Determinants and Matrices

Determinant of a Matrix

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and

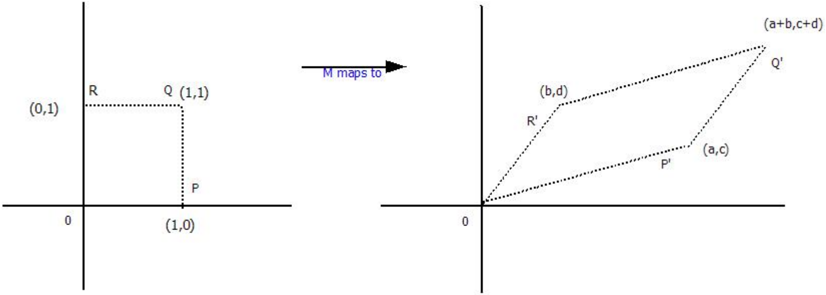
vectors, $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $R = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

Diagram:



$$M(0) = 0,\ M(P) = P',\ M(Q) = Q',\ M(R) = R'.$$

Change in Area?

The matrix, M, transforms the square $S = 0PQR$ to the parallelogram $M(S) = 0P'Q'R'$. By how much has the area changed? What is the ratio

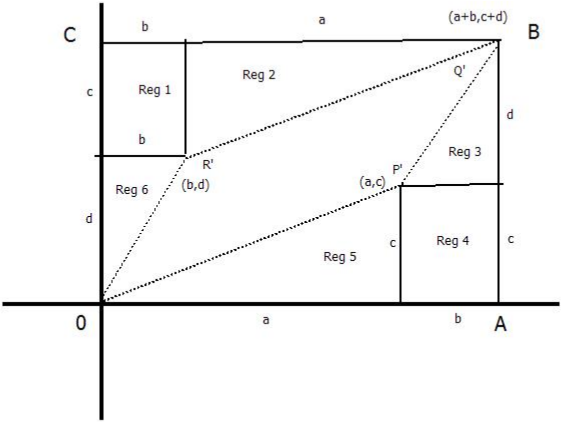
$$\frac{Area(0P'Q'R')}{Area(0PQR)}.$$

i.e. what is the ratio

$$\frac{Area(M(S))}{Areas(S)}$$

Since $area(S) = 1$, then $Area(M(S)) = Area(0P'Q'R')$. We can find the area of the parallelogram using geometry.

Parallelogram



$$Area\ 0ABC = (a+b)*(c+d)$$

Cont'd

From diagram,
$Reg\,1 = Reg\,4,\ Reg\,2 = Reg\,5, Reg\,3 = Reg\,6$

Area Parallelogram $0P'Q'R'$
$= Area\,0ABC - (2 * Reg\,1 + 2 * Reg\,2 + 2 * Reg\,3)$
$= (a + b) * (c + d) - 2 * (b * c + \frac{a}{2} * c + \frac{b}{2} * d)$
$= a * c + a * d + b * c + b * d - 2 * b * c - a * c - b * d$
{Simplifying}
$= a * d - b * c$

Determinant $|M|$; $Area\, M(R) = M| * Area\, R$

The determinant $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a * d - b * c$.
The 2×2 determinant $|M|$ of the matrix, M, represents the ratio of the change in area due to the transformation, M.
Let M be a Matrix and R a region on the plane,

$$\begin{aligned} \frac{Area\, M(R)}{Area\, R} &= |M| \\ Area\, M(R) &= |M| * Area\, R \end{aligned}$$

Notation: $det(M)$ is also used for $|M|$.

Let A and B be matrices then

$$|A * B| = |A| * |B|$$

"The Determinant of a Matrix Product is
the product of the determinants"

Reason:
Let $a = |A|$ and $b = |B|$. From above.
Transformation A multiplies Area by a and
Transformation B multiplies Area by b .
then since matrix multiplication corresponds to composition of transformations,
$A * B$ means apply transformation B first and then transformation A, i.e.
transformation B multiplies Area by b and then transformation A multiplies result by a.
Thus, the transformation $A * B$ transforms the Area by $a * b$ i.e. by $|A| * |B|$.

Determinant of Matrix Inverse

Id is the Identity matrix and $|Id| = 1$. The Identity, Id, does not change Area.
For a matrix, M, that has an inverse,

$$\begin{aligned} M * M^{-1} &= Id \\ |M * M^{-1}| &= |Id| \\ |M| * |M^{-1}| &= 1 \\ |M^{-1}| &= \frac{1}{|M|} \end{aligned}$$

"The Determinant of the inverse is the inverse of the determinant"

Singular Matrix

A Matrix, M, who Determinant is zero is a Singular matrix, i.e. if $|M| = 0$ then the matrix M is Singular. A Singular matrix has no Inverse, i.e. if the $|M| = 0$ then M has no Inverse.

Calculating 3x3 Determinants

A $(n+1)\times(n+1)$ Determinant is calculated in terms of an $n\times n$ Determinant. To calculate the 3×3 Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

we calculate:

$$a_{11} * \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} * \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} * \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Minor M_{ij} of a Matrix

The Minor M_{ij} of a $(n+1) \times (n+1)$ Matrix, M, is the determinant of the $n \times n$ matrix obtained by deleting row i and column j in M.
We can calculate the determinant

$$|M| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

by

$$a_{11} * M_{11} - a_{12} * M_{12} + a_{13} * M_{13}$$

Minor M_{ij}

where

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22} * a_{33} - a_{32} * a_{23}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21} * a_{33} - a_{31} * a_{23}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21} * a_{32} - a_{31} * a_{22}$$

Example: 3×3 Determinant

Calculate the Determinant

$$\begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= 1 * \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} - (-1) * \begin{vmatrix} 4 & 1 \\ 16 & 1 \end{vmatrix} + 1 * \begin{vmatrix} 4 & 2 \\ 16 & 4 \end{vmatrix} \\
&= 1 * (2 - 4) + (4 - 16) + (16 - 32) \\
&= -2 - 12 - 16 \\
&= -30
\end{aligned}$$

Determinant of 3×3 Matrix

Expanding

$$a_{11} * M_{11} - a_{12} * M_{12} + a_{13} * M_{13}$$

we get

$$\begin{aligned} a_{11} * (a_{22} * a_{33} - a_{32} * a_{23}) &\quad - \\ a_{12} * (a_{21} * a_{33} - a_{31} * a_{23}) &\quad + \\ a_{13} * (a_{21} * a_{32} - a_{31} * a_{22}) & \end{aligned}$$

Cont'd

Multiplying out, we get:

$$
\begin{aligned}
= \quad & a_{11} * a_{22} * a_{33} - a_{11} * a_{32} * a_{23} \\
- \quad & a_{12} * a_{21} * a_{33} + a_{12} * a_{31} * a_{23} \\
+ \quad & a_{13} * a_{21} * a_{32} - a_{13} * a_{31} * a_{22}
\end{aligned}
$$

Note:
For a 3×3 Determinant, there are $3! = 6$ terms and
for an $n \times n$ Determinant, there are $n!$ terms.
Calculating a Determinant is not an efficient calculation.

Re-Grouping Terms

We can regroup the terms into 'positive' and 'negative' terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$=$$

$$a_{11} * a_{22} * a_{33} + a_{12} * a_{23} * a_{31} + a_{13} * a_{21} * a_{32}$$
$$-(a_{31} * a_{22} * a_{13} + a_{32} * a_{23} * a_{11} + a_{33} * a_{21} * a_{12})$$

i.e. positive along the 'down diagonals' and negative on the 'up diagonals'.

Calculation by Up and Down Diagonals

Consider the following technique for calculating a 3×3 Determinant (only):
Repeat the first 2 columns to the right to get:

$$
\begin{array}{cccccc}
\searrow & \searrow & \searrow & & & \\
& a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
& a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
& a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
\nearrow & \nearrow & \nearrow & & &
\end{array}
$$

Cont'd

Down Diagonals (Positive, $+$):
Multiply down the diagonals and add to get:
$+(a_{11} * a_{22} * a_{33} + a_{12} * a_{23} * a_{31} + a_{13} * a_{21} * a_{32})$

Up Diagonals (Negative, $-$)
Multiply up the diagonals and add to get:
$-(a_{31} * a_{22} * a_{13} + a_{32} * a_{23} * a_{11} + a_{33} * a_{21} * a_{12})$

then add these 2 results together.

Example

Using this 'diagonal' technique calculate the determinant of M where

$$M = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}$$

Example Cont'd

Copy the first 2 columns to the right:

$$\begin{array}{ccccc} \searrow & \searrow & \searrow & \underline{\mathbf{32}} & \underline{\mathbf{4}} & \underline{\mathbf{-4}} \end{array}$$

$$\begin{array}{ccccc} 1 & -1 & 1 & 1 & -1 \\ 4 & 2 & 1 & 4 & 2 \\ 16 & 4 & 1 & 16 & 4 \end{array}$$

$$\begin{array}{cccccc} \nearrow & \nearrow & \nearrow & \underline{\mathbf{2}} & \underline{\mathbf{-16}} & \underline{\mathbf{16}} \end{array}$$

Down Diagonals $(+)$
$+(2 - 16 + 16) = 2$

Up Diagonals $(-)$
$-(32 + 4 - 4) = -32$

$\therefore |M| = 2 - 32 = -30$

The 6 permutations of $1, 2, 3$ can be listed as:

$$(1, 2, 3);\ (1, 3, 2);\ (2, 1, 3);\ (2, 3, 1);\ (3, 1, 2);\ (3, 2, 1)$$

In the terms of of the Determinant we can keep the 'first' indices fixed as 123 and then for the second index use a permutation of $1, 2, 3$ i.e.

$$a_{11} * a_{22} * a_{33} - a_{11} * a_{23} * a_{32} - a_{12} * a_{21} * a_{33}$$
$$+a_{12} * a_{23} * a_{31} + a_{13} * a_{21} * a_{32} - a_{13} * a_{22} * a_{31}$$

Determinant of a $n \times n$ Matrix

Let

$$M = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

then

$$\begin{aligned} |M| &= a_{11} * M_{11} - a_{12} * M_{12} + \cdots + (-1)^{1+n} * M_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} * a_{1j} * M_{1j} \end{aligned}$$

Also, we can 'expand' along row i,

$$|M| = \sum_{j=1}^{n} (-1)^{i+j} * a_{ij} * M_{ij}$$

Cramer's Rule

Cramer's Rule uses determinants to solve a system of linear equations that have a solution. Such a system has a solution if the determinant of the coefficients is not zero. Consider a system of n linear eequations in n unknowns:

$$\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & c_2 \\
\vdots & \vdots & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & & c_n
\end{array}$$

The determinant of the coefficients, D, is:

$$D = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

Cramer's Rule (Cont'd)

Cramer's Rule:

If the determinant, D, of the coefficients of a system of n linear equations in n unknowns is not zero ($D \neq 0$) then the equations have an unique solution. Each unknown may be expressed as a fraction of 2 determinants, with denoniminator (the bottom) the determinant, D, and with numerator (the top) obtained from D by replacing the column of co-efficients of the unknown in question by the constants, $c_1, c_2, \ldots, c_n$.

Let

$$D_{x_1} = \begin{vmatrix} c_1 & a_{12} & . & a_{1n} \\ c_2 & a_{21} & . & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_n & a_{n2} & . & a_{nn} \end{vmatrix} \dots D_{x_k} = \begin{vmatrix} & & k^{th} col & & \\ a_{11} & . & c_1 & . & a_{1n} \\ a_{21} & . & c_2 & . & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & . & c_n & . & a_{nn} \end{vmatrix}$$

The unknown, x_k, can be expressed as:

$$x_k = \frac{D_{x_k}}{D}$$

Note:
See previous lecture on 2×2 Determinants.

Example: Cramer's Rule

Assume the 3 points $(-1, 8)$, $(2, -1)$, $(4, 3)$ lie on the quadratic curve,

$$y = a * x^2 + b * x + c,$$

find the values of a, b and c.
Since $(-1, 8)$ lies of quadratic curve we have:

$$8 = a * (-1)^2 + b * (-1) + c$$

i.e.

$$a - b + c = 8$$

Similarly, for point $(2, -1)$,

$$4a + 2b + c = -1$$

and for point $(4, 3)$

$$16a + 4b + c = 3$$

From this system of linear equations we get the determinant of the coefficients:

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = -30$$

We find a, b and c using Cramer's Rule:

$$a = \frac{\begin{vmatrix} \mathbf{8} & -1 & 1 \\ -\mathbf{1} & 2 & 1 \\ \mathbf{3} & 4 & 1 \end{vmatrix}}{-30} = \frac{-30}{-30} = 1$$

Example: Cramer's Rule (Cont'd)

We find the values for b and c :

$$b = \frac{\begin{vmatrix} 1 & \mathbf{8} & 1 \\ 4 & -\mathbf{1} & 1 \\ 16 & \mathbf{3} & 1 \end{vmatrix}}{-30} = \frac{120}{-30} = -4$$

$$c = \frac{\begin{vmatrix} 1 & -1 & \mathbf{8} \\ 4 & 2 & -\mathbf{1} \\ 16 & 4 & \mathbf{3} \end{vmatrix}}{-30} = \frac{-90}{-30} = 3$$

Solution:
The 3 points $(-1, 8)$, $(2, -1)$, $(4, 3)$ lie on the quadratic curve, $y = x^2 + -4 * x + 3$.

Properties of Determinants

- $|A * B| = |A| * |B|$ (from before)
- Determinant of a Transpose: $|M^T| = |M|$
  A matrix and its transpose have the same determinant.
- If the determinant, D', is obtained by interchanging two rows in the determinant, D, then $D' = -D$.
- If all the items in a row (or column) are zero then the determinant is zero.
- If two rows are identical then the determinant is zero.
- If one row in a determinant, D, is k times another row then the determinant, $D = 0$.

Properties of Determinants (Cont'd)

- Multiply one row of a determinant. D, by a constant, k, equals $k * D$. e.g.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ k * b_1 & k * b_2 & k * b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k * \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Also,

$$\begin{vmatrix} k * a_1 & k * a_2 & k * a_3 \\ k * b_1 & k * b_2 & k * b_3 \\ k * c_1 & k * c_2 & k * c_3 \end{vmatrix} = k^3 * \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Determinants (Cont'd)

'Adding Rows' of two determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + d_1 & b_2 + d_2 & b_3 + d_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Determinants (Cont'd)

Add k times one row to another:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + k * c_1 & b_2 + k * c_2 & b_3 + k * c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + k * \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

but from above,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Properties of Determinants (Cont'd)

$\therefore$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + k * c_1 & b_2 + k * c_2 & b_3 + k * c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

i.e.
Adding k times one row to another does not change the determinant..

Properties of Determinants (Cont'd)

Triangular Determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 * b_2 * c_3$$

Since $|M^T| = |M|$ we also have:

$$\begin{vmatrix} a_1 & 0 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 * b_2 * c_3$$