Matrices, 2x2

Line, Plane, Space

Recall that:
$\mathbb{R}$ is the Real number Line
$\mathbb{R}^2$ is the Real Plane
$\mathbb{R}^3$ is the Real Space
$\mathbb{R}^n$ is the Real $n-$Space, the Space of n dimensions.

Vectors

In $\mathbb{R}^2$, given a fixed, Origin, 0 (zero), a Vector, v , may be considered as a point, (v_1, v_2), in $\mathbb{R}^2$. The Origin, 0, is the point, $(0, 0)$. In Physics/Engineering, a vector is an entity or quantitiy that has both size and direction, e.g. Force, Velocity etc.
Given a point, (v_1, v_2), in $\mathbb{R}^2$, we can consider it as a vector with the size as the length from $(0, 0)$ and the direction as the direction from the origin, $(0, 0)$, to the point, (v_1, v_2) . Similarly, for vectors in $\mathbb{R}^n$, given a fixed origin.
$\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$ for some n, are examples of Vector Spaces.

Equality of Vectors:

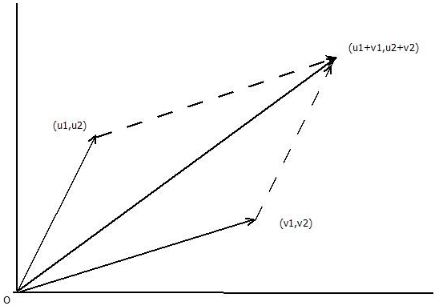
If the vector $u = (u_1, u_2)$ and the vector $v = (v_1, v_2)$ then
$u = v$ *iff* $(u_1, u_2) = (v_1, v_2)$
iff $u_1 = v_1$ *and* $u_2 = v_2$ i.e.
corresponding components are equal.

Multiplication by a Scalar

Given a number, α. which may regarded as a Scalar i.e. a quantity with no direction, i.e. with just a magniture, then
$\alpha * v$ is the vector that is $|\alpha|$ times longer than v .
If $\alpha > 0$ then $\alpha * v$ is in the same direction as v.
If $\alpha < 0$ then $\alpha * v$ is in the opposite direction as v.
If $\alpha = 0$ then $\alpha * v$ is the origin, 0.
If $v = (v_1, v_2)$ then $\alpha * v = (\alpha * v_1, \alpha * v_2)$.

Addition by Parallelogram Law

Addition of two vectors, $u = (u1, u2)$ and $v = (v1, v2)$ is achieved by the **Parallelogram Law** as in the diagram:



$$
\begin{aligned}
&u + v \\
&= (u1, u\,2) + (v1,\ v2) \\
&= (u1 + v1,\ u2 + v2)
\end{aligned}
$$

Properties of Vector Addition

For vectors u, v and w,

- Commutative: $u + v = v + u$
- Associative: $(u + v) + w = u + (v + w)$
- Identity for $+$: the origin or the zero vector, 0, is the identity for $+$, i.e.
  $v + 0 = 0 + v = v$.
- Additive inverse: for each vector, v, there is a vector. w, such that
  $v + w = w + v = 0$ where 0 is the origin or zero vector.
  The additive inverse of v can be written as $-v$ so that
  $v + (-v) = (-v) + v = 0$.
- Subtraction: $u - v = u + (-v)$.

Properties Cont'd

For scalars, α, β

- $\alpha * (u + v) = \alpha * u + \alpha * v$
- $(\alpha + \beta) * v = \alpha * v + \beta * v$
- $\alpha * (\beta * v) = (\alpha * \beta) * v$

In particular, $0 * v = 0$, the zero vector, and $1 * v = v$.

The 'unit' vectors $i = (1, 0)$ and $j = (0, 1)$ form a co-ordinate system or **Basis** for the Plane or Vector Space, $\mathbb{R}^2$. i.e.
each vector, $v = (x, y)$ can be expressed as a linear combination of the Basis vectors i and j
i.e. $(x, y) = x * i + y * j$ as
$x * i + y * j = x * (1, 0) + y * (0, 1) = (x, 0) + (0, y) = (x, y)$.

Linear Transformation

A transformation or mapping (function), T, on a Vector Space is Linear iff
for vectors u and v and scalar (number) α

- $T(u+v) = T(u) + T(v)$
- $T(\alpha * v) = \alpha * T(v)$

In particular, $T(0) = 0$ as we can let $\alpha = 0$.

Matrix Definition

Given a Basis (Co-ordinate system) i, j of the Plane, $\mathbb{R}^2$, we can associate with any Linear Transformation, T, a unique matrix, M, formed as follows.
The Basis vector, i, is mapped to $T(i)$ and Basis vector, j, is mapped to $T(j)$.
Since i and j form a Basis we can express both $T(i)$ and $T(j)$ as a linear combination of i and j i.e.

$$T(i) = a_{11} * i + a_{21} * j$$
$$T(j) = a_{12} * i + a_{22} * j$$

From this, for the transformation, T, we have the Matrix:

Matrix Definition, (Cont'd)

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The first column has the co-ordinates of $T(i)$ and the second column has the co-ordinates of $T(j)$.

We can write, M , more briefly as:

$$M = [a_{ij}]_{2\times 2}$$

Matrix and Vector

Given a vector, $v \in \mathbb{R}^2$, we can express $v = (x, y)$ as a linear combination of the Basis vectors i and j
i.e. $(x, y) = x * i + y * j$
where x and y are the co-ordinates of the vector, v.
Given a linear transformation, T, find where T maps the vector (x, y) to.
From above we have, for the Basis vectors i and j :
$T(i) = a_{11} * i + a_{21} * j$
$T(j) = a_{12} * i + a_{22} * j$

Matrix and Vector Cont'd

$$
\begin{aligned}
&T(x, y) \\
&= T(x * i + y * j) \\
&= x * T(i) + y * T(j) \\
&= x * (a_{11} * i + a_{21} * j) + y * (a_{12} * i + a_{22} * j) \\
&= x * a_{11} * i + x * a_{21} * j + y * a_{12} * i + y * a_{22} * j) \\
&= (a_{11} * x + a_{12} * y) * i + (a_{21} * x + a_{22} * y) * j
\end{aligned}
$$

i.e.

$$T(x, y) = (a_{11} * x + a_{12} * y,\ \ a_{21} * x + a_{22} * y)$$

Matrix and Vector Cont'd

The transformation, T i.e. the matrix, $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, maps the vector, $v = (x, y)$ to

$$\begin{aligned} T(v) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} a_{11} * x + a_{12} * y \\ a_{21} * x + a_{22} * y \end{bmatrix} \end{aligned}$$

Note:
In Matrix theory, we write vectors as Columns and a matrix corresponds to a linear transformation.

From above, we had for Basis, i, j

$$T(i) = a_{11} * i + a_{21} * j$$

$$T(j) = a_{12} * i + a_{22} * j$$

The vector i has the co-ordinates, $(1, 0)$ and
$T(i)$ has co-ordinates (a_{11}, a_{21})
i.e. as Column vectors:

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T(i) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

$$\therefore \; T(i) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}$$

i.e. $T(i) = a_{11} * i + a_{21} * j$.

Multiplying a vector by a Matrix (2x2)

Given a 2×2 Matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and vector $\begin{bmatrix} x \\ y \end{bmatrix}$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a * x + b * y \\ c * x + d * y \end{bmatrix}$$

e.g.

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 * 2 + 8 * 1 \\ 3 * 2 + 5 * 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

Consider the Transformation, H, the reflection in the line, $x = 0$
i.e. the reflection in the *y axis*. (Horizontal Reflection)
Every point on the *y axis* has 0 as the *x coordinate*.
A reflection in the *y axis* maps a vector (x, y) to the vector
$(-x, y)$, i.e.
$H(x, y) = (-x, y)$.
For example, $H(2, 1) = (-2, 1)$.
Similarly, consider a Transformation, V, the reflection in the line,
$y = 0$, i.e. in the *x axis*. (Vertical Reflection)
i.e. $V(x, y) = (x, -y)$.
For example, $V(2, 1) = (2, -1)$.

Reflection in the *y axis*

What is the Matrix corresponding to the Transformation, H, a reflection in the y *axis*.
As above, the vectors $i = (1, 0)$ and $j = (0, 1)$ are a Basis for $\mathbb{R}^2$.

$$H(1,0) = (-1,0) = (-1) * i + 0 * j$$

$$H(0,1) = (0,1) = 0 * i + 1 * j.$$

$\therefore$ the corresponding Matrix, H, is:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Generally, H maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} -x \\ y \end{bmatrix}$ i.e.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 * x + 0 * y \\ 0 * x + 1 * y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Reflection in the x *axis*

What is the Matrix corresponding to the Transformation V (Vertical Reflection)?
The corresponding Matrix for V is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 * x + 0 * y \\ 0 * x + -1 * y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Reflection in the line, $y = x$

The corresponding Matrix, is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as, for vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 * x + 1 * y \\ 1 * x + 0 * y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

Matrices and Linear Equations

We can rewrite the simultaneous equations

$$\begin{aligned} 5*x+8*y &= 18 \\ 3*x+5*y &= 11 \end{aligned}$$

in terms of a Matrix applying to a Vector as:

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 5*x+8*y \\ 3*x+5*y \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

i.e.

$$5*x+8*y = 18$$

$$3*x+5*y = 11$$

Cont'd

To solve the simultaneous equation

$$5 * x + 8 * y = 18$$

$$3 * x + 5 * y = 11$$

in terms of Matrices and Vectors, we find a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$\begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$$

Composition of Transformations

We can create a new Transformation by combining two others via the composition of Transformations. If S and T are Transformations then we can combine them by the compostion operator, $\circ$, where $S \circ T(v) = S(T(v))$ i.e. first apply T and then apply S.
$S \circ T$ is read as "S after T".
For example, what is the Transformation $V \circ H$. Consider a vector (x, y) then $V \circ H(x, y) = V(H(x, y)) = V(-x, y) = (-x, -y)$.
From Geometry, this Transformation is the Reflection through the origin. The matrix for this Transformation is:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \text{ as } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

Matrix Multiplication = Composition of Transformations

Matrix Multiplication corresponds to the composition of the corresponding Transformations.

2x2 Matrix Multiplication Rule

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} a*p+b*r & a*q+b*s \\ c*p+d*r & c*q+d*s \end{bmatrix}$$

Note: The composition operator, $\circ$, is not commutative. i.e. in general for functions/mappings f and g ; $f \circ g \neq g \circ f$.
Also, Matrix multiplication is not commutative, i.e. in general for matrices A and B , $A * B \neq B * A$.

Let V be the reflection in the *x axis* and H the reflection in the *y axis*. The composition of the Transformations V and H is the reflection in the origin. We can check this by matrix multiplication. From above:

Matrix for V is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Matrix for H is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$V \circ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

– Reflection in the Origin

With matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ then

$A * B$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= \begin{bmatrix} a*x + b*z & a*y + b*w \\ c*x + d*z & c*y + d*w \end{bmatrix}$$

Matrix Multiplication (Cont'd)

With matrices $A = [a_{ij}]_{2x2}$ i.e. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

and $B = [b_{ij}]_{2x2}$ i.e. $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

then

$A * B$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} * b_{11} + a_{12} * b_{21} & a_{11} * b_{12} + a_{12} * b_{22} \\ a_{21} * b_{11} + b_{22} * b_{21} & a_{21} * b_{12} + a_{22} * b_{22} \end{bmatrix}$$

i.e. $A * B = [c_{ij}]_{2x2}$ where

$$c_{ij} = a_{i1} * b_{1j} + a_{i2} * b_{2j}$$

e.g. $c_{12} = a_{11} * b_{12} + a_{12} * b_{22}$

Using summation notation:

$$c_{ij} = \sum_{k=1}^{2} a_{ik} * b_{kj}$$

Properties of Matrices

Let $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$

- Equality
  $A = B$ iff $a_{ij} = b_{ij}$ for each i and j. i.e.
  $A = B$ iff each item in A equals the corrresponding item in B.
- Addition/Subtraction
  $A + B = [a_{ij} + b_{ij}]_{n \times n}$
  i.e. each item in A is added to corresponding item in B.
  Similarly, for substracion.
- Product by a scalar/constant
  $k * [a_{ij}]_{n \times n} = [k * a_{ij}]_{n \times n}$

Matrix Addition

Properties of Matrix Addition

- $A + B = B + A$ //Addition is commutative
- $A + (B + C) = (A + B) + C$ // Addition is Associative
- The Identity matrix for $+$ is the matrix $[0]_{m \times n}$

Properties of Matrix Multiplication

- $A * (B * C) = (A * B) * C$ Multiplication is Associative

Note:
Multiplication is not **commutative**. In general
$A * B \neq B * A$.

Non-Square Matrices

If $A = [a_{ij}]_{m \times p}$ and $B = [b_{ij}]_{p \times n}$ then the matrix product $A * B = [c_{ij}]_{m \times n}$ where

$$c_{ij} = \sum_{k=1}^{p} a_{ik} * b_{kj}$$

Matrix Multiplication (Cont'd)

$$c_{ij} = \sum_{k=1}^{p} a_{ik} * b_{kj}$$

Diagram:

$$\begin{array}{c} \\ \textit{row } \mathbf{i} \\ \\ \end{array} \begin{bmatrix} & & \vdots & & \\ a_{i1} & \dots & a_{ij} & \dots & a_{ip} \\ & & \vdots & & \end{bmatrix} * \begin{array}{c} \quad \textit{col } \mathbf{j} \quad \\ \begin{bmatrix} & b_{1j} & \\ & \vdots & \\ \dots & b_{ij} & \dots \\ & \vdots & \\ & b_{pj} & \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \textit{col } \mathbf{j} \\ \begin{array}{c} \\ \textit{row } \mathbf{i} \\ \\ \end{array} & \begin{bmatrix} & \vdots & \\ \dots & c_{ij} & \dots \\ & \vdots & \end{bmatrix} \end{array}$$

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$ then

$$\begin{aligned} A * B &= \begin{bmatrix} 3*1+1*3+2*2 & 3+2+1*1+2*3 \\ 2*1+1*3+3*2 & 2*2+1*1+3*3 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 13 \\ 11 & 14 \end{bmatrix} \end{aligned}$$

Matrix Multiplication Example (Cont'd)

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ then

$$\begin{aligned} A * B &= \begin{bmatrix} 3+2+6 & 9+1+4 & 0+2+2 \\ 1+4+9 & 3+2+6 & 0+4+3 \\ 0+2+12 & 0+1+8 & 0+2+4 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 14 & 4 \\ 14 & 11 & 7 \\ 14 & 9 & 6 \end{bmatrix} \end{aligned}$$

Matrix Transpose

The Transpose M^T of a matrix, M, is where the rows and columns are transposed, i.e. the rows and columns are interchanged, i.e. the columns are written as rows.

If $M = \begin{bmatrix} -1 & 5 \\ 3 & -2 \end{bmatrix}$ then $M^T = \begin{bmatrix} -1 & 3 \\ 5 & -2 \end{bmatrix}$.

Matrices need not be square $\therefore$

if $M = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ then $M^T = \begin{bmatrix} 8 & 3 \\ 2 & 1 \\ 4 & 2 \end{bmatrix}$

In general, if $M = [a_{ij}]_{R \times C}$ then $M^T = [a_{ji}]_{C \times R}$.