Homework #3 of PH621, Due due on Monday, March 12

1. (30 points) A nasty bacterium in the shape of a spheroid with principle axes b, b, a and uniform density is spinning in free space about its axis of symmetry $\hat{e}_3$ with angular velocity ω_r (Figure 1). The symmetry axis of the bacterium is inclined at angle θ with respect to an axis OP fixed in space, and precesses around it with angular velocity ω_p.

   (a) Determine ω_p

   (b) The nasty bacterium finds life in space difficult and transforms itself into a spherical spore of the same uniform density and radius c. Assume no external forces or torque have acted on the nasty bacterium, find the rotational frequency of this sphere in terms of ω_r, a, b, and θ.

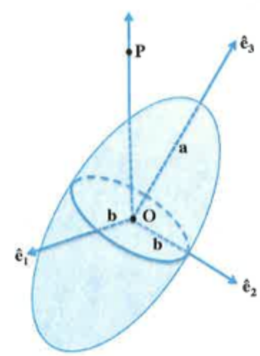

Figure 1

2. (30 points) A rigid body in the shape of a thumbtack formed from a think disk of mass M and radius a and a massless stem is placed on an inclined plane that makes an angle α with the horizontal. The point of the tack remains stationary at the point P, and the head rolls along a circle of radius b. Introduce a set of laboratory coordinates whose 3^0 axis is perpendicular to the inclined plane and whose 2^0 axis points down the plane, as well as a set of body-fixed principle axes with origin at the center of mass, whose 3 axis is perpendicular to the head of the tack pointing outward, whose 2 axis passes through the point of contact with the plane, and whose 1 axis is parallel to the surface. Introduce also the set of angles (θ, ϕ, γ) that specify the orientation of the tack, as indicated in Figure 2.

   (a) Show that in general the angular velocity of the role tack is given by

   $$\omega = \dot{\theta}\hat{e}_1 + \dot{\phi}\hat{e}_2 + (ba^{-1} - \sin\theta)\dot{\phi}\hat{e}_3.$$

   (b) Show that the kinetic energy of the tack is given by

   $$T = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\dot{\phi}^2\cos^2\theta + (2a^2)^{-1}I_3(b - a\sin\theta)^2\dot{\phi}^2 + \frac{1}{2}M(b - a\sin\theta)^2\dot{\phi}^2 + \frac{1}{2}Ma^2\dot{\theta}^2.$$

   (c) Show that the potential energy of the tack is given by

   $$V = -Mg\left[(b - a\sin\theta)\sin\alpha\cos\phi - a\cos\alpha\cos\theta\right]$$

   (d) Construct the Lagrangian $L(\phi, \theta; \dot{\phi}, \dot{\theta})$ and write the Lagrange's equations for ϕ and θ incorporating the constraint $\theta = \theta_0 = \arcsin(a/b)$ with a Lagrange multiplier λ_θ.

   (e) Show that ϕ satisfies the pendulum equation with angular frequency given by

   $$\Omega^2 = \frac{g}{a}\frac{\sin\alpha}{\cot\theta_0\cos\theta_0}\frac{4}{6 + \tan^2\theta_0}$$

   (f) Interpret λ_θ.

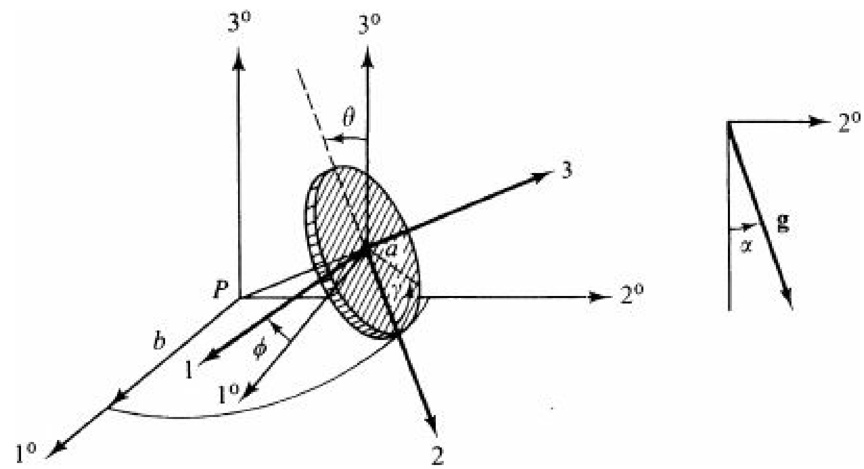

Figure 2