Constructing the Exponential Function

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1 Introduction

In this essay, we discuss the exponential function. In particular, we will show that there is a unique and continuous function $Exp : \mathbb{R} \to \mathbb{R}$ with the property that

$$Exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

In order to achieve this, we will have to start simpler, defining a function $\exp : \mathbb{Q} \to \mathbb{R}$ with the same properties. Continuity and uniqueness come to us easily after we explore certain properties of polynomial expansion of $\sum_{k=0}^{n} \frac{x^k}{k!}$ for some $n \in \mathbb{N}$. In particular, we will find that the limit of this summation as n approaches infinity does in fact exist for all $q \in \mathbb{Q}$. Moreover, we will find that this limit exists for all $x \in \mathbb{R}$ which will help us build our desired Exp function. We will finish by generalizing our work with the exponential to apply to all sequences of polynomials with similar properties.

Our ultimate goal of this essay is to prove that for any sequence of polynomials with properties similar to that of $\sum_{k=0}^{n} \frac{x^k}{k!}$, we can find a unique and continuous function that maps from $\mathbb{R}$ to $\mathbb{R}$.

2 $\{e^x_n\}_{n\in\mathbb{N}}$

First, assume that $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ have already been constructed. We start by defining a sequence of rational numbers $\{e^x_n\}_{n\in\mathbb{N}}$ as follows:

$$e^x_0 = 1 \quad e^x_{n+1} = e^x_n + \frac{x^{n+1}}{(n+1)!}$$

Where $x \in \mathbb{Q}$.
In order to better understand this sequence, we will start by examining some term in the sequence, e^x_n, where $n \in \mathbb{N}$.

2.1 Expressing e^x_n as a summation

Consider e^x_n for some $n \in \mathbb{N}$. In this section, we want to show that e^x_n can be written as some summation. This will ultimately make our lives easier when comparing different terms within $\{e^x_n\}_{n\in\mathbb{N}}$. We claim that:

$$e^x_n = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Proof.

In order to prove this, we will use the Principle of Mathematical Induction (PMI). Let $P(n)$ be the proposition $P(n) =$ "$e^x_n = \sum_{k=0}^{n} \frac{x^k}{k!}$".

Base Case: $P(0)$

Note that $e^x_0 = 1$ by definition, and

$$\sum_{k=0}^{0} \frac{x^k}{k!} = \frac{1}{1} = 1$$

Since $1 = 1$, we have that $P(0)$ holds.

Inductive Step:

Now suppose $P(n)$ holds for some $n \in \mathbb{N}$:

$$e^x_n = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Now consider the case for $P(n+1)$. By definition:

$$e^x_{n+1} = e^x_n + \frac{x^{n+1}}{(n+1)!}$$

Plugging in our assumed equation for e_n^x:

$$\begin{aligned} e_{n+1}^x &= \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!} \\ &= \sum_{k=0}^{n+1} \frac{x^k}{k!} \end{aligned}$$

Thus, we see that $(P(n) \implies P(n+1))$. Therefore, since $P(0)$ holds, and $(P(n) \implies P(n+1))$, then by PMI, we conclude that:

$$(\forall n \in \mathbb{N})(e_n^x = \sum_{k=0}^{n} \frac{x^k}{k!})$$

$\square$

2.2 $\{e_n^x\}_{n\in\mathbb{N}}$ is Cauchy

In order to show that $\{e_n^x\}_{n\in\mathbb{N}}$ is Cauchy, we must prove that:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n, m > N)(|e_n^x - e_m^x| < \varepsilon)$$

The proof of this will follow easily if we first prove the following proposition:

Proposition. *For all $x \in \mathbb{N}$, there exists $N_x \in \mathbb{N}$ with the following property:*

$$(\forall n \in \mathbb{N})(n > N_x \Longrightarrow x^n < n!)$$

Proof. Suppose $x \in \mathbb{N}$. Note that for some $n \in \mathbb{N}$:

$$x^n < n! \iff \frac{x^n}{n!} < 1$$

We will prove the right hand side of the equivalence. Now, fix some $k > x$. Note that $(\forall n > k)$, we have that:

$$x^n < k^n < n^n$$

Now, note that:

$$x^{n-k} < k^{n-k} < (k)(k+1)(k+2)...(n-2)(n-1)$$

since x^{n-k} is x multiplied by itself $n-k$ times, while the far right most side of the inequality is the product of $n-k$ terms, all of which are strictly greater than x. Therefore, their product must be strictly greater than the x^{n-k}.

Now multiplying the far left and far right side of the inequality by $\frac{x^k}{n!}$, we get:

$$\frac{x^k x^{n-k}}{n!} < \frac{x^k(k)(k+1)...(n-1)}{n!} = \frac{x^k(k)(k+1)...(n-1)}{(1)(2)...(k-1)(k)...(n-1)(n)}$$

which is nothing more than saying:

$$\frac{x^n}{n!} < \frac{x^k}{(k-1)!}\frac{1}{n}$$

Now since k is fixed, the term $\frac{x^k}{(k-1)!}$ is just a constant. Thus it follows that for all $(\varepsilon > 0)$, we can find $N \in \mathbb{N}$ such that for all $n > N$:

$$\frac{x^k}{(k-1)!}\frac{1}{n} < \varepsilon$$

and since $\frac{x^n}{n!} < \frac{x^k}{(k-1)!}\frac{1}{n}$, then we have found $N \in \mathbb{N}$ such that:

$$(\forall n \in \mathbb{N})(n > N \implies \frac{x^n}{n!} < 1)$$

by setting $\varepsilon = 1$. And this of course is the same as saying:

$$(\forall n \in \mathbb{N})(n > N \implies x^n < n!)$$

Since our choice of x was arbitrary, this applies to all $x \in \mathbb{N}$, so we are done. $\square$

Now we prove that $\{e_n^x\}_{n\in\mathbb{N}}$ is Cauchy:

Proof. Suppose $x \in \mathbb{Q}$. From our previous proposition, we have that:

$$(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})(n > N \implies x^n < n!)$$

Now consider $n+1$.

Claim: $\frac{x^n}{n!} > \frac{x^{n+1}}{(n+1)!}$ for all $n > N$

Note that
$$\frac{x^{n+1}}{(n+1)!} = \frac{x^n}{n!} \cdot \frac{x}{n+1}$$

From our previous induction proof, we took $n > k > x$. Therefore $n > x$, so $(n+1) > x$. Thus, it follows that:

$$\frac{x}{n+1} < 1$$

so multiplying both sides of the inequality by $\frac{x^n}{n!}$, we get:

$$\frac{x^n}{n!} \cdot \frac{x}{n+1} = \frac{x^{n+1}}{(n+1)!} < \frac{x^n}{n!}$$

Since we chose n arbitrarily, it follows that this statement holds for all $n > N$, and we've proven the claim.

This claim, along with the proposition that $x^n < n!$ for $n > N$ tells us that as n gets larger and larger, $\frac{x^n}{n!}$ gets smaller and smaller. Since x^n and $n!$ are non-negative, we can conclude that:

$$\lim_{n \Longrightarrow \infty} \frac{x^n}{n!} = 0$$

Putting this all together, we see that since $\{e_n^x\}_{n\in\mathbb{N}} = \sum_{k=0}^{n} \frac{x^n}{n!}$, and $\lim_{n\Longrightarrow\infty} \frac{x^n}{n!} = 0$, then it follows that we can always find some $N^* \in \mathbb{N}$ such that
$$(\forall \varepsilon > 0)(\forall n, m > N^*)(|e_n^x - e_m^x| < \varepsilon)$$

since if $n > m$, then
$$e_n^x - e_m^x = \sum_{k=m}^{n} \frac{x^k}{k!}$$

and we can choose n and m to be as large as we need. We conclude that $\{e_n^x\}_{n\in\mathbb{N}}$ is Cauchy. $\square$

2.3 The existence of exp: $\mathbb{Q} \to \mathbb{R}$

Since $\{e_n^x\}_{n \in \mathbb{N}}$ is Cauchy, then for every $x \in \mathbb{Q}$, there exists a limit as n approaches infinity. Since our limit is an infinite sum, we can conclude that our limit will be an element of $\mathbb{R}$. Thus, we can define a function exp as follows:

$$\exp : \mathbb{Q} \to \mathbb{R} \qquad x \mapsto \lim_{n \to \infty} e_n^x$$

Using exp: $\mathbb{Q} \to \mathbb{R}$, we can generalize this function even further; we claim that there exists a unique continuous function $Exp : \mathbb{R} \to \mathbb{R}$ with the property that $Exp(q) = \exp(q)$ for all $q \in \mathbb{Q}$.

In order to do this, we must split up our function into dealing with when our input is non-negative, and when it is negative, since the function behaves differently based on the input. We start with looking at the non-negative domain in the next section:

3 The function Exp with a non-negative domain

Our goal in this section is to show that there exists a unique and continuous function $Exp : [0, \infty] \to \mathbb{R}$ such that $Exp(q) = \exp(q)$ for all $q \in \mathbb{Q}$. In order to do this, We use the following sequence of polynomials:

$$e_0(x) = 1 \qquad e_{n+1}(x) = e_n(x) + \frac{x^{n+1}}{(n+1)!}$$

where we see that for some $n \in \mathbb{N}$, $e_n(x) = e_n^x$.

3.1 $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy over a closed, positive set

Consider the range $[0, b]$ where $b \in \mathbb{Q}$ and $b > 0$. Suppose $x_0 \in [0, b]$. Now consider $\{e_n(x_0)\}_{n\in\mathbb{N}}$. We claim that $\{e_n(x_0)\}_{n\in\mathbb{N}}$ is monotone increasing.

First, consider $e_m(x_0)$ for some $m \in \mathbb{N}$. Note that

$$e_{m+1}(x_0) = e_m(x_0) + \frac{{x_0}^{n+1}}{(n+1)!}$$

Since $x_0 \in [0, b]$, and every term in the polynomial $e_m(x_0)$ has positive coefficients, then $e_m(x_0)$ is just the sum of positive terms, thus $e_m(x_0) > 0$.

Since $\frac{{x_0}^{m+1}}{(m+1)!} > 0$, and $e_{m+1}(x_0) = e_m(x_0) + \frac{{x_0}^{m+1}}{(m+1)!}$, it follows that:

$$e_m(x_0) < e_{m+1}(x_0)$$

Since m and x_0 were arbitrarily chosen, we can conclude that $\{e_n(x)\}_{n\in\mathbb{N}}$ is monotone increasing for all $x \in [0, b]$.

Now, fix some $n \in \mathbb{N}$ and suppose $x_0 \in [0, b)$. We claim that $e_n(x_0) < e_n(b)$.

Since $x_0 \in [0, b)$, then we have that $x_0 < b$. Now consider $e_n(x_0)$ and $e_n(b)$:

$$e_n(x_0) = \sum_{k=0}^{n} \frac{(x_0)^k}{k!} = 1 + x_0 + \frac{(x_0)^2}{2!} + ... + \frac{(x_0)^n}{n!}$$

$$e_n(b) = \sum_{k=0}^{n} \frac{b^k}{k!} = 1 + b + \frac{b^2}{2!} + ... + \frac{b^n}{n!}$$

Since $0 \leq x_0 < b$, then it is true that $0 \leq x_0^k < b^k$ for some $k \in [0, n]$. Multiplying both sides by $\frac{1}{k!}$, we get that:

$$0 \leq \frac{(x_0)^k}{k!} < \frac{b^k}{k!}$$

Since $\frac{(x_0)^k}{k!} < \frac{b^k}{k!}$ is true for all $k \in [0, n]$, and since $\frac{(x_0)^k}{k!}$ and $\frac{b^k}{k!}$ are non-negative for all k, then we can conclude that $e_n(x_0) < e_n(b)$, since they are simply the summation of all of the k terms.

Thus we see that:

$$(\forall n \in \mathbb{N})(\forall x_0 \in [0, b))[e_n(x_0) < e_n(b)]$$

Since n was arbitrarily chosen, we have that this is true for all $n \in \mathbb{N}$. This is important because it tells us that the limit of $\{e_n(b)\}_{n\in\mathbb{N}}$ is an upper bound for the limit of $\{e_n(x_0)\}_{n\in\mathbb{N}}$.

Since we have just shown that $\{e_n(x_0)\}_{n\in\mathbb{N}}$ is monotone increasing and bounded, then it clearly follows that $\{e_n(x_0)\}_{n\in\mathbb{N}}$ must have a limit in $\mathbb{R}$. We can define a function for this limit of $e_n(x_0)$ as n approaches infinity as follows:

$$e_\infty : [0, b] \to \mathbb{R} \qquad e_\infty(x) = \lim_{n\to\infty} e_n(x)$$

Where we say $e_\infty(x)$ is the pointwise limit of $\{e_n(x_0)\}_{n\in\mathbb{N}}$.

Now, since $\{e_n(x_0)\}_{n\in\mathbb{N}}$ has a limit in $\mathbb{R}$ for all $x_0 \in [0, b]$, then we know that:

$$(\forall x_0 \in [0, b])\,\{e_n(x_0)\}_{n\in\mathbb{N}} \textit{ is Cauchy}$$

Note that $(\forall q \in \mathbb{Q})[e_\infty(q) = \exp(q)]$.

3.2 Uniform Convergence

In order to show that our desired function Exp is continuous, we must first show that there exists $n \in \mathbb{N}$ such that $\{e_n(x)\}$ converges uniformly to $e_\infty(x)$, i.e. we want to show that:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|e_\infty(x) - e_n(x)| < \varepsilon)$$

We ultimately want to show that this is true for all $x \in \mathbb{R}$, however since we have only defined e_∞ on a domain of $[0, b]$, we will start by showing that this is true for all $x \in [0, b]$. This will follow easily with the help of the following claim:

Claim: The supremum $||e_i(x) - e_j(x)|| := \max|e_i(x) - e_j(x)| = |e_i(b) - e_j(b)|$ for all $i, j \in \mathbb{N}$.

Proof.

Suppose $i, j \in \mathbb{N}$. Since $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy and monotone increasing for all $x \in [0, b]$, it follows that:

$$If\ i < j\ and\ x_0 \in [0, b],\ then\ e_i(x_0) < e_j(x_0)$$

and from the last section, we proved that:

$$If\ x \in [0, b)\ then\ e_i(x) < e_i(b)$$

Now consider $i, j \in \mathbb{N}$. If $i = j$, then everything just equals 0, so we are done. Thus, we may assume that $i \neq j$. Without loss of generality, suppose that $i < j$. Then

$$e_i(x_0) < e_j(x_0) = e_i(x_0) + (\frac{x_0^{i+1}}{(i+1)!} + ... + \frac{x_0^j}{j!})$$

Thus,

$$\begin{aligned} |e_j(x_0) - e_i(x_0)| &= e_j(x_0) - e_i(x_0) \\ &= \frac{x_0^{i+1}}{(i+1)!} + ... + \frac{x_0^j}{j!} \end{aligned}$$

We can denote a new function $d(x)$ to represent this difference:

$$d_{ij}(x) = |e_j(x) - e_i(x)|$$

where

$$d_{ij}(x_0) = \frac{x_0^{i+1}}{(i+1)!} + ... + \frac{x_0^j}{j!}$$

Now consider $||e_i - e_j|| = max|e_i(x) - e_j(x)|$. This is equivalent to $max|d_{ij}(x_0)|$. Now since $(\forall x_0 \in [0, b))[0 \leq e_i(x_0) < e_i(b)]$, then it follows that:

$$|e_i(x_0) - e_j(x_0)| < |e_i(b) - e_j(b)|$$

for all $x_0 \in [0, b)$. Thus,

$$max|d_{ij}(x)| = |d_{ij}(b)| = d_{ij}(b) = |e_i(b) - e_j(b)|$$

We conclude that $||e_i - e_j|| = |e_i(b) - e_j(b)|$.

$\square$

Now that we have this, we now know that over the interval $[0, b]$, the largest difference between i and j will be at the point b, so everything to the left of b will have a smaller difference. Thus, if we prove that the difference at b is less than some $\varepsilon > 0$, we have shown that the difference in all points of the interval is less than ε. Speaking in more mathematical language:

Since $||e_i - e_j|| = |e_i(b) - e_j(b)|$, then in order to prove that there exists $n \in \mathbb{N}$ such that $\{e_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x)$ for all $x \in [0, b]$, we simply have to show:

$$(\forall\varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|e_\infty(b) - e_n(b)| < \varepsilon)$$

which is easy since we know that $\{e_n(b)\}_{n\in\mathbb{N}}$ is Cauchy and converges to $e_\infty(b)$. Because of this, it is clear that for any $\varepsilon > 0$, we can choose an $n \in \mathbb{N}$ such that $|e_\infty(b) - e_n(b)| < \varepsilon$.

Now we want to expand our current domain of e_∞. We want to be able to choose any $x \in \mathbb{R}$ instead of any $x \in [0, b]$. Since b was chosen arbitrarily, we see that we can expand our domain to the left as far as we want for any $b \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we know that for any real number $r \in \mathbb{R}$:

$$(\exists q \in \mathbb{Q})\ r < q$$

Thus, we can conclude that our domain of e_∞ can be expanded to the left as far was we want, so it is unbounded to the left. So, we can redefine our new version of e_∞ as follows:

$$e^+_\infty : [0, \infty] \to \mathbb{R} \qquad e^+_\infty(x) = \lim_{n\to\infty} e_n(x)$$

4 The function Exp with a negative input

Even though the polynomial is a bit different when we plug in a negative value into $e_n(x)$, we can still prove that $\{e_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x)$ using inequalities.

Start by choosing some $x \in \mathbb{R}$ such that $x > 0$. Now consider $(-x)$. We see that the polynomial expansion of $e_n(-x)$ for some $n \in \mathbb{N}$ is as follows:

$$\begin{aligned}
e_n(-x) &= \sum_{k=0}^{n} \frac{(-x)^k}{k!} \\
&= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} ... + \frac{(-x)^n}{n!} \\
&= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ... \pm \frac{x^n}{n!}
\end{aligned}$$

where the $\pm$ in the end is determined on whether n is even or odd. Here we see that plugging in a negative value into $e_n(x)$ results in a polynomial where every term alternates with a plus and minus. Because of this, it is clear that $e_n(-x)$ is not monotone increasing.

But it turns out that that is not a problem, and here's why:

4.1 Uniform convergence of $\{e_n(x)\}_{n\in\mathbb{N}}$ with a negative input

In order to show uniform convergence, we must show that there exists a pointwise limit $e_\infty(x) = \lim_{n\to\infty} e_n(x)$. But we can easily do this by expanding the domain of our current $e_\infty(x) : [0,\infty) \to \mathbb{R}$ function. This will follow by showing that $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for all $x \in \mathbb{R}$

Claim: $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for all $x \in \mathbb{R}$
Since we already know that for the positives and 0, $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy, we only need to show that it is true for the negatives. This follows easily by looking at the sequence of coefficients for $\{e_n(x)\}_{n\in\mathbb{N}}$. We can write an element a_n in the sequence of coefficients as follows:

$$a_n \in \{a_n\}_{n\in\mathbb{N}} \qquad a_n = \frac{1}{n!}$$

Now notice that $\lim_{n\to\infty} a_n = 0$. Since every k^{th} term in the polynomial $e_n(x)$ is multiplied by the k^{th} element of $\{a_n\}_{n\in\mathbb{N}}$, it follows that $\{e_n(x)\}_{n\in\mathbb{N}}$ must converge to some limit, that we can denote as $e_\infty(x)$.

Since for all $x \in \mathbb{R}$, $\{e_n(x)\}_{n\in\mathbb{N}}$ has a limit, we conclude that $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for all $x \in \mathbb{R}$.

Now, just like for the positives, we want to show that there exists $N \in \mathbb{N}$ such that $\{e_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x)$:

$$(\forall\varepsilon > 0)(\forall n > N)(|e_\infty(x) - e_n(x)| < \varepsilon)$$

To do this, we again want to compare the difference between two values in $\{e_n(x)\}_{n\in\mathbb{N}}$:

Proof.

Suppose $x_0 \in \mathbb{R}$ such that $x_0 > 0$. Consider $-x_0 \in \mathbb{R}$. Let $i, j \in \mathbb{N}$. Of course, if $i = j$, then $|e_j(-x_0) - e_i(-x_0)| = 0$ which is less than any $\varepsilon > 0$, so we are done. Thus we may assume that $i \neq j$. Without loss of generality, suppose $i < j$.

Now consider $d_{ij}(-x_0) = |e_j(-x_0) - e_i(-x_0)|$:

$$\begin{aligned}
|e_i(-x_0) - e_j(-x_0)| &= |(1 - x_0 + \frac{(x_0)^2}{2!} - ... \pm \frac{(x_0)^i}{i!} \mp ... \pm \frac{(x_0)^j}{j!}) \\
&\quad - (1 - x_0 + \frac{(x_0)^2}{2!} - ... \pm \frac{(x_0)^i}{i!})| \\
&= | \pm \frac{(x_0)^{i+1}}{(i+1)!} \mp \frac{(x_0)^{i+2}}{(i+2)!} \pm ... \pm \frac{(x_0)^j}{j!}|
\end{aligned}$$

Now, consider taking the absolute value of each coefficients in the polynomial $d_{ij}(-x_0)$:

$$\begin{aligned}
|(| \pm \frac{x_0^{i+1}}{(i+1)!}|) + (| \mp \frac{x_0^{i+2}}{(i+2)!}|) + ... + (|\frac{x_0^j}{j!}|)| &= \frac{x_0^{i+1}}{(i+1)!} + \frac{x_0^{i+2}}{(i+2)!} + ... + \frac{x_0^j}{j!} \\
&= d_{ij}(x_0) \\
&= |e_j(x_0) - e_i(x_0)|
\end{aligned}$$

Since we took the absolute value of every term, it follows that the resulting sum cannot be less than the original sum. Thus, we conclude that:

$$|e_j(-x_0) - e_i(-x_0)| \leq |e_j(x_0) - e_i(x_0)|$$

Since i and j were chosen arbitrarily, then we have that this is true for all $i, j \in \mathbb{N}$.

Now, since we already found that $\{e_n(x_0)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x_0)$, we can find $N \in \mathbb{N}$ such that

$$(\forall\varepsilon > 0)(\forall n > N)(|e_\infty(x_0) - e_n(x_0)| < \varepsilon)$$

Since $|e_j(-x_0) - e_i(-x_0)| \leq |e_j(x_0) - e_i(x_0)|$ for all $i, j \in \mathbb{N}$, then it follows that:

$$\begin{gathered} |\lim_{i\to\infty} e_i(-x_0) - e_n(-x_0)| \leq |\lim_{i\to\infty} e_i(x_0) - e_n(x_0)| \\ \Updownarrow \\ |e_\infty(-x_0) - e_n(-x_0)| \leq |e_\infty(x_0) - e_n(x_0)| \\ < \varepsilon \end{gathered}$$

for some $\varepsilon > 0$.

Therefore, we have found some $N \in \mathbb{N}$ such that:

$$(\forall\varepsilon > 0)(\forall n > N)(|e_\infty(-x_0) - e_n(-x_0)| < \varepsilon)$$

which is nothing more than saying $\{e_n(-x_0)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(-x_0)$. Since our choice of x_0 was arbitrary, we can conclude that for all $x \in \mathbb{R}$ such that $x > 0$, $\{e_n(-x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(-x)$.

□

Thus, we have defined a new function e_∞^- as follows:

$$e_\infty^- : [-\infty, 0] \to \mathbb{R} \qquad e_\infty^-(x) = \lim_{n\to\infty} e_n(x)$$

5 Putting it all together: Exp: $\mathbb{R} \to \mathbb{R}$

In Sections 3 and 4, we defined the functions e_∞^+ and e_∞^- as follows:

$$e_\infty^+ : [0, \infty] \to \mathbb{R} \qquad e_\infty^+(x) = \lim_{n\to\infty} e_n(x)$$

$$e_\infty^- : [-\infty, 0] \to \mathbb{R} \qquad e_\infty^-(x) = \lim_{n\to\infty} e_n(x)$$

Since $(\forall x \in \mathbb{R})$ $\{e_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x)$, we can combine our two functions. We define:

$$Exp : \mathbb{R} \to \mathbb{R} \qquad Exp(x) = \lim_{n\to\infty} e_n(x) = \begin{cases} e_\infty^+(x) & \text{if } x \in [0, \infty) \\ e_\infty^-(x) & \text{if } x \in (\infty, 0) \end{cases}$$

Note that $(\forall q \in \mathbb{Q})$ we have:

$$Exp(q) = \lim_{n\to\infty} e_n(q) = exp(q)$$

Now, all we have to do is show that Exp is a unique and continuous function. But this is easy since we have already done most of the hard work.

5.1 Continuity of Exp

The proof that $Exp : \mathbb{R} \to \mathbb{R}$ will easily follow once we show that $e_n(x) : \mathbb{R} \to \mathbb{R}$ is continuous. That is, we want to show that for any sub-sequence of the domain $\{x_i\} \subset \mathbb{R}$ such that $\lim_{i\to\infty} x_i = x_0$ for some $x_0 \in \mathbb{R}$, we have that $\lim_{i\to\infty} e_n(x_i) = e_n(x_0)$.

Proof.

Let $\{x_i\} \subset \mathbb{R}$ such that $\lim_{i\to\infty} x_i = x_0$. Now consider $\lim_{i\to\infty} e_n(x_i)$ for some $n \in \mathbb{N}$:

$$\begin{aligned}
\lim_{i\to\infty} e_n(x_i) &= \lim_{i\to\infty} (1 + x_i + \frac{x_i^2}{2!} + ... + \frac{x_i^n}{n!}) \\
&= \lim_{i\to\infty} 1 + \lim_{i\to\infty} x_i + \lim_{i\to\infty} \frac{x_i^2}{2!} + ... + \lim_{i\to\infty} \frac{x_i^n}{n!} \\
&= 1 + x_0 + \frac{x_0^2}{2!} + ... + \frac{x_0^n}{n!} \\
&= e_n(x_0)
\end{aligned}$$

Since our choice of $\{x_i\}$ was arbitrary, this applies to all converging subsets of $\mathbb{R}$, so therefore our definition is satisfied and we conclude that $e_n(x)$ is continuous over $\mathbb{R}$. $\square$

Now we can show that Exp is also continuous over $\mathbb{R}$, since it is simply the limit function of $e_n(x)$. Since we know that $\{e_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $e_\infty(x)$, then it follows that $Exp : \mathbb{R} \to \mathbb{R}$ must be continuous.

5.2 Exp is a unique function

We will use a proof by contradiction to show that $Exp : \mathbb{R} \to \mathbb{R}$ is a unique function.

Proof.

Suppose, for the sake of contradiction, that there exists some function f defined as follows:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \lim_{n\to\infty} e_n(x)$$

where $f \neq Exp$. Then it follows that:

$$(\exists a \in \mathbb{R}) f(a) \neq Exp(a)$$

However, since $f(a) = \lim_{n\to\infty} e_n(a)$ and $Exp(a) = \lim_{n\to\infty} e_n(a)$, then we get:

$$\lim_{n\to\infty} e_n(a) \neq \lim_{n\to\infty} e_n(a)$$

which is clearly false, giving us our desired contradiction. Thus, we conclude that $Exp : \mathbb{R} \to \mathbb{R}$ is a unique function. $\square$

6 Generalizations from observing $e_n(x)$

Now that we have gone through and rigorously checked that there exists a unique and continuous function $Exp : \mathbb{R} \to \mathbb{R}$ with the property that $Exp(x) = \lim_{n\to\infty} e_n(x)$, we can now generalize this method that we used to achieve it, applying our strategy to other sequences of polynomials with similar properties.

6.1 The general strategy

Proposition: Suppose $\{f_n(x)\}_{n\in\mathbb{N}}$ is a sequence of polynomials defined as follows:

$$f_n(0) = a_0 \qquad f_{n+1}(x) = f_n(x) + a_{n+1}x^{n+1}$$

where each a_n is an element of a sequence of positive real numbers $\{a_n\} \subset \mathbb{R}$. If $\lim_{n\to\infty} a_n = 0$, then there exists a unique, continuous function f^* defined as follows:

$$f^* : \mathbb{R} \to \mathbb{R} \qquad f^*(x) = \lim_{n\to\infty} f_n(x)$$

Proof.

The proof will follow the same strategy we used to build e_∞.

Cauchy:

Since every k^{th} term in the polynomial $f_n(x)$ is multiplied by the k^{th} element in $\{a_n\}_{n\in\mathbb{N}}$, it follows that $f_n(x)$ must converge to a pointwise limit in $\mathbb{R}$, because $\lim_{n\to\infty} a_n = 0$. We will call this limit $f^*(x)$ for all $x \in \mathbb{R}$.

We can denote this limit function as follows:

$$f^* : \mathbb{R} \to \mathbb{R} \qquad f^*(x) = \lim_{n\to\infty} f^*_n(x)$$

Note that since $\{f_n(x)\}_{n\in\mathbb{N}}$ has a limit for all x, it is Cauchy for all x.

Uniform Convergence:

Now we want to prove uniform converge of $\{f_n(x)\}_{n\in\mathbb{N}}$ to $f^*(x)$ for all $x \geq 0$. We start by looking at when $x \in [0, b]$ for some $b \in \mathbb{R}$.

Note that we will have uniform convergence if we show that:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|f^*(x) - f_n(x)| < \varepsilon)$$

Claim: The supremum $||f_j - f_i|| = max|f_j(x) - f_i(x)| = |f_j(b) - f_i(b)|$.

Now, if $i = j$, then everything is 0, so clearly it is all equal. Thus, without loss of generality, suppose $i < j$. Let $x_0 \in [0, b)$. Then

$$f_i(x_0) < f_j(x_0) = f_i(x_0) + (a_{i+1}x_0^{i+1} + ... + a_j x_0^j)$$

Thus,

$$\begin{aligned} |f_j(x_0) - f_i(x_0)| &= f_j(x_0) - f_i(x_0) \\ &= a_{i+1}x_0^{i+1} + ... + a_j x_0^j \end{aligned}$$

We can denote a new function $g(x) = |f_j(x) - f_i(x)|$, where $g(x_0) = a_{i+1}x_0^{i+1} + ... + a_j x_0^j$.

Now consider $||f_j - f_i|| = max|f_j(x) - f_i(x)|$. This is equivalent to $max|g(x_0)|$. Now note that since $x_0 < b$, $f_n(x_0) < f_n(b)$ for all $n \in \mathbb{N}$. This is because $x_0^n < b^n$ for all $n \in \mathbb{N}$. Thus, it follows that $|f_j(b) - f_i(b)| > |f_j(x_0)) - f_i(x_0)|$ for all $x_0 \in [0, b)$. Therefore,

$$max|g(x)| = |g(b)| = g(b) = |f_j(b) - f_i(b)|$$

We conclude that $||f_j - f_i|| = |f_j(b) - f_i(b)|$.

Now that we know this, proving uniform convergence is easy. Since we know $\{f_n(b)\}_{n\in\mathbb{N}}$ is Cauchy and converges to $f^*(b)$, it follows that for any $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that:

$$(\forall n > N)(|f^*(b) - f_n(b)| < \varepsilon)$$

and since $|f_i(b) - f_j(b)| > |f_i(x_0)) - f_j(x_0)|$ for all $x_0 \in [0, b)$, it follows that all other differences will be less than epsilon too. Thus, we conclude that $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $f^*(x)$ for all $x \in [0, b]$.

Since b was arbitrarily chosen, we conclude that $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $f^*(x)$ for all $x \in [0, \infty)$. Or rather, we have built a function

$$f^* : [0, \infty) \to \mathbb{R} \qquad f^*(x) = \lim_{n\to\infty} f_n(x)$$

Now for the negatives, we see that for all $x_0 \in \mathbb{R}$ such that $x_0 > 0$, we have that for $i, j \in \mathbb{N}$ with $i < j$:

$$|f_j(-x_0) - f_i(-x_0)| = \pm a_{i+1}x_0^{i+1} \mp a_{i+2}x_0^{i+2} \pm ... \pm a_j x_0^j$$

which is an alternating series. However, taking the absolute value of each of the coefficients gives us:

$$\begin{aligned} |\pm a_{i+1}x_0^{i+1}| + |\mp a_{i+2}x_0^{i+2}| + ... + |\pm a_j x_0^j| &= a_{i+1}x_0^{i+1} + a_{i+2}x_0^{i+2} + ... + a_j x_0^j \\ &= |a_{i+1}x_0^{i+1} + a_{i+2}x_0^{i+2} + ... + a_j x_0^j| \\ &= |f_j(x_0) - f_i(x_0)| \end{aligned}$$

Now, note that since we took the absolute value of every term in $f_n(-x_0)$ and it gave us $f_n(x_0)$, it follows that $|f_j(-x_0) - f_i(-x_0)| < |f_j(x_0) - f_i(x_0)|$ for all $x > 0$.

Since we know that $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to $f^*(x_0)$, we can find $N \in \mathbb{N}$ such that

$$(\forall\varepsilon > 0)(\forall n > N)(|f^*(x_0) - f_n(x_0)| < \varepsilon)$$

Since $|f_j(-x_0) - f_i(-x_0)| \le |f_j(x_0) - f_i(x_0)|$ for all $i, j \in \mathbb{N}$, then we have that:

$$\begin{aligned} |\lim_{i\to\infty} f_i(-x_0) - f_n(-x_0)| &\le |\lim_{i\to\infty} f_i(x_0) - f_n(x_0)| \\ &\Updownarrow \\ |f_\infty(-x_0) - f_n(-x_0)| &\le |f_\infty(x_0) - f_n(x_0)| \\ &< \varepsilon \end{aligned}$$

for all $\varepsilon > 0$.

Therefore, we have found some $N \in \mathbb{N}$ such that:

$$(\forall\varepsilon > 0)(\forall n > N)(|f_\infty(-x_0) - f_n(-x_0)| < \varepsilon)$$

which is nothing more than saying $\{f_n(-x_0)\}_{n\in\mathbb{N}}$ converges uniformly to $f_\infty(-x_0)$. Since our choice of x_0 was arbitrary, we can conclude that for all $x \in \mathbb{R}$ such that $x > 0$, $\{f_n(-x)\}_{n\in\mathbb{N}}$ converges uniformly to $f_\infty(-x)$.

Thus, we have built a function f^* with the following properties:

$$f^* : \mathbb{R} \to \mathbb{R} \qquad f^*(x) = \lim_{n\to\infty} f_n(x)$$

Note that continuity comes easily from uniform convergence, and uniqueness comes easily because f^* maps from $\mathbb{R}$ to $\mathbb{R}$. Please see section 5 for more details.

And with that, we are done; we have built a unique and continuous function $f^* : \mathbb{R} \to \mathbb{R}$ with the property that $f^*(x) = \lim_{n\to\infty} f_n(x)$. $\square$

6.2 Checking our result

Now that we have this proposition, we want to make sure that it checks out with our example function of *Exp*. Here, we see that $\{e_n(x)\}_{n\in\mathbb{N}}$ is a sequence of polynomials where:

$$e_n(0) = 1 \qquad e_{n+1}(x) = e_n(x) + \frac{x^{n+1}}{(n+1)!}$$

where our sequence of coefficients is indeed a sequence of positive real numbers, let's denote it $\{a_n\}_{n\in\mathbb{N}}$ where for some $n \in \mathbb{N}$, we have that $a_n = \frac{1}{n!}$. Clearly, $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and it is easy to see that $\lim_{n\to\infty} a_n = 0$. Thus, we have satisfied all of the conditions of the proposition.

The proposition then claims that there exists a unique, continuous function f^* defined as follows:

$$f^* : \mathbb{R} \to \mathbb{R} \qquad f^*(x) = \lim_{n\to\infty} f_n(x)$$

This is, of course, exactly what we were able to find throughout this essay, with our function *Exp* satisfying this description. And the proof that *Exp* has these properties follows very closely to the proof of the proposition.

Thus, we conclude that this proposition holds for our example of $\{e_n(x)\}_{n\in\mathbb{N}}$.

7 Regrets

I would like to briefly discuss some more properties of the exponential function that I thought were interesting, but couldn't add to this essay because it would have strayed away from the main idea of this paper which was to find continuous limit functions for sequences of polynomials.

First, I would like to point out that once one has constructed the complex numbers, $\mathbb{C}$, it can be noted that our function $Exp : \mathbb{R} \to \mathbb{R}$ can be easily expanded to being a function $Exp_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, using a very similar argument to that for $\mathbb{R}$.

Once we have $Exp_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, then the derivation of Euler's formula practically falls in our laps. This is because we can plug in some ib into $Exp_{\mathbb{C}}$ where $i = \sqrt{-1}$ and $b \in \mathbb{R}$. Once one simply expands out the polynomial expansion for $Exp_{\mathbb{C}}(ib)$, then after some algebraic manipulation, one can arrive at the conclusion that:

$$Exp_{\mathbb{C}}(ib) = \lim_{n\to\infty} e_n(ib) = \cos(b) + i\sin(b)$$

which is Euler's formula.

With some more simple algebraic manipulation, one can even conclude the famous trigonometric identity:

$$cos^2 z + \sin^2 z = 1$$

where $z \in \mathbb{C}$.

Thus, we see that once we lay the groundwork for the exponential function, rigorously proving it's continuity, we are rewarded with numerous useful equations that help lay the foundation of trigonometry.

8 Conclusion

In this essay, we were able to find a unique and continuous function $Exp : \mathbb{R} \to \mathbb{R}$ with the property that

$$Exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

by simply starting with a sequence of polynomials $\{e_n(x)\}_{n\in\mathbb{N}}$. We did this by showing that $\{e_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for all $x \in \mathbb{R}$, which helped us prove that $\{e_n(x)\}_{n\in\mathbb{N}}$ uniformly converges to a limit we defined as $e_\infty(x)$. Uniqueness and continuity quickly followed from these results, which ultimately gave us our desired function.

We also showed that this strategy for finding a unique and continuous limit function for $\{e_n(x)\}_{n\in\mathbb{N}}$ can be applied to any sequence of polynomials with properties similar to those of $\{e_n(x)\}_{n\in\mathbb{N}}$.

From this essay, we have come out having learned more not just about the exponential function, but also about sequences of polynomials with continuous limit functions. Hopefully arguments from this paper can be expanded upon in order to learn more about different types of sequences of polynomials with continuous limit functions.

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