Exercise 2

PHYS4000 Advanced Computational Quantum Mechanics

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March 18, 2024

1 Question 1

Calculate the expectation value $\langle \hat{\mathbf{S}}_z \rangle$ (will be using bold & hat notation for operators because it looks nice in latex) for $|\psi\rangle = |z_-\rangle$:

Starting with the definition for the expectation value:

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = \langle \psi | \hat{\mathbf{S}_\mathbf{z}} | \psi \rangle \tag{1}$$

$$\langle \psi | \hat{\mathbf{S}_\mathbf{z}} | \psi \rangle = \frac{1}{2}(| \langle \psi | z_+ \rangle |^2 - | \langle \psi | z_- \rangle |^2) \tag{2}$$

for a general state $|\psi\rangle$, which we substitute with $|z_-\rangle$ to get

$$\langle z_- | \hat{\mathbf{S}_\mathbf{z}} | z_- \rangle = \frac{1}{2}(| \langle z_- | z_+ \rangle |^2 - | \langle z_- | z_- \rangle |^2) \tag{3}$$

We are aware that $\langle z_\pm | z_\mp \rangle = 0$ and $\langle z_\pm | z_\pm \rangle = 1$, subsequently

$$\langle z_- | \hat{\mathbf{S}_\mathbf{z}} | z_- \rangle = -\frac{1}{2}(-1)^2$$

$$\langle z_- | \hat{\mathbf{S}_\mathbf{z}} | z_- \rangle = -\frac{1}{2} \tag{4}$$

We expect this because its essentially asking what the expectation that we get a $|z_-\rangle$ state from an SG_z experiment is. And we know from Exp 1 that it ought to be $-1/2$.

2 Question 2

As in question one, for a general state $|\psi\rangle$, the expectation value $\hat{\mathbf{S}_\mathbf{z}}$

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = \langle \psi | \hat{\mathbf{S}_\mathbf{z}} | \psi \rangle \tag{5}$$

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = \frac{1}{2}(| \langle \psi | z_+ \rangle |^2 - | \langle \psi | z_- \rangle |^2) \tag{6}$$

There are two key terms $\langle \psi | z_+ \rangle$ and $\langle \psi | z_- \rangle$ which we compute:

$$\langle \psi | z_+ \rangle = -\frac{i}{\sqrt{3}} \langle z_+ | z_+ \rangle + 0 = -\frac{i}{\sqrt{3}} \tag{7}$$

$$\langle \psi | z_- \rangle = 0 + \sqrt{\frac{2}{3}} \langle z_- | z_- \rangle = \sqrt{\frac{2}{3}} \tag{8}$$

Which we apply to $\langle \hat{\mathbf{S}_\mathbf{z}} \rangle$

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = \frac{1}{2}(| - \frac{i}{\sqrt{3}}|^2 - |\sqrt{\frac{2}{3}}|^2) \tag{9}$$

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = \frac{1}{2}\left(\frac{1}{3} - \frac{2}{3}\right) \tag{10}$$

$$\langle \hat{\mathbf{S}_\mathbf{z}} \rangle = -\frac{1}{6} \tag{11}$$

3 Question 3

We start with the knowledge that a projection operator satisfies $P^2 = P$, and we apply this to I_y:

$$I_y = |y_+\rangle \langle y_+| + |y_-\rangle \langle y_-| \tag{12}$$

$$I_y^2 = (|y_+\rangle \langle y_+| + |y_-\rangle \langle y_-|)^2$$

$$I_y^2 = |y_+\rangle \langle y_+|y_-\rangle \langle y_-| + |y_-\rangle \langle y_-|y_+| \langle y_+| + (|y_+\rangle \langle y_+|)^2 + (|y_-\rangle \langle y_-|)^2$$

$$I_y^2 = 0 + 0 + |y_+\rangle \langle y_+|y_+\rangle \langle y_+| + |y_-\rangle \langle y_-|y_-\rangle \langle y_-|$$

$$I_y^2 = |y_+\rangle \langle y_+| + |y_-\rangle \langle y_-|$$

$$I_y^2 = |y_+\rangle \langle y_+| + |y_-\rangle \langle y_-| = I_y$$

The above shows that $I_y^2 = I_y$ and thus satisfies the condition of a projection operator.

4 Question 4

Showing that $I_y = I_z$ using 5.12.

5.12:

$$|y_\pm\rangle = \frac{1}{\sqrt{2}} |z_+\rangle \pm \frac{i}{\sqrt{2}} |z_-\rangle \tag{13}$$

Substitute directly into I_y:

$$I_y = |y_+\rangle \langle y_+| + |y_-\rangle \langle y_-|$$

$$I_y = \frac{1}{\sqrt{2}} |z_+\rangle \left(\frac{1}{\sqrt{2}} \langle z_+| - \frac{i}{\sqrt{2}} \langle z_-| \right)$$

$$+ \frac{i}{\sqrt{2}} |z_-\rangle \left(\frac{1}{\sqrt{2}} \langle z_+| - \frac{i}{\sqrt{2}} \langle z_-| \right)$$

$$+ \frac{1}{\sqrt{2}} |z_+\rangle \left(\frac{1}{\sqrt{2}} \langle z_+| + \frac{i}{\sqrt{2}} \langle z_-| \right)$$

$$+ \frac{i}{\sqrt{2}} |z_-\rangle \left(\frac{1}{\sqrt{2}} \langle z_+| + \frac{i}{\sqrt{2}} \langle z_-| \right)$$

$$I_y = |z_+\rangle \langle z_-| - \frac{i}{2} |z_+\rangle \langle z_-|$$

$$+\frac{i}{2} |z_-\rangle \langle z_+| + \frac{1}{2} |z_-\rangle \langle z_-|$$

$$+\frac{1}{2} |z_+\rangle \langle z_+| + \frac{i}{2} |z_+\rangle \langle z_-|$$

$$-\frac{i}{2} |z_-\rangle \langle z_+| - \frac{1}{2} |z_-\rangle \langle z_-|$$

$$I_y = |z_+\rangle \langle z_+| + |z_-\rangle \langle z_-| = I_z \tag{14}$$

5 Question 5

5.15:

$$0 = |\langle z_-|x_+\rangle|^2 |\langle x_+|z_+\rangle|^2 + |\langle z_-|x_-\rangle|^2 |\langle x_-|z_+\rangle|^2$$

$$+ \langle z_-|x_+\rangle \langle x_+|z_+\rangle \langle z_-|x_-\rangle^* \langle x_-|z_+\rangle^*$$

$$+ \langle z_-|x_+\rangle^* \langle x_+|z_+\rangle^* \langle z_-|x_-\rangle \langle x_-|z_+\rangle$$

We can note that there are really only four terms, and their respective conjugates. If we substitute $|x_\pm\rangle = \frac{1}{\sqrt{2}}(|z_+\rangle \pm |z_+\rangle)$ into the four repeating terms and evaluate them, we can simplify the RHS in terms of z.

$$\langle z_-|x_+\rangle = \langle z_-| \left(\frac{1}{\sqrt{2}} |z_+\rangle + \frac{1}{\sqrt{2}} |z_-\rangle \right) \tag{15}$$

$$\langle x_+|z_+\rangle = \left(\frac{1}{\sqrt{2}} |z_+\rangle + \frac{1}{\sqrt{2}} |z_-\rangle \right) |z_+\rangle \tag{16}$$

$$\langle z_-|x_-\rangle = \langle z_-| \left(\frac{1}{\sqrt{2}} |z_+\rangle - \frac{1}{\sqrt{2}} |z_-\rangle \right) \tag{17}$$

$$\langle x_-|z_+\rangle = \left(\frac{1}{\sqrt{2}} |z_+\rangle - \frac{1}{\sqrt{2}} |z_-\rangle \right) |z_+\rangle \tag{18}$$

Which evaluate respectively to

$$\langle z_-|x_+\rangle = \frac{1}{\sqrt{2}}\, , \langle x_+|z_+\rangle = \frac{1}{\sqrt{2}} \tag{19}$$

$$\langle z_-|x_-\rangle = -\frac{1}{\sqrt{2}}\, , \langle x_-|z_+\rangle = \frac{1}{\sqrt{2}} \tag{20}$$

Substituting the evaluated amplitudes into 5.15 we get

$$0 = |\frac{1}{\sqrt{2}}|^2|\frac{1}{\sqrt{2}}|^2 + |-\frac{1}{\sqrt{2}}|^2|\frac{1}{\sqrt{2}}|^2$$

$$+(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}})^*(\frac{1}{\sqrt{2}})^*$$

$$+(\frac{1}{\sqrt{2}})^*(\frac{1}{\sqrt{2}})^*(-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 0$$

RHS = 0 and LHS = RHS.

6 Question 6

Show that $(\hat{\mathbf{A}}^\dagger)^\dagger = \hat{\mathbf{A}}$.

For arbitrary $|\chi\rangle$

$$\langle\psi|\hat{\mathbf{A}}^\dagger|\chi\rangle = \langle\chi|\hat{\mathbf{A}}|\psi\rangle^* \tag{21}$$

It follows that

$$\langle\psi|\hat{\mathbf{A}}^\dagger|\chi\rangle^* = \langle\chi|\hat{\mathbf{A}}|\psi\rangle \tag{22}$$

These two facts imply that

$$\langle\psi|\hat{\mathbf{A}}^\dagger|\chi\rangle^* = \langle\chi|(\hat{\mathbf{A}}^\dagger)^\dagger|\psi\rangle \tag{23}$$

$$\langle\psi|\hat{\mathbf{A}}^\dagger|\chi\rangle^* = \langle\chi|\hat{\mathbf{A}}|\psi\rangle = \langle\chi|(\hat{\mathbf{A}}^\dagger)^\dagger|\psi\rangle \tag{24}$$

$$\therefore (\hat{\mathbf{A}}^\dagger)^\dagger = \hat{\mathbf{A}} \tag{25}$$

From this we can state:

$$\langle\phi|\,\hat{\mathbf{A}} = \langle\phi|\,(\hat{\mathbf{A}}^\dagger)^\dagger \tag{26}$$

$$\langle\phi|\,(\hat{\mathbf{A}}^\dagger)^\dagger = \langle\hat{\mathbf{A}}^\dagger\phi|\,/ \tag{27}$$

$$\text{Thus, } \langle\phi|\,\hat{\mathbf{A}} = \langle\hat{\mathbf{A}}^\dagger\phi|$$

7 Question 7

Show that $(\hat{\mathbf{A}}\hat{\mathbf{B}})^\dagger = \hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger$

$$\langle\phi|\hat{\mathbf{A}}\hat{\mathbf{B}}\psi\rangle \tag{28}$$

$$= \langle\hat{\mathbf{A}}^\dagger\phi|\hat{\mathbf{B}}\psi\rangle \tag{29}$$

$$= \langle\hat{\mathbf{B}}^\dagger\hat{\mathbf{A}}^\dagger\phi|\hat{\mathbf{B}}\psi\rangle \tag{30}$$

8 Question 8

If $\hat{\mathbf{A}}$ is Hermitian, then by definition $\hat{\mathbf{A}}^\dagger = \hat{\mathbf{A}}$.

The eigenvalue equation:

$$\hat{\mathbf{A}} |\psi\rangle = a |\psi\rangle \tag{31}$$

Lets pre-multiply by a generic $\langle\psi|$.

$$\langle\psi|\hat{\mathbf{A}}|\psi\rangle = a \langle\psi|\psi\rangle \tag{32}$$

If we do the same with the adjoint operator;

$$\langle\psi|\hat{\mathbf{A}}^\dagger|\psi\rangle = a^* \langle\psi|\psi\rangle \tag{33}$$

If we know that $\hat{\mathbf{A}}^\dagger = \hat{\mathbf{A}}$, then that implies that:

$$a \langle\psi|\psi\rangle = a^* \langle\psi|\psi\rangle \tag{34}$$

$$a = a^* \tag{35}$$

The only condition for which $a = a^*$ is if $a \in \mathbb{R}$. Thus the eigenvalues of Hermitian operators are real.