Research Statement

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1 Overview

1.1 Introduction

My field of study is set theory. The work I do focuses on consistency and independence results relating to axioms for infinite cardinal numbers. More specifically, I work with infinitary combinatorial principles at singular cardinals, and study the effects that these axioms have on the mathematical universe. My research aims to find theorems about the consistency strength of these principles, as well as to separate out various related principles more precisely.

1.2 Independence Results & Consistency Strength

In 1931 Gödel proved with his groundbreaking incompleteness theorems that sufficiently strong arithmetic systems of axioms can not be both complete and consistent, nor can they prove their own consistency. By thus showing that the search for new axioms is open ended, Gödel paved the way for consistency proofs. He first showed in 1940 that Cantor's famous Continuum Hypothesis (CH) that $2^{\aleph_0} = \aleph_1$, could coexist with the standard Zermelo-Fraenkel + Axiom of Choice (ZFC) axioms. Later in 1964, Paul Cohen demonstrated that the failure of CH was also consistent with ZFC, succesfully demonstrating the independence of CH and therefore the undecidability of Hilbert's first problem from the standard axioms of mathematics.

Thus the field of set theory was blown wide open, and many independence results soon followed. Of considerable interest has been the behavior of cardinal arithmetic at singular cardinals. König's lemma tells us how cardinal exponentiation works at regular cardinals, neatly closing the door for such problems. However the behavior of cardinal exponentiation at singular cardinals has proved to be far more difficult to pin down. Through the study of these independence results, a number of important infinite combinatorial principles were discovered which in turn became objects of study in their own right. Such principles include the Singular Cardinal Hypothesis (SCH), tree property, approachability property, square ($\square$) and diamond ($\Diamond$) principles. We will discuss these in more detail later.

Whether or not such principles can hold or fail at singular cardinals is often independent of ZFC, and requires we postulate the existence of large cardinals. Loosely speaking, large cardinals are infinite cardinal numbers κ with additional properties which are strong enough such that V_κ, the universe below κ, is a self-contained model of set theory. There are many kinds of large cardinals, and it generally follows that the the existence of stronger kinds of large cardinals, such as supercompact cardinals, implies the existence of weaker large cardinals, such as measurables. It turns out that these implications form a mostly linear ordering, and so these large cardinals are used as a way of measuring the consistency strength of theorems, that is, for measuring the strength of hypotheses needed for a theorem to hold.

2 Combinatorial Principles

The types of large cardinal axioms we choose to adopt can have drastic effects on set theory. Furthermore, many important combinatorial principles cannot coexist. For example we can relate the tree property to square principle with the following theorem [3].

Theorem 2.1 (Jensen) *There is a special κ^+-Aronzajn tree (that is, the tree property at κ^+ fails) if and only if the weak square property $\square^*_\kappa$ holds.*

We may also provide a result about the consistency strength of the tree property at $\aleph_{\omega+1}$ [10].

Theorem 2.2 (Magidor-Shelah) *It is consistent with ZFC and the existence of a huge cardinal with ω supercompact cardinals above it that the tree property at $\aleph_{\omega+1}$ fails.*

2.1 Square Principles

In the early nineties, Ernest Schimmerling [5] first proved that $\square_\kappa$ holds, from the assumption that there is no inner model of Mitchell order κ^{++}. On the other hand, if he allowed bigger large cardinals, he could not prove $\square_\kappa$, but could prove the weaker principle $\square_{\kappa,\mathrm{cf}(\kappa)}$ if one assumed that your inner model correctly computed κ^+. This result was improved in a later paper [11] in which he showed that in this extender model $L[\vec{E}]$, $\square_{\kappa,<\omega}$ holds.

Around the same time, Ronald Jensen proved that $\square_\kappa$ holds for all κ in the core model K, if it is assumed that K does not contain a strong cardinal. Jensen also proved that if $\kappa \geq \omega_1$ is regular, then $\square_\kappa$ does not follow from $\square_{\kappa,2}$. That is, using a Mahlo cardinal one can construct a forcing extension which separates these two concepts.

We would like to show that the result can follow at a small singular cardinal. Showing various independence results at small cardinals such as $\aleph_\omega$ is quite interesting as well as difficult, because many properties at such cardinals are provable in ZFC. In these situations we take a large cardinal at which a desired result holds, and then use a forcing argument to collapse our large cardinal down, doing so in such a way as to preserve the desired properties. In our case, to achieve such a result at $\aleph_\omega$ one must start with more than a Mahlo. It has been proved that this will take at least a strong cardinal, but it is believed that one needs much more.

In fact, this result was achieved by by James Cummings, Matthew Foreman and Menachem Magidor in 1991 in the following theorem [2].

Theorem 2.3 (Cummings-Foreman-Magidor) *Let κ be a supercompact cardinal and suppose $2^{\kappa^{+\omega}} = \kappa^{+\omega+1}$. Let $1 \leq \mu < \nu < \aleph_\omega$ be two cardinals. Then there is a generic extension satisfying $\square_{\aleph_\omega,\nu} + \neg\square_{\aleph_\omega,\mu}$.*

The argument here, however, relied on the influence supercompact cardinals have on the universe above. One collapses the supercompact cardinal in a way such that its ω^{th} successor becomes our singular cardinal of interest.

However, the existence of a supercompact cardinal is far higher in consistency strength than is necessary. In particular, to get $\square_\kappa$ to fail only requires that we influence the universe up to κ^+. Jensen isolated a much weaker property impling the failure of $\square_\kappa$ called subompactness [4]. This large cardinal axiom is very promising, as the following theorem suggests [6].

Theorem 2.4 (Schimmerling-Zeman) *In models of the form $L[\vec{E}]$, $\square_\kappa$ holds if and only if κ is not subcompact.*

In fact the theorem holds for $\square_{\kappa,<\kappa}$, so the model $L[\vec{E}]$ does not distinguish between weak and full square.

2.2 Results

We were able to use a slight strengthening of a similar large cardinal axiom known as quasicompactness, also introduced by Jensen, to get the following initial result [1].

Theorem 2.5 (Holben) *It is consistent with the existence of a cardinal which is a strengthening of quasicompactness that there is a model of ZFC in which $\square(\aleph_{\omega+1}, <\omega)$ fails.*

This modified large cardinal axiom we used is still weaker than κ^+-supercompactness. It was attained using a modified version of Prikry forcing to simultaneously change our cardinal's cofinality and collapse cardinals below, which is why measurability was required.

We have also achieved the following theorem which separates out square principles in much the same manner as Cummings, Foreman and Magidor [1].

Theorem 2.6 (Holben) *It is consistent with the existence of a cardinal which is both subcompact and measurable that there is a model of ZFC satisfying* $\square_{\kappa,2}+\neg\square_\kappa+cf(\kappa)=\omega$.

In this theorem I did a preparation which consisted of adding measure-1 many $\square_{\alpha,2}$ sequences below κ and then threading them, in such a way that κ stayed measurable. This forcing construction combined a method of Jensen with an Easton support iteration and our own methods. We then follow our preparation with a standard Prikry forcing, showing that we preserve our results.

3 Future Work

I am currently working on showing that my second result can be generalized to separate the principle $\square_{\aleph_\omega,n}$ for any $n \in \omega$. The proof is a combination of both of my thesis results, but likely will require a slight strengthening of subcompactness to go through. Additionally, one probably could remove the measurability requirement, but that would require the invention of a new forcing to collapse a subcompact cardinal into an ω-cofinal cardinal in a desirable way.

As a long term goal, I would also like to tackle finding the consistency strength of $\square^{\mathrm{SC}}$, global square at singulars. [8] The best known lower bound is approximately a stationary proper class of measurables of Mitchell order ω_1, and the best known upper bound is a κ which is λ-supercompact for some inaccessible $\lambda > \kappa$. Again this is a wide gap, and I believe the upper bound may be reduced to be something close to the known lower bound.

I am also interested in working with forcing axioms such as Martin's Maximum (MM) and the Proper Forcing Axiom (PFA). There are many known connections between forcing axioms and combinatorics at singulars, so this is a natural avenue of research. For example, [7] if PFA holds then $\square_\kappa$ fails for all $\kappa \geq \omega_2$, and [Magidor 8] if MM holds then weak square fails for all ω-cofinal cardinals.

Specifically, I would like to work with Weiß's two-cardinal ineffable tree property $ITP(\kappa,\lambda)$ [12] at ω_2 and study its connections with PFA and MM. In doing so I hope to help provide evidence that the consistency strengths of PFA, MM and stationary reflection (SR) are that of a supercompact cardinal. Additionally, I am interested in finding a combinatorial way to unify the proofs that SR and PFA each imply SCH.

4 References

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