OUR TITLE

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Contents

1. Introduction to Planar Graphs

Definition 1.1. A graph G is called a *planar graph* if G can be drawn in the plane without any two of its edges crossing [1]. If G is already drawn in the plane without crossings, then G is a *plane graph*.

Importantly, any graph isometric to a plane graph is therefore planar.

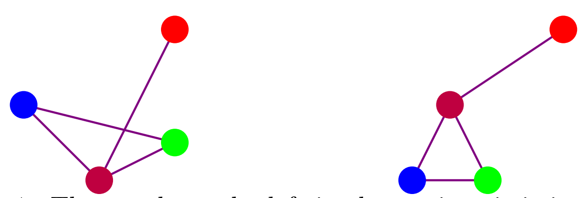

Figure 1. The graph on the left is planar since it is isomorphic to the plane graph on the right.

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From here on, when referring to planar graphs, we will be considering the plane graph that the graph is isomorphic to. Often, when working with planar graphs, one is concerned with whether or not a given graph is planar. This question appears often in contexts where there are connections on a $2D$ gid and intersections are impossible.

In order to solve this, we must discuss what properties define a planar graph. One important theorem is the Euler Identity.

Theorem 1.1 (The Euler Identity [1])**.** *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

In order to be able to understand this theorem, let us first discuss the regions of a graph. A *region* is an area bounded by the edges and vertices of a graph G. Additionally, there is an external region which is unbounded.

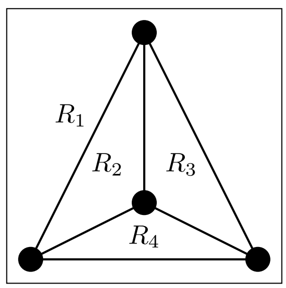

FIGURE 2. A planar graph with its regions denoted.

With this in mind, let us prove Euler's Identity.

Proof. Proceed by induction.

(1) *Base Case:* Consider the graph K_1, of order 1, size 0 and containing one region. Since there are no possible edges that could cross, the induction holds.
(2) *Inductive Hypothesis:* If we assume that the hypothesis holds for a graph of order n, size m and r regions, then we will show it holds for a graph of order n, size $m = 1$ and

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The Euler Identity is a powerful tool in characterizing planar graphs. However, it is difficult to determine the amount of regions in an arbitrary graph. Luckily, the Euler Identity leads to a result that no longer requires a region count. Since each edge is on the boundary of at most two regions in a graph G, we can use the Euler Identity to get a result in terms of the order and size of G.

If G is a planar graph of order $n \geq 3$ and size m, then

$$m \leq 3n - 6.$$

Equivalently, if G is of order $n \geq 5$ and size m such that

$$m > 3n - 6,$$

then G is nonplanar.

These results lead to two important nonplanar graphs that leas to a universal planarity criterion. Namely, both K_5 and $K_{3,3}$ are nonplanar. K_5 is of order 5 and size 20, and since $20 > 15 - 6$, it is therefore nonplanar by our newest result. $K_{3,3}$ is of order 6 and size 9, so we can't say anything using the size formulas. Instead, the Euler identity requires that $6 - 9 + r = 2$. Therefore, $K_{3,3}$ must have 5 regions to be nonplanar. However, since bipartite graphs have no odd cycles, each region of the graph requires at least four edges on the boundary (boundaries are cycles). Since each edge in $K_{3,3}$ is on a cycle, it is on the boundary of two regions – so each edge gets counted twice when constructing these regions. Therefore, the minimum size of $K_{3,3}$ must be $\frac{5\times 4}{2} = 10$. However, since $K_{3,3}$ is of size 9, this is impossible – sp $K_{3,3}$ must be nonplanar.

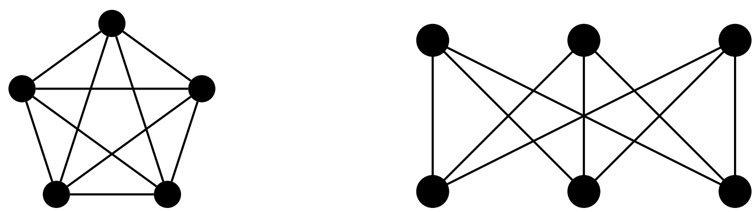

FIGURE 3. K_5 (left) and $K_{3,3}$ (right), two nonplanar graphs.

Put simply, the idea behind the universal criterion for planarity is the following: can we show that a given graph has the same kind of geometry as k_5 or $K_{3,3}$? Furthermore, we only have to show that a part of a graph has this geometry – as there only needs to be one instance of line crossing to have a nonplanar graph. To do this, we use the power of edge contractions.

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References

[1] Gary Chartrand, Linda Lesniak, and Ping Zhang, *Graphs and Digraphs*, CRC Press, Boca Raton, 2016.

[2] Richard Grassl and Oscar Levin, *More Discrete Mathematics via Graph Theory* (2018), `https://discrete.openmathbooks.org/more/mdm/mdm.html`. Accessed November 17, 2025.