OUR TITLE

LUCAS JOHNSON, NATHAN LACROSSE, AND NICK PROCTOR*

ABSTRACT. our abstract

Contents

1. Introduction to Planar Graphs

Definition 1.1. A graph G is called a *planar graph* if G can be drawn in the plane without any two of its edges crossing [1]. If G is already drawn in the plane without crossings, then G is a *plane graph*.

Importantly, any graph isometric to a plane graph is therefore planar.

FIGURE 1. The graph on the left is planar since it is isomorphic to the plane graph on the right.

From here on, when referring to planar graphs, we will be considering the plane graph that the graph is isomorphic to. Often, when working with planar graphs, one is concerned with whether or not a given graph is planar. This question appears often in contexts where there are connections on a $2D$ gid and intersections are impossible.

In order to solve this, we must discuss what properties define a planar graph. One important theorem is the Euler Identity.

Theorem 1.1 (The Euler Identity [1]). *For every connected plane graph of order n, size m and having r regions,*

$$n - m + r = 2.$$

In order to be able to understand this theorem, let us first discuss the regions of a graph. A *region* is an area bounded by the edges and vertices of a graph G. Additionally, there is an external region which is unbounded.
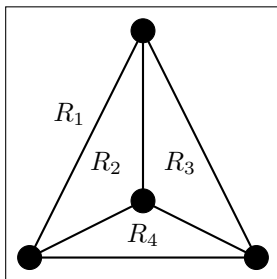

FIGURE 2. A planar graph with its regions denoted.

The Euler Identity is a powerful tool in characterizing planar graphs. However, it is difficult to determine the amount of regions in an arbitrary graph. Luckily, the Euler Identity leads to a result that no longer requires a region count. Since each edge is on the boundary of at most two regions in a graph G, we can use the Euler Identity to get a result in terms of the order and size of G.

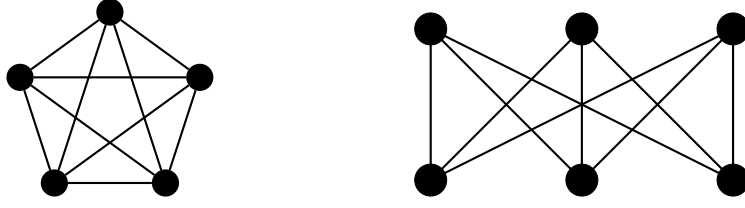If G is a planar graph of order $n \geq 3$ and size m, then

$$m \leq 3n - 6.$$

Equivalently, if G is of order $n \geq 5$ and size m such that
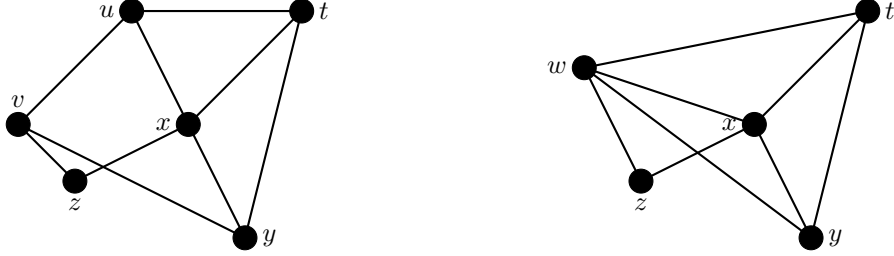
$$m > 3n - 6,$$

then G is nonplanar.

These results lead to two important nonplanar graphs that leas to a universal planarity criterion. Namely, both K_5 and $K_{3,3}$ are nonplanar. K_5 is of order 5 and size 20, and since $20 > 15 - 6$, it is therefore nonplanar by our newest result. $K_{3,3}$ is of order 6 and size 9, so we can't say anything using the size formulas. Instead, the Euler identity requires that $6 - 9 + r = 2$. Therefore, $K_{3,3}$ must have 5 regions to be nonplanar. However, since bipartite graphs have no odd cycles, each region of the graph requires at least four edges on the boundary (boundaries are cycles). Since each edge in $K_{3,3}$ is on a cycle, it is on the boundary of two regions – so each edge gets counted twice when constructing these regions. Therefore, the minimum size of $K_{3,3}$ must be $\frac{5 \times 4}{2} = 10$. However, since $K_{3,3}$ is of size 9, this is impossible – sp $K_{3,3}$ must be nonplanar.

FIGURE 3. K_5 (left) and $K_{3,3}$ (right), two nonplanar graphs.

Put simply, the idea behind the universal criterion for planarity is the following: can we show that a given graph has the same kind of geometry as k_5 or $K_{3,3}$? Furthermore, we only have to show that a part of a graph has this geometry – as there only needs to be one instance of line crossing to have a nonplanar graph.

To do this, we use the power of edge contractions. Given adjacent vertices u and v of a graph G, we define edge contraction as the process of "merging" the two vertices into a new vertex w, which is adjacent to all of the neighbors of u and v.

FIGURE 4. A graph G before and after the edge $u--v$ is contracted, creating G' and a new vertex w.

Any graph created by consecutively removing vertices, edges and performing edge contractions is called a *minor* of the graph G. So, in the figure above, G' is a valid minor of G. These minors lead into a powerful result that givs us a universal criterion for planarity.

Theorem 1.2 (Wagner's Theorem [1]). *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ is a minor of G.*

In other words, if a graph G can be simplified down, via vertex deletion, edge deletion, and edge contraction, into either K_5 or $K_{3,3}$, then it is a nonplanar graph. This is why proving that K_5 and $K_{3,3}$ were nonplanar was so important – they are the most basic nonplanar graphs that all nonplanar graphs can be reduced to. This theorem and its corollaries lead to powerful results in the world of planar graphs. For example, there are highly efficient planarity testing algorithms that can run in linear time.

2. TITLE

f

3. TITLE

f

4. APPLICATIONS OF PLANAR GRAPHS

When discussing applications of planar graphs, the first thing that might come to mind would resie in the field of civil engineering. Roads, bridges, and traffic flows are all operations of which general graph theory is very apparent. Intersections and roads can be thought of as graphs with edges and vertices. The crossing of edges, or in this case roads, requires a decision to be made whether to implement a new intersection or a bridge overpass. While both options are viable, bridges cost significantly more in every aspect.

So how do planar graphs make traffic planning optimal? Since planar graphs have no intersecting or crossing edges, civil engineers can optimize the placement of intersections and roads to understand if there exists somewhere a bridge is absolutely necessary or an opportunity to save time and money by avoiding bridge construction outright.

FIGURE 5. City roads represented as a graph. Colors inverted to match document style.

What other applications might exist? From a network engineer perspective, graphs can represent entire networks. Data being transferred by ethernet consists of eight electrical pulses through copper wire which are subject to electrical interference. If too many cables cross, packet loss may occur, leading to the user experience slowing down. Similarly, Electrical Engineers become subject to the same issue or interference. When designing a single layered Printed Circuit board (PCB), engineers must keep the layout planar, as adding in Vias leads to increased complexity, cost, and resistance on the board.

REFERENCES

[1] Gary Chartrand, Linda Lesniak, and Ping Zhang, *Graphs and Digraphs*, CRC Press, Boca Raton, 2016.

[2] Richard Grassl and Oscar Levin, *More Discrete Mathematics via Graph Theory* (2018), `https://discrete.openmathbooks.org/more/mdm/mdm.html`. Accessed November 17, 2025.

[3] Mateusz Praski, *Mapping the Jams: Traffic Analysis Using Graph Theory* (2023), `https://medium.com/data-science/mapping-the-jams-traffic-analysis-using-graph-theory-a387135ea748`. Accessed November 24, 2025.