Graduate Mathematical Analysis Notes

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Contents

Preface

In this course, two textbooks will be used as references:

- **Mathematical Analysis** by Tom M. Apostol,
- **Understanding Analysis** by Stephen Abbott.

Any references to sections or exercises will be from these books unless otherwise specified. Most of the content will be based on Abbott, but Apostol has some topics Abbott lacks.

Also, note, I'm an undergraduate student. I like to think I'm smart, but I make a lot of mistakes.

If you catch one, let me know. You can make it fun by proving me wrong, I love learning.

Also: Please note, GitHub copilot was used in this project. It's built into VSCode now, and it was used here and there for generating figures, since I don't know what I'm doing with Tikz and PGFPlots. It did try to generate theorems and proofs, but I didn't accept those edits. My definitions, theorems, and proofs are structured the way we did them in class, or by how Abbott/Apostol/I would do them.

Part 1

The Real Number System & Set Theory

Book References:
Apostol: 1.1 – 1.19, 2.1 – 2.15
Abbott: 1.1 – 1.7

In this section, we will cover:

- The Real Number System
- Bounds
- Functions
- Set Theory
- Cardinality

1.1 Topic 1

This is the content for Topic 1.

Theorem: Sample Theorem

This is a sample theorem in Topic 1.

Proof. Yes. ■

This concludes part 1 of Mathematical Analysis.

Part 6

Integration & The Fundamental Theorem of Calculus

Book References:
Abbott: Chapter 7

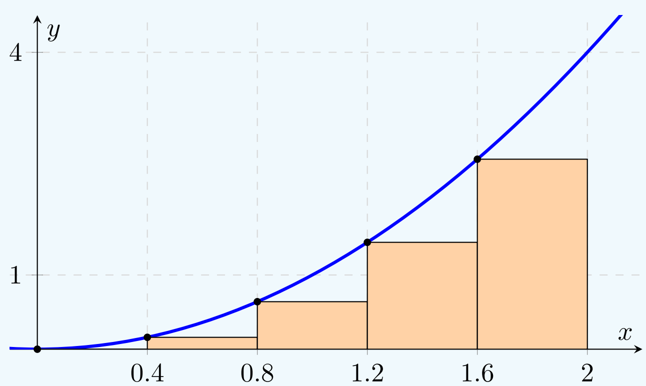
In this section, we will cover:

- The Reimann Integral
- Criteria for Integability
- Properties of the Integral
- The Fundamental Theorem of Calculus

6.1 The Reimann Integral

This is really the regular integral from Calculus! We'll end up making it a bit more rigorous, but it's going to work the way that you'd hope. First, let's start with a few examples from Calculus class.

Example:



Using really basic calculus, we can definitely integrate it. This is a Continuous curve, so it's integrable. We can use rectangles to compute the Riemann sum of this.
If we had a constant function, the Riemann sum wouldn't be an approximation, but exactly the integral.

In Analysis, though, we don't require continuity for integrability. Let's examine a fun function.

Example: Dirichlet Function

The Dirichlet function is defined as:

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

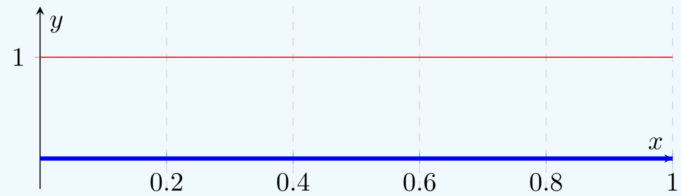
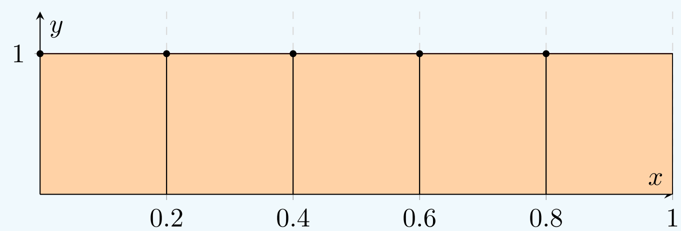


Figure. $D(x) = 1$ for $x \in \mathbb{Q}$ (thin red line) and $D(x) = 0$ for $x \notin \mathbb{Q}$ (bold blue line).

Let's try integrating it the calculus way.
We'd have to pick our endpoints, first. Let's have our rectangles start from the left, at 0. Let's pick 5 or so subintervals.



I mean, clearly,

$$\int_0^1 D(x)dx = 1 \tag{6.1}$$

Well, what if we change up the endpoints? It's entirely possible to pick irrationals for our endpoints, which lay on 0.
So, then,

$$\int_0^1 D(x)dx = 0. \tag{6.2}$$

We clearly have some disagreement here, and have another great example of the failure of Calculus classes.

The above example is great motivation for us to create a much better definition of an integral.

6.2 Criteria for Integrability

Definition: Integability

Let f be a bounded function on $[a,b]$. Create a partition P of $[a,b]$ such that

$$P = \{a = x_0, x_1, x_2, \ldots, x_n = b : x_0 < x_1 < \ldots < x_n\}$$

Which has a handy shorthand, $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$. Create the upper sum, $U(f,P)$,

$$U = \sum_{k=1}^{n} M_k \Delta x_k$$

where $M_k = \sup f(x) : x \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$.
And the lower sum, $L(f,P)$,

$$L = \sum_{k=1}^{n} m_k \Delta x_k$$

where $m_k = \inf f(x) : x \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$.
Define the upper integral as

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

Define the lower integral as

$$\underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

To say that f is integrable on $[a,b]$ means that $U(f) = L(f)$, and

$$\int_a^b f(x)dx = U(f) = L(f)$$

.

Wow, what a mouthful.

Let's talk about this in a slightly different register.

To say that a function is integrable means the lower and upper sums are going to the same number. That number is the value of the integral.

We may be able to think about this as a convergence (Which we can! We will prove this later!)

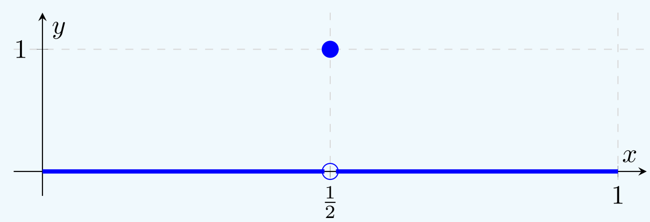
Note: Notice that this definition doesn't mention continuity, or antiderivatives.
Calculus class kind of gives you the idea that integration = antidifferentiation, but those are two separate concepts. We will make a connection between them later, with FTC, but they are still not the same thing.

Let's do a quick example.

Example: Line with Discontinuity Consider the function:

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2} \end{cases}$$

on $[0, 1]$.



Let's integrate this with our new technique.
Let's pick a partition, $P = \{0, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, 1\}$ where $\epsilon > 0$.
So, $U(f, P) = 0 \cdot 0 + |\frac{1}{2} + \epsilon - (\frac{1}{2} - \epsilon)| \cdot 1 + 0 \cdot 0 = 2\epsilon$.
And, $L(f, P) = 0 \cdot 0 + |\frac{1}{2} + \epsilon - (\frac{1}{2} - \epsilon)| \cdot 0 + 0 \cdot 0 = 0$.
Of course, $2\epsilon \neq 0$. But again, if we think about this as a limit, we can say ϵ is arbitrarily small, so this goes to 0.

Let's build upon this example with a theorem.

Theorem: ϵ-Criterion for Integrability

Let f be bounded on $[a, b]$.
Then, $\int_a^b f(x)dx$ exists $\iff$ for all $\epsilon > 0$, there is a partition P_ϵ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

This wasn't proven in class, so I'm not going to try to prove it here.
It's not difficult to prove, it relies heavily on the definitions.
Another integration theorem, it's very handy. It's not something we'll prove.

Theorem: Lebesgue's Theorem

Suppose f is bounded on $[a, b]$.
Then, $\int_a^b f(x)dx$ exists if and only if the set of points where f is discontinuous has measure zero.

What's measure zero, though? Not very helpful.

We can think about it like, if we remove all of the discontinuities, we still have basically the same set.

If we consider the above example, the function with the whole that goes up at a single point, by plucking out that one point we don't really change anything.

Here's a better definition, though, for measure zero.

Definition: Measure Zero

To say that $A \subseteq \mathbb{R}$ has measure zero means that
For all $\epsilon > 0$, there exists a countable collection of open intervals,

$$\{O_i : i \in I\}$$

Such that,

1. $A \subseteq \bigcup_{i \in I} O_i$,
2. $\sum_{i \in I} \text{length}(O_i) < \epsilon$.

So, if we can remove the discontinuities, and those discontinuities have a 1:1 correspondence with the naturals, we're good to integrate.

6.3 The Fundamental Theorem of Calculus

This is probably my favorite section of any analysis class.

The fundamental theorem of calculus connects the two different topics of integration, differentiation, and anti-differentiation.

Recall our definition of the integral. It has absolutely nothing to do with antiderivatives. Yet, Calculus Class seems to give us the idea that the two are connected, and that the definition of integral uses the antiderivative.

I seriously cannot stress enough how wrong that is. Calculus Class needs to do better at making the two distinct. Renaming the antiderivative to the "indefinite integral" was a huge mistake, and I will personally never use it.

Okay, without any further ado, here's the FTC.

Theorem: Fundamental Theorem of Calculus Part 1

Suppose f is integrable on $[a, b]$, and that $F'(x) = f(x)$ on $[a, b]$.
Then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The proof for this isn't the craziest thing in the world.

Dr. Shipman specifically says, this is "a good one to know...". It could absolutely be on **any** exam.

Proof. Let P be a partition of $[a, b]$.

On each subinterval, $[x_{k-1}, x_k]$, apply the Mean Value Theorem to F:

So,

$$\exists c_k \in (x_{k-1}, x_k) \text{ such that } \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(c_k) = f(c_k). \tag{6.3}$$

Rearrange this, we have:

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1}) = f(c_k)\Delta x_k.$$

Notice that for all k, $m_k \leq f(c_k) \leq M_k$.

So, summing this up:

$$\sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} f(c_k)\Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k.$$

From our rearrangement earlier, we have:

$$L(f, P) \leq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \leq U(f, P)$$

Notice that the middle term is a funny telescoping thing:

$$\sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = F(x_1) - F(x_0) + F(x_2) - F(x_1) + \ldots + F(x_n) - F(x_{n-1})$$

$$= F(x_n) - F(x_0) = F(b) - F(a).$$

Substituting this back in, we have:

$$L(f, P) \leq F(b) - F(a) \leq U(f, P).$$

Remember, this thing is integrable! So, $U(f) = L(f)$.

Thus,

$$U(f) = F(b) - F(a) = L(f)$$

Therefore, the result follows:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

■

Note: The proof for FTC part 1 really wasn't too crazy. It relies heavily on definitions, and the Mean Value Theorem is what's doing the heavy lifting.
Notice that we used the fact that $F' = f$ here, that way we can use MVT.
f being integrable isn't strong enough to use MVT on f itself.
The integrability of f was required for that fun upper/lower sum business.

Next, let's look at FTC part 2. Unfortunately, Dr. Shipman doesn't prove it, so I won't either. The proof has a few tricks, but if she doesn't cover it, I won't be tested on it, so I'm okay leaving it off. Perhaps I will try to prove it later, but I have 6 classes this semester, so I probably will not.

Theorem: Fundamental Theorem of Calculus Part 2

Suppose G is integrable on $[a, b]$.
Define $G(x) = \int_a^x g$.
Then,

1. G is continuous on $[a, b]$,
2. If g is continuous at $c \in [a, b]$, then G is differentiable at c, and $G'(c) = g(c)$.

This concludes the lecture notes on Part 6.

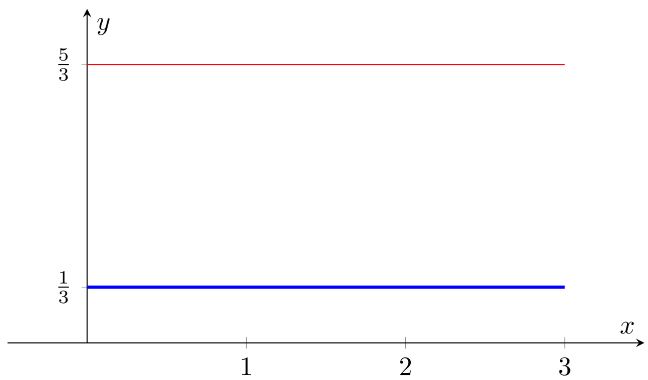
Next up, let's do some Homework. I did them before, I'll redo them as I type, and leave plenty of comments.

Homework 6

1. *Either give an example, or show that no such example exists.*

   A bounded function h on $[0,3]$ such that for every partition P of $[0,3]$, $U(h,P) = 5$ and $L(h,P) = 1$

   Solution. Consider: $h(x) = \begin{cases} \frac{5}{3}, & x \in \mathbb{Q} \\ \frac{1}{3}, & x \notin \mathbb{Q} \end{cases}$

   h is bounded by TREE(3). For all partitions, the upper sum is 5 and the lower sum is 1. ■

   A function g on $[0,1]$ and partitions P and Q of $[0,1]$ where $U(g,P) = L(g,P)$ but $U(g,Q) \neq L(g,Q)$.

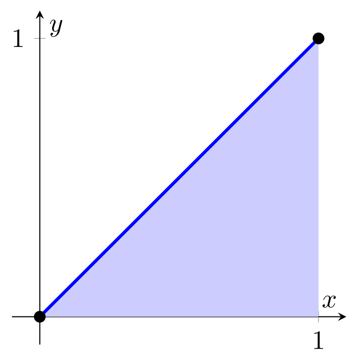
   Solution. Consider the function $g(x) = x$ on $[0,1]$.

   Let $P = \{0,1\}$ and $Q = \{0, \frac{1}{2}, 1\}$.

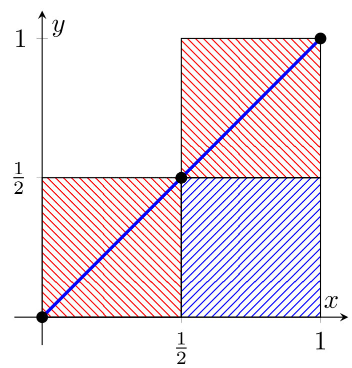
   For partition P, the upper sum $U(g,P) = L(g,P) = \frac{1}{2}$.

   For partition Q, the upper sum $U(g,Q) \neq L(g,Q)$.

Partition $P = \{0, 1\}$



Partition $Q = \{0, \frac{1}{2}, 1\}$



■

2. *True or False? Give a proof or a counter-example.*

______ If g is integrable on $[0,1]$, then so is $g(x^n)$, for all $n \in \mathbb{N}$.

Solution. True.

Assume g is integrable on $[0,1]$. Then, $\int_0^1 g(x)dx = U(g) = L(g)$.

For the case where $n = 1$, we have $g(x^1) = g(x)$, which is integrable.

Now, assume $g(x^k)$ is integrable for some $k \in \mathbb{N}$.

Need to show: $g(x^{k+1})$ is integrable.

Since $g(x^k)$ is integrable, for every $\epsilon > 0$, there exists a partition P of $[0,1]$ such that $U(g(x^k), P) - L(g(x^k), P) < \epsilon$.

Consider the same partition P for $g(x^{k+1})$.

Note that $x^{k+1} = x \cdot x^k$.

Since $x \in [0,1]$, we have $x^{k+1} \le x^k$ for all $x \in [0,1]$.

So, $U(g(x^{k+1}), P) \le U(g(x^k), P)$ and $L(g(x^{k+1}), P) \le L(g(x^k), P)$.

Therefore, $U(g(x^{k+1}), P) - L(g(x^{k+1}), P) \le U(g(x^k), P) - L(g(x^k), P) < \epsilon$.

By induction, $g(x^n)$ is integrable for all $n \in \mathbb{N}$. ■

> **Note:** The above problem makes intuitive sense, it really feels like it just *should* be true.
> All we're doing is shifting the x's (shrinking them, since we're in $[0,1]$), and adjusting the functional value accordingly.
> And since g is integrable, we shouldn't have any issue just moving the x's and functional values around, the result should still be integrable.
> And luckily for us, it is.

______ If $|f|$ is integrable on $[a,b]$, then so is f.

Solution. False.

Consider this modified Direchlet function on $[0,1]$:

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$$

So, $|f(x)| = 1$ for all $x \in [0,1]$.

Thus, $|f|$ is integrable on $[0,1]$.

But, the Direchlet function is not integrable. ■

______ If f and $|f|$ are integrable on $[a,b]$, then $|\int_a^b f| \le \int_a^b |f|$.

Solution. True.

Since f is integrable on $[a, b]$, we have:$\int_a^b f = L(f) = U(f)$ (1).

Since $|f|$ is integrable on $[a, b]$, we have: $\int_a^b |f| = L(|f|) = U(|f|)$ (2).

Notice: $f \leq |f|$ for all $x \in [a, b]$.

By (1): $|\int_a^b f| = |U(f)| = |\inf\{U(f, P) : P \text{ is a partition of } [a, b]\}|$.

By (2): $\int_a^b |f| = U(|f|) = \inf\{U(|f|, P) : P \text{ is a partition of } [a, b]\}$.

Each $U(f, P) \leq U(|f|, P) \Rightarrow U(f) \leq U(|f|) \Leftrightarrow |U(f)| \leq U(|f|)$ Since $|f| \geq 0$, so:

$|U(f)| \leq U(|f|)$,

Thus:

$$|\int_a^b f| \leq \int_a^b |f|$$

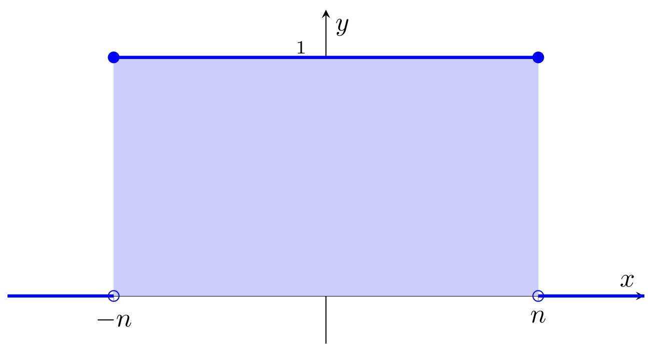
■

3. For every $n \in \mathbb{N}$ and for $x \in \mathbb{R}$, define $g_n(x) = \begin{cases} \frac{1}{n}, & \text{for } -n \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$.

Evaluate the integral $I_n = \int_{-100}^{100} g_n$.

> **Note:**
> Before we tackle this, we kinda need to do a bunch of work.
> We need to figure out what this function looks like, and what it's integrals will look like.
> Let's start figuring out what some of the g_n's look like, both graphically and functionally.
> Then, we can determine their "areas under the curve", which will help us figure out what we need to do to compute this integral.

Solution. First, let's look at a generic graph of g_n.

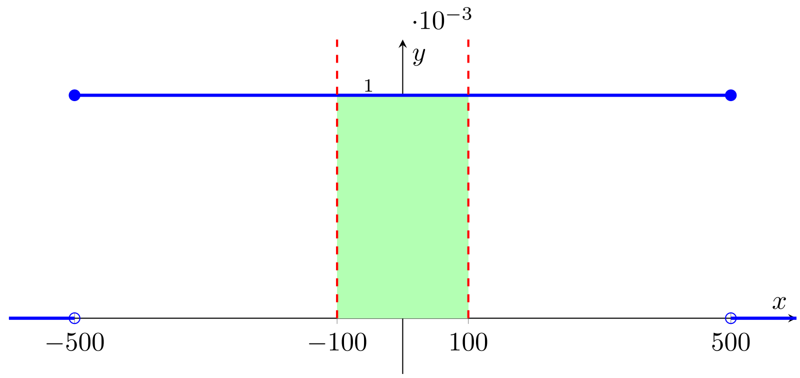


From this graph, we can see the area under that curve is $= (\frac{1}{n})(n+n) = \frac{1}{n}(2n) = 2$.

But what happens when we aren't integrating from $-n$ to n?

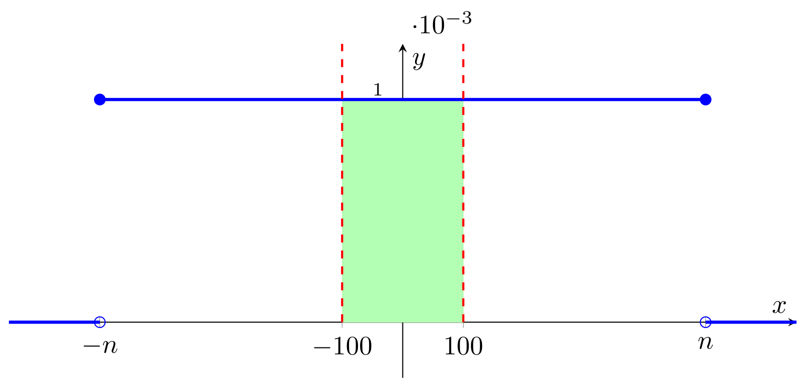
Let's say, $n = 500$, and we're working with the integral presented, I_n.

Then, our picture looks a little something like:



Which has an area far less than 2.

Okay, so let's look at one last picture, and generalize our integral from that.



Okay! We've got what we need!

Using a super simple area of a rectangle thing, we have:

Area = base · height.

So, when $n > 100$, the height is $\frac{1}{n}$, and the base is $|100 - (-100)| = 200$.

$$I_n = \begin{cases} 2, & n \leq 100 \\ \frac{200}{n}, & n > 100 \end{cases}$$

$\blacksquare$

Part 7

Series, Sequences of Functions, & Convergence

Book References:
Apostol: 8.1 – 8.8, 8.10 – 8.15, 8.17, 8.18, 9.1 – 9.5
Abbott: 6.1 – 6.3

In this section, we will cover:

- Infinite Series of Real Numbers
- Sequences of Functions
- Pointwise and Uniform Convergence
- Continuous Limit Theorem
- Convergence and Derivatives and Integrals

7.1 Topic 1

This is the content for Topic 1.

Theorem: Sample Theorem

This is a sample theorem in Topic 1.

Proof. Yes. ■

This concludes part 7 of Mathematical Analysis.