CS2010: Data Structures and Algorithms II

Shortest paths in graphs

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Outline

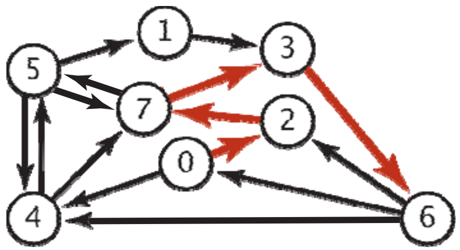
- Shortest paths
  - Single source shortest path
    - Topological sort – acyclic graphs but ok with negative weights
    - Dijkstra – non negative weights but ok with cycles
    - Bellman-Ford – non-negative cycles
  - Single-pair shortest path
    - A* search algorithm
  - All pairs shortest path
    - Floyd-Warshall
- Dynamic programming summary

Shortest paths in an edge-weighted digraph

Given an edge-weighted digraph, find the shortest path from s to t.

edge-weighted digraph

4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



shortest path from 0 to 6

0->2	0.26
2->7	0.34
7->3	0.39
3->6	0.52

Shortest path variants

Which vertices?

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source-sink: from one vertex s to another t.
- All pairs: between all pairs of vertices.

Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

Cycles?

- No directed cycles.
- No "negative cycles."

Dijkstra's shortest path algorithm

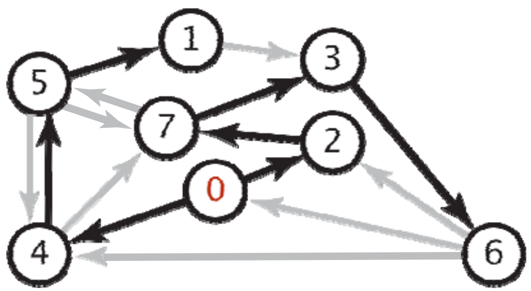
Data structures for single-source shortest paths

Goal. Find the shortest path from s to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists.

Consequence. Can represent the SPT with two vertex-indexed arrays:

- `distTo[v]` is length of shortest path from s to v.
- `edgeTo[v]` is last edge on shortest path from s to v.



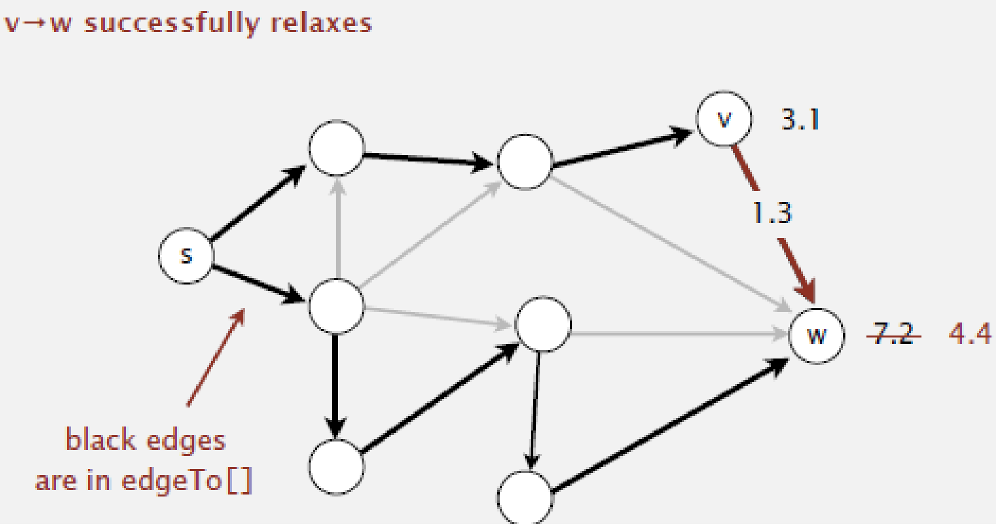
shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

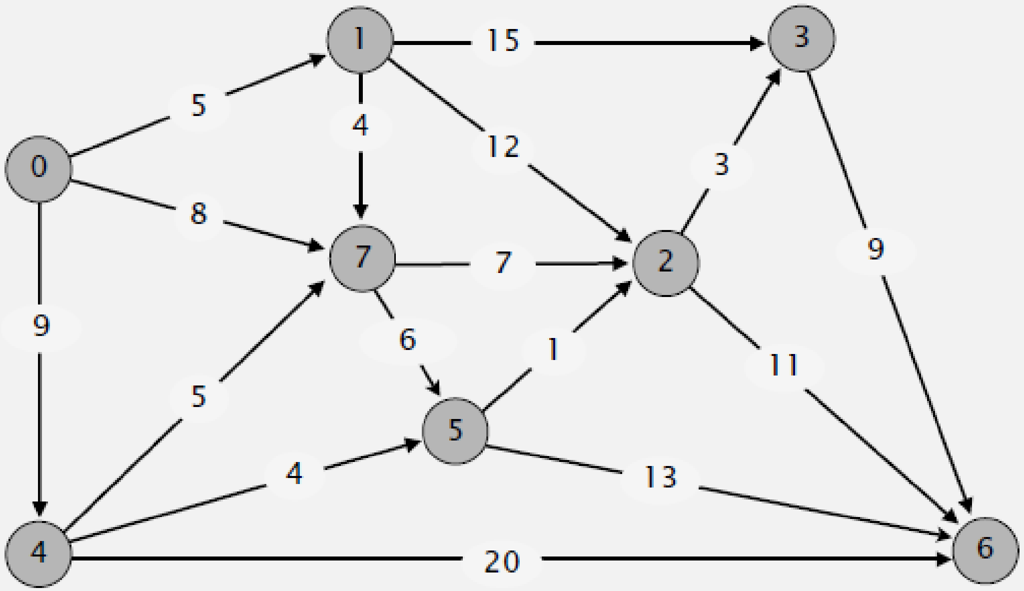
parent-link representation

Relax edge $e = v \rightarrow w$.

- `distTo[v]` is length of shortest known path from `s` to `v`.
- `distTo[w]` is length of shortest known path from `s` to `w`.
- `edgeTo[w]` is last edge on shortest known path from `s` to `w`.
- If `e = v→w` gives shorter path to `w` through `v`, update both `distTo[w]` and `edgeTo[w]`.



- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges pointing from that vertex.



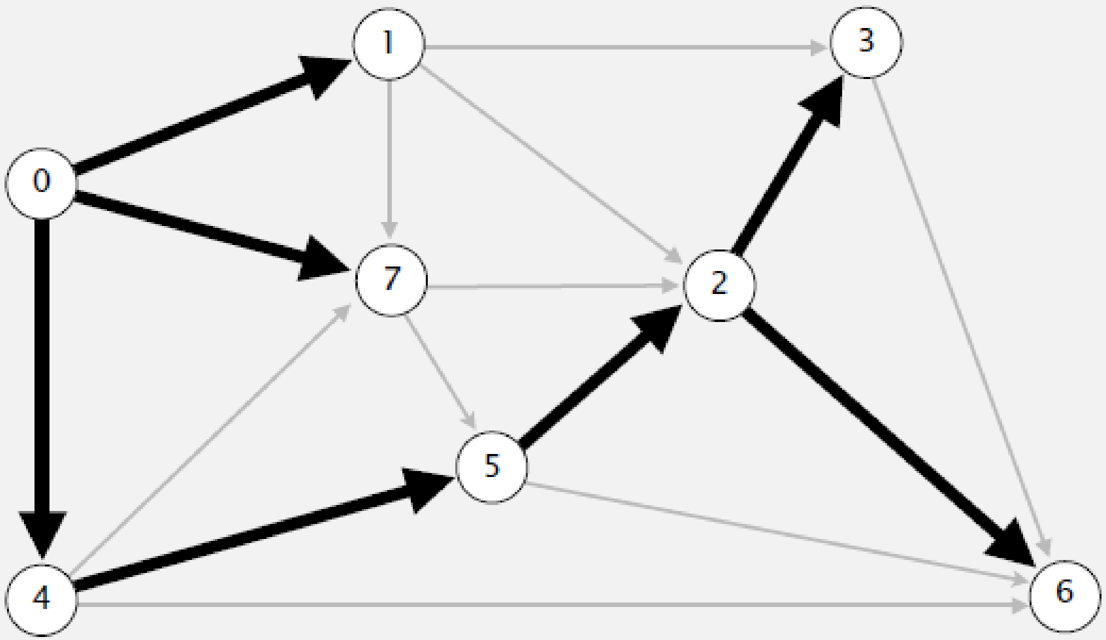
an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0

Dijkstra Demo

https://algs4.cs.princeton.edu/lectures/44DemoDijkstra.pdf

- Consider vertices in increasing order of distance from s
  (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges pointing from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

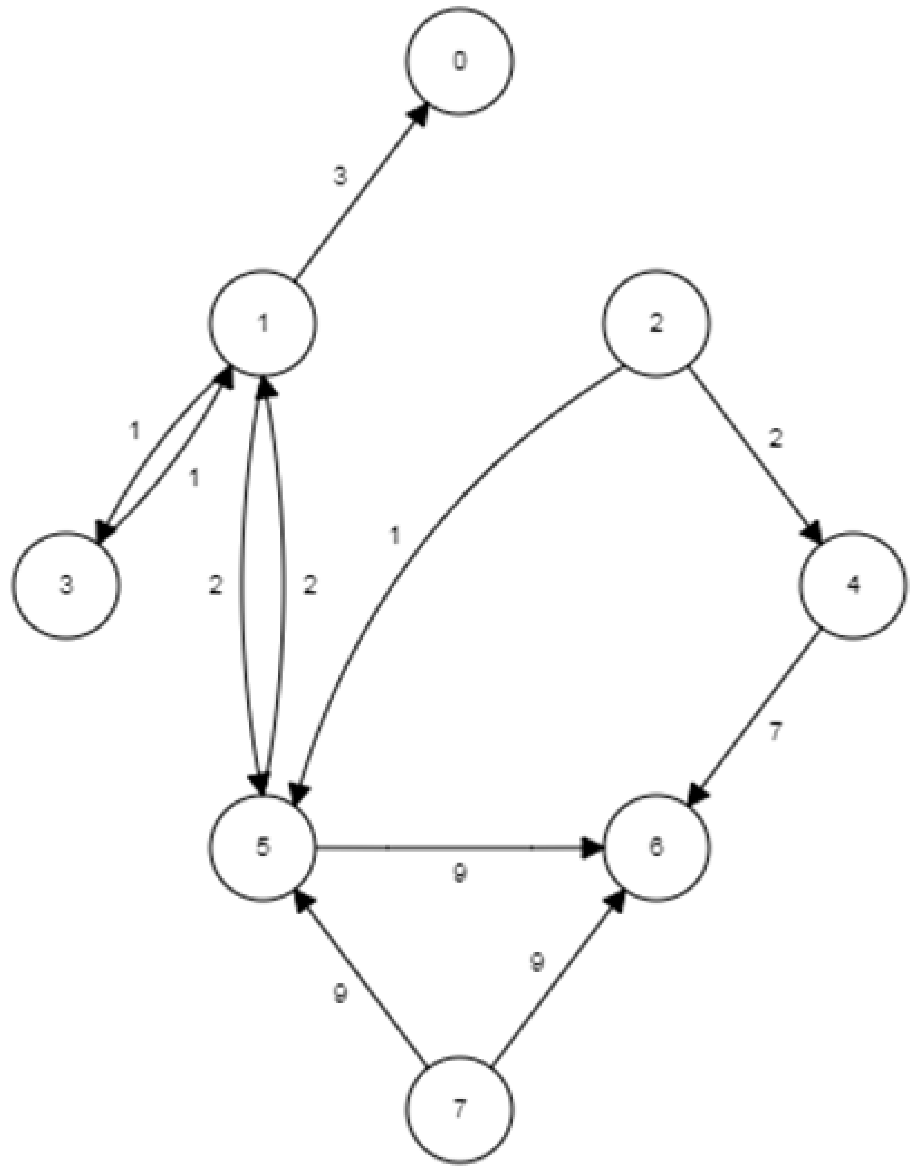
shortest-paths tree from vertex s

Dijkstra exercise

Start from vertex 1

v	DistTo[]	EdgeTo[]
0		
1		
2		
3		
4		
5		
6		
7		

Turning point – which vertice
Cant be reached from 1?



Solution

Vertex	Known	Cost	Path	
0	T	3	1	1 0
1	T	0	-1	1
2	F	INF	-1	No Path
3	T	1	1	1 3
4	F	INF	-1	No Path
5	T	2	1	1 5
6	T	11	5	1 5 6
7	F	INF	-1	No Path

Dijkstra performance

Dijkstra's algorithm: which priority queue?

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	1 †	$\log V$ †	1 †	$E + V \log V$

† amortized

Bottom line.

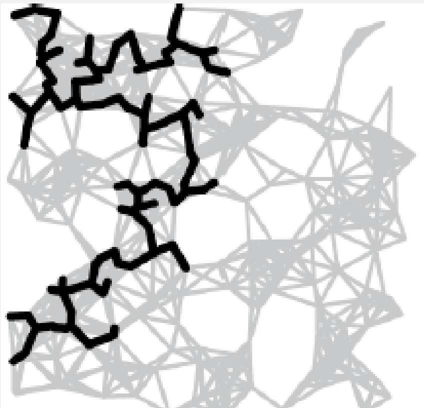
- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim: Closest vertex to the tree (via an undirected edge).
- Dijkstra: Closest vertex to the source (via a directed path).

Dijkstra – why can't work with negative weights?

- Example?

- Consquence: picking the shortest candidate edge (local optimality) always ends up being correct (global optimality).
  - Greedy algorithm!
  - Wouldn't be true if used dijkstra with negative weights – use a different algorithm instead!

- Can be modified to work with negative weights, i.e., vertex can be en-queued more than once -> exponential worst case running time though!

A*

A* shortest path

- Single source, single destination/goal
- Extension of Dijkstra's algorithm
- Uses heuristics to guide its search and achieve better performance
  - prioritizes paths that seem to be leading closer to the goal

Heuristic function

- solving a problem more quickly when classic methods are too slow
- finding an approximate solution when classic methods fail to find any exact solution
- trading optimality, completeness, accuracy, or precision for speed.
- a function that ranks alternatives in search algorithms at each branching step based on available information to decide which branch to follow
  - Eg approximate exact solution

Heuristic function h(n)

- The 2 most important properties:
  - relatively cheap to compute
  - relatively accurate estimator of the cost to reach a goal. Usually a “good” heuristic is if ½ opt(n) <h(n)<opt(n)
- Admissible heuristic
  - h(n) never overestimates the actual cost from n to goal

A*

- uses both the actual distance from the start and the estimated distance to the goal.
- only expands a node if it seems promising (only focuses on reaching the goal node, not every other node)
- A guided version of Dijkstra
- Evaluation function *f(n) = g(n) + h(n) where*

g(n) = cost so far to reach *n*

h(n) = estimated cost from *n* to goal

f(n) = estimated total cost of path through *n* to goal

A*

- How to pick a heuristic function?
- Domain knowledge – some information about a goal
- Eg in route planning:
  - Manhattan distance – on a square grid that allows 4 directions of movement
  - Diagonal distance – 8-direction square
  - Euclidean distance – any direction

  - A good example on route planning in games

  http://theory.stanford.edu/~amitp/GameProgramming/Heuristics.html

Topological sort shortest path

Shortest path in acyclic graphs

- Is it easier than in graphs with cycles?
- Yes! Linear time
- Use topological sort (which only works in DAGs – directional acyclic graphs) to find shortest path
- If we have an edge from x to y, the ordering visits x before y
- Shortest path visits/relaxes edges in topological order
- Topological sort – identify a vertex with no incoming edges, add it to the ordered list, remove it, repeat…
  - (decrease by 1 and conquer)

Topological sort shortest path demo

https://algs4.cs.princeton.edu/lectures/44DemoAcyclicSP.pdf

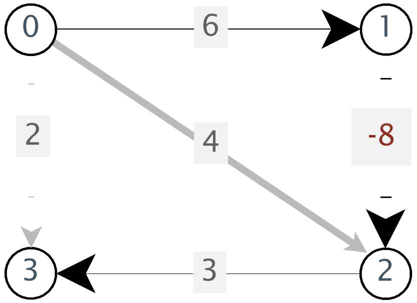
So far

- No negative weights – Dijkstra
- No cycles – Toplogical sort shortest path
- What if the graph has negative weights and cycles?
- What about negative cycles?

Bellman-Ford Shortest Path Algorithm

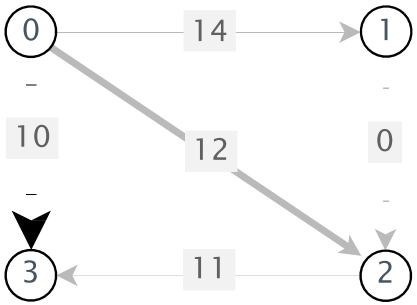
Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is 0→1→2→3.

Re-weighting. Add a constant to every edge weight doesn't work.



Adding 8 to each edge weight changes the shortest path from 0→1→2→3 to 0→3.

Conclusion. Need a different algorithm.

Negative cycles

Def. A **negative cycle** is a directed cycle whose sum of edge weights is negative.

digraph

4->5	0.35
5->4	-0.66
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
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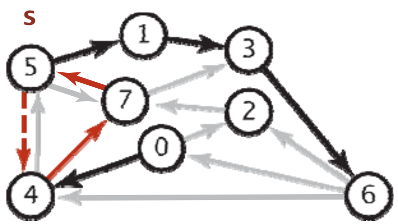


negative cycle (-0.66 + 0.37 + 0.28)
5->4->7->5

shortest path from 0 to 6
0->4->7->5->4->7->5...->1->3->6

Proposition. A SPT exists iff no negative cycles.

assuming all vertices reachable from s

Bellman-Ford algorithm

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:

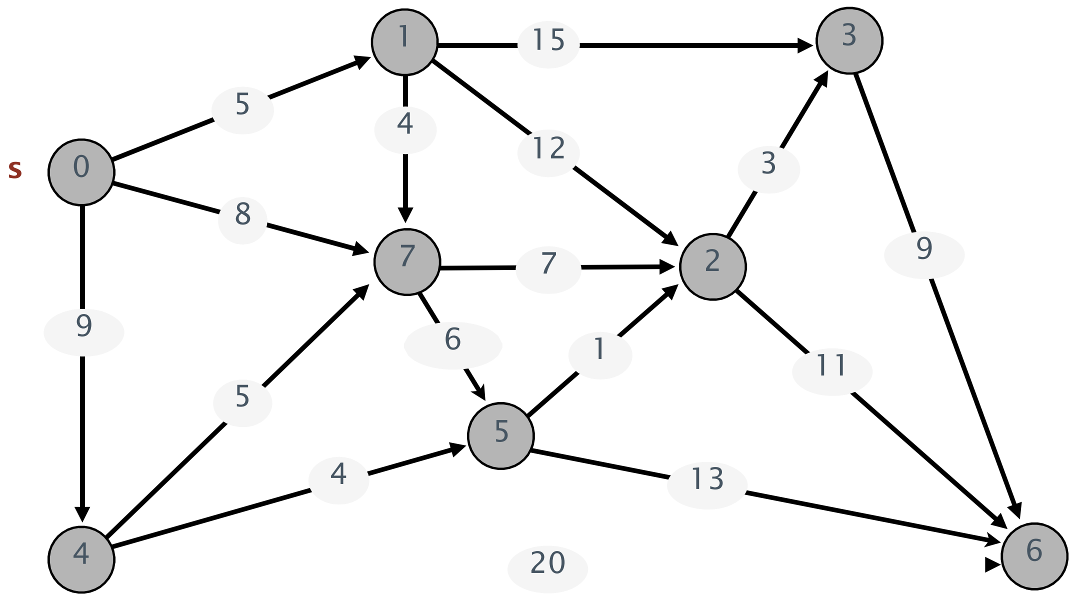
- **Relax each edge.**

```
for (int i = 0; i < G.V(); i++)
   for (int v = 0; v < G.V(); v++)
      for (DirectedEdge e : G.adj(v))
         relax(e);
```

pass i (relax each edge)

Bellman-Ford algorithm demo

Repeat V times: relax all E edges.



an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
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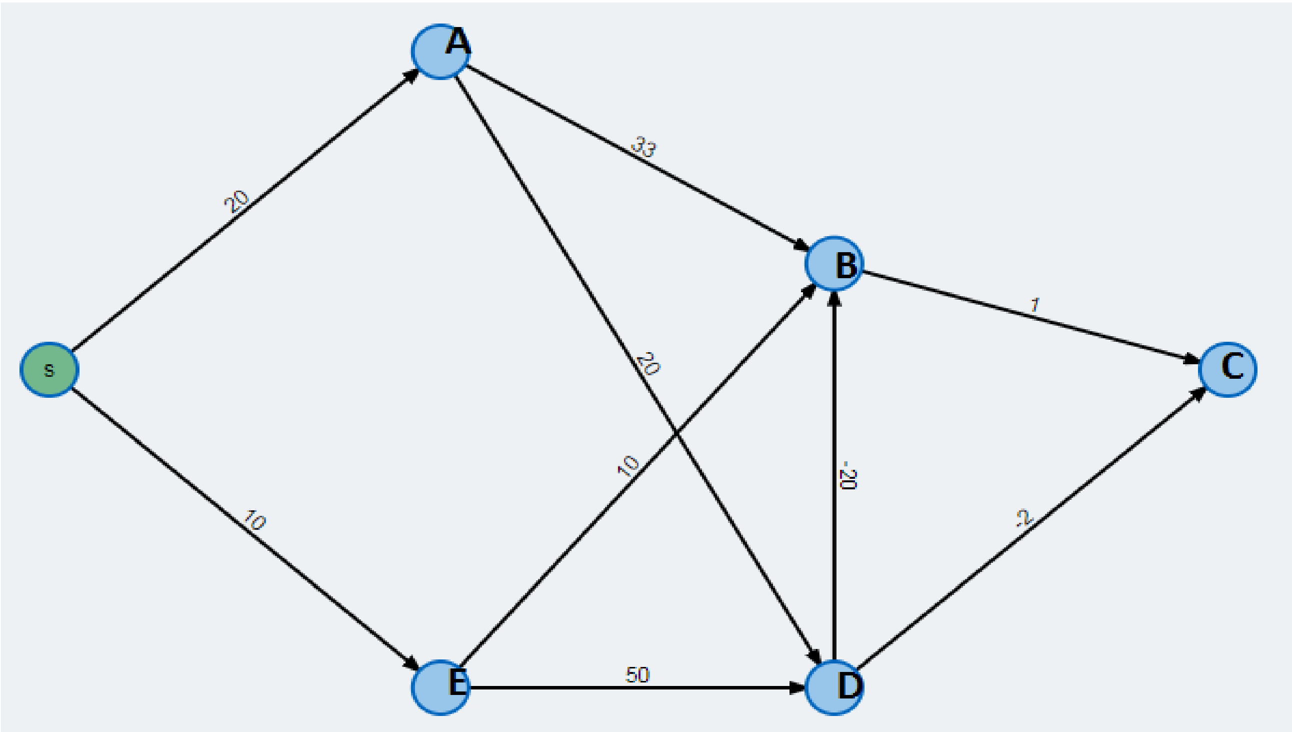
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Bellman-Ford Demo

- https://algs4.cs.princeton.edu/lectures/44DemoBellmanFord.pdf
- Only positive weights in this example – paper exercise with negative weights

Bellman-Ford Exercise

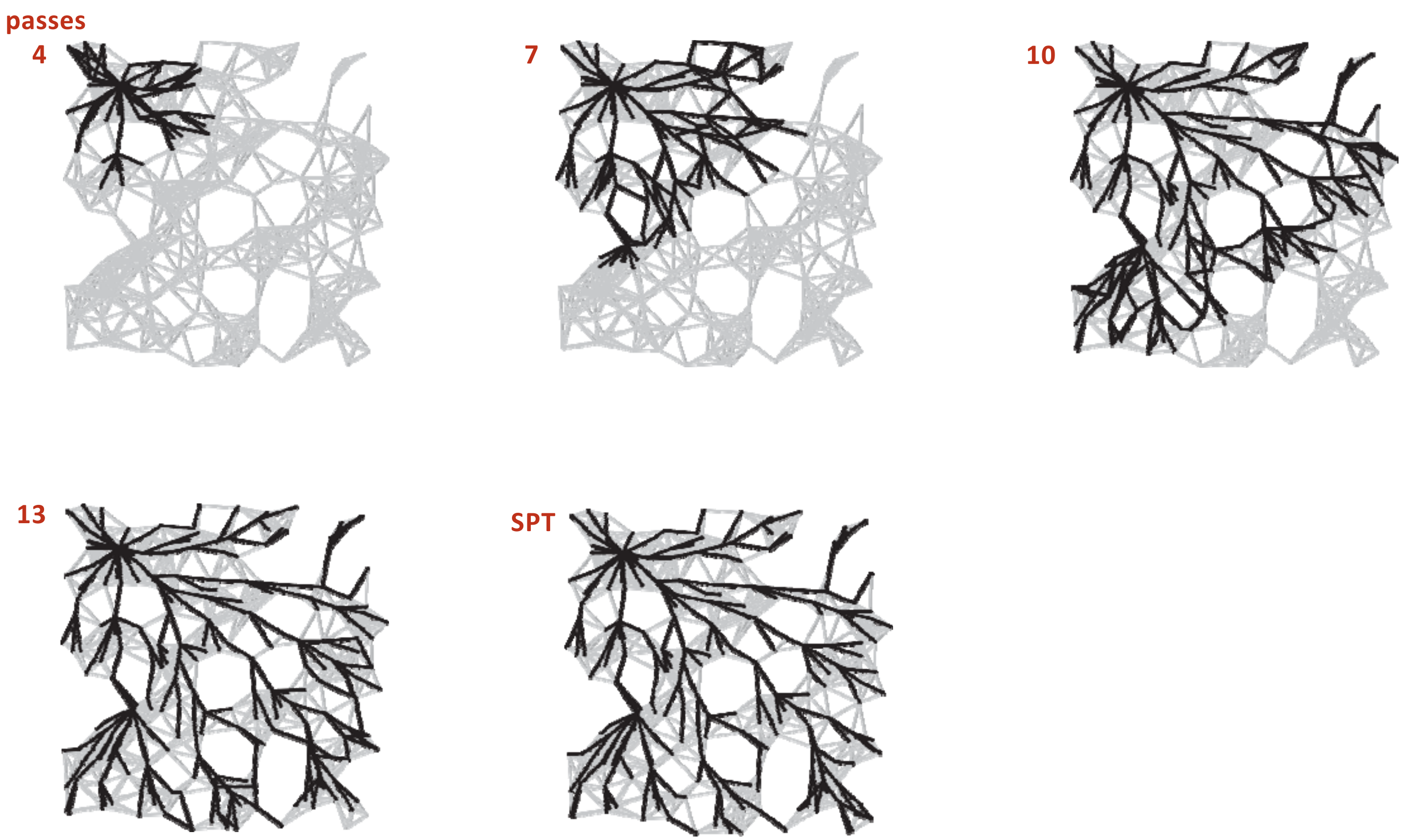


Bellman-Ford Exercise Table

Iteration	Dist to S	A	B	C	D	E
0	0	inf	inf	inf	inf	Inf
1						
2						
3						
4						
5						

Keep track of distance and parent node eg 5 (via B) for each node for each iteration

Bellman-Ford algorithm: visualization



Bellman-Ford algorithm: analysis

Bellman-Ford algorithm

Initialize distTo[s] = 0 and distTo[v] = ∞ for all other vertices.

Repeat V times:

- **Relax each edge.**

Proposition. Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to $E \times V$.

Pf idea. After pass `i`, found path that is at least as short as any shortest path containing `i` (or fewer) edges.

Bellman-Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass `i`, no need to relax any edge pointing from `v` in pass `i+1`.

FIFO implementation. Maintain queue of vertices whose `distTo[]` changed.

be careful to keep at most one copy of each vertex on queue (why?)

Overall effect.

- The running time is still proportional to $E \times V$ in worst case.
- But much faster than that in practice.

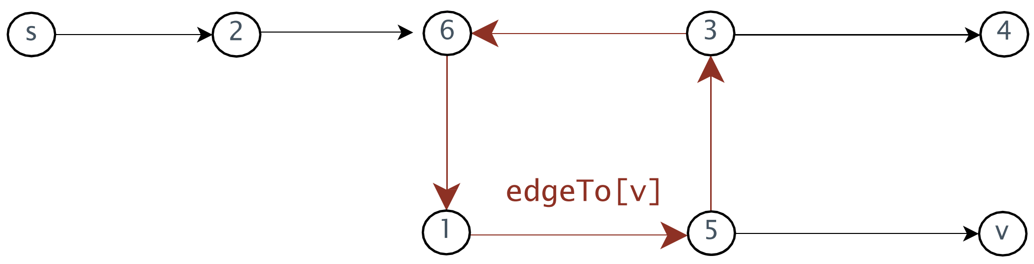
Single source shortest path summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	$E + V$	$E + V$	V
Dijkstra (binary heap)	no negative weights	$E \log V$	$E \log V$	V
Bellman-Ford	no negative cycles	$E V$	$E V$	V
Bellman-Ford (queue-based)		$E + V$	$E V$	V

Remark 1. Directed cycles make the problem harder.
Remark 2. Negative weights make the problem harder.
Remark 3. Negative cycles makes the problem intractable.

Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If any vertex `v` is updated in pass `V`, there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.

Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

	USD	EUR	GBP	CHF	CAD
USD	1	0.741	0.657	1.061	1.011
EUR	1.350	1	0.888	1.433	1.366
GBP	1.521	1.126	1	1.614	1.538
CHF	0.943	0.698	0.620	1	0.953
CAD	0.995	0.732	0.650	1.049	1

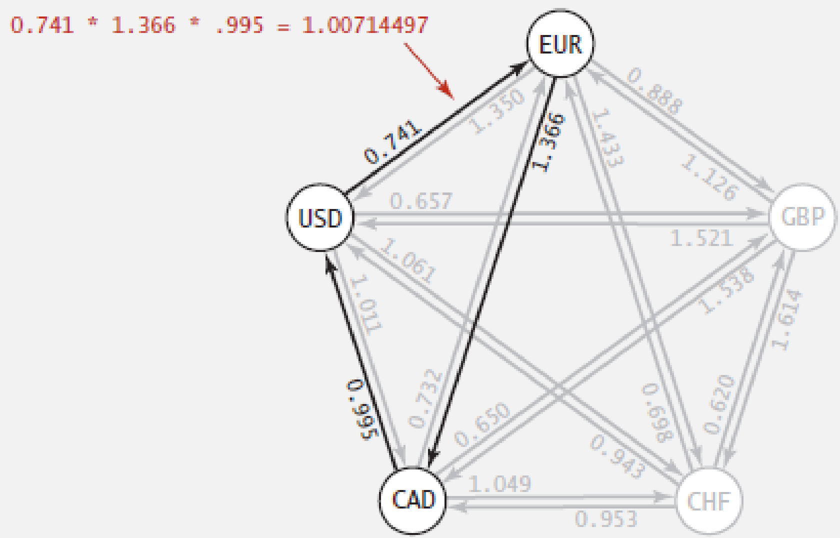
Ex. $1,000 ⇒ 741 Euros ⇒ 1,012.206 Canadian dollars ⇒ $1,007.14497.

$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$

Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is > 1.

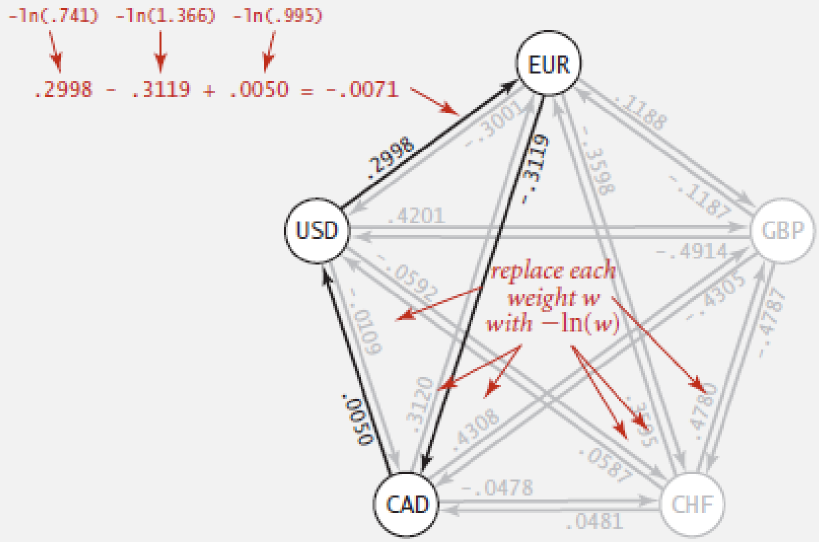


Challenge. Express as a negative cycle detection problem.

Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge $v \rightarrow w$ be $-ln$ (exchange rate from currency v to w).
- Multiplication turns to addition; > 1 turns to < 0.
- Find a directed cycle whose sum of edge weights is < 0 (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

Floyd-Warshall shortest path algorithm

All-pairs shortest path

- Run Dijkstra for each vertex?
- Run Bellman-Ford each vertex?
- Worst case scenario – assume fully connected graph, so E = V^2
- Dijkstra = V times V^2 log V = V^3 log V
- Bellman-Ford = V times V^3 = V^4
- Other options?
- Floyd-Warshall = V^3 (but V^2 space)

Floyd-Warshall all-pairs shortest path

- For a path p = {v1, v2, …, vl}, vertices v2 to vl-1 are intermediate vertices
- Path consisting of a single edge has no intermediate vertices
- $d_{ij}^{(k)}$ is a shortest path from i to j such that any intermediate vertices on the path are chosen from the set {1, 2, …, k} – in any order, any subset of them

Floyd-Warshall

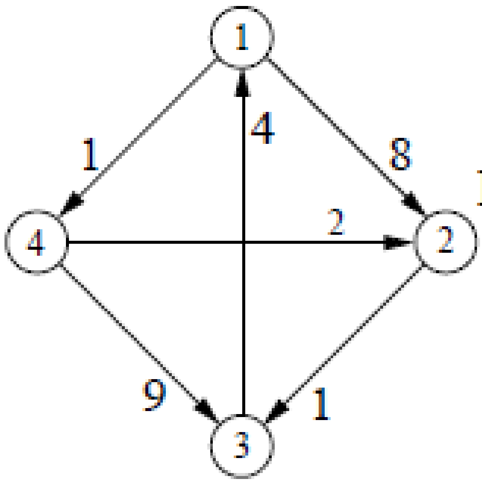
- Assume we know $d_{ij}^{(k-1)}$ – how do we calculate $d_{ij}^{(k)}$
- The path either goes through k, or it does not
  - If it does not, then shortest path $d_{ij}^{(k)} = d_{ij}^{(k-1)}$
  - If it does, it means it passes through it exactly once, and it consists of shortest path from i to k, and then the shortest path from k to j $d_{ij}^{(k)} = d_{ik}^{(k-1)-} + d_{kj}^{(k)-1}$

$$
\begin{aligned}
d_{ij}^{(0)} &= w_{ij}, \\
d_{ij}^{(k)} &= \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) \qquad \text{for } k \geq 1.
\end{aligned}
$$

Floyd-Warshall pseudocode

for $k = 1$ to n do
  for $i = 1$ to n do
    for $j = 1$ to n do
      if $(d[i,k] + d[k,j] < d[i,j])$
        $\{d[i,j] = d[i,k] + d[k,j];$
        $pred[i,j] = k;\}$
return $d[1..n, 1..n]$;

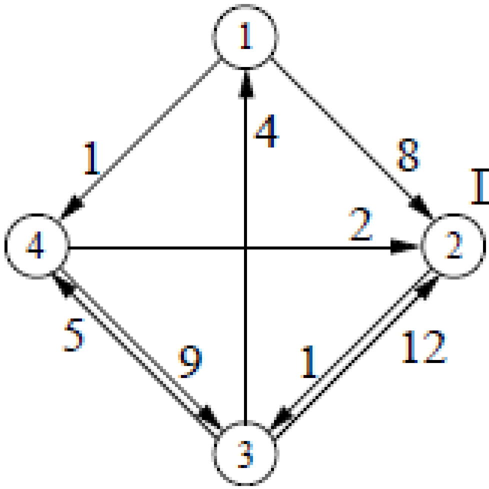
Floyd Warshall example



$$D^{(0)} = \begin{bmatrix} 0 & 8 & ? & 1 \\ ? & 0 & 1 & ? \\ 4 & ? & 0 & ? \\ ? & 2 & 9 & 0 \end{bmatrix}$$

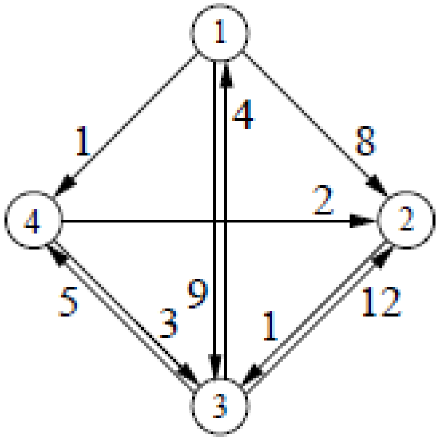
? = infinity

Floyd Warshall example



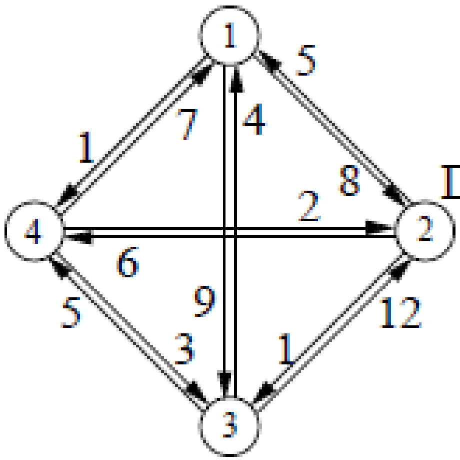
$$D^{(1)} = \begin{bmatrix} 0 & 8 & ? & 1 \\ ? & 0 & 1 & ? \\ 4 & 12 & 0 & 5 \\ ? & 2 & 9 & 0 \end{bmatrix}$$

Floyd Warshall example



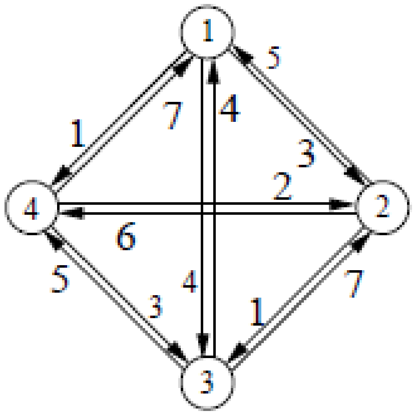
$$D^{(2)} = \begin{bmatrix} 0 & 8 & 9 & 1 \\ ? & 0 & 1 & ? \\ 4 & 12 & 0 & 5 \\ ? & 2 & 3 & 0 \end{bmatrix}$$

Floyd Warshall example



$$D^{(3)} = \begin{bmatrix} 0 & 8 & 9 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 12 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix}$$

Floyd Warshall example

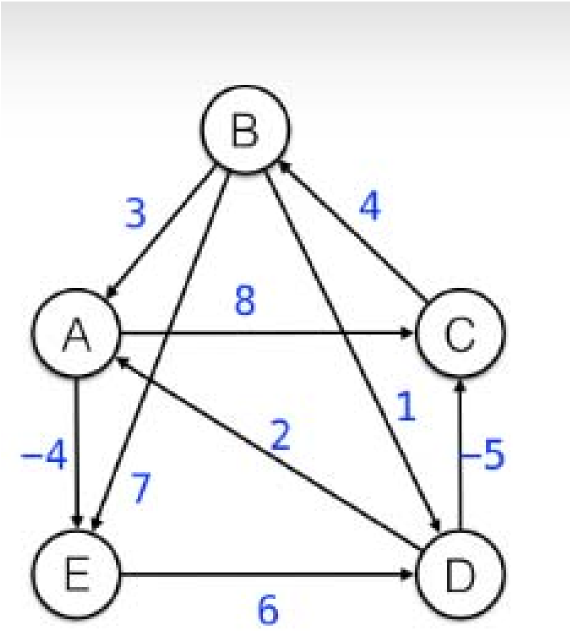


$$D^{(4)} = \begin{bmatrix} 0 & 3 & 4 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 7 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix}$$

Floyd-Warshall demo

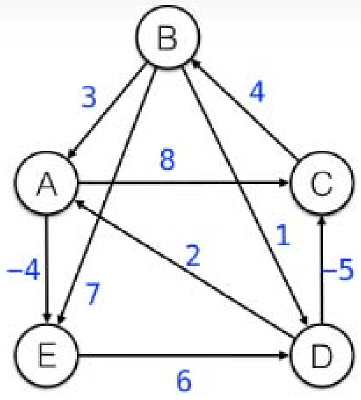
- https://www.cs.usfca.edu/~galles/visualization/Floyd.html

Floyd-Warshall exercise



D^0	A	B	C	D	E
A	0	3	8	INF	-4
B	INF	0	INF	1	7
C	INF	4	0	INF	INF
D	2	INF	-5	0	INF
E	INF	INF	INF	6	0

Exercise solution



D^5	A	B	C	D	E
A	0	0	1	-3	**-4**
B	3	0	-4	1	**-1**
C	7	4	0	5	**3**
D	2	-1	-5	0	**-2**
E	**8**	**5**	**1**	**6**	**0**

Dynamic Programming

Dynamic programming

- Algorithm Design, Kleinberg and Tardos, Pearson 2014
- Introduction to Design and Analysis of Algorithms, Levitin. Pearson 2012

Dynamic programming

- Examples:
  - Bellman-Ford
  - Floyd-Warshall

- Examines the full search space but implicitly, by breaking up the problem into a series of subproblems, and then building up the solution to larger and larger subproblems
- Typically overlapping subproblems – instead of over and over calculating solutions to a subproblem, record it in a table and look up when needed

- So why a fancy name if this is all that it's doing?
- Richard Bellman on the Birth of Dynamic Programming, by Stuart Dreyfus

https://pubsonline.informs.org/doi/pdf/10.1287/opre.50.1.48.17791

Dynamic programming

- Informal guidelines:
  - There are only a polynomial number of subproblems
  - The solution to the original problem can be easily computed from the solution of the subproblems
  - There is a natural order of subproblems from smallest to largest together with easy to compute recurrence that allows building a solution to a subproblem from smaller subproblems

Shortest paths challenges

› http://www.diag.uniroma1.it/challenge9/

› http://www.diag.uniroma1.it/challenge9/download.shtml