PLANAR GRAPHS AND PLANARITY

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1. Introduction to Planar Graphs

Definition 1.1. A graph G is called a *planar graph* if G can be drawn in the plane without any two of its edges crossing [1]. If G is already drawn in the plane without crossings, then G is a *plane graph*.

Importantly, any graph isometric to a plane graph is therefore planar.

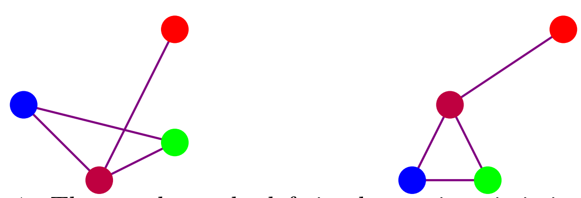

Figure 1. The graph on the left is planar since it is isomorphic to the plane graph on the right.

From here on, when referring to planar graphs, we will be considering the plane graph that the graph is isomorphic to. Often, when working with planar graphs, one is concerned with whether or not a given graph is planar. This question appears often in contexts where there are connections on a $2D$ grid and intersections are impossible.

In order to solve this, we must discuss what properties define a planar graph. One important theorem is the Euler Identity.

Date: December 4, 2025.

Theorem 1.1 (The Euler Identity [1, pp. 243])**.** *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

In order to be able to understand this theorem, let us first discuss the regions of a graph. A *region* is an area bounded by the edges and vertices of a graph G. Additionally, there is an external region which is unbounded.

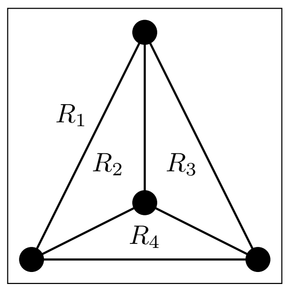

FIGURE 2. A planar graph with its regions denoted.

The Euler Identity is a powerful tool in characterizing planar graphs. However, it is difficult to determine the amount of regions in an arbitrary graph. Luckily, the Euler Identity leads to a result that no longer requires a region count. Since each edge is on the boundary of at most two regions in a graph G, we can use the Euler Identity to get a result in terms of the order and size of G.

If G is a planar graph of order $n \geq 3$ and size m, then

$$m \leq 3n - 6.$$

Equivalently, if G is of order $n \geq 5$ and size m such that

$$m > 3n - 6,$$

then G is nonplanar.

These results lead to two important nonplanar graphs that lead to a universal planarity criterion. Namely, both K_5 and $K_{3,3}$ are nonplanar. K_5 is of order 5 and size 20, and since $20 > 15-6$, it is therefore nonplanar by our newest result. $K_{3,3}$ is of order 6 and size 9, so we can't say anything using the size formulas. Instead, the Euler identity requires that $6-9+r = 2$. Therefore, $K_{3,3}$ must have 5 regions to be nonplanar. However, since bipartite graphs have no odd cycles, each region of the graph requires at least four edges on the boundary (boundaries are cycles). Since each edge in $K_{3,3}$ is on a cycle, it is on the boundary of two regions – so each edge gets counted twice when constructing these regions. Therefore, the minimum size of $K_{3,3}$ must be $\frac{5\times4}{2} = 10$. However, since $K_{3,3}$ is of size 9, this is impossible – so $K_{3,3}$ must be nonplanar.

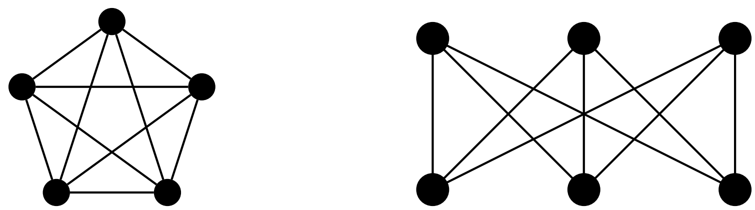

FIGURE 3. K_5 (left) and $K_{3,3}$ (right), two nonplanar graphs.

Put simply, the idea behind the universal criterion for planarity is the following: can we show that a given graph has the same kind of geometry as K_5 or $K_{3,3}$? Furthermore, we only have to show that a part of a graph has this geometry – as there only needs to be one instance of line crossing to have a nonplanar graph.

To do this, we use the power of edge contractions. Given adjacent vertices u and v of a graph G, we define edge contraction as the process of "merging" the two vertices into a new vertex w, which is adjacent to all of the neighbors of u and v.

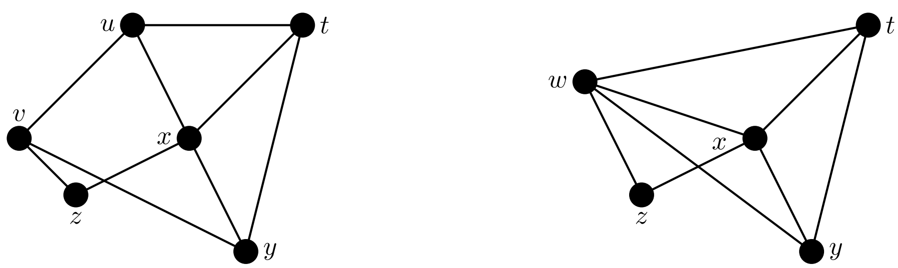

FIGURE 4. A graph G before and after the edge uv is contracted, creating G' and a new vertex w.

Any graph created by consecutively removing vertices, edges and performing edge contractions is called a *minor* of the graph G. So, in the figure above, G' is a valid minor of G. These minors lead into a powerful result that givs us a universal criterion for planarity.

Theorem 1.2 (Wagner's Theorem [1, pp. 260]). *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

In other words, if a graph G can be simplified down, via vertex deletion, edge deletion, and edge contraction, into either K_5 or $K_{3,3}$, then it is a nonplanar graph. This is why proving that K_5 and $K_{3,3}$ were nonplanar was so important – they are the most basic nonplanar graphs that all nonplanar graphs can be reduced to. This theorem and its corollaries lead to powerful results in the world of planar graphs. For example, there are highly efficient planarity testing algorithms that can run in linear time.

2. Planarity of Infinite Graphs

The graph $K_{1,n-1}$, the complete star of order n, is a graph defined as planar for all values of n.

2.1. **Simple Construction of: $K_{1,n-1}$.**

For a graph $K_{1,n-1},\ n \geq 2$, it follows that in radial coordinates, all vertices must exist at distinct degrees. That is, $\{\exists\ \theta_i\ \forall\ 0 \leq i, k < n-1 \in \mathbb{Z} \mid \theta_i \neq \theta_k \text{ and } \forall\ \theta_i, \exists \text{ radius } r_i\}$. Further, may it be assumed, for any value n, the distribution such that all values of θ_i are maximally spread from adjacent degrees: $\theta_{\text{mod}(i\pm 1,n-1)}$, is $\theta_i = \frac{2\pi}{n-1}i$. It follows, the non-origin vertices must then be:

$$v_i = \left(r_i \cos\left(\frac{2\pi}{n-1}i\right), r_i \sin\left(\frac{2\pi}{n-1}i\right)\right);\ r_i \neq 0$$

with edges:

$$E\left(v_{(0,0)}, v_i\right) = \left(t \cdot r_i \cos\left(\frac{2\pi}{n-1}i\right), t \cdot r_i \sin\left(\frac{2\pi}{n-1}i\right)\right);\ 0 \leq t \leq 1 \in \mathbb{R}$$

Thus, by construction, $K_{1,n-1}$ n-finite is planar.

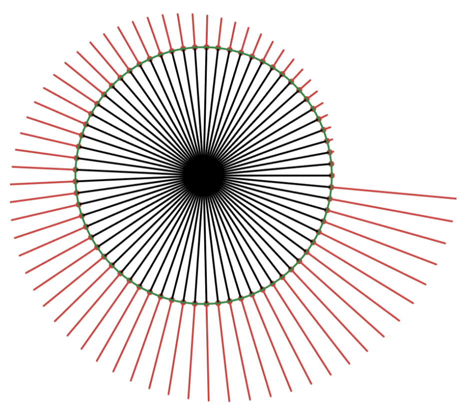

Figure 5. $K_{1,n-1}$ of monotonically increasing radii

2.2. **Planarity of $K_{1,\infty}$.**

The definition of planar requires edges to intersect only at their endpoints.
May it be self-evident that for any sequence of radii r_i', there exists another orientation of radii r_i that is monotonically increasing. That is, $r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$.
Pick any two distinct vertices in $K_{1,n-1}$.
That is, let $\{k, c \in \mathbb{Z} \mid 0 \leq k < n-1,\ 0 < c < n-1-k\}$. It follows that the edges created from the images of the vertices v_k and v_{k+c} may only intersect when the radii are the same. That is, $t_{k+c} = \frac{r_k}{r_{k+c}}t_k \mid 0 < \frac{r_k}{r_{k+c}}t_k \leq 1$.
May we evaluate $E\left(v_{(0,0)}, v_i\right) = E\left(v_{(0,0)}, v_{i+c}\right)$:

$$t_k \cdot r_k e^{i\frac{2\pi}{n-1}k} = t_{k+c} \cdot r_{k+c}\ e^{i\frac{2\pi}{n-1}(k+c)}$$

It follows that:

$$
\begin{aligned}
t_k \cdot r_k e^{i\frac{2\pi}{n-1}k} - t_{k+c} \cdot r_{k+c}\ e^{i\frac{2\pi}{n-1}(k+c)} &= 0 \\
r_{k+c} \cdot \left(\frac{r_k}{r_{k+c}} t_k e^{i\frac{2\pi}{n-1}k} - t_{k+c} e^{i\frac{2\pi}{n-1}}\ (k+c) \right) &= 0 \\
r_{k+c} \cdot \left(t_{k+c} e^{i\frac{2\pi}{n-1}k} - t_{k+c} e^{i\frac{2\pi}{n-1}(k+c)} \right) &= 0 \\
t_{k+c} \cdot r_{k+c} \cdot \left(e^{i\frac{2\pi}{n-1}k} - e^{i\frac{2\pi}{n-1}k} e^{i\frac{2\pi}{n-1}c} \right) &= 0 \\
t_{k+c} \cdot r_{k+c} \cdot e^{i\frac{2\pi}{n-1}k} \cdot \left(1 - e^{i\frac{2\pi}{n-1}c} \right) &= 0
\end{aligned}
$$

By the definition of Planar, an intersection at $t_{k+c} = 0$ maintains planarity, r_{k+c} is defined to be strictly greater than 0, and $e^{i\frac{2\pi}{n-1}k}$ is a point on the unit circle.

Consider however:

$$1 - e^{i\frac{2\pi}{n-1}c} = 0$$

Then:

$$
\begin{aligned}
e^{i\frac{2\pi}{n-1}c} &= 1 \\
\frac{2\pi}{n-1}c &= 0 + 2\pi m, m \in \mathbb{Z} \\
c &= m \cdot (n-1)
\end{aligned}
$$

Under the domain of c, $0 < c < n-1-k$, m can only ever be 0 or 1. Notice further, c has two strict inequalities for both 0 and the largest difference of index, $n-1$. That is, no finite n allows for the fulfillment of the above equality. We then introduce:

Corollary 2.1 (Paul Erdős' Corollary [8, pp. 305])**.**
If every finite subgraph of G is planar, G is planar.

As edges must connect two vertices and it is self-evident the removal of an edge or vertex on a planar graph cannot affect planarity, it is then sufficient to show that every finite induced subgraph is planar. As such, as $K_{1,\infty}[x_1, \ldots, x_n] \subseteq K_{1,n-1}$, $K_{1,\infty}$ is planar.

2.2.1. *Infinite Graphs.*
The order of a graph G is the cardinality of the vertex set. That is $|V(G)| = n$. A graph is then infinite when $|V(G)|$ or $|E(G)|$ is infinite. A popular method to organize a countably infinite graph G is via an increasing union of finite subgraphs [4]. That is, for a vertex set:

$$V(G) = \{x_1, x_2, \cdots\}$$

For each $n \in \mathbb{N}$ let the finite induced subgraph of G on n vertices be:

$$G_n \coloneqq G[x_1, \cdots, x_n]$$

It then follows:

$$
\begin{aligned}
G_1 \subseteq & G_2 \subseteq \cdots \\
G_\infty &= \bigcup_{n \in \mathbb{N}} G_n
\end{aligned}
$$

This method of organization, however, is not true for all graph sequences, but can be meaningful to be able to describe the behavior of a graph as it approaches a larger limit graph. When described as such, every finite induced subgraph of G_∞ is in some G_n [4, pp. 233]. Allowing G_∞ to be perceived as a graph of arbitrary scale. Unsurprisingly, it follows that if G_∞ of nested G_n is planar for all n, G_∞ is also planar as there is no induced subgraph that can contradict planarity. As it relates to planarity, the far more interesting object is the topological closure of an infinite graph.

Denoted via a bar over the top, the closure of an infinite graph is the union of the infinite graph and its limit points [5, pp. 97].

$$\overline{G_\infty} = G_\infty \cup G'_\infty$$

Limit points, often denoted with an apostrophe, are the set of points approached as sets approach infinity, with some points being a literal representation of ∞.
For example:

$$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$$

Graphs, however, are very odd objects; being merely a collection of vertices, without inherent spacial coordinates, and edges, being two tuples of vertices. To which, the closure of an infinite graph is merely that of the graph with an additional vertex: v_∞, or vertices, all with appropriate edges. Notice that as $\overline{\bigcup_{m=1}^{n} G_m} = \bigcup_{m=1}^{n} \overline{G_m}$ for n-finite and G_m being subsets of G_n, limit vertices and edges must be added to each appropriate finite and infinite subgraph, and if there are infinitely many subgraphs, as may occur with some trees or many constructions, an additional subgraph $G_{n\to\infty}$ may itself be needed.

Planarity is a property of the image of a graph. For the closure of an infinite graph to then be planar, it is self-evident the following equality must hold:

$$\Psi(\overline{G_\infty}) = \overline{\Psi(G_\infty)}$$

That is, $\Psi : G_m \to \mathbb{R}^2$ must be a topological embedding [6, Theorem 3.51]. A topological embedding f for topological spaces $(X, T), (Y, T')$ is a homeomorphism from X to Y [7, Definition 18.4]. A homeomorphism is a bijection (one-to-one and onto) between topologies: $f : A \to B$, where both f and f^{-1} are continuous [5, pp. 105]. (X, T) is a topological space iff T is a topology on X [7, Definition 12.2]. T is a topology on X iff: [7, Definition 12.1]

(1) $T \subseteq \wp(X)$ ($\wp(X)$ is the power set from discrete mathematics)
(2) $\emptyset \in T$ and $X \in T$
(3) $\forall\ S \subseteq T,\ \cup S \in T$
(4) $\forall\ U, V \in T,\ U \cap V \in T$

Notice: T is a set of sets and $\wp(S)$ of any set is a topology.

Or in other words, as $(X, \wp(X))$ is a topological space, a topological embedding is just a function between two domains that is one-to-one, onto, and not discontinuous.

2.2.2. *Planarity of $\overline{K_{1,\infty}}$.*
Let's revisit our chosen construction for $K_{1,n-1}$.
The image in $\mathbb{R}^2$ written in radial coordinates: (r,θ), as sets, follows as:

$$\Psi\left(V\left(K_{1,n-1}\right)\right)=\left\{(0,0),\left(r_0,0\right),\left(r_1,\frac{2\pi}{n-1}\right),\cdots,\left(r_{n-2},\frac{2\pi}{n-1}\left(n-2\right)\right)\right\}$$
$$\Psi\left(E\left(K_{1,n-1}\right)\right)=\left\{\left(r_0t_0,0\right),\left(r_1t_1,\frac{2\pi}{n-1}\right),\cdots,\left(r_{n-2}t_{n-2},\frac{2\pi}{n-1}\left(n-2\right)\right)\middle|\ 0\leq t_i\leq 1\right\}$$

For the sequence of degrees: $\theta_i=\{\frac{2\pi}{n-1}i|\ 0\leq i\leq n-2\}$

$$\lim_{n\to\infty}\theta_{n-2}=\frac{2\pi}{n-1}(n-2)=2\pi \tag{1}$$

That is, 2π is a limit point of $\Psi(V(K_{1,n-1}))$ and also therefore a limit point of $K_{1,\infty}$. Thus, under this construction:

$$\Psi\left(V\left(\overline{K_{1,\infty}}\right)\right)=\Psi\left(V\left(K_{1,\infty}\right)\right)\cup\left\{\left(r_{n-1},2\pi\right),\dots\right\}$$

That is, the points $(r_0,0)$ and $(r_{n-1},2\pi)$ are in the closure of this embedding. As both, $r_0,r_{n-1}>0$ and have edges to the origin. This would imply the edges intersect at all radii from $\min\{r_0,r_{n-1}\}\to 0$ **for this construction**.

Consider, however:

$$\Psi\left(V\left(K_{1,n-1}\right)\right)=\left\{(0,0),\left(r_0,0\right),\left(r_1,\frac{2\pi-\epsilon}{n-1}\right),\cdots,\left(r_{n-2},\frac{2\pi-\epsilon}{n-1}\left(n-2\right)\right)\right\}$$
$$\Psi\left(E\left(K_{1,n-1}\right)\right)=\left\{\left(r_0t_0,0\right),\left(r_1t_1,\frac{2\pi-\epsilon}{n-1}\right),\cdots,\left(r_{n-2}t_{n-2},\frac{2\pi-\epsilon}{n-1}\left(n-2\right)\right)\middle|\ 0\leq t_i\leq 1\right\}$$

The sequence of degrees is then: $\theta_i=\{\frac{2\pi-\epsilon}{n-1}i|\ 0\leq i\leq n-2\}$, with limit:

$$\lim_{n\to\infty}\theta_{n-2}=\frac{2\pi-\epsilon}{n-1}(n-2)=2\pi-\epsilon \tag{2}$$

Let $\epsilon\to 0$, and the domain of $\theta=[0,2\pi)\in\mathbb{R}$. With θ strictly less than 2π, this contradiction cannot occur. It may then be concluded $\overline{K_{1,\infty}}$ has a planar construction.

Notice: The domain of θ is actually continous in the real number system. While for any finite n, the values of θ are evenly spaced. For any rational number between 0 and 2π non-inclusive, we can choose the degree closest to it and assign it to a sequence S_n as value $S_i\ \forall i\leq n\in\mathbb{N}$. The result is a convergent infinite sequence of difference certainly less than 2π. By the definition of a real number, the chosen degree is in the domain of our degrees, and if it wasn't, it most certainly is in the closure.

2.3. **Planarity of Infinite Trees.**

2.3.1. *Planarity of $T_{n,m}$.*

Let $T_{n,m}$ be a tree with a central vertex v_c chosen such that the induced subgraph:

$$T_{n,m}[\{d(v_c,v_i)=j \cup d(v_c,v_i)=j-1\ \forall v_i \in V(T_{n,m})\}]$$

consists only of $K_{1,n-1}$ stars for $j>1$, and that no vertex in $T_{n,m}$ satisfies:

$$\{d(v_c,v_i)>m\ \forall v_i \in V(T_{n,m})\}$$

Construction of the image:

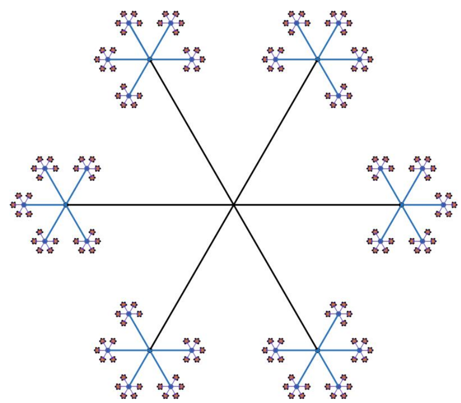

Figure 6. Planar image of $T_{6,5}$

Base Case: m = 1
Let $T_{n,1}$ be $K_{1,n-1}$ That is, for $\{k_0 \in \mathbb{Z} |\ 0 \leq k \leq n-1\}$:

$$\Psi(V(T_{n,1})) = \{e^{i\frac{2\pi}{n-1}k_0}\} \cup \{(0,0)\}$$

$$\Psi(E(T_{n,1})) = \{t_{k_0}e^{i\frac{2\pi}{n-1}k_0}\ 0 \leq t_k \leq 1 \in \mathbb{R}\}$$

$T_{n,1}$ is the star $K_{1,n-1}$; all stars are planar.
Inductive Hypothesis:
Let $\{k_q = \{\forall k \in \mathbb{Z} |\ 0 \leq k \leq n-2\};\ \forall q \leq m \in \mathbb{N}\}$.
That is, k indexed on q is a set containing $n-1$ values from 0 to $n-2$. When used in arithmetic, all combinations of all possible values must be used.
Let the image of $T_{n,m}$:

$$\Psi(V(T_{n,m})) = \bigcup_{p=0}^{m} \left\{ \sum_{q=0}^{p} \left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^q (-1)^q e^{i\frac{2\pi}{n-1}k_q} \middle| k_q \neq k_{q-1} \right\} \cup \{(0,0)\}$$

$$\Psi(E(T_{n,m})) = \bigcup_{p=0}^{m} \left\{ \left(\frac{-\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^p t_{k_p} e^{i\frac{2\pi}{n-1}k_p} + \sum_{q=0}^{p-1} \left(\frac{-\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^q e^{i\frac{2\pi}{n-1}k_q} \middle| k_q \neq k_{q-1} \right\}$$

be assumed planar. That is, let

$$\sum_{q=0}^{p} \left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^q (-1)^q e^{i\frac{2\pi}{n-1}k_q} | k_q \neq k_{q-1}$$

be the set of outermost vertices to appended at the p-th index of m; with the edge set consisting of the last term removed from the sum and added with an unique, individual time component t_{k_p} to create an edge from the p-th outermost vertices to the $(p-1)$-th outermost vertices, be assumed planar.

We want to show that unioning a new vertex and edge set of the same form will not break planarity.
Case 1: Scalar Collision

> As we no longer have vertices emanating from a single point, to ensure subsequent stars cannot intersect, a convergent infinite sequence less than half the distance from adjacent vertices is needed.
>
> Adding a series of $e^{i\theta}$ for some constant-in-context theta to some $e^{i\frac{2\pi}{n-1}k}$ only acts as a horizontal or vertical translation, thus the primary factor to consider is the scalar series applied to each subsequent star.
>
> First, let us find the distance between two adjecent vertices in $K_{1,n-1}$ with scalar λ.
> It follows that:

$$\begin{aligned}
d_{\mathbb{R}^2}(\lambda_m e^0, \lambda_m e^{i\frac{2\pi}{n-1}}) &= \sqrt{\left(\lambda_m - \lambda_m \cos\left(\frac{2\pi}{n-1}\right)\right)^2 + \left(0 - \lambda_m \sin\left(\frac{2\pi}{n-1}\right)\right)^2} \\
&= \lambda_m \sqrt{1 - 2\cos\left(\frac{2\pi}{n-1}\right) + \cos\left(\frac{2\pi}{n-1}\right)^2 + \sin\left(\frac{2\pi}{n-1}\right)^2} \\
&= \lambda_m \sqrt{2 - 2\cos\left(\frac{2\pi}{n-1}\right)} \\
&= \lambda_m \sqrt{2\left(1 - \cos\left(\frac{2\pi}{n-1}\right)\right)}
\end{aligned}$$

> May max-spread be assumed on the scalar radius. That is, may it be assumed no geometric constraints limit the sum of the sequence of scalars or:

$$\sum_{i=m+1}^{\infty} \lambda_i < \lambda_m \frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{2}$$

> Consider the sequence of scalars:

$$\lambda_m = \left\{ \left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^m \middle| \ 0 \le m \in \mathbb{Z} \right\}$$

It follows:

$$\sum_{i=m+1}^{\infty}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i = \sum_{i=0}^{\infty}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i - \sum_{i=0}^{m}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i$$

$$= \frac{1}{1-\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}} - \sum_{i=0}^{m}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i$$

Reconsidering the inequality:

$$\left(1-\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^{-1} - \sum_{i=0}^{m}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i < \left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^m \cdot \frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{2}$$

$$\left(1-\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^{-1} - \sum_{i=0}^{m}\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^i < 2\left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}\right)^{m+1}$$

Let $\lambda = \frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4}$

$$\begin{aligned}
(1-\lambda)^{-1} - \sum_{i=0}^{m}\lambda^i &< 2\lambda^{m+1} \\
1-(1-\lambda)\sum_{i=0}^{m}\lambda^i &< 2\lambda^{m+1}(1-\lambda) \\
1-2\lambda^{m+1}+\lambda^{m+2} &< \sum_{i=0}^{m}\lambda^i - \sum_{i=0}^{m}\lambda^{i+1} \\
&< \sum_{i=0}^{m}\lambda^i - \sum_{i=1}^{m+1}\lambda^i \\
&< \lambda^0 - \lambda^{m+1} \\
1-2\lambda^{m+1}+\lambda^{m+2} &< 1-\lambda^{m+1} \\
\lambda^{m+2}-\lambda^{m+1} &< 0 \\
\lambda^{m+1}(\lambda-1) &< 0
\end{aligned}$$

As $0 \le \lambda \le \frac{1}{2}$, this inequality holds for all $n > 2$ and all finite m.
The $n = 1$ case is non-graphical; as it implies the union of $K_{1,0}$ stars or 0 additional vertices.
The $n = 2$ case has a $\lambda = 0$, but can be reclaimed through an induced subgraph of the $n = 3$ case with the central and a chosen half of the vertices.
It may then be concluded λ_m satisfies:

$$\sum_{i=m+1}^{\infty}\lambda_i < \lambda_m\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{2}$$

Case 2: Self Collision

Under the definition of $T_{n,m}$, $T_{n,m}$ is defined as having a collection of $K_{1,n-1}$ stars for every induced subgraph of graphical distance of signed difference 1.
That is:

$$T_{n,m}[\{d(v_c, v_i) = j \cup d(v_c, v_i) = j - 1 \; \forall v_i \in V(T_{n,m})\}]$$

The optimal way to ensure this property follows is to ensure the union of subsequent $K_{1,n-1}$ stars have exactly 1 vertex intersecting with the edge of its parant star, and to remove the conflicting vertex.
That is, as all previous stars have the net effect solely on translation, may we look at the following:

$$te^{i\frac{2\pi}{n-1}k_1} = e^{i\frac{2\pi}{n-1}k_1} - e^{i\frac{2\pi}{n-1}k_2}$$

Notice: By the multiplication of a negative, $-te^{i\frac{2\pi}{n-1}k_1} = -e^{i\frac{2\pi}{n-1}k_1} + e^{i\frac{2\pi}{n-1}k_2}$ will also follow.
Similarly, this intersection is independent of radius, so may the simplist case be observed. That is, let $\lambda = 1$ and $t = 0$

$$\begin{aligned}
te^{i\frac{2\pi}{n-1}k_1} &= e^{i\frac{2\pi}{n-1}k_1} - e^{i\frac{2\pi}{n-1}k_2} \\
e^{i\frac{2\pi}{n-1}k_2} &= (1-t)e^{i\frac{2\pi}{n-1}k_1} \\
&= (1-0)e^{i\frac{2\pi}{n-1}k_1} \\
\frac{2\pi}{n-1}k_2 &= \frac{2\pi}{n-1}k_1 + 2\pi m \\
k_2 &= k_1 + 2\pi m(n-1) \\
m, n \in \mathbb{Z} &\to k_2 = k_1
\end{aligned}$$

By adding an alternating sign $(-1)^i$ and removing the index of k_2 equivalent to k_1 or more generally $k_i = k_{i-1}$, the properties of $T_{n,m}$ may be preserved without effecting planarity

Thus, by division into cases, $T_{n,m+1}$ with edges and vertices:

$$\Psi(V(T_{n,m+1})) = \Psi(V(T_{n,m})) \bigcup \left\{ \sum_{q=0}^{m+1} \left(\frac{\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^q (-1)^q e^{i\frac{2\pi}{n-1}k_q} \middle| k_q \neq k_{q-1} \right\}$$

$$\Psi(E(T_{n,m+1})) = \Psi(V(T_{n,m})) \bigcup$$

$$\left\{ \left(\frac{-\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^{m+1} t_{k_{m+1}} e^{i\frac{2\pi}{n-1}k_{m+1}} + \sum_{q=0}^{m} \left(\frac{-\sqrt{2(1-\cos(\frac{2\pi}{n-1}))}}{4} \right)^q e^{i\frac{2\pi}{n-1}k_q} \middle| k_i \neq k_{i-1} \right\}$$

preserves planarity, and by mathematical induction, $T_{n,m}$ is Planar.

2.3.2. $T_{\infty,\infty}$ *is planar.*

To use **Corollary 2.0.1**; the Paul Erdős Corollary, it must be shown that all finite trees are subgraphs of $T_{n,m}$.

Let T be a finite tree.
Let v_c be any vertex of T.
Let m be the maximum graphical distance from v_c and n be the maximum of the number of vertices at each distance m_i. That is:
Let m = $\max\{d(v_c, v_i);\ \forall v_i \in T\}$
Let n = $\max\{|\{\forall v_i \in V(T) \text{ st } d(v_c, v_i) = m_i \cap v_i \neq v_c\}|;\ \forall m_i \leq m \in \mathbb{N}\}$
As there are no vertices of distance greater than m with no more than n vertices existing at each subdistance m_i, $T \subseteq T_{n,m}$.

By **Corollary 2.0.1**; the Paul Erdős Corollary, the following graphs are planar:

$$T_{n,\infty}, T_{\infty,m}, T_{\infty,\infty}$$

2.4. **Remarks about infinite planarity.**
As should have been evident halfway through the section, there are far-simpler and far-easier means to prove infinite stars and trees are planar. Just use **Theorem 1.1** or **Theorem 1.2** properly through induction or even with just the general properties of a type of graph and **Corollary 2.0.1** will prove it's planar. Rather, there are certain apparent-contradictions I wanted to handle directly.

2.4.1. *A graph may have vertices of infinite degree and remain planar.*
That is, it is possible for 2 points to be so close together their distance is immeasurable, yet both points are distinct, their edges do not intersect, and the graph is still planar.

2.4.2. *An infinite graph may be closed and remain planar.*
One can then take those points that are immesurably close, note that the limit approaches 0, denote it a limit point in the closure, and said limit point will still be a distinct point from the non-limit points and will not violate planarity.

2.4.3. *An infinite graph may be recursively compacted.*
That is, one can take those immesurably close points and still find the room to shove the entire graph into itself between those points an indefinite number of times, and the graph will not be any less planar.

To some degree, planarity almost seems meaningless. You can take an infinite planar graph of infinite size and of infinite density (vertices per unit area), use a transformation like $f(u,v) = \left(\frac{1}{\sqrt{2\pi}}\int_0^u e^{-\frac{x^2}{2}}dx, \frac{1}{\sqrt{2\pi}}\int_0^v e^{-\frac{x^2}{2}}dx\right)$ to force the graph to occupy at most a 1x1 unit square making it an infinite planar graph of finite size. By **Theorem 1.1**, a planar graph has a region and that region will have an area. You can then take a copy of any other planar graph, including itself, and scale it to fit into that region, and as long as you connect it to the graph without violating planarity, you can keep going, and even if you do so infinitely, the resulting graph will still be planar.

3. Applications of Planar Graphs

When discussing applications of planar graphs, the first thing that might come to mind would reside in the field of civil engineering. Roads, bridges, and traffic flows are all operations of which general graph theory is very apparent. Intersections and roads can be thought of as graphs with edges and vertices. The crossing of edges, or in this case roads, requires a decision to be made whether to implement a new intersection or a bridge overpass. While both options are viable, bridges cost significantly more in every aspect.

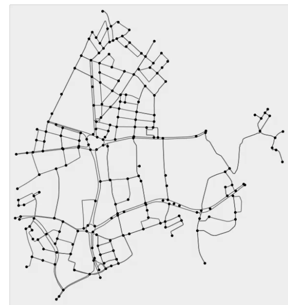

Figure 7. City roads represented as a graph. Colors inverted to match document style.

So how do planar graphs make traffic planning optimal? Since planar graphs have no intersecting or crossing edges, civil engineers can optimize the placement of intersections and roads to understand if there exists somewhere a bridge is absolutely necessary or an opportunity to save time and money by avoiding bridge construction outright.

What other applications might exist? From a network engineer perspective, graphs can represent entire networks. Data being transferred by ethernet consists of eight electrical pulses through copper wire which are subject to electrical interference. If too many cables cross, packet loss may occur, leading to the user experience slowing down. Similarly, electrical engineers become subject to the same issue or interference. When designing a single layered Printed Circuit board (PCB), engineers must keep the layout planar, as adding in Vias leads to increased complexity, cost, and resistance on the board.

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