $f'' > 0$ on the right, f is concave up on $x > 0$.

c. $f(x) = \frac{1}{1-x}, x \neq 1$

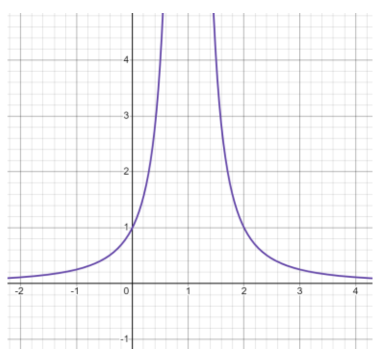
Set the derivative equal to 0 to find the critical points.

$$f(x) = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2} \cdot -1 = \frac{1}{(1-x)^2} = 0$$

The derivative cannot equal 0 so there are no local extrema.

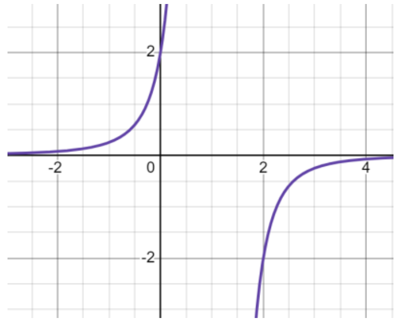
By looking at the graph we can see that f' is always positive [where defined], so f is increasing on its domain. Note: You will have to plot a point and show its sign as proof.



Now set the second derivative equal to 0 to find the possible inflection points.

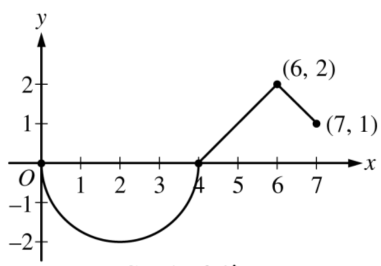
$$f''(x) = \frac{2}{(1-x)^3} = 0$$

Once again this function is not going to equal 0 so there are no inflection points in the domain of f. That does not mean the graph does not change concavity.



f'' is positive on the left and negative on the right, so f is concave up on $x < 1$ and concave down on $x > 1$. Note: You will have to plot points and show their signs as proof.

38. (4.5) Let f be a differentiable function. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure below.



Graph of f'

a. Find the x-coordiantes of all points of inflection on the graph of f for $0 < x < 7$. Justify your answer.
There is a point of inflection when the second derivative changes signs. That means that the first derivative changes from increasing to decreasing or vice versa. This occurs at $x = 2$ and $x = 6$.

b. Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
g is decreasing when its derivative g' is negative.

$$g'(x) = f'(x) - 1$$

$$f'(x) - 1 = 0$$

$$f'(x) = 1$$

$$x = 5, 7$$

Make a number line of $g'(x)$ to check around the critical point.

$g(x)$ is decreasing on $[0, 5]$ because $g'(x) < 0$.

39. (4.6) For the following functions, determine whether there is an asymptote at $x = a$. Explain.

a. $f(x) = \frac{x}{x-2}$, $a = 2$

There is a vertical asymptote at $x = 2$ since it makes the denominator 0 but not the numerator.

b. $f(x) = (x-1)^{-\frac{1}{3}}$, $a = 1$.

Rewriting the function:

$$f(x) = \frac{1}{\sqrt[3]{x-1}}$$

There is a vertical asymptote at $x = 1$ since it makes the denominator 0 but not the numerator.

40. (4.6) Evaluate the limit.

a. $\lim_{x\to\infty} \frac{1}{3x+6}$

Since the degree in the denominator is higher than the degree in the numerator, the limit equals 0. Since the numerator is just a number you can also do direct substitution with ∞.

$$\lim_{x\to\infty} \frac{1}{3x+6} = \frac{1}{3\infty+6} = \frac{1}{\infty} = 0$$

b. $\lim_{x\to\infty} \frac{x^2-2x+5}{x+2}$

Since the degree in the numerator is higher than the degree in the denominator, the limit equals either $-\infty$ or ∞. The leading term is squared with a positive coefficient, so its ∞.

$$\lim_{x\to\infty} \frac{x^2-2x+5}{x+2} = \infty$$

c. $\lim_{x\to-\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4}$

Rewrite the denominator in standard form so we can see it easier.

$$\lim_{x\to-\infty} \frac{x^4-4x^3+1}{-7x^4-2x^2+2}$$

The leading coefficients have x^4< which turns the $-\infty$ into ∞. Both the numerator and the denominator have the same degree (4), so the limit is the ratio between the leading coefficients.

$$\lim_{x\to-\infty} \frac{x^4-4x^3+1}{-7x^4-2x^2+2} = -\frac{1}{7}$$

d. $\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2}$

Rewrite the limit with the square root law.

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-1}}{x+2} = \lim_{x\to-\infty} \frac{\sqrt{4x^2}}{x} = \lim_{x\to-\infty} \frac{2x}{x}$$

Now the limit is easier to calculate and is just the ratio of the two leading coefficients. The only difference is that x approaches $-\infty$ so the limit value is actually -2.

$$\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}=-2$$

e. $\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}$

Rewrite the limit with the square root law.

$$\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2-1}}=\lim_{x\to-\infty}\frac{4x}{\sqrt{x^2}}=\lim_{x\to-\infty}\frac{4x}{x}$$

Similarly to the previous question, $x\to-\infty$, so

$$\lim_{x\to-\infty}\frac{4x}{x}=-4$$

41. (4.6) Find the horizontal and vertical asymptotes.

a. $f(x)=x-\frac{9}{x}$

A vertical asymptote occurs when the denominator equals 0 but not the numerator. To find this we need a common denominator.

$$\frac{x}{1}-\frac{9}{x}=\frac{x^2-9}{x}$$

$x=0$ makes the denominator 0 but the numerator is -9, so there is a vertical asymptote at $x=0$.

To check for horizontal asymptotes, the limit as x approcaches ∞ or $-\infty$ equals some value L. The degree in the numerator is bigger than the degree in the denominator, so plugging in ∞ and $-\infty$ would not result in a real number, so there are no horizontal asymptotes.

b. $f(x)=\frac{x^3}{4-x^2}$

Factor the denominator.

$$f(x)=\frac{x^3}{(2-x)(x+2)}$$

The vertical asymptotes make the denominator equal to 0 but not the numerator. So, there are vertical asymptotes at $x=2$ and $x=-2$.

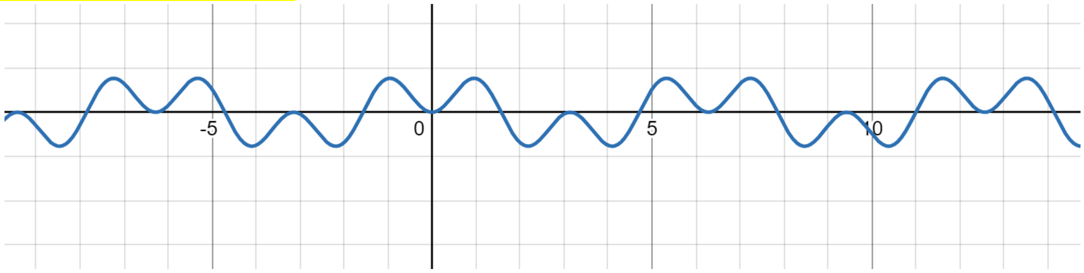
Horizontal asymptotes are calculated by taking the limit as x approaches $-\infty$ and ∞, but similarly to (a), the limit would not result in a real number, so there are no horizontal asymptotes.

c. $f(x)=\sin x\sin 2x$

This function has no denominator, so it can't have a vertical asymptote. $\sin x$ and $\sin 2x$ are oscillating functions that do not have limits. Recall the limit product law:

$$\lim_{x\to a}p(x)q(x)=\lim_{x\to a}p(x)\times\lim_{x\to a}q(x)$$

The limit of the overall function will therefore not exist, so there are no horizontal or vertical asymptotes.



d. $f(x) = \frac{x \sin x}{x^2 - 1}$

Factoring the denominator makes it easy to find the vertical asymptotes.

$$f(x) = \frac{x \sin x}{(x+1)(x-1)}$$

There are vertical asymptotes at $x = \pm 1$.

Take the limit as $x \to \pm\infty$ to find the horizontal asymptote(s).

$$\lim_{x\to\infty} \frac{x \sin x}{x^2 - 1}$$

This limit can be solved by dividing both terms by x in the denominator.

$$\lim_{x\to\infty} \frac{x \sin x}{x - \frac{1}{x}}$$

The limit is equal to the limit of the term with the greatest degree, so we can get rid of the $\frac{1}{x}$.

$$\lim_{x\to\infty} \frac{x \sin x}{x - \frac{1}{x}} = \lim_{x\to\infty} \frac{x \sin x}{x}$$

The resulting limit is a formula you should have memorized.

$$\lim_{x\to\infty} \frac{x \sin x}{x} = 0$$

The limit as $x \to -\infty$ is the same.

There is a horizontal asymptote at $y = 0$.

e. $f(x) = \frac{1}{x^3 + x^2}$

Factor the denominator.

$$\frac{1}{x^2(x+1)}$$

There are vertical asymptotes at $x = 0$ and $x = -1$ (see (a)).

Take the limit as $x \to \pm\infty$ to get the horizontal asymptotes.

$$\lim_{x\to\infty} \frac{1}{x^3 + x^2} = 0$$

The limit equals 0 because the degree is higher in the denominator. You will get the same result when $x \to -\infty$.

There is a horizontal asymptote at $y = 0$.

f. $f(x) = \frac{x^3+1}{x^3-1}$

There is a vertical asymptote at $x = 1$ because it makes the denominator equal 0 but not the numerator.

Take the limit as $x \to \pm\infty$ to get the horizontal asymptotes.

$$\lim_{x\to\infty} \frac{x^3+1}{x^3-1} = 1$$

The limit equals 1 because the numerator and denominator have the same degrees. The limit is the ratio of the leading coefficients, which is 1. You get the same result when $x \to -\infty$.

There is a horizontal asymptote at $y = 1$.

g. $f(x) = x - \sin x$

This function has no denominator, so there are no vertical asymptotes (see (c)).

Take the limit as $x \to \pm\infty$ to get the horizontal asymptotes.

$$\lim_{x\to\infty} x - \sin x$$

$\sin x$ oscillates so it does not have a limit. The limit result will depend on x.

$$\lim_{x\to\infty} x - \sin x = \infty$$

$$\lim_{x\to-\infty} x - \sin x = -\infty$$

None of these limits are real numbers, so there are no horizontal asymptotes.

42. (4.6) For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$ must have what relation?

$f(x)$ can have a vertical asymptote at $x = 0$ if it makes the denominator equal to 0 but not the numerator. Therefore, $P(0) \neq 0$ and $Q(0) = 0$.

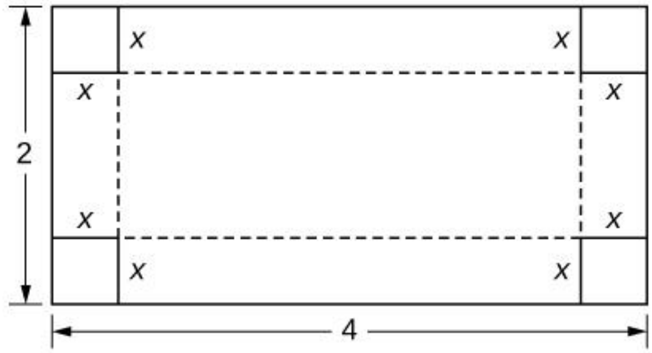
43. (4.7) Why do you need to check the sign of the derivative around the critical points in an optimization problem.

The goal of an optimization problem is to minimize or maximize a quantity. Without checking around the critical points, the critical point could be a minimum, maximum, or neither.

44. (4.7) True or false? For every continuous nonlinear function, you can find the value x that maximizes the function.

False. Some functions may only have a minimum. Ex: $y = x^2$

45. (4.7) You are constructing a cardboard box with the dimensions $2\ m$ by $4\ m$. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



Get a function for volume in terms of x so it can be differentiated and maximized.

$$V = l \cdot w \cdot h$$
$$h = x$$
$$l = 2 - 2x$$
$$w = 4 - 2x$$
$$V = (2 - 2x)(4 - 2x)(x)$$
$$V = x(4x^2 - 12x + 8) = 4x^3 - 12x^2 + 8x$$

Set the derivative equal to 0 and find the maximum.

$$\frac{dV}{dx} = 12x^2 - 24x + 8 = 0$$
$$4(3x^2 - 6x + 2) = 0$$

Use the quadratic formula.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

Obviously x cannot be less than 0 because that's impossible, and it can't equal 0 because there wouldn't be a box. $1 - \frac{\sqrt{3}}{3}$ is the only zero between 0 and 1.

Plug in $1 - \frac{\sqrt{3}}{3}$ into the volume equation to get the volume. It's better to plug the function in the calculator then solve at the x-value.

$$V = 4\left(1 - \frac{\sqrt{3}}{3}\right)^3 - 12\left(1 - \frac{\sqrt{3}}{3}\right)^2 + 8\left(1 - \frac{\sqrt{3}}{3}\right)$$
$$V = \frac{8\sqrt{3}}{9}$$

Plug $1 - \frac{\sqrt{3}}{3}$ into the definitions for l and w.

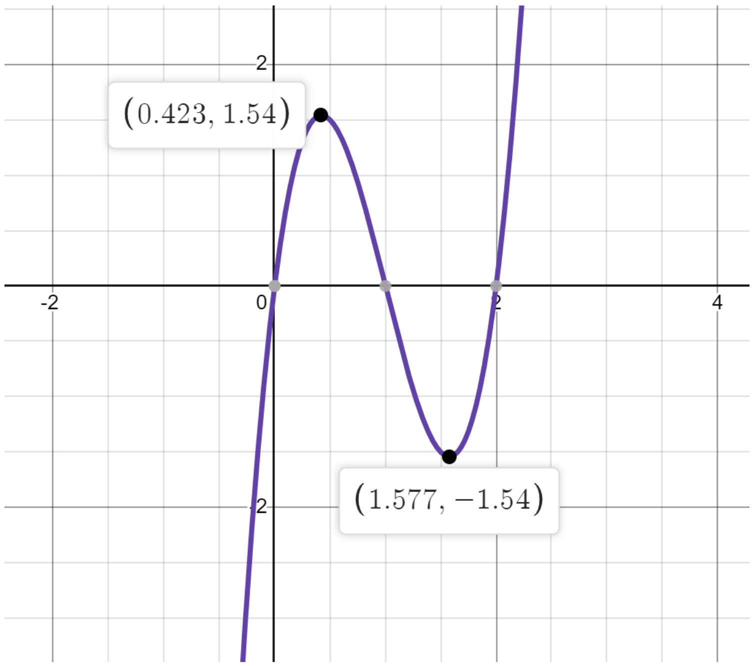
$$l = 2 - 2x = 2 - 2\left(1 - \frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}$$
$$w = 4 - 2x = 4 - 2\left(1 - \frac{\sqrt{3}}{3}\right) = 2 + \frac{2\sqrt{3}}{3}$$

$$l = \frac{2\sqrt{3}}{3}$$
$$w = 2 + \frac{2\sqrt{3}}{3}$$

Use the second derivative test to verify that the critical point is a maximum.

$$\frac{d^2V}{dx^2} = 24x - 24$$
$$\frac{d^2V}{dx^2}\Big|_{x=1-\frac{\sqrt{3}}{3}} = -8\sqrt{3} < 0$$

The volume function is concave down, so the critical point is a maximum.



$1 + \frac{\sqrt{3}}{3}$ had a minimum so we couldn't have used it anyway.

46. (4.7) You need to construct a fence around an area of 1600 ft^2. What are the dimensions of the rectangular pen to minimize the amount of material needed?

The amount of material needed is the perimeter. The equation for the perimeter of a rectangle is:

$$P = 2l + 2w$$

This is the equation that will be minimized. The problem is the two variables makes it impossible to know what to differentiate with respect to. Use the area equation to find a relationship between l and w.

$$A = 1600 = l \cdot w$$
$$l = \frac{1600}{w}$$

Now plug this relationship into the perimeter equation to get it in terms of one dimension.

$$P = 2\left(\frac{1600}{w} + w\right) = 2\left(\frac{1600 + w^2}{w}\right) = \frac{3200 + 2w^2}{w}$$

Now differentiate it with respect to w and set it equal to 0 to get the width that minimizes the perimeter.

$$\frac{dP}{dt} = \frac{w(4w) - (3200 + 2w^2)}{w^2} = \frac{4w^2 - 2w^2 - 3200}{w^2} = \frac{2w^2 - 3200}{w^2} = 0$$

$$2w^2 - 3200 = 0$$

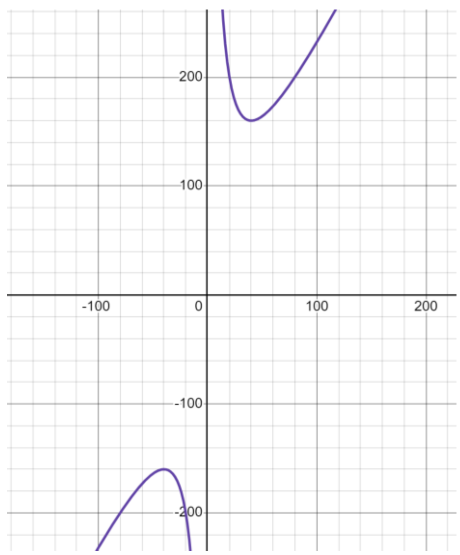
$$2w^2 = 3200$$

$$w^2 = 1600$$

$$w = 40\ ft$$

Use the first derivative test to test around the critical point to verify that it is a minimum.

$f'(39)$	40	$f'(41)$
< 0		> 0



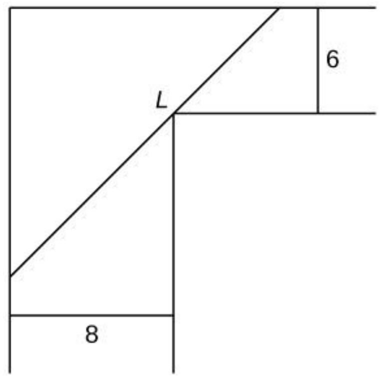
We do have a minimum at $w = 40$. Now calculate the length from the width and area.

$$l = \frac{1600}{w} = \frac{1600}{40}$$

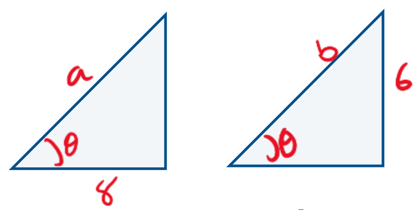
$$l = 40\ ft$$

So the dimensions of the pen are $40\ ft \times 40\ ft$.

47. (4.7) You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



On the right side of the item, there are two right triangles with a common angle θ, whose hypotenuses add to be L.



$$L = a + b$$

Use trig functions to get the values for a and b.

$$\cos\theta = \frac{8}{a}, a = \frac{8}{\cos\theta}$$

$$\sin\theta = \frac{6}{b}, b = \frac{6}{\sin\theta}$$

$$L = \frac{8}{\cos\theta} + \frac{6}{\sin\theta}$$

Set the derivative equal to 0 and get the maximum length.

$$\frac{dL}{d\theta} = \frac{8\sin\theta}{\cos^2\theta} - \frac{6\cos\theta}{\sin^2\theta} = 0$$

$$\frac{8\sin\theta}{\cos^2\theta} = \frac{6\cos\theta}{\sin^2\theta}$$

Cross multiply.

$$8\sin^3\theta = 6\cos^3\theta$$

$$\frac{3}{4} = \frac{\sin^3\theta}{\cos^3\theta} = \tan^3\theta$$

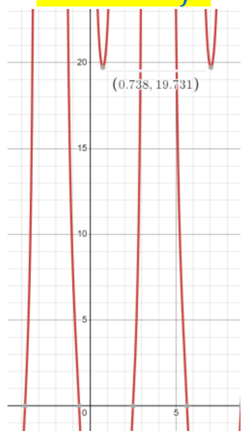
$$\tan\theta = \sqrt[3]{\frac{3}{4}}$$

$$\theta = \arctan\sqrt[3]{\frac{3}{4}}$$

Plug θ into the equation for L to get the maximum length.

$$L = \frac{8}{\cos\left(\arctan\sqrt[3]{\frac{3}{4}}\right)} + \frac{6}{\sin\left(\arctan\sqrt[3]{\frac{3}{4}}\right)}$$

$$L \approx 19.73\ ft$$



That may look like a minimum, and it is. This is because of what the function value is returning. The function value is actually returning the sum of two hypotenuses with one side length 8 and another side length 6. When we say that the critical point is a minimum, it means that it is the minimum length of the sum of the two hypotenuses. Of course, in real life we could have a shorter object go through the hallway, but it would not be the hypotenuses of the triangles in the problem because of the side length constraints. So, the minimum of the function is the maximum in real life.

48. (4.7) A Truck uses gas as $g(v) = av + \frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. Assuming a and b are positive, at what speed is fuel consumption minimized?

This problem involves no helper function, just find the minimum of the function.

$$g'(v) = a - \frac{b}{v^2} = 0$$

$$a = \frac{b}{v^2}$$

$$av^2 = b$$

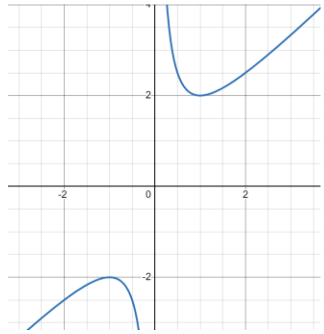
$$v^2 = \frac{b}{a}$$

$$v = \sqrt{\frac{b}{a}}$$

Use the second derivative test to verify that the critical point is a minimum.

$$g''(v) = 0 + \frac{2b}{v^3} = \frac{2b}{v^3}$$

If a and b are positive, the second derivative is positive, and the function g will always be concave up, so a minimum occurs at the critical point.



49. (4.7) Consider a limousine that gets $m(v) = \frac{120-2v}{5}$ miles per gallon at speed v, the chauffeur costs $\frac{\$15}{hr}$, and gas is $\frac{\$3.5}{gal}$. Find the cheapest driving speed.

We will need to minimize some function that represents the cost per mile as a function of speed. We aren't given this function so we will make it ourselves. This will take a lot of dimensional analysis/stoichiometry to convert the units to $\frac{\$}{mi}$. There are two factors of the cost: the price of gas and the cost of the chauffeur. They will have to be converted individually and then added together for the final function.

Starting with the price of gas:

$$\frac{\$3.50}{gal} \times \frac{1\ gal}{\frac{120-2v}{5}\,mi} = \frac{17.5}{120-2v}\frac{\$}{mi}$$

And now the cost of the chauffeur:

$$\frac{\$15}{hr} \times \frac{1\ hr}{v\ mi} = \frac{15}{v}\frac{\$}{mi}$$

Now that the units are the same, they can be added to one function that represents the total cost per mile.

$$C(v) = \frac{17.5}{120-2v} + \frac{15}{v}$$

Taket the derivative and find the minimum. Start by simplifying $C(v)$.

$$C(v) = \frac{17.5}{2(60-v)} + \frac{15}{v} = \frac{8.75}{60-v} + \frac{15}{v} = \frac{8.75v + 900 - 15v}{60v - v^2} = \frac{-6.25v + 900}{60v - v^2}$$

$$C'(v) = 0 = \frac{(60v - v^2)(-6.25) - (-6.25v + 900)(-2v + 60)}{(60v - v^2)^2}$$

$$0 = \frac{-375v + 6.25v^2 - (12.5v^2 - 375v - 1800v + 54000)}{(60v - v^2)^2}$$

$$0 = -375v + 6.25v^2 - 12.5v^2 + 375v + 1800v - 54000$$

$$0 = -6.25v^2 + 1800v - 54000$$

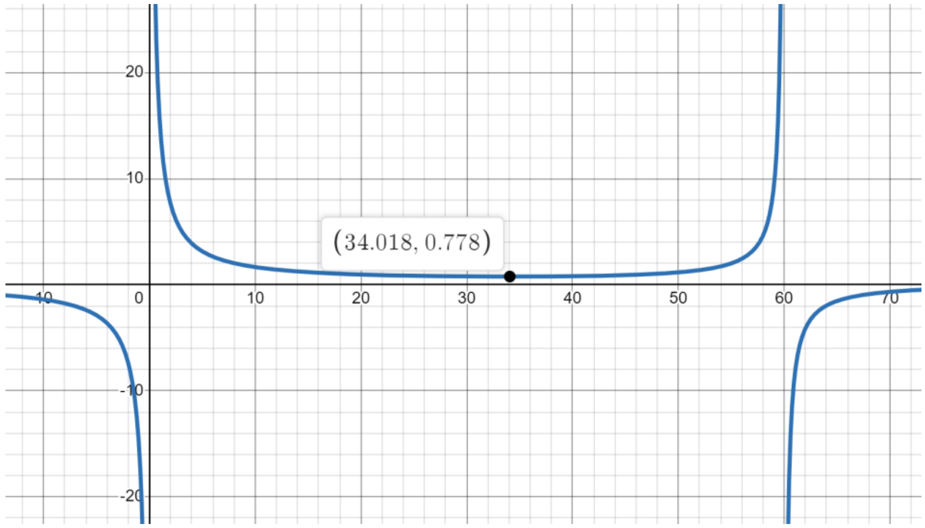
$$v \approx 34\ mph$$

Use the second derivative test to verify that the critical point is a minimum.

$$C''(v) = -12.5v + 1800$$

$$C''(34) = 1375 > 0$$

The graph of C is concave up, so a minimum occurs at the critical point.



The cheapest driving speed is about $34\ mph$.

50. (4.7) A pizzeria sells pizzas for a revenue of $R(x) = 10x$ and costs $C(x) = 2x + x^2$, where x represents the number of pizzas. How many pizzas sold maximizes the profit?

Profit is revenue minus cost.

$$P(x) = R(x) - C(x) = 10x - 2x - x^2 = 8x - x^2$$

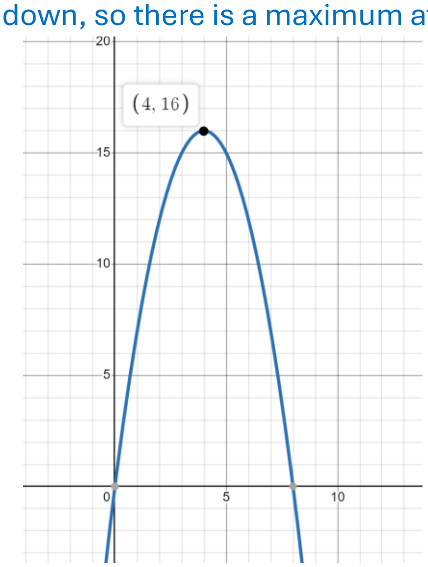
Take the derivative and get the critical points.

$$P'(x) = 8 - 2x = 0$$
$$8 = 2x$$
$$x = 4$$

Use the second derivative test to verify that the critical point is a maximum.

$$P''(x) = -2 < 0$$

The graph of P is concave down, so there is a maximum at $x = 4$.



51. (4.7) Consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x. Chose x to maximize the sum of their areas.

Set up an equation that splits the wire in two parts, and what each part represents.

$$4 = 2\pi r + 4x$$

One part is the circumference of a circle of radius r and the other is the perimeter of a square of side length x. Now we need the equation of the sum of the two areas, which will be maximized.

$$A_T = \pi r^2 + x^2$$

Rewrite the equation in terms of x so we can take the derivative. Use the relationship from the previous equation to solve for r.

$$r = \frac{4 - 4x}{2\pi}$$

$$A_T = \pi \left(\frac{4 - 4x}{2\pi}\right)^2 + x^2$$

Now find the critical points.

$$\frac{DA_T}{dt} = 2\pi\left(\frac{4 - 4x}{2\pi}\right)\left(-\frac{4}{2\pi}\right) + 2x = 0$$

$$0 = \frac{-16 + 6x}{2\pi} + 2x = \frac{-8 + 8x}{\pi} + 2x$$

$$-8 + 8x = -2\pi x$$

$$-8 = x(-2\pi - 8)$$

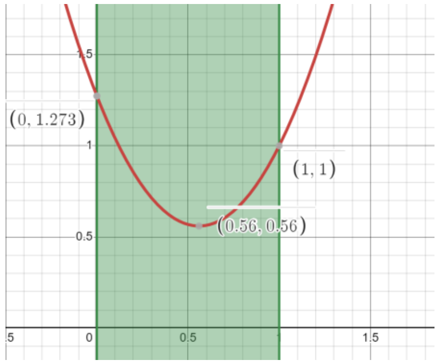
$$x = \frac{-8}{-2\pi - 8} = \frac{4}{\pi + 4}$$

The part of the string that forms the square can be between 0 and 4 feet long. That means x can be between 0 and 1 feet long. We must check these endpoints for x and the critical point.

x	$f(x)$
0	$\frac{4}{\pi} \approx 1.27$
$\frac{4}{\pi + 4}$	$\frac{4}{\pi + 4} \approx 0.56$
1	1

It looks like the maximum value over the interval $[0, 1]$ is $\frac{4}{\pi}$ which occurs at $x = 0$.

The side length that maximizes the area is $x = 0$.



If you look at the graph, there wasn't a maximum at $\frac{4}{\pi+4}$ anyway.

52. (4.7) Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Start by making and equation for the volume that relates h and r.

$$V = 16\pi = \pi r^2 h$$
$$16 = r^2 h$$

Now get the equation for the surface area of a cylinder that we will minimize.

$$S = 2\pi r(h + r)$$

Get the equation in terms of one variable so it can be differentiated.

$$S = 2\pi r\left(\frac{16}{r^2} + r\right) = \frac{32\pi}{r} + 2r^2 = 32r^{-1} + 2\pi r^2$$
$$\frac{dS}{dr} = -\frac{32\pi}{r^2} + 4\pi r = 0$$
$$4\pi r = \frac{32\pi}{r^2}$$
$$4r = \frac{32}{r^2}$$
$$r^3 = 8$$
$$r = 2$$

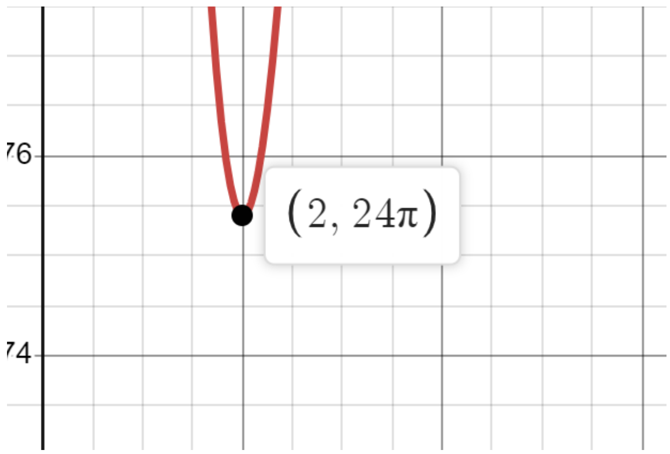
Plug r into the volume equation to get the height.

$$16 = r^2 h = 2^2 h$$
$$h = 4$$

Use the second derivative test to verify that the critical point is a minimum.

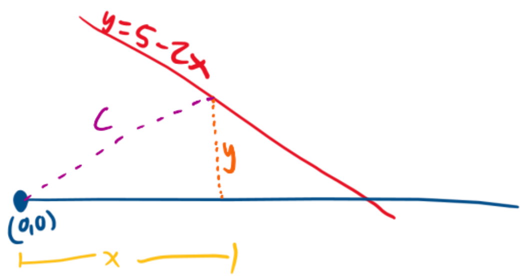
$$\frac{d^2S}{dr^2} = \frac{32\pi}{r^3} + 4\pi$$

The radius must always be positive, so the graph of S is always concave up, so there is a minimum at $r = 2$.



53. (4.7) Where is the line $y = 5 - 2x$ closest to the origin? (2, 1)

It helps to draw a picture of what you're looking for.



We are looking for c, the distance to the point on the function. The vertical line y represents the height, or y-value of the function. Given the horizontal distance x, the height y can be found, then Pythagorean's theorem can find the distance c. We will need to create a function for this so it can be differentiated and minimized.

$$c(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + (5 - 2x)^2} = \sqrt{5x^2 - 20x + 25}$$

This function takes the value of x and returns the distance to the function at that point. Now we only must differentiate with respect to x.

$$c'(x) = \frac{10x - 20}{2\sqrt{5x^2 - 20x + 25}} = 0$$
$$10x - 20 = 0$$
$$10x = 20$$
$$x = 2$$

Now plug the x value into the equation of the line to get the point.

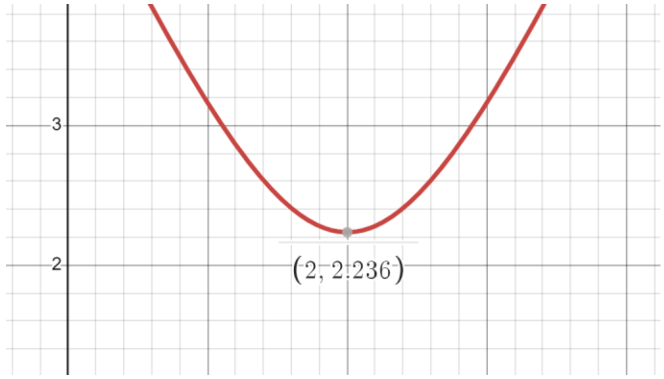
$$y = 5 - 2x = 5 - 2(2)$$
$$y = 1$$

The point closest to the origin is $(2, 1)$.

Use the first derivative test to check around the critical point to verify that it is a minimum.

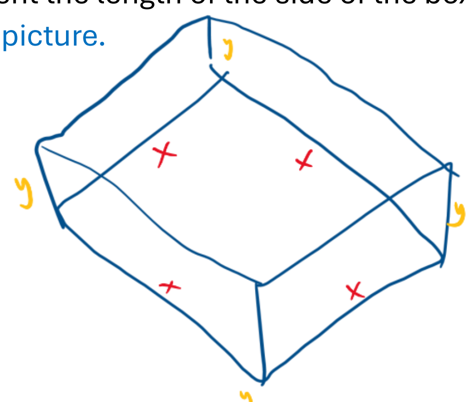
$c'(1)$	2	$c'(3)$
< 0		> 0

There is a minimum at $x = 2$.



54. (4.7) You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/ft^2$ and the material for the sides costs $\$2/ft^2$. You need a box with volume $4ft^3$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

It may help to draw a picture.



The cost of the bottom is the price multiplied by the area of the bottom:

$$\$5(x^2)$$

The cost of the sides is the area of each side (they are the same since the bottom is a square) multiplied by the price and the amount of sides.

$$\$2(xy) \cdot 4 = 8xy$$

So a total cost function would look like:

$$c(x) = 5x^2 + 8xy$$

It will be hard to differentiate with the y variable there, but a volume equation would relate the two variables.

$$V = 4 = x^2y$$

$$y = \frac{4}{x^2}$$

Now the cost function is

$$c(x) = 5x^2 + 8x\left(\frac{4}{x^2}\right) = 5x^2 + \frac{32}{x}$$

Set the derivative equal to 0 and get the minimum.

$$c'(x) = 0 = 10x - \frac{32}{x^2}$$

$$10x = \frac{32}{x^2}$$

$$x^3 = \frac{32}{10}$$

$$x = \sqrt[3]{\frac{32}{10}}$$

$$x = \sqrt[3]{\frac{16}{5}}$$

Plug x into the volume equation to get the height.

$$4 = x^2y = \left(\frac{16}{5}\right)^{\frac{2}{3}} y$$

$$y = \sqrt[3]{\frac{25}{4}}$$

We know that there is a minimum at the critical point because the function is clearly a parabola that opens up.

55. (4.7) You are the manager of an apartment complex with 50 units. When you set rent at \$800/month, all apartments are rented. As you increase rent by \$25/month, one fewer apartment is rented. Maintenance costs run \$50/month for each occupied unit. What is the rent that maximizes the total amount of profit?

To make a profit function, we need to make a cost and revenue function first. You can start by making a table and try to find the pattern.

x	# appts rented	Rent	Maint
0	50	800	2500
1	49	825	2450
2	48	850	2400
	$50 - x$	$800 + 25x$	$50(50 - x)$

We need to be very familiar with what these equations mean and what their units are. x is the number of apartments not rented. $50 - x$ is the number of apartments rented from the maximum of 50. $800 + 25x$ is the rent, but the units are in $\frac{\$}{apt}$, so it can't go straight into a revenue function, it must be multiplied by the number of units rented $50 - x$ to be a revenue function for everything.

$$R(x) = (50 - x)(800 + 25x)$$

$50(50-x)$ is the number of apartments rented multiplied by the maintenance cost per apartment. That is our cost function.
Our profit function would be

$$(50-x)(800+25x)-50(50-x)$$

When you factor out the $(50-x)$ you get

$$P(x)=(50-x)(800+25x-50)$$
$$P(x)=(50-x)(750+25x)=37500+1250x-750x-25x^2$$
$$P(x)=-25x^2+500x+37500$$

Now maximize the function.

$$P'^{(x)}=0=-50x+500$$
$$500=50x$$
$$x=10$$

We know a maximum occurs at $x=10$ because the function is a parabola opening down, but $x=10$ is the maximum number of apartments not rented, not the maximum rent, so plug it back into the rent equation.

$$800+25(10)=\$1050$$

56. (4.8) Evaluate the limit

a. $\lim\limits_{x\to\infty}\frac{e^x}{x^k}$

Using direct substitution, you get $\frac{\infty}{\infty}$.

$$\frac{e^\infty}{\infty^k}=\frac{\infty}{\infty}$$

Use L'Hôpital's rule.

$$\lim_{x\to\infty}\frac{e^x}{x^k}=\lim_{x\to\infty}\frac{e^x}{kx^{k-1}}$$

The differentiation is going to repeat forever, and eventually turn into

$$\lim_{x\to\infty}\frac{e^x}{k!}$$

As the limit is differentiated infinitely, the x^{k-1} term will become $\frac{1}{x^\infty}=0$ so it will not be included in the limit. All the constants multiplied in front are $k!$.
Since $k!$ is a constant and e^x is exponential:

$$\lim_{x\to\infty}\frac{e^x}{k!}=\infty$$

You could also have factored out the $\frac{1}{k!}$:

$$\frac{1}{k!}\lim_{x\to\infty}e^x$$

b. $\lim\limits_{x\to a}\frac{x-a}{x^2-a^2}$

Using direct substitution, you get $\frac{0}{0}$.

$$\frac{a-a}{a^2-a^2}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to a}\frac{x-a}{x^2-a^2}=\lim_{x\to a}\frac{1}{2x}$$

Now with direct substitution, the answer is

$$\lim_{x\to a}\frac{1}{2x}=\frac{1}{2a}$$

c. $\lim_{x\to a}\frac{x-a}{x^n-a^n}$

Using direct substitution you get $\frac{0}{0}$.

$$\frac{a-a}{a^n-a^n}$$

Use L'Hôpital's rule.

$$\lim_{x\to a}\frac{x-a}{x^n-a^n}=\lim_{x\to a}\frac{1}{nx^{n-1}}$$

This isn't going to repeat forever like (a) because we don't get indeterminate form again. Solve with direct substitution now.

$$\lim_{x\to a}\frac{1}{nx^{n-1}}=\frac{1}{na^{n-1}}$$

57. (4.8) Can you apply L'Hôpital's rule directly? Why or why not? Is there a way to alter the limit to apply L'Hôpital's rule?

a. $\lim_{x\to\infty} x^{1/x}$

No because there is no fraction. Use logs to bring out the exponent and put x in the denominator.

b. $\lim_{x\to 0}\frac{x^2}{1/x}$

No because it is not in proper indeterminate form. L'Hôpital's rule does not need to be applied if it is rewritten as $\lim_{x\to 0} x^3$.

58. (4.8) Evaluate the limits.

a. $\lim_{x\to 3}\frac{x^2-9}{x-3}$

You will get indeterminate form from direct substitution.

$$\frac{3^2-9}{3-3}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{2x}{1}=2(3)=6$$

b. $\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x}$

You can rewrite this to make it easier.

$$\lim_{x\to 0}\frac{\frac{1}{(1+x)^2}-1}{x}$$

Using direct substitution, you get

$$\frac{\frac{1}{(1+0)^2}-1}{0}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x}=\lim_{x\to 0}\frac{-2(1+x)^{-3}}{1}$$

Now we can do direct substitution.

$$\lim_{x\to 0}\frac{-2(1+x)^{-3}}{1}=\frac{\frac{-2}{(1+0)^3}}{1}=-2$$

c. $\lim_{x\to\pi}\frac{x-\pi}{\sin x}$

Direct substitution leads to indeterminate form.

$$\frac{\pi-\pi}{\sin\pi}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to\pi}\frac{x-\pi}{\sin x}=\lim_{x\to\pi}\frac{1}{\cos x}$$

Now use direct substitution.

$$\lim_{x\to\pi}\frac{1}{\cos x}=\frac{1}{\cos\pi}=\frac{1}{-1}=-1$$

d. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$

Direct substitution leads to indeterminate form.

$$\frac{(1+0)^n-1}{0}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{x\to 0}\frac{n(1+x)^{n-1}}{1}$$

The base of the exponent $(1+x)$ will always be 1, so the whole term $(1+x)^{n-?}$ will always be 1 no matter what n is.

$$\lim_{x\to 0}\frac{n(1+x)^{n-1}}{1}=n$$

e. $\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}$

Try direct substitution.

$$\frac{\sin 0-\tan 0}{0^3}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim_{x\to 0}\frac{\cos x-\sec^2 x}{3x^2}$$

Direct substitution returns indeterminate form again.

$$\frac{\cos 0-\sec^2 0}{3(0)^2}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{\cos x-\sec^2 x}{3x^2}=\lim_{x\to 0}\frac{-\sin x-2\sec^2 x\tan x}{6x}$$

It's indeterminate again. Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{-\sin x-2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\frac{-\cos x-2\sec^4 x+4\sec^2 x\tan^2 x}{6}$$

Try direct substitution:

$$\frac{-\cos 0-2\sec^4 0+4\sec^2 0\tan^2 0}{6}=\frac{-1-2(1)+4(0)}{6}=-\frac{3}{6}=-\frac{1}{2}$$

f. $\lim_{x\to 0}\frac{e^x-x-1}{x^2}$

Try direct substitution.

$$\frac{e^0-0-1}{0^2}=\frac{0}{0}$$

Use L'Hôpital's rule.

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2}=\lim_{x\to 0}\frac{e^x-1}{2x}$$

Direct substitution returns indeterminate form again.

$$\frac{e^0-1}{2(0)}=\frac{0}{0}$$

Use L'Hôpital's rule, then direct substitution.

$$\lim_{x\to 0}\frac{e^x-1}{2x}=\lim_{x\to 0}\frac{e^x}{2}=\frac{e^0}{2}=\frac{1}{2}$$

g. $\lim_{x\to 1}\frac{x-1}{\ln x}$

Try direct substitution.

$$\frac{1-1}{\ln 1}=\frac{0}{0}$$

Use L'Hôpital's rule, then direct substitution.

$$\lim_{x\to 1}\frac{x-1}{\ln x}=\lim_{x\to 1}\frac{1}{\frac{1}{x}}=\lim_{x\to 1}x=1$$

h. $\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Try direct substitution.

$$\frac{\sqrt{1}-\sqrt[3]{1}}{1-1}=\frac{0}{0}$$

Use L'Hôpital's rule, then direct substitution.

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1}=\lim_{x\to 1}\frac{\frac{1}{2\sqrt{x}}-\frac{1}{3\sqrt[3]{x^2}}}{1}=\frac{1}{2\sqrt{1}}-\frac{1}{3\sqrt[3]{1^2}}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

i. $\lim_{x\to\infty}x\sin\frac{1}{x}$

You can do this in your head, but I'll rewrite it to explain the concept.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} \sin\frac{1}{x}$$

The x term approaches ∞. The $\frac{1}{x}$ in the second half approaches 0, so $\sin 0 = 0$. This gives us the $0 \cdot \infty$ case. Chose any half whose reciprocal will go in the denominator. I'll do x because it's easy.

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}}$$

When you do direct substitution, you now get indeterminate form.

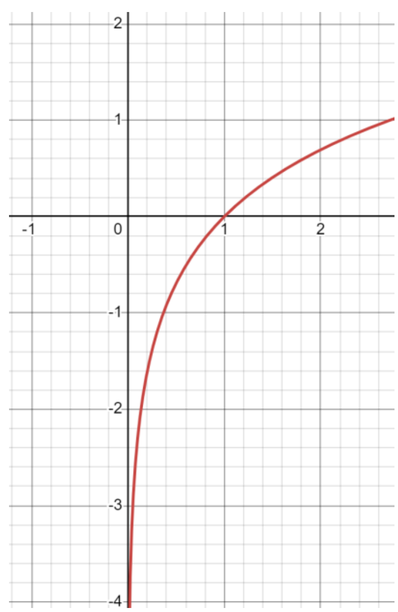
$$\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \frac{\sin\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{\sin 0}{0} = \frac{0}{0}$$

Use L'Hôpital's rule, simplify, then use direct substitution.

$$\lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\frac{1}{x}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \cos\frac{1}{x} = \cos\frac{1}{\infty} = \cos 0 = 1$$

j. $\lim_{x\to 0^+} x\ln(x^4)$

The x term approaches 0, and the x^4 in $\ln(x^4)$ also approaches 0, but $\ln 0$ doesn't exist. Since x approaches 0 from the right, we can say that the natural log will approach $-\infty$.



We now have the $0 \cdot -\infty$ case.

$$\lim_{x\to 0^+} x\ln(x^4) = 0 \cdot -\infty$$

Again, I'm choosing to put the reciprocal of x in the denominator. There is minimal difference in difficulty whatever you choose. Either way you will get indeterminate form:

$$\lim_{x\to 0^+} x\ln(x^4) = \lim_{x\to 0^+} \frac{\ln(x^4)}{\frac{1}{x}} = \frac{-\infty}{\infty}$$

Use L'Hôpital's rule, simplify, then use direct substitution.

$$\lim_{x\to 0^+} \frac{\ln(x^4)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x^4}\cdot 4x^3}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -4x = 0$$

k. $\lim_{x\to\infty} x^2e^{-x}$

This can be rewritten as

$$\lim_{x\to\infty} \frac{x^2}{e^x}$$

Direct substitution leads to indeterminate form.

$$\frac{\infty^2}{e^\infty} = \frac{\infty}{\infty}$$

Use L'Hôpital's rule, then use direct substitution.

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \frac{2\infty}{e^\infty} = \frac{\infty}{\infty}$$

Use L'Hôpital's rule again. This time it works.

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{e^\infty} = \frac{2}{\infty} = 0$$

l. $\lim_{x\to 0} \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

Fractions are confusing, so we will simplify it by multiplying both the numerator and denominator by x.

$$\lim_{x\to 0} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \lim_{x\to 0} \frac{x\left(1+\frac{1}{x}\right)}{x\left(1-\frac{1}{x}\right)} = \lim_{x\to 0} \frac{x+1}{x-1}$$

Then direct substitution can be used without having to use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{x+1}{x-1} = \frac{0+1}{0-1} = \frac{1}{-1} = -1$$

m. $\lim_{x\to\infty} xe^{1/x}$

Using direct substitution, you get

$$\infty\cdot e^{\frac{1}{\infty}} = \infty\cdot e^0 = \infty\cdot 1 = \infty$$

Again, you don't have to use L'Hôpital's rule.

n. $\lim_{x\to 0^+} x^{\frac{1}{x}}$

Using direct substitution, you get a special case:

$$0^{\frac{1}{0}} = 0^\infty$$

Use natural log to bring the exponent outside.

$$L = \lim_{x \to 0^+} \frac{1}{x} \ln x = \frac{1}{0^+} \cdot \ln 0^+ = \infty \cdot -\infty = -\infty$$

$\infty \cdot -\infty = -\infty$ because it's the same as $\infty \cdot \infty \cdot -1$.

The final answer is e^L.

$$e^{-\infty} = 0$$

o. $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

Try direct substitution.

$$\left(1 - \frac{1}{\infty}\right)^{\infty} = (1 - 0)^{\infty} = 1^{\infty}$$

This is a special case. Use natural log to bring the exponent outside.

$$L = \lim_{x \to \infty} x \ln\left(1 - \frac{1}{x}\right) = \infty \cdot \ln 1 = \infty \cdot 0$$

This is another special case. Put the reciprocals of one of the terms in the denominator.

$$\lim_{x \to \infty} x \ln\left(1 - \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

Use L'Hôpital's rule, then use direct substitution

$$\lim_{x \to \infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{1}{1 - \frac{1}{x}} \cdot \frac{1}{x^2}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{-1}{1 - \frac{1}{x}} = -\frac{1}{1 - \frac{1}{\infty}} = -\frac{1}{1} = -1$$

The final answer is e^L.

$$e^{-1} = \frac{1}{e}$$

59. (4.9) Write Newton's Formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

a. $f(x) = x^3 + 2x + 1$

$$f'(x) = 3x^3 + 2$$

$$x_{n+1} = F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

b. $f(x) = e^x$

$$f'(x) = e^x$$

$$x_{n+1} = F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

60. (4.9) Solve to four decimal places using Newton's method and a calculator. Chose an x_0 that is not the exact root.

You can get the exact solutions by calculating zeros and intercepts with the graphing calculator, or use math.

a. $x^4 - 100 = 0$

Using a calculator, the exact answer is $\pm\sqrt{10} \approx \pm 3.16228$.
A number near this is ± 3, so this will be our x_0.
We need the derivative for the formula:

$$4x^3$$

Make the formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$F(x_n) = x_n - \frac{x_n^4 - 100}{4x_n^3}$$

Plug in x_0 and keep repeating until x_n is accurate to four decimal places.

$$x_1 = x_0 - \frac{x_0^4 - 100}{4x_0^3} = 3 - \frac{3^4 - 100}{4(3)^3} \approx 3.1760$$

And for -3:

$$x_1 = x_0 - \frac{x_0^4 - 100}{4x_0^3} = -3 - \frac{(-3)^4 - 100}{4(-3)^3} \approx -3.1760$$

You can plug the formula $y = x - \frac{x^4 - 100}{4x^3}$ in your calculator to make it easier.

$$x_2 \approx \pm 3.1623$$

Don't count on the numbers being equal but opposite; always check them individually.

b. $x^3 - x = 0$

The exact answers are $0, -1, 1$. We'll choose $x_0 = -1.1, -0.1, 0.9$.
Get the derivative for the formula.

$$3x^2 - 1$$

Make the formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}$$

You can plug the formula into a calculator to make it easier.
Starting with $x_0 = -1.01$:

$$x_1 \approx -1.0122$$
$$x_2 \approx -1.0002$$
$$x_3 \approx -1.0000$$

Now for $x_0 = -0.1$:

$$x_1 \approx 0.0021$$
$$x_2 \approx 0.0000$$

And finally, $x_0 = 0.9$:

$$x_1 \approx 1.0196$$
$$x_2 \approx 1.0006$$

$$x_3 \approx 1.0000$$

c. $x + \tan x = 0$. Chose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

The exact answer is 0. We will choose $x_0 = 0.001$.

Get the derivative.

$$1 + \sec^2 x$$

Make the formula.

$$x_{n+1} = x_n - \frac{x + \tan x_n}{1 + \sec^2 x_n}$$

Plug the formula in the calculator to make it easier.

$$x_1 \approx 0.0005$$
$$x_2 \approx 0.0002$$
$$x_3 \approx 0.0001$$
$$x_4 \approx 0.0000$$

d. $1 + x + x^2 + x^3 + x^4 = 2$

The exact solutions are $x = -1.2907, 0.5188$. We'll choose $x_0 = -1.3, 0.5$.

Newton's methos finds zeros, so rewrite the function so it's something equal to 0.

$$x^4 + x^3 + x^2 + x - 1$$

Get the derivative.

$$4x^3 + 3x^2 + 2x + 1$$

Make the formula.

$$x_{n+1} = x_n - \frac{x_n^4 + x_n^3 + x_n^2 + x_n - 1}{4x_n^3 + 3x_n^2 + 2x_n + 1}$$

Plug the formula into the calculator to make it easier.

Starting with $x_0 = -1.3$:

$$x_1 \approx -1.2908$$
$$x_2 \approx -1.2907$$

Now do $x_0 = 0.5$:

$$x_1 \approx 0.5192$$
$$x_2 \approx 0.5188$$

e. $x = \sin^2 x$

The exact solution is $x = 0$. We'll choose $x_0 = 0.1$.

Newton's methos finds zeros, so rewrite the function so it's something equal to 0.

$$\sin^2 x - x = 0$$

Get the derivative:

$$2 \sin x \cos x - 1 = 0$$

Make the function.

$$x_{n+1} = x_n - \frac{\sin^2 x_n - x}{2 \sin x \cos x - 1}$$

Plug the formula into the calculator to make it easier.

$$x_1 \approx -0.0003$$
$$x_2 \approx 0.0000$$

61. (4.9) Use Newton's method to find the following. Round to three decimals.

Minimums and maximums are just zeros of the derivative. Get the derivatives and find their zeros using Newton's method.

a. Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Set the derivative equal to 0.

$$f'(x) = 9x^2 + 4x = 0$$

The exact solutions are $x = 0, -\frac{4}{9}$. We'll choose $x_0 = 0.1, -0.5$.

Get the second derivative.

$$f''(x) = 18x + 4$$

Make the formula.

$$x_{n+1} = x - \frac{9x_n^2 + 4x_n}{18x_n + 4}$$

Plug the formula into the calculator to make it easier.

Starting with $x_0 = -0.5$:

$$x_1 \approx -0.450$$
$$x_2 \approx -0.445$$
$$x_3 \approx -0.444$$

Now with $x_0 = 0.1$:

$$x_1 \approx 0.016$$
$$x_2 \approx 0.001$$
$$x_3 \approx 0.000$$

Use the second derivative test to check for concavity and determine which is a minimum and which is a maximum.

$$f''(x) = 18x + 4$$
$$f''(0) = 4 > 0$$
$$f''(0) \approx -3.992 < 0$$

The minimum occurs at $x \approx 0.000$.

b. Maximum of $f(x) = x + \frac{1}{x}$

Set the derivative equal to 0.

$$f'(x) = 1 - \frac{1}{x^2} = 0$$
$$1 = \frac{1}{x^2}$$

The exact solutions are $x = \pm 1$. We'll choose $x_0 = \pm 0.9$

Get the second derivative.

$$f''(x) = \frac{3}{x^3}$$

Make the formula.

$$x_{n+1} = x_n - \frac{1 - x^{-2}}{3x^{-3}}$$

Plug the formula into the calculator to make it easier.

$$x_1 \approx \pm 0.957$$
$$x_2 \approx \pm 0.984$$
$$x_3 \approx \pm 0.994$$
$$x_4 \approx \pm 0.998$$
$$x_5 \approx \pm 0.999$$
$$x_6 \approx \pm 1.000$$

Don't count on the numbers being equal but opposite; always check them individually.

The maximum occurs at either $x = 1$ or $x = -1$. Use the second derivative test to check.

$$f''(x) = \frac{3}{x^3}$$
$$f''(-1) = -3 < 0$$
$$f''(1) = 3 > 0$$

The maximum occurs at $x = -1$.

c. Maximum of $f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$

Set the derivative equal to 0.

$$f'(x) = \frac{x\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}\right) - \left(x^{\frac{1}{2}} - x^{\frac{1}{3}}\right)}{x^2}$$

The exact solution is approximately 5.619. We'll choose $x_0 = 5.618$.

Get the second derivative.

$$f''(x) = \frac{x^2\left(-\frac{1}{4}x^{-\frac{1}{2}} - \frac{4}{9}x^{-\frac{2}{3}}\right) - \left(-\frac{1}{2}x^{\frac{1}{2}} - \frac{4}{3}x^{\frac{1}{3}}\right)}{x^4}$$

Make the formula.

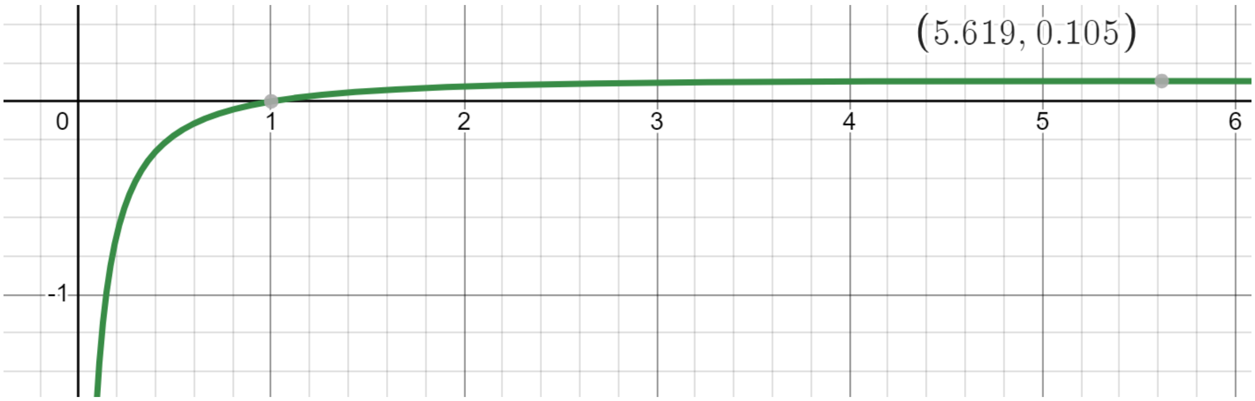
$$x_{n+1} = x_n - \frac{\dfrac{x\left(\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{3}x^{-\frac{2}{3}}\right) - \left(x^{\frac{1}{2}} - x^{\frac{1}{3}}\right)}{x^2}}{\dfrac{x^2\left(-\frac{1}{4}x^{-\frac{1}{2}} - \frac{4}{9}x^{-\frac{2}{3}}\right) - \left(-\frac{1}{2}x^{\frac{1}{2}} - \frac{4}{3}x^{\frac{1}{3}}\right)}{x^4}}$$

Plug the formula into the calculator to make it easier.

$$x_1 \approx 5.618$$
$$x_2 \approx 5.618$$
$$x_3 \approx 5.618$$
$$x_4 \approx 5.618$$

$x_5 \approx 5.619$

I chose x_0 really close because the function increases really slow.



62. (4.9) Can Newton's method be used to solve $0 = e^x$? Why or why not?

No, there is no solution to the equation. The x_n values are going to go $-\infty$ and not approach any number.

Note: Chapter 4.10 is included in the chapter 5 sheet.

MTH 200

Worksheet Answers

Problems found in OpenStax Calculus Volume II

Sections:

- 4.10 - Antiderivatives
- 5.1 – Approximating Areas
- 5.2 – The Definite Integral
- 5.3 – The Fundamental Theorem of Calculus
- 5.5 - Substitution
- 5.6 – Integrals Involving Exponential and Logarithmic Functions
- 5.7 – Integrals Resulting in Inverse Trigonometric Functions

1. (4.10) Find the antiderivative of the function.

a. $f(x) = e^x - 3x^2 + \sin x$

$$F(x) = e^x - 3x - \cos x + C$$

b. $f(x) = x - 1 + 4\sin 2x$

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$$

When integrating $4\sin 2x$, or anything with a differentiable function inside it, divide by the chain rule.

$$\int 4\sin 2x \cdot dx = 4 \cdot -\frac{\cos 2x}{2} = -2\cos 2x + C$$

c. $f(x) = x + 12x^2$

$$F(x) = \frac{x^2}{2} + \frac{12x^3}{3} + C$$

$$F(x) = \frac{1}{2}x^2 + 4x^3 + C$$

d. $f(x) = \left(\sqrt{x}\right)^3$

Rewrite this so there is only one power.

$$f(x) = \left(\sqrt{x}\right)^3 = \left(x^{\frac{1}{2}}\right)^3 = x^{\frac{3}{2}}$$

$$F(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2x^{\frac{5}{2}}}{5} + C$$

You could also write

$$F(x) = \frac{2}{5}\left(\sqrt{x}\right)^5 + C$$

e. $f(x) = \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}}$

When dividing powers of the same base, subtract the exponents.

$$f(x) = \frac{x^{\frac{1}{3}}}{x^{\frac{2}{3}}} = x^{\frac{1}{3}-\frac{2}{3}} = x^{-\frac{1}{3}}$$

$$F(x) = \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C$$

$$F(x) = \frac{3x^{\frac{2}{3}}}{2} + C$$

f. $f(x) = \sec^2 x + 1$

$$F(x) = \tan x + x + C$$

g. $f(x) = \sin^2 x \cos x$

When differentiating a trig function to a power, the chain rule must multiply by the derivative of the trig function. Notice that $\sin x$ is raised to a power, and $\cos x$

is its derivative. From this we can tell that the antiderivative is just the inverse power rule of $\sin^2 x$, which is $\frac{\sin^3 x}{3}$.

$$F(x) = \frac{\sin^3 x}{3} + C$$

h. $f(x) = \frac{1}{2}\csc^2 x + \frac{1}{x^2}$

The derivative of $-\cot x$ is $\csc^2 x$, so the integral of $\csc^2 x$ is $-\cot x$.

$$F(x) = -\frac{1}{2}\cot x + \frac{x^{-1}}{-1} + C$$

$$F(x) = -\frac{1}{2}\cot x - \frac{1}{x} + C$$

i. $f(x) = 4\csc x \cot x - \sec x \tan x$

The integral of $\csc x \cot x$ is $-\csc x + C$ and the integral of $\sec x \tan x$ is $\sec x + C$. The antiderivative becomes

$$F(x) = -4\csc x - \sec x + C$$

j. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

$$F(x) = \frac{1}{2}e^{-4x} \cdot \frac{1}{-4} - \cos x + C$$

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$$

2. (4.10) Evaluate the integral.

a. $\int \frac{3x^2+2}{x^2} \cdot dx$

Rewrite this as two separate fractions.

$$\int \frac{3x^2+2}{x^2} \cdot dx = \int \left(\frac{3x^2}{x^2} + \frac{2}{x^2}\right) \cdot dx = \int (3 + 2x^{-2}) \cdot dx$$

Now this can be integrated normally.

$$\int (3 + 2x^{-2}) \cdot dx = 3x + \frac{2x^{-1}}{-1} + C = 3x - \frac{2}{x} + C$$

b. $\int(4\sqrt{x} + \sqrt[4]{x}) \cdot dx$

Rewrite the roots as powers.

$$\int (4\sqrt{x} + \sqrt[4]{x}) \cdot dx = \int \left(4x^{\frac{1}{2}} + x^{\frac{1}{4}}\right) \cdot dx$$

Now integrate.

$$\int (4x^{1/2} + x^{1/4}) \cdot dx = \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{5}{4}}}{\frac{5}{4}} + C = \frac{8}{3}x^{\frac{3}{2}} + \frac{4}{5}x^{5/4} + C$$

c. $\int \frac{14x^3+2x+1}{x^3} \cdot dx$

Split the fractions just like (a).

$$\int \frac{14x^3 + 2x + 1}{x^3} \cdot dx = \int \frac{14x^3}{x^3} + \frac{2x}{x^3} + \frac{1}{x^3} \cdot dx = \int 14 + 2x^{-2} + x^{-3} \cdot dx$$

Now integrate.

$$\int 14 + 2x^{-2} + x^{-3} \cdot dx = 14x + \frac{2x^{-1}}{-1} + \frac{x^{-2}}{-2} + C = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

3. (4.10) Solve for $f(x)$, when $f'(x) = \cos x + \sec^2 x$ and $f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$.

We will have to integrate the function and plug in $f\left(\frac{\pi}{4}\right)$ to find the integration constant C.

$$\int (\cos x + \sec^2 x) \cdot dx = \sin x + \tan x + C$$

$$f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} + \tan\frac{\pi}{4} + C$$

$$2 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + 1 + C$$

$$C = 1$$

So $f(x) = \sin x + \tan x + 1$.

4. (4.10) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerated at a constant rate of $10\frac{ft}{sec^2}$. How long before the car stops?

We're given speed and acceleration values, so we'll make two functions $v(t)$ and $a(t)$ for velocity and acceleration as a function of time. 40 mph is an initial velocity. We'll label this as $v(0) = 40\ mph$. $10\frac{ft}{sec^2}$ is an acceleration value. It should be negative because it's decelerating, and it is also constant for every value of t, so we can make an acceleration function:

$$a(t) = -10$$

We have some mixed-up units so we'll convert mph into $\frac{ft}{sec}$.

$$v(0) = \frac{40\ mi}{1\ hr} \times \frac{5280\ ft}{1\ mi} \times \frac{1\ hr}{60\ min} \times \frac{1\ min}{60\ sec} = \frac{176}{3}\frac{ft}{sec}$$

We are asked to find when the car stops, so we're looking for t when $v(t) = 0$. Since velocity is the integral of acceleration, integrate the acceleration function and use the value $v(0)$ to find the integration constant C.

$$\int a(t) \cdot dt = \int -10 \cdot dt = -10t + C$$

$$v(t) = -10t + C$$

$$v(0) = \frac{176}{3} = -10(0) + C$$

$$C = \frac{176}{3}$$

Make the complete velocity function.

$$v(t) = -10t + \frac{176}{3}$$

Set it equal to 0 to find when the car stops.

$$-10t + \frac{176}{3} = 0$$
$$-10t = -\frac{176}{3}$$
$$t \approx 5.867$$

So, the car stops after 5.867 seconds.

5. (5.1) Compute the sum $\sum_{i=5}^{10} i^2$.

Use the rule $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. The problem is the bounds don't start at 1 but 5 so we'll have to use other algebraic methods.

$$\sum_{i=1}^{10} i^2 = \sum_{i=1}^{4} i^2 + \sum_{i=5}^{10} i^2$$

Or:

$$\sum_{i=5}^{10} i^2 = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{4} i^2$$

Now apply the rule.

$$\sum_{i=5}^{10} i^2 = \frac{10(10+1)(2(10)+1)}{6} - \frac{4(4+1)(2(4)+1)}{6} = 355$$

6. (5.1) Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. Compute the sum $\sum_{i=1}^{100} 5a_i + 4b_i$.

Use the addition rule, then factor out the constants from each sum.

$$\sum_{i=1}^{100} 5a_i + 4b_i = \sum_{i=1}^{100} 5a_i + \sum_{i=1}^{100} 4b_i = 5\sum_{i=1}^{100} a_i + 4\sum_{i=1}^{100} b_i = 5(15) + 4(-12) = 27$$

7. (5.1) Use summation properties and formulas to rewrite and evaluate $\sum_{k=1}^{25}[(2k)^2 - 100k]$.

Start by using the subtraction and constant multiple rules to isolate the k's.

$$\sum_{k=1}^{25}[(2k)^2 - 100k] = 4\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k$$

Now it is easy to use the power rules.

$$2\sum_{k=1}^{25} k^2 - 100\sum_{k=1}^{25} k = 4\left(\frac{25(25+1)(2(25)+1)}{6}\right) - 100\left(\frac{25(25+1)}{2}\right) = -10{,}400$$

8. (5.1) Compute the left or right endpoint Riemann sum.

a. R_4 for $g(x) = \cos \pi x$ on $[0, 1]$

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

Since we're using right endpoints, we're starting at $x = \frac{1}{4}$ and going to $x = 1$. The base lengths of the rectangles are the same $\left(\frac{1}{4}\right)$, and the heights will be different. Add up all the areas $b \cdot h_k$ and factor out the base.

$$R_4 = b \cdot h_1 + b \cdot h_2 + b \cdot h_3 + b \cdot h_4$$

$$b = \frac{1}{4}$$

$$R_4 = \frac{1}{4}\left[\cos\frac{\pi}{4} + \cos\frac{\pi}{2} + \cos\frac{3\pi}{4} + \cos\pi\right] = -\frac{1}{4}$$

b. L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

Find Δx.

$$\Delta x = \frac{b-a}{n} = \frac{2+2}{4} = 1$$

Since we're using left endpoints, we're starting at $x = -2$ and going to $x = 1$. Follow the same procedure as (a), adding the areas and factoring out the base length. Since the base length is 1, I'll ignore it.

$$L_4 = \frac{1}{(-2)^2+1} + \frac{1}{(-1)^2+1} + \frac{1}{0^2+1} + \frac{1}{1^2+1} = \frac{11}{5}$$

9. (5.1) Express the endpoint sums in sigma notation but do not evaluate them.

You will need to find Δx, x_k, and $f(x_k)$.

a. L_{10} for $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$

$$\Delta x = \frac{b-a}{n} = \frac{2+2}{10} = \frac{2}{5}$$

Since we're doing left endpoints, x_k uses $k-1$ instead of k.

$$x_k = a + (k-1)\Delta x = -2 + \frac{2(k-1)}{5}$$

$$f(x_k) = \sqrt{4 - \left(-2 + \frac{2(k-1)}{5}\right)^2}$$

Set up the sum:

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{10} \sqrt{4 - \left(-2 + \frac{2(k-1)}{5}\right)^2} \cdot \frac{2}{5}$$

b. R_{100} for $f(x) = \ln x$ on $[1, e]$

$$\Delta x = \frac{b-a}{n} = \frac{e-1}{100}$$

$$x_k = a + k\Delta x = 1 + k\left(\frac{e-1}{100}\right)$$

$$f(x_k) = \ln\left(1 + k\left(\frac{e-1}{100}\right)\right)$$

Set up the sum:

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{100} \ln\left(1 + k\left(\frac{e-1}{100}\right)\right)\left(\frac{e-1}{100}\right)$$

10. (5.1) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

It is summing amounts of rain, so the sum represents the total rainfall in January 2009 in Portland.

11. (5.2) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} (5(x_i^*)^2 - 3(x_i^*)^3)\Delta x$ over $[0, 2]$ as an integral.

This is infinitely summing the expression inside the parenthesis multiplied by the base length, which approaches 0 as $n \to \infty$. The integral must be:

$$\int_0^2 (5x^2 - 3x^3) \cdot dx$$

12. (5.2) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \cos^2(2\pi x_i^*)\, \Delta x$ over $[0, 1]$ as an integral.

Just like question 11,

$$\int_0^1 \cos^2 2\pi x \, dx$$

13. (5.2) Given R_n, express their limits as $n \to \infty$ as a definite integral.

Use the general form of a Riemann sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \left(\frac{b-a}{n}\right) = \int_a^b f(x) \cdot dx$$

a. $R_n = \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}$

Let the $\frac{1}{n}$ in front of the sum represent Δx, the width of each subinterval.

If $\Delta x = \frac{b-a}{n}$, then $b - a = 1$. According to the formula, $a = 0$, so $b = 1$ and the interval is $[0, 1]$.

The integrand appears to be $\frac{i}{n}$, which according to the formula is just the input of the function $f(x)$, so $f(x) = x$. Putting the integral together, we have:

$$\int_0^1 x \cdot dx$$

b. $R_n = \frac{3}{n}\sum_{i=1}^{n} \left(3 + 3\frac{i}{n}\right)$

By comparison with the Riemann sum formula, $b - a = 3$ and $a = 3$, so $b = 6$ and the interval is $[3, 6]$. $3 + 3\frac{i}{n}$ represents the input of the function $f(x)$. Since nothing is happening to it, $f(x) = x$ again. Putting the integral together, we have:

$$\int_3^6 x \cdot dx$$

c. $R_n = \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\log\left(\left(1+\frac{i}{n}\right)^2\right)$

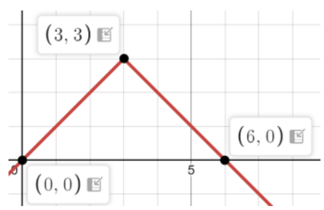
By comparison with the Riemann sum formula, $b-a=1$ and $a=1$, so $b=2$ and the interval is $[1,2]$. $1+\frac{i}{n}$ represents the input of the function $f(x)$, but this time there are other operations modifying it. If we think of $1+\frac{i}{n}$ as x, all the other operations define $f(x)$, so $f(x)=x\cdot\log(x^2)$. Thus, the definite integral is:

$$\int_1^2 x\cdot\log(x^2)\cdot dx$$

14. (5.2) Evaluate the integral using area formulas.

a. $\int_0^6 (3-|x-3|)\,dx$

This graph is a triangle, and the integral represents its area:

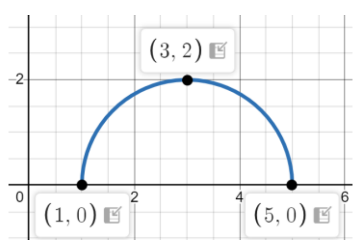


Use the formula for the area of a triangle:

$$\int_0^6 (3-|x-3|)dx = \frac{1}{2}\cdot b\cdot h = \frac{1}{2}\cdot 6\cdot 3 = 9$$

b. $\int_1^5 \sqrt{4-(x-3)^2}\,dx$

The graph is a semicircle of radius 2 centered at (3,0) and the integral represents its area:



Use the formula for the area of a circle and divide it by 2:

$$\int_1^5 \sqrt{4-(x-3)^2}dx = \frac{1}{2}\pi\cdot r^2 = \frac{1}{2}\pi\cdot 2^2 = 2\pi$$

If you wanted to figure out the graph algebraically:

$$y=\sqrt{4-(x-3)^2}$$

$$y^2 + (x-3)^2 = 4$$

The equation of a circle is $(y-h)^2 + (x-k)^2 = r^2$, where r is the radius and the circle is centered at (h,k).

So, the radius is 2 and the circle is centered at $(3,0)$. With this information you can find the bounds.

The reason it's a semicircle is that when y is isolated, you get the positive case of the square root, so everything is above the x-axis.

15. (5.2) Suppose $\int_0^4 f(x)dx = 5$, $\int_0^2 f(x)dx = -3$, $\int_0^4 g(x)dx = -1$, and $\int_0^2 g(x)dx = 2$. Evaluate the integrals.

a. $\int_2^4 \big(f(x) + g(x)\big)dx$

Initial decomposition of the integral yields:

$$\int_2^4 f(x)\cdot dx + \int_2^4 g(x)\cdot dx$$

The integral over $[2,4]$ is not given, but it can be expressed as the integral over $[0,2]$ subtracted from the integral over $[0,4]$ (since integrals represent area, you can remove a part of the area and solve for the other part).

$$\int_0^4 f(x)\cdot dx - \int_0^2 f(x)\cdot dx + \int_0^4 g(x)\cdot dx - \int_0^2 g(x)\cdot dx$$

Now plug everything in:

$$5 - (-3) + (-1) - 2 = 5$$

a. $\int_2^4 \big(f(x) - g(x)\big)dx$

It's the same as (a), but with subtraction in the middle.

$$5 - (-3) - \big((-1) - 2\big) = 11$$

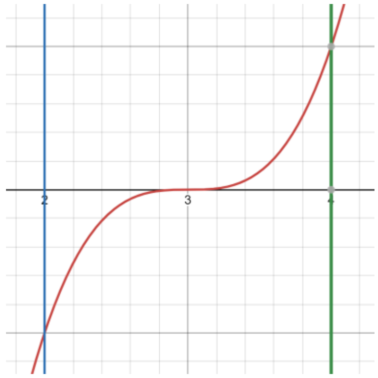
b. $\int_2^4 \big(4f(x) - 3g(x)\big)dx$

It's the same as (b), but multiplied by constants.

$$4\big(5 - (-3)\big) - 3\big((-1) - 2\big) = 41$$

16. (5.2) Find the net signed area between the function and the x-axis. $\int_2^4 (x-3)^3 dx$.
Hint: look at the graph.

The function $y = (x-3)^3$ is just x^3 shifted horizontally 3 units to the right.



To solve this integral by looking at the graph, notice that this function is symmetrical on either side of $x = 3$, so the areas under the curve will also be symmetrical. Since the left half is under the x-axis, its area is equal and opposite of the area under the right side.

By symmetry with respect to $x - 3$, $\int_2^4 (x-3)^3 \cdot dx = 0$.

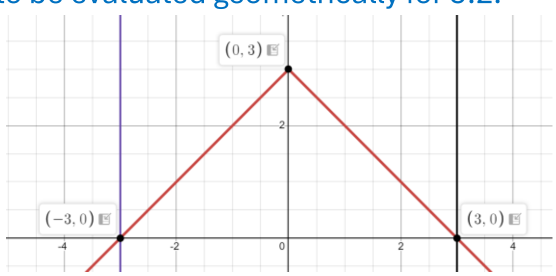
17. (5.2) Find the average value f_{ave} of $f(x)$ between a and b and find a point c where $f(c) = f_{ave}$.

$$f(x) = (3 - |x|),\ a = -3,\ b = 3.$$

The formula for average value is:

$$\frac{1}{b-a}\int_a^b f(x) \cdot dx = \frac{1}{3-(-3)}\int_{-3}^{3} (3 - |x|) \cdot dx$$

The integral will have to be evaluated geometrically for 5.2.



The average value is just $\frac{1}{6}$ times the area of the triangle.

$$f_{ave} = \frac{1}{6} \cdot \left(\frac{1}{2} \cdot 6 \cdot 3\right) = \frac{3}{2}$$

To find a value c such that $f(c) = f_{ave}$, plug $\frac{3}{2}$ into the function and solve.

$$\frac{3}{2} = 3 - |c|$$

$$-|c| = -\frac{3}{2}$$

$$c = \pm\frac{3}{2}$$

18. (5.3) Let $F(x) = \int_1^x (1-t)dt$. Find $F'(2)$ and the average value of F' over $[1,2]$.

To find $F'(x)$, just replace t with x because the derivative of x is 1 and it is not necessary to show the chain rule.

$$F'(x) = 1 - x$$
$$F'(2) = 1 - 2 = -1$$

To find the average value of $F'(x)$, you have to integrate it.

$$F'_{ave} = \frac{1}{2-1}\int_1^2 (1-x)\cdot dx = \left[x - \frac{1}{2}x^2\right]_1^2 = 0 - \frac{1}{2} = -\frac{1}{2}$$

19. (5.3) Use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

a. $\frac{d}{dx}\int_1^x e^{\cos t} dt$

$$\frac{d}{dx}\int_1^x e^{\cos t} dt = e^{\cos x}$$

b. $\frac{d}{dx}\int_3^x \frac{ds}{\sqrt{16-s^2}}$

$$\frac{d}{dx}\int_3^x \frac{ds}{\sqrt{16-s^2}} = \frac{1}{\sqrt{16-x^2}}$$

c. $\frac{d}{dx}\int_0^{\sqrt{x}} t dt$

$$\frac{d}{dx}\int_0^{\sqrt{x}} t dt = \sqrt{x}\cdot\frac{d}{dx}[\sqrt{x}] = \sqrt{x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2}$$

d. $\frac{d}{dx}\int_{\cos x}^1 \sqrt{1-t^2} dt$

You will need the $\cos x$ as the top bound, so just flip the bounds and bring out the -1.

$$\frac{d}{dx}\int_{\cos x}^1 \sqrt{1-t^2} dt = -\frac{d}{dx}\int_1^{\cos x}\sqrt{1-t^2} dt = -1\left[\sqrt{1-\cos^2 x}\cdot\frac{d}{dx}[\cos x]\right]$$
$$-1\left[\sqrt{\sin^2 x}\cdot -\sin x\right] = |\sin x|\cdot\sin x$$

The $|\sin x|$ comes from taking the square root of the square, but the other $\sin x$ is not necessarily positive, so it cannot be rewritten as $\sin^2 x$.

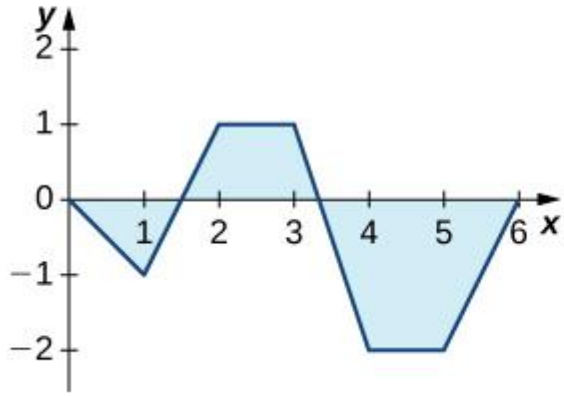
e. $\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t} dt$

$$\frac{d}{dx}\int_1^{x^2} \frac{\sqrt{t}}{1+t} dt = \frac{\sqrt{x^2}}{1+x^2}\cdot\frac{d}{dx}[x^2] = 2x\cdot\frac{|x|}{1+x^2} = \frac{2x|x|}{1+x^2}$$

f. $\frac{d}{dx}\int_1^{e^x} \ln u^2 \, du$

$$\frac{d}{dx}\int_1^{e^x} \ln u^2 \, du = \ln((e^x)^2)\cdot\frac{d}{dx}[e^x] = \ln(e^{2x})\cdot e^x = 2xe^x$$

20. (5.3) The graph $y = \int_0^x f(t)dt$, where f is a piecewise constant function, is shown here.



a. Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
y represents the antiderivative of $f(x)$, $F(x)$. To find where $f(x)$ is positive, negative, and zero, look at the sign of the slope of $y = F(x)$, since $f(x)$ is the derivative of $F(x)$.
$f(x) > 0$ over $(1,2) \cup (5,6)$ since $F'(x) > 0$ (the slope is positive)
$f(x) < 0$ over $(0,1) \cup (3,4)$ since $F'(x) < 0$ (the slope is negative)
$f(x) = 0$ over $(2,3) \cup (4,5)$ since $F'(x) = 0$ (the slope is zero)
These bounds must be exclusive because the $F(x)$ is not differentiable at the bounds' endpoints.

b. What are the minimum and maximum values of f?
We are asked to find the min and max values of the derivative of $F(x)$, so we just have to look at the slopes again.
The highest slope in the graph is 2 and the lowest is -3.
f has a maximum value of 2 and a minimum value of -3.

c. What is the average value of f?
Over the interval $[0,6]$, the average value of f is defined as:
$$\frac{1}{6-0}\int_0^6 f(t)dt$$
The integral $\int_0^6 f(t)dt$ is easy to find; it is just $F(6)$.
$$f_{ave} = \frac{1}{6}(0) = 0$$

21. (5.3) Evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

a. $\int_{-2}^{3}(x^2 + 3x - 5)dx$

$$\int_{-2}^{3}(x^2+3x-5)dx=\left[\frac{1}{3}x^3+\frac{3}{2}x^2-5x\right]_{-2}^{3}=\frac{15}{2}-\frac{40}{3}=-\frac{35}{6}$$

b. $\int_2^3 (t^2-9)(4-t^2)dt$

FOIL the integrand first.

$$\int_2^3 (t^2-9)(4-t^2)dt=\int_2^3(-t^4+13t^2-36)dt=\left[-\frac{1}{5}t^5+\frac{13}{3}t^3-36t\right]_2^3$$

$$=-\frac{198}{5}+\frac{656}{15}=\frac{62}{15}$$

c. $\int_0^1 x^{99}dx$

$$\int_0^1 x^{99}dx=\frac{1}{100}x^{100}|_0^1=\frac{1}{100}$$

d. $\int_{1/4}^{4}\left(x^2-\frac{1}{x^2}\right)dx$

Remember that the integral of $\frac{1}{x}$ is $\ln|x|$.

$$\int_{\frac{1}{4}}^{4}\left(x^2-\frac{1}{x^2}\right)dx=\left[\frac{1}{3}x^3+\frac{1}{x}\right]_{\frac{1}{4}}^{4}=\frac{259}{12}-\frac{769}{192}=\frac{1125}{64}$$

e. $\int_1^4 \frac{1}{2\sqrt{x}}dx$

$$\int_1^4 \frac{1}{2\sqrt{x}}dx=\int_1^4\frac{1}{2}x^{-\frac{1}{2}}dx=\left[\frac{2}{2}x^{\frac{1}{2}}\right]_1^4=\left[\sqrt{x}\right]_1^4=2-1=1$$

f. $\int_1^{16}\frac{dt}{t^{\frac{1}{4}}}$

$$\int_1^{16}\frac{dt}{t^{\frac{1}{4}}}=\int_1^{16}t^{-\frac{1}{4}}dt=\left[\frac{4}{3}t^{\frac{3}{4}}\right]_1^{16}=\frac{32}{3}+\frac{4}{3}=\frac{28}{3}$$

g. $\int_0^{\frac{\pi}{2}}\sin\theta\, d\theta$

$$\int_0^{\frac{\pi}{2}}\sin\theta\, d\theta=\left[-\cos\theta\right]_0^{\frac{\pi}{2}}=-(0-1)=1$$

h. $\int_0^{\frac{\pi}{4}}\sec\theta\tan\theta\, d\theta$

$$\int_0^{\frac{\pi}{4}}\sec\theta\tan\theta\, d\theta=\left[\sec\theta\right]_0^{\frac{\pi}{4}}=\sqrt{2}-1$$

i. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 \theta \, d\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 \theta \, d\theta = [-\cot\theta]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 0 - (-1) = 1$$

j. $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt$

$$\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt = \int_{-2}^{-1} (t^{-2} - t^{-3}) dt = \left[-\frac{1}{t} + \frac{1}{2x^2}\right]_{-2}^{-1} = \frac{3}{2} - \frac{5}{8} = \frac{7}{8}$$

22. (5.3) Express the integral as a function $F(x)$.

a. $\int_1^x e^t \, dt$

$$\int_1^x e^t dt = e^t|_1^x = e^x - e^1 = e^x - e$$

b. $\int_{-x}^{x} \sin t \, dt$

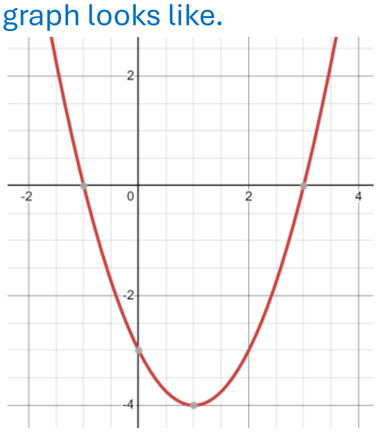
$$\int_{-x}^{x} \sin t \, dt = -\cos t \,|_{-x}^{x} = -\cos x + (-x) = -\cos x + \cos x = 0$$

You can also say that by symmetry, the areas cancel and equal zero.

23. (5.3) Identify the roots of the integrand to remove absolute values. Then evaluate the definite integral.

$$\int_{-2}^{4} |t^2 - 2t - 3| dt$$

We are finding the total area, not net signed area.
You can factor $t^2 - 2t - 3$ into $(t+1)(t-3)$ to find the zeros.
We need the zeros to know which areas are negative so they can be made positive. Based on the location of the zeros and that the parabola is opening up, you can visualize what the graph looks like.



From -1 to 3 the area will be negative so we will have to take the absolute value of that part of the integral.

$$\int_{-2}^{-1}(t^2-2t-3)dt+\left|\int_{-1}^{3}(t^2-2t-3)dt\right|+\int_{3}^{4}(t^2-2t-3)dt$$
$$=\frac{7}{3}+\left|-\frac{32}{3}\right|+\frac{7}{3}=\frac{46}{3}$$

24. (5.3) Suppose the rate of gasoline consumption over the course of a year in the United States can be modeled by a sinusoidal function of the form

$$\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\times 10^9 \text{ gal/mo.}$$

a. What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?

We will need to calculate the average value of the function:

$$\frac{1}{12}\int_0^{12}\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\cdot 10^9 dx$$
$$\frac{1}{12}\left[11.21t-\frac{\sin\left(\frac{\pi t}{6}\right)}{\frac{\pi}{6}}\right]_0^{12}\times 10^9=11.21\times 10^9\frac{gal}{mo}$$

To find the time that this average value occurs, set the average value equal to the function.

$$11.21\times 10^9=\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\times 10^9$$

Clearly $\cos\left(\frac{\pi t}{6}\right)$ must be equal to zero, so $\frac{\pi t}{6}=\frac{\pi}{2},\frac{3\pi}{2}$.

Therefore, $t=3$ and $t=9$.

b. What is the number of gallons of gasoline consumed in the United States in a year?

To calculate this, you can either take the entire integral over $[0,12]$ or multiply the average value by the duration (12).

$$(11.21\times 10^9)(12)=1.35\times 10^{11}\,gal$$

If you evaluate the following integral, you get the same result:

$$\int_0^{12}\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\cdot 10^9 dx$$

But since we already evaluated it when calculating the average value, it's easier just to multiply the average value by 12.

c. Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April $(t=3)$, and the end of September $(t=9)$.

This is similar to the average calculation in (a), but on the interval $[3,9]$.

$$\frac{1}{9-3}\int_3^9\left(11.21-\cos\left(\frac{\pi t}{6}\right)\right)\cdot 10^9 dx=\frac{1}{6}\left[11.21t-\frac{\sin\left(\frac{\pi t}{6}\right)}{\frac{\pi}{6}}\right]_3^9\times 10^9$$

$$=11.85\times 10^9\frac{gal}{mo}$$

25. (5.5) Find the antiderivative using the indicated substitution.

a. $\int(x+1)^4dx; u=x+1$

$$u=x+1$$
$$du=1dx\Rightarrow dx=du$$
$$\int(x+1)^4dx=\int u^4\cdot du=\frac{1}{5}u^5+C=\frac{1}{5}(x+1)^5+C$$

b. $\int(2x-3)^{-7}\,dx; u=2x-3$

$$u=2x-3\Rightarrow du=2dx\Rightarrow dx=\frac{1}{2}du$$
$$\int(2x-3)^{-7}dx=\int u^{-7}\cdot\frac{1}{2}du=\frac{1}{2}\left[-\frac{1}{6u^6}+C\right]=-\frac{1}{12u^6}+C$$
$$=-\frac{1}{12(2x-3)^6}+C$$

c. $\int\frac{x}{\sqrt{x^2+1}}dx; u=x^2+1$

$$u=x^2+1.\Rightarrow du=2xdx\Rightarrow dx=\frac{1}{2x}du$$
$$\int\frac{x}{\sqrt{x^2+1}}dx=\int\frac{x}{\sqrt{u}}\cdot\frac{du}{2x}=\frac{1}{2}\left[2\sqrt{u}+C\right]=\sqrt{u}+C=\sqrt{x^2+1}+C$$

Multiplying the constant C with other constants like $\frac{1}{2}$ are still constants that could be anything, so the C is the same.

d. $\int(x-1)(x^2-2x)^3dx; u=x^2-2x$

$$u=x^2-2x$$
$$du=(2x-2)dx$$
$$dx=\frac{du}{2x-2}$$

Initially rewriting the integral yields:

$$\int(x-1)(x^2-2x)^3dx=\int(x-1)u^3\frac{du}{2x-2}$$

But the $2x-2$ can be rewritten as something that includes $x-1$:

$$\int(x-1)u^3\frac{du}{2(x-1)}$$

Now the $(x-1)$'s cancel.

$$\frac{1}{2}\int u^3du=\frac{1}{2}\left[\frac{1}{4}u^4\right]+C=\frac{1}{8}(x^2-2x)^4+C$$

26. (5.5) Determine the indefinite integral.

a. $\int x(1-x)^{99}dx$

$$u = 1 - x \Rightarrow du = -dx$$

$$-\int x(1-x)^{99}dx = -\int x \cdot u^{99}du$$

We can rewrite the remaining x in terms of u, then distribute:

$$u = 1 - x \Rightarrow x = 1 - u$$

$$-\int (1-u)\cdot u^{99}du = -\int (u^{99} - u^{100})du = -\frac{1}{100}u^{100} + \frac{1}{101}u^{101} + C$$

$$= -\frac{1}{100}(1-x)^{100} + \frac{1}{101}(1-x)^{101} + C$$

b. $\int (11x-7)^{-3}dx$

$$u = 11x - 7 \Rightarrow du = 11dx$$

$$\int (11x-7)^{-3}dx = \frac{1}{11}\int u^{-3}du = -\frac{1}{22}u^{-2} + C = -\frac{1}{22(11x+7)^2} + C$$

c. $\int \cos^3\theta \sin\theta\, d\theta$

$$u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta \Rightarrow d\theta = \frac{du}{-\sin\theta}$$

$$\int \cos^3\theta \sin\theta\, d\theta = \int u^3 sin\theta \cdot \frac{du}{-\sin\theta} = -\int u^3 du = -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cos^4\theta + C$$

d. $\int t\sin(t^2)\cos(t^2)\,dt$

You can make u either $\sin(t^2)$ or $\cos(t^2)$. The only difference is the negative sign.

$$u = \sin(t^2) \Rightarrow du = 2t\cos(t^2)\,dt \Rightarrow dt = \frac{du}{2t\cos(t^2)}$$

$$\int t\sin(t^2)\cos(t^2)\,dt = \int t \cdot u \cdot \cos(t^2)\cdot \frac{du}{2t\cos(t^2)} = \frac{1}{2}\int u \cdot du = \frac{1}{4}u^2 + C$$

$$= \frac{1}{4}\sin^2(t^2) + C$$

If you set $u = \cos(t^2)$, you will get $-\frac{1}{4}\cos^2(t^2) + C$ as an answer, but they are equivalent because of the Pythagorean identity.

e. $\int \frac{x^2}{(x^3-3)^2}dx$

$$u = x^3 - 3 \Rightarrow du = 3x^2dx \Rightarrow dx = \frac{du}{3x^2}$$

$$\int \frac{x^2}{(x^3-3)^2}dx = \int \frac{x^2}{u^2}\cdot\frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{u^2}du = \frac{1}{3}(-u^{-1}) + C = -\frac{1}{3u} + C$$

$$= -\frac{1}{3(x^3-3)} + C$$

27. (5.5) Evaluate the definite integral.

$$\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}d\theta$$

$$u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta \Rightarrow d\theta = \frac{du}{-\sin\theta}$$

$$\theta = 0 \Rightarrow u = \cos 0 = 1$$

$$\theta = \frac{\pi}{4} \Rightarrow u = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta} d\theta = \int_1^{\frac{\sqrt{2}}{2}} \frac{\sin\theta}{\mathrm{u}^4} \cdot \frac{du}{-\sin\theta} = -\int_1^{\frac{\sqrt{2}}{2}} \frac{1}{u^4} du = -\left[\frac{1}{3}u^{-3}\right]_1^{\frac{\sqrt{2}}{2}} = -\left[\frac{2\sqrt{2}}{3} - \frac{1}{3}\right]$$

$$= \frac{1}{3}\left(2\sqrt{2} - 1\right)$$

28. (5.6) Compute the indefinite integral.

a. $\int e^{-3x} dx$

You can use substitution here, but it's easier to do it in your head. If you know that the derivative and integral of e^x is itself, the only difference is dividing by the derivative of $-3x$ to undo the chain rule. When you differentiate to check your work, the -3s will cancel.

$$\frac{e^{-3x}}{-3} + C$$

If you want to use substitution:

$$u = -3x, du = -3dx, dx = \frac{du}{-3}$$

$$\int e^u \cdot \frac{du}{-3} = \frac{1}{-3} e^u + C = \frac{1}{-3} e^{-3x} + C = \frac{e^{-3x}}{-3} + C$$

b. $\int 3^{-x} dx$

$$\int 3^{-x} dx = \frac{3^{-x}}{-\ln 3} + C$$

The -1 comes from dividing by the derivative of $-x$ to undo the chain rule.

c. $\int \frac{2}{x} dx$

Do this in your head, please.

$$\int \frac{2}{x} dx = 2\int \frac{1}{x} dx = 2\ln|x| + C$$

An equivalent answer is also $\ln x^2 + C$.

d. $\int \frac{dx}{x(\ln x)^2}$

Use u-substitution.

$$u = \ln x \Rightarrow du = \frac{1}{x} dx \Rightarrow dx = x \cdot du$$

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{1}{x \cdot u^2} \cdot x \cdot du = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

e. $\int \frac{\cos x - x\sin x}{x\cos x} dx$

$$u = x\cos x \Rightarrow du = (-x\sin x + \cos x)dx \Rightarrow dx = \frac{du}{(-x\sin x + \cos x)}$$

I decided to set $u = x\cos x$ because using the product rule, its derivative looks close to the numerator of the integrand. It turns out that the derivative is exactly the same, so the choice of u was correct.

$$\int \frac{\cos x - x\sin x}{x\cos x}dx = \int \frac{\cos x - x\sin x}{u}\cdot\frac{du}{(-x\sin x + \cos x)} = \int \frac{1}{u}du = \ln|u| + C$$
$$= \ln|x\cos x| + C$$

f. $\int \ln(\cos x)\tan x\,dx$

It helps to rewrite trig functions to determine the appropriate u-substitution.

$$\int \ln(\cos x)\tan x\,dx = \int \ln(\cos x)\cdot\frac{\sin x}{\cos x}\cdot dx$$

I will set $u = \ln(\cos x)$ because when taking the derivative, we have both $\cos x$ and $\sin x$ present with chain rule that will cancel out.

$$u = \ln(\cos x) \Rightarrow du = -\frac{\sin x}{\cos x}dx \Rightarrow dx = -\frac{\cos x}{\sin x}du$$
$$\int \ln(\cos x)\tan x\,dx = \int u\cdot\frac{\sin x}{\cos x}\cdot -\frac{\cos x}{\sin x}du = -\int u\cdot du = -\frac{1}{2}(\ln|\cos x|)^2 + C$$

g. $\int x^2 e^{-x^3}dx$

$$u = -x^3 \Rightarrow du = -3x^2dx \Rightarrow dx = -\frac{du}{3x^2}$$
$$\int x^2 e^{-x^3}dx = \int x^2 e^u\cdot\frac{du}{-3x^2} = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{-x^3} + C$$

29. (5.6) Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$. Evaluate the integral.

$$\int_0^{\ln x} e^t dt$$
$$= e^t|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

30. (5.6) Express the trig integrals in terms of compositions with logarithms.

a. $\int \frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}dx$

$$u = \sin(3x) + \cos(3x) \Rightarrow du = 3(\cos(3x) - \sin(3x))dx$$
$$dx = \frac{du}{3(\cos(3x) - \sin(3x))}$$
$$\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)}dx = \int \frac{\sin(3x) - \cos(3x)}{u}\cdot\frac{du}{3(\cos(3x) - \sin(3x))}$$
$$= \frac{1}{3}\int \frac{\sin(3x) - \cos(3x)}{u}\cdot\frac{du}{-(\cos(3x) - \sin(3x))} = -\frac{1}{3}\int \frac{1}{u}du$$
$$= -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|\sin(3x) + \cos(3x)| + C$$

b. $\int \ln(\csc x)\cot x\,dx$

$$u = \ln(\csc x) \Rightarrow du = \frac{-\csc x \cot x}{\csc x} dx = -\cot x\, dx \Rightarrow dx = \frac{du}{-\cot x}$$
$$\int \ln(\csc x) \cot x\, dx = \int u \cdot \cot x \cdot \frac{du}{-\cot x} = -\int u \cdot du = -\frac{1}{2}u^2 + C$$
$$= -\frac{1}{2}\ln(\csc x)^2 + C$$

31. (5.6) Evaluate the definite integral.
$$\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$$
$$u = 3x + 3x^2 + x^3 \Rightarrow du = (3 + 6x + 3x^2)dx \Rightarrow dx = \frac{du}{3(1 + 2x + x^2)}$$
$$u(x = 1) = 7$$
$$u(x = 2) = 26$$
$$\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx = \int_1^2 \frac{1 + 2x + x^2}{u} \cdot \frac{du}{3(1 + 2x + x^2)} = \frac{1}{3}\int_7^{26} \frac{1}{u} du = \frac{1}{3}\ln|u|_7^{26}$$
$$= \frac{1}{3}[\ln 26 - \ln 7] = \frac{1}{3}\ln\frac{26}{7}$$

32. (5.6) Integrate using the indicated substitution.

a. $\int \frac{y-1}{y+1} dy\,; u = y + 1$
$$u = y + 1 \Rightarrow du = dy$$
$$u - 2 = y + 1 - 2 = y - 1$$
$$\int \frac{y - 1}{y + 1} dy = \int \frac{y - 1}{u} du = \int \frac{u - 2}{u} du$$
Split the integrand into two fractions that we know how to integrate.
$$\int \left[\frac{u}{u} - \frac{2}{u}\right] du = u - 2\ln|u| + C = y + 1 - 2\ln|y + 1| + C = y - 2\ln|y + 1| + C$$

b. $\int \ln(x) \frac{\sqrt{1-(\ln x)^2}}{x} dx\,; u = \ln x$
$$u = \ln x \Rightarrow du = \frac{1}{x} dx \Rightarrow dx = gx \cdot du$$
$$\int \ln(x) \frac{\sqrt{1 - (\ln x)^2}}{x} dx = -\int u \cdot \frac{\sqrt{1 - u^2}}{x} \cdot x \cdot du = -\int u\sqrt{1 - u^2} du$$
We will need a second substitution.
$$v = 1 - u^2 \Rightarrow dv = -2u \cdot du \Rightarrow du = \frac{dv}{-2u}$$
$$-\int u\sqrt{1 - u^2} du = -\int u\sqrt{v} \cdot \frac{dv}{-2u} = \frac{1}{2}\int \sqrt{v} dv = \frac{1}{2} \cdot \frac{2}{3} v^{\frac{3}{2}} + C = \frac{1}{3}(1 - u^2)^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(1 - \ln^2 x)^{\frac{3}{2}} + C$$
The $\ln^2 x$ just means $(\ln x)^2$, just like on trig functions.

33. (5.7) Evaluate the integral in terms of an inverse trig function.

a. $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \,|_0^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

b. $\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2}$

$$\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2} = \arctan x \,|_{\sqrt{3}}^{1} = \frac{\pi}{4} - \frac{\pi}{3} = -\frac{\pi}{12}$$

c. $\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

$$\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}} = \operatorname{arcsec} x \,|_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

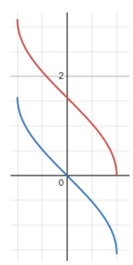
34. (5.7) Given the following relationship, does $\arccos t = -\arcsin t$?

$$-\arccos t + C = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C$$

No. This relationship yields:

$$-\arccos t = \arcsin t + C$$

This does not mean they equal each other, it means they have the same shape but are shifted vertically from each other. They actually differ by $\frac{\pi}{2}$.



35. (5.7) What is wrong with this integral?

$$\int_1^2 \frac{dt}{\sqrt{1-t^2}}$$

$\frac{1}{\sqrt{1-t^2}}$ is not completely defined on $[1,2]$ because after $t = 1$ there is a negative number in the square root.

36. (5.7) Find the antiderivative using substitutions.

a. $\int \frac{\arcsin t}{\sqrt{1-t^2}} dt$

$$u = \arcsin t \Rightarrow du = \frac{1}{\sqrt{1-t^2}}dt \Rightarrow dt = \sqrt{1-t^2}du$$

$$\int \frac{\arcsin t}{\sqrt{1-t^2}}dt = \int \frac{u}{\sqrt{1-t^2}} \cdot \sqrt{1-t^2}du = \int u \cdot du = \frac{1}{2}u^2 + C = \frac{1}{2}\arcsin^2 t + C$$

b. $\int \frac{\arctan(2t)}{1+4t^2}dt$

$$u = \arctan(2t) \Rightarrow du = \frac{2}{1+(2t)^2}dt \Rightarrow dt = \frac{1}{2}(1+4t^2)du$$

$$\int \frac{\arctan(2t)}{1+4t^2}dt = \frac{1}{2}\int \frac{u}{1+4t^2} \cdot (1+4t^2)du = \frac{1}{2}\int u \cdot du = \frac{1}{4}u^2 + C$$

$$= \frac{1}{4}\arctan^2(2t) + C$$

c. $\int \frac{\operatorname{arcsec}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}dt$

$$u = \operatorname{arcsec}\left(\frac{t}{2}\right) \Rightarrow du = \frac{2}{|t|\sqrt{t^2-4}}dt \Rightarrow dt = \frac{1}{2}\left(|t|\sqrt{t^2-4}\right)du$$

$$\int \frac{\operatorname{arcsec}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}dt = \frac{1}{2}\int \frac{u}{|t|\sqrt{t^2-4}}\left(|t|\sqrt{t^2-4}\right)du = \frac{1}{4}u^2 + C = \frac{1}{4}\operatorname{arcsec}^2\left(\frac{t}{2}\right) + C$$

d. $\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

$$u = \ln t \Rightarrow du = \frac{1}{t}dt \Rightarrow dt = t \cdot du$$

$$\int \frac{dt}{t\sqrt{1-\ln^2 t}} = \int \frac{1}{t\sqrt{1-u^2}}t \cdot du = \int \frac{1}{\sqrt{1-u^2}}du = \arcsin(u) + C$$

$$= \arcsin(\ln t) + C$$

37. (5.7) Evaluate the definite integral.

a. $\int_0^{\frac{1}{2}} \frac{\tan(\arcsin t)}{\sqrt{1-t^2}}dt$

This involves integrating tangent. If you didn't memorize it, I went through the steps again. You will then need two different substitutions and change the bounds twice also.

$$u = \arcsin t \Rightarrow du = \frac{1}{\sqrt{1-t^2}}dt \Rightarrow dt = \sqrt{1-t^2}du$$

$$\int_0^{\frac{\pi}{6}} \frac{\tan(\arcsin t)}{\sqrt{1-t^2}} dt = \int_0^{\frac{\pi}{6}} \frac{\tan u}{\sqrt{1-t^2}} \sqrt{1-t^2} du = \int_0^{\frac{\pi}{6}} \tan u \, du = \int_0^{\frac{\pi}{6}} \frac{\sin u}{\cos u} du$$

$$v = \cos u \Rightarrow dv = -\sin u \, du \Rightarrow du = \frac{dv}{-\sin u}$$

$$\int_0^{\frac{\pi}{6}} \frac{\sin u}{\cos u} du = \int_1^{\frac{\sqrt{3}}{2}} \frac{\sin u}{v} \cdot \frac{dv}{-\sin u} = -\int_1^{\frac{\sqrt{3}}{2}} \frac{1}{v} dv = -\ln|v| \Big|_1^{\frac{\sqrt{3}}{2}} = -\ln\frac{\sqrt{3}}{2}$$

This can be simplified further to $\frac{1}{2}\ln\left(\frac{4}{3}\right)$.

b. $\int_0^{\frac{1}{2}} \frac{\sin(\arctan t)}{1+t^2} dt$

$$u = \arctan t \Rightarrow du = \frac{1}{1+t^2} dt \Rightarrow dt = (1+t^2) du$$

$$\int_0^{\arctan\frac{1}{2}} \frac{\sin u}{1+t^2} \cdot (1+t^2) du = \int_0^{\arctan\frac{1}{2}} \sin u \, du = -\cos u \Big|_0^{\arctan\frac{1}{2}}$$

$$= -\cos\left(\arctan\frac{1}{2}\right) + \cos 0 = 1 - \cos\left(\arctan\frac{1}{2}\right) = 1 - \frac{2\sqrt{5}}{5}$$

Finding $\cos(\arctan(1/2))$ involves creating a right triangle with the known side lengths and solving for the hypotenuse to then perform the cosine.