CS2010: ALGORITHMS AND DATA STRUCTURES

Lecture 2: Analysis of Algorithms

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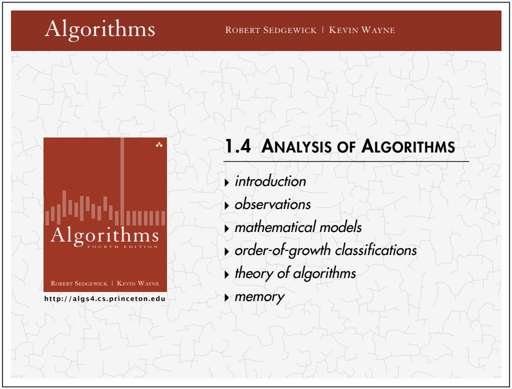
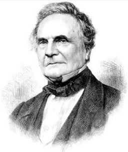
THIS LECTURE

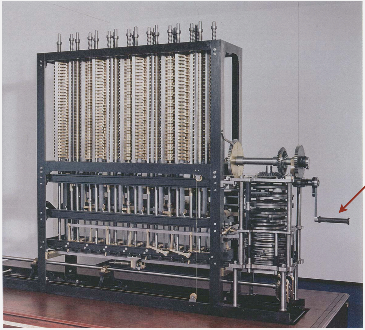


- Parts from S&W 1.4
- Estimate the performance of algorithms by
  - Experiments & Observations
  - Precise Mathematical Calculations

Running time

" As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise—By what course of calculation can these results be arrived at by the machine in the shortest time? " — *Charles Babbage (1864)*





how many times do you have to turn the crank?

Analytic Engine

WHY ANALYSE ALGORITHMS?

→ **Good programmer:** to predict the performance of our programs.

→ **Good client:** to choose between alternative algorithms/implementations.

→ **Good manager:** to provide guarantees to clients / avoid client complaints.

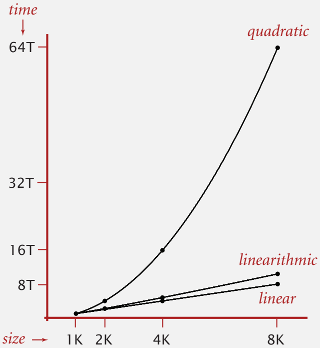
→ **Good theoritician:** to understand the nature of computing.

Some algorithmic successes

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ...
- Brute force: N^2 steps.
- FFT algorithm: $N \log N$ steps, enables new technology.

Friedrich Gauss
1805

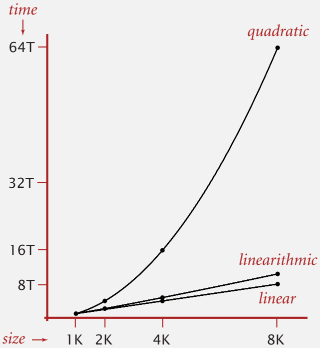


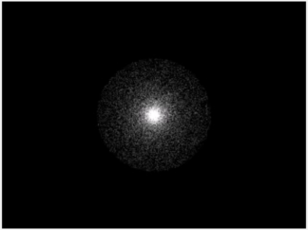
Some algorithmic successes

N-body simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut algorithm: $N \log N$ steps, enables new research.

Andrew Appel
PU '81





The challenge

Q. Will my program be able to solve a large practical input?

Insight. [Knuth 1970s] Use scientific method to understand performance.

Scientific method applied to analysis of algorithms

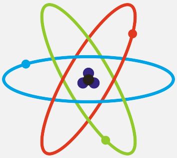
A framework for predicting performance and comparing algorithms.

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments must be reproducible.
- Hypotheses must be falsifiable.



Feature of the natural world. Computer itself.

Experimental Approach:

Measuring Precise Running Time

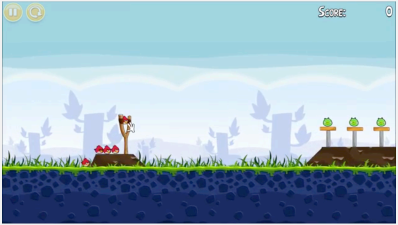
Example: 3-SUM

3-SUM. Given N distinct integers, how many triples sum to exactly zero?

```
% more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

% java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum
1	30	-40	10	0
2	30	-20	-10	0
3	-40	40	0	0
4	-10	0	10	0



Context. Deeply related to problems in computational geometry.

3-SUM: brute-force algorithm

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int N = a.length;
      int count = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }

   public static void main(String[] args)
   {
      In in = new In(args[0]);
      int[] a = in.readAllInts();
      StdOut.println(count(a));
   }
}
```

check each triple

for simplicity, ignore integer overflow

The input of `ThreeSum` is an array of size N.

Suppose we care only about 100-element arrays.

There are many different 100-element arrays.

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int N = a.length;
      int count = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }
```

FORWARD THINKING QUESTION

The input of `ThreeSum` is an array of size *N*.

Suppose we care only about 100-element arrays.

There are many different 100-element arrays.

Q. Is the running time of `ThreeSum` dependent on which 100-element array we provide as input?

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int N = a.length;
      int count = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }
```

Measuring the running time

Q. How to time a program?

A. Manual.

`% java ThreeSum 1Kints.txt`

tick tick tick

`70`

`% java ThreeSum 2Kints.txt`

tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

`528`

`% java ThreeSum 4Kints.txt`

tick tick tick tick tick tick tick tick
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`4039`

Measuring the running time

Q. How to time a program?

A. Automatic.

public class Stopwatch	(part of `stdlib.jar`)	
	`Stopwatch()`	*create a new stopwatch*
`double`	`elapsedTime()`	*time since creation (in seconds)*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   int[] a = in.readAllInts();
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
   StdOut.println("elapsed time " + time);
}
```

Empirical analysis

Run the program for various input sizes and measure running time.

```
% 
```

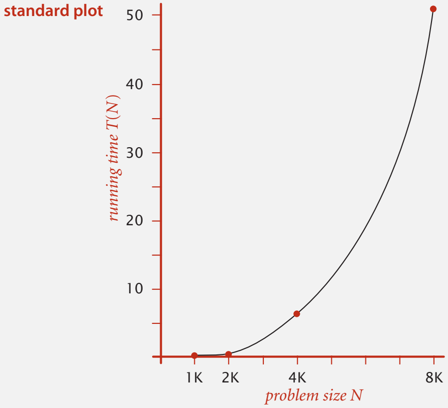
Empirical analysis

Run the program for various input sizes and measure running time.

N	time (seconds) †
250	0.0
500	0.0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

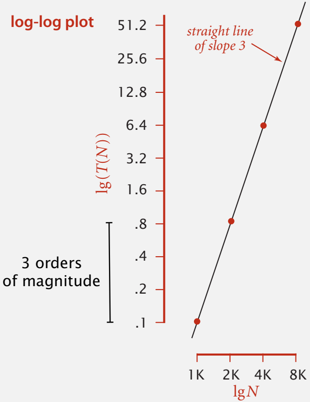
Data analysis

Standard plot. Plot running time $T(N)$ vs. input size N.



Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



$$\lg(T(N)) = b \lg N + c$$

$$b = 2.999$$

$$c = -33.2103$$

$$T(N) = a N^b, \text{ where } a = 2^c$$

power law

Regression. Fit straight line through data points: $a N^b$.

slope

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Try out the experimental analysis:

https://docs.google.com/spreadsheets/d/
1WnihyK6g1pYdcT2ndZOqNNRkTitXkWKnOrTgCnM-bw8/edit?usp=sharing

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

"order of growth" of running time is about N^3 [stay tuned]

Predictions.

- 51.0 seconds for $N = 8{,}000$.
- 408.1 seconds for $N = 16{,}000$.

Observations.

N	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Run program, doubling the size of the input.

N	time (seconds) †	ratio	lg ratio
250	0.0		–
500	0.0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8.0	3.0
8,000	51.1	8.0	3.0

lg (6.4 / 0.8) = 3.0

seems to converge to a constant b ≈ 3

$$\frac{T(2N)}{T(N)} = \frac{a(2N)^b}{aN^b} = 2^b$$

Hypothesis. Running time is about $a N^b$ with $b = \lg$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of N) and solve for a.

N	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1

$51.1 = a \times 8000^3$

$\Rightarrow \; a = 0.998 \times 10^{-10}$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times N^3$ seconds.

almost identical hypothesis
to one obtained via linear regression

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent in power law

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

determines constant in power law

Bad news. Difficult to get precise measurements.

Good news. Much easier and cheaper than other sciences.

e.g., can run huge number of experiments

This was the experimental approach to algorithm analysis.

Is there a mathematical approach where we can do **calculations** instead of experiments?

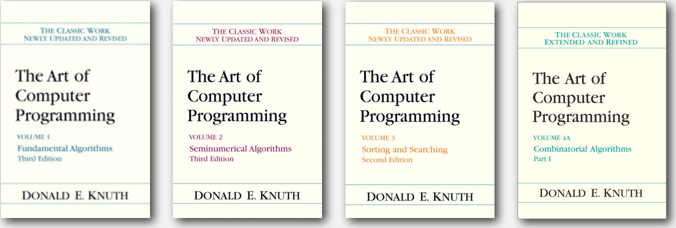
Mathematical Approach 1:

Calculating Precise Running Time

Mathematical models for running time

Total running time: sum of cost × frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.



Donald Knuth
1974 Turing Award

In principle, accurate mathematical models are available.

Cost of basic operations

Challenge. How to estimate constants.

operation	example	nanoseconds †
integer add	`a + b`	2.1
integer multiply	`a * b`	2.4
integer divide	`a / b`	5.4
floating-point add	`a + b`	4.6
floating-point multiply	`a * b`	4.2
floating-point divide	`a / b`	13.5
sine	`Math.sin(theta)`	91.3
arctangent	`Math.atan2(y, x)`	129.0
...	...	...

† Running OS X on Macbook Pro 2.2GHz with 2GB RAM

Cost of basic operations

Observation. Most primitive operations take constant time.

operation	example	nanoseconds †
variable declaration	`int a`	c_1
assignment statement	`a = b`	c_2
integer compare	`a < b`	c_3
array element access	`a[i]`	c_4
array length	`a.length`	c_5
1D array allocation	`new int[N]`	$c_6\ N$
2D array allocation	`new int[N][N]`	$c_7\ N^2$

Caveat. Non-primitive operations often take more than constant time.

novice mistake: abusive string concatenation

Example: 1-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0)
      count++;
```

N array accesses

operation	frequency
variable declaration	2
assignment statement	2
less than compare	$N + 1$
equal to compare	N
array access	N
increment	N to $2N$

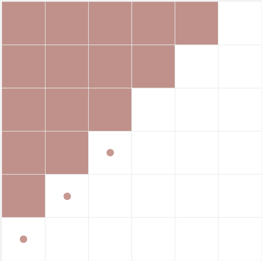
Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) \;=\; \frac{1}{2} N (N-1) \;=\; \binom{N}{2}$$

Pf. [n even]



$$0 + 1 + 2 + \ldots + (N-1) \;=\; \underbrace{\frac{1}{2}N^2}_{\text{half of square}} \;-\; \underbrace{\frac{1}{2}N}_{\text{half of diagonal}}$$

String theory infinite sum

$$1 + 2 + 3 + 4 + \ \ldots = \ -\frac{1}{12}$$

http://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html

Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2} (N+1)(N+2)$
equal to compare	$\frac{1}{2} N (N-1)$
array access	$N (N-1)$
increment	$\frac{1}{2} N (N-1)$ to $N (N-1)$

tedious to count exactly

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

Where

c_1 :cost of array access	A :number of array accesses
c_2 :cost of integer addition	B :number of integer additions
c_3 :cost of integer comparison	C :number of integer comparisons
c_4 :cost of increment	D :number of increments
c_5 :cost of assignment	E :number of assignments

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

Where

c_1 :cost of array access	A :number of array accesses
c_2 :cost of integer addition	B :number of integer additions
c_3 :cost of integer comparison	C :number of integer comparisons
c_4 :cost of increment	D :number of increments
c_5 :cost of assignment	E :number of assignments

Q. Advantages / Disadvantages?