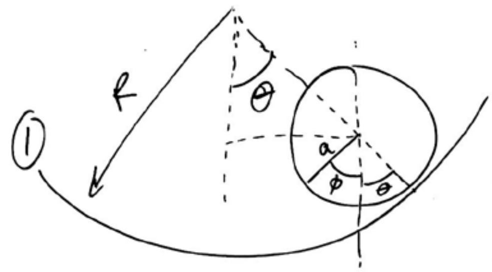
Problem 1: Setup and solving for the Lagrangian.



ARC LENGTHS ARE =

$S = R\theta = a(\phi+\theta)$

$\theta(R-a) = a\phi$

$\phi = \theta\frac{(R-a)}{a}$

$\dot\phi = \dot\theta\frac{(R-a)}{a}$

$x = (R-a)\sin\theta$

$\dot x = (R-a)\cos\theta\dot\theta$

$y = (R-a)\cos\theta$

$\dot y = -(R-a)\sin\theta\dot\theta$

$\omega = \dot\phi = \dot\theta\frac{(R-a)}{a}$

Expand

$\cos\theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4)$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$L = \frac{1}{2}m(\dot x^2 + \dot y^2) + \frac{1}{2}I\omega^2 - mgR(1-\cos\theta)$

$L = \frac{1}{2}m((R-a)^2\dot\theta^2) + \frac{1}{2}(\frac{1}{2}m\cancel{a^2})\dot\theta^2\frac{(R-a)^2}{\cancel{a^2}} - mgR(1-(1-\frac{\theta^2}{2}))$

$$\boxed{L = \frac{3}{4}m(R-a)^2\dot\theta^2 - \frac{mgR}{2}\theta^2}$$

Solving for the equation of motion and eigenfrequencies.

$$L = \frac{3}{4}m(R-a)^2\dot\theta^2 - \frac{mgR}{2}\theta^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{3}{2}\cancel{m}(R-a)^2\ddot\theta + \cancel{m}gR\theta = 0$$

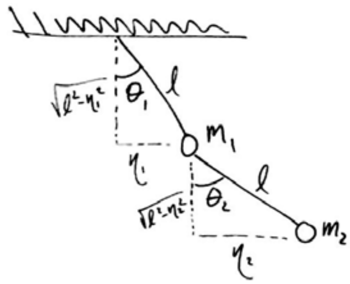
EQUATION OF MOTION (SMALL OSILLATIONS)

$$\boxed{\ddot\theta = -\frac{gR}{\frac{3}{2}(R-a)^2}\theta}$$

FREQUENCY

$$\boxed{\omega = \sqrt{\frac{2gR}{3(R-a)^2}}}$$

Problem 2: Setup and solving for the Lagrangian.



Expand $\cos\theta_\sigma$

$$\cos\theta_\sigma = \frac{\sqrt{\ell^2-\eta_\sigma^2}}{\ell} = \frac{1}{\ell}\left(\ell^2-\eta_\sigma^2\right)^{\frac{1}{2}}$$

$$= \left(1-\left(\frac{\eta_\sigma}{\ell}\right)^2\right)^{\frac{1}{2}} = \boxed{1+\frac{1}{2}\left(\frac{\eta_\sigma}{\ell}\right)^2}$$

$$\sin\theta_\sigma = \theta_\sigma - \frac{\theta_\sigma^3}{6} + \mathcal{O}(\theta_\sigma^5)$$

$$X_1 = \eta_1 \qquad X_2 = \eta_1+\eta_2$$

$$\dot{X}_1 = \dot{\eta}_1 \qquad \dot{X}_2 = \dot{\eta}_1+\dot{\eta}_2$$

$$y_1 = \ell(1-\cos\theta_1) \qquad y_2 = \ell(2-\cos\theta_1-\cos\theta_2)$$

$$\dot{y}_1 = \ell\ (\sin\theta_1\dot{\theta}_1) \qquad \dot{y}_2 = \ell\ (\sin\theta_1\dot{\theta}_1+\sin\theta_2\dot{\theta}_2)$$

Small: $\dot{y}_1^2 \propto \dot{\theta}_1^2 \simeq 0$ Small: $\dot{y}_2^2 \propto \dot{\theta}_2^2+\dot{\theta}_1^2 \simeq 0$

$$L = \frac{1}{2}m_1\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{1}{2}m_2\left[\left(\dot{x}_2+\dot{x}_1\right)^2+\left(\dot{y}_2+\dot{y}_1\right)^2\right]$$
$$-\left[m_1 g\ell(1-\cos\theta_1)+m_2 g\ell(2-\cos\theta_1-\cos\theta_2)\right]$$

$$L = \frac{1}{2}m_1\left(\dot{\eta}_1^2+\underset{0}{\cancel{\text{small}}}\right)+\frac{1}{2}m_2\left(\dot{\eta}_2+\dot{\eta}_1\right)^2$$
$$-\left[m_1 g\ell\left(1-\left(1+\frac{1}{2}\left(\frac{\eta_1}{\ell}\right)^2\right)\right)+m_2 g\ell\left(2-\left(1+\frac{1}{2}\left(\frac{\eta_1}{\ell}\right)^2\right)-\left(1+\frac{1}{2}\left(\frac{\eta_2}{\ell}\right)^2\right)\right)\right]$$

$$L = \frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2(\dot{\eta}_1+\dot{\eta}_2)^2+\frac{m_1 g}{2\ell^2}\eta_1^2+\frac{m_2 g}{2\ell^2}\left(\eta_1^2+\eta_2^2\right)$$

$$\boxed{L = \frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2\left(\dot{\eta}_1^2+2\dot{\eta}_1\dot{\eta}_2+\dot{\eta}_2^2\right)-\frac{g}{2\ell}\left[(m_1+m_2)\eta_1^2+m_2\eta_2^2\right]}$$

Solving for mass matrix, potential matrix, and eigenfrequencies.

$$L = \frac{1}{2}m_1\dot{\eta}_1^2+\frac{1}{2}m_2\underbrace{\left(\dot{\eta}_1+\dot{\eta}_2\right)^2}_{\left(\dot{\eta}_1^2+2\dot{\eta}_1\dot{\eta}_2+\dot{\eta}_2^2\right)}-\frac{g}{2\ell}\left[(m_1+m_2)\eta_1^2+m_2\eta_2^2\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\eta}_\sigma}\right)-\frac{\partial L}{\partial\eta_\sigma}=0$$

$\sigma=1$: $$m_1\ddot{\eta}_1+m_2\ddot{\eta}_1+m_2\ddot{\eta}_2+\frac{g}{\ell}(m_1+m_2)\eta_1 = 0$$

$\sigma=2$: $$m_2\ddot{\eta}_1+m_2\ddot{\eta}_2+\frac{g}{\ell}m_2\eta_2 = 0$$

$$\underline{M} = \begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$$

$$\underline{V} = \frac{g}{\ell}\begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$-\omega_s^2\begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}+\frac{g}{\ell}\begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}=0$$

$$\text{let } \gamma^2=\frac{m_2}{m_1+m_2}$$

$$\det\begin{vmatrix} -\omega_s^2(m_1+m_2)+\frac{g}{\ell}(m_1+m_2) & -\omega_s^2 m_2 \\ -\omega_s^2 m_2 & -\omega_s^2 m_2+\frac{g}{\ell}m_2 \end{vmatrix}=0$$

$(m_1+m_2)(g/\ell-\omega_s^2)$ $\quad$ $m_2(g/\ell-\omega_s^2)$

$$\frac{\left(\frac{g}{\ell}-\omega_s^2\right)^2 m_2(m_1+m_2)-\omega_s^4 m_2^2=0}{(m_1+m_2)^2}$$

$$\left(\frac{g}{\ell}-\omega_s^2\right)^2\cancel{\gamma^2}-\omega_s^4\gamma^{\cancel{4}\,2}=0$$

$$\omega_s^4\gamma^2=\left(\frac{g}{\ell}-\omega_s^2\right)^2$$

$$\pm\omega_s^2\gamma=\frac{g}{\ell}-\omega_s^2$$

$$\omega_s^2(1\pm\gamma)=\frac{g}{\ell}$$

(b) $$\boxed{\omega_s^2=\frac{g/\ell}{1\pm\gamma}}$$

Solving for the eigenvectors ρ_1 and ρ_2 and normalization:

$$\omega_s^2 = \frac{g/\ell}{1 \pm \gamma} \quad ; \quad \gamma^2 \equiv \frac{m_2}{m_1 + m_2}$$

Solving for ρ_2 in terms of ρ_1 for ω_+ & ω_-.

$$-\omega_s^2(m_1+m_2)\rho_1 - \omega_s^2 m_2 \rho_2 + \frac{g}{\ell}(m_1+m_2)\rho_1 = 0$$

$$+\left(-\omega_s^2 m_2 \rho_1 - \omega_s^2 m_2 \rho_2 + \frac{g}{\ell} m_2 \rho_2 = 0\right)$$

$$-\omega_s^2 \rho_1 \underbrace{(m_1 + m_2 + m_2)} - \omega_s^2 \rho_2 \, 2m_2 + \frac{g}{\ell}\rho_1 (m_1+m_2) + \frac{g}{\ell} m_2 \rho_2 = 0$$

$$\overset{m_1+m_2}{}$$

$$-\omega_s^2 \rho_1 (1 + \gamma^2) - \omega_s^2 \rho_2 \, 2\gamma^2 + \frac{g}{\ell}\rho_1 + \frac{g}{\ell}\rho_2 \gamma^2 = 0$$

$$\left[-\frac{\cancel{g/\ell}}{1\pm\gamma}\rho_1(1+\gamma^2) - \frac{\cancel{g/\ell}}{1\pm\gamma}\rho_2 \, 2\gamma^2 + \cancel{g/\ell}\,\rho_1 + \cancel{g/\ell}\,\rho_2\gamma^2 = 0\right](1\pm\gamma)$$

$$-\rho_1(1+\gamma^2) - \rho_2(2\gamma^2) + \rho_1(1\pm\gamma) + \rho_2\gamma^2(1\pm\gamma) = 0$$

$$\rho_2\left(-2\gamma^2 + \gamma^2(1\pm\gamma)\right) = \rho_1\left(1+\gamma^2 - (1\pm\gamma)\right)$$

ω_+: $\rho_2(-2\gamma^2 + \gamma^2 + \gamma^3) = \rho_1(\cancel{1} + \gamma^2 - (\cancel{1}+\gamma))$

$$\rho_2 \gamma^{\cancel{2}}\cancel{(\gamma - 1)} = \rho_1 \cancel{\gamma}\cancel{(\gamma-1)}$$

$$\boxed{\rho_2 = \frac{1}{\gamma}\rho_1}$$

ω_-: $\rho_2(-2\gamma^2 + \gamma^2(1-\gamma)) = \rho_1(1+\gamma^2 - (1-\gamma))$

$$\rho_2(-2\gamma^2 + \gamma^2 - \gamma^3) = \rho_1(\cancel{1} + \gamma^2 - \cancel{1} + \gamma)$$

$$-\rho_2\gamma^{\cancel{2}}\cancel{(\gamma+1)} = \rho_1\cancel{\gamma}\cancel{(\gamma+1)}$$

$$\boxed{\rho_2 = \frac{\rho_1}{-\gamma}}$$

$$\rho_+ = \begin{pmatrix}\gamma \\ 1\end{pmatrix} \quad \rho_- = \begin{pmatrix}-\gamma \\ 1\end{pmatrix}$$

For ω_+, the pendula swing together, but with different amplitudes by a factor of γ. For ω_-, they swing opposite one another, the amplitudes again different by a factor of γ.

$$\rho_+^T \, \underline{m} \, \rho_+ = 1$$

$$\begin{pmatrix}\gamma & 1\end{pmatrix}\begin{pmatrix}m_1+m_2 & m_2 \\ m_2 & m_2\end{pmatrix}\begin{pmatrix}\gamma \\ 1\end{pmatrix} = \frac{1}{C^2}$$

$$\begin{pmatrix}\gamma & 1\end{pmatrix}\begin{pmatrix}\gamma(m_1+m_2) + m_2 \\ \gamma m_2 + m_2\end{pmatrix}$$

$$\gamma^2(m_1+m_2) + \gamma m_2 + \gamma m_2 + m_2 = \frac{1}{C^2}$$

$$\cancel{\gamma^2}\frac{1}{\cancel{\gamma^2}} \quad 2\gamma + 1 = \frac{1}{C^2 m_2}$$

$$2(\gamma+1) = \frac{1}{C^2 m_2}$$

$$\boxed{C^2 = \frac{1}{2m_2(\gamma+1)}}$$

Normalization and solving for ρ_1 and ρ_2 to match Weihong's solution.

CHANGING ρ_+ & ρ_- to match Weihong Solution.

$$C = \frac{1}{\sqrt{2m_2(1+\gamma)}}$$

$$\gamma^2 = \frac{m_2}{m_1+m_2}$$

$$\gamma^2 m_1 + \gamma^2 m_2 = m_2$$

$$\gamma^2 m_1 = (1-\gamma^2)m_2$$

$$m_2 = \frac{\gamma^2}{1-\gamma^2}m_1$$

$$\frac{1}{\sqrt{2\cancel{(1+\gamma)}\left(\frac{\gamma^2}{\cancel{(1+\gamma)}(1-\gamma)}\right)m_1}}$$

$$\boxed{C_+ = \frac{\sqrt{1-\gamma}}{\gamma\sqrt{2m_1}}}$$

$$\boxed{C_- = \frac{\sqrt{1+\gamma}}{\gamma\sqrt{2m_1}}}$$

$$\rho_+ = \frac{\sqrt{1-\gamma}}{\gamma\sqrt{2m_1}}\begin{pmatrix}\gamma \\ 1\end{pmatrix} = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma}\end{pmatrix}$$

$$\rho_- = \frac{\sqrt{1+\gamma}}{\gamma\sqrt{2m_1}}\begin{pmatrix}-\gamma \\ 1\end{pmatrix} = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}-\sqrt{1+\gamma} \\ \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}$$

The modal matrix A, and showing that $A^T MA = Identity\ Matrix$

Normalized Eigenvectors:

$$\underline{\rho}_+ = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma}\\ \frac{\sqrt{1-\gamma}}{\gamma}\end{pmatrix};\quad \underline{\rho}_- = \frac{1}{\sqrt{2m_1}}\begin{pmatrix}-\sqrt{1+\gamma}\\ \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}$$

MODAL MATRIX:

$$A \equiv \frac{1}{\sqrt{2m_1}}\begin{pmatrix}\sqrt{1-\gamma} & -\sqrt{1+\gamma}\\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}$$

Showing that the modal matrix Diagonalize $\underline{m}$ and $\underline{v}$.

PROVE: $A^T\underline{m}A = \mathbb{I}$

$$A^T\underline{m}A = \frac{1}{2m_1}\begin{pmatrix}\sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma}\\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}\begin{pmatrix}m_1+m_2 & m_2\\ m_2 & m_2\end{pmatrix}\begin{pmatrix}\sqrt{1-\gamma} & -\sqrt{1+\gamma}\\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}$$

$$= \frac{m_2}{\sqrt{2m_1}}\left(A^T\right)\begin{pmatrix}\frac{1}{\gamma^2} & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\sqrt{1-\gamma} & -\sqrt{1+\gamma}\\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}$$

$$= \frac{m_2}{\sqrt{2m_1}}\left(A^T\right)\begin{pmatrix}\frac{\sqrt{1-\gamma}}{\gamma^2}+\frac{\sqrt{1-\gamma}}{\gamma} & -\frac{\sqrt{1+\gamma}}{\gamma^2}+\frac{\sqrt{1+\gamma}}{\gamma}\\ \sqrt{1-\gamma}+\frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}-\sqrt{1+\gamma}\end{pmatrix}$$

$$\frac{m_2}{2m_1}\begin{pmatrix}\sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma}\\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{1-\gamma}}{\gamma^2}+\frac{\sqrt{1-\gamma}}{\gamma} & -\frac{\sqrt{1+\gamma}}{\gamma^2}+\frac{\sqrt{1+\gamma}}{\gamma}\\ \sqrt{1-\gamma}+\frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma}-\sqrt{1+\gamma}\end{pmatrix}$$

$$= \frac{m_2}{2m_1}\begin{pmatrix}\frac{1-\gamma}{\gamma^2}+\frac{1-\gamma}{\gamma}+\frac{1-\gamma}{\gamma}+\frac{1-\gamma}{\gamma^2} & \cancel{-\frac{\sqrt{(1-\gamma)(1+\gamma)}}{\gamma^2}}+\cancel{\frac{\sqrt{(1-\gamma)(1+\gamma)}}{\gamma}}-\cancel{\frac{\sqrt{(1+\gamma)(1-\gamma)}}{\gamma}}+\cancel{\frac{\sqrt{(1-\gamma)(1+\gamma)}}{\gamma^2}}\\ \cancel{-\frac{\sqrt{(1+\gamma)(1-\gamma)}}{\gamma^2}}-\cancel{\frac{\sqrt{(1+\gamma)(1-\gamma)}}{\gamma}}+\cancel{\frac{\sqrt{(1+\gamma)(1-\gamma)}}{\gamma}}+\cancel{\frac{\sqrt{(1+\gamma)(1-\gamma)}}{\gamma^2}} & \frac{1+\gamma}{\gamma^2}-\frac{1+\gamma}{\gamma}-\frac{1+\gamma}{\gamma}+\frac{1+\gamma}{\gamma^2}\end{pmatrix}$$

$$= \frac{m_2}{2m_1}\begin{pmatrix}\frac{1-\cancel{\gamma}}{\gamma^2}+\frac{\cancel{\gamma}-\gamma^2}{\gamma^2}+\frac{\cancel{\gamma}-\gamma^2}{\gamma^2}+\frac{1-\cancel{\gamma}}{\gamma^2} & 0\\ 0 & \frac{1+\cancel{\gamma}}{\gamma^2}-\frac{\cancel{\gamma}+\gamma^2}{\gamma^2}-\frac{\cancel{\gamma}+\gamma^2}{\gamma^2}+\frac{1+\cancel{\gamma}}{\gamma^2}\end{pmatrix} = \frac{m_2}{\cancel{2}m_1\gamma^2}\begin{pmatrix}\cancel{2}(1-\gamma^2) & 0\\ 0 & \cancel{2}(1-\gamma^2)\end{pmatrix}$$

$$\frac{\cancel{m_2}(m_1+m_2)}{m_1\cancel{m_2}}\begin{pmatrix}1-\gamma^2 & 0\\ 0 & 1-\gamma^2\end{pmatrix}$$

$$\frac{m_1+m_2}{m_1}\begin{pmatrix}1-\frac{m_2}{m_1+m_2} & 0\\ 0 & 1-\frac{m_2}{m_1+m_2}\end{pmatrix}$$

$$\frac{m_1+m_2}{m_1}-\frac{m_2\cancel{(m_1+m_2)}}{m_1\cancel{(m_1+m_2)}}$$

$$= \frac{m_1+\cancel{m_2}-\cancel{m_2}}{m_1}$$

$$= \boxed{1}$$

$$\boxed{A^T\underline{m}A = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}$$

Diagonalizing the potential matrix.

$$\underline{A}^T \underline{V} \underline{A} = \frac{g}{\ell\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & \frac{\sqrt{1-\gamma}}{\gamma} \\ -\sqrt{1+\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix}$$

$$= \frac{g\, m_2}{\ell\sqrt{2m_1}} \left(\underline{A}^T\right) \begin{pmatrix} \frac{1}{\gamma^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix}$$

$$= \frac{g\, m_2}{\ell\sqrt{2m_1}} \underline{A}^T \begin{pmatrix} \frac{\sqrt{1-\gamma}}{\gamma^2} & -\frac{\sqrt{1+\gamma}}{\gamma^2} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} = \frac{g\, m_2}{\ell\sqrt{2m_1}} \begin{pmatrix} \frac{1-\gamma}{\gamma^2} + \frac{1-\gamma}{\gamma^2} & \cancel{\frac{\sqrt{1-\gamma}\left(-\sqrt{1+\gamma}\right)}{\gamma^2}} + \cancel{\frac{\sqrt{1-\gamma}\sqrt{1+\gamma}}{\gamma^2}} \\ \cancel{-\frac{\sqrt{1+\gamma}\sqrt{1-\gamma}}{\gamma^2}} + \cancel{\frac{\sqrt{1+\gamma}\sqrt{1-\gamma}}{\gamma^2}} & \frac{(1+\gamma)}{\gamma^2} + \frac{1+\gamma}{\gamma^2} \end{pmatrix} = \frac{g\, m_2}{\ell\sqrt{2m_1}} \begin{pmatrix} 2(1-\gamma) & 0 \\ 0 & 2(1+\gamma) \end{pmatrix}$$

Eigenfrequencies: $\omega_\pm = \frac{g}{\ell}\sqrt{\frac{2m_2}{m_1}}\left(1 \mp \sqrt{\frac{m_2}{m_1+m_2}}\right) = \boxed{\frac{g}{\ell}\left(\sqrt{\frac{2m_2}{m_1}} \mp m_2\sqrt{\frac{2}{m_1+m_2}}\right) = \omega_\pm}$

$$\boxed{A^T \underline{V} A = \frac{g}{\ell}\sqrt{\frac{2m_2}{m_1}} \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix}}$$

Solving for the normal modes $\xi(t)$

$$\eta(t) = \sum_s C_s \underline{\rho_s} \cos(\omega_s t + \phi_s)$$

$$\eta(t) = \begin{bmatrix} C_+\rho_1^{(+)}\cos(\omega_+ t + \phi_+) + C_-\rho_1^{(-)}\cos(\omega_- t + \phi_-) \\ C_+\rho_2^{(+)}\cos(\omega_+ t + \phi_+) + C_-\rho_2^{(-)}\cos(\omega_- t + \phi_-) \end{bmatrix}$$

The NORMAL COORDINATES:

$$\eta(t) \equiv \underline{A}\, \underline{\xi}(t)$$

$$\underline{A}^T \underline{m}\, \eta(t) = \underline{A}^T \underline{m}\, \underline{A}\, \xi(t) = \underline{\mathbb{1}}\, \xi(t)$$

$$\xi(t) = \underline{A}^T \underline{m}\, \underline{\eta}(t)$$

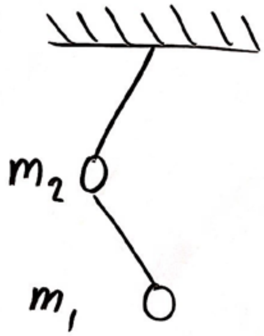
$$= \frac{1}{\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix} \eta$$

$$= \frac{m_2}{\sqrt{2m_1}} \begin{pmatrix} \sqrt{1-\gamma} & -\sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma} & \frac{\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} \frac{1}{\gamma^2} & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \frac{m_2}{\sqrt{2m_1}} \begin{pmatrix} \frac{\sqrt{1-\gamma}}{\gamma^2} - \sqrt{1+\gamma} & \sqrt{1-\gamma} - \sqrt{1+\gamma} \\ \frac{\sqrt{1-\gamma}}{\gamma^3} + \frac{\sqrt{1+\gamma}}{\gamma} & \frac{\sqrt{1-\gamma}+\sqrt{1+\gamma}}{\gamma} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

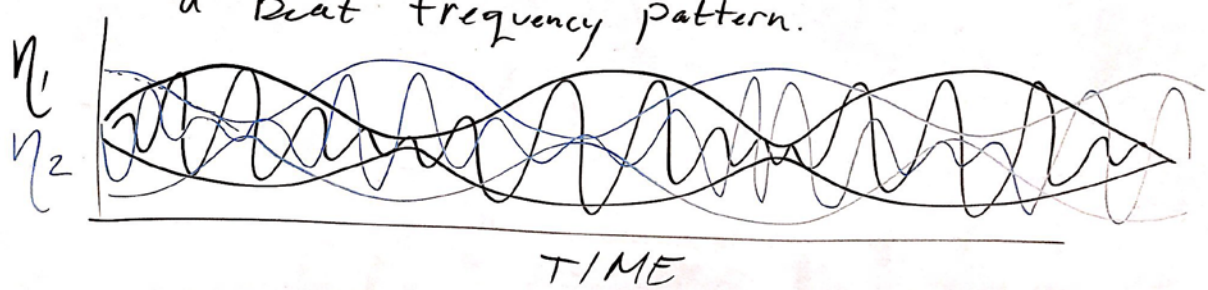
$$\xi(t) = \frac{m}{\sqrt{2m_1}} \begin{pmatrix} \left(\frac{\sqrt{1-\gamma}}{\gamma^2} - \sqrt{1+\gamma}\right)\eta_1 + \left(\sqrt{1-\gamma} - \sqrt{1+\gamma}\right)\eta_2 \\ \left(\frac{\sqrt{1-\gamma}}{\gamma^3} + \frac{\sqrt{1+\gamma}}{\gamma}\right)\eta_1 + \left(\frac{\sqrt{1-\gamma}+\sqrt{1+\gamma}}{\gamma}\right)\eta_2 \end{pmatrix}$$

Solving for particular initial conditions. There's not much work shown here, I didn't have a good idea of how to get the solutions $\eta_\sigma(t)$ quickly. I may try in Mathematica/Wolfram next time. I do however know that the system will follow a beat phenomena, where the pendula trade momentum back and forth, each following a beat function.

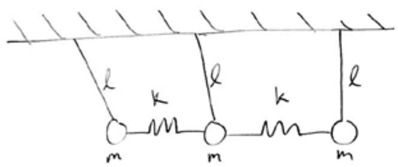


Eigenfrequencies (MODES) $W_\pm = \sqrt{\dfrac{g}{\ell(1\pm\gamma)}}$

If the upper mass is displaced slightly, the system will oscillate with a beat frequency pattern.



Problem 2 setup



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\eta_\sigma}\right) - \frac{\partial L}{\partial \eta_\sigma} = 0 \qquad \sigma = 1,2,3$$

$$\left\{\begin{array}{l} m\ddot\eta_1 + \frac{mg\eta_1}{\ell} + k(\eta_1 - \eta_2) = 0 \\ m\ddot\eta_2 + \frac{mg\eta_2}{\ell} + k(-\eta_1 + 2\eta_2 - \eta_3) = 0 \\ m\ddot\eta_3 + \frac{mg\eta_3}{\ell} + k(-\eta_2 + \eta_3) = 0 \end{array}\right\}$$

$$T = \frac{1}{2}m\left(\dot\eta_1^2 + \dot\eta_2^2 + \dot\eta_3^2\right)$$

$$U = \frac{mg}{2\ell}\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right) + \frac{1}{2}k\left[(\eta_1-\eta_2)^2 + (\eta_2-\eta_3)^2\right]$$

$$L = \frac{m}{2}\left(\dot\eta_1^2 + \dot\eta_2^2 + \dot\eta_3^2\right) - \frac{mg}{2\ell}\left(\eta_1^2 + \eta_2^2 + \eta_3^2\right) - \frac{1}{2}k\left[(\eta_1-\eta_2)^2 + (\eta_2-\eta_3)^2\right]$$

$$- \frac{1}{2}k\left[\eta_1^2 - 2\eta_1\eta_2 + \underbrace{\eta_2^2 + \eta_2^2}_{2\eta_2^2} - 2\eta_2\eta_3 + \eta_3^2\right]$$

After solving the Euler-Lagrange equation for all σ, we get a set of three equations that contain η_1, η_2, η_3. If we assume that η_σ will be a simple harmonic function, then we can rewrite $\ddot{\eta}(t) = -\omega_s \eta(t)$. Then our system can succinctly be written in the following way:

$$\left[-\omega_s \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} + \begin{pmatrix} \frac{mg}{\ell} + k & -k & 0 \\ -k & \frac{mg}{\ell} + 2k & -k \\ 0 & -k & \frac{mg}{\ell} + k \end{pmatrix} \right] \begin{bmatrix} \rho_1^{(s)} \\ \rho_2^{(s)} \\ \rho_3^{(s)} \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By taking the determinant, I solved for the eigenfrequecies then wrote what modes they correspond to. The determinant was solved in Mathematica.

Solve for eigen frequencies / eigen vectors

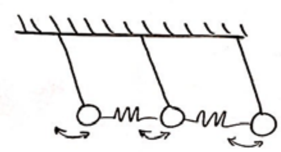
$$\det \begin{vmatrix} -\omega_s^2 m + \frac{mg}{\ell} + k & -k & 0 \\ -k & -\omega_s^2 m + \frac{mg}{\ell} + 2k & -k \\ 0 & -k & -\omega_s^2 m + \frac{mg}{\ell} + k \end{vmatrix} = 0$$

Mathematica solved for characteristic frequencies

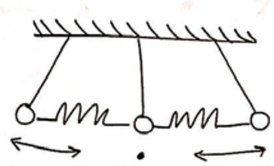
$$\boxed{\omega^2 = \frac{g}{\ell},\ \frac{k}{m} + \frac{g}{\ell},\ \frac{3k}{m} + \frac{g}{\ell}}$$

The first mode is:

$\omega_1 = \pm\sqrt{\frac{g}{\ell}}$ → All three pendula swing together and there is no compression in any of the springs.

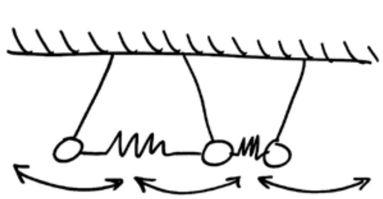


The second mode has one spring compressing on each side while the center mass is stationary.



$\omega_2 = \frac{g}{\ell} + \frac{k}{m}$

The third mode has the two outside springs oscillating while all thee masses swing in alternating directions.



$\omega_3 = \frac{g}{\ell} + \frac{3k}{m}$