期貨與選擇權 – 作業四

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I. Basics (15%) Derivation

End-of-Chapter exercise 15.15在第三頁

II. Hedging Performance (55%)

(i) (40%) Delta Hedging
Please use the settings in Table 19.2 and Table 19.3 to simulate performance of delta hedging. The performance measure is the ratio of the standard deviation of the cost of hedging the option to the theoretical price the option. In this problem, please use the Black-Scholes model to compute the theoretical price of the option. Please set the number of trials (sample paths of stock prices) as 1000. Change the time between hedge rebalancing and try to duplicate similar results as those in Table 19.4.

```
Delta Hedging
#############
Rebalacing (weeks) : 0.25, 0.5 ,  1  ,  2  ,  4  ,  5
----------------------------------------------------------
Performance measure: 0.10, 0.14, 0.20, 0.28, 0.36, 0.40
```

圖一、**HW4_311707006_1.m**執行結果

(ii) (15%) Stop-loss Strategy
Please use the same settings to simulate the performance of stop-loss strategy. Change the time between hedge rebalancing and try to duplicate results similar to as those in Table 19.1.

```
Stop loss
#########
Rebalacing (weeks) : 0.25, 0.5 ,  1  ,  2  ,  4  ,  5
----------------------------------------------------------
Performance measure: 0.80, 0.78, 0.81, 0.87, 0.96, 1.05
```

圖二、**HW4_311707006_1.m**執行結果

III. Greek Letters (30%)

Consider a non-dividend-paying stock with current stock price S_0 =\$50, volatility σ =0.3, strike price K=\$52, time to maturity T=2 years, interest rate r=5%.

(i) (10%) Please use the closed-form solutions from the Black-Scholes model to compute the Greek letters of the European put option, including delta, theta, gamma, vega and rho. You may find the closed-form solutions in Table 19.6.

```
Delta: -0.361149
Gamma: 0.017655
Theta: -0.745354
Vega : 26.483105
Rho  : -49.635146
```

圖三、HW4_311707006_GreekLetters_1.m執行結果

(ii) (20%) Please use binomial model to compute the Greek letters of this European put options. Please set the time step as 1 day, i.e., Δt =1 day (252*T steps). While computing vega and rho, please set the change of σ and r as one basis point (0.01%).

```
Delta: -0.361256
Gamma: 0.017682
Theta: -0.748159
Vega : 0.002637
Rho  : -0.004963
```

圖四、HW4_311707006_GreekLetters_2.m執行結果

I. 15.15

(a) what is $N'(X)$?

$$N(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} e^{\frac{-a^2}{2}} da \quad a \text{ stun}$$

$$N'(X) = \frac{dN(x)}{dX} = \frac{1}{\sqrt{2\pi}} \left(e^{\frac{-X^2}{2}} - 0\right) = \frac{1}{\sqrt{2\pi}} e^{\frac{-X^2}{2}} \#$$

(b) $\dfrac{N'(d_1)}{N'(d_2)} = e^{-\frac{1}{2}(d_1^2 - d_2^2)}$, by (a)

$$\Rightarrow d_1^2 - d_2^2 = (d_1 + d_2)(d_1 - d_2) = \frac{2\ln(S/K) + 2r(T-t)}{\sigma\sqrt{T-t}} \times \frac{\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$= 2\left(\ln\left(\frac{S}{K}\right) + r(T-t)\right)$$

$$\Rightarrow e^{\frac{-1}{2}(d_1^2 - d_2^2)} = e^{\ln\left(\frac{K}{S}\right) - r(T-t)} = \frac{K}{S} \cdot e^{-r(T-t)}$$

$$\Rightarrow \frac{N'(d_1)}{N'(d_2)} = \frac{K}{S} \cdot e^{-r(T-t)}$$

$$\Rightarrow S \cdot N'(d_1) = K \cdot e^{-r(T-t)} \cdot N'(d_2) \#$$

(C) $$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{S}{K}\right) + \frac{(r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{T-t}} \left(\frac{1}{K} \cdot \frac{K}{S}\right) = \frac{1}{S \cdot \sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S \cdot \sigma\sqrt{T-t}} \#$$

(d) $C = S \cdot N(d_1) - K \cdot e^{-r(T-t)} \cdot N(d_2)$

$$\frac{\partial N(X)}{\partial t} = \frac{\partial N(X)}{\partial X} \cdot \frac{\partial X}{\partial t} \Rightarrow \frac{\partial N(d_1)}{\partial t} = N'(d_1) \cdot \frac{\partial d_1}{\partial t}$$

$$\Rightarrow \frac{\partial N(d_2)}{\partial t} = N'(d_2) \cdot \frac{\partial (d_1 - \sigma\sqrt{T-t})}{\partial t}$$

$$= N'(d_2) \cdot \left(\frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}\right)$$

$$\frac{\partial C}{\partial t} = -rKe^{-r(T-t)} \cdot N(d_2) + S \cdot \frac{\partial N(d_1)}{\partial t} - Ke^{-r(T-t)} \cdot \frac{\partial N(d_2)}{\partial t}$$

$$= -rKe^{-r(T-t)} \cdot N(d_2) + \underbrace{\left(S \cdot N'(d_1) \cdot \frac{\partial d_1}{\partial t} - Ke^{-r(T-t)} \cdot N'(d_2) \cdot \frac{\partial d_1}{\partial t}\right)}_{\hookrightarrow 0 \text{ by } b} - \underbrace{Ke^{-r(T-t)} \cdot N'(d_2)}_{\hookrightarrow S \cdot N'(d_1) \text{ by } b} \cdot \frac{\sigma}{2\sqrt{T-t}}$$

$$= -rK e^{-r(T-t)} \cdot N(d_2) - S \cdot N'(d_1) \cdot \frac{\sigma}{2\sqrt{T-t}} \ \#$$

e. $C = S \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2)$

$$\frac{\partial N(X)}{\partial S} = \frac{\partial N(X)}{\partial X} \cdot \frac{\partial X}{\partial S} \Rightarrow \frac{\partial N(d_1)}{\partial S} = N'(d_1) \cdot \frac{\partial d_1}{\partial S}$$

$$\Rightarrow \frac{\partial N(d_2)}{\partial S} = N'(d_2) \cdot \frac{\partial d_2}{\partial S} = N'(d_2) \cdot \frac{\partial d_1}{\partial S}$$

$$\frac{\partial C}{\partial S} = N(d_1) + S \cdot \frac{\partial N(d_1)}{\partial S} - Ke^{-r(T-t)} \cdot \frac{\partial N(d_2)}{\partial S}$$

$$= N(d_1) + \frac{\partial d_1}{\partial S} \underbrace{\left(S \cdot N'(d_1) - Ke^{-r(T-t)} \cdot N'(d_2)\right)}_{\text{by } b, \ \hookrightarrow 0}$$

$$= N(d_1) \ \#$$

f、BSM differential equation for f

$$\Rightarrow \frac{\partial f}{\partial t} + rS \cdot \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

用c带入，$c = S \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2)$

$$\Rightarrow \frac{\partial c}{\partial t} + rS \cdot \frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \cdot \frac{\partial^2 c}{\partial S^2}$$

$$= -rKe^{-r(T-t)} \cdot N(d_2) - S \cdot N'(d_1) \cdot \frac{\sigma}{2\sqrt{T-t}} + rS \cdot N(d_1) + \frac{1}{2}\sigma^2 S^2 \cdot N'(d_1) \cdot \frac{1}{S \cdot \sigma \cdot \sqrt{T-t}}$$

$$= r(\underbrace{S \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2)}_{c}) \underbrace{- S \cdot N'(d_1) \cdot \frac{\sigma}{2\sqrt{T-t}} + \frac{1}{2}\sigma S \cdot N'(d_1) \cdot \frac{1}{\sqrt{T-t}}}_{0}$$

$= rc + 0 = rc$，符合 BSM differential equation 结果 #

g、 2 cases

(1) If $S \geq K$

$d_1, d_2 \Rightarrow \infty$ as $t \to T$

$\Rightarrow N(d_1) = N(d_2) = 1 \Rightarrow c = S - K$

(2) If $S < K$

$d_1, d_2 \Rightarrow -\infty$ as $t \to T$

$\Rightarrow N(d_1) = N(d_2) = 0 \Rightarrow c = 0$

由2点可得證 c满足 European call 的 boundary condition #