Set Properties

Set Properties

Sets have properties similar but not the same as Arithmetic.
Let U be the Universal set of elements of interest.
Let $X, Y, Z \subseteq U$
The basic operators on sets are:

- Complement: $\overline{X}$
- Intersection: $X \cap Y$
- Union $X \cup Y$

Set Props. Cont'd

Fundamental Properties of Set Theory Operators

Identity

$$X \cap U = X \qquad X \cup \{\} = X$$

Anihilation

$$X \cap \{\} = \{\} \qquad X \cup U = U$$

Complement

$$X \cap \overline{X} = \{\} \qquad X \cup \overline{X} = U$$

Idempotent

$$X \cap X = X \qquad X \cup X = X$$

Commutativity

$$X \cap Y = Y \cap X \qquad X \cup Y = Y \cup X$$

Set Props Cont'd

Associativity

$(X \cap Y) \cap Z = X \cap (Y \cap Z) \quad (X \cup Y) \cup Z = X \cup (Y \cup Z)$

Distributivity: $\cap$ over $\cup$

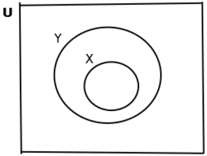
$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

Distributivity: $\cup$ over $\cap$

$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Elementary Properties of Sets

- $\overline{\overline{X}} = X$
- $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$
- $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$
- $X \subseteq Y \equiv \overline{Y} \subseteq \overline{X}$



- Also $Y \subseteq X \equiv \overline{X} \subseteq \overline{Y}$

Elementary Properties (Cont'd)

- $X = Y \equiv \overline{X} = \overline{Y}$

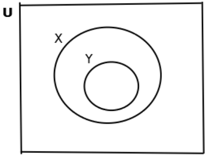
Proof:

$X = Y$
$\equiv X \subseteq Y \text{ and } Y \subseteq X$
$\equiv \overline{Y} \subseteq \overline{X} \text{ and } \overline{X} \subseteq \overline{Y}$
$\equiv \overline{X} = \overline{Y}$

Set Theory Theorems

Set Theory Theorems

- $Y \subseteq X \quad \equiv \quad X \cup Y = X$
- $Y \subseteq X \quad \equiv \quad X \cap Y = Y$



$Y \subseteq X \equiv X \cup Y = X$

Show $Y \subseteq X \equiv X \cup Y = X$

1. $Y \subseteq X \rightarrow X \cup Y = X$
2. $X \cup Y = X \rightarrow Y \subseteq X$

Proof.

(1.)
Assume $Y \subseteq X$,
show $X \cup Y = X$ i.e. $X \cup Y \subseteq X$ and $X \subseteq X \cup Y$
Show $X \cup Y \subseteq X$
let $z \in X \cup Y$
$\therefore z \in X$ or $z \in Y$ $\square$

Cont'd

Proof.

Case $z \in X$

$\therefore z \in X$

Case $z \in Y$

$\{$assuming $Y \subseteq X$ $\}$

$\therefore z \in X$.

Show $X \subseteq X \cup Y$

True, from properties of $\cup$. $\square$

$Y \subseteq X \equiv X \cup Y = X$ (Cont'd)

Show(2.) $X \cup Y = X \rightarrow Y \subseteq X$

Proof.

(2.)
Assume $X \cup Y = X$, show $Y \subseteq X$
let $z \in Y$,
$\therefore z \in X \cup Y$
{assuming $X \cup Y = X$ }
$\therefore z \in X$ $\square$

$Y \subseteq X \equiv X \cap Y = Y$

Show $Y \subseteq X \equiv X \cap Y = Y$

i.e. Show

1. $Y \subseteq X \rightarrow X \cap Y = Y$
2. $X \cap Y = Y \rightarrow Y \subseteq X$

Proof.

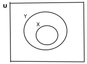
Exercise □

$X \subseteq Y \equiv X \cap \overline{Y} = \{\}$

Theorem

$X \subseteq Y \equiv X \cap \overline{Y} = \{\}$

Diagram:



Show $X \subseteq Y \rightarrow X \cap \overline{Y} = \{\}$

Assume $X \subseteq Y$

i.e. from above (swapping X and Y): $X \cap Y = X$ $\therefore$

$X \cap \overline{Y}$

$= (X \cap Y) \cap \overline{Y}$

$= X \cap (Y \cap \overline{Y})$

$= X \cap \{\}$

$= \{\}$

Show $X \cap \overline{Y} = \{\} \rightarrow X \subseteq Y$

Show $X \cap \overline{Y} = \{\} \rightarrow X \subseteq Y$

Assume $X \cap \overline{Y} = \{\}$
As $X \subseteq Y \equiv X \cap Y = X$, show $X = X \cap Y$

$$\begin{aligned} X &= X \cap U \\ &= X \cap \left(Y \cup \overline{Y}\right) \\ &= (X \cap Y) \cup \left(X \cap \overline{Y}\right) \\ &= (X \cap Y) \cup \{\} \\ &= X \cap Y \end{aligned}$$

De Morgan's Laws

De Morgan's Laws

1. $\overline{(X \cap Y)} = \overline{X} \cup \overline{Y}$ – De Morgan 1
2. $\overline{(X \cup Y)} = \overline{X} \cap \overline{Y}$ – De Morgan 2

De Morgan 1 Veitch Diagram

$$X \cap Y = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-3} & 0 & 0 \\ \cline{2-3} X & 0 & 1 \\ \cline{2-3} \end{array} \quad \therefore \quad \overline{X \cap Y} = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$$\overline{X} = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 0 & 0 \\ \cline{2-3} \end{array} \quad \overline{Y} = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-3} & 1 & 0 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array}$$

$\therefore$

$$\overline{X} \cup \overline{Y} = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-3} & 1 & 1 \\ \cline{2-3} X & 1 & 0 \\ \cline{2-3} \end{array} = \overline{X \cap Y}$$

Proof of De Morgan's Law 1

Proof of De Morgan 1 $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$

1. Show $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$
2. Show $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Theorem

$\overline{X} \cap \overline{X \cap Y} = \overline{X}$

Proof.

$X \cap Y \subseteq X$
$\{ A \subseteq B \equiv \overline{B} \subseteq \overline{A} \}$
$\equiv \overline{X} \subseteq \overline{X \cap Y}$
$\{ A \subseteq B \equiv A \cap B = A \}$
$\equiv \overline{X} \cap \overline{X \cap Y} = \overline{X}$ $\square$

Corollary

$\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$

Show 1. $\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$ (Cont'd)

Theorem

$\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$

Recall: $A \subseteq B \equiv A \cap B = A$

Proof.

$\overline{X} \cup \overline{Y} \subseteq \overline{X \cap Y}$
$\equiv (\overline{X} \cup \overline{Y}) \cap \overline{X \cap Y} = \overline{X} \cup \overline{Y}$
{ $\cap$ Distributes over $\cup$ }
$\equiv (\overline{X} \cap \overline{X \cap Y}) \cup (\overline{Y} \cap \overline{X \cap Y}) = \overline{X} \cup \overline{Y}$
{ by Thms. $\overline{X} \cap \overline{X \cap Y} = \overline{X}$ and $\overline{Y} \cap \overline{X \cap Y} = \overline{Y}$ }
$\equiv \overline{X} \cup \overline{Y} = \overline{X} \cup \overline{Y}$
$\equiv$ *True* $\square$

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Theorem

$$\overline{\overline{X} \cup \overline{Y}} \cup X = X$$

Proof.

$$\overline{X} \subseteq \overline{X} \cup \overline{Y}$$
$$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X}}$$
$$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X$$
$$\equiv \overline{\overline{X} \cup \overline{Y}} \cup X = X \qquad \square$$

Corollary

$$\overline{\overline{X} \cup \overline{Y}} \cup Y = Y$$

Show 2. $\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$ (Cont'd)

Theorem

$\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

Proof.

$\overline{X \cap Y} \subseteq \overline{X} \cup \overline{Y}$

$\{\ A \subseteq B \equiv \overline{B} \subseteq \overline{A}\ \}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq \overline{\overline{X \cap Y}}$

$\equiv \overline{\overline{X} \cup \overline{Y}} \subseteq X \cap Y$

$\equiv \overline{\overline{X} \cup \overline{Y}} \cup (X \cap Y) = X \cap Y$

$\equiv \left(\overline{\overline{X} \cup \overline{Y}} \cup X\right) \cap \left(\overline{\overline{X} \cup \overline{Y}} \cup Y\right) = X \cap Y$

$\equiv X \cap Y = X \cap Y$

$\equiv$ *True* $\square$

Prove De Morgan 2 $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Theorem

$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

Proof.

$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$

$\{\ A = B \equiv \overline{A} = \overline{B}\ \}$

$\equiv \overline{\overline{X \cup Y}} = \overline{\overline{X} \cap \overline{Y}}$

$\equiv X \cup Y = \overline{\overline{X} \cap \overline{Y}}$

$\{$ De Morgan 1 $\}$

$\equiv X \cup Y = \overline{\overline{X}} \cup \overline{\overline{Y}}$

$\equiv X \cup Y = X \cup Y$

$\equiv \mathit{True}$ $\square$

Cardinality of Sets

Disjoint Sets

Sets X and Y are disjoint iff $X \cap Y = \{\}$.
We define $|X|$ as the size of set X,
i.e. $|X|$ is the number of elements in X .
Sometimes $\#X$ is used instead of $|X|$.
With $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$, $|A| = 8$.

Lemma 1

$$B \subseteq A \to |A - B| = |A| - |B|$$

Lemma 2

$$A \cap B = \{\} \to |A \cup B| = |A| + |B|$$

Cardinalty Cont'd

Theorem $|A \cup B| = |A| + |B| - |A \cap B|$

We can split $A \cup B$ into disjoint sets:
i.e. $A \cup B = (A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)$

Proof.

$$
\begin{aligned}
|A \cup B| &= |(A - (A \cap B)) \cup (B - (A \cap B)) \cup (A \cap B)| \\
&\quad \{\textit{all these disjoint}\} \\
&= |A - (A \cap B)| + |B - (A \cap B)| + |A \cap B| \\
&\quad \{A \cap B \subseteq A \textit{ and } A \cap B \subseteq B\} \\
&= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\
&= |A| + |B| - |A \cap B|
\end{aligned}
$$

□

Cardinality of Sets

Cardinality $A \cup B \cup C$

$$
\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= \{\textit{Set Theory distributive law}\} \\
&\quad |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| - |A \cap B| + |C| \\
&\quad -(|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)| \\
&= |A| + |B| + |C| \\
&\quad -(|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad +|A \cap B \cap C|
\end{aligned}
$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example

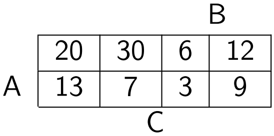
Students pass the year if they pass all 3 exams A, B, C.
For a particular year it was found that

- 3% failed all 3 papers
- 9% failed papers B and C
- 10% failed papers A and C
- 12% failed papers A and B
- 32% failed paper A
- 30% failed paper B
- 46% failed paper C

1. What percentage of students passed the year
2. What percentage failed exactly one paper.

Solution

Solution:



Power Set

The Power Set, $P(S)$, of a set S, is the set of subsets of S,
i.e. $x \in P(S) \equiv x \subseteq S$.
If $|S| = n$ then $|P(S)| = 2^n$.

Example

$S = \{a, b, c\}$
$P(S) = \{\, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \,\}$
where $\emptyset$ is the empty set, i.e. $\emptyset = \{\}$.

In forming the subsets of S e.g.$S = \{0, 1, 2, 3, \ldots, n-1\}$, we have 2 choices for each element; to include it or exclude it.
2 choices for 0, 2 choices for 1, 2 choices for 2 etc.
Total #choices $= 2 * 2 * \cdots * 2$ (n times) $= 2^n$.
There is a natural correspondence between the subsets of $\{0, 1, 2, 3, \ldots, n-1\}$ and binary numbers.

Subsets and Binary

subset	$n-1$	$\dots$	k	$\dots$	3	2	1	0
$\{\}$	0	$\dots$	0	$\dots$	0	0	0	0
$\{0\}$	0	$\dots$	0	$\dots$	0	0	0	1
$\{1\}$	0	$\dots$	0	$\dots$	0	0	1	0
$\{0,1\}$	0	$\dots$	0	$\dots$	0	0	1	1
$\vdots$								
$\{\dots, k, \dots\}$		$\dots$	1	$\dots$				
$\vdots$								
$\{0, 1, 2, \dots k, \dots n\}$	1	$\dots$	1	$\dots$	1	1	1	1

- 0 in column, k, indicates that k is not in the subset
- 1 in column, k, indicates that k is in the subset.

Binary and Decimal

Binary	decimal		
$0\dots0$	0	$=$	$0*2^{n-1}+\cdots+0*2^2+0*2^1+0*2^0$
$0\dots1$	1	$=$	$0*2^{n-1}+\cdots+0*2^2+0*2^1+1*2^0$
$0\dots10$	2	$=$	$0*2^{n-1}+\cdots+0*2^2+1*2^1+0*2^0$
$0\dots11$	3	$=$	$0*2^{n-1}+\cdots+0*2^2+1*2^1+1*2^0$
	$\vdots$		$\vdots$
$1\dots1$	2^n-1	$=$	$1*2^{n-1}+\cdots+1*2^2+1*2^1+1*2^0$

$|P(S)| = 2^{|S|}$ Proof by Induction

$|P(S)| = 2^{|S|}$

Let $|S| = n$. Proof by induction on n.
Base Case:
$n = 0$
If $|S| = 0$ then $S = \emptyset \therefore P(S) = \{\emptyset\}$.
$|\{\emptyset\}| = 1$ tf $|P(S)| = 1 = 2^0 = 2^{|S|}$.
Induction Step:
Assume true for n, show true for $n + 1$.
i.e. Assume (if $|A| = n$ then $|P(A)| = 2^n$),
show (if $|S| = n + 1$ then $|P(S)| = 2^{n+1}$).

Induction Step

Assume $|S| = n + 1$.
Consider an element, x, of S, i.e. $x \in S$.
Discard x , then we have $S - \{x\}$ and $\therefore |S - \{x\}| = n$.
By induction, $|P(S - \{x\})| = 2^n$.
The original subsets of S consist of

- those that do not have the element, x, i.e. the subsets of $S - \{x\}$. and $|P(S - \{x\})| = 2^n$.
- those that do have the element, x, which are the subsets of of $S - \{x\}$ with the element, x, added in, giving 2^n subsets.

$\therefore |P(S)| = 2^n + 2^n = 2^{n+1}$.

Cantor's Theorem, $|\mathbb{N}| \neq |P(\mathbb{N})|$

Cardinality of Sets

Let $S = \{0, 1, 2\}$ then
$P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}, \therefore$
$|S| = 3$ and $|P(S)| = 8$ and in this case $|S| \neq |P(S)|$.
For any finite set, S, $|S| \neq |P(S)|$.

Sets with same Cardinality

Two sets have the same cardinality iff there is a one to one, 1-1, correspondence between both sets.
Let $A = \{a, b, c, d, e, \ldots, x, y, z\}$and $B = \{1, 2, 3, \ldots 26\}$ then $|A| = |B|$ as we have the 1-1 correspondence

A	a	b	c	$\ldots$	y	z
B	1	2	3	$\ldots$	25	26

$|\mathbb{N}| = |Even|$

$|\mathbb{N}| = |Even|$

Consider infinite sets:
Infinite sets S_1 and S_2 have the same cardinality if there is a one to one, 1-1, correspondence between both sets.
Let *Even* be the set of even natural numbers then $|\mathbb{N}| = |Even|$ as:

Even	0	2	4	6	...	$2 * n$	...
$\mathbb{N}$	0	1	2	3	...	n	...

There is a 1-1 correspondence between the two sets $\mathbb{N}$ and *Even*.
The sets $\mathbb{N}$ and *Even* have the same cardinality i.e. $|\mathbb{N}| = |Even|$, even though $Even \subseteq \mathbb{N}$ and $Even \neq \mathbb{N}$.

$|\mathbb{N}| = |\mathbb{Z}|$

$|\mathbb{N}| = |\mathbb{Z}|$

Consider a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{Z}$,

$\mathbb{N}$	n	$\ldots$	$2*k-1$	$\ldots$	3	1	0	2	4	$\ldots$	$2*k$	$\ldots$
$\mathbb{Z}$	$f(n)$	$\ldots$	$-k$	$\ldots$	-2	-1	0	1	2	$\ldots$	k	$\ldots$

The odd natural numbers are in 1-1 correspondence with the negative integers
and the even natural numbers are in 1-1 correspondence with the positive integers.
The function, $f(n)$, can be defined as:

$$f(n) = if\ even(n)\ then\ \frac{n}{2}\ else\ \frac{-(n+1)}{2}$$

e.g. $f(2*k-1) = \frac{-((2*k-1)+1)}{2} = \frac{-2*k}{2} = -k$

$|\text{Naturals}|=|\text{Positive Rationals}|$

Let $\mathbb{Q}^+$ be the set of positive Rational numbers (positive fractions).

$|\mathbb{N}| = |\mathbb{Q}^+|$

Let $f \ : \ \mathbb{N} \to \mathbb{Q}^+$ such that

$\mathbb{N}$	n	0	1	2	3	4	5	...	...
$\mathbb{Q}^+$	$\frac{a}{b}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{3}$	$\frac{2}{2}$	$\frac{3}{1}$	...	...

We can list all fractions using the following:

List all fractions $\frac{a}{b}$ such that $a+b=2$

List all fractions $\frac{a}{b}$ such that $a+b=3$

List all fractions $\frac{a}{b}$ such that $a+b=4$

etc.

|Naturals|=|Positive Rationals| Cont'd

Consider listing the positive Rationals in matrix form:
Each row is infinite and there are infinite rows.

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[2ex]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\[2ex]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\[2ex]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

List the Rationals along the diagonals.

|Naturals|=|Naturals x Naturals|

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Consider $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that

$\mathbb{N}$	n	0	1	2	3	4	...
$\mathbb{N}\times\mathbb{N}$	$f(n)$	(0,0)	(0,1)	(1,0)	(0,2)	(1,1)	...

We can list all pairs from $\mathbb{N}\times\mathbb{N}$ by:
listing all pairs (a,b) such that a+b = 0, i.e. (0,0)
listing all pairs (a,b) such that a+b = 1, i.e. (0,1), (1,0)
listing all pairs (a,b) such that a+b = 2, i.e. (0,2), (1,1), (2,0)
etc.

$|\mathbb{N}| = |\mathbb{N}|^k$

Note: *Consider a different 1-1 function*

The function, $g : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}^+$ such that

$$g(m, n) = 2^{m-1} * (2n - 1)$$

is a 1-1 function.

Exercise: Find m and n such that $g(m, n) = 80$.

$|\mathbb{N}| = |\mathbb{N}|^k$

Since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| * |\mathbb{N}|$ we get that $|\mathbb{N}| = |\mathbb{N}|^2$.
Similarly, $|\mathbb{N}|^3 = |\mathbb{N}| * |\mathbb{N}|^2$ therefore $|\mathbb{N}| = |\mathbb{N}|^3$,
therefore, for any finite natural $k > 0$, $|\mathbb{N}| = |\mathbb{N}|^k$.

Proof of Cantor's Theorem $|\mathbb{N}| \neq |P(\mathbb{N})|$

Cantor's Theorem

$|\mathbb{N}| \neq |P(\mathbb{N})|$

Proof is by contradiction.
Assume $|\mathbb{N}| = |P(\mathbb{N})|$ $\therefore$ there is a 1-1 correspondence between $\mathbb{N}$ and $P(\mathbb{N})$.

$\mathbb{N}$	0	1	$\dots$	n	$\dots$
$P(\mathbb{N})$	$sub(0)$	$sub(1)$	$\dots$	$sub(n)$	$\dots$

where $sub(n)$ is the subset corresponding to n.
Also, for each subset, S, of $\mathbb{N}$ there is a matching element in $\mathbb{N}$, i.e. for each element $S \in P(\mathbb{N})$, there is an element, $k \in \mathbb{N}$, such that $sub(k) = S$.
Recall: $S \in P(\mathbb{N})$ iff $S \subseteq \mathbb{N}$.

Cantor's Thm. (Cont'd)

For each subset, $sub(n)$, of $\mathbb{N}$, either $n \in sub(n)$ or $n \notin sub(n)$.
Define a subset D of $\mathbb{N}$, such that

$$D = \{k \in \mathbb{N} : \ k \notin sub(k)\}$$

i.e. for $k \in \mathbb{N}$,

$$k \in D \equiv k \notin sub(k)$$

Note similarity with Russell Set, R, where
$R = \{x \,|\, x \notin x\}$
i.e. $x \in R \equiv x \notin x$.

Cantor's Thm. (Cont'd)

Since $D \subseteq \mathbb{N}$, i.e. $D \in P(\mathbb{N})$,
there is an element, $d \in \mathbb{N}$, such that $sub(d) = D$, $\therefore$.

$$d \in sub(d) \equiv d \in D$$

but from the definition of D,

$$d \in D \equiv d \notin sub(d)$$

and so $d \in sub(d) \equiv d \notin sub(d)$, a contradiction.
This contradiction arose due to assuming that $|\mathbb{N}| = |P(\mathbb{N})|$ $\therefore$.
$|\mathbb{N}| \neq |P(\mathbb{N})$.

$|(0,1)| = |P(\mathbb{N})|$

In Real Number Theory, the notation $(0,1)$ is used to denote the set of Real numbers between 0 and 1 i.e. $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$. The notation, $(0,1)$, denotes an **open interval**, i.e. the end points are not included while the notation $[0,1]$ denotes the **closed interval** that does include both end points.

Consider $x \in (0,1)$ in binary notation.
$0.5 \textit{ in decimal} = 0.1 \textit{ in binary}$
as $0.5 \textit{ in decimal} = 5 * \frac{1}{10} = \frac{1}{2}$ and $0.1 \textit{ in binary} = 1 * \frac{1}{2} = \frac{1}{2}$.
Every $x \in (0,1)$ can be written in binary as: $x = 0.b_0 b_1 b_2 \ldots$
where $b_i = 0 \textit{ or } 1$.

$|(0,1)| = |P(\mathbb{N})|$ Cont'd

$|(0,1)| = |P(\mathbb{N})|$

The 1-1 function $s : (0,1) \to P(\mathbb{N})$ is defined as follows. For every (binary) $x \in (0,1)$ where $x = 0.b_0 b_1 b_2 \ldots$ there corresponds exactly one subset, $s(x) \subseteq \mathbb{N}$, where, for $k \in \mathbb{N}$,

$$k \in s(x) \quad \textit{iff} \quad b_k = 1$$

Corollary: It can be shown that $|\mathbb{R}| = |(0,1)|$ and therefore $|\mathbb{R}| = |P(\mathbb{N})|$

$|(0,1)| = |P(\mathbb{N})|$ Cont'd

Eample $x = \frac{5}{8}$

In binary,

$$x = 0.10100\ldots$$

therefore

$$s(x) = \{0,2\}$$

Example.

$$x = 0.00100100\ldots$$

therefore

$$s(x) = \{2,5\}$$