Review of Previous Lecture

Formulated Max s-t Flow, Min Capacity s-t Cut, Minimum Weight Spanning Tree and Satisfiability as ILP or LP.

Saw that if all capacities are integer then the Max Flow ILP and Max Flow LP have the same solution.

Discussed Rock,Papers, Scissors. Saw its Payoff Matrix and determined the optimal strategy was to randomly choose each possibility with probability 1/3.

COMP 360 14/10/2018 Lecture Solving LPS Via The Simplex Method I

BEFORE STARTING THIS LECTURE WE FINISHED THE MATERIAL ON MATRIX GAMES WHICH CAN BE FOUND IN THE POWERPOINT FOR THE LECTURE OF SEPTEMBER 11th (AFTER THE SLIDE SAYING: WE GOT TO HERE).

LPs in Standard Form

An LP is in standard form if we are maximizing the objective function subject to a set of linear constraints, which consists of a non-negativity constraint for each variable and some constraints each insisting a linear function is at most some constant:

n variables $x_1,\ldots,x_n$ and m constraints yields:

Maximize $\sum_{j=1}^{n} c_j x_j$

Subject to:

For i=1 to m: $\sum_{j=1}^{n} a_{i,j} x_j \leq b_i$

For j=1 to n: $x_j \geq 0$

Reduction of All LPs to LPs in Standard Form
+Algorithm for LPs in Standard Form
=Algorithm for All LPs

Reducing LPs to LPS in standard form.

Reducing LPs to LPS in standard form.

We can replace minimize z by maximize $-z$

Reducing LPs to LPS in standard form.

We can replace minimize z by maximize –z

We can replace x by (x'-x'') where we insist each of x' and x'' are nonegative.

Reducing LPs to LPS in standard form.

We can replace minimize z by maximize –z

We can replace x by (x'-x'') where we insist each of x' and x'' are nonegative.

We can replace f=b by f le b and f ge b

Reducing LPs to LPS in standard form.

We can replace minimize z by maximize –z

We can replace x by (x'-x'') where we insist each of x' and x'' are nonegative.

We can replace f=b by f le b and f ge b

We can replace f ge b by –f le -b.

Solving LPs in Standard Form

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 \\
4x_1 + x_2 + 2x_3 &\leq 11 \\
3x_1 + 4x_2 + 2x_3 &\leq 8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 \\
4x_1 + x_2 + 2x_3 &\leq 11 \\
3x_1 + 4x_2 + 2x_3 &\leq 8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$2x_1 + 3x_2 + x_3 \le 5 \qquad x_4 = 5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \ge 0)$$

$$4x_1 + x_2 + 2x_3 \le 11$$

$$3x_1 + 4x_2 + 2x_3 \le 8$$

$$x_1, x_2, x_3 \ge 0$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 & \qquad x_4 &= 5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \geq 0)\\
4x_1 + x_2 + 2x_3 &\leq 11 & \qquad x_5 &= 11 - 4x_1 - x_2 - 2x_3 \quad (x_5 \geq 0)\\
3x_1 + 4x_2 + 2x_3 &\leq 8 & &\\
x_1, x_2, x_3 &\geq 0 & &
\end{aligned}
$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$2x_1 + 3x_2 + x_3 \leq 5 \qquad x_4=5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \geq 0)$$

$$4x_1 + x_2 + 2x_3 \leq 11 \qquad x_5=11 - 4x_1 - x_2 - 2x_3 \quad (x_5 \geq 0)$$

$$3x_1 + 4x_2 + 2x_3 \leq 8 \qquad x_6=8 - 3x_1 - 4x_2 - 2x_3 \quad (x_6 \geq 0)$$

$$x_1,x_2,x_3 \geq 0$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 \\
4x_1 + x_2 + 2x_3 &\leq 11 \\
3x_1 + 4x_2 + 2x_3 &\leq 8 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \geq 0) \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \quad (x_5 \geq 0) \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \quad (x_6 \geq 0)
\end{aligned}
$$

========================

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$

$$x_1, x_2, x_3 \geq 0$$

The Slack Variables, The Initial Dictionary,

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 \qquad & x_4 &= 5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \geq 0)\\
4x_1 + x_2 + 2x_3 &\leq 11 \qquad & x_5 &= 11 - 4x_1 - x_2 - 2x_3 \quad (x_5 \geq 0)\\
3x_1 + 4x_2 + 2x_3 &\leq 8 \qquad & x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \quad (x_6 \geq 0)\\
x_1, x_2, x_3 &\geq 0 & &
\end{aligned}
$$

========================

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$

$$x_1, x_2, x_3 \geq 0$$

The Slack Variables, The Initial Dictionary, & corresponding feasible solution (0,0,0,5,11,8)

Maximize $5x_1+4x_2+3x_3$

Subject to:

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &\leq 5 \\
4x_1 + x_2 + 2x_3 &\leq 11 \\
3x_1 + 4x_2 + 2x_3 &\leq 8 \\
x_1,x_2,x_3 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= 5 - 2x_1 - 3x_2 - x_3 \quad (x_4 \geq 0) \\
x_5 &= 11 - 4x_1 - x_2 - 2x_3 \quad (x_5 \geq 0) \\
x_6 &= 8 - 3x_1 - 4x_2 - 2x_3 \quad (x_6 \geq 0)
\end{aligned}
$$

========================

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$

$$x_1,x_2,x_3 \geq 0$$

Choosing A Decision Variable To Increase From Zero

$x_4 = 5 - 2x_1 - 3x_2 - x_3$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

========================

$z = 0 + 5x_1 + 4x_2 + 3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Increasing any decision variable increases the objective function.

We choose to increase x_3 (text makes a different choice)

Bounding The Increase in x_3 leaving x_1=x_2=0 We must insure none of x_4,x_5,x_6 go negative.

The Initial Dictionary

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$
$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$
$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

========================

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Bounding The Increase in x_3 leaving x_1=x_2=0 We must insure none of x_4,x_5,x_6 go negative.

The Initial Dictionary

$x_4 = 5 - 2x_1 - 3x_2 - x_3$ $\qquad$ $x_3 \leq 5$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

========================

$z = 0 + 5x_1 + 4x_2 + 3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Bounding The Increase in x_3 leaving $x_1=x_2=0$ We must insure none of x_4,x_5,x_6 go negative.

The Initial Dictionary

$x_4=5 - 2x_1 - 3x_2 - x_3$ $\qquad$ $x_3 \leq 5$

$x_5=11- 4x_1 - x_2 - 2x_3$ $\qquad$ $x_3 \leq 11/2$

$x_6=8 - 3x_1 - 4x_2 - 2x_3$

=========================

$z= 0+ 5x_1 +4x_2 +3x_3$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

Bounding The Increase in x_3 leaving $x_1=x_2=0$ We must insure none of x_4,x_5,x_6 go negative.

The Initial Dictionary

$x_4=5 - 2x_1 - 3x_2 - x_3$ $\qquad x_3 \leq 5$

$x_5=11- 4x_1 - x_2 - 2x_3$ $\qquad x_3 \leq 11/2$

$x_6=8 - 3x_1 - 4x_2 - 2x_3$ $\qquad x_3 \leq 4$

========================

$z= 0+ 5x_1 +4x_2 +3x_3$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

Bounding The Increase in x_3 leaving x_1=x_2=0 We must insure none of x_4,x_5,x_6 go negative.

The Initial Dictionary

$x_4 = 5 - 2x_1 - 3x_2 - x_3$	$x_3 \leq 5$
$x_5 = 11 - 4x_1 - x_2 - 2x_3$	$x_3 \leq 11/2$
$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$	$x_3 \leq 4$
========================	
$z = 0 + 5x_1 + 4x_2 + 3x_3$	Set x_3=4, so x_6=0
$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$	

Creating A New Dictionary

The Old Dictionary

$x_4 = 5 \ - 2x_1 \ - 3x_2 - \ \ x_3$

$x_5 = 11 - 4x_1 \ - \ \ x_2 - 2x_3$

$x_6 = 8 \ - 3x_1 \ - 4x_2 \ - 2x_3$

========================

$z = 0 + 5x_1 \ +4x_2 \ +3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Creating A New Dictionary

The Old Dictionary

$x_4 = 5 - 2x_1 - 3x_2 - x_3$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

========================

$z = 0 + 5x_1 + 4x_2 + 3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Rewrite equation for x_6 as

$x_3 = 4 - 3x_1/2 - 2x_2 - x_6/2$

Creating A New Dictionary

The Old Dictionary

$x_4 = 5 - 2x_1 - 3x_2 - x_3$

$x_5 = 11 - 4x_1 - x_2 - 2x_3$

$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$

========================

$z = 0 + 5x_1 + 4x_2 + 3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

And Substitute

$x_4 = 1 - x_1/2 - x_2 + x_6/2$

$x_3 = 4 - 3x_1/2 - 2x_2 - x_6/2$

Creating A New Dictionary

The Old Dictionary

$x_4 = 5 \ - 2x_1 \ - 3x_2 - \ x_3$

$x_5 = 11 - 4x_1 \ - \ x_2 - 2x_3$

$x_6 = 8 \ - 3x_1 \ - 4x_2 \ - 2x_3$

========================

$z = 0 + 5x_1 \ + 4x_2 \ + 3x_3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

And Substitute

$x_4 = 1 - x_1/2 - \ x_2 + x_6/2$

$x_5 = 3 \ - x_1 \ + \ 3x_2 + x_6$

$x_3 = 4 - 3x_1/2 - \ 2x_2 - x_6/2$

Creating A New Dictionary

The Old Dictionary

$$x_4=5 - 2x_1 - 3x_2 - x_3$$
$$x_5=11- 4x_1 - x_2 - 2x_3$$
$$x_6=8 - 3x_1 - 4x_2 - 2x_3$$

========================

$$z= 0+ 5x_1 +4x_2 +3x_3$$

$$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$$

And Substitute

$$x_4=1 -x_1/2 - x_2 + x_6/2$$
$$x_5=3 -x_1 + 3x_2+ x_6$$
$$x_3=4-3x_1/2 - 2x_2 - x_6/2$$

=================

$$z=12+x_1/2 -2x_2 -3x_6/2$$

$$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$$

Creating A New Dictionary & Solution(0,0,4,1,3,0)

The Old Dictionary

$x_4=5 - 2x_1 - 3x_2 - x_3$

$x_5=11- 4x_1 - x_2 - 2x_3$

$x_6=8 - 3x_1 - 4x_2 - 2x_3$

========================

$z= 0+ 5x_1 +4x_2 +3x_3$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

And Substitute

$x_4=1 -x_1/2 - x_2 + x_6/2$

$x_5=3 -x_1 + 3x_2+ x_6$

$x_3=4-3x_1/2 - 2x_2 - x_6/2$

=================

$z=12+x_1/2 -2x_2 -3x_6/2$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

Creating A New Dictionary & Solution(0,0,4,1,3,0)

The Old Dictionary

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$
$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$
$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

=========================

$$z = 0 + 5x_1 + 4x_2 + 3x_3$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Rewrite equation for x_6 as

$$x_4 = 1 - x_1/2 - x_2 + x_6/2$$
$$x_5 = 3 - x_1 + 3x_2 + x_6$$
$$x_3 = 4 - 3x_1/2 - 2x_2 - x_6/2$$

=================

$$z = 12 + x_1/2 - 2x_2 - 3x_6/2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Choosing which variable to increase from zero

$x_4 = 1 - x_1/2 - \quad x_2 + x_6/2$

$x_5 = 3 \ - x_1 \quad + \ 3x_2 + x_6$

$x_3 = 4 - 3x_1/2 - \ 2x_2 - x_6/2$

==================

$z = 12 + x_1/2 - 2x_2 - 3x_6/2$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Increasing x_2 or x_6 decreases z

so we choose to increase x_1

Bounding The Increase in x_1 leaving x_2=x_6=0

Current Dictionary:

$X_4 = 1 - x_1/2 - x_2 + x_6/2$ $x_1 \leq 2$

$x_5 = 3 - x_1 + 3x_2 + x_6$ $x_1 \leq 3$

$x_3 = 4 - 3x_1/2 - 2x_2 - x_6/2$ $x_1 \leq 8/3$

==================

$z = 12 + x_1/2 - 2x_2 - 3x_6/2$ Set $x_6=2$, so $x_1=0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Creating A New Dictionary

Current Dictionary:

$x_4=1 - x_1/2 - x_2 + x_6/2$

$x_5=3 - x_1 + 3x_2 + x_6$

$x_3=4-3x_1/2 - 2x_2 - x_6/2$

=================

$z=12+x_1/2 - 2x_2 - 3x_6/2$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

Rewrite first equation as

$x_1=2 - 2x_2 - 2x_4 + x_6$

$x_5=1 + 5x_2 + 2x_4$

$x_3=1 + x_2 + 3x_4 - 2x_6$

=================

$z=13 - 3x_2 - x_4 - x_6$

$x_1,x_2,x_3,x_4,x_5,x_6 \geq 0$

Creating A New Dictionary

Current Dictionary:

$x_4 = 1 - x_1/2 - x_2 + x_6/2$

$x_5 = 3 - x_1 + 3x_2 + x_6$

$x_3 = 4 - 3x_1/2 - 2x_2 - x_6/2$

==================

$z = 12 + x_1/2 - 2x_2 - 3x_6/2$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Rewrite first equation and substitute:

$x_1 = 2 - 2x_2 - 2x_4 + x_6$

$x_5 = 1 + 5x_2 + 2x_4$

$x_3 = 1 + x_2 + 3x_4 - 2x_6$

==================

$z = 13 - 3x_2 - x_4 - x_6$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

An Optimal Dictionary & Solution (2,0,1,0,1,0)

Current Dictionary

$x_1 = 2 - 2x_2 - 2x_4 + x_6$

$x_5 = 1 + 5x_2 + 2x_4$

$x_3 = 1 + x_2 + 3x_4 - 2x_6$

=================

$z = 13 - 3x_2 - x_4 - x_6$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Since each of the x_i are non-negative in a feasible solution we see that z can never be greater than 13 and we have an optimal solution. We stop.

Dictionaries

Slack Variables are x_{n+1} to x_{n+m} where x_{n+i} is the extent by which the RHS of the ith constraint exceeds its LHS. I.e. $x_{n+i} = b_i - \sum_{j=1}^{n} a_{i,j} x_j$

We introduce a variable z which we set equal to $\sum_{j=1}^{n} c_j x_j$.

Thus we have a set of m+1 equations whose non-negative solutions are in 1-1 correspondence to the feasible solutions to the LP.

Any other set of m+1 linear equations in our n+m+1 variables is a *dictionary* provided

(i) its solutions are in 1-1 correspondence with the solutions to this set of m+1 equations,

(ii) It must express z and each of m of the other variables as a linear function of the remaining n variables.

Basic Variables and Feasible Dictionaries

A variable other than z is *basic* for a dictionary if it is one of the other m variables expressed as a linear function of the remaining n variables. The other variables are *non-basic*.

A solution $(x^*_1, x^*_2, \ldots, x^*_{n+m})$ to the set of equations for a dictionary is feasible if each x^*_l is non-negative (and hence the solution corresponds to a feasible solution to the LP).

A dictionary is *feasible* if setting all the non-basic variables to zero and assigning values to the basic variable yields a feasible solution. Equivalently the constant terms in all the linear functions of the dictionary setting out the value of the basics variables are nonnegative.

Each feasible dictionary yields a (unique) feasible solution with the nonbasic variables all zero. Any feasible solution obtained in this way is *basic.*

The Simplex Method

Step 1: Find a feasible dictionary to start at.

Step 2: If there is no nonbasic variable with a positive cost coefficient in the function for z in the current feasible dictionary then terminate, the feasible solution given by the dictionary is optimal.

Step 3: Choose a non-basic variable x_e with a positive cost coefficient in the function for z. This is the entering variable.

Step 4: Let x_l be (one of) the basic variable(s) whose nonnegativity gives the most stringent upper bound on the possible values of x_e. x_l is the leaving variable.

Step 5: Rewrite the equation for x_l in the dictionary as an equation for x_e.

Step 6: Replace x_e by the RHS of this equation in all the other equations of the dictionary to obtain a new feasible dictionary.

Step 7: Go back to Step 2.

The Simplex Method

Step 1: Find a Feasible Dictionary to start at.

Step 2: If there is no nonbasic variable with a positive cost coefficient in the function for z in the current feasible dictionary then terminate, the feasible solution given by the dictionary is optimal.

Step 3: Choose a variable x_e with a positive cost coefficient in the function for z. This is the entering variable.

Step 4: Determine if there is a basic variable x_l whose nonnegativity gives an upper bound on the possible values of x_e if we increase it while leaving all other nonbasic variables at 0. If not, the problem is unbounded, return this fact.

Step 5: Otherwise let x_l be (one of) the basic variable(s) whose nonnegativity gives the most stringent upper bound on the possible values of x_e. x_l is the leaving variable.

Step 6: Rewrite the equation for x_l in the dictionary as an equation for x_e.

Step 7: Replace x_e by the RHS of this equation in all the other equations of the dictionary to obtain a new feasible dictionary.

Step 8: Go back to Step 2.

The Simplex Method

Step 1: Find a Feasible Dictionary to start at.

Step 2: If there is no nonbasic variable with a positive cost coefficient in the function for z in the current feasible dictionary then terminate, the feasible solution given by the dictionary is optimal.

Step 3: Choose a variable x_e with a positive cost coefficient in the function for z. This is the entering variable.

Step 4: Determine if there is a basic variable x_l whose nonnegativity gives an upper bound on the possible values of x_e if we increase it while leaving all other nonbasic variables at 0. If not, the problem is unbounded, return this fact.

Step 5: Otherwise let x_l be (one of) the basic variable(s) whose nonnegativity gives the most stringent upper bound on the possible values of x_e. x_l is the leaving variable.

Step 6: Rewrite the equation for x_l in the dictionary as an equation for x_e.

Step 7: Replace x_e by the RHS of this equation in all the other equations of the dictionary to obtain a new feasible dictionary.

Step 8: Go back to Step 2.