Review of Last Week: We formulated ILPS for two scheduling problems and ILPS and LPS for the Knapsack Problem and its variants.

Maximize $8x_1+14x_2+11x_3+4x_4+12x_5+7x_6+4x_7+13x_8+9x_9$

Subject to:

$x_1+x_2+x_3 = 480$; $x_4+x_5+x_6 = 400$; $x_7+x_8+x_9 = 230$

$x_2+x_5+x_8 \leq 420$; $x_3+x_6+x_9 \leq 250$;

$x_1,\ldots,x_9 \geq 0 x_1,\ldots,x_9$ integer

Maximize $15r_1+13r_2+11r_3+9r_4-200w_1-200w_2-200w_3-200w_4-100t_1-100t_2-100t_3-100t_4$

Subject To:

$r_1+r_2+r_3+r_4 = 20{,}000$, $r_1 \leq 50a_1$, $r_2 \leq 50a_2$, $r_3 \leq 50a_3$, $r_4 \leq 50a_4$

$t_1 \leq 3Ins_1$, $t_2 \leq 3Ins_2$, $t_3 \leq 3Ins_3$, $t_4 \leq 3Ins_4$,

$w_1+t_1=w_2$, $w_2+t_2=w_3$, $w_3+t_3=w_4$

$a_1+Ins_1 \leq w_1$, $a_2+Ins_2 \leq w_2$, $a_3+Ins_3 \leq w_3$, $a_4+Ins_4 \leq w_4$

All variables ≥ 0 and integer

Knapsack:

Maximize $\sum_{i=1}^{n} v_i x_i$ Subject to:

$\sum_{i=1}^{n} w_i x_i \leq$ W

For i with $1 \leq i \leq n$, $x_i \geq 0$, $x_i \leq 1$, x_i integer

Multi-Knpasack:

Maximize $\sum_{i=1}^{n} v_i x_i$ Subject to:

$\sum_{i=1}^{n} w_i x_i \leq$ W

For i with $1 \leq i \leq n$, $x_i \geq 0$, x_i integer

Fractional Multi-Knapsack

Maximize $\sum_{i=1}^{n} v_i x_i$ Subject to:

$\sum_{i=1}^{n} w_i x_i \leq$ W

For i with $1 \leq i \leq n$, $x_i \geq 0$

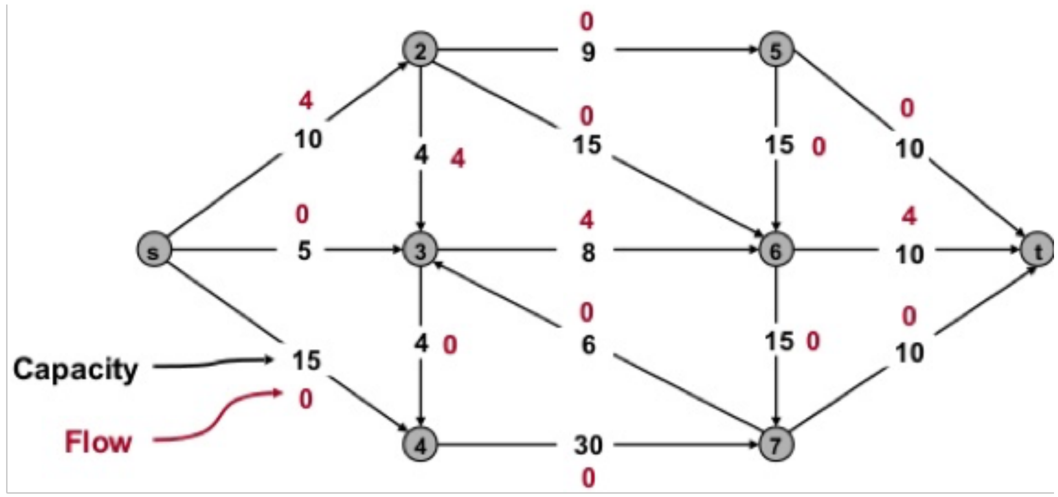
COMP 360 11/10/2018 Lecture Formulating Problems as LPs II

Maximum Volume s-t Flow

For a directed graph G with edge set e, a capacity u(e) for each edges, a vertex s which has no arcs entering it and a node t which has no arcs leaving it, an s-t flow is a function f from E(G) to the nonnegative reals so that for every edge e, the flow along e is at most u(e), and for every node v other than s and t, the flow into v is equal to the flow out of v.

Its volume is the total flow out on the edges out of s.

Our problem is to find a maximum volume s-t flow.



Maximum Volume s-t Flow as an LP

Input Directed graph G=(V,E). A capacity u(e) for each (directed) arc e.
A source s such that there is no arc vs in G and a sink t such that there is no arc tv in G.

Variables: For all e in E have a variable f_e specifying the flow through e.

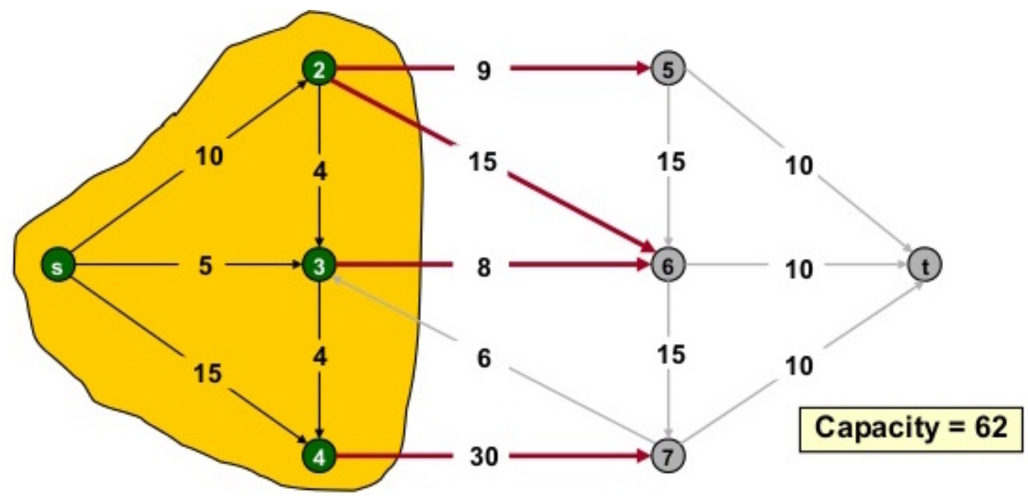
Maximize $\sum_{\{e\ in\ E\,|\,e=sv\}} f_e$

s.t. $\forall v\ in$ V-s-t, $\sum_{\{e\ in\ E|e=uv\}} f_e$-$\sum_{\{e\ in\ E|e=vw\}} f_e$=0 (flow conservation)
$\forall e\ in\ E, f_e \leq u(e)(capacity), f_e \geq 0$ (nonnegativity)

Minimum Capacity s-t Cut

For a directed graph G with edge set e, a capacity u(e) for each edges, a vertex s which has no arcs entering it and a node t which has no arcs leaving it, an s-t cut is a subset of V containing s but not t. Its capacity is the sum of the capacities on the edges leaving C.

We want to find a minimum capacity s-t cut.



Min Capacity Cut as an ILP

Input Directed G=(V,E). A source s s.t there is no arc vs in G and a sink t s. t. there is no arc tv in G. A capacity u(e) for each (directed) arc e.

Variables: $\forall v \; in \; V,$ InC_v which is 1 if v is in C and 0 otherwise.

$\forall e \; in \; E,$ x_e which is 1 if e goes from a vertex v in C to a vertex w in V-C and either 0 or 1 otherwise.

Min $\sum_{e \; in \; E} u(e) \, x_e$

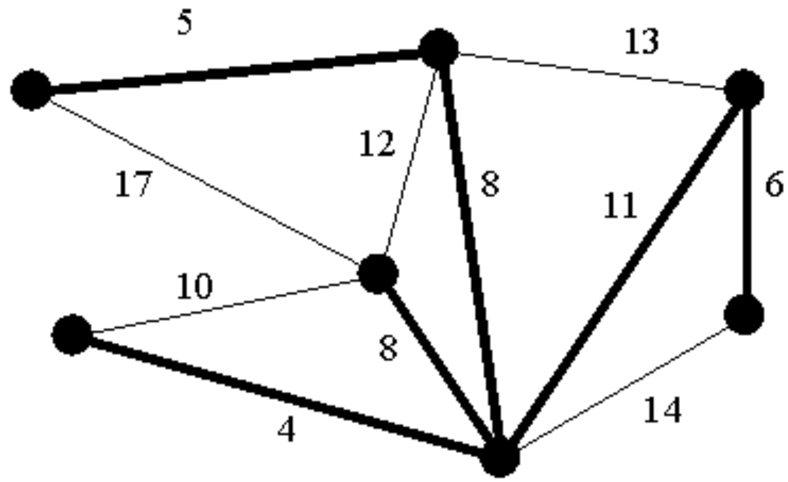
s.t. All variables non-negative, integer, and at most 1,

$inC_s=1, inC_t=0$

$\forall e \; in \; E, if \;\; e = vw \; then \; x_e - InC_v + InC_w \geq 0$

Minimum Weight Spanning Tree

Recall that a tree is a graph which contains no cycles. A spanning tree of an (undirected) connected graph G is a subgraph of G which is a tree and contains all of the vertices of G. Given a connected graph G=(V,E) and a weight w(e) for each edge e of G find a minimum weight spanning tree.



MWST as an ILP

Input: Connected graph G=(V,E), weight w(e) for each edge e of G.

Variables: ∀e in E, x_e which is 1 if e is in the tree and 0 otherwise.

Minimize $\sum_{e \ in \ E} w(e)x_e$

s.t. All variables integer, nonnegative, and at most one.

$\sum_{e \ in \ E} x_e = |V(G)| - 1$, For all cycles C of G,$\sum_{e \ in \ E(C)} x_e \leq$|V(C)|-1.

Satisfiability

Boolean Variables: are TRUE or FALSE

The literals corresponding to a Boolean variable x are x and its negation not(x)- which is true precisely if x is false.

A clause is the OR of literals.

A satisfiability instance consists of a family X of Boolean variables and a boolean formula which is the AND of a family F of clauses such that each literal in these clauses corresponds to a variable in X.

We are asked to determine if we can choose a truth assignment to the variables of X so that the formula evaluates to true. I.e. so that each clause contains a true literal.

Which Formulas are satisfiable?

(x_1 OR x_2) AND (x_1 OR not(x_2)) AND (not(x_1) OR not(x_2)) AND (not(x_1) OR x_2)

(x_1 OR x_2) AND (x_1 OR x_3) AND (x_1 OR x_4) AND (not(x_2) OR not(x_3)) AND (not(x_2) OR not(x_4)) AND (x_3 OR x_4) AND (not(x_3) OR not(x_4))

(x_1 OR not(x_2)) AND (not(x_1) OR not(x_2)) AND (x_1 OR not(x_3)) AND (not(x_1) OR not(x_3)) AND (x_1 OR not(x_4)) AND (not(x_1) OR not(x_4)) AND (x_1 OR not(x_5)) AND (not(x_1) OR not(x_5)) AND (x_1 OR not(x_6)) AND (not(x_1) OR not(x_6)) AND (x_2 OR x_3 OR x_4 OR x_5 OR x_6)

$(x_1$ OR $x_2)$ AND $(x_1$ OR not$(x_2))$ AND (not(x_1) OR not$(x_2))$ AND (not(x_1) OR $x_2)$

Clauses: $C_1 = x_1$ OR x_2, $C_2 = x_1$ OR not(x_2), $C_3 =$ not(x_1) OR not(x_2), $C_4 =$ not(x_1) OR x_2

x_1 and x_2 both true => clause C_3 and hence whole formula is false.

x_1 and x_2 both false => clause C_1 and hence whole formula is false.

x_1 true and x_2 false => clause C_4 and hence whole formula is false.

x_1 false and x_2 true => clause C_2 and hence whole formula is false.

NO SATISFYING TRUTH ASSIGNMENT.

Letting C^+_i be k such that x_k is in C_i and C^-_i be k such that not(x_k) is in C_i we have:

$C^+_1=\{1,2\}$, $C^-_1=\emptyset$, $C^+_2=\{1\}$, $C^-_2=\{2\}$, $C^+_3=\emptyset$, $C^-_3=\{1,2\}$, $C^+_4=\{2\}$, $C^-_4=\{1\}$

$(x_1$ OR $x_2)$ AND $(x_1$ OR $x_3)$ AND $(x_1$ OR $x_4)$ AND (not(x_2) OR not(x_3)) AND (not(x_2) OR not(x_4)) AND $(x_3$ OR $x_4)$ AND (not(x_3) OR not(x_4))

Clause 1: x_1 OR x_2,Clause 2: x_1 OR x_3, Clause 3: x_1 OR x_4, Clause 4: not(x_2) OR not(x_3)

Cluase 5: not(x_2) OR not(x_4), Clause 6: x_3 OR x_4, Clause 7: not(x_3) OR not(x_4).

Under the truth assignment x_1=T, x_2=F, x_3=T,x_4=F. Formula evaluates to:

(T OR F) AND (T OR T) AND (T or F) AND (T OR F) AND (T OR T) AND (T OR F) AND (F OR T) = T AND T AND T AND T AND T AND T AND T=T.

Letting C^+_i be k such that x_k is in C_i and C^-_i be k such that not(x_k) is in C_i we have:

$C^+_1=\{1,2\}$, $C^-_1=\emptyset$, $C^+_2=\{1,3\}$, $C^-_2=\emptyset$, $C^+_3=\{1,4\}$, $C^-_3=\emptyset$, $C^+_4=\emptyset$, $C^-_4=\{2,3\}$,

$C^+_5=\emptyset$, $C^-_5=\{2,4\}$, $C^+_6=\{3,4\}$, $C^-_6=\emptyset$, $C^+_7=\emptyset$, $C^-_7=\{3,4\}$

(x_1 OR not(x_2)) AND (not(x_1) OR not(x_2)) AND (x_1 OR not(x_3)) AND (not(x_1) OR not(x_3)) AND (x_1 OR not(x_4)) AND (not(x_1) OR not(x_4)) AND (x_1 OR not(x_5)) AND (not(x_1) OR not(x_5)) AND (x_1 OR not(x_6)) AND (not(x_1) OR not(x_6)) AND (x_2 OR x_3 OR x_4 OR x_5 OR x_6)

Clause 1: x_1 OR not(x_2), Clause 2: not(x_1) OR not(x_2),Clause 3: x_1 OR not(x_3),

Clause 4: not(x_1) OR n not(x_3), Clause 5: x_1 OR not(x_4), Clause 6: not(x_1) OR not(x_4),

Clause 7: x_1 OR not(x_5), Clause 8: not(x_1) OR not(x_5), Clause 9: x_1 OR not(x_6),

Clause 10: not(x_1) OR not(x_6), Clause 11: x_2 OR x_3 OR x_4 OR x_5 OR x_6.

In any truth assignment, if all of x_2 through x_6 are false then so is Clause 11 and the formula. Otherwise there is a j between 2 and 6 which is true. Now if x_1 is true then Clause 2j-2 is false while if x_1 is false so is Clause 2j-3.So far any truth assignment some clause is false and so is the formula.

NO SATISFYING TRUTH ASSIGNMENT

Letting C^+_i be k such that x_k is in C_i and C^-_i be k such that not(x_k) is in C_i we have: C^+_1={1}, C^-_1 ={2}, C^+_2= Ø, C^-_2 ={1,2}, C^+_3={1}, C^-_3 ={3}, C^+_4= Ø, C^-_4 ={1,3}, C^+_5={1}, C^-_5 ={4}, C^+_6= Ø,C^-_6 ={1,4}, C^+_7={1}, C^-_7 ={5}, C^+_8= Ø, C^-_8 ={1,5},C^+_9={1}, C^-_9 ={6}, C^+_{10}= Ø, C^-_{10} ={1,6}, C^+_{11}={2,3,4,5,6}, C^-_{11}= Ø

Satisfiability as an ILP

Input: Set $X=\{x_1,\ldots,x_l\}$ of Boolean variables. Set $C_1, \ldots, C_j$ of Clauses

Specified by $C^+_i=\{k \mid x_k \text{ in } C_i\}$ and $C^-_i=\{k \mid \text{not}(x_k) \text{ in } C_i\}$ for $1 \leq i \leq j$.

Variable y_i which is 1 if x_i is true and o if it is false.

Maximize 0

Subject to:

For every $1 \leq i \leq j$: $\sum_{k \text{ in } C_i^+} y_k + \sum_{k \text{ in } C_i^-} (1 - y_k) \geq 1$

For every $1 \leq i \leq l$: $0 \leq y_k, y_k \leq 1$, y_k integer.

(x_1 OR x_2) AND (x_1 OR not(x_2)) AND (not(x_1) OR not(x_2)) AND (not(x_1) OR x_2) AS AN ILP

Maximize 0

Subject to:

$y_1+y_2 \geq 1$

$y_1+(1-y_2) \geq 1 \Rightarrow y_1-y_2 \geq 0$

$(1-y_1)+(1-y_2) \geq 1 \Rightarrow -y_1-y_2 \geq -1$

$y_2+(1-y_1) \geq 1 \Rightarrow y_2-y_1 \geq 0$

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0,$

$y_1 \leq 1,\ y_2 \leq 1,\ y_3 \leq 1,\ y_4 \leq 1$

$y_{1,},y_2,y_3,y_4$ integer.

(x_1 OR x_2) AND (x_1 OR x_3) AND (x_1 OR x_4) AND (not(x_2) OR not(x_3)) AND (not(x_2) OR not(x_4)) AND (x_3 OR x_4) AND (not(x_3) OR not(x_4)) AS AN ILP

Maximize 0

Subject to:

$y_1+y_2 \geq 1$, $y_1+y_3 \geq 1$, $y_1+y_4 \geq 1$

$(1-y_2)+(1-y_3) \geq 1 \Rightarrow -y_2-y_3 \geq -1$

$(1-y_2)+(1-y_4) \geq 1 \Rightarrow -y_2-y_4 \geq -1$

$y_3+y_4 \geq 1$, $(1-y_3)+(1-y_4) \geq 1 \Rightarrow -y_3-y_4 \geq -1$

$y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$,

$y_1 \leq 1$, $y_2 \leq 1$, $y_3 \leq 1$, $y_4 \leq 1$

$y_{1,}, y_2, y_3, y_4$ integer.

$(x_1$ OR not$(x_2))$ AND (not(x_1) OR not$(x_2))$ AND $(x_1$ OR not$(x_3))$ AND (not(x_1) OR not$(x_3))$ AND $(x_1$ OR not$(x_4))$ AND (not(x_1) OR not$(x_4))$ AND $(x_1$ OR not$(x_5))$ AND (not(x_1) OR not$(x_5))$ AND $(x_1$ OR not$(x_6))$ AND (not(x_1) OR not$(x_6))$ AND $(x_2$ OR x_3 OR x_4 OR x_5 OR $x_6)$ AS AN ILP

Maximize 0

Subject to:

$y_1 - y_2 \geq 0$, $-y_1 - y_2 \geq -1$, $y_1 - y_3 \geq 0$

$-y_1 - y_3 \geq -1$, $y_1 - y_4 \geq 0$, $-y_1 - y_4 \geq -1$,

$y_1 - y_5 \geq 0$, $-y_1 - y_5 \geq -1$, $y_1 - y_6 \geq 0$

$-y_1 - y_6 \geq -1$, $y_2 + y_3 + y_4 + y_5 + y_6 \geq 1$.

$y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, $y_4 \geq 0$, $y_5 \geq 0$, $y_6 \geq 0$,

$y_1 \leq 1$, $y_2 \leq 1$, $y_3 \leq 1$, $y_4 \leq 1$,, $y_5 \leq 1$, $y_6 \leq 1$

$y_{1,}, y_2, y_3, y_{4,} y_5, y_6$ integer.

After A Billion Consecutive Defeats, Scissors Beats Rock!

Rock Paper Scissors

Each player shows Rock, Paper, or Scissors.

Scissors (cuts and) beats Paper.

Paper (wraps around and) beats Rock.

Rock (smashes and) beats Scissors.

Rock Paper Scissors Matrix

The Donald

		R	P	S
	R	0	-1	1
Mr. Mueller	P	1	0	-1
	S	-1	1	0

Mr. Mueller's winnings

The Best Strategy For Rock, Paper Scissors?

Play randomly with each choice equally likely.

We got to here.

Morra and its Payoff Matrix
What is the Best Strategy?

Each player hides one or two loonies and at the same time guesses the number of loonies hidden by the other player. If only one player guesses right she wins the total amount hidden. Otherwise no money changes hand.

We use [I,j] to mean hides I guesses j.

		The Donald			
		[1,1]	[1,2]	[2,1]	[2,2]
	[1,1]	0	2	-3	0
Mr. Mueller	[1,2]	-2	0	0	3
	[2,1]	3	0	0	-4
	[2,2]	0	-3	4	0

Mr.Mueller's winnings

A Lower Bound on The Expected Winnings of The Row Player in Morra

Maximize z

Subject to:

$x_1+x_2+x_3+x_4=1$

$x_1,x_2,x_3,x_4 \geq 0$

$-2x_2 + 3x_3 - z \geq 0$

$2x_1 - 3x_4 - z \geq 0$

$-3x_1 + 4x_4 - z \geq 0$

$3x_2 - 4x_3 - z \geq 0$

		The Donald			
		[1,1]	[1,2]	[2,1]	[2,2]
	[1,1]	0	2	-3	0
Mr. Mueller	[1,2]	-2	0	0	3
	[2,1]	3	0	0	-4
	[2,2]	0	-3	4	0

An Upper Bound on The Expected Loss of The Column Player in Morra

Minimize z

Subject to:

$x_1+x_2+x_3+x_4=1$

$x_1,x_2,x_3,x_4 \geq 0$

$2x_2 - 3x_3 - z \leq 0$

$-2x_1 + 3x_4 - z \leq 0$

$3x_1 - 4x_4 - z \leq 0$

$-3x_2 + 4x_3 - z \leq 0$

		The Donald			
		[1,1]	[1,2]	[2,1]	[2,2]
	[1,1]	0	2	-3	0
Mr. Mueller	[1,2]	-2	0	0	3
	[2,1]	3	0	0	-4
	[2,2]	0	-3	4	0

Zero Sum Matrix Games

Real-valued Payoff Matrix A with m rows and n columns.

$a_{i,j}$ is the entry in the intersection of the ith row and the jth column.

In each turn, the Row player selects a row i and the Column player selects a column j, and the row player wins $a_{i,j}$ dollars with the column player losing the same amount.

Choosing Strategies For Matrix Games.

The row player can ensure that her expected winnings are at least the solution to:

Maximize z subject to

$\sum_{i=1}^{m} x_i$ =1 , $x_1,\ldots,x_m \geq 0$

For $1 \leq j \leq n$, $\sum_{i=1}^{m} a_{i,j} x_i - \mathrm{z} \geq 0$

The column player can ensure that the row player's expected winnings are at most the solution to:

Minimize z' subject to

$\sum_{j=1}^{n} y_j$ =1 , $y_1,\ldots,y_n \geq 0$

For $1 \leq i \leq m$, $\sum_{j=1}^{n} a_{i,j} y_j - \mathrm{z} \leq 0$