CS2010: ALGORITHMS AND DATA STRUCTURES

Lecture 3: Cost Models of Running Time

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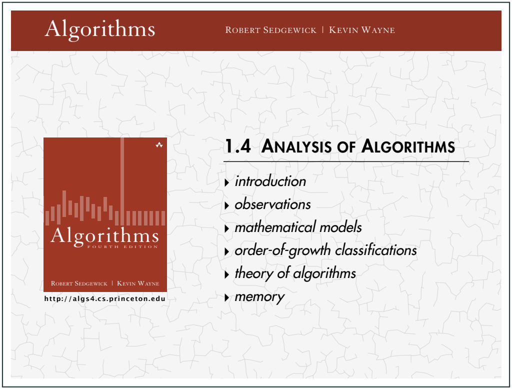
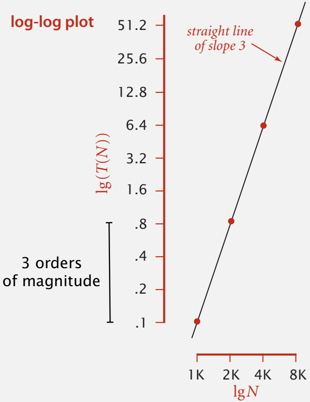
LAST LECTURE



- → Estimate the performance of algorithms by
  - → Experiments & Observations
    - \+ Easy experiments
    - − Works only for running times of the form $T(N) = aN^b$
  - → Precise Mathematical Calculations
    - \+ Works for any running time function
    - − Tedious & difficult

Data analysis

Log-log plot. Plot running time $T(N)$ vs. input size N using log-log scale.



$\lg(T(N)) = b \lg N + c$

$b = 2.999$

$c = -33.2103$

$T(N) = a N^b$, where $a = 2^c$

power law

slope

Regression. Fit straight line through data points: $a N^b$.

Hypothesis. The running time is about $1.006 \times 10^{-10} \times N^{2.999}$ seconds.

Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

$$\boxed{T(N) = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E + c_6 F}$$

operation	frequency	
variable declaration	$N + 2$	$= A$
assignment statement	$N + 2$	$= B$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$	$= C$
equal to compare	$\frac{1}{2} N (N - 1)$	$= D$
array access	$N (N - 1)$	$= E$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$= F$

tedious to count exactly

"best case" vs "worst case" input of size N

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

costs (depend on machine, compiler)

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

$A =$ array access

$B =$ integer add

$C =$ integer compare

$D =$ increment

$E =$ variable assignment

frequencies (depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(N) \sim c N^3$.

1. Approximate **Calculations** using different Cost Models

2. Classifying algorithms based on **order of growth**

Cost Model 1: all constant costs = 1

COST MODEL 1: ALL CONSTANT COSTS = 1

→ New generation computers have smaller constants than previous generation

$$\boxed{c_i = 1}$$

$$T_N = A + B + C + D + E$$

Where

$$\begin{aligned} A &: \text{number of array accesses} \\ B &: \text{number of integer additions} \\ C &: \text{number of integer comparisons} \\ D &: \text{number of increments} \\ E &: \text{number of assignments} \end{aligned}$$

Careful!

There are operations that **do not** have a constant cost:

- Naive string concatenation: `s = str + "ABCDEFG";`

- Method calls: `max = Collections.max(myList);`

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  - when efficiency is important use `StringBuilder`
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There are operations that **do not** have a constant cost:

→ Naive string concatenation: `s = str + "ABCDEFG";`

  → the cost of this operation is linear to the size of `str`

  → when efficiency is important use `StringBuilder`

→ Method calls: `max = Collections.max(myList);`

  → the cost of this operation is the cost of running the algorithm in `Collections.max` with an input of size `myList.size()`

Example: 2-SUM

Q. How many instructions as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
assignment statement	$N + 2$
less than compare	$\frac{1}{2} (N+1)(N+2)$
equal to compare	$\frac{1}{2} N (N-1)$
array access	$N (N-1)$
increment	$\frac{1}{2} N (N-1)$ to $N (N-1)$

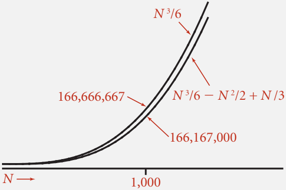
tedious to count exactly

Estimate performance by adding up frequencies

Cost model 2: only highest order terms count

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care



Leading-term approximation

Ex 1. $\frac{1}{6} N^3 + 20 N + 16 \quad \sim \quad \frac{1}{6} N^3$

Ex 2. $\frac{1}{6} N^3 + 100 N^{4/3} + 56 \quad \sim \quad \frac{1}{6} N^3$

Ex 3. $\frac{1}{6} N^3 - \underbrace{\frac{1}{2} N^2 + \frac{1}{3} N}_{\text{discard lower-order terms}} \quad \sim \quad \frac{1}{6} N^3$

(e.g., N = 1000: 166.67 million vs. 166.17 million)

Technical definition. $f(N) \sim g(N)$ means $\displaystyle \lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Simplification 2: tilde notation

- Estimate running time (or memory) as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

operation	frequency	tilde notation
variable declaration	$N + 2$	$\sim N$
assignment statement	$N + 2$	$\sim N$
less than compare	$\frac{1}{2} (N + 1) (N + 2)$	$\sim \frac{1}{2} N^2$
equal to compare	$\frac{1}{2} N (N - 1)$	$\sim \frac{1}{2} N^2$
array access	$N (N - 1)$	$\sim N^2$
increment	$\frac{1}{2} N (N - 1)$ to $N (N - 1)$	$\sim \frac{1}{2} N^2$ to $\sim N^2$

Estimate performance by adding up **simplified** frequencies

Cost model 3: count only SOME operations

Simplifying the calculations

" It is convenient to have a measure of the amount of work involved in a computing process, even though it be a very crude one. We may count up the number of times that various elementary operations are applied in the whole process and then given them various weights. We might, for instance, count the number of additions, subtractions, multiplications, divisions, recording of numbers, and extractions of figures from tables. In the case of computing with matrices most of the work consists of multiplications and writing down numbers, and we shall therefore only attempt to count the number of multiplications and recordings. " — Alan Turing

ROUNDING-OFF ERRORS IN MATRIX PROCESSES

By A. M. TURING

(*National Physical Laboratory, Teddington, Middlesex*)

[Received 4 November 1947]

SUMMARY

A number of methods of solving sets of linear equations and inverting matrices are discussed. The theory of the rounding-off errors involved is investigated for some of the methods. In all cases examined, including the well-known 'Gauss elimination process', it is found that the errors are normally quite moderate: no exponential build-up need occur.

Simplification 1: cost model

Cost model. Use some basic operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N (N-1) = \binom{N}{2}$$

operation	frequency
variable declaration	$N + 2$
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less than compare	$\frac{1}{2} (N+1)(N+2)$
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array access	$\boxed{N (N-1)}$
increment	$\frac{1}{2} N (N-1)$ to $N (N-1)$

cost model = array accesses

(we assume compiler/JVM do not optimize any array accesses away!)

Performance estimate = (array accesses) $\times c_{\text{arraccess}}$

Careful!

Make sure that the operations you are not counting add up to a factor **lower** than the operations you do count.

Combinations of Cost Models

Each cost model makes a **simplification** in the calculation of running time.

$\Longrightarrow$ approximation of running time.

We can even **combine** the assumptions of different cost models.

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (N-1) = \frac{1}{2} N(N-1) = \binom{N}{2}$$

A. $\sim N^2$ array accesses.

Performance estimate = simplified number of array accesses

Bottom line. Use cost model and tilde notation to simplify counts.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size N?

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

$$\binom{N}{3} = \frac{N(N-1)(N-2)}{3!} \sim \frac{1}{6}N^3$$

A. $\sim \frac{1}{2} N^3$ array accesses.

Bottom line. Use cost model and tilde notation to simplify counts.

Diversion: estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 1. $1 + 2 + \ldots + N.$

$$\sum_{i=1}^{N} i \;\sim\; \int_{x=1}^{N} x\, dx \;\sim\; \frac{1}{2} N^2$$

Ex 2. $1^k + 2^k + \ldots + N^k.$

$$\sum_{i=1}^{N} i^k \;\sim\; \int_{x=1}^{N} x^k dx \;\sim\; \frac{1}{k+1} N^{k+1}$$

Ex 3. $1 + 1/2 + 1/3 + \ldots + 1/N.$

$$\sum_{i=1}^{N} \frac{1}{i} \;\sim\; \int_{x=1}^{N} \frac{1}{x} dx \;=\; \ln N$$

Ex 4. 3-sum triple loop.

$$\sum_{i=1}^{N} \sum_{j=i}^{N} \sum_{k=j}^{N} 1 \;\sim\; \int_{x=1}^{N} \int_{y=x}^{N} \int_{z=y}^{N} dz\, dy\, dx \;\sim\; \frac{1}{6} N^3$$

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course.

A2. Replace the sum with an integral, and use calculus!

Ex 4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

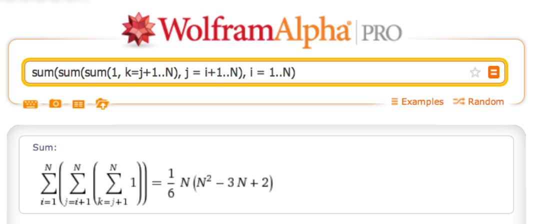
$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

Caveat. Integral trick doesn't always work!

Estimating a discrete sum

Q. How to estimate a discrete sum?

A3. Use Maple or Wolfram Alpha.



wolframalpha.com

```
[wayne:nobel.princeton.edu] > maple15
    |\^/|     Maple 15 (X86 64 LINUX)
._|\|   |/|_. Copyright (c) Maplesoft, a division of Waterloo Maple Inc. 2011
 \  MAPLE  /  All rights reserved. Maple is a trademark of
 <____ ____>  Waterloo Maple Inc.
      |       Type ? for help.
> factor(sum(sum(sum(1, k=j+1..N), j = i+1..N), i = 1..N));

                          N (N - 1) (N - 2)
                          -----------------
                                  6
```

Mathematical models for running time

In principle, accurate mathematical models are available.

In practice,

- Formulas can be complicated.
- Advanced mathematics might be required.
- Exact models best left for experts.

costs (depend on machine, compiler)

$$T_N = c_1 A + c_2 B + c_3 C + c_4 D + c_5 E$$

$A =$ array access

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frequencies (depend on algorithm, input)

Bottom line. We use approximate models in this course: $T(N) \sim c\,N^3$.

http://algs4.cs.princeton.edu

1.4 Analysis of Algorithms

- *introduction*
- *observations*
- *mathematical models*
- *order-of-growth classifications*
- *theory of algorithms*
- *memory*

Common order-of-growth classifications

Definition. If $f(N) \sim c\, g(N)$ for some constant $c > 0$, then the order of growth of $f(N)$ is $g(N)$.

- Ignores leading coefficient.
- Ignores lower-order terms.

Ex. The order of growth of the running time of this code is N^3.

```
int count = 0;
for (int i = 0; i < N; i++)
   for (int j = i+1; j < N; j++)
      for (int k = j+1; k < N; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

Typical usage. With running times.

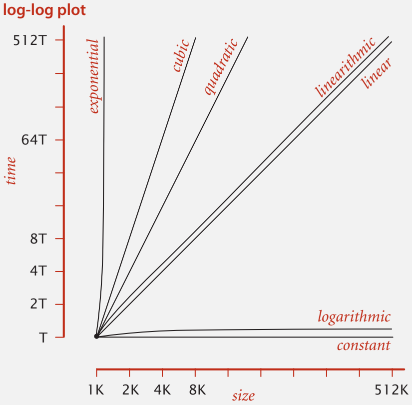
where leading coefficient depends on machine, compiler, JVM, ...

Common order-of-growth classifications

Good news. The set of functions

$$1, \ \log N, \ N, \ N \log N, \ N^2, \ N^3, \text{ and } 2^N$$

suffices to describe the order of growth of most common algorithms.



Typical orders of growth

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2N) / T(N)$
1	**constant**	`a = b + c;`	statement	add two numbers	1
$\log N$	**logarithmic**	`while (N > 1)` `{   N = N / 2;  ...   }`	divide in half	binary search	~ 1
N	**linear**	`for (int i = 0; i < N; i++)` `{  ...       }`	loop	find the maximum	2
$N \log N$	**linearithmic**	[see mergesort lecture]	divide and conquer	mergesort	~ 2

N^2	**quadratic**	`for (int i = 0; i < N; i++)`
		` for (int j = 0; j < N; j++)`
		`{  ...       }`
			double loop	check all pairs	4

N^3	**cubic**	`for (int i = 0; i < N; i++)`
		` for (int j = 0; j < N; j++)`
		`  for (int k = 0; k < N; k++)`
		`{  ...       }`
			triple loop	check all triples	8

2^N	**exponential**	[see combinatorial search lecture]	exhaustive search	check all subsets	$T(N)$

Binary search demo

Goal. Given a sorted array and a key, find index of the key in the array?

Binary search. Compare key against middle entry.

- Too small, go left.
- Too big, go right.
- Equal, found.

successful search for 33

6	13	14	25	33	43	51	53	64	72	84	93	95	96	97
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
↑ lo														↑ hi

Binary search: Java implementation

Trivial to implement?

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006.

```
public static int binarySearch(int[] a, int key)
{
   int lo = 0, hi = a.length-1;
   while (lo <= hi)
   {
      int mid = lo + (hi - lo) / 2;
      if      (key < a[mid]) hi = mid - 1;
      else if (key > a[mid]) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

one "3-way compare"

Invariant. If `key` appears in the array `a[]`, then `a[lo]` ≤ `key` ≤ `a[hi]`.

Binary search: mathematical analysis

Proposition. Binary search uses at most $1 + \lg N$ key compares to search in a sorted array of size N.

Def. $T(N)$ = # key compares to binary search a sorted subarray of size ≤ N.

Binary search recurrence. $T(N) \le T(N/2) + 1$ for $N > 1$, with $T(1) = 1$.

left or right half (floored division)

possible to implement with one 2-way compare (instead of 3-way)

Pf sketch. [assume N is a power of 2]

$$
\begin{aligned}
T(N) &\le T(N/2) + 1 && \text{[given]} \\
&\le T(N/4) + 1 + 1 && \text{[apply recurrence to first term]} \\
&\le T(N/8) + 1 + 1 + 1 && \text{[apply recurrence to first term]} \\
&\vdots \\
&\le T(N/N) + 1 + 1 + \ldots + 1 && \text{[stop applying, } T(1) = 1 \text{]} \\
&= 1 + \lg N
\end{aligned}
$$

An $N^2 \log N$ algorithm for 3-SUM

Algorithm.

- Step 1: Sort the N (distinct) numbers.
- Step 2: For each pair of numbers `a[i]` and `a[j]`, binary search for `-(a[i] + a[j])`.

Analysis. Order of growth is $N^2 \log N$.

- Step 1: N^2 with insertion sort.
- Step 2: $N^2 \log N$ with binary search.

Remark. Can achieve N^2 by modifying binary search step.

input

```
30 -40 -20 -10 40 0 10 5
```

sort

```
-40 -20 -10 0 5 10 30 40
```

binary search

(-40, -20)	60
(-40, -10)	50
(-40, 0)	40
(-40, 5)	35
(-40, 10)	30
⋮	⋮
(-20, -10)	30
⋮	⋮
(-10, 0)	10
⋮	⋮
(10, 30)	~~-40~~
(10, 40)	-50
(30, 40)	-70

only count if a[i] < a[j] < a[k] to avoid double counting

Comparing programs

Hypothesis. The sorting-based $N^2 \log N$ algorithm for 3-SUM is significantly faster in practice than the brute-force N^3 algorithm.

N	time (seconds)
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1

ThreeSum.java

N	time (seconds)
1,000	0.14
2,000	0.18
4,000	0.34
8,000	0.96
16,000	3.67
32,000	14.88
64,000	59.16

ThreeSumDeluxe.java

Guiding principle. Typically, better order of growth $\Rightarrow$ faster in practice.