15

Matrix Games

In this chapter, we shall discuss "finite two-person zero-sum games," also called "matrix games" for short. The first attempts to formalize a theory of such games were made by E. Borel (1921, 1924, 1927); a solid foundation of the theory was laid down by J. von Neumann (1928) who proved the celebrated "Minimax Theorem." His original proof, involving Brouwer's fixed-point theorem, was rather complicated; some twenty years later, von Neumann also pointed out that solving matrix games may be reduced to solving certain linear programming problems. Eventually, through the work of G. B. Dantzig, D. Gale, H. W. Kuhn, A. W. Tucker, and others, the study of matrix games became a part of linear programming. In particular, it turned out that the Minimax Theorem follows easily from the Duality Theorem.

AN INTRODUCTORY EXAMPLE: THE GAME OF MORRA

We shall begin our presentation with the game of *Morra*, which is played by two. Its rules are simple: each player hides one or two francs and tries to guess (aloud) how many francs the other player has hidden. If *only one* player makes the correct guess, then this player wins from the other player an amount of money equal to the *total* amount that has been hidden; in all the other cases, the result is a draw and no money changes hands. (For example, suppose that Trucula hides two francs and guesses two, whereas Claude hides two and guesses one. In that case, Claude has to give Trucula four francs.) Trivially, each player has the choice of four courses of action:

- Hide one, guess one.
- Hide one, guess two.
- Hide two, guess one.
- Hide two, guess two.

These courses of action are called pure *strategies*. We shall denote them, in the above order, by [1, 1], [1, 2], [2, 1], and [2, 2]; thus $[x, y]$ denotes "hide x, guess y."

Now suppose that the two players played a very long match. Claude either stuck to one of his pure strategies in every round or used different pure strategies in different rounds, with or without a discernible pattern in his choices; all we know is that he played [1, 1] in c_1 rounds, [1, 2] in c_2 rounds, [2, 1] in c_3 rounds, and [2, 2] in c_4 rounds. Trucula, however, secretly flipped a coin in each round; then she played either [1, 2] if the coin showed heads or [2, 1] if the coin showed tails. If her coin behaved as an unbiased coin should, then she countered Claude's [1, 1] by her own [1, 2] in $c_1/2$ rounds, countering by [2, 1] in the remaining $c_1/2$ rounds. In fact, she countered each of Claude's pure strategies by [1, 2] half the time and by [2, 1] in the remaining half. Thus a detailed record of the match goes as follows.

In $c_1/2$ rounds, Claude played [1, 1] and Trucula played [1, 2], losing 2 francs.
In $c_1/2$ rounds, Claude played [1, 1] and Trucula played [2, 1], winning 3 francs.
In $c_2/2$ rounds, Claude played [1, 2] and Trucula played [1, 2]: a draw.
In $c_2/2$ rounds, Claude played [1, 2] and Trucula played [2, 1]: a draw.
In $c_3/2$ rounds, Claude played [2, 1] and Trucula played [1, 2]: a draw.
In $c_3/2$ rounds, Claude played [2, 1] and Trucula played [2, 1]: a draw.
In $c_4/2$ rounds, Claude played [2, 2] and Trucula played [1, 2], winning 3 francs.
In $c_4/2$ rounds, Claude played [2, 2] and Trucula played [2, 1], losing 4 francs.

Trucula's total winnings come to $(c_1 - c_4)/2$ francs. This number may be negative: if Claude played [2, 2] more often than [1, 1], then Trucula actually lost. Nevertheless, her average loss per round does not exceed half a franc. We conclude that Trucula can protect herself from expected losses greater than half a franc per round by mixing her pure strategies [1, 2] and [2, 1] in the proportion 1:1. (Of course, she must do so in an unpredictable way exhibiting no regularity; otherwise the consequences would be disastrous. For instance, if she played [1, 2] every odd round and [2, 1] every even round, then Claude would catch on, countering every [1, 2] by [1, 1] and every [2, 1] by [2, 2]. Flipping the coin helps to mix the two pure strategies in the desired proportion and yet creates no discernible pattern.) Could she protect herself even better by using a different mixture of her pure strategies? We shall answer this question in a general setting.

MATRIX GAMES

Every matrix $\mathbf{A} = (a_{ij})$ defines a game for two. In each round, the *row player* selects one of the rows $i = 1, 2, \ldots, m$ and the *column player* selects one of the columns $j = 1, 2, \ldots, n$; the resulting *payoff* to the row player is a_{ij}. (That is to say, the row player receives a_{ij} monetary units from the column player. Of course, if a_{ij} is negative, then it is the row player who pays: receiving a negative amount means paying.) Each player makes a choice unaware of the opponent's choice; however, the *payoff matrix* A is known to both players. Clearly, Morra fits into this format; its payoff matrix is as follows:

$$
\begin{array}{cc}
 & \text{Claude's pure strategies} \\
 & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\
\begin{array}{c} \text{Trucula's} \\ \text{pure strategies} \end{array}
\begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} &
\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}.
\end{array}
$$

In a long match, the row player may decide to mix her m pure strategies so that each row i will be selected with a probability x_i in every round. The column player may respond in a regular or random manner; over a long period of time, he will choose the jth column with some relative frequency y_j. (In our example, we considered $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{2}$, $x_4 = 0$, and $y_j = c_j/N$ with $N = c_1 + c_2 + c_3 + c_4$.) Thus, the row i and the column j will be selected in $x_i y_j N$ of the total N rounds. The resulting average payoff (to the row player) per round equals

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

or, in matrix notation, $\mathbf{xAy}$. Here $\mathbf{x}$ stands for the row vector with components $x_1, x_2, \ldots, x_m$ and y stands for the column vector with components $y_1, y_2, \ldots, y_n$. These two vectors share a characteristic feature: their components are nonnegative, with the sum equal to one. Such vectors are called *stochastic.*

Whenever the row player adopts a *mixed strategy* described by a stochastic row vector $\mathbf{x}$, he assures himself of winning at least

$$\min_{\mathbf{y}} \mathbf{xAy}$$

per round on the average, with the minimum taken over all stochastic column vectors $\mathbf{y}$. For instance, by adopting the mixed strategy described by $\mathbf{x} = [0, \frac{1}{2}, \frac{1}{2}, 0]$, Trucula assures herself of winning at least -0.5 francs (that is, losing at most 0.5 francs) per round on the average. Thus, a row player desiring the best possible *guarantee* that her expected losses will be curbed and/or her expected winnings kept high should look for a mixed strategy $\mathbf{x}$ that maximizes the quantity min $\mathbf{xAy}$; such a strategy is called *optimal.* Let us note at once that

$$\min_{\mathbf{y}} \mathbf{xAy} = \min_{j} \sum_{i=1}^{m} a_{ij}x_i. \tag{15.1}$$

In words, identity (15.1) asserts that among the most effective replies $\mathbf{y}$ to the row player's mixed strategy $\mathbf{x}$, there is always at least one pure strategy. This claim is not difficult to justify in intuitive terms; a formal proof is as follows. If t stands for the right-hand side in (15.1) and if $\mathbf{y}$ is an arbitrary stochastic column vector of length n, then

$$\mathbf{xAy} = \sum_{j=1}^{n} y_j\left(\sum_{i=1}^{m} a_{ij}x_i\right) \geq \sum_{j=1}^{n} y_j t = t$$

and so the left-hand side of (15.1) is at least the right-hand side. On the other hand, since each $\mathbf{y}$ with one component equal to one and the remaining components equal to zero is a candidate for minimizing $\mathbf{xAy}$, we have

$$\min_{\mathbf{y}} \mathbf{xAy} \leq \sum_{i=1}^{m} a_{ij}x_i$$

for each $j = 1, 2, \ldots, n$. Hence the left-hand side of (15.1) is at most the right-hand side.

By virtue of (15.1), the problem of finding the row player's optimal strategy reduces to the form

$$\begin{aligned} \text{maximize} \quad & \min_{j} \sum_{i=1}^{m} a_{ij}x_i \\ \text{subject to} \quad & \sum_{i=1}^{m} x_i = 1 \\ & x_i \geq 0 \qquad (i = 1, 2, \ldots, m). \end{aligned} \tag{15.2}$$

The key observation of this chapter is that (15.2) is equivalent to the linear programming problem

$$\begin{array}{lll} \text{maximize} & z & \\ \text{subject to} & z - \sum_{i=1}^{m} a_{ij}x_i \le 0 & (j = 1, 2, \ldots, n) \\ & \sum_{i=1}^{m} x_i = 1 & \\ & x_i \ge 0 & (i = 1, 2, \ldots, m). \end{array} \tag{15.3}$$

[To see the equivalence, note that every optimal solution $z^*, x_1^*, \ldots, x_m^*$ of (15.3) satisfies at least one of the constraints $z - \sum a_{ij}x_i \le 0$ with the sign of equality, and so $z^* = \min \sum a_{ij}x_j^*$. A similar trick was used in Chapter 12 and Chapter 14.] Thus the row player can find his optimal strategy by applying the simplex method to (15.3). For instance, since $z^* = 0$, $x_1^* = 0$, $x_2^* = \frac{3}{5}$, $x_3^* = \frac{2}{5}$, $x_4^* = 0$ is one of the optimal solutions of the problem

$$\begin{array}{ll} \text{maximize} & z \\ \text{subject to} & z \qquad\quad + 2x_2 - 3x_3 \qquad\quad \le 0 \\ & z - 2x_1 \qquad\qquad\qquad + 3x_4 \le 0 \\ & z + 3x_1 \qquad\qquad\qquad - 4x_4 \le 0 \\ & z \qquad\quad - 3x_2 + 4x_3 \qquad\quad \le 0 \\ & \qquad x_1 + x_2 + x_3 + x_4 = 1 \\ & \qquad\qquad x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

one of Trucula's optimal strategies is $[0, \frac{3}{5}, \frac{2}{5}, 0]$. Note that by adopting this mixed strategy, Trucula protects herself from positive expected losses.

Similarly, whenever the column player adopts a mixed strategy described by a stochastic column vector $\mathbf{y}$, he assures himself of losing no more than

$$\max_{\mathbf{x}} \mathbf{xAy}$$

per round on the average, with the maximum taken over all stochastic row vectors $\mathbf{x}$; a mixed strategy $\mathbf{y}$ that minimizes the quantity is called *optimal*. Since

$$\max_{\mathbf{x}} \mathbf{xAy} = \max_{i} \sum_{j=1}^{n} a_{ij}y_j$$

[which can be proved analogously to (15.1)], the problem of finding the column player's optimal strategy reads

$$\text{minimize} \qquad \max_{i} \sum_{j=1}^{n} a_{ij}y_j$$

$$\begin{aligned} \text{subject to} \qquad & \sum_{j=1}^{n} y_j = 1 \\ & y_j \geq 0 \qquad (j = 1, 2, \ldots, n) \end{aligned}$$

or, in the linear programming form,

$$\begin{aligned} \text{minimize} \qquad & w \\ \text{subject to} \qquad & w - \sum_{j=1}^{n} a_{ij} y_j \geq 0 \qquad (i = 1, 2, \ldots, m) \\ & \sum_{j=1}^{n} y_j = 1 \\ & y_j \geq 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{15.4}$$

Now the main theorem of this chapter can be proved instantaneously.

THE MINIMAX THEOREM

THEOREM 15.1 (The Minimax Theorem). For every $m \times n$ matrix $\mathbf{A}$ there is a stochastic row vector $\mathbf{x}^*$ of length m and a stochastic column vector $\mathbf{y}^*$ length n such that

$$\min_{\mathbf{y}} \mathbf{x}^*\mathbf{A}\mathbf{y} = \max_{\mathbf{x}} \mathbf{x}\mathbf{A}\mathbf{y}^* \tag{15.5}$$

with the minimum taken over all stochastic column vectors $\mathbf{y}$ of length n and the maximum taken over all stochastic row vectors $\mathbf{x}$ of length m.

PROOF. Note that (15.3) and (15.4) are duals of each other and that each of them has feasible solutions. Hence the Duality Theorem guarantees that (15.3) has an optimal solution $z^*, x_1^*, \ldots, x_m^*$ and (15.4) has an optimal solution $w^*, y_1^*, \ldots, y_n^*$ such that $z^* = w^*$. Since z^* equals the left-hand side in (15.5) and w^* equals the right-hand side in (15.5), the desired conclusion follows. ■

When $\mathbf{A}$ is thought of as defining a game, the common value v of the two sides in (15.5) is referred to as the *value* of that game. By adopting the mixed strategy $\mathbf{x}^*$, the row player assures himself of winning at least v units per round on the average. On the other hand, the column player can assure himself of losing no more than v units per round on the average by adopting the mixed strategy $\mathbf{y}^*$. Thus *fair games* have value zero. Games such as Morra, where the roles of the two players are interchangeable, are clearly fair. Such games are called *symmetric*; their payoff matrices satisfy $a_{ij} = -a_{ji}$ for all i and j.

FURTHER REMARKS AND EXAMPLES

The Minimax Theorem has an interesting corollary: as long as your mixed strategy is optimal, you can reveal it to your opponent without hurting your future prospects. This conclusion may seem inconsistent with the mystique of gambling. Apparently Borel found it hard to accept: even though he had proved the theorem for symmetric games of size 3×3 and 5×5, he was led to believe that it may be false for large games. On the subject of symmetric games, he speculated that "Whatever the manner of playing of the second player may be . . . once that manner of playing is determined, the first player can arrange to win for sure; if he knows the manner of playing of the second player, i.e., the probability that the second player plays in such and such a manner." And again, "The player who does not observe the psychology of his partner, and does not modify his manner of playing must necessarily lose against an adversary whose mind is sufficiently flexible to vary his play while taking account of that of the adversary." [From an English translation by J. L. Savage.]

For a further illustration of the power of the theory, let us return to Morra. Unless the two players write their guesses down, they may find it awkward to announce them simultaneously. Eventually, they may agree that Claude will always announce his guess first. That may give Trucula the edge: having heard Claude's guess, she can still adjust her own. However, by simply announcing his *guess*, Claude gives away no information as to the number of coins he has *hidden*. Thus, one may be led to believe that the game remains fair. To find out which is right, let us first construct the payoff matrix for the new version. In addition to the original four pure strategies, Trucula now has four pure strategies that take Claude's guess into account:

- Hide one, make the same guess as Claude.
- Hide one, make a guess different from Claude's.
- Hide two, make the same guess as Claude.
- Hide two, make a guess different from Claude's.

We shall denote these pure strategies by $[1, S]$, $[1, D]$, $[2, S]$, $[2, D]$. The resulting payoff matrix is as follows:

$$\begin{array}{ccc} & & \text{Claude's pure strategies} \\ & & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\ \begin{array}{c} \text{Trucula's} \\ \text{pure strategies} \end{array} & \begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \\ [1,S] \\ [1,D] \\ [2,S] \\ [2,D] \end{array} & \begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & -3 & 3 \\ -2 & 2 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & 4 & -4 \end{bmatrix} \end{array}.$$

By adopting the mixed strategy $[0, 56/99, 40/99, 0, 0, 2/99, 0, 1/99]$, Trucula assures herself of winning at least 4/99 francs per round on the average. On the other hand, by adopting the mixed strategy $[28/99, 30/99, 21/99, 20/99]^T$, Claude assures himself of losing no more than 4/99 francs per round on the average. Thus the value of this game is 4/99.

Bluffing and Underbidding

In card games such as poker, the players sometimes *bluff* by challenging their opponents to a bet even though they are bound to lose if the challenge is accepted. On the other hand, they may also *underbid* by refraining from making such a challenge even though they are sure to win in an open confrontation. In this section, we present an example in which these stratagems are justified as perfectly rational.

The example is a game invented and analyzed by H. W. Kuhn (1950). It is played with a deck of three cards numbered 1, 2, 3. At the beginning of a play, each of the two players bets an *ante* of one unit and receives a card. Then the players take turns either *betting* one additional unit or *passing* without further betting. The play terminates as soon as a bet is answered by a bet, or a pass by a pass, or bet by a pass. The first two eventualities lead to a confrontation in which the player holding the higher card wins the total amount bet by his opponent; a player answering a bet by a pass chooses to lose his ante. Each play takes one of the following five courses:

A passes, B passes . . . payoff 1 to holder of higher card.
A passes, B bets, A passes . . . payoff 1 to B.
A passes, B bets, A bets . . . payoff 2 to holder of higher card.
A bets, B passes . . . payoff 1 to A.
A bets, B bets . . . payoff 2 to holder of higher card.

Once the cards have been dealt, A may proceed along one of three lines:

1. Pass; if B bets, pass again.
2. Pass; if B bets, bet.
3. Bet.

Each complete set of instructions telling A unequivocally what to do in each situation may be described by a triple $x_1x_2x_3$ such that x_j is the line to be used when holding j. For example, 3 1 2 directs A to bet on a 1 in the first round, always pass with a 2, and wait till the second round to bet on a 3. These triples $x_1x_2x_3$ are A's pure strategies. Similarly, B has four different lines:

1. Pass no matter what A did.
2. If A passes, pass; if A bets, bet.
3. If A passes, bet; if A bets, pass.
4. Bet no matter what A did.

Each of B's pure strategies will be denoted by a triple $y_1y_2y_3$ such that y_j is the line to be used when holding j. To evaluate the payoffs for each pair of pure strategies, we have to assume that each of the six possible deals (A holding 1 and B holding 2, A holding 1 and B holding 3, and so on) is equally likely. For example, if A uses 3 1 2 and B uses 1 2 4 then there are six possible outcomes:

A holds 1, B holds 2 . . .	A bets, B bets	. . . payoff to A $= -2$.
A holds 1, B holds 3 . . .	A bets, B bets	. . . payoff to A $= -2$.
A holds 2, B holds 1 . . .	A passes, B passes	. . . payoff to A $= +1$.
A holds 2, B holds 3 . . .	A passes, B bets, A passes	. . . payoff to A $= -1$.
A holds 3, B holds 1 . . .	A passes, B passes	. . . payoff to A $= +1$.
A holds 3, B holds 2 . . .	A passes, B passes	. . . payoff to A $= +1$.

$$\text{Average payoff to A} = \frac{1}{6}(-2-2+1-1+1+1) = -\frac{1}{3}.$$

Obviously, A has $3 \times 3 \times 3 = 27$ pure strategies whereas B has $4 \times 4 \times 4 = 64$ pure strategies. A straightforward analysis of the 27×64 payoff matrix would be quite tedious. Fortunately, we can use common sense to reduce the payoff matrix down to the size of only 8×4. To begin with, note that a player holding a 1 would lose an extra unit if he answered a bet by a bet, rather than a pass. Similarly, a player holding a 3 would lose for no good reason if he answered a bet by a pass; in addition, he cannot go wrong if he answers a pass by a bet. Hence A has at least one optimal mixed strategy in which:

Holding 1, he refrains from line 2.

Holding 3, he refrains from line 1.

Similarly, B has at least one optimal mixed strategy in which:

Holding 1, he refrains from lines 2 and 4.

Holding 3, he refrains from lines 1, 2, 3.

Now we may pretend that the pure strategies $2x_2x_3$ and x_1x_21 are simply unavailable to A and that the pure strategies $2y_2y_3$, $4y_2y_3$, y_1y_21, y_1y_22, y_1y_23 are unavailable to B. Even though some optimal strategies may become lost, at least one optimal strategy for A and at least one optimal strategy for **B** will remain preserved. In particular, the value of the game will not change.

These eliminations reduce the number of A's pure strategies to 12 and the number of B's pure strategies to 8; in addition, they create possibilities for further simplification. If A holds 2, then he might as well pass in the first round; since B refrains from line 2 when holding 1, and from lines 1, 3 when holding 3, A's line 2 is now as good as 3. Thus we may eliminate A's pure strategies x_13x_3. Similarly, if B holds 2, then his line 1 is as good as 3 and his line 2 is as good as 4. This observation eliminates B's pure strategies y_13y_3 and y_14y_3. The resulting matrix of payoffs to A is as follows:

$$\begin{array}{c} \\ 1\,1\,2 \\ 1\,1\,3 \\ 1\,2\,2 \\ 1\,2\,3 \\ 3\,1\,2 \\ 3\,1\,3 \\ 3\,2\,2 \\ 3\,2\,3 \end{array}
\begin{array}{c} \begin{array}{cccc} 1\,1\,4 & 1\,2\,4 & 3\,1\,4 & 3\,2\,4 \end{array} \\
\begin{bmatrix}
0 & 0 & -\frac{1}{6} & -\frac{1}{6} \\
0 & \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\
-\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{6} & 0 & 0 & \frac{1}{6} \\
\frac{1}{6} & -\frac{1}{3} & 0 & -\frac{1}{2} \\
\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} \\
0 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{6} \\
0 & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6}
\end{bmatrix} \end{array}.$$

Considering A's mixed strategy $[\frac{1}{3}, 0, 0, \frac{1}{2}, \frac{1}{6}, 0, 0, 0]$ and B's mixed strategy $[\frac{2}{3}, 0, 0, \frac{1}{3}]^T$, we conclude that both of the strategies are optimal and the value of the game is $-\frac{1}{18}$. Our optimal strategy for A may be broken down into the following simple instructions:

- Holding 1, mix lines 1 and 3 in the proportion 5:1.
- Holding 2, mix lines 1 and 2 in the proportion 1:1.
- Holding 3, mix lines 2 and 3 in the proportion 1:1.

Note that these instructions call for bluffing (that is, betting on a 1 in the first round) once out of every six available times, and for underbidding (that is, passing on a 3 in the first round) half the available time. Similarly, our optimal strategy for B breaks down as follows:

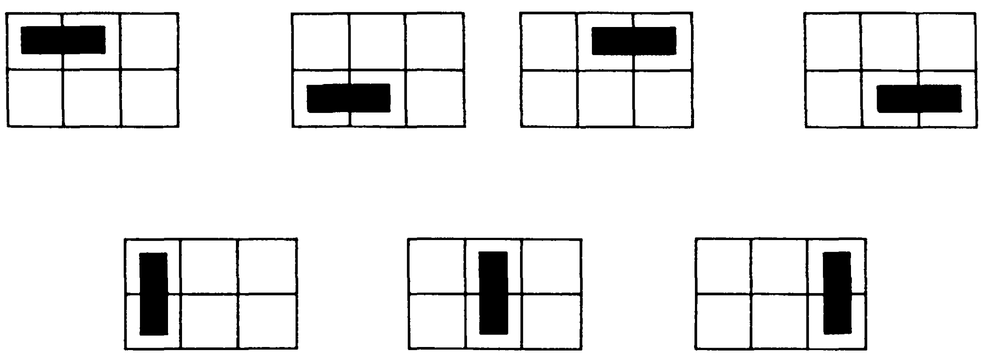
- Holding 1, mix lines 1 and 3 in the proportion 2:1.
- Holding 2, mix lines 1 and 2 in the proportion 2:1. (15.6)
- Holding 3, always use line 4.

Hence B should use one-third of his opportunities to bluff; underbidding is not available to him.
A further discussion of this game is deferred to problems 15.9 and 15.10.

PROBLEMS

△ **15.1** Each of two players hides either a nickel or a dime. If the two coins match, A gets both; if they don't match, B gets both. What are the optimal strategies? Is this game fair? What about games with coins of arbitrary but fixed denominations x and y?

15.2 A domino piece can be placed on a 2×3 checkerboard in seven different ways:



The first player places the domino and the second player selects one of the six squares. If the selected square is covered by the domino, then the second player wins; otherwise the first player wins. Is this game fair? What are the optimal strategies? Can you exploit the symmetries? How do your findings generalize to larger boards?

△ **15.3** Both you and your opponent choose an integer between 1 and 1,000 inclusive. If your number x is smaller than your opponent's number y, then you win, except for $x = y - 1$ in which case you lose. If your number x is larger than your opponent's number, then you lose, except for $x = y + 1$ in which case you win. If $x = y$ then the play is a draw.

15.4 Consider the variant of Morra in which each player can hide one, two, or three coins; for simplicity, assume that the players announce their guesses simultaneously. What are the optimal strategies?

15.5 A row r of the payoff matrix is said to *dominate* a row s if $a_{rj} \geq a_{sj}$ for all $j = 1, 2, \ldots, n$. Similarly, a column r of the payoff matrix is said to *dominate* a column s if $a_{ir} \geq a_{is}$ for all $i = 1, 2, \ldots, m$. Prove:

(i) If a row r is dominated by another row, then the row player has at least one optimal strategy $\mathbf{x}^*$ in which $x_r^* = 0$. In particular, if row r is deleted from the payoff matrix, then the value of the game does not change.

(ii) If a column s dominates another column, then the column player has at least one optimal strategy $\mathbf{y}^*$ in which $y_s^* = 0$. In particular, if column s is deleted from the payoff matrix, then the value of the game does not change.

Use these facts to reduce the following payoff matrix to size 2×2:

$$\begin{bmatrix} -2 & 3 & 0 & -6 & -3 \\ 0 & -4 & 9 & 2 & 1 \\ 6 & -2 & 7 & 4 & 5 \\ 7 & -3 & 8 & 3 & 2 \end{bmatrix}.$$

15.6 Prove that the row player's mixed strategy $\mathbf{x}$ and the column player's mixed strategy $\mathbf{y}$ are simultaneously optimal if and only if

$$x_i = 0 \qquad \text{whenever} \quad \sum_{j=1}^{n} a_{ij}y_j < \max_k \sum_{j=1}^{n} a_{kj}y_j$$

and

$$y_j = 0 \qquad \text{whenever} \quad \sum_{i=1}^{m} a_{ij}x_i > \min_k \sum_{i=1}^{m} a_{ik}x_i.$$

15.7 Use the result of problem 15.6 to describe all optimal strategies for Morra.

15.8 In the game with the payoff matrix

$$\begin{bmatrix} 3 & 2 & 0 & -1 & 5 & -2 \\ -2 & -3 & 2 & 4 & 0 & 4 \\ 5 & -3 & 4 & 0 & 4 & 7 \\ 1 & 3 & 3 & 2 & -6 & 5 \end{bmatrix}$$

the row player's mixed strategy $[\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}]$ is optimal. Describe all the optimal strategies of the column player.

15.9 In Kuhn's simplified poker, different mixed strategies may lead to the same explicit instructions. For example, note that B's optimal strategies

$$\begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

lead to the same instructions (15.6). Prove that every optimal strategy for B is described by (15.6). Furthermore, prove that every optimal strategy for A breaks down as follows:

Holding 1, mix lines 1 and 3 in the proportion $(3 - t):t$.

Holding 2, mix lines 1 and 2 in the proportion $(2 - t):(t + 1)$.

Holding 3, mix lines 2 and 3 in the proportion $(1 - t):t$.

Here t is an arbitrary but fixed number such that $0 \leq t \leq 1$. (Note that the instructions in the text correspond to $t = \frac{1}{2}$.)

15.10 Solve the following variants of simplified poker:

(i) Ante = 2, bet = 1.

(ii) Ante = 2, bet = 3.

(iii) Ante = 2, bet = 4.

15.11 Find necessary and sufficient conditions for the rth pure strategy of the row player and the sth strategy of the column player to be simultaneously optimal.

15.12 The Minimax Theorem is sometimes stated as

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}A\mathbf{y} = \min_{\mathbf{y}} \max_{\mathbf{x}} \mathbf{x}A\mathbf{y}.$$

Prove this identity.

15.13 [G. B. Dantzig (1951a).] Describe the relationship between the linear programming problem

$$\begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

and the game with the payoff matrix

$$\begin{bmatrix} 0 & -c_1 & -c_2 & -c_3 & b_1 & b_2 \\ c_1 & 0 & 0 & 0 & -a_{11} & -a_{21} \\ c_2 & 0 & 0 & 0 & -a_{12} & -a_{22} \\ c_3 & 0 & 0 & 0 & -a_{13} & -a_{23} \\ -b_1 & a_{11} & a_{12} & a_{13} & 0 & 0 \\ -b_2 & a_{21} & a_{22} & a_{23} & 0 & 0 \end{bmatrix}.$$

15.14 On page 128 of R. C. MacLagan (1901), there is the following description of an old Scottish game.

> Cheap, Middling, or Dear
>
> This also is played by two. The letters C, M, D, representing respectively the words from which the game is named, are written on a slate, with some interval between them. Under C the figures 1, 2, 3 are placed, under M 4, 5, 6, and under D 7, 8, 9, thus:
>
> | C | M | D |
> |---|---|---|
> | 1, 2, 3 | 4, 5, 6 | 7, 8, 9 |
>
> Player A, who is to play first, marks one of the figures from any of the groups, concealing it from player B, whom he challenges to guess to which group it belongs, saying "My father bought a horse at a fair." B asks, "Cheap, middling, or dear?" A answers him, naming the group from which he has selected his figure. Thus if his figure were 5, the answer would be "middling." B then guesses one of the three numbers, and if he hits upon 5, that is a gain to him of 5, but if he says 4 or 6, then the 5 is scored to A. In any case the 5 is blotted out. B then leads, each playing in turn, till all the figures have been expunged. The total marks credited to each are then ascertained, and he who has the highest number is the winner.

What are the optimal strategies in the next-to-last round? [For results of a complete analysis, see V. Chvátal (1981).]