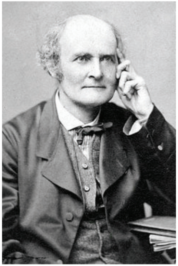
Linear Equations
Consistent and Inconsistent

Invention of Matrices

How old are matrices?

- Matrices were invented by the British mathematician Arthur Cayley (1821-1895).
- Cayley presented a paper giving the rule for matrix operations and the conditions under which a matrix has an inverse to the Royal Society in 1858.
- Cayley's friend, James Joseph Sylvester (1814-1897), was the person who first used the term "matrix" in 1850.



Arthur Cayley (1821-1895)

James Joseph Sylvester (1814-1897)

m Linear Equations in n unknowns

A system of m Linear Equations in n unknowns

$$\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= c_1 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= c_2 \\
\vdots \quad & \quad \vdots \\
a_{m1} * x_1 + a_{m2} * x_2 + \cdots + a_{mn} * x_n &= c_m
\end{aligned}$$

is:

- **Consistent**, if it has a least one solution
- **Inconsistent**, if it has no solution

The i^{th} equation of the m equations above may be rewritten as:

$$\sum_{j=1}^{n} a_{ij} x_j = c_i$$

Linear Equations in Matrix Form

The above system of m linear equations in n unknowns can be written in matrix form as:

$$A * x = c$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

Note:
m may not be same as n .
The vector x of unknowns has length n while the vector c of constants has length m.
When row i 'dives' into the column vector, x, we get the sum: $\sum_{j=1}^{n} a_{ij} x_j$.

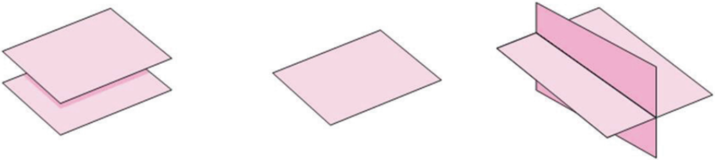
Geometric view

With $n = 3$, the equations have 3 unknowns. We could write an equation in 3 unknowns, x, y and z as:

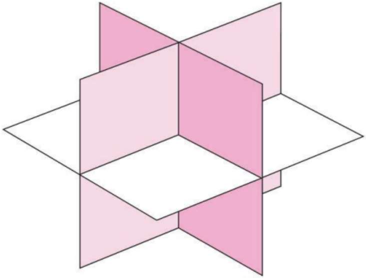
$$a * x + b * y + c * z = d.$$

(a, b and c are not zero constants)
This equation represents a plane in $\mathbb{R}^3$. With one equation, there are infinite solutions.

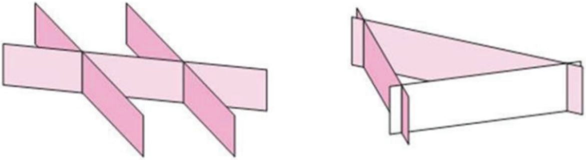
With 2 linear equations in 3 unknowns:
No solution or infinite solutions.



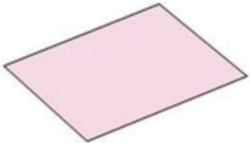
With 3 linear equations in 3 unknowns:
Unique solution:

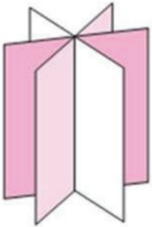


With 3 linear equations in 3 unknowns:
No solution:



With 3 linear equations in 3 unknowns:
Infinite solutions:
when the planes intersect either in overlapping planes or intersect in a line.





Augmented Matrix

From the system of m Linear Equations in n unknowns

$$\begin{aligned}
a_{11} * x_1 + a_{12} * x_2 + \cdots + a_{1n} * x_n &= c_1 \\
a_{21} * x_1 + a_{22} * x_2 + \cdots + a_{2n} * x_n &= c_2 \\
\vdots \qquad & \quad \vdots \\
a_{m1} * x_1 + a_{n2} * x_2 + \cdots + a_{mn} * x_n &= c_m
\end{aligned}$$

we have the augmented matrix of the system as:

Augmented Matrix (Cont'd)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{array}\right]$$

or dropping the 'dividing line'

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{bmatrix}$$

Gauss/Jordan Elimination Method

The Gauss/Jordan method uses elementary row operations to reduce an augmented matrix to a simpler form so that the solution set of the orginal system of linear equations can be found, if there is a solution.
The elementary row operations are:
(let R_k name the k^{th} row)

Allowed row operations

1. Interchange (swap) rows R_i and R_j
2. Multiply a row by a non-zero scalar (number): $R_i := k * R_i$.
3. Add a multiple of one row to another: $R_i := R_i + k * R_j$ where $i \neq j$.

Row Equivalent Matrices

Matrices A and B are row equivalent if one is obtained from the other by elementary row operations.
If the augmented matrices A and B are row equivalent then they have the same solution set.
Given an initial augumented matrix A for a system of linear equations and the matrix A is reduced by row operations to a row equivalent form B which is simpler then B can be used to find the solution set for the initial matrix A.

Row Equivalent Matrices (Cont'd)

The Gauss/Jordan method reduces an initial augmented matrix A to a row equivalent simpler form B using elementary row operations.

The Gauss/Jordan method reduces the matrix to **Reduced Row Echelon Form**.

This is done in two stages:

1. Reduce matrix A to **Row Echelon Form**
   then
2. Reduce this result further to **Reduced Row Echelon Form**

Row Equivalent Matrices (Cont'd)

The matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{bmatrix}$$

is first converted to the form

$$\begin{bmatrix} 1 & a_{12}^* & \cdots & a_{1n}^* & c_1^* \\ 0 & 1 & \cdots & a_{2n}^* & c_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_m^* \end{bmatrix}$$

where the 'starred' entries are the final values in **row echelon form**. Then the matrix is further converted to **Reduced Row Echelon Form** i.e.

Row Equivalent Matrices (Cont'd)

$$\begin{bmatrix} 1 & a_{12}^* & \cdots & a_{1n}^* & c_1^* \\ 0 & 1 & & a_{2n}^* & c_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{m2} & \cdots & 1 & c_m^* \end{bmatrix}$$

is further reduced to the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_1^{**} \\ 0 & 1 & \cdots & 0 & c_2^{**} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_m^{**} \end{bmatrix}$$

From the **Reduced Row Echelon Form**, the soluton set for the initial system of linear equations can be found.

Row-Echelon Form

(From Dictionary: ***Echelon***
a formation of troops, ships, aircraft, or vehicles in parallel rows with the end of each row projecting further than the one in front.)

Row Echelon Form of a Matrix

1. All zero rows (if any) are at the bottom of the matrix
2. In a non-zero row, the first non zero entry is 1 (the leading 1)
3. Each leading 1 in a row is to the right of the leading 1 in the row above it.
   As a consequence, all entries, if any, in the column below the leading 1 are zeros.

Example: Row-Echelon Form

The Row Echelon Form of the 3×6 matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Reduced Row-Echelon Form

A Matrix is in Reduced Row-Echelon Form if:

1. it is Row-Echelon Form
2. Each column that has a leading 1 has zeros elsewhere in the column.

Reduced Row-Echelon Form (Cont'd)

The **Reduced Row Echelon Form** of

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

It is in Row Echelon Form and each column that has a leading 1 in a row has zeros elsewhere.

Algorithm for Row Echelon Form

Given an $m \times n$ matrix, A. (Using $:=$ for assignment)

```
i := 1;
while ( A is not in Row Echelon Form )
{
    Locate the leftmost non-zero column;
    If top of column is zero, swap with another row
    to bring an entry, k ≠ 0, to the top;
    If ( k ≠ 1 )
        R_i := R_i / k; // row i has leading 1
    Use suitable multiples of R_i to add to other rows
    below so that entries below the leading 1 are zeros;
    // Ignore Row i
    i := i+1;
    // Consider remaining submatrix
}
```

Example Row Echelon Form

Reduce

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to Row Echelon Form.
As $A_{11} = 0$, swap $R1$ and $R2$

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Example (Cont'd)

$$R1 := \frac{R1}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

$$R3 := R3 - 2 * R1$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Ignore $R1$, and process submatrix

$$R2 := \frac{R2}{-2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

$$R3 := R3 - 5 * R2$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Ignore $R2$, process submatrix

Example (Cont'd)

$$R3 := 2 * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Matrix is in Row Echelon Form

Algorithm: Reduced Row Echelon Form

First convert a matrix to Row Echelon Form:

```
Begin with  last nonzero row;
Work upwards;
Add suitable multiples of each row to the rows
above to introduce zeros above the leading 1;
```

Example: Reduced Row Echelon Form

From above we have the Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & \frac{-7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Starting with the 3^{rd} row and 5^{th} column, work upwards:

Example (Cont'd)

$$R2 := R2 + \frac{7}{2} * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R1 := R1 - 6 * R3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$R1 := R1 + 5 * R2$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Matrix is now in

Reduced Row Echelon Form

The initial augmented matrix:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

represents the linear system:

$$\begin{aligned} -2 * x_3 + 7 * x_5 &= 12 \\ 2 * x_1 + 4 * x_2 - 10 * x_3 + 6 * x_4 + 12 * x_5 &= 28 \\ 2 * x_1 + 4 * x_2 - 5 * x_3 + 6 * x_4 + -5 * x_5 &= -1 \end{aligned}$$

Cont'd

The Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

represents the linear system:

$$\begin{aligned} x_1 + 2 * x_2 + 7 * x_4 &= 7 \\ x_3 &= 1 \\ x_5 &= 2 \end{aligned}$$

No Solution Example

Consider the augmented matrix:

$$\begin{bmatrix} 0 & 0 & 4 & 0 \\ 5 & 5 & -1 & 5 \\ 2 & 2 & -2 & 5 \end{bmatrix}$$

swap $R1$ and $R3$

$$\begin{bmatrix} 2 & 2 & -2 & 5 \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Cont'd

$$R1 := \frac{R1}{2}$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 5 & 5 & -1 & 5 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R2 := \frac{R2}{4}$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R2 := R2 - 5 * R1$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 4 & \frac{-15}{2} \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

$$R3 := R3 - 4 * R2$$
$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix}$$

Linear System with No Solution

$$R3 := \frac{2}{15} * R3$$

$$\begin{bmatrix} 1 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{-15}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix is now in Row Echelon Form.

At this stage it can detected that there is no solution.

Linear System with No Solution (Cont'd)

Rewriting the Row Echelon Form of the augmented matrix back in terms of the original unknown variables:

$$\begin{aligned} x_1 + x_2 - x_3 &= \frac{5}{2} \\ x_3 &= \frac{-15}{9} \\ 0 &= 1 \end{aligned}$$

There is a contradiction with $0 = 1$ and so the initial system has no solution.

By reducing the initial augmented matrix to Row Echelon Form, it was found that the initial system was **inconsistent** and so has no solution.

Consistent System of Linear Equations

While it is possible for a system of Linear Equations to have either an unique solution or no solution, it is also possible for a system to have an infinite number of solutions.
A system of Linear Equations is **consistent** if is has either an unique solution or an infinite number of solutions.
Consider the system of Linear Equations:
(4 equations in 5 unknowns)

$$\begin{array}{ccccccccccc} 2*x_1 & + & 2*x_2 & - & 6*x_3 & & & + & 6*x_5 & = & 2 \\ 4*x_1 & + & 4*x_2 & - & 11*x_3 & + & 2*x_4 & + & 11*x_5 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & 8*x_4 & + & x_5 & = & 11 \\ -3*x_1 & - & 3*x_2 & + & 11*x_3 & + & 4*x_4 & & & = & 12 \end{array}$$

Consistent System: Infinite Solutions

The Augmented matrix for this system is:

$$\begin{bmatrix} 2 & 2 & -6 & 0 & 6 & 2 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{bmatrix}$$

We convert this Matrix to Reduced Row Echelon Form to determine if it has an infinite set of solutions.

Cont'd

$$\begin{bmatrix} 2 & 2 & -6 & 0 & 6 & 2 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{bmatrix}$$

$$R1 := \frac{R1}{2}$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 4 & 4 & -11 & 2 & 11 & 6 \\ 1 & 1 & 1 & 8 & 1 & 11 \\ -3 & -3 & 11 & 4 & 0 & 12 \end{bmatrix}$$

Cont'd

$$R2 := R2 - 4 * R1$$
$$R3 := R3 - R1$$
$$R4 := R4 + 3 * R1$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 4 & 8 & -2 & 10 \\ 0 & 0 & 2 & 4 & 9 & 15 \end{bmatrix}$$

Cont'd

$$R3 := R3 - 4 * R2$$
$$R4 := R4 - 2 * R2$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{bmatrix}$$

Cont'd

$$R3 := \frac{R3}{2}$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 11 & 11 \end{bmatrix}$$

$$R4 := R4 - 11 * R3$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix is now in Row Echelon Form.

Reduced Row Echelon

Continue to convert the matrix to Reduced Row Echelon Form.

$$R2 := R2 + R3$$
$$R1 := R1 - 3 * R3$$

$$\begin{bmatrix} 1 & 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R1 := R1 + 3 * R2$$

$$\begin{bmatrix} 1 & 1 & 0 & 6 & 0 & 7 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix is now in Reduced Row Echelon Form

Rewriting at Linear Equation

We can rewrite this Reduced Row Echelon Form matrix back into a system of linear equations:

$$\begin{aligned} x_1 + x_2 + 6 * x_4 &= 7 \dots Eq1 \\ x_3 + 2 * x_4 &= 3 \dots Eq2 \\ x_5 &= 1 \dots Eq3 \end{aligned}$$

The last row of all zeros corresponds to
$0 * x_1 + 0 * x_2 + 0 * x_3 + 0 * x_4 + 0 * x_5 = 0$
which is true and adds no additional information.
We can refer to the unknown variables, x_1, x_3 and x_5 as lead variables as they correspond to the leading 1 in the Reduced Row Echelon Form.

Cont'd

From equation, $Eq3$, we have $x_5 = 1$. We have no equation to determine x_4 in terms of x_5 and no equation to determine x_2 in terms of x_3, x_4, x_5.
Since we have 3 equations in 5 unknowns, we cannot detemine an unique solution but we can describe an infinite solution set by:
$x_1 = 7 - x_2 - 6 * x_4$
$x_3 = 3 - 2 * x_4$
$x_5 = 1.$
For readability, we introduce parameters s and t for x_2 and x_4 and so in terms of the parameters s and t, we have the infinite solution set:

Solution Set

$x_1 = 7 - s - 6 * t$;
$x_2 = s$
$x_3 = 3 - 2 * t$
$x_4 = t$
$x_5 = 1.$

While s and t can have any value, once s and t are fixed a solution is determined.
While there are infinite solutions, the solution set is determined by the above equations involving s and t.

Introducing Parameters

Introducing Parameters

- Introduce parameters when the system of linear equations from the Reduced Row Echelon Form has few equations than the numbers of unknowns.
- Introduce parameters for the non leading unknowns; in the above for $x_2$and x_4.
- When one or more parameters are introduced, this implies there are infinitely many solutions.

Instructive Example

Consider the following system of linear equations which includes a fixed but unknown constant, k.

$$\begin{aligned} x_1 + 2 * x_2 - 3 * x_3 &= 4 \\ 5 * x_1 + 3 * x_2 - x_3 &= 10 \\ 9 * x_1 + 4 * x_2 + (k^2 - 15) * x_3 &= k + 12 \end{aligned}$$

For which values of k does this system have:

1. No solution
2. Infinite number of solutions
3. A unique solution

Instructive Example (Cont'd)

Augmented Matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 5 & 3 & -1 & 10 \\ 9 & 4 & (k^2-15) & (k+12) \end{bmatrix}$$

$$R2 := R2 - 5 * R1$$

$$R3 := R3 - 9 * R1$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -14 & (k^2+12) & (k-24) \end{bmatrix}$$

$$R2 := \frac{R2}{-7}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -14 & (k^2+12) & (k-24) \end{bmatrix}$$

$$R3 := R3 + 14 * R2$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & (k^2-16) & (k-4) \end{bmatrix}$$

Matrix is in Row Echelon Form

Instructive Example (Cont'd)

We can further reduce the matrix towards a Reduced Row Echelon Form:

$$R1 := R1 - 2 * R2$$

$$\begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & (k^2 - 16) & (k - 4) \end{bmatrix}$$

We can rewrite this back as a system of Linear Equations.

Instructive Example (Cont'd)

$$\begin{aligned} x_1 + x_3 &= \frac{8}{7} \ldots Eq1 \\ x_2 - 2 * x_3 &= \frac{10}{7} \ldots Eq2 \\ (k^2 - 16) * x_3 &= k - 4 \ldots Eq3 \end{aligned}$$

Inconsistency of the system depends on $Eq3$ as the unknown constant, k, appears only in this equation.
The system is **Inconsistent** when:

$$0 * x_3 = c \text{ where } c \neq 0$$

i.e. when:

$$k^2 - 16 = 0 \text{ and } k - 4 \neq 0$$

No Solution

Solving $k^2 - 16 = 0$ we have $k = 4$ or $k = -4$.
When $k - 4 \neq 0$ then $k \neq 4$.
The system is **Inconsistent** when $k^2 - 16 = 0$ and $k - 4 \neq 0$
i.e. when ($k = 4$ or $k = -4$.) and $k \neq 4$
{ From Logic: (P or Q) and (not P) = Q and (not P) }
i.e. when $k = -4$ and $k \neq 4$
i.e. when $k = -4$.

- There is no solution to the system when $k = -4$.

Instructive Example (Cont'd)

Infinite number of solutions

In *Eq*3 above, there is an infinite number of solutions when:
$0 * x_3 = 0$
i.e. when $k^2 - 16 = 0$ and $k - 4 = 0$
i.e. when ($k = 4$ or $k = -4$) and $k = 4$
{ From Logic: (P or Q) and P = P }
i.e. when $k = 4$.

Unique Solution

In $Eq3$ above we have:

$$(k^2 - 16) * x_3 = k - 4$$

If $k^2 - 16 \neq 0$ then we can divide across by $k^2 - 16$ to get:

$$x_3 = \frac{k-4}{k^2-16}$$

Note: if $k = 4$ or $k = -4$ then $\frac{k-4}{k^2-16}$ is not defined.

- There is an unique solution when $k \neq 4$ and $k \neq -4$ i.e. when $k \notin \{-4, 4\}$.

Example of unique solution

Rewriting the equations above:

$$
\begin{aligned}
x_1 &= -x_3 + \frac{8}{7} \\
x_2 &= 2 * x_3 + \frac{10}{7} \\
x_3 &= \frac{k-4}{k^2-16}
\end{aligned}
$$

For example, with $k = 0$, $x_3 = \frac{1}{4}$
$x_1 = -\frac{1}{4} + \frac{8}{7}$ i.e.

$$x_1 = \frac{25}{28}$$

$x_2 = \frac{1}{2} + \frac{10}{7}$ i.e.

$$x_2 = \frac{27}{14}$$