CS2010: ALGORITHMS AND DATA STRUCTURES

Lecture 12: Recursion vs Iteration

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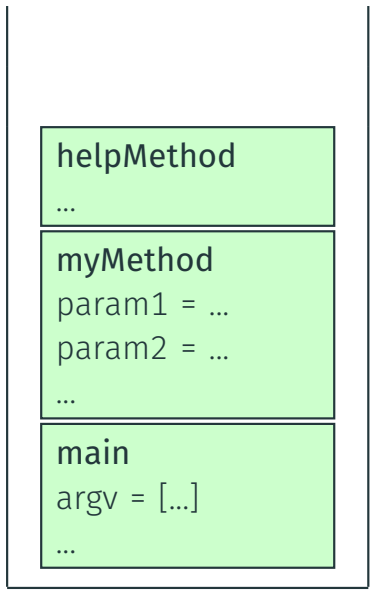
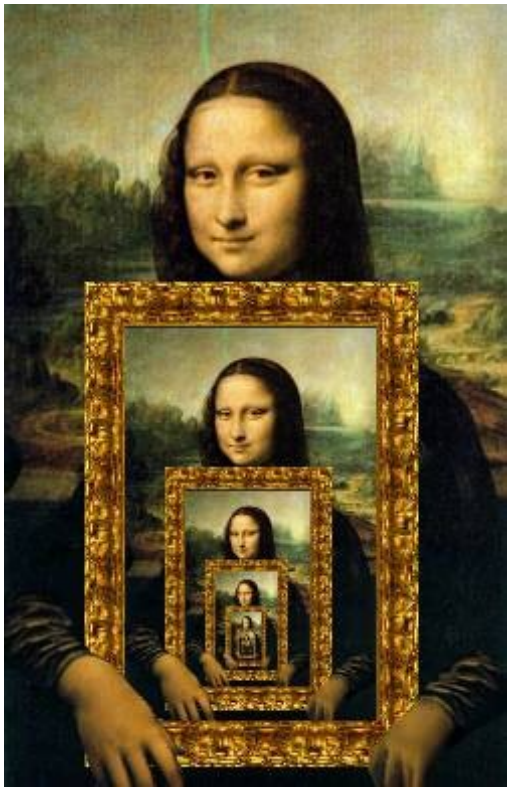
HOW METHODS EXECUTE

- → Call stack: is a stack maintained by the Java runtime system
- → One call stack frame (aka activation record) for each running instance of a method: contains all information necessary to execute the method
  - **references** to parameter values and local objects, return address etc.
- → Objects themselves are stored in another part of memory: the heap
- → every time a method is called, a new stack frame is pushed on the call stack.
- → every time a method returns, the top-most stack frame is popped.



Recursion: when something is defined in terms of itself.

Infinite Recusion



Well-founded Recursion

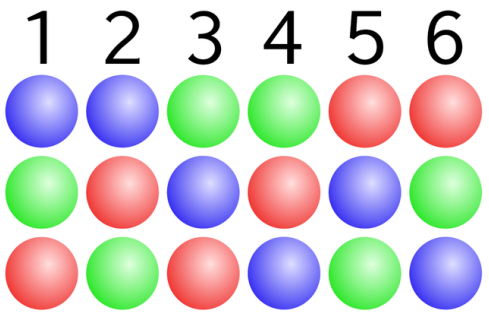
Principle: A method is recursive when its definition **calls the method itself**.

A correct recursive method should be well-founded: it should terminate.

Classic example: factorial – in math written as: $n!$

Math definition:

$$0! = 1$$
$$n! = n \cdot (n-1)! \quad \text{when } n > 0$$



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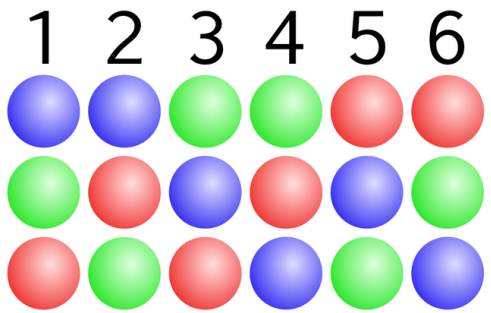

Classic example: Fibonacci numbers

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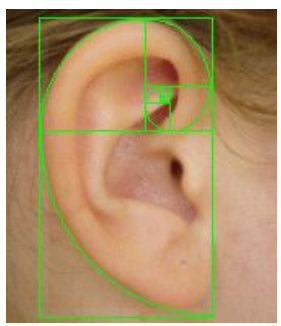

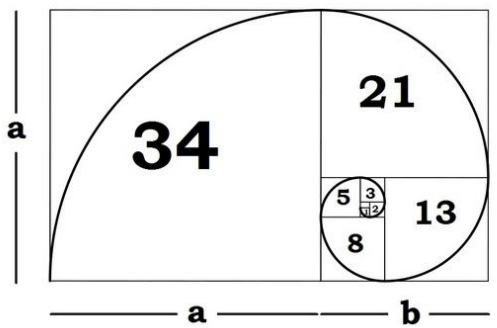

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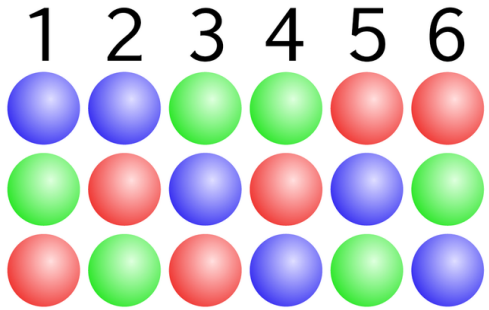

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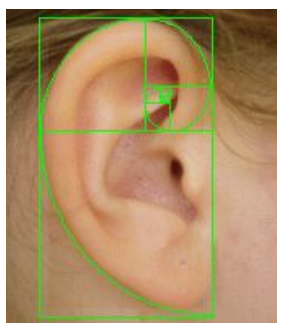

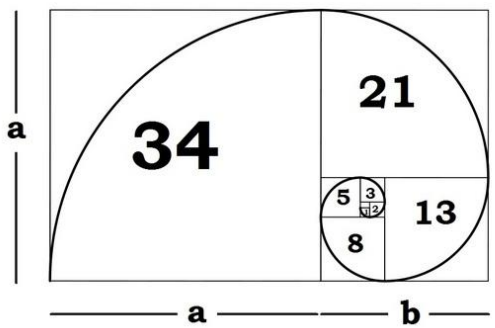

It is convenient to implement recursive math definitions using recursive methods.

IMPLEMENTATION OF FACTORIAL

Recursive implementation:

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int fac(int n) {



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IMPLEMENTATION OF FACTORIAL

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- → the rest of the cases are the recursive cases (here when $n > 0$)
- → specify how smaller solutions **compose** into the solutions of recursive cases
- → it is usually a **top-down calculation**

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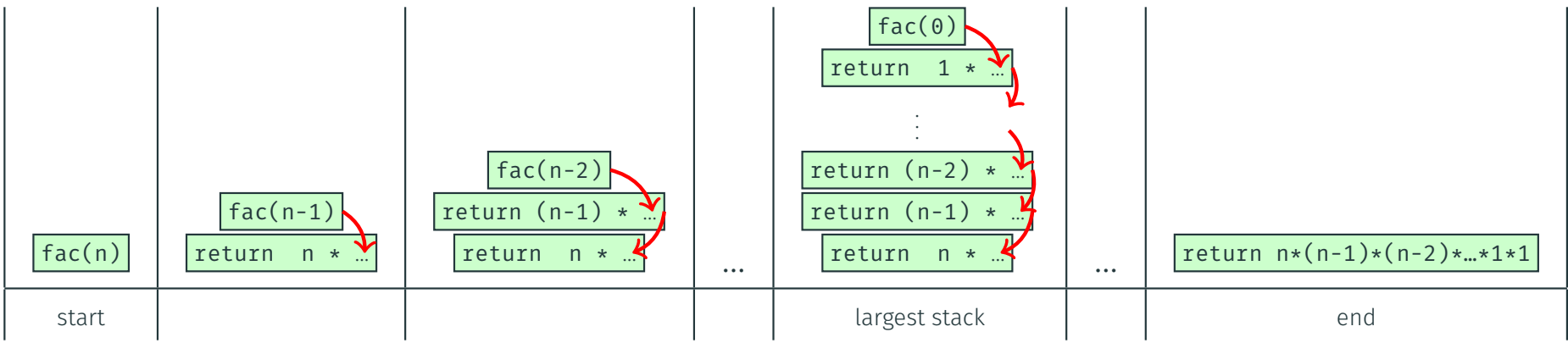
→ Q: memory space for call stack frames? (max frames on call stack)
A: $\Theta(n)$ space
All this call stack space is needed because of the `return n *` …



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- → In other, mainly functional, languages (ML, Lisp, Haskell, ...) the compiler runs this using $\Theta(1)$ stack space.
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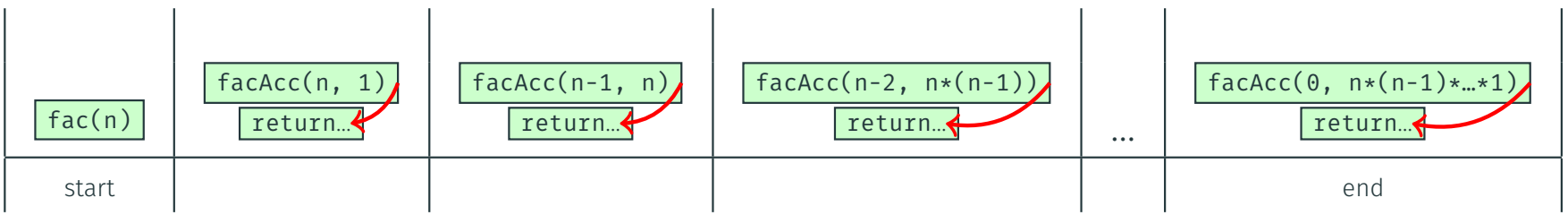

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From recursive implementation w/ accumulator $\longrightarrow$ iterative implementation:

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→ Running time of iterative implementation: $\Theta(n)$

→ Stack space of iterative implementation: $\Theta(1)$

In functional languages this simple translation is done by the compiler!

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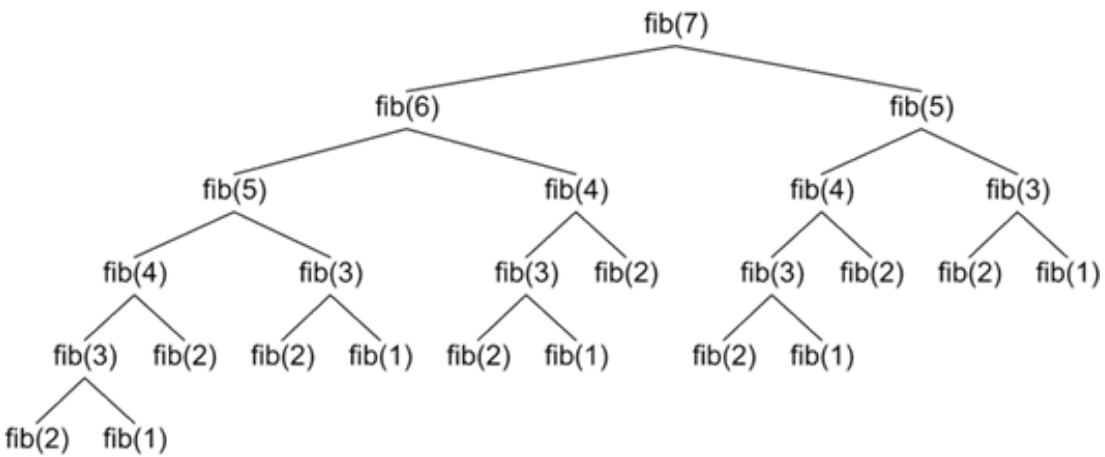
→ Running time: non-tight upper bound: $O(2^n)$ tight bound: $\Theta(fib(n))$

→ # of recursive calls of fib(n) is the size of the binary tree of recursive calls with n levels ($\leq 2^n$); however this is not a complete tree; thus $O(2^n)$.



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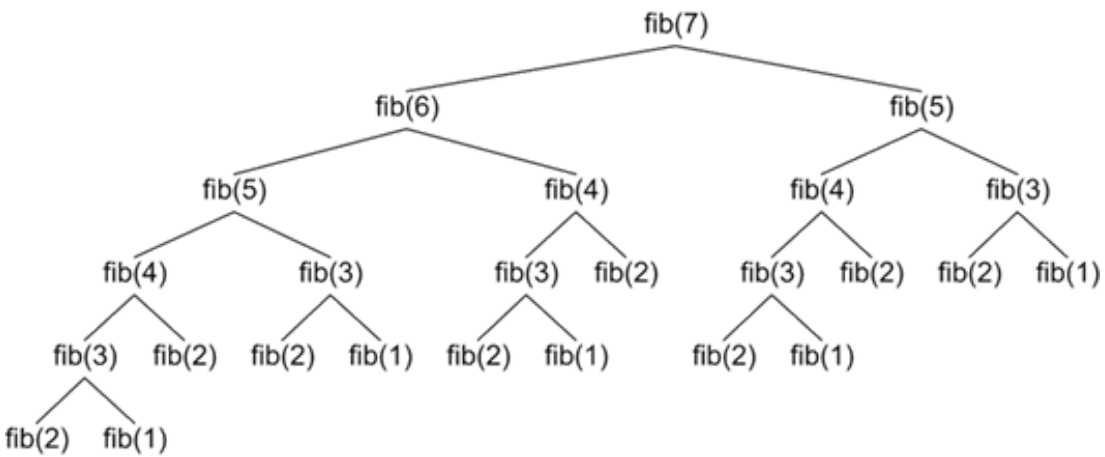

→ Call stack space: $\Theta(n)$

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Recursive implementation with accumulator (tail recursion):

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→ This is much trickier! Read it again off-line and understand why it works.

→ It is a bottom-up calculation using two accumulators.

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→ Running time: $\Theta(n)$

→ Call stack space: $\Theta(n)$ in Java and $\Theta(1)$ in other languages.

FIBONACCI NUMBERS

Tail-recursive implementation $\longrightarrow$ iterative implementation

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```
int fib(int n) {
  int last = 1; int secondToLast = 1;
  for ( ; !(n <= 1); n--) {
    int tmpLast = last;
    last = last + secondToLast;
    secondToLast = tmpLast;
  }
  return last;
}
```

→ Worst Case Asymptotic Running time: $\Theta(n)$

→ Call stack space: $\Theta(1)$

GREATEST COMMON DIVISOR (GCD)

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Attempt 1: try all numbers n from **+inf** down to 1 until you find one that has the above property.

Attempt 2: try all numbers n from **x** down to 1 until you find one that has the above property (because gcd will necessarily be $\leq min(x, y)$).

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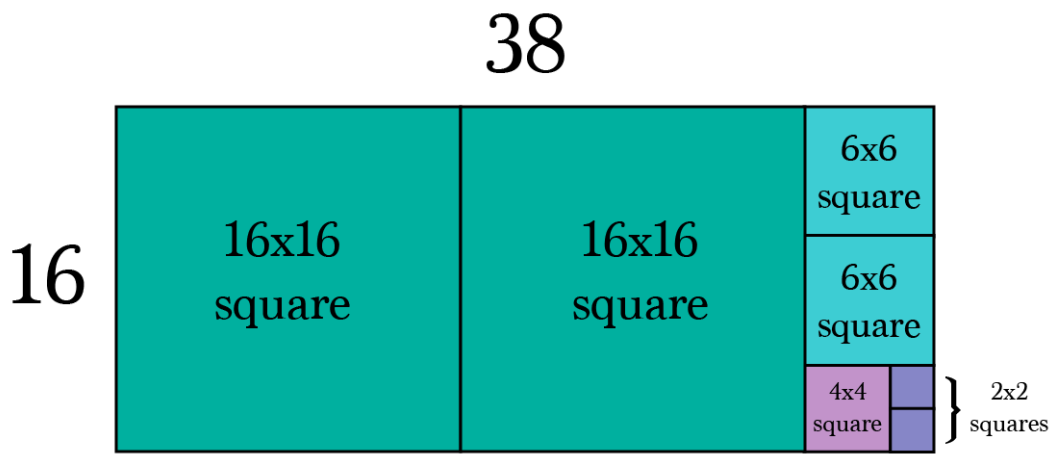
Attempt 3: Euclid's algorithm a Divide & Conquer approach:

Euclid's Theorem: gcd(x,y) = gcd(y, x % y).



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→ Q: How many iterations in the worst case?

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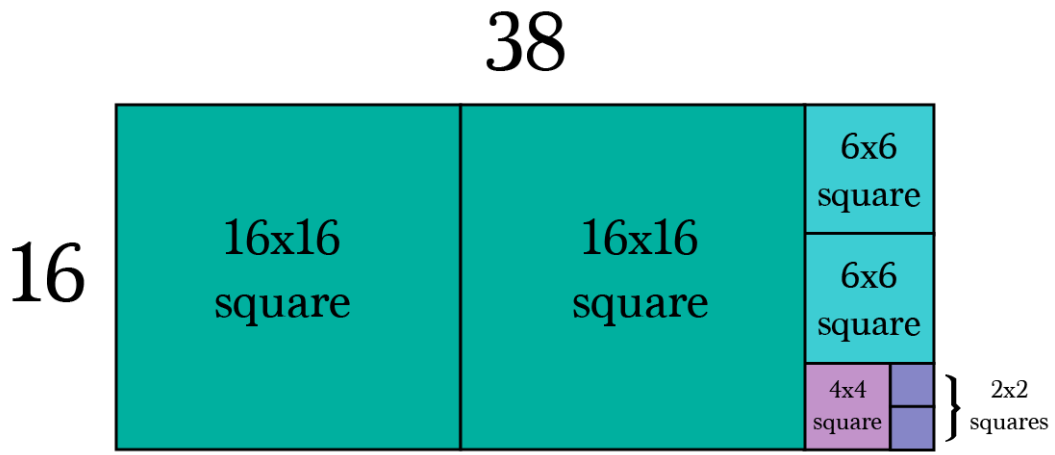

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- → Q: How many iterations in the worst case?
- → **Gabriel Lame's Theorem (1844):** #iterations $< 5 \cdot h$
  Where h = digits of $min(x, y)$ (here this is x) in base 10.
- → A: $O(lg(x))$

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- → Iterative algorithms usually have easier proofs of memory usage (and in Java use less memory)

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- → Iterative algorithms usually have easier proofs of memory usage (and in Java use less memory)
- → Recursive algorithms are sometimes easier to implement when we have complex data structures.
  - → because the compiler maintains a stack of previous calls for us.
  - → We will use recursive implementations for most operations over **trees**.

RECURSION VS ITERATION: WHICH ONE TO CHOOSE?

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- → Iterative algorithms usually have easier proofs of memory usage (and in Java use less memory)
- → Recursive algorithms are sometimes easier to implement when we have complex data structures.
  - → because the compiler maintains a stack of previous calls for us.
  - → We will use recursive implementations for most operations over **trees**.
- → We *could* have used recursive implementations for operations over lists & arrays but:
  - → they are not simpler than the iterative implementations
  - → Java will need $O(n)$ call stack space to execute them.

HOMEWORK (OPTIONAL)

Homework 1: Implement Euclid's algorithm using

- → A recursive method.
- → An iterative method.

Homework 2: give recursive implementations for:

- → binary search over an array
- → linear search over a linked list

Homework 3: Implement **search** on a Binary Search Tree (next lecture) using recursion and compare with the iterative version in the book.

Homework 4:** give an iterative implementation for method **put** on binary search tree.