Homework 3

Binomial Trees

I. Basics

(a) Derivation of (Eq.13.2 and Eq.13.3) (15%)

In the binomial model, suppose that the initial stock price is S_0, and the life of the option is T. S_0 can either move up from S_0 to a new level, $S_0 u$, where $u > 1$, or down to a new level, $S_0 d$, where $0 < d < 1$. Suppose the payoff from option is f_u in the up state, and is f_d in the down state. Denote the risk-free rate by r.

Please construct a riskless portfolio in a one-step tree and show **<u>in detail</u>** that $f = e^{-rT}[pf_u + (1-p)f_d]$ where $p = \dfrac{e^{rT} - d}{u - d}$

(b) (10%) End-of-Chapter exercise 21.7.

II. Computing Option Prices Using Binomial Model

Consider a non-dividend-paying stock with current stock price S_0 =\$50, volatility σ =0.3, strike price K=\$52, time to maturity T=2 years, interest rate r=5%.

Please use binomial model to price European put options. You may refer to the materials on page 475 of the textbook. Consider the following three alternative settings of time steps: Δt =1 month (12*T steps); 1 week (52*T steps); and 1 day (252*T steps).

(a) (5%) First compute the up step size *u*, the down step size *d*, and the probability of up move *p* under these three settings.

(b) (30%) Use binomial model to compute the put option prices under these three settings. Report your results and compare them with that of the Black-Scholes formula. Briefly explain your findings.

(c) (10%) Change the number of time steps from 1 to 2 to 3 all the way to 252. Plot your results as well as the Black-Scholes closed form solution. Briefly explain your findings.

(d) (10%) For 6, 12, and 52 time steps, compute the terminal stock prices as well as their corresponding probabilities. Plot the terminal stock price distribution. Briefly explain your findings.

(e) (20%) Modify your program in (b) to compute the American put option values. Report your result.

Matlab function and syntax:

1. To plot terminal stock distribution, you may use matlab function plot().

    e.g.

    ```
    plot(ST,Prob,'-o');
    ```

    where ST is a vector of terminal stock prices and Prob is a vector of their corresponding probabilities. '-o' is the LineSpec option that specifies the line type, marker symbol and color: LineStyle is Solid line ('-') and marker type is circle ('o').

2. nchoosek(): Binomial coefficient or all combinations

*You have to submit your homework and **programs by e3**. Your computer program is part of this assignment.

1、Basics

(a) S_0 = initial stock price

T = the life of the option

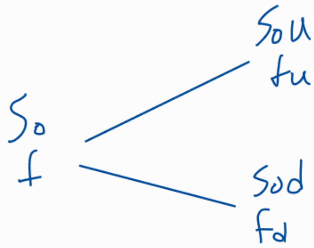
S_0u = move up from S_0 , $u>1$

S_0d = down to a new level , $0<d<1$

f_u = up state

f_d = down state

r = risk-free rate



Δ的推導：

$$S_0u\Delta - f_u = S_0d\Delta - f_d$$

$$S_0u\Delta - S_0d\Delta = f_u - f_d$$

$$\Delta(S_0u - S_0d) = f_u - f_d$$

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Value of the portfolio today

$\Rightarrow (S_0 u\Delta - f_u)e^{-rT}$

Another portfolio today's portfolio

$\Rightarrow S_0\Delta - f$

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT}$$

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

$$f = \cancel{S_0}\frac{f_u - f_d}{\cancel{S_0}u - \cancel{S_0}d} - \left(\cancel{S_0}u\frac{f_u - f_d}{\cancel{S_0}u - \cancel{S_0}d} - f_u\right)e^{-rT}$$

$$= \frac{f_u - f_d}{u-d} - \left(u\frac{f_u - f_d}{u-d} - f_u\right)e^{-rT}$$

$$= \left(\frac{f_u - f_d}{u-d}e^{rT} - \frac{uf_u - uf_d}{u-d} + \frac{f_u(u-d)}{u-d}\right)e^{-rT}$$

$$= e^{-rT}\left(\frac{(f_u - f_d)e^{rT} - \cancel{uf_u} + uf_d + \cancel{uf_u} - df_u}{u-d}\right)$$

$$= e^{-rT}\left(\frac{(f_u - f_d)e^{rT} + uf_d - df_u}{u-d}\right)$$

$$= e^{-rT}\left(\frac{f_u e^{rT} - f_d e^{rT} + uf_d - df_u}{u-d}\right)$$

$$= e^{-rT}\left(\frac{f_u e^{rT} - df_u}{u-d} + \frac{-f_d e^{rT} + uf_d}{u-d}\right)$$

$$= e^{-rT}\left(\frac{e^{rT} - d}{u-d}f_u + \left(\frac{-e^{rT} + u}{u-d}\right)f_d\right)$$

$$= e^{-rT}\left(\frac{e^{rT} - d}{u-d}f_u + \left(1 - \frac{e^{rT} - d}{u-d}\right)f_d\right)$$

$$= e^{-rT}\left[p f_u + (1-p) f_d\right] \text{ where } p = \frac{e^{rT} - d}{u-d} \ \#$$

$$p = \frac{e^{rT} - d}{u-d}$$

$$1 - p = \frac{(u - \cancel{d}) - e^{rT} + \cancel{d}}{u-d}$$

$$= \frac{u - e^{rT}}{u-d}$$

(b) 21.7.

CRR model之機率可表示如下：

$$p = \frac{a-d}{u-d}, \quad 1-p = \frac{u-a}{u-d}$$

則：$d > a$ or $a > u$，機率為負

$$\Rightarrow e^{-\sigma\sqrt{\Delta t}} > e^{(r-q)\Delta t} \text{ or } e^{(r-q)\Delta t} > e^{\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \sigma\sqrt{\Delta t} < (q-r)\Delta t \text{ or } (r-q)\Delta t > \sigma\sqrt{\Delta t}$$

$$\Rightarrow \sigma < (q-r)\sqrt{\Delta t} \text{ or } (r-q)\sqrt{\Delta t} > \sigma$$

$$\Rightarrow \sigma < |(r-q)\sqrt{\Delta t}| \ \#$$

期貨與選擇權 – 作業三

311707006 汪文豪

II. Computing Option Prices Using Binomial Model

Consider a non-dividend-paying stock with current stock price S_0 =\$50, volatility σ=0.3, strike price *K*=\$52, time to maturity *T*=2 years, interest rate *r*=5%.

Please use binomial model to price European put options. You may refer to the materials on page 475 of the textbook. Consider the following three alternative settings of time steps: Δt =1 month (12*T steps); 1 week (52*T steps); and 1 day (252*T steps).

(a) (5%) First compute the up step size *u*, the down step size *d*, and the probability of up move *p* under these three settings.

dt = T/NT (根據題目設定，會有三種結果。T = 2; NT = 12, 52, 252)

(根據計算dt = 0.1667, 0.0385, 0.0079 三種結果)

u = exp(sigma*sqrt(dt))

d = 1/u

a = exp(r*dt)

p = (a-d)/(u-d)

依據上述公式以及題目給予的初始值設定計算結果如下

1. dt = 0.1667:

u = 1.1303

d = 0.8847

a = 1.0084

p = 0.5035

2. dt = 0.0385:

u = 1.0606

d = 0.9429

a = 1.0019

p = 0.5016

3. dt = 0.0079:

u = 1.0271

d = 0.9736

a = 1.0004

p = 0.5007

(b) (30%) Use binomial model to compute the put option prices under these three settings. Report your results and compare them with that of the Black-Scholes formula. Briefly explain your findings.

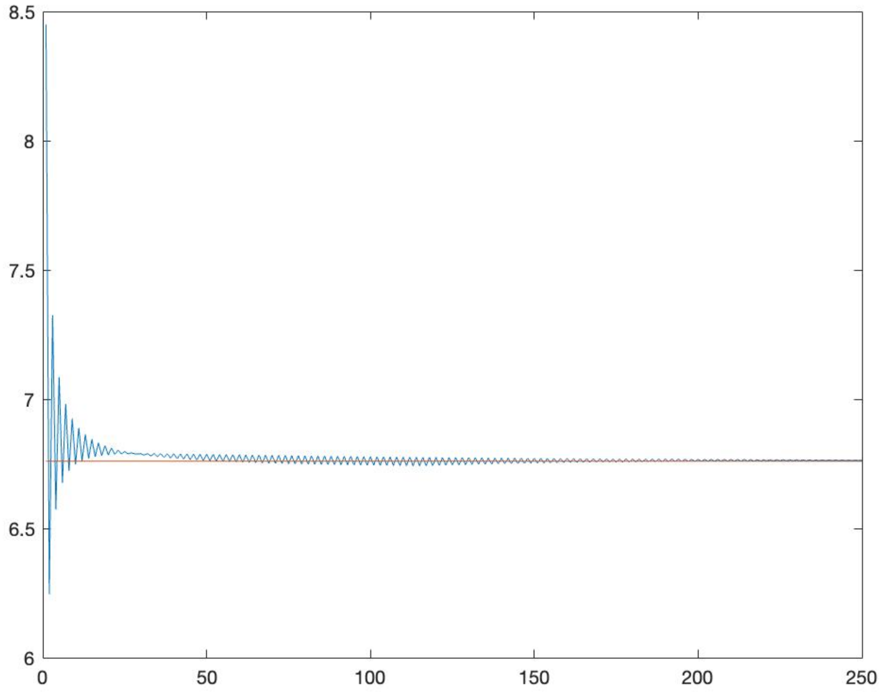
```
-----NT == 12-----
Option value by binomial: 6.761719
Option value by BS: 6.760140
-----NT == 52-----
Option value by binomial: 6.786838
Option value by BS: 6.760140
-----NT == 252-----
Option value by binomial: 6.763850
Option value by BS: 6.760140 >>
```

HW3_311707006_b.m 執行結果

Explain:

從這三項設定中可看出根據不同的Number of time steps之設定會有不同的結果，但光從目前這三項之設定還看不出是否會逐漸收斂於BS公式之結果。

(c) (10%) Change the number of time steps from 1 to 2 to 3 all the way to 252. Plot your results as well as the Black-Scholes closed form solution. Briefly explain your findings.

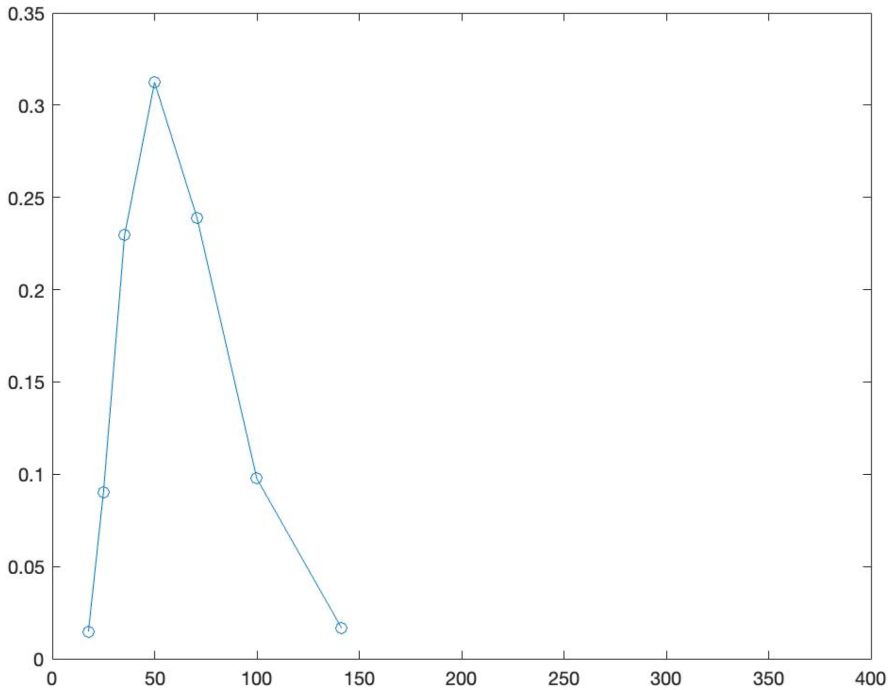


HW3_311707006_c.m 執行結果

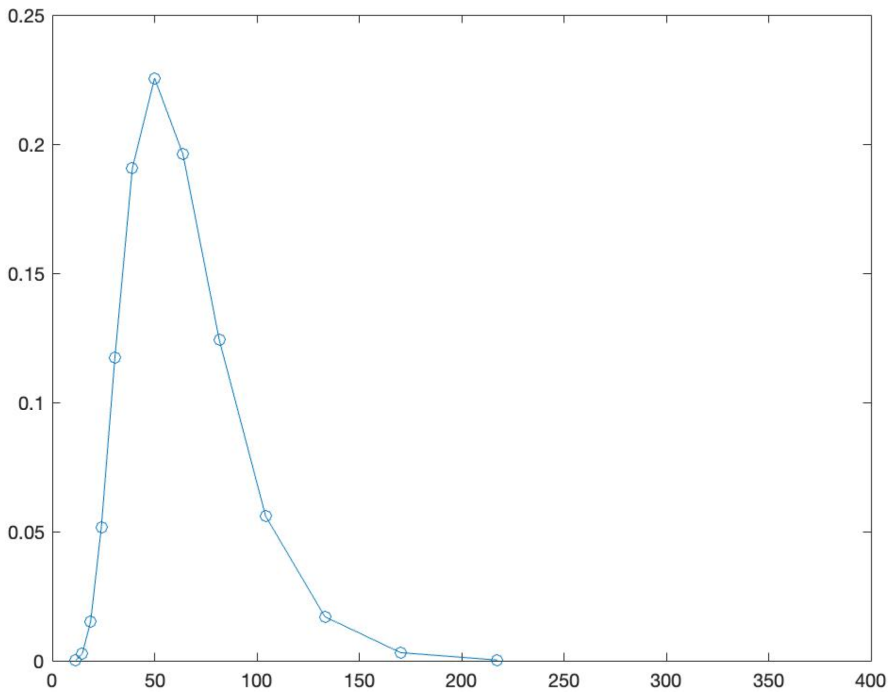
Explain:

由此圖可看出，當Number of time steps越大時，其結果會愈趨穩定且收斂於其**BS**公式所計算出來之結果。

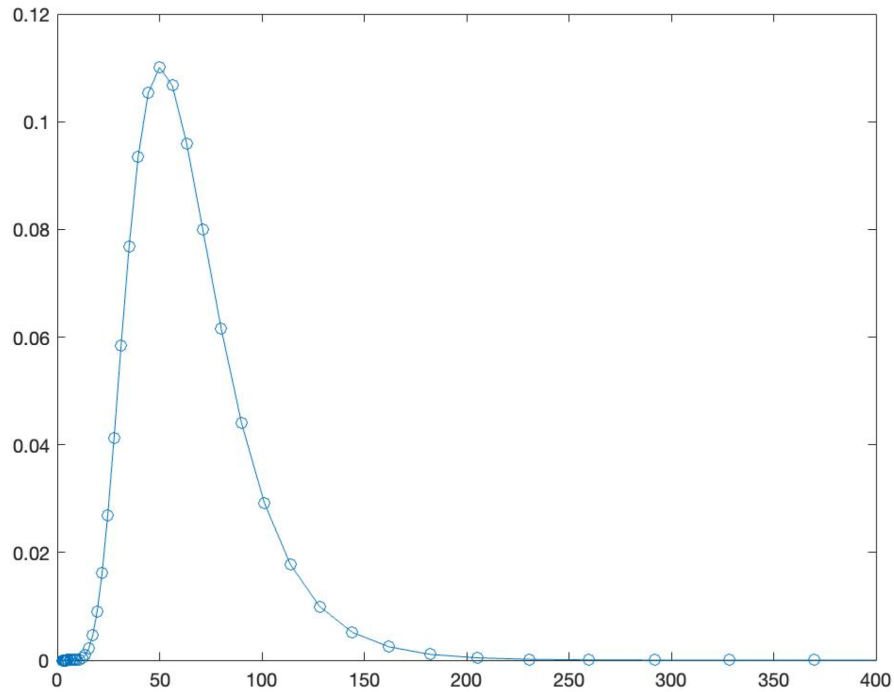
(d) (10%) For 6, 12, and 52 time steps, compute the terminal stock prices as well as their corresponding probabilities. Plot the terminal stock price distribution. Briefly explain your findings.



HW3_311707006_d_6.m 執行結果 (6 time steps)



HW3_311707006_d_12.m 執行結果 (12 time steps)



HW3_311707006_d_52.m 執行結果 (52 time steps)

Explain:

當time steps愈大時，將其各點做連線時，就可以看出其機率分佈將會趨近於log normal distribution.

(e) (20%) Modify your program in (b) to compute the American put option values. Report your result.

```
-----NT == 12-----
Option value by binomial: 7.518059
Option value by BS: 6.760140
-----NT == 52-----
Option value by binomial: 7.495838
Option value by BS: 6.760140
-----NT == 252-----
Option value by binomial: 7.474798
Option value by BS: 6.760140 >>
```

HW3_311707006_e.m 執行結果

Explain:

與HW3_311707006_b.m 執行結果相比，可看出美式賣權價值較歐式賣權高一些。