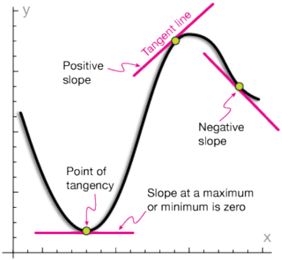
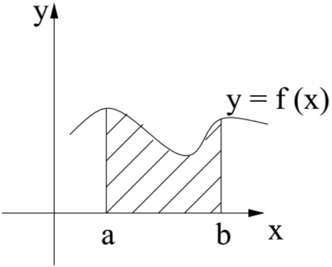
Differentiation and Integration

Geometric View

Differentiation	Integration
	
Slope of curve at point p = slope of tangent at point p	Area under the curve between (a, b) $= \int_a^b f(x)dx$

Derivative Notation

Notation

Let $y = f(x)$ then

$f'(x) = \frac{dy}{dx}$
i.e.
$f'(x) = \frac{d(f(x))}{dx}$

e.g.
let $f(x) = sin\, x$ then
$f'(x) = \frac{d(sin\, x)}{dx} = cos\, x$

Slope at point (x, y) .

With $y = f(x)$ then
$f'(x)$ is the slope of the curve of f at the point (x, y)

Anti-Derivative = Integration

Anti-Derivative = Integration

Show $f(x) = \int f'(x)\,dx$ i.e. $f(x) = \int \frac{d}{dx}(f(x))\,dx$

Let $y = f(x)$ then
$\frac{dy}{dx} = f'(x)$ $\therefore$
$dy = f'(x)\,dx \therefore$
$\int dy = \int f'(x)\,dx$ $\therefore$
$y = \int f'(x)\,dx$
$f(x) = \int f'(x)\,dx$

Example

$\frac{d(\sin x)}{dx} = \cos x$ $\therefore$
$d(\sin x) = (\cos x)\,dx$ $\therefore$
$\int d(\sin x) = \int (\cos x)\,dx$
$\sin x = \int (\cos x)\,dx$ i.e.
if $\frac{d(\sin x)}{dx} = \cos x$ then $\sin x = \int (\cos x)\,dx$

Cont'd

Integration reverses the action of differentiation i.e.

- Integrating the entry in the left column gives the entry in the right column and
- differentiating the entry in the right column gives the entry in the left column.

$f(x)$	$\frac{\rightarrow integrate}{differentiate \leftarrow}$	$\int f(x)dx$
$x^n,\ n \neq -1$		$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$ i.e. x^{-1}		$ln\, x$
$cos\, x$		$sin\, x$
$sin\, x$		$-cos\, x$
$tan\, x$		$ln(sec\, x)$
$\frac{1}{1+x^2}$		$tan^{-1} x$

Note: $y = tan^{-1} x$ iff $tan\, y = x$
also, $ln\, x = log_e x$, the natural logarithm

Indefinite Integral

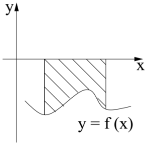
An indefinite integral, $\int f(x)dx$, is an anti-derivative and results in a function.
e.g $\int(\cos x)dx = \sin x$

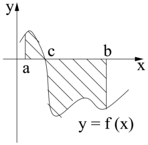
- **Constant of Integration**
  Since the derivative of a constant, C, is 0, the anti-derivative can involve a constant so that:
  $\int(\cos x)dx = \sin x + C$ as $\frac{d(\sin x + C)}{dx} = \cos x$

Definite Integral

The Definite Integral is motivated from the 'area under a curve' view of integration.
$\int_a^b f(x)dx$ is a number which can correspond to the 'area under a curve'.

- 'Negative Area'
  A negative area corresponds to area below the $x - axis$





$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Definite Integral (Cont'd)

Given $F(x) = \int f(x)dx$ i.e. $\frac{d(F(x))}{dx} = f(x)$ i.e. $F'(x) = f(x)$ we evalulate the definite integral

$$\begin{aligned}\int_a^b f(x)dx &= F(x)|_a^b \\ &= F(b) - F(a)\end{aligned}$$

Examples

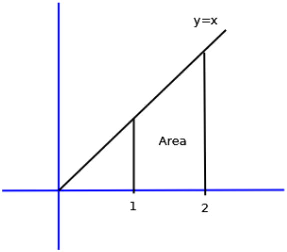
- $\int_1^2 x\,dx$

$$\begin{aligned}\int_1^2 x\,dx &= \left.\frac{x^2}{2}\right|_1^2 \\ &= \frac{2^2}{2} - \frac{1^2}{2} \\ &= \frac{3}{2}\end{aligned}$$

Cont'd

$\int_1^2 x\,dx$

Area
$= (\text{Area of } \triangle \text{ from } 0 \text{ to } 2) - (\text{Area of } \triangle \text{ form } 0 \text{ to } 1)$
$= (\frac{1}{2} * 2) * 2 - (\frac{1}{2} * 1) * 1$
$= 2 - \frac{1}{2}$
$= \frac{3}{2}$



Cont'd

- $\int_1^2 \frac{1}{x} dx$

$$\begin{aligned} \int_1^2 \frac{1}{x} dx &= ln\, x|_1^2 \\ & \quad ln\, 2 - ln\, 1 \\ &= ln\, 2 \quad as\ ln\, 1 = 0 \end{aligned}$$

- $\int_1^2 \frac{1}{x^2} dx$

$$\begin{aligned} \int \frac{1}{x^2} dx &= \int x^{-2} dx \\ &= \frac{x^{-2+1}}{-2+1} \\ &= \frac{x^{-1}}{-1} \\ &= \frac{-1}{x} \end{aligned}$$

Cont'd

$$\begin{aligned}\int_1^2 \frac{1}{x^2}dx &= \left.\frac{-1}{x}\right|_1^2 \\ &= \frac{-1}{2} - \frac{-1}{1} \\ &= \frac{-1}{2} + 1 \\ &= \frac{1}{2}\end{aligned}$$

Some Properties of Integration

- Integral of a sum = sum of the integrals

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

- Factor a constant outside the integral; let c be a constant

$$\int (c * f(x))dx = c * \int f(x)dx$$

Example

- $\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4 sin\, x\right) dx$

$$\begin{aligned}
\int \left(3x^{\frac{1}{2}} - \frac{2}{x} + 4 sin\, x\right) dx \\
&= \int 3x^{\frac{1}{2}} dx - \int \frac{2}{x} dx + \int 4 sin\, x dx \\
&= 3\int x^{\frac{1}{2}} dx - 2\int \frac{1}{x} dx + 4\int sin\, x dx \\
&= 3\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) - 2 ln\, x + 4(-cos\, x) \\
&= 2x^{\frac{3}{2}} - 2 ln\, x - 4 cos\, x
\end{aligned}$$

Integration by Substitution

Consider differentiating e^{x^3} i.e. find $\frac{dy}{dx}$ when

$$y = e^{x^3}$$

Use Chain Rule: Let $u = x^3$ then

$$\begin{array}{rcl|l|rcl} y &=& e^u & & u &=& x^3 \\ \frac{dy}{du} &=& e^u & & \frac{du}{dx} &=& 3x^2 \end{array}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} * \frac{du}{dx} \\ &= e^u * 3x^2 \\ &= 3x^2 e^{x^3} \end{aligned}$$

Since integration is anti-derivative

$$\int 3x^2 e^{x^3} dx = e^{x^3}$$

This is the basis of the Integration technique:
Integration by Substitution

Cont'd

Integration by Substitution

Consider the problem of evaluating

$$\int 3x^2 e^{x^3} dx$$

We use a **Substitution**, we use the subsitution, $u = x^3$.
Since $u = x^3$, $\frac{du}{dx} = 3x^2$ i.e. using differentials, $du = 3x^2 dx$
We can rewrite $3x^2 e^{x^3}$ as $e^{x^3} 3x^2$
Using the substitution

$$\begin{aligned} \int e^{x^3} 3x^2 dx &= \int e^u du \\ &= e^u \\ &= e^{x^3} \end{aligned}$$

i.e.

$$\int 3x^2 e^{x^3} dx = e^{x^3}$$

Integrals of the form:$g(f(x))f'(x)$

$e^{x^3}3x^2$ is of the form $g(f(x))f'(x)$ (a constant item may be involved) where $f'$is the derivative of f and $g(u) = e^u$.
Since multiplication is commutative, $g(f(x))f'(x) = f'(x)g(f(x))$.
Evaluate $\int g(f(x))f'(x)dx$
Let $u = f(x)$ then $du = f'(x)dx$

$$\int g(f(x))f'(x)dx = \int g(u)du$$

Evaluating $\int g(f(x))f'(x)dx$ has been reduced to evaluating $\int g(u)du$.

Cont'd

- $\int x^3\sqrt{16+x^4}\ dx$

Consider evaluating $\int x^3\sqrt{16+x^4}\ dx$
$x^3\sqrt{16+x^4}$ is of form $f'(x)g(f(x))$ where $g(u)=\sqrt{u}$.

$f(x)=16+x^4 \therefore f'(x)=4x^3$ (a constant term is involved)

Let $u=16+x^4 \therefore \frac{du}{dx}=4x^3 \therefore du=4x^3dx$ i.e. $x^3dx=\frac{1}{4}du$

Example

$\int x^3\sqrt{16+x^4}\ dx$

Let $u = 16 + x^4 \therefore \frac{du}{dx} = 4x^3 \therefore du = 4x^3 dx$ i.e. $x^3 dx = \frac{1}{4} du$

$$\begin{aligned}
\int x^3\sqrt{16+x^4}\ dx &= \frac{1}{4}\int \sqrt{u}\, du \\
&= \frac{1}{4}\int u^{\frac{1}{2}}\, du \\
&= \frac{1}{4}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{1}{6}(16+x^4)^{\frac{3}{2}} + C
\end{aligned}$$

Note: Differentiating $\frac{1}{6}(16+x^4)^{\frac{3}{2}} + C$ returns $x^3\sqrt{16+x^4}$

Cont'd

Consider integrals of the form:

- $\int \frac{f'(x)}{f(x)} dx$ i.e. of the form $g(f(x))f'(x)$ where $g(u) = \frac{1}{u}$

Example

$\int \frac{x^4}{x^5+2} dx$

let $u = x^5 + 2$, then $\frac{du}{dx} = 5x^4 \therefore \frac{du}{5} = x^4 dx$

$$\begin{aligned} \int \frac{x^4}{x^5+2} dx &= \frac{1}{5} \int \frac{1}{u} du \\ &= \frac{1}{5}(\ln u) + C \\ &= \frac{1}{5} \ln(x^5+2) + C \end{aligned}$$

Example

$\int x^3(x^4+4)^5\, dx$

let $u = x^4 + 4$, then $\frac{du}{dx} = 4x^3 \therefore \frac{du}{4} = x^3 dx$

$$\begin{aligned}
\int x^3(x^4+4)^5\, dx &= \int (x^4+4)^5 x^3 dx \\
&= \frac{1}{4}\int u^5 du \\
&= \frac{1}{4}\frac{u^6}{6} + C \\
&= \frac{(x^4+4)^6}{24} + C
\end{aligned}$$

Trigonometry Functions

$f'(x)$	$\underset{Integrate \rightarrow}{\xleftarrow{Differentiate}}$	$f(x)$
$cos\, x$		$sin\, x$
$sin\, x$		$-cos\, x$
$tan\, x$		$ln(sec\, x) = -ln(cos\, x)$
		Note: $sec\, x = \frac{1}{cos\, x}$

Example

$\int sin\, 6x\;\, dx$

Let $u = 6x,\; du = 6dx$

$$\begin{aligned} \int sin\, 6x\;\, dx &= \frac{1}{6} \int sin\, u\;\, du \\ &= \frac{1}{6} * (-cos\, u) \\ &= -\frac{1}{6} cos\, 6x \end{aligned}$$

Cont'd

- $\int \frac{sin\,x}{2+cos\,x}dx$

This is of the form $g(f(x))f'(x)$ where $g(u) = \frac{1}{u}$ and $f(x) = 2 + cos\,x$
Let $u = 2 + cos\,x$, then $\frac{du}{dx} = -sin\,x$, $\therefore\, du = -sin\,x\,dx$

$$\begin{aligned}
\int \frac{sin\,x}{2+cos\,x}dx &= \int \frac{sin\,x\,dx}{u} \\
&= -\int \frac{du}{u} \\
&= -ln\,u + C \\
&= -ln(2 + cos\,x) + C
\end{aligned}$$

- $\int (\sin x)(\cos^2 x)\, dx$

Let $u = \cos x \therefore du = -\sin x \ dx$

$$\begin{aligned}
\int (\sin x)(\cos^2 x)\, dx &= \int (\cos^2 x) \sin x dx \\
&= -\int u^2\, du \\
&= -\frac{u^3}{3} + C \\
&= -\frac{1}{3}\cos^3 x + C
\end{aligned}$$

Cont'd

Prove that $\int tan\, x \; dx = -ln(cos\, x) + C$

- **Method 1**. Differentiate Right Hand Side.
  Let $y = -ln(cos\, x)$.
  Substitution: let $u = cos\, x$ then $y = -ln\, u \therefore$
  $\frac{dy}{du} = -\frac{1}{u}$ and $\frac{du}{dx} = -sin\, x$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= \left(-\frac{1}{u}\right)(-sin\, x) \\
&= \frac{sin\, x}{cos\, x} \\
&= tan\, x
\end{aligned}
$$

- **Method 2.** Evaluate $\int tan\,x\;dx$
  $tan\,x = \frac{sin\,x}{cos\,x}$
  Let $u = cos\,x \therefore du = -sin\,x\;dx$

$$\begin{aligned}\int tan\,x\;dx &= \int \frac{sin\,x}{cos\,x}\;dx \\ &= -\int \frac{1}{cos\,x}(-sin\,x)dx \\ &= -\int \frac{1}{u}du \\ &= -ln\,u + C \\ &= -ln(cos\,x) + C\end{aligned}$$

Note: $ln(\frac{1}{x}) = ln(x^{-1}) = -ln\,x \therefore -ln(cos\,x) = ln(\frac{1}{cos\,x}) \therefore$ $\int tan\,x\;dx = ln(\frac{1}{cos\,x}) = ln(sec\,x)$ as $sec\,x = \frac{1}{cos\,x}$.

- $\int sin^3x\ dx$

$$sin^3x = (sin^2x)(sin\,x) = (1 - cos^2x)(sin\,x) = sin\,x - (cos^2x)(sin\,x)$$

$$\begin{aligned}\int sin^3x\ dx &= \int (sin\,x - (cos^2x)(sin\,x))\,dx \\ &= \int sin\,x\ dx - \int (cos^2x)(sin\,x)\,dx \\ &= -cos\,x - \int (cos^2x)(sin\,x)\,dx \\ &\quad \{previous\ example\} \\ &= -cos\,x + \frac{cos^3x}{3} + C\end{aligned}$$

$\int sin^2 x \; dx$

- $\int sin^2 x \; dx$

From Trigonometry:
$sin^2 x = \frac{1}{2}(1 - cos\, 2x)$, also
$cos^2 x = \frac{1}{2}(1 + cos\, 2x)$

$$\begin{aligned} \int sin^2 x \; dx &= \int \frac{1}{2}(1 - cos\, 2x)dx \\ &= \frac{1}{2}\int dx - \frac{1}{2}\int cos\, 2x \, dx \\ &= \frac{x}{2} - \frac{sin\, 2x}{4} + C \end{aligned}$$

$\int \frac{x}{\sqrt{2x+2}}dx$
This is not of the form $g(f(x))f'(x)$
To evaluate this integral:
let $u = 2x + 2$ then $du = 2dx$

$$\int \frac{x}{\sqrt{2x+2}}dx = \frac{1}{2}\int \frac{x}{\sqrt{u}}du$$

Since $u = 2x + 2 \therefore x = \frac{u-2}{2} \therefore$

$$\begin{aligned}\frac{1}{2}\int \frac{x}{\sqrt{u}}du &= \frac{1}{4}\int \frac{u-2}{\sqrt{u}}du \\ &= \frac{1}{4}\int \left(\sqrt{u} - \frac{2}{\sqrt{u}}\right) du \\ &= \frac{1}{4}\int (u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})du\end{aligned}$$

Cont'd

$$\begin{aligned}
\frac{1}{4}\int (u^{\frac{1}{2}} - 2u^{-\frac{1}{2}})du &= \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - 2\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)\right) \\
&= \frac{1}{6}u^{\frac{3}{2}} - u^{\frac{1}{2}} \\
&= \frac{1}{6}(2x+2)^{\frac{3}{2}} - (2x+2)^{\frac{1}{2}} + C
\end{aligned}$$