Example: Finding Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

1 Solution:

First, the eigenvalues are found and then for each of the eigenvalues the corresponding eigenvectors are found.

1.1 Find the Eigenvalues

Find the values of λ that satisfy the characteristic equation:

$$|A - \lambda * Id| = 0$$

where Id is the 3×3 identity matrix.

$A - \lambda * Id$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \lambda * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{bmatrix}$$

1.1.1 Calculate $|A - \lambda * Id|$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix}$$

$$=(1-\lambda) * \begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3) * \begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3 * \begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix}$$

$$
\begin{aligned}
&= (1-\lambda) * ((-5-\lambda) * (4-\lambda) \\
&\quad +3 * (3 * (4-\lambda) - 6 * 3) \\
&\quad +3 * (3 * (-6) - 6 * (-5-\lambda))
\end{aligned}
$$

$= (1-\lambda)*((-20+5*\lambda-4*\lambda+\lambda^2)+18)+3*((12-3*\lambda)-18)+3*(-18-(-30-6*\lambda))$

$= (1-\lambda) * (-2+\lambda+\lambda^2) + 3 * (-6-3*\lambda) + 3 * (12+6*\lambda)$

$= -2+\lambda+\lambda^2+2*\lambda-\lambda^2-\lambda^3-18-9*\lambda+36+18*\lambda$

$= 16+12*\lambda-\lambda^3$

$\therefore |A-\lambda * Id| = 16+12*\lambda-\lambda^3$
the characteristic polynomial of the Matrix A.

1.1.2 Find integer solutions for λ in $\lambda^3 - 12*\lambda - 16 = 0$

An integer solution for $\lambda^3 - 12*\lambda - 16 = 0$ divides the constant 16. Possible factors of 16 are:
$\pm1, \pm2, \pm4, \pm8, \pm16$
By trial we find that $4^3 - 12*4 - 16 = 0$ and so $\lambda - 4$ is a factor of $\lambda^3 - 12*\lambda - 16$
Divide $\lambda^3 - 12*\lambda - 16$ by $\lambda - 4$

$$
\begin{array}{ll}
 & \lambda^2 + 4*\lambda + 4 \\
\cline{2-2}
\lambda - 4 & \multicolumn{1}{|l}{\lambda^3 - 12*\lambda - 16} \\
 & \lambda^3 - 4*\lambda^2 \\
\cline{2-2}
\text{subtract} & 4*\lambda^2 - 12*\lambda - 16 \\
 & 4*\lambda^2 - 16*\lambda \\
\cline{2-2}
\text{subtract} & 4*\lambda - 16 \\
 & 4*\lambda - 16 \\
\cline{2-2}
\text{subtract} & 0
\end{array}
$$

$\therefore$
$\lambda^3 - 12*\lambda - 16 = (\lambda - 4) * (\lambda^2 + 4*\lambda + 4)$

Factors of $\lambda^2 + 4*\lambda + 4$

By inspection $\lambda^2 + 4*\lambda + 4 = (\lambda+2)^2$ or by formula;
the formula for the roots of $a*\lambda^2 + b*\lambda + c$

$$
\begin{aligned}
\lambda &= \frac{-b \pm \sqrt{b^2 - 4*a*c}}{2*a} \\
&= \frac{-4 \pm 0}{2}
\end{aligned}
$$

Factors of $\lambda^3 - 12*\lambda - 16$ are $\lambda - 4$, $\lambda + 2$ and $\lambda + 2$
i.e. $\lambda^3 - 12*\lambda - 16 = (\lambda - 4) * (\lambda + 2) * (\lambda + 2)$

The roots of $\lambda^3 - 12 * \lambda - 16$ are 4 and -2, where -2 is a repeated root.
The eigenvalues of the matrix A are $\lambda_1 = 4$ and $\lambda_2 = -2$.
Recall, $\lambda^3 - 12 * \lambda - 16 = 0$ is the **characteristic equation** of the matrix A.

1.2 Finding the Eigenvectors

We find the eigenvectors corresponding to the eigenvalues.

- For each eigenvalue, λ, we have $(A - \lambda * Id) * x = 0$,
  where the vectors, x, are the corresponding eigenvectors for the eigenvalue, λ.

- Find the vectors, x, by Gauss/Jordan elimination, i.e. reduce the augmented matrix, $[A - \lambda * x \,|\, 0]$ to reduced row echelon form and solve the associated linear system.

1.2.1 Case when the eigenvalue is $\lambda_1 = 4$

Find vectors, x, which satisfy $(A - 4 * Id) * x = 0$

$$A - 4 * Id = \begin{bmatrix} 1-4 & -3 & 3 \\ 3 & -5-4 & 3 \\ 6 & -6 & 4-4 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{bmatrix}$$

$R1 := \frac{R1}{-3}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{bmatrix}$$

$R2 := R2 - 3$*R1
$R3 := R3 - 6 * R1$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{bmatrix}$$

$R2 := \frac{R2}{-12}$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -12 & 6 & 0 \end{bmatrix}$$

$R3 := R3 + 12 * R2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R1 := R1 - R2$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewriting this augmented matrix back as a linear system we get:

$$\begin{aligned} x_1 - \frac{1}{2} * x_3 &= 0 \\ x_2 - \frac{1}{2} * x_3 &= 0 \end{aligned}$$

x_1 and x_2 are lead variables and we let t be the parameter for x_3.

We get the eigenvector, x, corresponding to the eigenvalue, $\lambda_1 = 4$:

$$x = \begin{bmatrix} \frac{1}{2} * t \\ \frac{1}{2} * t \\ t \end{bmatrix} = t * \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

With $t = 2$ we get an eigenvector, $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, for A corresponding to the eigenvalue, 4.

Check:

$$A{*}x = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1*1-3*1+3*2 \\ 3*1-5*1+3*2 \\ 6*1-6*1+4*2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4{*} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

1.2.2 Case when the eigenvalue is $\lambda_2 = -2$

$$A + 2 * Id = \begin{bmatrix} 1+2 & -3 & 3 \\ 3 & -5+2 & 3 \\ 6 & -6 & 4+2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$$

Augmented Matrix

$$\begin{bmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix}$$

$$R1 := \frac{R1}{3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{bmatrix}$$

$$R2 := R2 - 3 * R1$$

$$R3 := R3 - 6 * R1$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewritng the augmented matrix back as a linear system:

$$x_1 - x_2 + x_3 = 0$$

i.e.

$$x_1 = x_2 - x_3$$

x_1 is the lead variable and let s and t be parameters for x_2 and x_3.
The eigenvectors x, corresponding to the eigenvalue, -2 have the form:

$$\begin{aligned} x &= \begin{bmatrix} s-t \\ s \\ t \end{bmatrix} \\ &= s * \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t * \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

In particular, with $s = 1 = t$ we have an eigenvector corresponding to the eigenvalue, -2:

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Check:

$A * x$

$$= \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1*0-3*1+3*1 \\ 3*0-5*1+3*1 \\ 6*0-6*1+4*1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$= -2 * \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$